CONCERNING IDEALS OF POINTFREE FUNCTION RINGS

by

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submitted in accordance with the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in the subject

MATHEMATICS

at the

UNIVERSITY OF SOUTH AFRICA

SUPERVISOR: PROF T A DUBE

NOVEMBER 2013

Declaration

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I declare that *Concerning Ideals of Pointfree Function Rings* is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

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Abstract

We study ideals of pointfree function rings. In particular, we study the lattices of z-ideals and d-ideals of the ring $\mathcal{R}L$ of continuous real-valued functions on a completely regular frame L. We show that the lattice of z-ideals is a coherently normal Yosida frame; and the lattice of d-ideals is a coherently normal frame. The lattice of z-ideals is demonstrated to be flatly projectable if and only if the ring $\mathcal{R}L$ is feebly Baer. On the other hand, the frame of d-ideals is projectable precisely when the frame is cozero-complemented.

These ideals give rise to two functors as follows: Sending a frame to the lattice of these ideals is a functorial assignment. We construct a natural transformation between the functors that arise from these assignments. We show that, for a certain collection of frame maps, the functor associated with z-ideals preserves and reflects the property of having a left adjoint.

A ring is called a UMP-ring if every maximal ideal in it is the union of the minimal prime ideals it contains. In the penultimate chapter we give several characterisations for the ring $\mathcal{R}L$ to be a UMP-ring. We observe, in passing, that if a UMP ring is a $\mathbb{Q}$-algebra, then each of its ideals when viewed as a ring in its own right is a UMP-ring. An example is provided to show that the converse fails.

Finally, piggybacking on results in classical rings of continuous functions, we show that, exactly as in $C(X)$, n^{th} roots exist in $\mathcal{R}L$. This is a consequence of an earlier proposition that every reduced f-ring with bounded inversion is the ring of fractions of its bounded part relative to those elements in the bounded part which are units in the bigger ring. We close with a result showing that the frame of open sets of the structure space of $\mathcal{R}L$ is isomorphic to βL.

Keywords: frame, ring of continuous functions, d-ideal, z-ideal, functor, f-ring.

Contents

Acknowledgements

I greatly acknowledge my advisor and promoter, Professor Themba Dube, for excellently guiding and advising me through the preparation of this thesis.

I express my sincere gratitude to Dr Enunuwe Ochonogor who advised me to register with the University of South Africa. I extend my gratitude to Prof Inderasan Naidoo, Prof Yorick Hardy, Dr Mack Matlabyana, Mr Charles Msipha, and Mr Soweto Kubeka for assisting me with regard to installation and usage of LaTeX. I would also like to express my gratitude to my fellow students at Unisa, especially Mrs Maggie Aphane and Mr Jissy Nsonde Nsayi. Further thanks go to Ms Busi Zwane for her excellent arrangements of seminars and colloquia. Mrs Stella Mugisha gave motherly support throughout my studies, and for that I thank her most heartily.

I also acknowledge the Department of Mathematical Sciences of the University of South Africa for financial assistance in the form of a contract position as a research assistant during my studies.

I also wish to express my sincere gratitude to my husband, Peter Ighedo, our children, Ikuesiri, Efuo and Precious for their patience for all the times that I have been away from them during the course of this study. I also want to express my profound gratitude to my mother who actually encouraged me to do Mathematics and took care of my children when I was away. Of course I will not forget my brothers and sisters, and my inlaws for all their support.

Finally I want to express my sincere gratitude to the Almighty God for His grace and favour upon me and my family.

Chapter 1

Introduction and preliminaries

1.1 A brief history on z-ideals and d-ideals

Throughout this thesis, the term "ring" means a commutative ring with identity. An ideal I of a ring A is a *z-ideal* if whenever two elements of A are in the same set of maximal ideals and I contains one of the elements, then it also contains the other. This algebraic definition of z-ideal was coined in the context of rings of continuous functions by Kohls in [49] and is also recorded as Problem 4A.5 in the text *Rings of Continuous Functions* by Gillman and Jerison.

A study of z-ideals in rings generally has been carried out by Mason in the article [58]. In pointfree topology, z-ideals were introduced by Dube in [30] where he showed that the algebraic definition agrees with the "topological" definition in terms of the cozero map. The study of lattices of z-ideals in $C(X)$, for compact Hausdorff spaces, was undertaken by Martínez and Zenk in [56]. They showed in this article that, for any compact Hausdorff space X, the lattice of z-ideals of $C(X)$ is a coherent normal Yosida frame when ordered by inclusion.

Let A be a ring, $a \in A$ and $S \subseteq A$. We denote the annihilator of S by $\mathrm{Ann}(S)$ or $S^{\perp}$, and the annihilator of the singleton $\{a\}$ is abbreviated as $\mathrm{Ann}(a)$ or $a^{\perp}$. Double annihilators will be written as $\mathrm{Ann}^2(S))$ or $S^{\perp\perp}$, and $\mathrm{Ann}^2(a))$ or $a^{\perp\perp}$. An ideal I of A is called a *d-ideal* if $a^{\perp\perp} \subseteq I$, for every $a \in I$. These ideals have been studied in rings by Mason [59]. Both z-ideals and d-ideals have been studied in the context of Riesz space in [44] and [45], and by de Pagter in [61].

1.2 Synopsis of the thesis

The thesis is mainly about the study of z-ideals and d-ideals of pointfree function rings and how they are related. It consists of seven chapters. Chapter 1 is introductory. It is where we present relevant definitions pertaining to frames and give relevant background for the other chapters.

In Chapter 2 we show that the lattice $\mathrm{Zid}(\mathcal{R}L)$ of z-ideals of $\mathcal{R}L$ is a normal coherent Yosida frame, which extends the corresponding $C(X)$ result of Martínez and Zenk in [56]. This we do by exhibiting $\mathrm{Zid}(\mathcal{R}L)$ as a quotient of $\mathrm{Rad}(\mathcal{R}L)$, the frame of radical ideals of $\mathcal{R}L$. The saturation quotient of $\mathrm{Zid}(\mathcal{R}L)$ is shown to be isomorphic to the Stone-Čech compactification of L. Given a morphism $h\colon L \to M$ in **CRegFrm**, Zid creates a coherent frame homomorphism $\mathrm{Zid}(h)\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$ whose right adjoint maps as $(\mathcal{R}h)^{-1}$, for the induced ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$. Thus, $\mathrm{Zid}(h)$ is an s-map, in the sense of Martìnez [54], precisely when $\mathcal{R}(h)$ contracts maximal ideals to maximal ideals.

In Chapter 3 we let A be a reduced commutative f-ring with identity and bounded inversion, and A^* its subring of bounded elements. We establish that the frame of d-ideals of A is a coherent frame. By first observing that A is the ring of fractions of A^* relative to the subset of A^* consisting of elements which are units in the bigger ring, we show that the frames $\mathrm{Did}(A)$ and $\mathrm{Did}(A^*)$ of d-ideals of A and A^*, respectively, are isomorphic, and that the isomorphism witnessing this is precisely the restriction of the extension map $I \mapsto I^e$ which takes a radical ideal of A^* to the ideal it generates in A. We also show that the extension of any d-ideal of A^* is a d-ideal of A and the contraction of any d-ideal of A is a d-ideal of A^*.

In Chapter 4 we give characterisations of d-ideals of $\mathcal{R}L$. We show that the frame $\mathrm{Did}(\mathcal{R}L)$ is a quotient of $\mathrm{Zid}(\mathcal{R}L)$. We observe that, for any coherent frame M, the d-nucleus on M is codense precisely if the only dense compact element of M is the top. As a consequence, the d-nucleus on $\mathrm{Zid}(\mathcal{R}L)$ is codense if and only if L is an almost P-frame. Hence we deduce that the frame $\mathrm{Did}(\mathcal{R}L)$ is compact if L is an almost P-frame. We show that if L is a quasi F-frame, then the saturation quotient of $\mathrm{Did}(\mathcal{R}L)$ is isomorphic to βL.

We also give results on some commutative squares associated with d-ideals.

Also investigated are projectability properties of $\mathrm{Did}(\mathcal{R}L)$ and $\mathrm{Zid}(\mathcal{R}L)$. We give characterisations for when $\mathrm{Did}(\mathcal{R}L)$ is flatly projectable. We show that $\mathrm{Zid}(\mathcal{R}L)$ is flatly projectable precisely when $\mathcal{R}L$ is a feebly Baer ring. Quite easily, $\mathrm{Zid}(\mathcal{R}L)$ is projectable if and only if L is basically disconnected. Less obvious is that $\mathrm{Did}(\mathcal{R}L)$ is projectable if and only if L is cozero-complemented.

In Chapter 5 we show that the assignments $L \mapsto \mathrm{Zid}(\mathcal{R}L)$ and $L \mapsto \mathrm{Did}(\mathcal{R}L)$ are functorial from **CRegFrm** to **CohFrm**. Writing δ_L for the frame homomorphism $\mathrm{Zid}(\mathcal{R}L) \to \mathrm{Did}(\mathcal{R}L)$ induced by the d-nucleus on $\mathrm{Zid}(\mathcal{R}L)$, we show that $L \mapsto \delta_L$ defines a natural transformation between these functors. Both functors preserve and reflect skeletality and $*$-density. A λ-map is a morphism $h\colon L \to M$ in **CRegFrm** with the property that $h^\lambda \cdot (\lambda_L)_* = (\lambda_M)_* \cdot h$, where $h^\lambda\colon \lambda L \to \lambda M$ is the lift of h to the Lindelöf coreflections. We prove that, for λ-maps, the functor induced by z-ideals preserves and reflects openness.

In Chapter 6 we define nearly round quotient maps, and use them to characterise completely regular frames L for which every maximal ideal of $\mathcal{R}L$ is the union of the minimal prime ideals it contains. All such frames are almost P-frames, and an Oz-frame is of this kind precisely if it is an almost P-frame. If L is perfectly normal (and hence if L is metrisable), then every maximal ideal of $\mathcal{R}L$ is the union of the minimal prime ideals it contains if and only if L is Boolean. Call a ring with the feature just stated a UMP-ring. We show that if A is a UMP-ring which is a $\mathbb{Q}$-algebra, then every ideal of A, when viewed as a ring in its own right, is a UMP-ring.

Chapter 7 consists of miscellaneous results. We show that if A is an f-ring with bounded inversion and every element of A^* (the bounded of A) has an n^{th} root in A^* for every odd $n \in \mathbb{N}$, then every element of A has an n^{th} root for every odd $n \in \mathbb{N}$. Also, if every positive element of A^* has an n^{th} root in A^* for every $n \in \mathbb{N}$, then every positive element of A has an n^{th} root in A for every $n \in \mathbb{N}$. Specialising to the ring $\mathcal{R}L$, we obtain a generalisation of a result of Banaschewski [11] which states that every positive element of $\mathcal{R}L$ is a square. We show that an ideal of $\mathcal{R}L$ is a z-ideal if and only if its radical is a z-ideal. We also show that an ideal of $\mathcal{R}L$ is a z-ideal if and only if every prime ideal minimal over it is a z-ideal.

Banaschewski has shown that every radical ideal of $\mathcal{R}L$ is absolutely convex. His proof, which is recorded in [32, Lemma 3.5], uses the theory of uniform frames. We give a different, purely algebraic proof of this result. We also show that the frame $\mathfrak{O}(\mathrm{Max}\,\mathcal{R}L)$ of open sets of the space $\mathrm{Max}\,\mathcal{R}L$ with the Zariski topology is isomorphic to βL. This we show by actually constructing a frame isomorphism $\beta L \to \mathfrak{O}(\mathrm{Max}\,\mathcal{R}L)$.

1.3 Frames and their homomorphisms

In this section we collect a few facts about frames and their homomorphisms, and also recall how the ring of continuous real-valued functions on a frame is constructed. Our reference for frames are [46], [62] and [63]; and our reference for the ring $\mathcal{R}L$, the ring of continuous real-valued functions on a frame L, are [6] and [11].

A *frame* is a complete lattice L in which the distributive law

$$a \wedge \bigvee S = \bigvee \{a \wedge x \mid x \in S\}$$

holds for all $a \in L$ and $S \subseteq L$. We denote the top element and the bottom element of L by 1_L and 0_L respectively, dropping the subscripts if L is clear from the context. An example is the frame of open subsets of a topological space X, which is denoted by $\mathfrak{O}X$. The *closed quotient* (resp. *open quotient*) of a frame L induced by $a \in L$ is the frame ${\uparrow}a$ (resp. ${\downarrow}a$).

A *frame homomorphism* is a map $h\colon L \to M$ between frames which preserves all joins and all finite meets. Every frame homomorphism $h\colon L \to M$ has a *right adjoint*, denoted h_*. It is given by

$$h_*(b) = \bigvee \{a \in L \mid h(a) \leq b\}.$$

An element a of L is said to be *rather below* an element b, written $a \prec b$, in case there is an element s, called a *separating* element, such that $a \wedge s = 0$ and $s \vee b = 1$. On the other hand, a is *completely below* b, written $a \prec\!\!\prec b$, if there are elements (x_r) indexed by rational numbers $\mathbb{Q} \cap [0, 1]$ such that $a = x_0, x_1 = b$ and $x_r \prec x_s$ for $r < s$. The frame L is said to be *regular* if $a = \bigvee \{x \in L \mid x \prec a\}$ for each $a \in L$, and *completely regular* if $a = \bigvee \{x \in L \mid x \prec\!\!\prec a\}$ for each $a \in L$. A frame L is *normal* if for any elements $a, b \in L$ such that $a \vee b = 1$, there are elements $c, d \in L$ such that $c \wedge d = 0$ and $a \vee c = 1 = b \vee d$.

An element c of a frame L is said to be *compact* if for any $S \subseteq L$, $c \leq \bigvee S$ implies $c \leq \bigvee T$, for some finite $T \subseteq S$. If the top of L is compact we say the frame itself is compact. We denote the set of all compact elements of L by $\mathfrak{K}(L)$. A frame is *algebraic* if it is generated by its compact elements. If L is a compact algebraic frame such that $a \wedge b \in \mathfrak{K}(L)$ for all $a, b \in \mathfrak{K}(L)$, then L is called *coherent*. A frame homomorphism $h \colon L \to M$ between coherent frames is called *coherent* in case it takes compact elements to compact elements.

By a *point* of L we mean an element $p < 1$ such that $x \wedge y \leq p$ implies $x \leq p$ or $y \leq p$. Points of a frame are also called *prime elements*. The points of any regular frame L are precisely those elements which are maximal in the poset $L \setminus \{1\}$. We denote the set of all points of L by $\mathrm{Pt}(L)$. A frame *has enough points* if every element is a meet of points above it. Every compact regular frame has enough points if one assumes (as we do throughout) the Prime Ideal Theorem. Frames that have enough points are also said to be *spatial*. In [56] a frame is called a *Yosida frame* if each of its compact elements is a meet of maximal elements.

We regard the Stone-Čech compactification of L, denoted βL, as the frame of completely regular ideals of L. We denote the right adjoint of the join map $j_L \colon \beta L \to L$ by r_L, and recall that $r_L(a) = \{x \in L \mid x \prec\!\!\prec a\}$.

1.4 Rings and f-rings

As stated in the Introduction, all rings are assumed to be commutative with identity. Let A be a ring. The *annihilator* of $S \subseteq A$ is the ideal

$$\mathrm{Ann}(S) = \{a \in A \mid as = 0 \text{ for every } s \in S\}.$$

A ring A is called a *Gelfand ring* if $a+b = 1$ in A implies that $(1+ar)(1+bs) = 0$ for some $r, s \in A$. The *Jacobson radical* of A, denoted $\mathrm{Jac}(A)$, is the intersection of all maximal ideals of A. We shall write $\mathrm{Max}(A)$ for the set of all maximal ideals of A. For an ideal I of A we write

$$\mathfrak{M}(I) = \{M \in \mathrm{Max}(A) \mid M \supseteq I\},$$

and abbreviate $\mathfrak{M}(\{a\})$ as $\mathfrak{M}(a)$. The Zariski topology on $\mathrm{Max}(A)$ is defined as follows. For each ideal I of A let

$$\mathfrak{M}'(I) = \{M \in \mathrm{Max}(A) \mid M \not\supseteq I\}.$$

Then the *Zariski topology* on $\mathrm{Max}(A)$ is topology given by

$$\mathfrak{O}(\mathrm{Max}(A)) = \{\mathfrak{M}'(I) \mid I \text{ is an ideal of } A\},$$

with the understanding that the improper ideal is included in this defining condition.

An *f-ring* is a lattice-ordered ring A in which the identity

$$(a \wedge b)c = (ac) \wedge (bc)$$

holds for all $a, b \in A$ and $c \geq 0$ in A.

1.5 The rings $\mathcal{R}L$ and $\mathcal{R}^*L$

Pointfree function rings can be studied starting with $\mathfrak{O}\mathbb{R}$, as in [6], or, as in [11], starting with the *frame of reals* $\mathfrak{L}(\mathbb{R})$. We follow the latter approach. The frame $\mathfrak{L}(\mathbb{R})$ is defined by generators, which are pairs (p, q) of rationals, and the relations (R1) through (R4) below:

(R1) $(p, q) \wedge (r, s) = (p \vee r, q \wedge s)$,

(R2) $(p, q) \vee (r, s) = (p, s)$ whenever $p \leq r < q \leq s$,

(R3) $(p, q) = \bigvee\{(r, s) \mid p < r < s < q\}$,

(R4) $1_{\mathfrak{L}(\mathbb{R})} = \bigvee\{(p, q) \mid p, q \in \mathbb{Q}\}$.

A *continuous real-valued function* on L is a frame homomorphism $\mathfrak{L}(\mathbb{R}) \to L$. The ring $\mathcal{R}L$ has as its elements continuous real-valued functions on L, with operations determined by the operations of $\mathbb{Q}$ viewed as a lattice-ordered ring as follows:

For $\diamond \in \{+, \cdot, \wedge, \vee\}$ and $\alpha, \beta \in \mathcal{R}L$,

$$\alpha \diamond \beta = \bigvee\{\alpha(r, s) \wedge \beta(t, u) \mid \langle r, s\rangle \diamond \langle t, u\rangle \subseteq \langle p, q\rangle\},$$

where $\langle\cdot,\cdot\rangle$ denotes the open interval in $\mathbb{Q}$, and the given condition means that $x \diamond y \in \langle p, q\rangle$ for any $x \in \langle r, s\rangle$ and $y \in \langle t, u\rangle$.

For any $\alpha \in \mathcal{R}L$ and $p, q \in \mathbb{Q}$,

$$(-\alpha)(p,q) = \alpha(-q,-p),$$

and for any $r \in \mathbb{R}$ the *constant function* $\mathbf{r}$ is the member of $\mathcal{R}L$ given by

$$\mathbf{r}(p,q) = \begin{cases} 1 & \text{if } p < r < q \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathcal{R}L$ becomes a reduced archimedean f-ring with bounded inversion. Furthermore, the correspondence $L \mapsto \mathcal{R}L$ is functorial, where, for any frame homomorphism $h\colon L \to M$, the ℓ-ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ is given by $\mathcal{R}h(\alpha) = h \cdot \alpha$; the centre dot designating composition.

An important link between a frame and its ring of real-valued continuous functions is given by the *cozero map* $\operatorname{coz} : \mathcal{R}L \to L$ defined by

$$\operatorname{coz}\varphi = \bigvee\{\varphi(p,0) \vee \varphi(0,q) \mid p, q \in \mathbb{Q}\} = \varphi\big((-,0) \vee (0,-)\big),$$

where, for any $r \in \mathbb{Q}$,

$$(-,r) = \bigvee\{(p,r) \mid p < r \text{ in } \mathbb{Q}\} \qquad \text{and} \qquad (r,-) = \bigvee\{(r,q) \mid q > r \text{ in } \mathbb{Q}\}.$$

The cozero map has several known properties (see [6] and [11]) that we shall use freely.

When we say an element $\alpha \in \mathcal{R}L$ is *positive*, we shall mean that $\alpha \geq \mathbf{0}$. An element of $\mathcal{R}L$ is *bounded* if there exist $p, q \in \mathbb{Q}$ such that $\alpha(p,q) = 1_L$. The subring of $\mathcal{R}L$ consisting of bounded elements is denoted by $\mathcal{R}^*L$. For any topological space X, the rings $C(X)$ and $\mathcal{R}(\mathfrak{O}X)$ are isomorphic.

A ring is said to be *reduced* if it has no nonzero nilpotent element. An f-ring has *bounded inversion* if every $a \geq 1$ is invertible. It is shown in [11] that $\mathcal{R}L$ is a reduced f-ring with bounded inversion. Every frame homomorphism $h\colon L \to M$ induces a ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ which sends an element α of $\mathcal{R}L$ to the composite $h \cdot \alpha$. Furthermore, $\operatorname{coz}(h \cdot \alpha) = h(\operatorname{coz}\alpha)$.

A *cozero element* of L is an element of the form $\operatorname{coz}\alpha$ for some $\alpha \in \mathcal{R}L$. The *cozero part* of L, denoted $\operatorname{Coz} L$, is the regular sub-σ-frame consisting of all the cozero elements of L. General properties of $\operatorname{Coz} L$ can be found in [20].

We recall from [33] the following types of ideals of $\mathcal{R}L$. For any $I \in \beta L$, put

$$\boldsymbol{M}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\},$$

and observe that $\boldsymbol{M}^I$ is an ideal of $\mathcal{R}L$ which is proper if and only if $I \neq 1_{\beta L}$. For any $a \in L$ we abbreviate $\boldsymbol{M}^{r_L(a)}$ as $\boldsymbol{M}_a$, and observe that

$$\boldsymbol{M}_a = \{\alpha \in \mathcal{R}L \mid \operatorname{coz}\alpha \leq a\}.$$

Since, for any $I, J \in \beta L$, $\boldsymbol{M}^I = \boldsymbol{M}^J$ implies $I = J$ (see proof of [33, Lemma 4.15]), it follows that, for any $a, b \in L$, $\boldsymbol{M}_a = \boldsymbol{M}_b$ if and only if $a = b$. Maximal ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}^I$, for $I \in \operatorname{Pt}(\beta L)$. For any prime ideal P, there is a unique point I of βL such that $\boldsymbol{O}^I \subseteq P \subseteq \boldsymbol{M}^I$ (see [33]).

Chapter 2

The frame of z-ideals of $\mathcal{R}L$

Our aim in this chapter is to study z-ideals of the ring $\mathcal{R}L$. We remark that z-ideals of $\mathcal{R}L$ already appear in [33]. In this paper the author defines z-ideals by means of the cozero map, and then shows that his definition agrees with the algebraic definition of Mason's [58]. In the first section of this chapter we recall the algebraic definition of z-ideals by Mason, and give a different proof to that in [33] that these ideals are characterisable in terms of the cozero map.

2.1 Characterisation of z-ideals of $\mathcal{R}L$

Following [58] we define for any $a \in A$ the set

$$\mathfrak{M}(a) = \{M \in \operatorname{Max} A \mid a \in M\}.$$

An ideal I of A is called a *z-ideal* in case for any $a, b \in A$, $a \in I$ and $\mathfrak{M}(a) = \mathfrak{M}(b)$ imply $b \in I$. This is equivalent to saying $a \in I$ and $\mathfrak{M}(a) \supseteq \mathfrak{M}(b)$ imply $b \in I$. It is apposite to remark that although a number of authors seem to attribute this definition to Mason (emanating from his paper [58]), it already appears in Kohl's paper [49].

As mentioned above, Dube [33] showed that an ideal Q of $\mathcal{R}L$ is a z-ideal if and only if for any $\alpha, \beta \in \mathcal{R}L$, $\alpha \in Q$ and $\operatorname{coz}\alpha = \operatorname{coz}\beta$ imply $\beta \in Q$. Here we give an alternative proof which brings to the fore a number of other noteworthy observations. We start with the following lemma.

Lemma 2.1.1. *Let $\{I_\lambda \mid \lambda \in \Lambda\}$ be a subset of βL. Then*

$$\bigcap_\lambda \boldsymbol{M}^{I_\lambda} = \boldsymbol{M}^I,$$

where $I = \bigwedge_\lambda I_\lambda$.

Proof. For any $\alpha \in \mathcal{R}L$ we have

$$\begin{aligned} \alpha \in \bigcap_\lambda \boldsymbol{M}^{I_\lambda} &\iff \alpha \in \boldsymbol{M}^{I_\lambda} \text{ for every } \lambda \\ &\iff r_L(\operatorname{coz}\alpha) \leq I_\lambda \text{ for every } \lambda \\ &\iff r_L(\operatorname{coz}\alpha) \leq \bigwedge_\lambda I_\lambda \\ &\iff \alpha \in \boldsymbol{M}^I, \end{aligned}$$

which proves the result. □

Next we recall the following notation. Let A be a ring and $a \in A$. The ideal $M(a)$ is defined by

$$M(a) = \bigcap\{N \in \operatorname{Max}(A) \mid a \in N\}.$$

In the case of $\mathcal{R}L$, the ideals $M(\alpha)$ are expressible in terms of $\boldsymbol{M}$-ideals as shown in the following result.

Corollary 2.1.1. *An ideal of $\mathcal{R}L$ is an intersection of maximal ideals iff it is of the form $\boldsymbol{M}^I$, for some $I \in \beta L$.*

Proof. That an intersection of maximal ideals is of the form $\boldsymbol{M}^I$ for some $I \in \beta L$ follows immediately from the preceding lemma. On the other hand, let $I \in \beta L$. If $I = 1_{\beta L}$, then $\boldsymbol{M}^I = \mathcal{R}L$, so that it is the empty meet of maximal ideals. So suppose $I < 1_{\beta L}$. Since βL is spatial, $I = \bigwedge\{J \in \operatorname{Pt}(\beta L) \mid I \leq J\}$. Thus, by Lemma 2.1.1,

$$\boldsymbol{M}^I = \bigcap\{\boldsymbol{M}^J \mid J \in \operatorname{Pt}(\beta L) \text{ and } J \geq I\},$$

an intersection of maximal ideals. □

Lemma 2.1.2. *For any $a, b \in L$, $\boldsymbol{M}_a \subseteq \boldsymbol{M}_b$ if and only if $a \leq b$.*

Proof. The "only if" part is trivial. Concerning the "if" part, observe that, by complete regularity, we have

$$\begin{aligned} a &= \bigvee\{c \in \operatorname{Coz} L \mid c \le a\} \\ &= \bigvee\{\operatorname{coz}\gamma \mid \operatorname{coz}\gamma \le a\} \\ &= \bigvee\{\operatorname{coz}\gamma \mid \gamma \in \boldsymbol{M}_a\} \\ &\le \bigvee\{\operatorname{coz}\gamma \mid \gamma \in \boldsymbol{M}_b\} \\ &= \bigvee\{\operatorname{coz}\gamma \mid \operatorname{coz}\gamma \le b\} \\ &= b, \end{aligned}$$

which establishes the result. □

Corollary 2.1.2. *For any $a, b \in L$, $\boldsymbol{M}_a = \boldsymbol{M}_b$ if and only if $a = b$.*

One last observation before we state the desired characterisation.

Lemma 2.1.3. *For any $\alpha, \beta \in \mathcal{R}L$, the following are equivalent:*

(1) $\mathfrak{M}(\alpha) = \mathfrak{M}(\beta)$.

(2) $\boldsymbol{M}_{\operatorname{coz}\alpha} = \boldsymbol{M}_{\operatorname{coz}\beta}$.

(3) $\operatorname{coz}\alpha = \operatorname{coz}\beta$.

Proof. The implication (1) ⇒ (2) and the equivalence of (2) and (3) follow from what has gone before. To show that (3) implies (1), let $\boldsymbol{M}^I$, I a point of $\operatorname{Pt}(\beta L)$, be a maximal ideal of $\mathcal{R}L$ containing α. Then $r_L(\operatorname{coz}\alpha) \subseteq I$, which implies $r_L(\operatorname{coz}\beta) \subseteq I$, which further implies $\beta \in \boldsymbol{M}^I$. Therefore $\mathfrak{M}(\alpha) \subseteq \mathfrak{M}(\beta)$, and hence equality by symmetry. □

Lemma 2.1.4. *The following are equivalent for an ideal Q of $\mathcal{R}L$.*

(1) *Q is a z-ideal.*

(2) *For any $\alpha, \beta \in \mathcal{R}L$, $\alpha \in Q$ and $\operatorname{coz}\alpha = \operatorname{coz}\beta$ imply $\beta \in Q$.*

(3) *For any $\alpha, \beta \in \mathcal{R}L$, $\alpha \in Q$ and $\operatorname{coz}\beta \le \operatorname{coz}\alpha$ imply $\beta \in Q$.*

(4) $Q = \bigcup\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \alpha \in Q\}$.

Proof. The equivalence of (1) and (2) is shown in [30, Corollary 3.8].

(2) $\Rightarrow$ (3): Assume $\alpha \in Q$ and $\operatorname{coz}\beta \leq \operatorname{coz}\alpha$. Then $\operatorname{coz}\beta = \operatorname{coz}\alpha \wedge \operatorname{coz}\beta = \operatorname{coz}(\alpha\beta)$. Since Q is an ideal and $\alpha \in Q$, we have that $\alpha\beta \in Q$. Therefore, by (2), $\beta \in Q$.

(3) $\Rightarrow$ (4): Clearly $Q \subseteq \bigcup\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \alpha \in Q\}$ because for any $\tau \in \mathcal{R}L, \tau \in \boldsymbol{M}_{\operatorname{coz}\tau}$. To see the reverse inclusion let $\alpha \in Q$, and consider any $\beta \in \boldsymbol{M}_{\operatorname{coz}\alpha}$. This means $\operatorname{coz}\beta \leq \operatorname{coz}\alpha$, so that, by (3), $\beta \in Q$ showing that $\boldsymbol{M}_{\operatorname{coz}\alpha} \subseteq Q$, and hence the desired inclusion.

(4) $\Rightarrow$ (2): Let $\alpha \in Q$ and β be an element of $\mathcal{R}L$ with $\operatorname{coz}\alpha = \operatorname{coz}\beta$. Then

$$\beta \in \boldsymbol{M}_{\operatorname{coz}\beta} = \boldsymbol{M}_{\operatorname{coz}\alpha} \subseteq Q,$$

and hence (2) follows. $\square$

For any ideal Q of $\mathcal{R}L$, denote by Q_z the z-ideal

$$Q_z = \bigcup\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \alpha \in Q\}.$$

We observe that Q_z is the smallest z-ideal containing Q. Indeed, suppose H is a z-ideal containing Q. Then for any $\alpha \in Q$ and $\tau \in \boldsymbol{M}_{\mathrm{COZ}\,\alpha}, \operatorname{coz}\tau \leq \operatorname{coz}\alpha$, and hence $\tau \in H$ since $\alpha \in H$ and H is a z-ideal. Therefore $Q_z \subseteq H$.

Remark 2.1.1. As an aside, we use Lemma 2.1.1 to observe that the Jacobson radical of $\mathcal{R}L$ is the zero ideal. Indeed, in light of βL being spatial, $\bigwedge \mathrm{Pt}(\beta L) = 0_{\beta L}$, and hence

$$\mathrm{Jac}(\mathcal{R}L) = \bigcap\{\boldsymbol{M}^I \mid I \in \mathrm{Pt}(\beta L)\} = \boldsymbol{M}^{0_{\beta L}} = \{\mathbf{0}\}.$$

2.2 The frame Zid($\mathcal{R}L$)

For any completely regular frame L, we denote by Zid($\mathcal{R}L$) the lattice of z-ideals of $\mathcal{R}L$ partially ordered by inclusion. We shall establish some preliminary results and hence show that Zid($\mathcal{R}L$) is a normal coherent Yosida frame. Recall that a *nucleus* on a frame L is a closure operator $\ell : L \to L$ such that $\ell(a \wedge b) = \ell(a) \wedge \ell(b)$ for all $a, b \in L$. The set

$$\mathrm{Fix}(\ell) = \{a \in L \mid \ell(a) = a\}$$

is a frame with meet as in L and join given by

$$\bigvee_{\mathrm{Fix}(\ell)} S = \ell\Big(\bigvee S\Big)$$

for every $S \subseteq \mathrm{Fix}(\ell)$. Further, the map $L \to \mathrm{Fix}(\ell)$, effected by ℓ, is a surjective frame homomorphism. Banaschewski shows in [8, Lemma 2] that if L is compact and ℓ is codense (meaning that it sends only the top to the top), then $\mathrm{Fix}(\ell)$ is also compact. Recall that an ideal I of a ring A is a *radical* ideal if for any $a \in A, a^2 \in I$ implies $a \in I$. As usual, we let $\mathrm{Rad}(\mathcal{R}L)$ denote the frame of radical ideals of $\mathcal{R}L$. Since $\mathcal{R}L$ is a Gelfand ring in the sense of [12], $\mathrm{Rad}(\mathcal{R}L)$ is a normal coherent frame (see [12]). We prove that $\mathrm{Zid}(\mathcal{R}L)$ is a normal coherent Yosida frame by showing that $\mathrm{Zid}(\mathcal{R}L) = \mathrm{Fix}(z)$, where $z\colon \mathrm{Rad}(\mathcal{R}L) \to \mathrm{Rad}(\mathcal{R}L)$ denotes the z-nucleus on $\mathrm{Rad}(\mathcal{R}L)$. Let us recall the pertinent definitions from [55]. Observe that every z-ideal is a radical ideal.

We remark that z-ideals of $\mathcal{R}L$ are, in the language of Martínez and Zenk [55], precisely the z-elements of the frame $\mathrm{Rad}(\mathcal{R}L)$. We write $\mathrm{Max}(L)$ for the set of maximal elements of a frame L, and remind the reader that "maximal" is understood to mean maximal different from the top. Recall that, for an algebraic frame L, Martínez and Zenk define the *archimedean nucleus* $\mathrm{ar}\colon L \to L$ by

$$\mathrm{ar}(x) = \bigwedge\{m \in \mathrm{Max}(L) \mid x \le m\},$$

and the *z-nucleus* $z\colon L \to L$ by

$$z(x) = \bigvee\{\mathrm{ar}(c) \mid c \le x,\ c \text{ compact}\}.$$

Elements of $\mathrm{Fix}(z)$ are then called *z-elements* of L. As observed in [52, Definitions and Remarks 3.5],

$$x \in \mathrm{Fix}(z) \quad\Longleftrightarrow\quad \text{for every } c \in \mathfrak{K}(L),\ \mathrm{ar}(c) \le x \text{ whenever } c \le x.$$

We use this characterisation to show that z-ideals of $\mathcal{R}L$ are precisely the z-elements of $\mathrm{Rad}(\mathcal{R}L)$.

Lemma 2.2.1. $\mathrm{Zid}(\mathcal{R}L) = \mathrm{Fix}(z)$*, for the z-nucleus on* $\mathrm{Rad}(\mathcal{R}L)$.

Proof. Observe that, by Lemma 2.1.1, $\mathrm{ar}(\boldsymbol{M}_{\mathrm{coz}\,\alpha}) = \boldsymbol{M}_{\mathrm{coz}\,\alpha}$, for every $\alpha \in \mathcal{R}L$. Recall that the compact elements of $\mathrm{Rad}(\mathcal{R}L)$ are precisely the finitely generated radical ideals. Let

K be a compact element of $\operatorname{Rad}(\mathcal{R}L)$, generated by $\alpha_1, \ldots, \alpha_m$, say. For brevity, write $\alpha = \alpha_1^2 + \cdots + \alpha_m^2$, and note that $K \subseteq \boldsymbol{M}_{\operatorname{coz}\alpha}$, so that

$$\operatorname{ar}(K) \subseteq \operatorname{ar}(\boldsymbol{M}_{\operatorname{coz}\alpha}) = \boldsymbol{M}_{\operatorname{coz}\alpha}.$$

Next, let I be a point of βL with $K \subseteq \boldsymbol{M}^I$. Since $\alpha \in K$, we have $r_L(\operatorname{coz}\alpha) \subseteq I$, and hence $\boldsymbol{M}_{\operatorname{coz}\alpha} = \boldsymbol{M}^{r_L(\operatorname{coz}\alpha)} \subseteq \boldsymbol{M}^I$. Therefore

$$\boldsymbol{M}_{\operatorname{coz}\alpha} \subseteq \bigcap\{M \in \operatorname{Max}(\mathcal{R}L) \mid K \subseteq M\} = \operatorname{ar}(K),$$

and hence $\operatorname{ar}(K) = \boldsymbol{M}_{\operatorname{coz}\alpha}$.

Now if Q is a z-ideal and K a compact element of $\operatorname{Rad}(\mathcal{R}L)$ with $K \subseteq Q$, then, an argument as above shows that $\operatorname{ar}(K) \subseteq Q$. Therefore Q is a z-element of $\operatorname{Rad}(\mathcal{R}L)$. Conversely, suppose $Q \in \operatorname{Rad}(\mathcal{R}L)$ is a z-element. Let $\alpha \in Q$, and consider the radical ideal $[\alpha]$ of $\mathcal{R}L$ generated by α. It is a compact element of $\operatorname{Rad}(\mathcal{R}L)$ with $[\alpha] \subseteq Q$. Since Q is a z-element, we have $Q \supseteq \operatorname{ar}([\alpha]) = \boldsymbol{M}_{\operatorname{coz}(\alpha^2)} = \boldsymbol{M}_{\operatorname{coz}\alpha}$. But this implies Q is a z-ideal, by Lemma 2.1.4. □

Proposition 2.2.1. $\operatorname{Zid}(\mathcal{R}L)$ *is a normal coherent Yosida frame with*

$$\mathfrak{K}(\operatorname{Zid}(\mathcal{R}L)) = \{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \alpha \in \mathcal{R}L\}.$$

Proof. That $\operatorname{Zid}(\mathcal{R}L)$ is a normal coherent Yosida frame follows from the properties of the z-nucleus which are summarised in [54, Definition & Remarks 3.3.1]. Furthermore,

$$\mathfrak{K}(\operatorname{Zid}(\mathcal{R}L)) = z[\mathfrak{K}(\operatorname{Rad}(\mathcal{R}L))].$$

Now, for any $K \in \mathfrak{K}(\operatorname{Rad}(\mathcal{R}L))$, we have

$$z(K) = \bigvee\{\operatorname{ar}(T) \mid T \in \mathfrak{K}(\operatorname{Rad}(\mathcal{R}L)), T \leq K\} = \operatorname{ar}(K) = \boldsymbol{M}_{\operatorname{coz}\alpha},$$

for some $\alpha \in \mathcal{R}L$, as we observed in the foregoing proof. Also, for any $\beta \in \mathcal{R}L$,

$$\boldsymbol{M}_{\operatorname{coz}\alpha} = \operatorname{ar}([\beta]) = z([\beta]);$$

and so $\mathfrak{K}(\operatorname{Zid}(\mathcal{R}L)) = \{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \alpha \in \mathcal{R}L\}$. □

2.3 Zid($\mathcal{R}L$) is coherently normal.

We show next that Zid($\mathcal{R}L$) has a property which is a stronger version of normality. In [10], Banaschewski calls a frame L *coherently normal* if it is coherent and, for each compact $c \in L$, the frame $\downarrow c$ is normal. We show below that Zid($\mathcal{R}L$) is coherently normal. We will use [19, Lemma 1] which (paraphrased) states:

> *For any elements a and b of a σ-frame L, there exist u and v in L such that $u \wedge v = 0$ and $a \vee u = b \vee v = a \vee b$.*

In the proof of this result it is clear that $u \leq b$ and $v \leq a$. An algebraic frame L is said to have the *finite intersection property* (abbreviated FIP) if the meet of any two compact elements in L is compact. Martínez [52] says an algebraic frame L *has disjointification* – a property equivalent to coherent normality for algebraic frames with FIP – if for each pair of compact elements $a, b \in L$, there exist disjoint compact elements $c \leq a$ and $d \leq b$ in L with $a \vee b = a \vee d = b \vee c$, and remarks that if L has FIP, then it is coherently normal if and only if it has disjointification.

Let us observe an easy lemma for use in the upcoming result and later.

Lemma 2.3.1. *For any $c, d \in \operatorname{Coz} L$, $\boldsymbol{M}_c \vee \boldsymbol{M}_d = \boldsymbol{M}_{c\vee d}$.*

Proof. Clearly, $\boldsymbol{M}_{c\vee d}$ is an upper bound for the set $\{\boldsymbol{M}_c, \boldsymbol{M}_d\}$. Now let H be a z-ideal containing $\boldsymbol{M}_c$ and $\boldsymbol{M}_d$. Take positive $\alpha, \beta \in \mathcal{R}L$ with $\operatorname{coz}\alpha = c$ and $\operatorname{coz}\beta = d$. If $\gamma \in \boldsymbol{M}_{c\vee d}$, then $\operatorname{coz}\gamma \leq \operatorname{coz}(\alpha+\beta)$, implying $\gamma \in H$ as H is a z-ideal. □

Proposition 2.3.1. Zid($\mathcal{R}L$) *is coherently normal.*

Proof. Let $\alpha \in \mathcal{R}L$ and suppose $Q \vee R = \boldsymbol{M}_{\operatorname{coz}\alpha}$ for some $Q, R \in \mathrm{Zid}(\mathcal{R}L)$. Thus

$$\boldsymbol{M}_{\operatorname{coz}\alpha} = \bigvee_{\tau\in Q} \boldsymbol{M}_{\operatorname{coz}\tau} \vee \bigvee_{\rho\in R} \boldsymbol{M}_{\operatorname{coz}\rho},$$

so that, by compactness, there exist $\gamma \in Q$ and $\delta \in R$ such that $\boldsymbol{M}_{\operatorname{coz}\alpha} = \boldsymbol{M}_{\operatorname{coz}\gamma} \vee \boldsymbol{M}_{\operatorname{coz}\delta}$. Consequently, $\operatorname{coz}\alpha = \operatorname{coz}\gamma \vee \operatorname{coz}\delta$. By the result quoted above from [19], there exist $\mu, \nu \in \mathcal{R}L$ such that $\operatorname{coz}\mu \leq \operatorname{coz}\gamma$, $\operatorname{coz}\nu \leq \operatorname{coz}\delta$ and

$$\operatorname{coz}\mu \wedge \operatorname{coz}\nu = 0 \quad \text{and} \quad \operatorname{coz}\gamma \vee \operatorname{coz}\nu = \operatorname{coz}\delta \vee \operatorname{coz}\mu = \operatorname{coz}\gamma \vee \operatorname{coz}\delta.$$

Since Q and R are z-ideals, $\boldsymbol{M}_{\operatorname{coz}\mu} \subseteq Q$ and $\boldsymbol{M}_{\operatorname{coz}\nu} \subseteq R$. Thus, $\boldsymbol{M}_{\operatorname{coz}\mu}$ and $\boldsymbol{M}_{\operatorname{coz}\nu}$ are elements of $\operatorname{Zid}(\mathcal{R}L)$ which witness the normality of $\downarrow\boldsymbol{M}_{\operatorname{coz}\alpha}$. $\square$

For what follows, we first recall the definition of the *saturation nucleus*, $s_L\colon L \to L$, on a compact frame L (see, for instance, [10] or [54]). For any $x, a \in L$, the element x is said to be *a-small* if, for any $y \in L$, $x \vee y = 1$ implies $a \vee y = 1$. The map s_L is then defined by

$$s_L(a) = \bigvee\{x \in L \mid x \text{ is } a\text{-small}\}.$$

As is tradition, we will write SL for the frame $\operatorname{Fix}(s_L)$. When confusion is unlikely, we will drop the subscript on the nucleus s_L.

It is known that for any coherent frame L and any $x \in L$

$$x = s(x) \qquad \Longleftrightarrow \qquad x = \bigwedge\{m \in \operatorname{Max}(L) \mid x \le m\}.$$

It follows therefore from Corollary 2.1.1 that the saturation of $\operatorname{Rad}(\mathcal{R}L)$ is

$$S\big(\operatorname{Rad}(\mathcal{R}L)\big) = \{\boldsymbol{M}^I \mid I \in \beta L\}.$$

Since $\operatorname{Rad}(\mathcal{R}L)$ and $\operatorname{Zid}(\mathcal{R}L)$ have exactly the same maximal elements, we deduce immediately that

$$S\big(\operatorname{Rad}(\mathcal{R}L)\big) = S\big(\operatorname{Zid}(\mathcal{R}L)\big).$$

Now observe that the map $I \mapsto \boldsymbol{M}^I$ is a frame homomorphism from βL into $S(\operatorname{Rad}(\mathcal{R}L)$. As remarked in the Preliminaries, this map is one-one. It is also clearly onto. We therefore have the following result.

Proposition 2.3.2. *$S(\operatorname{Zid}(\mathcal{R}L)) = S(\operatorname{Rad}(\mathcal{R}L)) \cong \beta L$.*

Remark 2.3.1. For any Gelfand ring A, let $\operatorname{JRad}(A)$ be the frame of its Jacobson radical ideals. Banaschewski [12] observes that $\operatorname{JRad}(A) = S(\operatorname{Rad}(A))$, for any Gelfand ring A. The foregoing proposition can therefore also be deduced from the work of Banaschewski and Sioen [25] in which they show that the frame $\operatorname{JRad}(\mathcal{R}L)$ of Jacobson radical ideals of $\mathcal{R}L$ is the compact completely regular coreflection of L.

We close this section by investigating when the frame $\mathrm{Zid}(\mathcal{R}L)$ is regular. We will use Banaschewski's characterisation [10, Lemma1.5] of regularity in normal coherent frames. Recall that a *P-frame* is one in which every cozero element is complemented. Throughout, the top and bottom elements of $\mathrm{Zid}(\mathcal{R}L)$ will be denoted by $\top$ and $\bot$, respectively. They are, of course, the ideal L and the zero ideal.

Proposition 2.3.3. $\mathrm{Zid}(\mathcal{R}L)$ *is regular if and only if* L *is a* P*-frame.*

Proof. Suppose L is P-frame, and consider any positive $\alpha \in \mathcal{R}L$. Since L is a P-frame, there exists a positive $\beta \in \mathcal{R}L$ such that $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = 0$ and $\operatorname{coz}\alpha \vee \operatorname{coz}\beta = 1$. Then $\boldsymbol{M}_{\operatorname{coz}\alpha} \wedge \boldsymbol{M}_{\operatorname{coz}\beta} = \{\mathbf{0}\}$ and $\boldsymbol{M}_{\operatorname{coz}\alpha} \vee \boldsymbol{M}_{\operatorname{coz}\beta} = \boldsymbol{M}_{\operatorname{coz}(\alpha+\beta)} = \top$. Therefore every compact element of $\mathrm{Zid}(\mathcal{R}L)$ is complemented, hence $\mathrm{Zid}(\mathcal{R}L)$ is regular in view of [10, Lemma 1.5].

Conversely, Suppose $\mathrm{Zid}(\mathcal{R}L)$ is regular. Then, by [10, Lemma 1.5] again, $\boldsymbol{M}_{\operatorname{coz}\alpha}$ is complemented in $\mathrm{Zid}(\mathcal{R}L)$, for any positive $\alpha \in \mathcal{R}L$. Pick $Q \in \mathrm{Zid}(\mathcal{R}L)$ such that

$$\boldsymbol{M}_{\operatorname{coz}\alpha} \wedge Q = \{\mathbf{0}\} \quad \text{and} \quad \boldsymbol{M}_{\operatorname{coz}\alpha} \vee Q = \top.$$

The latter implies

$$\boldsymbol{M}_{\operatorname{coz}\alpha} \vee \bigvee \{\boldsymbol{M}_{\operatorname{coz}\gamma} \mid \gamma \in Q\} = \top,$$

so that, by compactness, there is a positive $\beta \in Q$ such that

$$\top = \boldsymbol{M}_{\operatorname{coz}\alpha} \vee \boldsymbol{M}_{\operatorname{coz}\beta} = \boldsymbol{M}_{\operatorname{coz}\alpha \vee \operatorname{coz}\beta} = \boldsymbol{M}_{\operatorname{coz}(\alpha+\beta)},$$

which implies $\operatorname{coz}\alpha \vee \operatorname{coz}\beta = 1$. But now $\boldsymbol{M}_{\operatorname{coz}\alpha} \wedge \boldsymbol{M}_{\operatorname{coz}\beta} = \{\mathbf{0}\}$, since $\boldsymbol{M}_{\operatorname{coz}\beta} \subseteq Q$, and so $\operatorname{coz}\alpha \wedge \operatorname{coz}\beta = 0$, showing that $\operatorname{coz}\beta$ is a complement of $\operatorname{coz}\alpha$. Therefore L is a P-frame. □

Since a frame L is a P-frame if and only if every ideal of $\mathcal{R}L$ is a z-ideal [30, Proposition 3.9], we have the following corollary.

Corollary 2.3.1. *If* $\mathrm{Zid}(\mathcal{R}L)$ *is regular, then* $\mathrm{Zid}(\mathcal{R}L) = \mathrm{Rad}(\mathcal{R}L)$.

2.4 Some commutative squares associated with z-ideals.

Given a completely regular frame L, we wish to establish a frame map $\sigma_L\colon \mathrm{Zid}(\mathcal{R}L) \to L$ in such a way that, for any frame homomorphism $h\colon L \to M$, the wedge

$$\begin{tikzcd}
\mathrm{Zid}(\mathcal{R}L) \arrow[d, "\sigma_L"'] & \mathrm{Zid}(\mathcal{R}M) \arrow[d, "\sigma_M"] \\
L \arrow[r, "h"'] & M
\end{tikzcd} \tag{2.1}$$

is completable to a commutative square

$$\begin{tikzcd}
\mathrm{Zid}(\mathcal{R}L) \arrow[r, "\mathrm{Zid}(h)"] \arrow[d, "\sigma_L"'] & \mathrm{Zid}(\mathcal{R}M) \arrow[d, "\sigma_M"] \\
L \arrow[r, "h"'] & M
\end{tikzcd} \tag{‡}$$

with a coherent homomorphism $\mathrm{Zid}(h)\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$.

Lemma 2.4.1. *For any $L \in$ **CRegFrm**, the map $\sigma_L\colon \mathrm{Zid}(\mathcal{R}L) \to L$ given by*

$$\sigma_L(Q) = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\}$$

is a dense onto frame homomorphism.

Proof. Clearly, σ_L takes the bottom to the bottom, and the top to the top. Let $Q, R \in \mathrm{Zid}(\mathcal{R}L)$. Then, by the properties of the cozero map, we have

$$\begin{aligned}
\sigma_L(Q) \wedge \sigma_L(R) &= \bigvee_{\alpha\in Q}\operatorname{coz}\alpha \wedge \bigvee_{\beta\in R}\operatorname{coz}\beta \\
&= \bigvee\{\operatorname{coz}(\alpha\beta) \mid \alpha \in Q, \beta \in R\} \\
&\leq \bigvee\{\operatorname{coz}\gamma \mid \gamma \in Q \cap R\} \\
&= \sigma_L(Q \cap R).
\end{aligned}$$

Since σ_L clearly preserves order, it follows that σ_L preserves binary meets. Next, let $\{Q_i \mid i \in I\} \subseteq \mathrm{Zid}(\mathcal{R}L)$, and put $a = \bigvee_{i\in I} \sigma_L(Q_i)$. For any $i \in I$, if $\alpha \in Q_i$, then $\operatorname{coz}\alpha \le a$, and hence $Q_i \subseteq \boldsymbol{M}_a$. Since $\boldsymbol{M}_a \in \mathrm{Zid}(\mathcal{R}L)$, it follows that $\bigvee_{i\in I} Q_i \le \boldsymbol{M}_a$. Thus,

$$\sigma_L\Big(\bigvee_{i\in I} Q_i\Big) \le \sigma_L(\boldsymbol{M}_a) = a,$$

the latter in view of complete regularity. Consequently $\sigma_L(\bigvee_{i\in I} Q_i) = \bigvee_{i\in I} \sigma_L(Q_i)$, and hence σ_L is a frame homomorphism, which is clearly dense. It is onto since, for any $b \in L$, $\sigma_L(\boldsymbol{M}_b) = b$. □

Remark 2.4.1. The homomorphism σ_L maps precisely as that employed by Banaschewski [11, Proposition 12] in showing that the frame of closed ℓ-ideals of $\mathcal{R}^*L$ realizes the Stone-Čech compactification of L. There should therefore be no wonder that our proof (in certain places) is modelled on that of Banaschewski.

For use in the upcoming proposition, we recall from Johnstone [46, page 64] that if A and B are coherent frames, then any lattice homomorphism $\mathfrak{K}(A) \to \mathfrak{K}(B)$ extends uniquely to a coherent frame homomorphism $A \to B$ because, as Johnstone remarks, A – being coherent – is freely generated by $\mathfrak{K}(A)$.

Proposition 2.4.1. *For any morphism $h\colon L \to M$ in* **CRegFrm**, *the map*

$$\mathrm{Zid}(h)\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M) \quad \textit{given by} \quad \mathrm{Zid}(h)(Q) = \bigvee\{\boldsymbol{M}_{\operatorname{coz}(h\cdot\alpha)} \mid \alpha \in Q\}$$

is the unique frame homomorphism making the square (‡) *above commute.*

Proof. Define $\bar{h}\colon \mathfrak{K}(\mathrm{Zid}(\mathcal{R}L)) \to \mathfrak{K}(\mathrm{Zid}(\mathcal{R}M))$ by $\bar{h}(\boldsymbol{M}_{\operatorname{coz}\alpha}) = \boldsymbol{M}_{\operatorname{coz}(h\cdot\alpha)}$. A routine calculation shows that this is a lattice homomorphism. Its extension to a frame homomorphism $\mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$ is precisely the map $\mathrm{Zid}(h)$. So we are left with verifying commutativity of the diagram. By coherence, it suffices to show that $\sigma_M \cdot \mathrm{Zid}(h)$ agrees with $h \cdot \sigma_L$

on $\mathfrak{K}(\mathrm{Zid}(\mathcal{R}L))$. For any $\alpha \in \mathcal{R}L$,

$$\begin{aligned}
\big(\sigma_M \cdot \mathrm{Zid}(h)\big)(\boldsymbol{M}_{\mathrm{coz}\,\alpha}) &= \bigvee\{\mathrm{coz}\,\gamma \mid \gamma \in \boldsymbol{M}_{\mathrm{coz}\,(h\cdot\alpha)}\} \\
&= \mathrm{coz}\,(h \cdot \alpha) \\
&= h(\mathrm{coz}\,\alpha) \\
&= h\Big(\bigvee\{\mathrm{coz}\,\tau \mid \tau \in \boldsymbol{M}_{\mathrm{coz}\,\alpha}\}\Big) \\
&= \big(h \cdot \sigma_L\big)(\boldsymbol{M}_{\mathrm{coz}\,\alpha}).
\end{aligned}$$

Finally, to show uniqueness, suppose $g\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$ is a coherent map with $\sigma_M \cdot g = h \cdot \sigma_L$. We shall be done if we can show that g agrees with $\mathrm{Zid}(h)$ on compact elements. Let $\alpha \in \mathcal{R}L$, and, by coherence, pick $\gamma \in \mathcal{R}M$ such that $g(\boldsymbol{M}_{\mathrm{coz}\,\alpha}) = \boldsymbol{M}_{\mathrm{coz}\,\gamma}$. Then $\big(\sigma_M \cdot g\big)(\boldsymbol{M}_{\mathrm{coz}\,\alpha}) = \big(h \cdot \sigma_L)(\boldsymbol{M}_{\mathrm{coz}\,\alpha}\big)$ implies $\mathrm{coz}\,\gamma = \mathrm{coz}\,(h \cdot \alpha)$, so that $g(\boldsymbol{M}_{\mathrm{coz}\,\alpha}) = \mathrm{Zid}(h)(\boldsymbol{M}_{\mathrm{coz}\,\alpha})$. This completes the proof. $\square$

In [54], Martínez calls a frame homomorphism $h\colon L \to M$ between compact frames an *s-map* in case there is a frame homomorphism $S(h)\colon SL \to SM$ making the square below commute:

$$\begin{tikzcd}
L \arrow[r, "h"] \arrow[d, "s_L"'] & M \arrow[d, "s_M"] \\
SL \arrow[r, "S(h)"'] & SM
\end{tikzcd}$$

Letting ϱL stand for the subframe of L generated by the regular subframes of L, and denoting the inclusion map $\varrho L \to L$ by ϱ_L, he then shows that there is a frame homomorphism $\varrho(h)\colon \varrho L \to \varrho M$ which makes the square

$$\begin{tikzcd}
\varrho L \arrow[r, "\varrho(h)"] \arrow[d, "\varrho_L"'] & \varrho M \arrow[d, "\varrho_M"] \\
L \arrow[r, "h"'] & M
\end{tikzcd}$$

commute. Furthermore, in Proposition 3.3 he shows that a homomorphism $h\colon L \to M$ is an s-map if and only if the square has the property that when the downward morphisms are replaced with their right adjoints, the resulting square is also commutative.

Here we will find a necessary and sufficient condition on a frame homomorphism $h\colon L \to M$ between completely regular frames for the square

$$\begin{tikzcd}
\mathrm{Zid}(\mathcal{R}L) \arrow[r, "\mathrm{Zid}(h)"] \arrow[d, "\sigma_L"'] & \mathrm{Zid}(\mathcal{R}M) \arrow[d, "\sigma_M"] \\
L \arrow[r, "h"'] & M
\end{tikzcd} \tag{‡}$$

to have similar properties. Actually we will do a little more. Recall that a frame homomorphism $h\colon L \to M$ is called *perfect* if its right adjoint preserves directed joins. This is equivalent to saying its right adjoint preserves joins of ideals of M. We will show, among other things, that the diagram obtained from (‡) by replacing the horizontal morphisms with their right adjoints commutes if and only if h is a perfect map.

A word of caution is in order. Whereas in Martínez's case the maps $(\varrho_L)_*$ are also frame homomorphisms, no such claim is made here. Our diagrams sporting arrows which are right adjoints are not necessarily in **Frm**.

To start, observe that the right adjoint of $\sigma_L\colon \mathrm{Zid}(\mathcal{R}L) \to L$ is given by

$$(\sigma_L)_*(a) = \boldsymbol{M}_a.$$

Indeed, $\boldsymbol{M}_a$ is an element of $\mathrm{Zid}(\mathcal{R}L)$ mapped under a (actually mapped to a) by σ_L, and if Q is a member of $\mathrm{Zid}(\mathcal{R}L)$ with $\sigma_L(Q) \leq a$, then $\bigvee\limits_{\alpha\in Q} \operatorname{coz}\alpha \leq a$, which clearly implies $Q \subseteq \boldsymbol{M}_a$.

Recall from [50] that Lindelöf frames are coreflective in **CRegFrm**. The coreflection of L is the frame λL of σ-ideals of $\operatorname{Coz} L$ with the coreflection map $\lambda_L\colon \lambda L \to L$ given by join. For any $a \in L$ let $[a]$ be the σ-ideal of $\operatorname{Coz} L$ given by

$$[a] = \{c \in \operatorname{Coz} L \mid c \leq a\}.$$

The right adjoint of λ_L is the map $(\lambda_L)_*(a) = [a]$. Every homomorphism $h\colon L \to M$ has a

λ-*lift* $h^\lambda\colon \lambda L \to \lambda M$, which is the unique frame homomorphism making the diagram

$$\begin{tikzcd} \lambda L \arrow[r, "h^\lambda"] \arrow[d, "\lambda_L"'] & \lambda M \arrow[d, "\lambda_M"] \\ L \arrow[r, "h"'] & M \end{tikzcd}$$

commute. The homomorphism h^λ maps as follows: For any $s \in \operatorname{Coz} M$ and $I \in \lambda L$,

$$s \in h^\lambda(I) \iff s \le h(c) \text{ for some } c \in I.$$

In [37], a homomorphism $h\colon L \to M$ is called a λ-*map* if the diagram

$$\begin{tikzcd} \lambda L \arrow[r, "h^\lambda"] & \lambda M \\ L \arrow[u, "(\lambda_L)_*"] \arrow[r, "h"'] & M \arrow[u, "(\lambda_M)_*"'] \end{tikzcd}$$

commutes; that is, if $(\lambda_M)_* \cdot h = h^\lambda \cdot (\lambda_L)_*$. Since the comparison

$$h^\lambda \cdot (\lambda_L)_* \le (\lambda_M)_* \cdot h$$

always holds, it follows that h is a λ-map if and only if $[h(a)] \subseteq h^\lambda([a])$ for every $a \in L$.

Proposition 2.4.2. *The square*

$$\begin{tikzcd} \mathrm{Zid}(\mathcal{R}\mathrm{L}) \arrow[r, "\mathrm{Zid(h)}"] & \mathrm{Zid}(\mathcal{R}\mathrm{M}) \\ L \arrow[u, "(\sigma_L)_*"] \arrow[r, "h"'] & M \arrow[u, "(\sigma_M)_*"'] \end{tikzcd}$$

commutes if and only if h is a λ-map.

Proof. ($\Leftarrow$) Suppose h is a λ-map. Since $\sigma_M \cdot \mathrm{Zid}(h) = h \cdot \sigma_L$, the comparison

$$\mathrm{Zid}(h) \cdot (\sigma_L)_* \le (\sigma_M)_* \cdot h$$

does hold. So we need only show that, for any $a \in L$,

$$(\sigma_M)_* h(a) \leq \mathrm{Zid}(h)(\sigma_L)_*(a).$$

The left side of this inequality is $\boldsymbol{M}_{h(a)}$, and the right side is

$$\mathrm{Zid}(h)(\boldsymbol{M}_a) = \bigvee\{\boldsymbol{M}_{\mathrm{coz}(h\cdot\alpha)} \mid \alpha \in \boldsymbol{M}_a\} = \bigcup\{\boldsymbol{M}_{\mathrm{coz}(h\cdot\alpha)} \mid \mathrm{coz}\,\alpha \leq a\},$$

since the join is directed, and $\alpha \in \boldsymbol{M}_a$ if and only if $\mathrm{coz}\,\alpha \leq a$. Let $\gamma \in \boldsymbol{M}_{h(a)}$. Then $\mathrm{coz}\,\gamma \leq h(a)$, and hence $\mathrm{coz}\,\gamma \in [h(a)]$. Since h is a λ-map, by hypothesis, $[h(a)] \subseteq h^{\lambda}([a])$, and hence there is a $\delta \in \mathcal{R}L$ such that

$$\mathrm{coz}\,\delta \leq a \quad \text{and} \quad \mathrm{coz}\,\gamma \leq h(\mathrm{coz}\,\delta) = \mathrm{coz}(h\cdot\delta).$$

This shows that γ is in the ideal on the right side of the desired inequality.

($\Rightarrow$) Let $a \in L$, and consider any $c \in \mathrm{Coz}\,M$ with $c \leq h(a)$. Pick $\gamma \in \mathcal{R}M$ with $c = \mathrm{coz}\,\gamma$. Then $\gamma \in \boldsymbol{M}_{h(a)}$, so that, by the current hypothesis, $\gamma \in \boldsymbol{M}_{h(\mathrm{coz}\,\alpha)}$ for some $\alpha \in \mathcal{R}L$ with $\mathrm{coz}\,\alpha \leq a$. This shows that $c \in h^{\lambda}([a])$, whence h is a λ-map. $\square$

In the proposition that follows it is the horizontal morphisms in the diagram (‡) that we replace with their right adjoints. We shall need to know how the right adjoint of $\mathrm{Zid}(h)$ maps. To calculate it, we recall that coherent maps are perfect. Because we do not have reference for this fact, we give a proof.

Lemma 2.4.2. *A frame homomorphism $\phi\colon A \to B$ between coherent frames is perfect.*

Proof. Let $D \subseteq B$ be directed. We must show that $\phi_*(\bigvee D) = \bigvee\{\phi_*(d) \mid d \in D\}$. Since ϕ_* preserves order, so that the inequality $\geq$ holds, we need only show the other. Let $s \in \mathfrak{K}(A)$ with $s \leq \phi_*(\bigvee D)$. Then $\phi(s) \leq \phi\phi_*(\bigvee D) \leq \bigvee D$. Since ϕ is coherent and $s \in \mathfrak{K}(A)$, $\phi(s) \in \mathfrak{K}(B)$. So there exist finitely many elements $d_1, \ldots, d_m$ in D such that $\phi(s) \leq d_1 \vee \cdots \vee d_m$. Since D is directed, there exists $d \in D$ such that $d_1 \vee \cdots \vee d_m \leq d$. So $\phi(s) \leq d$. Therefore, $s \leq \phi_*(d) \leq \bigvee\{\phi_*(d) \mid d \in D\}$. Since $\phi_*(\bigvee D)$ is the join of all compact elements of A below it, it follows that $\phi_*(\bigvee D) \leq \bigvee \phi_*[D]$. Hence the result. $\square$

Now, the equality $\sigma_M \cdot \mathrm{Zid}(h) = h \cdot \sigma_L$ implies $\mathrm{Zid}(h)_* \cdot (\sigma_M)_* = (\sigma_L)_* \cdot h_*$, so that, for any $a \in M$,

$$\mathrm{Zid}(h)_*(\boldsymbol{M}_a) = \boldsymbol{M}_{h_*(a)}.$$

Thus, for any $Q \in \mathrm{Zid}(\mathcal{R}M)$,

$$\begin{aligned}\mathrm{Zid}(h)_*(Q) &= \mathrm{Zid}(h)_*\Big(\bigvee_{\alpha\in Q} \boldsymbol{M}_{\operatorname{coz}\alpha}\Big)\\ &= \bigvee_{\alpha\in Q} \mathrm{Zid}(h)_*(\boldsymbol{M}_{\operatorname{coz}\alpha})\\ &= \bigvee_{\alpha\in Q} \boldsymbol{M}_{h_*(\operatorname{coz}\alpha)}.\end{aligned}$$

Since the last join is directed, we can also express this as

$$\mathrm{Zid}(h)_*(Q) = \bigcup\{\boldsymbol{M}_{h_*(\operatorname{coz}\alpha)} \mid \alpha \in Q\} = (\mathcal{R}h)^{-1}[Q].$$

The last equality is verified by a routine calculation.

Proposition 2.4.3. *For any morphism $h\colon L \to M$ in* **CRegFrm***, the square*

$$\begin{tikzcd} \mathrm{Zid}(\mathcal{R}L) \arrow[d, "\sigma_L"'] & \mathrm{Zid}(\mathcal{R}M) \arrow[l, "\mathrm{Zid}(h)_*"'] \arrow[d, "\sigma_M"] \\ L & M \arrow[l, "h_*"] \end{tikzcd}$$

commutes if and only if h is a perfect map.

Proof. ($\Leftarrow$) Assume h is a perfect map. We must show that $\sigma_L \cdot \mathrm{Zid}(h)_* = h_* \cdot \sigma_M$. For any $Q \in \mathrm{Zid}(\mathcal{R}M)$, we have

$$h_*\sigma_M(Q) = h_*\Big(\bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\}\Big) = \bigvee\{h_*(\operatorname{coz}\alpha) \mid \alpha \in Q\},$$

since the join is directed. On the other hand,

$$\sigma_L\big(\mathrm{Zid}(h)_*(Q)\big) = \sigma_L\Big(\bigvee\{\boldsymbol{M}_{h_*(\operatorname{coz}\alpha)} \mid \alpha \in Q\}\Big) = \bigvee\{h_*(\operatorname{coz}\alpha) \mid \alpha \in Q\}.$$

($\Rightarrow$) Assume $h_* \cdot \sigma_M = \sigma_L \cdot \mathrm{Zid}(h)_*$. Let I be an ideal of M, and define $Q \in \mathrm{Zid}(\mathcal{R}M)$ by

$$Q = \bigvee\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \operatorname{coz}\alpha \in I\} = \bigcup\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \operatorname{coz}\alpha \in I\}.$$

Observe that

$$\sigma_M(Q) = \bigvee\{\operatorname{coz}\alpha \mid \operatorname{coz}\alpha \in I\} = \bigvee I,$$

by complete regularity. Thus

$$\begin{aligned}
h_*\Big(\bigvee I\Big) &= (h_* \cdot \sigma_M)(Q) \\
&= (\sigma_L \cdot \mathrm{Zid}(h)_*)(Q) \\
&= \bigvee\{\operatorname{coz}\rho \mid \rho \in (\mathcal{R}h)^{-1}[Q]\} \\
&= \bigvee\{\operatorname{coz}\rho \mid h \cdot \rho \in Q\} \\
&\leq \bigvee\{h_* h(\operatorname{coz}\rho) \mid h \cdot \rho \in Q\} \\
&= \bigvee\{h_*\big(\operatorname{coz}(h \cdot \rho)\big) \mid h \cdot \rho \in Q\} \\
&\leq \bigvee\{h_*(x) \mid x \in I\} \qquad \text{since } \tau \in Q \Rightarrow \operatorname{coz}\tau \in I.
\end{aligned}$$

It follows therefore that $h_*(\bigvee I) = \bigvee h_*[I]$, whence h is a perfect map. □

Martínez [53] says that a frame homomorphism $\phi\colon A \to B$ is *weakly closed* if for every $a \in A$ and $b \in B$, $\phi(a) \vee b = 1_B$ implies $a \vee \phi_*(b) = 1_A$. In [54, Proposition 3.2.2] he shows that a frame homomorphism between normal compact frames is an s-map if and only if it is weakly closed if and only if its right adjoint maps maximal elements to maximal elements. Since maximal elements of $\mathrm{Zid}(\mathcal{R}L)$ are precisely the maximal ideals of $\mathcal{R}L$, and since $\mathrm{Zid}(h) = (\mathcal{R}h)^{-1}$, we have the following result.

Proposition 2.4.4. *Let $h\colon L \to M$ be a morphism in* **CRegFrm**. *The following are equivalent*

(1) $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ *contracts maximal ideals to maximal ideals.*

(2) $\mathrm{Zid}(h)$ *is an s-map.*

(3) $\mathrm{Zid}(h)$ *is weakly closed.*

Remark 2.4.2. A frame homomorphism $h\colon L \to M$ between completely regular frames is called a *W-map* [35] if $h^\beta r_L(c) = r_M h(c)$ for every $c \in \operatorname{Coz} L$. It is shown in [35, Proposition 4.9] that h is a W-map if and only if $\mathcal{R}h$ contracts maximal ideals to maximal ideals. It follows therefore that another condition equivalent to $\mathrm{Zid}(h)$ being an s-map is that h be a W-map.

2.5 A note on flatness

We remind the reader that a frame homomorphism $h\colon L \to M$ is *flat* if h is onto and $h_*\colon M \to L$ is a lattice homomorphism [13]. Weakening this, we say h is *coz-flat* if $h_*(0) = 0$ and $h_*(a \vee b) = h_*(a) \vee h_*(b)$ for all $a, b \in \operatorname{Coz} L$. Observe that coz-flatness is a genuine weakening of flatness. Indeed, for any non-normal completely regular frame L, the join map $\beta L \to L$ is coz-flat, but not flat. We aim to show that for a homomorphism h whose right adjoint sends cozero elements to cozero elements, $\operatorname{Zid}(h)$ is flat precisely when h is coz-flat. We need a lemma.

Lemma 2.5.1. *Let $h\colon L \to M$ be a morphism in* **CRegFrm**. *For all $S, T \in \operatorname{Zid}(\mathcal{R}L)$ and $Q, R \in \operatorname{Zid}(\mathcal{R}M)$ we have:*

(1) $S \vee T = \bigvee\{\boldsymbol{M}_{\operatorname{coz}\gamma} \mid \gamma \in S + T\} = \bigcup\{\boldsymbol{M}_{\operatorname{coz}\gamma} \mid \gamma \in S + T\}$

(2) $\operatorname{Zid}(h)_*(Q \vee R) = \bigvee\{\boldsymbol{M}_{h_*(\operatorname{coz}\tau)} \mid \tau \in Q + R\}$.

Proof. (1) Observe that the join is directed, and hence equals the union. The rest is easy to check.

(2) Again, observe that the join is directed, and hence, by the first part,

$$\begin{aligned}
\operatorname{Zid}(h)_*(Q \vee R) &= (\mathcal{R}h)^{-1}\Big(\bigcup\{\boldsymbol{M}_{\operatorname{coz}\tau} \mid \tau \in Q + R\}\Big) \\
&= \bigcup\{(\mathcal{R}h)^{-1}(\boldsymbol{M}_{\operatorname{coz}\tau}) \mid \tau \in Q + R\} \\
&= \bigvee\{\boldsymbol{M}_{h_*(\operatorname{coz}\tau)} \mid \tau \in Q + R\}.
\end{aligned}$$

□

Proposition 2.5.1. *Let $h\colon L \to M$ be a morphism in* **CRegFrm**. *Consider the following statements.*

(1) $\operatorname{Zid}(h)$ *is flat.*

(2) $\operatorname{Zid}(h)_*$ *is a frame homomorphism.*

(3) h *is coz-flat.*

We have that (1) $\Leftrightarrow$ (2) $\Rightarrow$ (3). *Furthermore, if* h_* *takes cozero elements to cozero elements, then all three statements are equivalent.*

Proof. Since $\mathrm{Zid}(h)$ is a coherent map, and hence its right adjoint preserves directed joins, it follows that (1) and (2) are equivalent.

(1) $\Rightarrow$ (3): Banaschewski [14, Lemma 2] has shown that h is dense if and only if $\mathcal{R}h$ is one-one. Thus, h is dense if and only if $(\mathcal{R}h)^{-1}\{\mathbf{0}\} = \{\mathbf{0}\}$, that is, if and only if $\mathrm{Zid}(h)_*(\bot) = \bot$. So we need only show preservation of binary joins of cozero elements. Let $a, b \in \mathrm{Coz}\, M$. By (1),

$$\begin{aligned}
\boldsymbol{M}_{h_*(a\vee b)} = \mathrm{Zid}(h)_*(\boldsymbol{M}_{a\vee b}) &= \mathrm{Zid}(h)_*(\boldsymbol{M}_a \vee \boldsymbol{M}_b) \\
&= \mathrm{Zid}(h)_*(\boldsymbol{M}_a) \vee \mathrm{Zid}(h)_*(\boldsymbol{M}_b) \quad \text{since } \mathrm{Zid}(h) \text{ is flat} \\
&= \boldsymbol{M}_{h_*(a)} \vee \boldsymbol{M}_{h_*(b)}.
\end{aligned}$$

Applying the map σ_L yields $h_*(a \vee b) = h_*(a) \vee h_*(b)$, as required.

(3) $\Rightarrow$ (1): Let $Q, R \in \mathrm{Zid}(\mathcal{R}M)$. Then

$$\begin{aligned}
\mathrm{Zid}(h)_*(Q \vee R) &= \bigvee\{\boldsymbol{M}_{h_*(\mathrm{coz}\,\gamma)} \mid \gamma \in Q + R\} \quad \text{by the lemma above} \\
&= \bigvee\{\boldsymbol{M}_{h_*(\mathrm{coz}\,(\alpha+\beta))} \mid \alpha \in Q, \beta \in R\} \\
&\leq \bigvee\{\boldsymbol{M}_{h_*(\mathrm{coz}\,\alpha \vee \mathrm{coz}\,\beta)} \mid \alpha \in Q, \beta \in R\} \\
&= \bigvee\{\boldsymbol{M}_{h_*(\mathrm{coz}\,\alpha) \vee h_*(\mathrm{coz}\,\beta)} \mid \alpha \in Q, \beta \in R\} \quad \text{since } h \text{ is coz-flat} \\
&= \bigvee\{\boldsymbol{M}_{h_*(\mathrm{coz}\,\alpha)} \vee \boldsymbol{M}_{h_*(\mathrm{coz}\,\beta)} \mid \alpha \in Q, \beta \in R\} \quad \text{since } h_*[\mathrm{Coz}\, M] \subseteq \mathrm{Coz}\, L \\
&= \bigvee_{\alpha\in Q} \boldsymbol{M}_{h_*(\mathrm{coz}\,\alpha)} \vee \bigvee_{\beta \in R} \boldsymbol{M}_{h_*(\mathrm{coz}\,\beta)} \\
&= \mathrm{Zid}(h)_*(Q) \vee \mathrm{Zid}(h)_*(R),
\end{aligned}$$

which proves the nontrivial inequality of the desired equality. □

We end with the following observation. We remarked above that coz-flatness is strictly weaker than flatness. However,

> *a coz-flat perfect homomorphism into a completely regular frame is flat.*

To see this, let $\phi\colon A \to B$ be such a homomorphism, and let $b_1, b_2 \in B$. By complete regularity,

$$\phi_*(b_1 \vee b_2) = \phi_*\Big(\bigvee\{c \vee d \mid c, d \in \operatorname{Coz} B, c \leq b_1 \text{ and } d \leq b_2\}\Big).$$

The set whose join is displayed is directed, so

$$\begin{aligned}
\phi_*(b_1 \vee b_2) &= \bigvee\{\phi_*(c \vee d) \mid c, d \in \operatorname{Coz} B, c \leq b_1 \text{ and } d \leq b_2\} \quad \text{since } \phi \text{ is perfect} \\
&= \bigvee\{\phi_*(c) \vee \phi_*(d) \mid c, d \in \operatorname{Coz} B, c \leq b_1 \text{ and } d \leq b_2\} \quad \text{since } \phi \text{ is coz-flat} \\
&= \bigvee\{\phi_*(c) \mid c \in \operatorname{Coz} B, c \leq b_1\} \vee \bigvee\{\phi_*(d) \mid d \in \operatorname{Coz} B, d \leq b_2\} \\
&= \phi_*(b_1) \vee \phi_*(b_2).
\end{aligned}$$

2.6 Contracting z-ideals

In this section we investigate if z-ideals of $\mathcal{R}L$ contract to z-ideals of $\mathcal{R}^*L$, and if z-ideals of the smaller ring extend to z-ideals of the bigger ring. Recall that if $\phi\colon A \to B$ is a ring homomorphism and I is an ideal of B, then $\phi^{-1}[I]$ is an ideal of A called the *contraction* of I, and frequently denoted by I^c. On the other hand, if J is an ideal of A, the (possibly improper) ideal of B generated by $\phi[J]$ is called an *extension* of J and denoted by J^e. In the event that A is a subring of B and ϕ the inclusion map, then $I^c = I \cap A$.

In what follows we shall make use of the well-known fact that $\mathcal{R}^*L \cong \mathcal{R}(\beta L)$ [22], and $\mathcal{R}(\beta L) \cong C(X)$ for some topological space X. For later use, let us recall how a ring isomorphism can be constructed.

(1) If $h\colon L \to M$ is a dense frame homomorphism, then the ring homomorphism $\mathcal{R}h\colon \mathcal{R}L \to \mathcal{R}M$ is one-one ([14, Lemma 2]).

(2) The frame homomorphism $j_L\colon \beta L \to L$ is, in the terminology of [6], a C^*-quotient map ([6, Corollary 8.2.7]), meaning that for every $\alpha \in \mathcal{R}^*L$ there is a (necessarily unique) element of $\mathcal{R}(\beta L)$, which, as in the classical case [39], we shall denote by α^β, such that the triangle

$$\begin{tikzcd} & \mathfrak{L}(\mathbb{R}) \arrow[dl, "\alpha^\beta"'] \arrow[dr, "\alpha"] & \\ \beta L \arrow[rr, "j_L"] & & L \end{tikzcd} \tag{2.2}$$

commutes. The reason α^β is unique is that j_L is dense, and hence monic.

(3) For any $\varphi \in \mathcal{R}(\beta L)$, $\mathcal{R}(j_L)(\varphi)$ is bounded because

$$\bigvee_{n\in\mathbb{N}}(-n,n) = 1_{\mathfrak{L}(\mathbb{R})} \quad \text{implies} \quad \bigvee_{n\in\mathbb{N}}\varphi(-n,n) = 1_{\beta L},$$

so that, by compactness, $\varphi(-m,m) = 1_{\beta L}$ for some $m \in \mathbb{N}$, and hence $(j_L\varphi)(-m,m) = 1_L$.

Therefore the (one-one) ring homomorphism $\mathcal{R}(j_L)\colon \mathcal{R}(\beta L) \to \mathcal{R}L$ maps into $\mathcal{R}^*L$. But it is onto by what we have mentioned above. We therefore have a ring isomorphism

$$\mathfrak{t}_L\colon \mathcal{R}^*L \to \mathcal{R}(\beta L) \quad \text{given by} \quad \mathfrak{t}_L(\alpha) = \alpha^\beta.$$

We now have the following characterisation of z-ideals of $\mathcal{R}^*L$ in terms of the cozero map. Observe that any ring isomorphism sends z-ideals to z-ideals, that is, if $\phi : A \to B$ is a ring isomorphism and I is a z-ideal in A, then $\phi[I]$ is a z-ideal of B.

Proposition 2.6.1. *An ideal Q of $\mathcal{R}^*L$ is a z-ideal if and only if for any $\alpha, \gamma \in \mathcal{R}^*L$, $\operatorname{coz}(\alpha^\beta) = \operatorname{coz}(\gamma^\beta)$ and $\alpha \in Q$ imply $\gamma \in Q$.*

Proof. ($\Rightarrow$) Let Q be a z-ideal in $\mathcal{R}^*L$. Suppose that $\alpha \in Q$ and $\operatorname{coz}(\alpha^\beta) = \operatorname{coz}(\gamma^\beta)$ for some $\gamma \in \mathcal{R}^*L$. We must show that $\gamma \in Q$. Since $\mathfrak{t}_L : \mathcal{R}^*L \to \mathcal{R}(\beta L)$ is a ring isomorphism, $\mathfrak{t}_L[Q]$ is a z-ideal in $\mathcal{R}(\beta L)$. Since $\mathfrak{t}_L(\alpha) = \alpha^\beta$, it follows that $\alpha^\beta \in \mathfrak{t}_L[Q]$. Since $\operatorname{coz}(\gamma^\beta) = \operatorname{coz}(\alpha^\beta)$ by hypothesis, it follows from Lemma 2.1.4 that $\gamma^\beta \in \mathfrak{t}_L[Q]$. But $\gamma^\beta = \mathfrak{t}_L(\gamma)$, therefore $\mathfrak{t}_L(\gamma) \in \mathfrak{t}_L[Q]$, which implies $\gamma \in \mathfrak{t}_L^{-1}\mathfrak{t}_L[Q] = Q$ because $\mathfrak{t}_L$ is an isomorphism.

($\Leftarrow$) Let Q be an ideal of $\mathcal{R}^*L$ with the hypothesized property. To show that Q is a z-ideal, it suffices to show that $\mathfrak{t}_L[Q]$ is a z-ideal in $\mathcal{R}(\beta L)$. So consider any τ and ρ in

$\mathcal{R}(\beta L)$ with $\operatorname{coz}\tau = \operatorname{coz}\rho$ and $\tau \in \mathfrak{t}_L[Q]$. Since $\tau = (\mathfrak{t}_L^{-1}(\tau))^\beta$, and similarly for ρ, we have that

$$\operatorname{coz}(\mathfrak{t}_L^{-1}(\tau))^\beta = \operatorname{coz}(\mathfrak{t}_L^{-1}(\rho))^\beta \quad \text{and} \quad \mathfrak{t}_L^{-1}(\tau) \in Q.$$

So, by the hypothesis on $Q, \mathfrak{t}_L^{-1}(\rho) \in Q$, which implies $\rho \in \mathfrak{t}_L[Q]$. Therefore, by Lemma 2.1.4, $\mathfrak{t}_L[Q]$ is a z-ideal in $\mathcal{R}(\beta L)$, and hence Q is a z-ideal in $\mathcal{R}^*L$. $\square$

Corollary 2.6.1. *The contraction of every z-ideal of $\mathcal{R}L$ is a z-ideal of $\mathcal{R}^*L$.*

Proof. Let Q be a z-ideal in $\mathcal{R}L$. We must show that $Q^c = Q \cap \mathcal{R}^*L$ is a z-ideal in $\mathcal{R}^*L$. Let $\alpha, \gamma \in \mathcal{R}^*L$ be such that $\operatorname{coz}\alpha^\beta = \operatorname{coz}\gamma^\beta$ and $\alpha \in Q^c$. By commutativity of diagram 2.2, $j_L \cdot \alpha^\beta = \alpha$ and $j_L \cdot \gamma^\beta = \gamma$. Therefore

$$\operatorname{coz}\alpha = \operatorname{coz}(j_L \cdot \alpha^\beta) = j_L(\operatorname{coz}\alpha) = j_L(\operatorname{coz}\gamma^\beta) = \operatorname{coz}\gamma.$$

Since $\alpha \in Q^c \subseteq Q$, then $\alpha \in Q$. But Q is a z-ideal in $\mathcal{R}L$, so $\gamma \in Q$. But $\gamma \in \mathcal{R}^*L$, therefore $\gamma \in Q^c$. So by Proposition 2.6.1, Q^c is a z-ideal in $\mathcal{R}^*L$. $\square$

Concluding Remarks 2.6.1. (a) Professor F. Azarpanah has shown us a draft [3] of work he has done in the lattice of z-ideals of $C(X)$. Our work on z-ideals does not overlap with his – neither in style nor content.

(b) In $C(X)$ the sum of two z-ideals is a z-ideal. We have not been able to determine if the same holds in $\mathcal{R}L$.

(c) We have not been able to determine if z-ideals of $\mathcal{R}^*L$ extend to z-ideals in $\mathcal{R}L$. However, proper z-ideals of $\mathcal{R}^*L$ do not always extend to proper ideals. For instance let $f \in C^*(\mathbb{R})$ be given by $f(x) = \frac{1}{1+x^2}$. Then f is not invertible in $C^*(\mathbb{R})$. So any maximal ideal of $C^*(\mathbb{R})$ which contains f is a proper z-ideal which extends to the entire ring $C(\mathbb{R})$.

Chapter 3

The frame of d-ideals of an f-ring

3.1 Coherence of the frame of d-ideals of an f-ring

We recall that an ideal of a ring A is *singular* if it consists entirely of zero-divisors. For any $a \in A$, let P_a denote the intersection of all minimal prime ideals of A containing a. It is shown in [59] that $P_a = \mathrm{Ann}^2(a)$. An ideal I of A is called a *d-ideal* if $\mathrm{Ann}^2(a) \subseteq I$, for every $a \in I$. Examples of d-ideals abound (see, for instance, [4]). It is clear that the union of a directed family of d-ideals is a d-ideal. As stated in the Introduction, we shall at times write the annihilator of a set S as $S^\perp$, and that of an element a as $a^\perp$.

Lemma 3.1.1. *Let A be a reduced f-ring and I be a singular ideal of A. Then the set*

$$J = \bigcup \{a^{\perp\perp} \mid a \in I\}$$

is the smallest d-ideal of A containing I.

Proof. Let us show first that the family $\{a^{\perp\perp} \mid a \in I\}$ is directed. Let $a, b \in I$. We claim that $a^{\perp\perp} \cup b^{\perp\perp} \subseteq (a^2 + b^2)^{\perp\perp}$. To verify this it suffices to show that $(a^2 + b^2)^\perp \subseteq a^\perp \cap b^\perp$. Let $r \in (a^2 + b^2)^\perp$. Then $r(a^2 + b^2) = 0$, which implies $(ra)^2 + (rb)^2 = 0$. Since squares are positive in f-rings, this implies $(ra)^2 = (rb)^2 = 0$, and hence $ra = rb = 0$ since A is reduced. Therefore $r \in a^\perp \cap b^\perp$. Thus, J is d-ideal which clearly contains I. To show that it is the smallest such, consider any d-ideal K of A which contains I. Let $u \in J$. Then $u \in a^{\perp\perp}$ for some $a \in I$. But $a \in K$ since K is a d-ideal, so $u \in K$, and hence $J \subseteq K$. $\square$

Recall that an ideal of A is called *radical* if it does not contain squares of non-members. As usual, we denote by $\mathrm{Rad}(A)$ the coherent frame of radical ideals of A. Its compact elements are precisely the finitely generated radical ideals. Since annihilator ideals are d-ideals, it is easy to see that, for any ideal I of A

$$I \text{ is a } d\text{-ideal} \qquad \Longleftrightarrow \qquad I = \bigcup\{a^{\perp\perp} \mid a \in I\}.$$

Observe that, for any radical ideal I of A, $I^\perp$ is the pseudocomplement of I in $\mathrm{Rad}(A)$. Thus, if I is a d-ideal of A, then

$$\begin{aligned} I &= \bigvee_{\mathrm{Rad}(A)} \{a^{\perp\perp} \mid a \in I\} \\ &\subseteq \bigvee_{\mathrm{Rad}(A)} \{K^{\perp\perp} \mid K \in \mathfrak{K}(\mathrm{Rad}(A)), K \subseteq I\} \\ &\subseteq I, \end{aligned}$$

which shows that $\mathrm{Did}(A) \subseteq d(\mathrm{Rad}(A))$. On the other hand, if I is a radical ideal such that

$$I = \bigvee_{\mathrm{Rad}(A)} \{K^{\perp\perp} \mid K \in \mathfrak{K}(\mathrm{Rad}(A)), K \subseteq I\},$$

then observe that the join is directed, so that it is a directed union of d-ideals, and hence is a d-ideal. Thus, $d(\mathrm{Rad}(A)) = \mathrm{Did}(A)$. Since $\mathrm{Rad}(A)$ is coherent, we can therefore deduce that:

Proposition 3.1.1. $\mathrm{Did}(A)$ *is a coherent frame.*

Remark 3.1.1. The definition of d-ideal we have used is the traditionally algebraic one (see, for instance, [59]). In [41] the authors define d-ideals to be d-elements of the frame $\mathrm{Rad}(A)$, no doubt based on the fact that the two notions agree.

3.2 Extending and contracting d-ideals

Recall that all our rings are commutative with identity element 1. We remind the reader that an f-ring A has *bounded inversion* if any $a \geq 1$ in A is invertible in A. Let us observe that in any f-ring, the inverse of a positive invertible element is positive. For, if a is such an element, then the inequalities

$$a \geq 0 \;\text{ and }\; (a^{-1})^2$$

imply

$$a^{-1} = a(a^{-1})^2 \geq 0.$$

Given an f-ring A, we write A^* for the subring of *bounded* elements. That is,

$$A^* = \{a \in A \mid |a| \leq n, \text{ for some } n \in \mathbb{N}\}.$$

Lemma 3.2.1. *Let A be an f-ring with bounded inversion. For any $a \in A$, the elements $\frac{a}{1+|a|}$ and $\frac{1}{1+|a|}$ are in A^*.*

Proof. Since $-1 \leq 0$, we have

$$-1 - |a| \leq -|a| \leq a \leq 1 + |a|,$$

which implies

$$-(1 + |a|) \leq a \leq 1 + |a|.$$

Multiplying throughout with the positive element $\frac{1}{1+|a|}$ we get

$$-1 \leq \frac{1}{1 + |a|} \leq 1,$$

proving the first result. For the second, since $0 \leq 1$, we have

$$0 \leq 1 \leq 1 + |a|.$$

Multiply throughout with $\frac{1}{1+|a|}$ and we get

$$0 \leq \frac{1}{1 + |a|} \leq 1,$$

which completes the proof. □

Now let $S = \{a \in A^* \mid a \text{ is a unit in } A\}$, and consider the ring $A^*[S^{-1}]$ of fractions of A with respect to S. Since for any $a \in A$ we have

$$a = \Big(\frac{1}{1 + |a|}\Big)^{-1} \cdot \frac{a}{1 + |a|},$$

standard algebraic considerations (see, for instance [1]) combined with the lemma just proved establish the following result.

Corollary 3.2.1. *Let A be a reduced f-ring with bounded inversion. Then $A = A^*[S^{-1}]$. That is, A is the ring of fractions of A^* with respect to the set of members of A^* which are invertible in A.*

Emanating from Corollary 3.2.1, we note that ideals of A are precisely the ideals

$$I^e = \{us^{-1} \mid u \in I \text{ and } s \in S\},$$

for I an ideal of A^*. As in [41], we let $\varepsilon\colon \operatorname{Rad}(A^*) \to \operatorname{Rad}(A)$ be the coherent map which takes a radical ideal in A^* to the smallest radical ideal of A containing its extension. Our goal is to show that the restriction of ε to $\operatorname{Did}(A^*)$ is precisely the extension map $I \mapsto I^e$, and that it is an isomorphism onto $\operatorname{Did}(A)$. We need intermediate results.

Lemma 3.2.2. *The extension of any radical ideal of A^* is a radical ideal in A.*

Proof. Let I be a radical ideal in A^*, and suppose a is an element of A with $a^2 \in I^e$. Then $\left(\frac{a}{1+|a|}\right)^2 \in I^e$, and hence we can choose $u \in I$ and $s \in S$ such that $\left(\frac{a}{1+|a|}\right)^2 = us^{-1}$. This implies $s \cdot \left(\frac{a}{1+|a|}\right)^2 \in I$, so that $\left(\frac{sa}{1+|a|}\right)^2 \in I$ because s is an element of A^*. Since I is a radical ideal in A^* and $\frac{sa}{1+|a|}$ is an element of A^* whose square is in I, it follows that $\frac{sa}{1+|a|} \in I$. Now,

$$a = \frac{sa}{1+|a|} \cdot \left(\frac{s}{1+|a|}\right)^{-1} \in I^e$$

since $\frac{s}{1+|a|} \in S$. □

Next, let us recall some facts from [12]. For any $a \in A$, the principal radical ideal generated by a is given by

$$[a] = \{x \in A \mid x^n \in \langle a \rangle \text{ for some } n\},$$

where $\langle \cdot \rangle$ denotes ordinary ideal-generation in A. The compact elements of $\operatorname{Rad}(A)$ are precisely the ideals $[a_1] \vee \cdots \vee [a_n]$, for some finitely many $a_1, \ldots, a_n$ in A. We shall write $[\cdot]_*$ to signify the principal radical ideal contemplated in A^*.

Proposition 3.2.1. *The frame homomorphism $\varepsilon\colon \operatorname{Rad}(A^*) \to \operatorname{Rad}(A)$ is given by $I \mapsto I^e$, and it is dense onto.*

Proof. That ε is precisely the map $I \mapsto I^e$ follows from Lemma 3.2.2. The density of the homomorphism is obvious, so we show surjectivity. We do this by showing that every compact element of $\mathrm{Rad}(A)$ is the image of some compact element of $\mathrm{Rad}(A^*)$. For this, it clearly suffices to show that for every $a \in A$, there is a compact element of $\mathrm{Rad}(A^*)$ mapped to $[a]$. We claim that $\varepsilon\big(\big[\frac{a}{1+|a|}\big]_*\big) = [a]$. If $x \in \big(\big[\frac{a}{1+|a|}\big]_*\big)^e$, then $x = us^{-1}$ for some $u \in \big[\frac{a}{1+|a|}\big]_*$ and some $s \in S$. Take an integer m and $r \in A^*$ such that $u^m = \frac{ra}{1+|a|}$. Then

$$x^m = \frac{ra}{1+|a|} \cdot (s^m)^{-1} \in [a].$$

Since $[a]$ is a radical ideal, this implies $x \in [a]$, establishing the inclusion $\subseteq$. To reverse the inclusion, let $z \in [a]$. Take an integer n and $t \in A$ such that $z^n = ta$. Then

$$z^n = \frac{t}{1+|t|} \cdot \Big(\frac{1}{1+|t|}\Big)^{-1} \cdot \Big(\frac{1}{1+|a|}\Big)^{-1} \cdot \frac{a}{1+|a|}.$$

Since $\frac{t}{1+|t|} \in A^*$ and $\frac{1}{1+|t|} \cdot \frac{1}{1+|a|} \in S$, it follows that

$$z^n \in \Big(\Big[\frac{a}{1+|a|}\Big]_*\Big)^e = \varepsilon\Big(\Big[\frac{a}{1+|a|}\Big]_*\Big).$$

But this ideal is radical, so $z \in \varepsilon\big(\big[\frac{a}{1+|a|}\big]_*\big)$, and this concludes the proof. $\square$

In the process of the proof of the foregoing proposition it has come to light that $\varepsilon\colon \mathrm{Rad}(A^*) \to \mathrm{Rad}(A)$ is rigid in the sense that for every $b \in \mathfrak{K}(M)$ there is an $a \in \mathfrak{K}(L)$ such that $b^{**} = h(a)^{**}$. Since this map is coherent and dense, we deduce from [41, Proposition 2.6] the following corollary.

Corollary 3.2.2. *The map $d(\varepsilon)\colon \mathrm{Did}(A^*) \to \mathrm{Did}(A)$ is an isomorphism.*

In order to reach our goal we need only show that $d(\varepsilon)$ takes an ideal in $\mathrm{Did}(A^*)$ to its extension, which we must show to be an element of $\mathrm{Did}(A)$. Furthermore, we must show that contracting a d-ideal of A takes us to a d-ideal of A^*. We need the following lemma. In the first part we will use the characterisation (see [59, Theorem 2.3(c)]) that I is a d-ideal if and only if $a^\perp = b^\perp$ and $a \in I$ imply $b \in I$. We write Ann and Ann_* for annihilation in A and A^*, respectively.

Lemma 3.2.3. *Let A be a reduced f-ring with bounded inversion.*

(*a*) *The extension of any d-ideal of A^* is a d-ideal of A.*

(b) *The contraction of any d-ideal of A is a d-ideal of A^*.*

Proof. (a) Let J be a d-ideal of A^*. Consider an arbitrary $s^{-1}u \in J^e$ with $u \in J$ and $s \in S$. Suppose a is an element of A such that $\mathrm{Ann}(s^{-1}u) = \mathrm{Ann}(a)$. We claim that $\mathrm{Ann}_*(u) = \mathrm{Ann}_*\left(\frac{a}{1+|a|}\right)$. Let $b \in \mathrm{Ann}_*(u)$. Then b is an element of A^* with $bu = 0$, and hence $ba = 0$, whence $\frac{ba}{1+|a|} = 0$; establishing the containment $\subseteq$. To reverse the inclusion, let $c \in \mathrm{Ann}_*(\frac{a}{1+|a|})$. Then $ca = 0$, and hence $c(s^{-1}u) = 0$, since $\mathrm{Ann}(a) = \mathrm{Ann}(s^{-1}u)$. Thus, $cu = 0$, as desired. Since $u \in J$ and J is a d-ideal of A^*, $\frac{a}{1+|a|} \in J$. But

$$a = \frac{a}{1+|a|} \cdot \left(\frac{1}{1+|a|}\right)^{-1} \in J^e,$$

so J^e is d-ideal in A.

(b) Observe that, for any $a \in A^*$, $\mathrm{Ann}_*^2(a) \subseteq \mathrm{Ann}^2(a)$. To verify this, let $r \in \mathrm{Ann}_*^2(a)$. Consider any $x \in \mathrm{Ann}(a)$. Then $\frac{x}{1+|x|}$ is an element of A^* with $\frac{ax}{1+|x|} = 0$, so that $\frac{x}{1+|x|} \in \mathrm{Ann}_*(a)$. Thus, $\frac{rx}{1+|x|} = 0$, and hence $rx = 0$. Therefore $r \in \mathrm{Ann}^2(a)$, as required. Now if I is a d-ideal of A and $a \in I^c = I \cap A^*$, then $a \in I$, and hence $\mathrm{Ann}^2(a) \subseteq I$, which implies

$$\mathrm{Ann}_*^2(a) \subseteq \mathrm{Ann}^2(a) \cap A^* \subseteq I \cap A^* = I^c.$$

Therefore I^c is a d-ideal of A^*. □

Proposition 3.2.2. *The map $I \mapsto I^e$ is a frame isomorphism* $\mathrm{Did}(A^*) \to \mathrm{Did}(A)$ *whose inverse is the contraction map $J \mapsto J^c$.*

Proof. Denote by $\mathfrak{e}\colon \mathrm{Did}(A^*) \to \mathrm{Did}(A)$ the extension map. We show that $d(\varepsilon) = \mathfrak{e}$. Let $I \in \mathrm{Did}(A^*)$. Then

$$\begin{aligned}
d(\varepsilon)(I) &= \bigvee_{\mathrm{Rad}(A)} \{\varepsilon(K)^{\perp\perp} \mid K \in \mathfrak{K}(\mathrm{Rad}(A^*)), K \subseteq I\} \\
&= \bigcup \{\varepsilon(K)^{\perp\perp} \mid K \in \mathfrak{K}(\mathrm{Rad}(A^*)), K \subseteq I\} \quad \text{since the join is directed} \\
&= \bigcup \{\varepsilon(K^{\perp\perp}) \mid K \in \mathfrak{K}(\mathrm{Rad}(A^*)), K \subseteq I\} \quad \text{since } \varepsilon \text{ is dense onto} \\
&\subseteq \varepsilon(I) \quad \text{since } K^{\perp\perp} \subseteq I \text{ for compact } K \subseteq I \\
&= \mathfrak{e}(I).
\end{aligned}$$

On the other hand, if $x \in \mathfrak{e}(I) = I^e$, then $x = us^{-1}$ for some $u \in I$ and $s \in S$. But $u \in [u]_*$, so

$$us^{-1} \in ([u]_*)^e = \varepsilon([u]_*) \subseteq \varepsilon([u]_*)^{\perp\perp} \subseteq d(\varepsilon)(I),$$

which shows that $\mathfrak{e}(I) \subseteq d(\varepsilon)(I)$. Therefore we have equality. That contraction is the inverse of $\mathfrak{e}$ follows from the fact that it is the right adjoint of ε, and therefore certainly the right adjoint $\mathfrak{e}$, by the (b) part of Lemma 3.2.3. $\square$

Chapter 4

The frame of d-ideals of $\mathcal{R}L$

In this chapter we specialise to the f-ring $\mathcal{R}L$ regarding d-ideals. Thus, all the resources of the previous chapter are applicable. We denote by $\mathrm{Did}(\mathcal{R}L)$ the lattice of d-ideals of $\mathcal{R}L$. We highlight that, for any nonzero $a \in L$, $\boldsymbol{M}_{a^{**}}$ is a d-ideal of $\mathcal{R}L$. Here are characterisations of d-ideals in terms of the cozero map.

4.1 Characterisation of d-ideals of $\mathcal{R}L$

Proposition 4.1.1. *The following are equivalent for a singular ideal Q of $\mathcal{R}L$.*

(1) *Q is a d-ideal.*

(2) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $(\operatorname{coz}\alpha)^* = (\operatorname{coz}\beta)^*$, then $\beta \in Q$.*

(3) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $(\operatorname{coz}\alpha)^* \leq (\operatorname{coz}\beta)^*$, then $\beta \in Q$.*

(4) *For any $\alpha, \beta \in \mathcal{R}L$, if $\alpha \in Q$ and $\operatorname{coz}\beta \leq (\operatorname{coz}\alpha)^{**}$, then $\beta \in Q$.*

Proof. In view of the fact that, for any $\gamma \in \mathcal{R}L$, $\gamma^{\perp} = \boldsymbol{M}_{(\operatorname{coz}\alpha)^*}$, the equivalence of (1) and (2) follows from [4, Proposition 1.4].

(2) $\Rightarrow$ (3): Assume (2), and suppose that $\alpha \in Q$ and $(\operatorname{coz}\alpha)^* \leq (\operatorname{coz}\beta)^*$. Now observe the following. If a and b are elements of L with $a^* \leq b^*$, then $(a \wedge b)^* \leq b^*$. To see this, note that $b^{**} \leq a^{**}$, so that $(a \wedge b)^* \wedge b \leq (a \wedge b)^* \wedge a^{**}$. But we also have

$$(a \wedge b)^* \wedge b \leq (a \wedge b)^* \wedge b^{**},$$

as a consequence of which

$$(a \wedge b)^* \wedge b \leq (a \wedge b)^* \wedge (a^{**} \wedge b^{**}) = (a \wedge b)^* \wedge (a \wedge b)^{**} = 0,$$

from which assertion follows. Now suppose Q satisfies (2), and let $\alpha, \beta \in \mathcal{R}L$ be such that $\alpha \in Q$ and $(\operatorname{coz}\alpha)^* \leq (\operatorname{coz}\beta)^*$. Then $\alpha\beta$ is an element of Q such that

$$\begin{aligned} (\operatorname{coz}\alpha\beta)^* &= (\operatorname{coz}\alpha \wedge \operatorname{coz}\beta)^* \\ &\leq (\operatorname{coz}\beta)^* \quad \text{by what we have just observed} \\ &\leq (\operatorname{coz}\alpha\beta)^*. \end{aligned}$$

Therefore $(\operatorname{coz}\alpha\beta)^* = (\operatorname{coz}\beta)^*$, and hence, by (2), $\beta \in Q$. So (3) holds.

(3) $\Rightarrow$ (4): Suppose Q satisfies (3), and let $\alpha \in Q$ and $\beta \in \mathcal{R}L$ be such that $\operatorname{coz}\beta \leq (\operatorname{coz}\alpha)^{**}$. Then $(\operatorname{coz}\alpha)^* \leq (\operatorname{coz}\beta)^*$, so that, by (3), $\beta \in Q$. Therefore Q satisfies (4).

(4) $\Rightarrow$ (1): Let $\alpha \in Q$ and $\gamma \in P_\alpha = \boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}}$. Then $\operatorname{coz}\gamma \leq (\operatorname{coz}\alpha)^{**}$. So, by (4), $\gamma \in Q$, and hence $P_\alpha \subseteq Q$. Therefore Q is a d-ideal. $\square$

4.2 The frame Did($\mathcal{R}L$)

We know from the previous chapter that Did($\mathcal{R}L$) is the frame of d-elements of Rad($\mathcal{R}L$). In [51, Theorem 4.2] Martínez shows that if L is an algebraic frame with FIP and has disjointification, then dL is also coherently normal. Now observe that:

(a) For any ideal Q in $\mathcal{R}L$, Ann(Q) is a z-ideal. We show that, in fact, Ann(Q) is the pseudocomplement of Q in Zid($\mathcal{R}L$). Recall from [35, Lemma 3.1] that

$$\operatorname{Ann}(Q) = \boldsymbol{M}_{a^*} \quad \text{where} \quad a = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\}.$$

Now if $\gamma \in Q \cap \boldsymbol{M}_{a^*}$, then $\operatorname{coz}\gamma \leq a$ and $\operatorname{coz}\gamma \leq a^*$, so that $\operatorname{coz}\gamma = 0$, and hence $\gamma = \mathbf{0}$. Therefore $Q \wedge \boldsymbol{M}_{a^*} = \bot$. Next, suppose H is a z-ideal with $Q \wedge H = \bot$. Let $\rho \in H$ and consider any $\alpha \in Q$. Then $\rho\alpha = \mathbf{0}$, and therefore $\operatorname{coz}\rho \wedge \operatorname{coz}\alpha = 0$. So $\operatorname{coz}\rho \wedge \bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\} = 0$, which implies $\operatorname{coz}\rho \leq a^*$, and hence $\rho \in \boldsymbol{M}_{a^*}$. Therefore $H \subseteq \boldsymbol{M}_{a^*}$, which proves the claim.

(b) In view of the result in (a), $\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}}$ is the double pseudocomplement in $\operatorname{Zid}(\mathcal{R}L)$ of the element $\boldsymbol{M}_{\operatorname{coz}\alpha}$ of $\operatorname{Zid}(\mathcal{R}L)$.

(c) If Q is a z-ideal in $\mathcal{R}L$, then, for any $\alpha \in \mathcal{R}L$, $\boldsymbol{M}_{\operatorname{coz}\alpha} \subseteq Q$ if and only if $\alpha \in Q$.

Consequently, the d-nucleus on $\operatorname{Zid}(\mathcal{R}L)$ takes the form

$$d(Q) = \bigvee_{\operatorname{Zid}(\mathcal{R}L)} \{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \mid \alpha \in Q\} = \bigcup \{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \mid \alpha \in Q\},$$

the join being equal to the union because it is directed. Since an ideal Q of $\mathcal{R}L$ is a d-ideal if and only if $Q = \bigcup \{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \mid \alpha \in Q\}$, we have the following proposition.

Proposition 4.2.1. *For any completely regular frame L:*

(a) $\operatorname{Did}(\mathcal{R}L) = d(\operatorname{Zid}(\mathcal{R}L))$*, and is therefore a coherently normal frame.*

(b) $\mathfrak{K}(\operatorname{Did}(\mathcal{R}L)) = \{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \mid \alpha \in \mathcal{R}L\} = \{\boldsymbol{M}_{c^{**}} \mid c \in \operatorname{Coz} L\}$.

We should note that in the case of $\operatorname{Did}(A)$, for an arbitrary ring A, no normality was claimed. In the case of $\operatorname{Did}(\mathcal{R}L)$, not only do we have normality, but we also have a much stronger property.

Remark 4.2.1. In [8], Banaschewski shows that if j is a *codense* nucleus (meaning that the only element it takes to the top is the top) on a compact frame, then $\operatorname{Fix}(j)$ is also compact. Now one might wonder if the compactness of $\operatorname{Did}(\mathcal{R}L)$ is perhaps not also delivered by codensity of the d-nucleus. That is not the case. Indeed, for any coherent M, the d-nucleus $d\colon M \to M$ is codense if and only if the only dense compact element of M is the top. For the forward implication, if $c \in \mathfrak{K}(M)$ is dense, then $d(c) = c^{**} = 1$, implying $c = 1$ by codensity. For the converse, if $d(a) = 1$ for $a \in M$, then, by compactness of M and the way $d(a)$ is defined, there is a compact $c \leq a$ with $c^{**} = 1$, implying $c = 1$, and hence $a = 1$. Applying this to $\mathcal{R}L$, recall that L is an *almost P-frame* if $c = c^{**}$ for every $c \in \operatorname{Coz} L$. This is equivalent to saying the only dense cozero element of L is the top [33, Proposition 3.3]. Thus we have the following result.

Corollary 4.2.1. *For a completely regular frame L, the nucleus $d\colon \operatorname{Zid}(\mathcal{R}L) \to \operatorname{Zid}(\mathcal{R}L)$ is codense if and only if L is an almost P-frame.*

Concerning extension and contraction of d-ideals of $\mathcal{R}L$, the results below follow from Lemma 3.2.3.

Corollary 4.2.2. *For any completely regular frame L:*

(a) The extension of any d-ideal of $\mathcal{R}^*L$ is a d-ideal of $\mathcal{R}L$.

(b) The contraction of any d-ideal of $\mathcal{R}L$ is a d-ideal of $\mathcal{R}^*L$.

We end the section with a word on the saturation quotient of $\mathrm{Did}(\mathcal{R}L)$. We showed in Proposition 2.3.2 that the saturation quotient of $\mathrm{Zid}(\mathcal{R}L)$ is isomorphic to βL. In the case of $\mathrm{Did}(\mathcal{R}L)$ we have the same result for certain types of frames. In [6] Ball and Walters-Wayland define a *quasi F-frame* to be a completely regular frame L which satisfies a condition they show to be equivalent to saying for all $a, b \in \mathrm{Coz}\, L$, if $a \wedge b = 0$ and $a \vee b$ is dense, then there exist $c, d \in \mathrm{Coz}\, L$ such that $a \wedge c = b \wedge d = 0$ and $c \vee d = 1$. This generalises the spatial notion of *quasi F-spaces*, which are those Tychonoff spaces X such that every dense cozero-subset is C^*-embedded. In [5] it is shown that X is a quasi F-space if and only if the sum of two d-ideals in $C(X)$ is a d-ideal. In the proof that follows (a proof parts of which piggyback on $C(X)$ results) we use Banaschewski's result [13, Proposition 2.2] which states that if $h\colon L \to M$ is a flat coherent map between compact normal frames, then the map $Sh\colon SL \to SM$, induced by h, is an isomorphism.

Proposition 4.2.2. *If L is a completely regular quasi-F frame, then $\mathcal{S}(\mathrm{Did}(\mathcal{R}L)) \cong \beta L$.*

Proof. Since $\mathrm{Zid}(\mathcal{R}L)$ and $\mathrm{Did}(\mathcal{R}L)$ are compact normal frames, the proposition will follow from the result of Banaschewski mentioned above if we can show that $d_L\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Did}(\mathcal{R}L)$ is flat. This is where the piggyback on $C(X)$ kicks in. If L is a quasi-F frame, then so is βL [36, Proposition 3.6]. Since βL is spatial (modulo AC), $\beta L \cong \mathfrak{O}X$ for X equal to the spectrum of βL. Then X is a quasi F-space, and therefore the sum of two d-ideals in $C(X)$ is a d-ideal. But now

$$C(X) \cong \mathcal{R}(\mathfrak{O}X) \cong \mathcal{R}(\beta L) \cong \mathcal{R}^*L,$$

so the sum of any two d-ideals in $\mathcal{R}^*L$ is a d-ideal. We use this to show that the sum of any two d-ideals in $\mathcal{R}L$ is a d-ideal. Let Q_1 and Q_2 be d-ideals in $\mathcal{R}L$. Then, by Lemma

3.2.3, Q_1^c and Q_2^c are d-ideals in $\mathcal{R}^*L$, and so $(Q_1^c + Q_2^c)^e$ is a d-ideal in $\mathcal{R}L$. Since $\mathcal{R}L$ is a ring of fractions of $\mathcal{R}^*L$, $I = I^{ce}$ for any ideal I of $\mathcal{R}L$. So

$$(Q_1^c + Q_2^c)^e = Q_1^{ce} + Q_2^{ce} = Q_1 + Q_2,$$

which shows that $Q_1 + Q_2$ is a d-ideal. Denote by $\sqcup$ and by $\vee$ the binary join in $\mathrm{Did}(\mathcal{R}L)$ and $\mathrm{Zid}(\mathcal{R}L)$, respectively. What we have just shown tells us that, for any $P, Q \in \mathrm{Did}(\mathcal{R}L)$,

$$P \sqcup Q = P + Q = P \vee Q.$$

Since the right adjoint of d_L is the inclusion map, we conclude that

$$(d_L)_*(P \sqcup Q) = P + Q = P \vee Q = (d_L)_*(P) \vee (d_L)_*(Q),$$

showing that d_L is flat. $\square$

4.3 Projectability properties

In this section we seek conditions on L or $\mathcal{R}L$ which make $\mathrm{Zid}(\mathcal{R}L)$ and $\mathrm{Did}(\mathcal{R}L)$ satisfy certain variants of projectability. Let us recall the definitions from [48].

Definition 4.3.1. An algebraic frame L is said to be*:*

(a) *projectable* if for every $c \in \mathfrak{K}(L)$, $c^{\perp\perp}$ is a complemented element.

(b) *feebly projectable* if whenever $a, b \in \mathfrak{K}(L)$ and $a \wedge b = 0$, then there exists a $c \in \mathfrak{K}(L)$ such that $c^{\perp\perp}$ is complemented and $a \leq c^{\perp\perp}$, $b \leq c^{\perp}$.

(c) *flatly projectable* if whenever $a, b \in \mathfrak{K}(L)$ and $a \wedge b = 0$, then there exists a complemented $c \in L$ such that $a \leq c$ and $b \leq c^{\perp}$.

We shall also need the following definition from [47].

Definition 4.3.2. A ring A is a *feebly Baer ring* if whenever $a, b \in A$ and $ab = 0$, there is an idempotent $e \in A$ such that $a \in eA$ and $b \in (1 - e)A$.

For the next result, we need to recall that a frame L is *basically disconnected* if $c^* \vee c^{**} = 1$ for $c \in \operatorname{Coz} L$. The following quick lemma is known, but we include the short proof nevertheless.

Lemma 4.3.1. *Let L be an algebraic frame with a compact top. If $c \in L$ is complemented, then c^* is compact.*

Proof. Let $c^* = \bigvee d_i$ for some elements d_i in L. Now $1 = c \vee c^* = c \vee \bigvee d_i$. Since L is compact, there exist finitely many indices $i_1, \ldots, i_m$ such that $c \vee d_{i_1} \vee \cdots \vee d_{i_m} = 1$. Therefore, $c^* \leq d_{i_1} \vee \cdots \vee d_{i_m}$ which shows that c^* is compact. □

Recall from Chapter 2 the map

$$\sigma_L \colon \operatorname{Zid}(\mathcal{R}L) \to L \quad \text{given by} \quad \sigma_L(Q) = \bigvee\{\operatorname{coz} \alpha \mid \alpha \in Q\}.$$

Proposition 4.3.1. *Let L be a completely regular frame. Then the following are equivalent.*

(1) $\operatorname{Zid}(\mathcal{R}L)$ *is projectable.*

(2) L *is basically disconnected.*

Proof. (1) $\Rightarrow$ (2): Let $a \in \operatorname{Coz} L$. Since $\boldsymbol{M}_a \in \mathfrak{K}(\operatorname{Zid}(\mathcal{R}L)$, by Lemma 4.3.1 there is an element $b \in \operatorname{Coz} L$ such that

$$\boldsymbol{M}_{a^{**}} \vee \boldsymbol{M}_b = \top \quad \text{and} \quad \boldsymbol{M}_{a^{**}} \wedge \boldsymbol{M}_b = \bot.$$

On applying the homomorphism σ_L, we have

$$a^{**} \vee b = 1 \quad \text{and} \quad a^{**} \wedge b = 0,$$

showing that a^{**} is complemented. Therefore L is basically disconnected.

(2) $\Rightarrow$ (1): Let $Q \in \mathfrak{K}(\operatorname{Zid}(\mathcal{R}L)$ and find $a \in \operatorname{Coz} L$ such that $Q = \boldsymbol{M}_a$. Since L is basically disconnected by the current hypothesis, $a^{**} \vee a^* = 1$. Thus, a^{**} and a^* are cozero elements because they are complemented. So, by Lemma 2.3.1,

$$\boldsymbol{M}_{a^{**}} \vee \boldsymbol{M}_{a^*} = \boldsymbol{M}_{a^{**} \vee a^*} = \boldsymbol{M}_1 = \top,$$

which shows that $Q^{**} \vee Q^* = \top$. Therefore Q^{**} is complemented, and thus $\operatorname{Zid}(\mathcal{R}L)$ is projectable. □

Let us recall from [34, Proposition 2.2] that if $c \in L$ is complemented, then there is an idempotent $\gamma \in \mathcal{R}L$ such that $\operatorname{coz}\gamma = c$. Also, by [33, Lemma 4.4], if $\operatorname{coz}\alpha \lll \operatorname{coz}\gamma$, then α is a multiple of γ. Now here is the second of the projectability results.

Proposition 4.3.2. *The following are equivalent for a completely regular frame L.*

(1) $\operatorname{Zid}(\mathcal{R}L)$ *is flatly projectable.*

(2) $\mathcal{R}L$ *is a feebly Baer ring.*

(3) *For every $a, b \in \operatorname{Coz} L$ with $a \wedge b = 0$, there exists a complemented $c \in \operatorname{Coz} L$ such that $a \leq c$ and $b \leq c^*$.*

Proof. (1) $\Rightarrow$ (2): Suppose $\alpha\beta = \mathbf{0}$ in $\mathcal{R}L$. Then $\boldsymbol{M}_{\operatorname{coz}\alpha} \wedge \boldsymbol{M}_{\operatorname{coz}\beta} = \bot$, and so, by (1) and the result cited from [34], there is an idempotent η in $\mathcal{R}L$ such that

$$\boldsymbol{M}_{\operatorname{coz}\alpha} \leq \boldsymbol{M}_{\operatorname{coz}\eta} \ \text{ and } \ \boldsymbol{M}_{\operatorname{coz}\beta} \leq \boldsymbol{M}_{\operatorname{coz}(\mathbf{1}-\eta)}.$$

This implies that

$$\operatorname{coz}\alpha \leq \operatorname{coz}\eta \ \text{ and } \ \operatorname{coz}\beta \leq \operatorname{coz}(\mathbf{1}-\eta) \lll \operatorname{coz}(\mathbf{1}-\eta),$$

so that, by [34, Lemma 2.1], $\alpha \in \langle \eta \rangle$ and $\beta \in \langle \mathbf{1} - \eta \rangle$. Therefore $\mathcal{R}L$ is feebly Baer.

(2) $\Rightarrow$ (3): If $a \wedge b = 0$ in $\operatorname{Coz} L$ and α, β are elements of $\mathcal{R}L$ with $a = \operatorname{coz}\alpha$ and $b = \operatorname{coz}\beta$, then $\alpha\beta = \mathbf{0}$, so that, by (2), there is an idempotent $\eta \in \mathcal{R}L$ with $\alpha \in \langle \eta \rangle$ and $\beta \in \langle \mathbf{1} - \eta \rangle$. Therefore $c = \operatorname{coz}\eta$ is a cozero element with $a \leq c$ and $b \leq c^*$.

(3) $\Rightarrow$ (1): Let $Q, R \in \mathfrak{K}(\operatorname{Zid}(\mathcal{R}L)$ be such that $Q \wedge R = \bot$. Pick $a, b \in \operatorname{Coz} L$ such that $Q = \boldsymbol{M}_a$ and $R = \boldsymbol{M}_b$. Then $a \wedge b = 0$, and so by (3) there is a complemented $c \in \operatorname{Coz} L$ such that $a \leq c$ and $b \leq c^*$. This implies $\boldsymbol{M}_c$ is a complemented element of $\operatorname{Zid}(\mathcal{R}L)$ with $\boldsymbol{M}_a \leq \boldsymbol{M}_c$ and $\boldsymbol{M}_b \leq \boldsymbol{M}_{c^*}$. Therefore $\operatorname{Zid}(\mathcal{R}L)$ is flatly projectable. $\square$

Flat projectability of $\operatorname{Did}(\mathcal{R}L)$ has a similar characterisation. To present it, note that, as in $\operatorname{Rad}(A)$, the pseudocomplement of any $I \in \operatorname{Did}(A)$ computed in $\operatorname{Did}(A)$ is $I^{\perp}$. Recall that the *Booleanization* of a frame L is the frame $\mathfrak{B}L$ whose underlying set is $\{a^{**} \mid a \in L\}$ with the meet calculated as in L, and join $\bigsqcup$ given by

$$\bigsqcup S = \Big(\bigvee\nolimits_L S\Big)^{**}$$

for any $S \subseteq \mathfrak{B}L$. The map $\flat\colon L \to \mathfrak{B}L$ given by $x \mapsto x^{**}$ is a dense onto frame homomorphism.

We shall need the following proposition in a number of places.

Proposition 4.3.3. *The map* $\tau_L\colon \mathrm{Did}(\mathcal{R}L) \to \mathfrak{B}L$ *defined by*

$$\tau_L(Q) = \Big(\bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\}\Big)^{**}$$

is a dense onto frame homomorphism.

Proof. Observe that τ_L is the restriction of $\flat_L \cdot \sigma_L$ to $\mathrm{Did}(\mathcal{R}L)$, and therefore preserves binary meets in $\mathrm{Did}(\mathcal{R}L)$ because they are calculated exactly as in $\mathrm{Zid}(\mathcal{R}L)$. Regarding joins, let $\{Q_i \mid i \in I\}$ be a collection of elements of $\mathrm{Did}(\mathcal{R}L)$. We need only show that $\tau_L\big(\bigvee_i Q_i\big) \leq \bigvee_i \tau_L(Q_i)$ since τ_L preserves order. Write $a = \bigvee_i \tau_L(Q_i)$, and keep in mind that $a = a^{**}$, so that $\boldsymbol{M}_a \in \mathrm{Did}(\mathcal{R}L)$. We show that each $Q_j \subseteq \boldsymbol{M}_a$. Observe that, for any $b \in L$, complete regularity implies that

$$\tau_L\big(\boldsymbol{M}_{b^{**}}\big) = (b^{**})^{**} = b^{**}.$$

Since Q_j is a d-ideal, $Q_j = \bigcup\{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \mid \alpha \in Q_j\}$. For any $\alpha \in Q_j$ we have

$$(\operatorname{coz}\alpha)^{**} = \tau_L\big(\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}}\big) \leq \tau_L\big(Q_j\big) \leq a,$$

which shows that $Q_j \subseteq \boldsymbol{M}_a$, and hence $\bigvee_i Q_i \subseteq \boldsymbol{M}_a$. Thus, $\tau_L\big(\bigvee_i Q_i\big) \leq a$, and therefore τ_L is a frame homomorphism. It is clearly dense, and it is surjective because for any $b \in \mathfrak{B}L$, $\boldsymbol{M}_b$ is an element of $\mathrm{Did}(\mathcal{R}L)$ mapped to b by τ_L. $\square$

Lemma 4.3.2. *For any $a, b \in \operatorname{Coz} L$, we have:*

(1) $\boldsymbol{M}_{a^{**}} \wedge \boldsymbol{M}_{b^{**}} = \boldsymbol{M}_{(a\wedge b)^{**}}$, *and*

(2) $\boldsymbol{M}_{a^{**}} \sqcup \boldsymbol{M}_{b^{**}} = \boldsymbol{M}_{(a\vee b)^{**}}$.

Proof. (1) We apply the homomorphism $\tau_L\colon \mathrm{Did}(\mathcal{R}L) \to \mathfrak{B}L$ described above. Since $\mathrm{Did}(\mathcal{R}L)$ is a coherent frame, there is a $c \in \operatorname{Coz} L$ such that $\boldsymbol{M}_{a^{**}} \wedge \boldsymbol{M}_{b^{**}} = \boldsymbol{M}_{c^{**}}$. Applying the map τ_L to this equality yields $c^{**} = a^{**} \wedge b^{**} = (a \wedge b)^{**}$, which then proves the result.

(2) As before, there is $c \in \operatorname{Coz} L$ such that $\boldsymbol{M}_{a^{**}} \sqcup \boldsymbol{M}_{b^{**}} = \boldsymbol{M}_{c^{**}}$. Apply the map τ_L to obtain

$$c^{**} = a^{**} \sqcup b^{**} = (a^{**} \vee b^{**})^{**} \geq (a \vee b)^{**}.$$

But clearly $\boldsymbol{M}_{a^{**}} \sqcup \boldsymbol{M}_{b^{**}} \leq \boldsymbol{M}_{(a\vee b)^{**}}$, which then implies $c^{**} \leq (a \vee b)^{**}$, whence $c^{**} = (a \vee b)^{**}$, thus proving the result. $\square$

To prove the next result, we recall from [36, Lemma 3.8 and Lemma 3.9] respectively that, for any $\gamma \in \mathcal{R}L$, $\operatorname{Ann}^2(\gamma) = \boldsymbol{M}_{(\operatorname{coz}\gamma)^{**}}$, and that an element α of $\mathcal{R}L$ is not a zero-divisor if and only if $\operatorname{coz}\alpha$ is dense.

Proposition 4.3.4. *The following are equivalent for any completely regular frame L.*

(1) $\operatorname{Did}(\mathcal{R}L)$ *is flatly projectable.*

(2) *For any $a, b \in \operatorname{Coz} L$ with $a \wedge b = 0$, there are elements $c, d \in \operatorname{Coz} L$ such that $c \wedge d = 0, c \vee d$ is dense and $a \leq c^{**}, b \leq d^{**}$.*

(3) *For every $\alpha, \beta \in \mathcal{R}L$ with $\alpha\beta = \mathbf{0}$, there exist positive $\gamma, \delta \in \mathcal{R}L$ such that $\gamma\delta = \mathbf{0}$ and $\gamma + \delta$ is a non zero divisor, and $\alpha \in \operatorname{Ann}^2(\gamma)$ and $\beta \in \operatorname{Ann}^2(\delta)$.*

Proof. (1) $\Rightarrow$ (2): If $a \wedge b = 0$ in $\operatorname{Coz} L$, then $\boldsymbol{M}_{a^{**}}$ and $\boldsymbol{M}_{b^{**}}$ are compact elements of $\operatorname{Did}(\mathcal{R}L)$ with zero meet. Denote the join in $\operatorname{Did}(\mathcal{R}L)$ by $\sqcup$. Since a complemented element in a compact frame is compact by Lemma 4.3.1, the present hypothesis implies that there are elements $c, d \in \operatorname{Coz} L$ such that

$$\boldsymbol{M}_{c^{**}} \wedge \boldsymbol{M}_{d^{**}} = \bot \ \text{ and } \ \boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{d^{**}} = \top,$$

and $\boldsymbol{M}_{a^{**}} \leq \boldsymbol{M}_{c^{**}}, \boldsymbol{M}_{b^{**}} \leq \boldsymbol{M}_{d^{**}}$. The first equality above implies $c \wedge d = 0$. On applying the homomorphism τ_L to the second, and keeping in mind that it maps into $\mathfrak{B}L$, we have $(c^{**} \vee d^{**})^{**} = 1$, which implies $c \vee d$ is dense. Now, $a \leq c^{**}$ and $b \leq d^{**}$, so (2) holds.

(2) $\Rightarrow$ (3): Suppose $\alpha\beta = \mathbf{0}$ for $\alpha, \beta \in \mathcal{R}L$. Put $a = \operatorname{coz}\alpha$ and $b = \operatorname{coz}\beta$. Then $a \wedge b = 0$. By (2) there exist $c, d \in \operatorname{Coz} L$ such that $c \wedge d = 0$ and $c \vee d$ is dense, $a \leq c^{**}, b \leq d^{**}$. Pick positive $\gamma, \delta \in \mathcal{R}L$ such that $c = \operatorname{coz}\gamma$ and $d = \operatorname{coz}\delta$. These imply that $\gamma\delta = \mathbf{0}$, and by [36, Lemma 3.9] $\gamma + \delta$ is a non zero-divisor. Since $\operatorname{coz}\alpha \leq (\operatorname{coz}\gamma)^{**}$ and $\operatorname{coz}\beta \leq (\operatorname{coz}\delta)^{**}$, $\alpha \in \boldsymbol{M}_{(\operatorname{coz}\gamma)^{**}}$ and $\beta \in \boldsymbol{M}_{(\operatorname{coz}\delta)^{**}}$; that is $\alpha \in \operatorname{Ann}^2(\gamma)$ and $\beta \in \operatorname{Ann}^2(\delta)$.

(3) $\Rightarrow$ (1): Let $\boldsymbol{M}_{a^{**}}, \boldsymbol{M}_{b^{**}} \in \mathfrak{K}(\mathrm{Did}(\mathcal{R}L))$ with $a, b \in \mathrm{Coz}\, L$ be such that $\boldsymbol{M}_{a^{**}} \wedge \boldsymbol{M}_{b^{**}} = \bot$. Take $\alpha, \beta \in \mathcal{R}L$ with $a = \mathrm{coz}\, \alpha$ and $b = \mathrm{coz}\, \beta$. Then $\alpha \in \boldsymbol{M}_{(\mathrm{coz}\, \alpha)^{**}}$ and $\beta \in \boldsymbol{M}_{(\mathrm{coz}\, \beta)^{**}}$. This implies that

$$\alpha\beta \in \boldsymbol{M}_{(\mathrm{coz}\, \alpha)^{**}} \cap \boldsymbol{M}_{(\mathrm{coz}\, \beta)^{**}} = \boldsymbol{M}_{a^{**}} \wedge \boldsymbol{M}_{b^{**}} = \bot.$$

Hence $\alpha\beta = \mathbf{0}$. By (3) we can pick positive $\gamma, \delta \in \mathcal{R}L$ such that $\gamma\delta = \mathbf{0}$, $\gamma + \delta$ is a not a zero-divisor, $\alpha \in \mathrm{Ann}^2(\gamma)$ and $\beta \in \mathrm{Ann}^2(\delta)$. For brevity, put $\mathrm{coz}\, \gamma = c$ and $\mathrm{coz}\, \delta = d$. By [36, Lemma 3.9] we have that $\mathrm{coz}(\gamma + \delta)$ is dense. Since

$$\mathrm{coz}(\gamma + \delta) = \mathrm{coz}\, \gamma \vee \mathrm{coz}\, \delta = c \vee d,$$

it follows that $(c \vee d)^{**} = 1$, which implies

$$\top = \boldsymbol{M}_{(c \vee d)^{**}} = \boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{d^{**}}$$

by Lemma 4.3.2. Since $c \wedge d = 0$, we have $c^{**} \wedge d^{**} = 0$, so that, by Lemma 4.3.2 again, $\boldsymbol{M}_{c^{**}} \wedge \boldsymbol{M}_{d^{**}} = \bot$. Therefore $\boldsymbol{M}_{c^{**}}$ is complemented with $\boldsymbol{M}_{d^{**}}$ as its complement. Now, $\alpha \in \mathrm{Ann}^2(\gamma)$ implies $\mathrm{coz}\, \alpha \leq (\mathrm{coz}\, \gamma)^{**}$, that is, $a \leq c^{**}$, which implies $a^{**} \leq c^{**}$. Similarly, $b^{**} \leq d^{**}$. Hence

$$\boldsymbol{M}_{a^{**}} \leq \boldsymbol{M}_{c^{**}} \quad \text{and} \quad \boldsymbol{M}_{b^{**}} \leq \boldsymbol{M}_{d^{**}},$$

which shows that $\mathrm{Did}(\mathcal{R}L)$ is flatly projectable. $\square$

In order to characterize when $\mathrm{Did}(\mathcal{R}L)$ is projectable, we recall the following definition. A frame L is *cozero-complemented* if for every $c \in \mathrm{Coz}\, L$, there is a $d \in \mathrm{Coz}\, L$ such that $c \wedge d = 0$ and $c \vee d$ is dense.

Proposition 4.3.5. $\mathrm{Did}(\mathcal{R}L)$ *is projectable if and only if L is cozero-complemented.*

Proof. Let L be cozero-complemented, and consider any compact element $\boldsymbol{M}_{c^{**}}$ of $\mathrm{Did}(\mathcal{R}L)$, with $c \in \mathrm{Coz}\, L$, of course. We must show that $\boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{c^{*}} = \top$. Take $d \in \mathrm{Coz}\, L$ such that $c \wedge d = 0$ and $c \vee d$ dense. Then $d \leq c^*$, so that $\boldsymbol{M}_{c \vee d}$ is a compact element of $\mathrm{Zid}(\mathcal{R}L)$ below $\boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{c^{*}}$, hence its double pseudocomplement, which is $\top$, is also below this element. Thus, $\boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{c^{*}} = \top$.

Conversely, suppose $\mathrm{Did}(\mathcal{R}L)$ is projectable. Let $c \in \mathrm{Coz}\, L$. Then $\boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{c^*} = \top$. Thus,

$$\boldsymbol{M}_{c^{**}} \sqcup \bigvee_{\mathrm{Zid}(\mathcal{R}L)} \{K^{\perp\perp} \mid K \in \mathfrak{K}(\mathrm{Zid}(\mathcal{R}L)), K \leq \boldsymbol{M}_{c^*}\} = \top,$$

and so, by compactness of the frame $\mathrm{Did}(\mathcal{R}L)$, there is a $d \in \mathrm{Coz}\, L$ such that

$$\boldsymbol{M}_d \leq \boldsymbol{M}_{c^*} \quad \text{and} \quad \boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{d^{**}} = \top.$$

The inequality implies $d \leq c^*$, so that $c \wedge d = 0$, and, as in the proof of Proposition 4.3.4, the equality implies $c \vee d$ is dense. Therefore L is cozero-complemented. $\square$

Keeping with the terminology above, we say an algebraic frame L is *strongly projectable* if for every $a \in L, a^* \vee a^{**} = 1$. In usual frame theoretic parlance, frames with this property are called *extremally disconnected*. In the result that follows we show that $\mathrm{Zid}(\mathcal{R}L)$ is strongly projectable if and only if L is extremally disconnected. We observe further that the strong projectability of $\mathrm{Zid}(\mathcal{R}L)$ implies that of $\mathrm{Did}(\mathcal{R}L)$, and conversely if L is an almost P-frame. We need a lemma.

Lemma 4.3.3. *Let L be an algebraic frame with FIP. If L is strongly projectable, then dL is strongly projectable. The converse holds if the top of L is the only dense compact element.*

Proof. Assume L is strongly projectable, and let $a \in dL$. Denote the join in dL by $\sqcup$ and the pseudocomplement by $(\,)^{\perp}$. Since $a \in L$ and L is strongly projectable, $a^* \vee a^{**} = 1$. Recall that pseudocomplements of L are members of dL, and pseudocomplements of elements of dL are exactly their pseudocomplements in L. Thus

$$\begin{aligned} 1 = d(a^* \vee a^{**}) &= d(a^*) \sqcup d(a^{**}) \\ &= d(a)^{\perp} \sqcup d(a)^{\perp\perp} \quad \text{since } d \text{ is dense onto} \\ &= a^{\perp} \sqcup a^{\perp\perp}. \end{aligned}$$

Therefore dL is strongly projectable.

Now assume dL is strongly projectable and that the top of L is its only dense compact element. Then, as observed in Remark 4.2.1, d is codense. Let $a \in L$. So, by the strong projectability of dL,

$$1 = d(a)^{\perp} \sqcup d(a)^{\perp\perp} = d(a^* \vee a^{**}),$$

which implies $a^* \vee a^{**} = 1$ by the codensity of d. Thus, L is strongly projectable. $\square$

Proposition 4.3.6. *Consider the following conditions on a completely regular frame L.*

(1) *L is extremally disconnected.*

(2) $\mathrm{Zid}(\mathcal{R}L)$ *is strongly projectable.*

(3) $\mathrm{Did}(\mathcal{R}L)$ *is strongly projectable.*

Statements (1) *and* (2) *are equivalent, and they imply statement* (3)*. If L is an almost P-frame, then all the three statements are equivalent.*

Proof. (1) $\Rightarrow$ (2): Let $Q \in \mathrm{Zid}(\mathcal{R}L)$, and put $a = \bigvee\{\mathrm{coz}\,\alpha \mid \alpha \in Q\}$. Since L is extremally disconnected, $a^* \vee a^{**} = 1$. Furthermore, a^* and a^{**} are cozero elements in L because they are complemented. Thus

$$\begin{aligned} \top = \boldsymbol{M}_1 &= \boldsymbol{M}_{a^* \vee a^{**}} \\ &= \boldsymbol{M}_{a^*} \vee \boldsymbol{M}_{a^{**}} \quad \text{by Lemma 2.3.1} \\ &= Q^* \vee Q^{**}. \end{aligned}$$

Therefore $\mathrm{Zid}(\mathcal{R}L)$ is strongly projectable.
(2) $\Rightarrow$ (1): Let $a \in L$ and consider the z-ideal $\boldsymbol{M}_{a^*}$. By the present hypothesis, $\boldsymbol{M}_{a^*} \vee \boldsymbol{M}_{a^{**}} = \top$. Applying the frame homomorphism σ_L, we have

$$1 = \sigma_L(\boldsymbol{M}_{a^*} \vee \boldsymbol{M}_{a^{**}}) = a^* \vee a^{**},$$

which shows that L is extremally disconnected.

(2) $\Rightarrow$ (3): This follows from the first part of the foregoing lemma.

(3) $\Rightarrow$ (2): If L is an almost P-frame, then the d-nucleus on $\mathrm{Zid}(\mathcal{R}L)$ is codense. Then this implication follows from foregoing lemma. $\square$

The following example shows that the condition that L be an almost P-frame cannot be relaxed in the implication (3) $\Rightarrow$ (2) in this proposition. Recall that an *Oz-frame* is a frame in which every regular element is a cozero element.

Example 4.3.1. Let L be an Oz-frame which is not an almost P-frame, and also not extremally disconnected. An example of such a frame is $\mathfrak{O}\mathbb{R}$. We show that $\mathrm{Did}(\mathcal{R}L)$ is strongly projectable. Let $Q \in \mathrm{Did}(\mathcal{R}L)$. Then $Q^{\perp}$, the pseudocomplement of Q in $\mathrm{Did}(\mathcal{R}L)$, is $\mathrm{Ann}(Q)$. Put $a = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\}$. Then

$$\begin{aligned} Q^{\perp} \sqcup Q^{\perp\perp} &= \boldsymbol{M}_{a^*} \sqcup \boldsymbol{M}_{a^{**}} \\ &= \boldsymbol{M}_{(a^*)^{**}} \sqcup \boldsymbol{M}_{(a^{**})^{**}} \\ &= \boldsymbol{M}_{(a^* \vee a^{**})^{**}} \quad \text{by Lemma 4.3.2 since } a^*, a^{**} \in \operatorname{Coz} L. \\ &= \boldsymbol{M}_1 \\ &= \top. \end{aligned}$$

We observed in Lemma 3.1.1 how the smallest d-ideal containing a given singular ideal of a reduced f-ring is described. Applying this lemma to $\mathcal{R}L$, we have that for any singular ideal Q of $\mathcal{R}L$, the smallest d-ideal containing Q is

$$\bigcup\{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \mid \alpha \in Q\}.$$

We end this chapter by showing that if L is a quasi F-frame and Q a singular ideal in $\mathcal{R}L$, then there is a largest d-ideal contained in Q. This we will do by actually describing this d-ideal.

Proposition 4.3.7. (cf. [5, Proposition 3.9]) *Let L be a quasi F-frame and Q be a singular ideal in $\mathcal{R}L$. The set*

$$D = \bigcup\{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \mid \boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \subseteq Q\} = \{\gamma \in \mathcal{R}L \mid \boldsymbol{M}_{(\operatorname{coz}\gamma)^{**}} \subseteq Q\}$$

is the largest d-ideal contained in Q.

Proof. It is easy to see that the two sets displayed above coincide. So what we need to show is that D is a d-ideal, and the largest one contained in Q. Since each of the sets $\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}}$ is a d-ideal, to show that D is a d-ideal it suffices to show that the collection

$$\{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \mid \boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \subseteq Q\}$$

is directed. Consider any $\alpha, \beta \in \mathcal{R}L$ such that $\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \subseteq Q$ and $\boldsymbol{M}_{(\operatorname{coz}\beta)^{**}} \subseteq Q$. Without loss of generality, we may assume α and β are positive. Since L is a quasi F-frame,

$\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} + \boldsymbol{M}_{(\operatorname{coz}\beta)^{**}}$ is a d-ideal (see proof of Proposition 4.2.2), and hence the smallest d-ideal containing these two d-ideals. In other words, $\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} + \boldsymbol{M}_{(\operatorname{coz}\beta)^{**}}$ is the join in $\operatorname{Did}(\mathcal{R}L)$ of the ideals $\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}}$ and $\boldsymbol{M}_{(\operatorname{coz}\beta)^{**}}$. But by Lemma 4.3.2 the join of these ideals in $\operatorname{Did}(\mathcal{R}L)$ is

$$\boldsymbol{M}_{(\operatorname{coz}\alpha\vee\operatorname{coz}\beta)^{**}} = \boldsymbol{M}_{(\operatorname{coz}(\alpha+\beta))^{**}}.$$

This shows that $\boldsymbol{M}_{(\operatorname{coz}(\alpha+\beta))^{**}} \subseteq Q$, and hence the collection is directed. Therefore D is a d-ideal contained in Q. Now suppose H is a d-ideal with $D \subseteq H \subseteq Q$. Let $\alpha \in H$. Since H is d-ideal, $\alpha^{\perp\perp} = \boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \subseteq H \subseteq Q$. Therefore $\alpha \in D$, and hence $D = H$. This completes the proof. $\square$

Chapter 5

Two functors induced by z- and d-ideals

We start by collecting some data from Chapter 2 and Chapter 4. We showed that $\mathrm{Zid}(\mathcal{R}L)$ is a normal coherent frame, and that $\mathrm{Did}(\mathcal{R}L) = \mathrm{Fix}(d)$, for the d-nucleus on $\mathrm{Zid}(\mathcal{R}L)$; so that, by [51, Theorem 4.2], $\mathrm{Did}(\mathcal{R}L)$ is also a normal coherent frame. The compact elements of these coherent frames are given by

$$\mathfrak{K}\big(\mathrm{Zid}(\mathcal{R}L)\big) = \{\boldsymbol{M}_c \mid c \in \mathrm{Coz}\, L\} \quad \text{and} \quad \mathfrak{K}\big(\mathrm{Did}(\mathcal{R}L)\big) = \{\boldsymbol{M}_{c^{**}} \mid c \in \mathrm{Coz}\, L\}.$$

Recall from Proposition 2.4.1 that, for any frame homomorphism $h\colon L \to M$, the map $\mathrm{Zid}(\mathcal{R}L) \to \mathrm{Zid}(\mathcal{R}M)$, defined by

$$Q \mapsto \bigvee \{\boldsymbol{M}_{h(\mathrm{coz}\,\alpha)} \mid \alpha \in Q\},$$

is a coherent map, which we denoted by $\mathrm{Zid}(h)$ in Chapter 2. We shall here denote it by $\bar{h}$; not to be confused with the map $\bar{h}$ in Chapter 2. For purposes of computation it is helpful to note that

$$\bar{h}(Q) = \bigcup \{\boldsymbol{M}_{h(\mathrm{coz}\,\alpha)} \mid \alpha \in Q\},$$

and that, for each $c \in \mathrm{Coz}\, L$,

$$\bar{h}(\boldsymbol{M}_c) = \boldsymbol{M}_{h(c)}.$$

What we have just discussed suggests how the functor based on z-ideals should be defined. It must send an object $L \in \mathbf{CRegFrm}$ to $\mathrm{Zid}(\mathcal{R}L)$, and a morphism $h \in \mathbf{CRegFrm}$ to the morphism $\bar{h}$ in $\mathbf{CohFrm}$. We will formalise this shortly; but first we clear the ground for the construction of the functor based on d-ideals.

This functor will send a completely regular frame L to the coherent frame $\mathrm{Did}(\mathcal{R}L)$. To define its action on morphisms we will associate with every frame homomorphism $h\colon L \to M$ a coherent map $\tilde{h}\colon \mathrm{Did}(\mathcal{R}L) \to \mathrm{Did}(\mathcal{R}M)$, roughly along the same lines as how $\bar{h}$ is defined.

5.1 The functors Z and D

Definition 5.1.1. *We define* $\mathsf{Z}\colon \mathbf{CRegFrm} \to \mathbf{CohFrm}$ *by*

$$\mathsf{Z}(L) = \mathrm{Zid}(\mathcal{R}L) \quad \textit{and} \quad \mathsf{Z}(h) = \bar{h}.$$

Lemma 5.1.1. *Given a frame homomorphism* $h\colon L \to M$*, the map* $\phi\colon \mathfrak{K}(\mathrm{Did}(\mathcal{R}L)) \to \mathfrak{K}(\mathrm{Did}(\mathcal{R}M))$ *given by*

$$\phi(\boldsymbol{M}_{c^{**}}) = \boldsymbol{M}_{h(c)^{**}}$$

is a lattice homomorphism.

Proof. It is clear that ϕ preserves the bottom and the top. We show that it preserves binary joins. The proof that it preserves meets is similar. Let $a, b \in \operatorname{Coz} L$. Then

$$\begin{aligned}
\phi\big(\boldsymbol{M}_{a^{**}} \sqcup \boldsymbol{M}_{b^{**}}\big) &= \phi\big(\boldsymbol{M}_{(a\vee b)^{**}}\big) \\
&= \boldsymbol{M}_{(h(a\vee b))^{**}} \\
&= \boldsymbol{M}_{(h(a)\vee h(b))^{**}} \\
&= \boldsymbol{M}_{h(a)^{**}} \sqcup \boldsymbol{M}_{h(b)^{**}} \\
&= \phi\big(\boldsymbol{M}_{a^{**}}\big) \sqcup \phi\big(\boldsymbol{M}_{b^{**}}\big),
\end{aligned}$$

which shows that ϕ preserves binary joins. □

We now invoke the fact that if A and B are coherent frames, then any lattice homomorphism $\mathfrak{k}(A) \to \mathfrak{k}(B)$ extends uniquely to a coherent map $A \to B$ (see [46, page 64]). Let $h\colon L \to M$ be a frame homomorphism, and consider any $Q \in \mathrm{Did}(\mathcal{R}L)$. Since

$$Q = \bigvee\{\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}} \mid \alpha \in Q\},$$

the map $\tilde{h}\colon \mathrm{Did}(\mathcal{R}L) \to \mathrm{Did}(\mathcal{R}M)$ defined by

$$\tilde{h}(Q) = \bigvee\{\boldsymbol{M}_{(h(\operatorname{coz}\alpha))^{**}} \mid \alpha \in Q\}$$

is the unique coherent map extending the map ϕ defined above. We are now equipped to define the two functors.

Definition 5.1.2. We define $\mathsf{Z}\colon \mathbf{CRegFrm} \to \mathbf{CohFrm}$ by setting $\mathsf{Z}(L) = \mathrm{Zid}(\mathcal{R}L)$ and $\mathsf{Z}(h) = \bar{h}$; and we define $\mathsf{D}\colon \mathbf{CRegFrm} \to \mathbf{CohFrm}$ by setting $\mathsf{D}(L) = \mathrm{Did}(\mathcal{R}L)$ and $\mathsf{D}(h) = \tilde{h}$.

For any frame L, we write δ_L for the frame homomorphism $\delta_L\colon \mathrm{Zid}(\mathcal{R}L) \to \mathrm{Did}(\mathcal{R}L)$ induced by the d-nucleus on $\mathrm{Zid}(\mathcal{R}L)$. We emphasize that $\delta_L(\boldsymbol{M}_c) = \boldsymbol{M}_{c^{**}}$, for any $c \in \mathrm{Coz}\,L$, because the d-nucleus sends a compact element to its double pseudocomplement, and the double pseudocomplement of $\boldsymbol{M}_c$, contemplated in $\mathrm{Zid}(\mathcal{R}L)$, is $\boldsymbol{M}_{c^{**}}$.

Proposition 5.1.1. *The following statements about Z and D hold.*

(a) Z and D are functors.

(b) Both Z and D are faithful.

(c) The assignment $L \mapsto \delta_L$ is a natural transformation $\mathsf{Z} \to \mathsf{D}$.

Proof. (a) We give a proof only for Z since that for D is similar, with minor adjustments such as writing double pseudocomplements where appropriate. That Z preserves identities follows easily from the fact that, for any $Q \in \mathrm{Zid}(\mathcal{R}L)$,

$$Q = \bigvee_{\mathrm{Zid}(\mathcal{R}L)} \{\boldsymbol{M}_{\mathrm{coz}\,\alpha} \mid \alpha \in Q\} = \bigcup \{\boldsymbol{M}_{\mathrm{coz}\,\alpha} \mid \alpha \in Q\}.$$

Let $h\colon L \to M$ and $g\colon M \to N$ be morphisms in $\mathbf{CRegFrm}$. We must show that $\mathsf{Z}(g \cdot h) = \mathsf{Z}(g) \cdot \mathsf{Z}(h)$. For any $Q \in \mathrm{Zid}(\mathcal{R}L)$,

$$\mathsf{Z}(g \cdot h)(Q) = \bigcup \{\boldsymbol{M}_{(g\cdot h)(\mathrm{coz}\,\gamma)} \mid \gamma \in Q\} = \bigcup \{\boldsymbol{M}_{\mathrm{coz}\,(g\cdot h\cdot\gamma)} \mid \gamma \in Q\},$$

and

$$\begin{aligned}
\mathsf{Z}(g) \cdot \mathsf{Z}(h)(Q) &= \mathsf{Z}(g)\Big(\bigcup \{\boldsymbol{M}_{\mathrm{coz}\,(h\cdot\alpha)} \mid \alpha \in Q\}\Big) \\
&= \bigcup \Big\{\boldsymbol{M}_{\mathrm{coz}\,(g\cdot\tau)} \mid \tau \in \bigcup \{\boldsymbol{M}_{\mathrm{coz}\,(h\cdot\alpha)} \mid \alpha \in Q\}\Big\} \\
&= \bigcup \{\boldsymbol{M}_{\mathrm{coz}\,(g\cdot\tau)} \mid \tau \in R\},
\end{aligned}$$

where, for brevity, we write $R = \bigcup\{\boldsymbol{M}_{\mathrm{coz}\,(h\cdot\alpha)} \mid \alpha \in Q\}$. We must show that

$$\bigcup\{\boldsymbol{M}_{\mathrm{coz}\,(g\cdot h\cdot\gamma)} \mid \gamma \in Q\} = \bigcup\{\boldsymbol{M}_{\mathrm{coz}\,(g\cdot\tau)} \mid \tau \in R\}. \tag{5.1}$$

Let $\gamma \in Q$. Then $h \cdot \gamma \in R$ because $\boldsymbol{M}_{\mathrm{coz}\,(h\cdot\gamma)} \subseteq R$ and $h \cdot \gamma \in \boldsymbol{M}_{\mathrm{coz}\,(h\cdot\gamma)}$. Now,

$$\mathrm{coz}\,((g \cdot h) \cdot \gamma) = \mathrm{coz}\,(g \cdot (h \cdot \gamma)) = \mathrm{coz}\,(g \cdot \tau),$$

for $\tau = h \cdot \gamma$. Therefore

$$\boldsymbol{M}_{\mathrm{coz}\,(g\cdot h\cdot\gamma)} = \boldsymbol{M}_{\mathrm{coz}\,(g\cdot\tau)} \subseteq \bigcup\{\boldsymbol{M}_{\mathrm{coz}\,(g\cdot\rho)} \mid \rho \in R\},$$

which shows that the left side of (5.1) is contained in the right side. To show the reverse inclusion, let $\beta \in \bigcup\{\boldsymbol{M}_{\mathrm{coz}\,(g\cdot\tau)} \mid \tau \in R\}$. Pick a $\tau \in R$ such that $\beta \in \boldsymbol{M}_{\mathrm{coz}\,(g\cdot\tau)}$. Next, pick $\alpha \in Q$ such that $\tau \in \boldsymbol{M}_{\mathrm{coz}\,(h\cdot\alpha)}$. Since $\beta \in \boldsymbol{M}_{\mathrm{coz}\,(g\cdot\tau)}$ and $\mathrm{coz}\,\tau \leq \mathrm{coz}\,(h \cdot \alpha)$, we have

$$\mathrm{coz}\,\beta \leq \mathrm{coz}\,(g \cdot \tau) = g(\mathrm{coz}\,\tau) \leq g(\mathrm{coz}\,(h \cdot \alpha)) = \mathrm{coz}\,(g \cdot h \cdot \alpha),$$

which implies

$$\beta \in \boldsymbol{M}_{\mathrm{coz}\,(g\cdot h\cdot\alpha)} \subseteq \bigcup\{\boldsymbol{M}_{\mathrm{coz}\,(g\cdot h\cdot\gamma)} \mid \gamma \in Q\}.$$

Thus, the desired equality holds, and hence Z is a functor.

(b) We prove the faithfulness of D only because that of Z is similar; and, in fact, more straightforward. We will use the fact that if $x \prec\!\!\prec a$, then $x^{**} \leq a$. Let $h\colon L \to M$ and $g\colon L \to M$ be two morphisms in **CRegFrm** such that $\mathsf{D}(h) = \mathsf{D}(g)$. Then, for any $c \in \mathrm{Coz}\,L$, $\mathsf{D}(h)(\boldsymbol{M}_{c^{**}}) = \mathsf{D}(g)(\boldsymbol{M}_{c^{**}})$, which implies $\boldsymbol{M}_{h(c)^{**}} = \boldsymbol{M}_{g(c)^{**}}$, and consequently, $h(c)^{**} = g(c)^{**}$. Let $a \in L$. Then, by complete regularity,

$$a = \bigvee\{c \in \mathrm{Coz}\,L \mid c \prec\!\!\prec a\},$$

and hence

$$\begin{aligned} h(a) &= \bigvee\{h(c) \mid c \in \mathrm{Coz}\,L \text{ and } c \prec\!\!\prec a\} \\ &\leq \bigvee\{h(c)^{**} \mid c \in \mathrm{Coz}\,L \text{ and } c \prec\!\!\prec a\} \\ &= \bigvee\{g(c)^{**} \mid c \in \mathrm{Coz}\,L \text{ and } c \prec\!\!\prec a\} \\ &\leq g(a) \qquad \text{since } g(c) \prec\!\!\prec g(a) \text{ whenever } c \prec\!\!\prec a. \end{aligned}$$

By symmetry, we conclude that $h(a) = g(a)$, so that $h = g$. Therefore D is faithful.

(c) To prove the claimed naturality, we need to show that, for any frame morphism $h\colon L \to M$ in **CRegFrm**, the square

$$\begin{tikzcd}
\mathrm{Zid}(\mathcal{R}L) \arrow[r, "\mathsf{Z}(h)"] \arrow[d, "\delta_L"'] & \mathrm{Zid}(\mathcal{R}M) \arrow[d, "\delta_M"] \\
\mathrm{Did}(\mathcal{R}L) \arrow[r, "\mathsf{D}(h)"] & \mathrm{Did}(\mathcal{R}M)
\end{tikzcd} \tag{5.2}$$

commutes. Since $\mathrm{Zid}(\mathcal{R}L)$ is generated by its compact elements, it suffices to show that, for any $c \in \mathrm{Coz}\, L$,

$$\delta_M \mathsf{Z}(h)(\boldsymbol{M}_c) = \mathsf{D}(h)\delta_L(\boldsymbol{M}_c).$$

Since $h(c) \in \mathrm{Coz}\, M$, so that $\mathsf{D}(h)(\boldsymbol{M}_{c^{**}}) = \boldsymbol{M}_{h(c)^{**}}$, the desired equality follows easily from the way the involved homomorphisms map. $\square$

We close this section by putting Lemma 4.3.2 to another good use; this time to characterise those frames L for which $\mathrm{Did}(\mathcal{R}L)$ is regular. Recall from [55, Corollary 2.6] that a coherent frame is regular if and only if every compact element in it has a compact complement.

Proposition 5.1.2. $\mathrm{Did}(\mathcal{R}L)$ *is regular if and only if* L *is cozero complemented.*

Proof. Suppose L is cozero complemented, and let $\boldsymbol{M}_{c^{**}}$ be a compact element of $\mathrm{Did}(\mathcal{R}L)$, with $c \in \mathrm{Coz}\, L$. Take $d \in \mathrm{Coz}L$ such that $c \wedge d = 0$ and $c \vee d$ is dense. Then

$$\boldsymbol{M}_{c^{**}} \sqcup \boldsymbol{M}_{d^{**}} = \boldsymbol{M}_{(c\vee d)^{**}} = \top \quad \text{and} \quad \boldsymbol{M}_{c^{**}} \wedge \boldsymbol{M}_{d^{**}} = \boldsymbol{M}_{(c\wedge d)^{**}} = \bot,$$

which says $\boldsymbol{M}_{c^{**}}$ is complemented with complement $\boldsymbol{M}_{d^{**}}$. Therefore $\mathrm{Did}(\mathcal{R}L)$ is regular.

Conversely, suppose $\mathrm{Did}(\mathcal{R}L)$ is regular, and let $c \in \mathrm{Coz}L$. Then the compact element $\boldsymbol{M}_{c^{**}}$ of $\mathrm{Did}(\mathcal{R}L)$ has a compact complement, say $\boldsymbol{M}_{d^{**}}$, for some $d \in \mathrm{Coz}L$. A calculation as above shows that $(c \wedge d)^{**} = 0$ and $(c \vee d)^{**} = 1$, so that $c \wedge d = 0$ and $c \vee d$ is dense. Therefore L is cozero complemented. $\square$

Remark 5.1.1. In Proposition 2.3.3, it is shown that $\mathrm{Zid}(\mathcal{R}L)$ is regular if and only if L is a P-frame, which is to say every cozero element is complemented. Every P-frame is cozero complemented, but not conversely, as, for instance, the frame of reals attests.

5.2 Replacing morphisms with their right adjoints

Following [38], we say that a commutative square in **Frm**, like the one on the left of the squares

$$\begin{tikzcd} L \arrow[r, "h"] \arrow[d, "\mathfrak{a}"'] & M \arrow[d, "\mathfrak{b}"] \\ N \arrow[r, "g"] & K \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L \arrow[r, "h"] & M \\ N \arrow[u, "\mathfrak{a}_*"] \arrow[r, "g"] & K, \arrow[u, "\mathfrak{b}_*"'] \end{tikzcd} \tag{5.3}$$

is $(\mathfrak{a}, \mathfrak{b})$-*round* (or simply *round*) if the one on the right, which is obtained by replacing the downward morphisms with their right adjoints, is also commutative. We do not require that the square on the right be in **Frm**. In a commutative square in **Frm** we can replace any pair of parallel morphisms with their right adjoints. So let us agree that when we say a square such as the one on the left of (5.3) is round we shall be meaning that it is the downward morphisms that are to be replaced with their right adjoints.

Proposition 5.2.1. *The square*

$$\begin{tikzcd} \mathrm{Zid}(\mathcal{R}L) \arrow[r, "\mathsf{Z}(h)"] \arrow[d, "\delta_L"'] & \mathrm{Zid}(\mathcal{R}M) \arrow[d, "\delta_M"] \\ \mathrm{Did}(\mathcal{R}L) \arrow[r, "\mathsf{D}(h)"] & \mathrm{Did}(\mathcal{R}M) \end{tikzcd} \tag{5.4}$$

is round if and only if $h(c^{**}) = h(c)^{**}$ for every $c \in \mathrm{Coz}\,L$.

Proof. Suppose the square is round, and let $c \in \mathrm{Coz}\,L$. Keep in mind that $(\delta_L)_*$ and $(\delta_M)_*$ are inclusion maps. Since $\boldsymbol{M}_{c^{**}} \in \mathrm{Did}(\mathcal{R}L)$, we have

$$(\delta_M)_*\mathsf{D}(h)(\boldsymbol{M}_{c^{**}}) = \mathsf{Z}(h)(\delta_L)_*(\boldsymbol{M}_{c^{**}}),$$

by roundness of the square, which implies $\boldsymbol{M}_{h(c)^{**}} = \boldsymbol{M}_{h(c^{**})}$, whence $h(c)^{**} = h(c^{**})$.

Conversely, suppose the condition holds. Since for any $Q \in \mathrm{Did}(\mathcal{R}L)$,

$$Q = \bigvee\{\boldsymbol{M}_{(\mathrm{coz}\,\alpha)^{**}} \mid \alpha \in Q\}$$

we have

$$\begin{aligned}(\delta_M)_*\mathsf{D}(h)(Q) = \mathsf{D}(h)(Q) &= \bigvee\{\mathsf{D}(h)(\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}}) \mid \alpha \in Q\} \\ &= \bigvee\{\boldsymbol{M}_{h(\operatorname{coz}\alpha)^{**}} \mid \alpha \in Q\}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\mathsf{Z}(h)(\delta_L)_*(Q) = \mathsf{Z}(h)(Q) &= \bigvee\{\mathsf{Z}(h)(\boldsymbol{M}_{(\operatorname{coz}\alpha)^{**}}) \mid \alpha \in Q\} \\ &= \bigvee\{\boldsymbol{M}_{h((\operatorname{coz}\alpha)^{**})} \mid \alpha \in Q\} \\ &= \bigvee\{\boldsymbol{M}_{h(\operatorname{coz}\alpha)^{**}} \mid \alpha \in Q\} \quad \text{by the hypothesis on } h.\end{aligned}$$

Therefore $(\delta_M)_*\mathsf{D}(h) = \mathsf{Z}(h)(\delta_L)_*(Q)$, as required. □

Remark 5.2.1. The condition $h(c^{**}) = h(c)^{**}$ for each $c \in \operatorname{Coz} L$ brings to mind *nearly open* maps, which are frame homomorphisms $\varphi\colon L \to M$ such that $\varphi(a^*) = \varphi(a)^*$ for each $a \in L$. It is shown in [24] that φ is nearly open if and only if $\varphi(a^{**}) = \varphi(a)^{**}$ for every $a \in L$.

We have not been able to find a condition on h alone which ensures that the diagram resulting from the square (5.4) by replacing the horizontal morphisms with their right adjoints is commutative. However, if the domain of h is assumed to be *perfectly normal*, which is to say $\operatorname{Coz} L = L$ because our frames are completely regular, then we have a condition on h which guarantees commutativity of the resulting diagram. That is the content of the next result. Observe that, for any frame homomorphism $h\colon L \to M$, and any $a \in M$,

$$\mathsf{Z}(h)_*(\boldsymbol{M}_a) = \boldsymbol{M}_{h_*(a)},$$

as an easy calculation reveals. It is actually explicitly shown in Chapter 2 that, for any $Q \in \operatorname{Zid}(\mathcal{R}M)$,

$$\mathsf{Z}(h)_*(Q) = \bigvee\{\boldsymbol{M}_{h_*(\operatorname{coz}\alpha)} \mid \alpha \in Q\}.$$

We need also to know how the right adjoint of $\mathsf{D}(h)$ maps. Since right adjoints of coherent maps preserve directed joins, and every element of $\operatorname{Did}(\mathcal{R}M)$ is a directed join of compact elements, it suffices to know how $\mathsf{D}(h)_*$ acts on compact elements.

Lemma 5.2.1. *Let $h\colon L \to M$ be a frame homomorphism. For any $c \in \mathit{Coz}\, M$,*

$$\mathsf{D}(h)_*\big(\boldsymbol{M}_{c^{**}}\big) = \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \mathit{Coz}\, L,\ a^{**} \le h_*(c^{**})\}.$$

Proof. Since the square (5.2) commutes, $(\delta_L)_* \cdot \mathsf{D}(h)_* = \mathsf{Z}(h)_* \cdot (\delta_M)_*$, so that, by the surjectivity of δ_L, $\mathsf{D}(h)_* = \delta_L \cdot \mathsf{Z}(h)_* \cdot (\delta_M)_*$. Since $(\delta_M)_*$ is the inclusion map, for any $c \in \mathrm{Coz}\, M$ we have

$$\begin{aligned}
\mathsf{D}(h)_*\big(\boldsymbol{M}_{c^{**}}\big) &= \delta_L\big(\boldsymbol{M}_{h_*(c^{**})}\big)\\
&= \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \mathrm{Coz}\, L,\ \boldsymbol{M}_{a^{**}} \subseteq \boldsymbol{M}_{h_*(c^{**})}\}\\
&= \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \mathrm{Coz}\, L,\ a^{**} \le h_*(c^{**})\}.
\end{aligned}$$

□

Proposition 5.2.2. *Let $h\colon L \to M$ be a frame homomorphism with L perfectly normal. The square*

$$\begin{tikzcd}
\mathrm{Zid}(\mathcal{R}L) \arrow[d, "\delta_L"'] & \mathrm{Zid}(\mathcal{R}M) \arrow[l, "\mathsf{Z}(h)_*"'] \arrow[d, "\delta_M"] \\
\mathrm{Did}(\mathcal{R}L) & \mathrm{Did}(\mathcal{R}M) \arrow[l, "\mathsf{D}(h)_*"]
\end{tikzcd}$$

commutes if and only if $h_*(c)^{**} = h_*(c^{**})^{**}$ for every $c \in \mathit{Coz}\, M$.

Proof. Observe first that, in view of L being perfectly normal, for any $x \in L$ we have

$$\bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \mathrm{Coz}\, L,\ a \le x\} = \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \mathrm{Coz}\, L,\ a^{**} \le x^{**}\} = \boldsymbol{M}_{x^{**}}, \tag{5.5}$$

where the joins are reckoned in $\mathrm{Did}(\mathcal{R}L)$. Now suppose the square commutes, and let $c \in \mathrm{Coz}\, M$. Then, by commutativity of the square,

$$\begin{aligned}
\mathsf{D}(h)_*\delta_M(\boldsymbol{M}_c) &= \delta_L\mathsf{Z}(h)_*(\boldsymbol{M}_c)\\
&= \delta_L(\boldsymbol{M}_{h_*(c)})\\
&= \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \mathrm{Coz}\, M,\ \boldsymbol{M}_a \le \boldsymbol{M}_{h_*(c)}\}\\
&= \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \mathrm{Coz}\, M,\ a \le h_*(c)\}\\
&= \boldsymbol{M}_{h_*(c)^{**}} \qquad \text{by (5.5).}
\end{aligned}$$

Now, since

$$\mathsf{D}(h)_*\delta_M(\boldsymbol{M}_c) = \mathsf{D}(h)_*(\boldsymbol{M}_{c^{**}}) = \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \operatorname{Coz} L,\ a^{**} \leq h_*(c^{**})\},$$

it follows that

$$\boldsymbol{M}_{h_*(c)^{**}} = \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \operatorname{Coz} L,\ a^{**} \leq h_*(c^{**})\}.$$

Applying the map τ_L to this equality, and recalling how the join is calculated in $\mathfrak{B}L$, we obtain

$$h_*(c)^{**} = \left(\bigvee\{a^{**} \mid a \in \operatorname{Coz} L,\ a^{**} \leq h_*(c^{**})\}\right)^{**} = h_*(c^{**})^{**},$$

since the join equals $h_*(c^{**})$, by complete regularity.

For the converse, note first that the equality $\mathsf{Z}(h)_* \cdot (\delta_M)_* = (\delta_L)_* \cdot \mathsf{D}(h)_*$ and the surjectivity of δ_L imply

$$\delta_L \cdot \mathsf{Z}(h)_* \leq \delta_L \cdot \mathsf{Z}(h)_* \cdot (\delta_M)_* \cdot \delta_M = \mathsf{D}(h)_* \cdot \delta_M.$$

So we need only show that $\mathsf{D}(h)_* \cdot \delta_M \leq \delta_L \cdot \mathsf{Z}(h)_*$. Since each of these maps preserves directed joins, and any member of $\operatorname{Did}(\mathcal{R}M)$ is a directed join of compact elements, it suffices to show that

$$\mathsf{D}(h)_*\delta_M(\boldsymbol{M}_c) \leq \delta_L\mathsf{Z}(h)_*(\boldsymbol{M}_c) \quad \text{for all } c \in \operatorname{Coz} M.$$

Indeed,

$$\begin{aligned}
\mathsf{D}(h)_*\delta_M(\boldsymbol{M}_c) &= \mathsf{D}(h)_*(\boldsymbol{M}_{c^{**}}) \\
&= \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \operatorname{Coz} L, a^{**} \leq h_*(c^{**})\} \\
&\leq \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \operatorname{Coz} L, a^{**} \leq h_*(c^{**})^{**}\} \\
&= \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \operatorname{Coz} L, a^{**} \leq h_*(c)^{**}\} \quad \text{by the hypothesis on } h \\
&= \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \operatorname{Coz} L, a \leq h_*(c)\} \quad \text{by (5.5)} \\
&= \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \operatorname{Coz} L, \boldsymbol{M}_a \leq \boldsymbol{M}_{h_*(c)}\} \\
&= \bigvee\{\boldsymbol{M}_{a^{**}} \mid a \in \operatorname{Coz} L, \boldsymbol{M}_a \leq \mathsf{Z}(h)_*(\boldsymbol{M}_c)\} \\
&= \delta_L(\mathsf{Z}(h)_*(\boldsymbol{M}_c)).
\end{aligned}$$

This completes the proof. $\square$

5.3 Some commutative squares associated with d-ideals

Proposition 5.3.1. *For any completely regular frame L, the square*

$$\begin{tikzcd} \mathrm{Zid}(\mathcal{R}L) \arrow[r, "\delta_L"] \arrow[d, "\sigma_L"'] & \mathrm{Did}(\mathcal{R}L) \arrow[d, "\tau_L"] \\ L \arrow[r, "\flat_L"'] & \mathfrak{B}L \end{tikzcd} \tag{5.6}$$

is commutative.

Proof. Since $\mathrm{Zid}(\mathcal{R}L)$ is algebraic, it suffices to show that $\tau_L \cdot \delta_L$ and $\flat_L \cdot \sigma_L$ agree on compact elements. For any $c \in \mathrm{Coz}\, L$,

$$\tau_L(\delta_L(\boldsymbol{M}_c)) = \tau_L(\boldsymbol{M}_{c^{**}}) = (c^{**})^{**} = c^{**} = \flat_L(\sigma_L(\boldsymbol{M}_c)),$$

which proves the result. □

We recall that a frame homomorphism is *skeletal* if it sends dense elements to dense elements. By a result of Banaschewski and Pultr in [23], $h\colon L \to M$ is skeletal if and only if $h(a^{**}) \leq h(a)^{**}$ for every $a \in L$.

Proposition 5.3.2. *Let $h\colon L \to M$ be a skeletal frame homomorphism between completely regular frames. Then in the diagram*

$$\begin{tikzcd} \mathrm{Zid}(\mathcal{R}L) \arrow[rrr, "Z(h)"] \arrow[dr, "\delta_L"] \arrow[ddd, "\sigma_L"'] & & & \mathrm{Zid}(\mathcal{R}M) \arrow[dl, "\delta_M"'] \arrow[ddd, "\sigma_M"] \\ & \mathrm{Did}(\mathcal{R}L) \arrow[r, "D(h)"] \arrow[d, "\tau_L"'] & \mathrm{Did}(\mathcal{R}M) \arrow[d, "\tau_M"] & \\ & \mathfrak{B}L \arrow[r, "\mathfrak{B}(h)"'] & \mathfrak{B}M & \\ L \arrow[ur, "\flat_L"] \arrow[rrr, "h"'] & & & M \arrow[ul, "\flat_M"'] \end{tikzcd}$$

every quadrilateral is commutative.

Proof. We already know that the outer square, the lower trapezoid and the trapezoids on the left and the right are commutative. Let us show that the upper trapezoid is commutative. Since $\mathrm{Zid}(\mathcal{R}L)$ is algebraic, it suffices to show that $D(h) \cdot d_L$ and $d_M \cdot Z(h)$ agree on compact elements. For this simply observe that, for any $c \in \operatorname{Coz} L$,

$$D(h)(\boldsymbol{M}_{c^{**}}) = \boldsymbol{M}_{h(c)^{**}}.$$

To see the commutativity of the inner square we again compare the composites at compact elements. For any $c \in \operatorname{Coz} L$,

$$\tau_M D(h)(\boldsymbol{M}_{c^{**}}) = \tau_M(\boldsymbol{M}_{h(c)^{**}}) = h(c)^{**}.$$

On the other hand,

$$\mathfrak{B}(h)\tau_L(\boldsymbol{M}_{c^{**}}) = \mathfrak{B}(h)(c^{**}) = h(c^{**})^{**} = h(c)^{**}$$

since h is dense onto. Therefore $\tau_M \cdot D(h) = \mathfrak{B}(h) \cdot \tau_L$. $\square$

Proposition 5.3.3. *For a completely regular frame L, the square*

$$\begin{tikzcd}
\mathrm{Zid}(\mathcal{R}L) \arrow[r, "\delta_L"] & \mathrm{Did}(\mathcal{R}L) \\
L \arrow[u, "(\sigma_L)_*"] \arrow[r, "\flat_L"'] & \mathfrak{B}L \arrow[u, "(\tau_L)_*"']
\end{tikzcd}$$

is a commutative diagram.

Proof. We start by determining how $(\tau_L)_*$ maps. Since $\tau_L \cdot \delta_L = \flat_L \cdot \sigma_L$ (refer to diagram 5.6), we have $(\delta_L)_* \cdot (\tau_L)_* = (\sigma_L)_* \cdot (\flat_L)_*$, which, by surjectivity of δ_L, implies

$$(\tau_L)_* = \delta_L \cdot (\sigma_L)_* \cdot (\flat_L)_*.$$

Therefore for any $x \in \mathfrak{B}L$,

$$(\tau_L)_*(x) = \delta_L((\sigma_L)_*(x))$$

because $(\flat_L)_*$ is the inclusion $\mathfrak{B}L \to L$. Recall from Chapter 2 that $(\sigma_L)_*(x) = \boldsymbol{M}_x$ for every $x \in L$. Therefore

$$(\tau_L)_*(x) = \delta_L(\boldsymbol{M}_x) = \boldsymbol{M}_{x^{**}}.$$

Now for any $a \in L$, $\delta_L(\sigma_L)_*(a) = \delta_L(\boldsymbol{M}_a) = \boldsymbol{M}_{a^{**}}$ and

$$(\tau_L)_*\flat_L(a) = (\tau_L)_*(a^{**}) = \boldsymbol{M}_{(a^{**})^{**}} = \boldsymbol{M}_{a^{**}}.$$

Therefore the diagram commutes. $\square$

Proposition 5.3.4. *Let L be a completely regular frame, the square below is a commutative diagram if and only if $\bigvee_{\alpha\in Q}\operatorname{coz}\alpha = (\bigvee_{\alpha\in Q}\operatorname{coz}\alpha)^{**}$ for every $Q \in \operatorname{Did}(\mathcal{R}L)$.*

$$\begin{tikzcd}
\operatorname{Zid}(\mathcal{R}L) \arrow[d, "\sigma_L"'] & \operatorname{Did}(\mathcal{R}L) \arrow[l, "(\delta_L)_*"'] \arrow[d, "\tau_L"] \\
L & \mathfrak{B}L \arrow[l, "(\flat_L)_*"]
\end{tikzcd}$$

Proof. For any $Q \in \operatorname{Did}(\mathcal{R}L)$

$$(\flat_L)_*\tau_L(Q) = \tau_L(Q) = \Big(\bigvee_{\alpha\in Q}\operatorname{coz}\alpha\Big)^{**}.$$

On the other hand,

$$\sigma_L(\delta_L)_*(Q) = \sigma_L(Q) = \bigvee_{\alpha\in Q}\operatorname{coz}\alpha.$$

Therefore $(\flat_L)_* \cdot \tau_L = \sigma_L \cdot (\delta_L)_*$ if and only if $\left(\bigvee_{\alpha\in Q}\operatorname{coz}\alpha\right)^{**} = \bigvee_{\alpha\in Q}\operatorname{coz}\alpha$. $\square$

Remark 5.3.1. Since $\mathfrak{K}(\operatorname{Did}\mathcal{R}L) = \boldsymbol{M}_{a^{**}}$, a straightforward diagram chase shows that in $\mathfrak{K}(\operatorname{Did}\mathcal{R}L) = \boldsymbol{M}_{a^{**}}$, $(\flat_L)_* \cdot \tau_L$ and $\sigma_L \cdot (\delta_L)_*$ agree. Observe also that the diagram commutes whenever L is Boolean.

In Proposition 5.2.1 we encountered the condition $h(c^{**}) = h(c)^{**}$ for all $c \in \operatorname{Coz} L$. Weakening this condition and also the notion of skeletality, we may say a frame homomorphism $h\colon L \to M$ is *coz-skeletal* if $h(c^{**}) \le h(c)^{**}$ for every $c \in \operatorname{Coz} L$. For certain frames this agrees with the condition that h should send dense cozero elements to dense elements, as we show below, thus justifying in a way the choice of terminology.

Proposition 5.3.5. *Let L be a cozero-complemented frame. Then any frame homomorphism $h: L \to M$ is coz-skeletal if and only if $h(c)$ is dense for $c \in \operatorname{Coz} L$.*

Proof. $(\Rightarrow)$: Let $c \in \operatorname{Coz} L$ be dense. Then $c^{**} = 1$, so that $1 = h(c^{**}) \leq h(c)^{**}$, which implies that $h(c)$ is dense for every $c \in \operatorname{Coz} L$.

$(\Leftarrow)$: Let $c \in \operatorname{Coz} L$. Since L is cozero-complemented, there exists a $d \in \operatorname{Coz} L$ such that $c \wedge d = 0$ and $c \vee d$ is dense. Now, $c \wedge d = 0$ implies that $c^{**} \wedge d = 0$, hence $h(c^{**}) \wedge h(d) = 0$, so that $h(c^{**}) \leq h(d)^*$. Since $c \vee d$ is dense and is a cozero element, by the present hypothesis we have that $h(c) \wedge h(d) = 0$ and $h(c \vee d)$ is dense. So

$$0 = (h(c) \vee h(d))^* = h(c)^* \wedge h(d)^*,$$

which implies $h(d)^* \leq h(c)^{**}$. Thus $h(c^{**}) \leq h(d)^* \leq h(c)^{**}$, whence $h(c^{**}) \leq h(c)^{**}$. Therefore h is coz-skeletal. □

Remark 5.3.2. If $h: L \to M$ is coz-skeletal, then $h(c^{**})^{**} = h(c)^{**}$ for every $c \in \operatorname{Coz} L$. Indeed, $h(c) \leq h(c^{**})$, which implies that $h(c)^{**} \leq h(c^{**})^{**}$. For the reverse inequality, coz-skeletality implies $h(c^{**}) \leq h(c)^{**}$, so that $h(c^{**})^{**} \leq h(c)^{****} = h(c)^{**}$.

5.4 Preservation and reflection of certain properties

In the study of algebraic frames, skeletal and $*$-dense maps have recently played prominent rôles (see, for instance, [41] and other recent articles of Jorge Martínez). In this section we show that skeletality and $*$-density are preserved and reflected by the functors Z and D. We remind the reader that dense onto homomorphisms are skeletal, and surjective homomorphisms are $*$-dense.

Recall that if $\phi: A \to B$ is a dense onto frame homomorphism, then $\phi_*(x^*) = (\phi_*(x))^*$, for all $x \in B$; so that ϕ_* takes dense elements to dense elements. In preparation for one of the upcoming propositions, we recall from Lemma 2.4.1 that, for any completely regular frame L, the map $\sigma_L: \operatorname{Zid}(\mathcal{R}L) \to L$ defined by

$$\sigma_L(Q) = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in Q\}$$

is a dense onto frame homomorphism such that the following square commutes.

$$\begin{tikzcd}
\mathrm{Zid}(\mathcal{R}L) \arrow[r, "\mathrm{Zid}(h)"] \arrow[d, "\sigma_L"'] & \mathrm{Zid}(\mathcal{R}M) \arrow[d, "\sigma_M"] \\
L \arrow[r, "h"] & M
\end{tikzcd} \tag{5.7}$$

Lemma 5.4.1. *Suppose that the square*

$$\begin{tikzcd}
L \arrow[r, "h"] \arrow[d, "\mathfrak{a}"'] & M \arrow[d, "\mathfrak{b}"] \\
N \arrow[r, "g"'] & K
\end{tikzcd} \tag{5.8}$$

in **Frm** *is commutative, and the downward morphisms are dense onto. Then:*

(*a*) *h is skeletal if and only if g is skeletal.*

(*b*) *If h is $*$-dense, then so is g.*

Proof. (a) If h is skeletal, then since $g \cdot \mathfrak{a} = \mathfrak{b} \cdot h$ and $\mathfrak{a}$ is onto, we have $g = \mathfrak{b} \cdot h \cdot \mathfrak{a}_*$, which is a composite of maps each of which takes dense elements to dense elements. Therefore g is skeletal. Conversely, suppose g is skeletal, and let $a \in L$ be dense. Now, since $\mathfrak{b}$ is dense onto and $g \cdot \mathfrak{a}$ is skeletal,

$$\mathfrak{b}\big(h(a)^*\big) = \big(\mathfrak{b}h(a)\big)^* = \big(g\mathfrak{a}(a)\big)^* = 0,$$

which implies $h(a)^* = 0$ by density of $\mathfrak{b}$. Therefore h is skeletal.

(b) From the commutativity of the diagram, we have $\mathfrak{a}_* \cdot g_* = h_* \cdot \mathfrak{b}_*$, which, by the surjectivity of $\mathfrak{a}$, implies $g_* = \mathfrak{a} \cdot h_* \cdot \mathfrak{b}_*$. Now consider any $x \in K$ with $g_*(x) = 0$. The density of $\mathfrak{a}$ implies $h_*\mathfrak{b}_*(x) = 0$, so that $\mathfrak{b}_*(x) = 0$ since h is $*$-dense, by hypothesis, and hence $x = 0$ since $\mathfrak{b}$ is $*$-dense, as it is onto. □

Remark 5.4.1. If M is regular, then the converse statement in (b) also holds. For, if $h_*(z) = 0$ for $z \in M$, then for any $x \prec z$ we have $\mathfrak{b}_*\mathfrak{b}(x) \leq z$, so that $0 = h_*\mathfrak{b}_*\mathfrak{b}(x) = \mathfrak{a}_*g_*\mathfrak{b}(x)$, implying $\mathfrak{b}(x) = 0$, so that $x = 0$ since b is dense, whence $z = 0$, by regularity.

Proposition 5.4.1. *The following are equivalent for a frame homomorphism $h\colon L \to M$.*

(*a*) *h is skeletal.*

(*b*) *$\mathsf{Z}(h)$ is skeletal.*

(*c*) *$\mathsf{D}(h)$ is skeletal.*

Proof. This follows from Lemma 5.4.1, in view of the diagrams 5.7 and 5.4. □

Proposition 5.4.2. *The following are equivalent for any frame homomorphism $h\colon L \to M$.*

(1) *h is $*$-dense.*

(2) *$\mathsf{D}(h)$ is $*$-dense.*

(3) *$\mathsf{Z}(h)$ is $*$-dense.*

Proof. (1) ⇒ (2): Consider any $Q \in \mathrm{Did}(\mathcal{R}L)$ with $\mathsf{D}(h)_*(Q) = 0$. Now

$$Q = \bigvee\Big\{\boldsymbol{M}_{(\mathrm{coz}\,\alpha)^{**}} \mid \alpha \in Q\Big\},$$

since the join is directed and right adjoints of coherent maps preserve directed joins, we have

$$\mathsf{D}(h)_*\Big(\bigvee\{\boldsymbol{M}_{(\mathrm{coz}\,\alpha)^{**}} \mid \alpha \in Q\}\Big) = \bigvee\Big\{\mathsf{D}(h)_*\big(\boldsymbol{M}_{(\mathrm{coz}\,\alpha)^{**}}\big) \mid \alpha \in Q\Big\} = 0,$$

which implies $\mathsf{D}(h)_*(\boldsymbol{M}_{(\mathrm{coz}\,\alpha)^{**}}) = 0$ for each $\alpha \in Q$. Then, in view of how $\mathsf{D}(h)_*$ is computed (Lemma 5.2.1), what we have just said implies $\boldsymbol{M}_{a^{**}} = \{\mathbf{0}\}$ for each $a \in \mathrm{Coz}\, L$ with $a^{**} \le h_*((\mathrm{coz}\,\alpha)^{**})$, so that $a^{**} = 0$ for each $a \in \mathrm{coz}\, L$ with $a^{**} \le h_*((\mathrm{coz}\,\alpha)^{**})$. Since, by complete regularity, $h_*((\mathrm{coz}\,\alpha)^{**})$ is the join of all $a^{**} \in \mathrm{Coz}L$ with $a^{**} \le h_*((\mathrm{coz}\,\alpha)^{**})$, it follows that $h_*((\mathrm{coz}\,\alpha)^{**}) = 0$, whence, by $*$-density of h, $\mathrm{coz}\,\alpha = 0$, implying $\alpha = \mathbf{0}$. Therefore $Q = 0$, as required.

(2) ⇒ (3): This follows from Lemma 5.4.1, in light of the commutative square (5.4).

(3) ⇒ (1): Let $a \in M$ be such that $h_*(a) = 0$. For any $c \in \mathrm{Coz}L$ with $c \prec\!\!\prec a$, $c^{**} \le a$, and therefore $h_*(c^{**}) = 0$, which implies $\mathsf{Z}(h)_*(\boldsymbol{M}_{c^{**}}) = 0$, whence $c^{**} = 0$ by $*$-density of $\mathsf{Z}(h)$. Consequently $a = 0$, and therefore h is $*$-dense. □

5.5 Preservation and reflection of openness by the functor Z

We now turn to openness, and briefly investigate conditions under which Z preserves and reflects openness. The conditions for D that we have found are too stringent and do not seem worthy of inclusion. Recall that a frame homomorphism $h\colon L \to M$ is *open* if h has a left adjoint $h_!\colon M \to L$ which satisfies the Frobenius identity

$$h_!(h(a) \wedge b) = a \wedge h_!(b),$$

for all $a \in L$ and $b \in M$. If L is regular, then the equation above holds automatically (see, for instance, [62, p. 84]). In any event, since $h_!$ preserves order and $h_!h \leq id_L$, the inequality $h_!(h(a) \wedge b) \leq a \wedge h_!(b)$ always holds, so that in order to check the equation, it suffices to check that

$$a \wedge h_!(b) \leq h_!(h(a) \wedge b) \text{ for all } a \in L \text{ and } b \in M.$$

We recall again from [37] that a homomorphism $h\colon L \to M$ is a λ-*map* if the diagram

$$\begin{tikzcd} \lambda L \arrow[r, "h^\lambda"] & \lambda M \\ L \arrow[u, "(\lambda_L)_*"] \arrow[r, "h"'] & M \arrow[u, "(\lambda_M)_*"'] \end{tikzcd}$$

is round; that is, if $(\lambda_M)_* \cdot h = h^\lambda \cdot (\lambda_L)_*$. Since the comparison

$$h^\lambda \cdot (\lambda_L)_* \leq (\lambda_M)_* \cdot h$$

always holds, it follows that h is a λ-map if and only if $[h(a)] \subseteq h^\lambda([a])$ for every $a \in L$; that is, if and only if for any $a \in L$ and $z \in \operatorname{Coz} M$,

$$z \leq h(a) \implies z \leq h(c) \text{ for some } c \in \operatorname{Coz} L \text{ with } c \leq a.$$

The map Zid always satisfies $\mathrm{Zid}(h)(\boldsymbol{M}_c) = \boldsymbol{M}_{h(c)}$, for $c \in \operatorname{Coz} L$. Below we shall need to know when this holds for all $c \in L$. The next Lemma tells us when.

Lemma 5.5.1. $\mathsf{Z}(h)(\boldsymbol{M}_a) = \boldsymbol{M}_{h(a)}$ *for every* $a \in L$ *if and only if* h *is a* λ*-map.*

Proof. We show only the right-to-left implication; which is, in fact, the one we need. The other can be demonstrated similarly. It is clear from the definition of $\mathsf{Z}(h)$ that $\mathsf{Z}(h)(\boldsymbol{M}_a) \subseteq \boldsymbol{M}_{h(a)}$, for every $a \in L$. To reverse the inclusion, let $\alpha \in \boldsymbol{M}_{h(a)}$. Then $\operatorname{coz}\alpha \leq h(a)$. Since h is a λ-map, there is a $\gamma \in \mathcal{R}L$ such that

$$\operatorname{coz}\gamma \leq a \quad \text{and} \quad \operatorname{coz}\alpha \leq h(\operatorname{coz}\gamma).$$

Thus, $\alpha \in \boldsymbol{M}_{h(\operatorname{coz}\gamma)} \subseteq \mathsf{Z}(h)(\boldsymbol{M}_a)$. $\square$

We now have the following result which shows that the functor Z preserves and reflects the property of having a left adjoint for λ-maps.

Proposition 5.5.1. *A* λ*-map* $h\colon L \to M$ *has a left adjoint if and only if* $\mathsf{Z}(h)$ *has a left adjoint.*

Proof. ($\Rightarrow$) Assume h has a left adjoint, say $h_!\colon M \to L$. Define a map $\mathsf{Z}(h)_!\colon \operatorname{Zid}(\mathcal{R}M) \to \operatorname{Zid}(\mathcal{R}L)$ by

$$\mathsf{Z}(h)_!(R) = \bigvee\{\boldsymbol{M}_{h_!(\operatorname{coz}\alpha)} \mid \alpha \in R\} = \bigcup\{\boldsymbol{M}_{h_!(\operatorname{coz}\alpha)} \mid \alpha \in R\}; \tag{5.9}$$

the join being equal to the union because it is directed as $h_!$ preserves joins. We show that $\mathsf{Z}(h)_! \cdot \mathsf{Z}(h) \leq \mathrm{id}_{\operatorname{Zid}(\mathcal{R}L)}$. Let $Q \in \operatorname{Zid}(\mathcal{R}L)$. Then

$$\begin{aligned} \mathsf{Z}(h)_!(\mathsf{Z}(h)(Q)) &= \mathsf{Z}(h)_!\Big(\bigcup\{\boldsymbol{M}_{h(\operatorname{coz}\alpha)} \mid \alpha \in Q\}\Big) \\ &= \bigcup\Big\{\boldsymbol{M}_{h_!(\operatorname{coz}\tau)} \mid \tau \in \bigcup\{\boldsymbol{M}_{h(\operatorname{coz}\alpha)} \mid \alpha \in Q\}\Big\}. \end{aligned}$$

Consider any $\boldsymbol{M}_{h_!(\operatorname{coz}\tau)}$ with $\tau \in \bigcup\{\boldsymbol{M}_{h(\operatorname{coz}\alpha)} \mid \alpha \in Q\}$. Pick $\alpha \in Q$ such that $\tau \in \boldsymbol{M}_{h(\operatorname{coz}\alpha)}$. Since $\tau \in \boldsymbol{M}_{h(\operatorname{coz}\alpha)}$, $\operatorname{coz}\tau \leq h(\operatorname{coz}\alpha)$. Let $\rho \in \boldsymbol{M}_{h_!(\operatorname{coz}\tau)}$. Then

$$\operatorname{coz}\rho \leq h_!(\operatorname{coz}\tau) \leq h_!h(\operatorname{coz}\alpha) \leq \operatorname{coz}\alpha,$$

which implies $\rho \in Q$ since Q is a z-ideal. Therefore $\mathsf{Z}(h)_!(\mathsf{Z}(h)(Q)) \subseteq Q$, implying that $\mathsf{Z}(h)_! \cdot \mathsf{Z}(h) \leq \mathrm{id}_{\operatorname{Zid}(\mathcal{R}L)}$.

Next, let $P \in \mathrm{Zid}(\mathcal{R}M)$. Since $P = \bigvee\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \alpha \in P\}$,

$$\begin{aligned}
\mathsf{Z}(h)\mathsf{Z}(h)_!(P) &= \mathsf{Z}(h)\Big(\bigvee\{\boldsymbol{M}_{h_!(\operatorname{coz}\alpha)} \mid \alpha \in P\}\Big) \\
&= \bigvee\{\mathsf{Z}(h)(\boldsymbol{M}_{h_!(\operatorname{coz}\alpha)}) \mid \alpha \in P\} \\
&= \bigvee\{\boldsymbol{M}_{hh_!(\operatorname{coz}\alpha)} \mid \alpha \in P\} \quad \text{by Lemma 5.5.1 since } h \text{ is a } \lambda\text{-map} \\
&\geq \bigvee\{\boldsymbol{M}_{\operatorname{coz}\alpha} \mid \alpha \in P\} \quad \text{since } hh_! \geq \mathrm{id}_M \\
&= P.
\end{aligned}$$

This shows that $\mathrm{id}_{\mathrm{Zid}(\mathcal{R}M)} \leq \mathsf{Z}(h) \cdot \mathsf{Z}(h)_!$. Therefore $\mathsf{Z}(h)_!$ is left adjoint to $\mathsf{Z}(h)$.

($\Leftarrow$) Denote by $\mathsf{Z}(h)_!$ the left adjoint of $\mathsf{Z}(h)$, and define a map $h_!\colon M \to L$ by

$$h_!(b) = \bigvee\{\operatorname{coz}\alpha \mid \alpha \in \mathsf{Z}(h)_!(\boldsymbol{M}_b)\}.$$

Clearly, $h_!$ is order preserving. We will show that it is left adjoint to h. For any $b \in M$ we have

$$\begin{aligned}
h_!h(a) &= \bigvee\{\operatorname{coz}\alpha \mid \alpha \in \mathsf{Z}(h)_!(\boldsymbol{M}_{h(a)})\} \\
&= \bigvee\{\operatorname{coz}\alpha \mid \alpha \in \mathsf{Z}(h)_!\mathsf{Z}(h)(\boldsymbol{M}_a)\} \quad \text{by Lemma 5.5.1 since } h \text{ is a } \lambda\text{-map} \\
&\leq \bigvee\{\operatorname{coz}\alpha \mid \alpha \in \boldsymbol{M}_a\} \quad \text{since } \mathsf{Z}(h)_!\mathsf{Z}(h)(\boldsymbol{M}_a) \subseteq \boldsymbol{M}_a \\
&= \bigvee\{\operatorname{coz}\alpha \mid \operatorname{coz}\alpha \leq a\} \\
&= a.
\end{aligned}$$

Next, let $b \in M$. Then

$$hh_!(b) = \bigvee\{h(\operatorname{coz}\alpha) \mid \alpha \in \mathsf{Z}(h)_!(\boldsymbol{M}_b)\},$$

since h is a frame homomorphism. Since $\mathsf{Z}(h)_! \dashv \mathsf{Z}(h)$,

$$\boldsymbol{M}_b \subseteq \mathsf{Z}(h)\mathsf{Z}(h)_!(\boldsymbol{M}_b) = \bigcup\{\boldsymbol{M}_{h(\operatorname{coz}\rho)} \mid \rho \in \mathsf{Z}(h)_!(\boldsymbol{M}_b)\}.$$

Now consider any $\gamma \in \mathcal{R}M$ with $\operatorname{coz}\gamma \leq b$, so that $\gamma \in \boldsymbol{M}_b$. Then, in view of the foregoing containment, $\gamma \in \boldsymbol{M}_{h(\operatorname{coz}\rho)}$, for some $\rho \in \mathsf{Z}(h)_!(\boldsymbol{M}_b)$. Thus, $\operatorname{coz}\gamma \leq h(\operatorname{coz}\rho)$. Since h is a λ-map, there is a $\tau \in \mathcal{R}L$ such that $\operatorname{coz}\tau \leq \operatorname{coz}\rho$ and $\operatorname{coz}\gamma \leq h(\operatorname{coz}\tau)$. Since $\mathsf{Z}(h)_!(\boldsymbol{M}_b)$ is a z-ideal, $\tau \in \mathsf{Z}(h)_!(\boldsymbol{M}_b)$. This shows that $\operatorname{coz}\gamma \leq hh_!(b)$, and since b is the join of all such $\operatorname{coz}\gamma$, we conclude that $b \leq hh_!(b)$. Thus, $h_!$ is left adjoint to h. $\square$

Corollary 5.5.1. *A λ-map $h\colon L \to M$ is open if and only if $\mathsf{Z}(h)$ is open.*

Proof. If $\mathsf{Z}(h)$ is open, then it has a left adjoint, hence h has a left adjoint, whence it is open because its domain is regular.

Conversely, suppose h has a left adjoint, $h_!$. By the foregoing proposition, $\mathsf{Z}(h)$ does have a left adjoint, say $\mathsf{Z}(h)_!$. So we need to show that $\mathsf{Z}(h)_!$ satisfies the Frobenius identity. Let $P \in \mathrm{Zid}(\mathcal{R}L)$ and $Q \in \mathrm{Zid}(\mathcal{R}M)$. It suffices to show that

$$P \cap \mathsf{Z}(h)_!(Q) \subseteq \mathsf{Z}(h)_!\Big(\mathsf{Z}(h)(P) \cap Q\Big).$$

Let α be in the ideal on the left. Recall from the previous proof how $\mathsf{Z}(h)_!$ maps, and also recall that

$$\mathsf{Z}(h)(Q) = \bigcup\{\boldsymbol{M}_{\mathrm{coz}\,(h\cdot\beta)} \mid \beta \in Q\}.$$

Since $\alpha \in \mathsf{Z}(h)_!(Q)$, there is a $\beta \in Q$ such that $\mathrm{coz}\,\alpha \le h_!(\mathrm{coz}\,\beta)$, by (5.9). Thus,

$$\begin{aligned} \mathrm{coz}\,\alpha = h_!(\mathrm{coz}\,\beta) \wedge \mathrm{coz}\,\alpha &= h_!\big(h(\mathrm{coz}\,\alpha) \wedge \mathrm{coz}\,\beta\big) \qquad \text{since } h_! \dashv h \\ &= h_!\big(\mathrm{coz}\,(h\cdot\alpha) \wedge \mathrm{coz}\,\beta\big) \\ &= h_!\Big(\mathrm{coz}\,\big(\beta(h\cdot\alpha)\big)\Big). \end{aligned}$$

Since $\beta \in Q$ and Q is an ideal, $\beta(h\cdot\alpha) \in Q$. Also, $h\cdot\alpha \in \mathsf{Z}(h)(P)$ since $\alpha \in P$. Thus, $\beta(h\cdot\alpha) \in \mathsf{Z}(h)(P)$, and hence $\alpha \in \mathsf{Z}(h)_!\Big(\mathsf{Z}(h)(P) \cap Q\Big)$. This completes the proof. $\square$

We have an example which shows that, in general, Z does not reflect openness. In order to present it, we need to know that Z does not reflect isomorphisms. Recall that a frame homomorphism $h\colon L \to M$ is said to be *coz-surjective* if, for every $d \in \mathrm{Coz}\,M$, there is a $c \in \mathrm{Coz}\,L$ with $h(c) = d$. On the other hand, h is called *coz-faithful* if it is one-one on $\mathrm{Coz}\,L$. This is equivalent to saying the only cozero element it takes to the top is the top. Let us recall from [9, Lemma 1] that a coherent frame homomorphism is one-one whenever it is one-one on the sublattice of compact elements. Observe as well that if a frame homomorphism $\phi\colon A \to B$ between coherent frames is onto, then for every $b \in \mathfrak{K}(B)$ there exists $a \in \mathfrak{K}(A)$ such that $\phi(a) = b$. For, if t is an element of L with $\phi(t) = b$, then $b = \bigvee \phi[C]$ for some $C \subseteq \mathfrak{K}(A)$, so that, by compactness, $\phi(c) = b$ for some $c \in C$. Also recall from [66] that a frame L is pseudocompact if and only if the join map $j_L\colon \beta L \to L$ is coz-faithful.

Proposition 5.5.2. *For any morphism $h\colon L \to M$ in* **CRegFrm**, *we have:*

(a) $\mathsf{Z}(h)$ *is one-one if and only if h is coz-faithful.*

(b) $\mathsf{Z}(h)$ *is onto if and only if h is coz-surjective.*

Hence, $\mathsf{Z}(h)$ *is an isomorphism if and only if h is coz-faithful and coz-surjective.*

Proof. (a) For any $\alpha, \beta \in \mathcal{R}L$ we have

$$\mathsf{Z}(h)(\boldsymbol{M}_{\operatorname{coz}\alpha}) = \mathsf{Z}(h)(\boldsymbol{M}_{\operatorname{coz}\beta}) \iff h(\operatorname{coz}\alpha) = h(\operatorname{coz}\beta),$$

whence we deduce that $\mathsf{Z}(h)$ is one-one if and only if h is coz-faithful.

(b) If h is coz-surjective, then for any $\beta \in \mathcal{R}M$ there is an $\alpha \in \mathcal{R}L$ such that $\operatorname{coz}\beta = h(\operatorname{coz}\alpha)$, so that $\mathsf{Z}(h)(\boldsymbol{M}_{\operatorname{coz}\alpha}) = \boldsymbol{M}_{\operatorname{coz}\beta}$, implying $\mathsf{Z}(h)$ is onto.

Conversely, if $\mathsf{Z}(h)$ is onto, then, being a coherent map, given any $\beta \in \mathcal{R}M$, there exists an $\alpha \in \mathcal{R}L$ such that

$$\mathsf{Z}(h)(\boldsymbol{M}_{\operatorname{coz}\alpha}) = \boldsymbol{M}_{\operatorname{coz}(h\cdot\alpha)} = \boldsymbol{M}_{\operatorname{coz}\beta},$$

which implies $h(\operatorname{coz}\alpha) = \operatorname{coz}\beta$, and thus showing that h is coz-surjective. $\square$

Example 5.5.1. Let X be a pseudocompact Tychonoff space which is not locally compact. See, for instance, [43, Example 2.2] for such a space. Let $L = \mathfrak{O}X$. The map $j_L\colon \beta L \to L$ is coz-surjective by [20, Corollary 5], and it is coz-faithful since L is pseudocompact. Thus, by the result quoted above, $\operatorname{Zid}(j_L)$ is an isomorphism, and hence open. However, j_L is not open. Indeed, because X is not locally compact, the inclusion map $X \hookrightarrow \beta X$ is not open, and hence the induced frame homomorphism $\mathfrak{O}(\beta X) \to \mathfrak{O}X$ is not open. But $\mathfrak{O}(\beta X) \cong \beta(\mathfrak{O}X)$, so the claim is established.

The characterisation that $h\colon L \to M$ is a λ-map if and only if for any $a \in L$ and $z \in \operatorname{Coz} M$,

$$z \le h(a) \quad \Longrightarrow \quad z \le h(c) \text{ for some } c \in \operatorname{Coz} L \text{ with } c \le a,$$

shows that the composite of λ-maps is a λ-map as shown in the following result.

Lemma 5.5.2. *Let the homomorphisms $h\colon L \to M$ and $g\colon M \to N$ be λ-maps. Then the composite $g \cdot h$ is a λ-map.*

Proof. Take any $a \in L$ and $z \in \operatorname{Coz} N$ such that $z \leq gh(a)$. Since g is a λ-map there exists a $w \in \operatorname{Coz} M$ such that $w \leq h(a)$ and $z \leq g(w)$. This implies that $z \leq g(h(a))$. Since h is a λ-map there exists a $c \in \operatorname{Coz} L$ with $c \leq a$ and $w \leq h(c)$. Therefore $z \leq gh(c)$, which shows that $g \cdot h$ is a λ-map. □

We thus have the category $\mathbf{CRegFrm}_\lambda$, consisting of completely regular frames with λ-maps as the morphisms. Observe that a λ-map $h\colon L \to M$ is an isomorphism if and only if it is coz-faithful and coz-surjective. Indeed, any isomorphism is coz-faithful and coz-surjective. Conversely, if h is coz-surjective, then it is surjective since our frames are completely regular. Now suppose $h(a) = 1$ for some $a \in L$. Then, in light of h being a λ-map, there is a $c \in \operatorname{Coz} L$ such that $c \leq a$ and $1 \leq h(c)$. Coz-density implies $c = 1$, whence $a = 1$, making h codense and hence one-one.

Now, letting $\mathsf{Z}_\lambda\colon \mathbf{CRegFrm}_\lambda \to \mathbf{CohFrm}$ be the functor acting as Z, we can restate some of the results above as follows:

(a) Z_λ *reflects isomorphisms.*

(b) Z_λ *preserves and reflects the property of having a left adjoint.*

(c) Z_λ *preserves and reflects open maps.*

Chapter 6

Covering maximal ideals

Our aim in this chapter is to extend the work of Banerjee, Ghosh and Henriksen in [7] where they characterise Tychonoff spaces X for which $C(X)$ is a UMP-ring (see definition below). The authors utilise, among other things, the notion of a nearly round subset of βX which was introduced in [40]. We also extend this concept. We want to state that in [31] and [32], respectively, round and almost round quotient maps were introduced. Our main goal in this chapter is to give analogous characterisations for frames.

We denote by $\mathrm{Min}(A)$ the set of minimal prime ideals of a commutative ring A. A maximal ideal M of A is a *UMP-ideal* if

$$M = \bigcup\{P \in \mathrm{Min}(A) \mid P \subseteq M\}.$$

If every maximal ideal of A is a UMP-ideal, we say A is a *UMP-ring*. Banaschewski [16] has shown that the class of the function rings $\mathcal{R}L$ contains strictly the classical function rings $C(X)$ in the sense that, although for any Tychonoff space X the rings $C(X)$ and $\mathcal{R}(\mathfrak{O}X)$ are isomorphic, there are frames L for which $\mathcal{R}L$ is isomorphic to no $C(X)$. Indeed, there are Boolean frames L such that $\mathcal{R}L$ is isomorphic to no $C(X)$ [16]. Since, as we shall see below, $\mathcal{R}L$ is a UMP-ring for every Boolean frame L, we shall here be dealing with more rings than in the classical case.

The maximal ideals of $\mathcal{R}L$ are described in [33] as follows. For any quotient map $h\colon \beta L \to M$ we define the ideals $\boldsymbol{M}^h$ and $\boldsymbol{O}^h$ of $\mathcal{R}L$ by

$$\boldsymbol{M}^h = \{\alpha \in \mathcal{R}L \mid h(r_L(\operatorname{coz}\alpha)) = 0\} \quad \text{and} \quad \boldsymbol{O}^h = \{\alpha \in \mathcal{R}L \mid h(r_L(\operatorname{coz}\alpha)^*) = 1\}.$$

In particular, if $I \in \beta L$ and $h \colon \beta L \to \uparrow I$ is the closed quotient map determined by I, then we denote $\boldsymbol{M}^h$ and $\boldsymbol{O}^h$ by $\boldsymbol{M}^I$ and $\boldsymbol{O}^I$, respectively. Thus,

$$\boldsymbol{M}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \subseteq I\} \quad \text{and} \quad \boldsymbol{O}^I = \{\alpha \in \mathcal{R}L \mid r_L(\operatorname{coz}\alpha) \prec I\}.$$

The maximal ideals of $\mathcal{R}L$ are precisely the ideals $\boldsymbol{M}^I$, for $I \in \mathrm{Pt}(\beta L)$. If $I \in \mathrm{Pt}(\beta L)$ and P is a prime ideal such that $P \subseteq \boldsymbol{M}^I$, then it is shown in [33] that $\boldsymbol{O}^I \subseteq P$.

6.1 Variants of roundness

One of the characterisations in the main result (Theorem 6.2.1 below) is in terms of what we call nearly round quotient maps. These are defined as generalisations of nearly round subspaces [40]. The notion of a round quotient of the Stone-Čech compactification of a frame was defined in [33] as follows. A quotient map $h \colon \beta L \to M$ is *round* if $\boldsymbol{M}^h = \boldsymbol{O}^h$.

Next we recall from [32] that a collection $\mathcal{F}$ of minimal prime ideals of $\mathcal{R}L$ is said to be *adequate* for a quotient map $h \colon \beta L \to M$ if for every $I \in \mathrm{Pt}(\beta L)$ with $h(I) < 1$, there exists $Q \in \mathcal{F}$ such that $Q \subseteq \boldsymbol{M}^I$. We wish to cast this definition in a slightly different but equivalent way which will make it easier for our purposes to work with. To this end, let us observe that if $h \colon K \to M$ is a quotient map, then

$$\{p \in \mathrm{Pt}(K) \mid h(p) < 1\} = \{h_*(q) \mid q \in \mathrm{Pt}(M)\}.$$

Indeed, if p is in the set on the left, then $h(p) \in \mathrm{Pt}(M)$ since h is onto, and hence from the inequality $p \leq h_*h(p)$ we have $p = h_*h(p)$ by maximality, showing that p is in the set on the right. The other inclusion is immediate. Thus,

> *a collection $\mathcal{F}$ of minimal prime ideals of $\mathcal{R}L$ is adequate for a quotient map $h \colon \beta L \to M$ if and only if, for every $p \in \mathrm{Pt}(M)$, there is a $Q \in \mathcal{F}$ such that $Q \subseteq \boldsymbol{M}^{h_*(p)}$.*

Since it is possible that $\mathrm{Pt}(M) = \emptyset$, we need to exercise care when we speak of adequate families of minimal prime ideals. In order to avoid being entangled with "empty collections of minimal primes", we shall consider only those quotient maps whose codomains have at least one point.

Following [40], we formulate the following definition.

Definition 6.1.1. A quotient map $h\colon \beta L \to M$ into a frame with at least one point is *nearly round* if whenever α is an element of $\mathcal{R}L$ with $\alpha \in \bigcap\{\boldsymbol{M}^{h_*(p)} \mid p \in \mathrm{Pt}(M)\}$, then there is a collection $\mathcal{F}$ of minimal prime ideals of $\mathcal{R}L$ which is adequate for h such that $\alpha \in \bigcap \mathcal{F}$.

In the proposition that follows we show that if a quotient map $h\colon \beta L \to M$ is round and $\bigwedge \mathrm{Pt}(M) = 0$, then h is nearly round. Observe that frames M for which $\bigwedge \mathrm{Pt}(M) = 0$ include the spatial ones, and the inclusion is strict, as the following example attests.

Example 6.1.1. Let X be realcompact Tychonoff space which is not Lindelöf, and consider the Lindelöf coreflection $\lambda\colon \lambda(\mathfrak{O}X) \to \mathfrak{O}X$ of $\mathfrak{O}X$. Since right adjoints preserve meets, we have

$$0 = \lambda_*(0) = \lambda_*\Big(\bigwedge \mathrm{Pt}(\mathfrak{O}X\Big) = \bigwedge\{\lambda_*(p) \mid p \in \mathrm{Pt}(\mathfrak{O}X)\}.$$

Since $\lambda_*(p) \in \mathrm{Pt}(\lambda(\mathfrak{O}X))$ for each $p \in \mathrm{Pt}(\mathfrak{O}X)$, it follows that $\bigwedge \mathrm{Pt}(\lambda(\mathfrak{O}X)) = 0$. The frame $\lambda(\mathfrak{O}X)$ is non-spatial, for otherwise it would be isomorphic to $\upsilon(\mathfrak{O}X)$, and hence to $\mathfrak{O}X$, whence X would be Lindelöf.

We recall from [32] that a quotient map $h\colon \beta L \to M$ is *almost round* if whenever $\mathcal{F}$ is adequate for h, then $\bigcap \mathcal{F} \subseteq \bigcap\{\boldsymbol{O}^{h_*(p)} \mid p \in \mathrm{Pt}(M)\}$. In [32] it is shown that every round quotient map $\beta L \to \mathbf{2}$ is almost round. The following proposition strengthens this.

Proposition 6.1.1. *Let $h\colon \beta L \to M$ be a quotient map where $\bigwedge \mathrm{Pt}(M) = 0$. Consider the following statements about h:*

(1) *h is round.*

(2) *h is nearly round and almost round.*

Then (1) *implies* (2)*. If M is spatial, then the two statements are equivalent.*

Proof. (1) $\Rightarrow$ (2): We show first that h is nearly round. Take any $\alpha \in \mathcal{R}L$ such that

$$\alpha \in \bigcap\{\boldsymbol{M}^{h_*(p)} \mid p \in \mathrm{Pt}(M)\}.$$

Then $r_L(\operatorname{coz}\alpha) \leq h_*(p)$ for every $p \in \mathrm{Pt}(M)$, so that

$$r_L(\operatorname{coz}\alpha) \leq \bigwedge_{p\in\mathrm{Pt}(M)} h_*(p) = h_*\Big(\bigwedge_{p\in\mathrm{Pt}(M)} p\Big) = h_*(0).$$

Therefore $h(r_L(\operatorname{coz}\alpha)) = 0$. Since h is round, this implies $h(r_L(\operatorname{coz}\alpha)^*) = 1$. Let $p \in \mathrm{Pt}(M)$ and suppose, by way of contradiction, that $r_L(\operatorname{coz}\alpha)^* \vee h_*(p) < 1_{\beta L}$. Since $h_*(p)$ is a point in βL, this implies $r_L(\operatorname{coz}\alpha)^* \leq h_*(p)$, and hence

$$1_M = h(r_L(\operatorname{coz}\alpha)^*) \leq hh_*(p) = p,$$

which is a contradiction. Thus, $r_L(\operatorname{coz}\alpha) \prec h_*(p)$ for every $p \in \mathrm{Pt}(M)$, which shows that

$$\alpha \in \bigcap\{\boldsymbol{O}^{h_*(p)} \mid p \in \mathrm{Pt}(M)\}.$$

For each $q \in \mathrm{Pt}(M)$ take any minimal prime ideal $P_q \subseteq \boldsymbol{M}^{h_*(q)}$. Then the family

$$\mathcal{F} = \{P_q \mid q \in \mathrm{Pt}(M)\}$$

is adequate for h and its intersection contains α because $\boldsymbol{O}^{h_*(q)} \subseteq P_q$, for any $q \in \mathrm{Pt}(M)$. Therefore h is nearly round.

To show that h is almost round, let $\mathcal{F}$ be adequate for h, and let $\alpha \in \bigcap \mathcal{F}$. Let $p \in \mathrm{Pt}(M)$, and, by adequacy of $\mathcal{F}$, take $Q \in \mathcal{F}$ with $Q \in \boldsymbol{M}^{h_*(p)}$. Thus, $\alpha \in \boldsymbol{M}^{h_*(p)}$, and hence $r_L(\operatorname{coz}\alpha) \leq h_*(p)$ for every $p \in \mathrm{Pt}(M)$, which, exactly as above, implies $h(r_L(\operatorname{coz}\alpha)) = 0$. A calculation as above shows that $\alpha \in \bigcap\{\boldsymbol{O}^{h_*(p)} \mid p \in \mathrm{Pt}(M)\}$, implying that h is almost round.

(2) $\Rightarrow$ (1) if M is spatial: Assume h is nearly round and almost round. Let $\alpha \in \boldsymbol{M}^h$. Then $h(r_L(\operatorname{coz}\alpha)) = 0$, which implies $r_L(\operatorname{coz}\alpha) \leq h_*(0) \leq h_*(p)$, for every $p \in \mathrm{Pt}(M)$, whence $\alpha \in \bigcap\{\boldsymbol{M}^{h_*(p)} \mid p \in \mathrm{Pt}(M)\}$. Because h is nearly round, there is a collection $\mathcal{F} \subseteq \mathrm{Min}(\mathcal{R}L)$ which is adequate for h such that $\alpha \in \bigcap \mathcal{F}$. Since h is almost round, $\alpha \in \bigcap\{\boldsymbol{O}^{h_*(p)} \mid p \in \mathrm{Pt}(M)\}$. Therefore $r_L(\operatorname{coz}\alpha) \prec h_*(p)$, which implies $r_L(\operatorname{coz}\alpha)^* \vee h_*(p) = 1_{\beta L}$, and hence $h(r_L(\operatorname{coz}\alpha)^*) \vee p = 1$, for every $p \in \mathrm{Pt}(M)$. Thus, there is no point of M above $h(r_L(\operatorname{coz}\alpha)^*)$, and so, by spatiality, $h(r_L(\operatorname{coz}\alpha)^*) = 1$. Therefore $\alpha \in \boldsymbol{O}^h$, and hence h is round. $\square$

Recall from [6] that L is called an *F-frame* if, for every $c \in \operatorname{Coz} L$, the open quotient map $L \to {\downarrow}c$ is a C^*-quotient map. A *P-frame* is a frame in which every cozero element is complemented. We refer to [32] and [30] for some characterisations of F-frames and P-frames, respectively.

Corollary 6.1.1. *An F-frame is a P-frame if and only if every quotient map out of its Stone-Čech compactification into a frame with at least one point is nearly round.*

Proof. Every quotient of the Stone-Čech compactification of a P-frame is round by [33, Proposition 4.17]. So the left-to-right implication follows from the foregoing proposition.

Conversely, let L be an F-frame with the hypothesised feature. Let $I \in \mathrm{Pt}(\beta L)$. By [30, Proposition 3.9], it suffices to show that $\boldsymbol{M}^I = \boldsymbol{O}^I$. Consider the quotient map $\xi\colon \beta L \to \mathbf{2}$ given by

$$\xi(J) = 0 \iff J \leq I.$$

For any $\alpha \in \mathcal{R}L$ we have

$$\begin{aligned} \alpha \in \boldsymbol{M}^\xi &\iff \xi(r_L(\operatorname{coz}\alpha)) = 0 \\ &\iff r_L(\operatorname{coz}\alpha) \leq I \\ &\iff \alpha \in \boldsymbol{M}^I. \end{aligned}$$

On the other hand

$$\begin{aligned} \alpha \in \boldsymbol{O}^\xi &\iff \xi(r_L(\operatorname{coz}\alpha)^*) = 1 \\ &\iff r_L(\operatorname{coz}\alpha)^* \nleq I \\ &\iff r_L(\operatorname{coz}\alpha)^* \vee I = 1 \qquad\qquad \text{since } I \text{ is a point} \\ &\iff \alpha \in \boldsymbol{O}^I. \end{aligned}$$

Since L is an F-frame, Proposition 3.11 in [32] tells us that ξ is almost round. But by the current hypothesis it is also nearly round, so it is round by the foregoing proposition because its codomain is spatial. Therefore $\boldsymbol{O}^I = \boldsymbol{O}^\xi = \boldsymbol{M}^\xi = \boldsymbol{M}^I$, which implies that L is a P-frame. □

6.2 When $\mathcal{R}L$ is a UMP-ring

We shall now characterise those frames L for which every maximal ideal of $\mathcal{R}L$ is the union of minimal prime ideals it contains. Our characterisations extend Theorem 2.4 of [7]. There is among them a characterisation which is not immediately seen to be an extension of one of those in the stated theorem from [7]. We shall explain why it is. One of our characterisations is in terms of the Lindelöf coreflection, and hence cannot have a $C(X)$ analogue. We remark that the equivalence of (1) and (3) in [7, Theorem 2.4] holds for any reduced ring.

Let us fix terminology so that it accords with that in [7]. A point I of βL is a *UMP-point* if the maximal ideal $\boldsymbol{M}^{\boldsymbol{I}}$ is a UMP-ideal. Thus, $\mathcal{R}L$ is a UMP-ring precisely if every point of βL is a UMP-point. We shall also say L is a *UMP-frame* if $\mathcal{R}L$ is a UMP-ring. Finally, we shall at times use "UMP" as an adjective.

Examples 6.2.1. (a) Recall that a reduced ring is Von Neumann regular if and only if every prime ideal in it is maximal. It follows therefore that every Von Neumann regular ring is UMP. Thus, every P-frame is a UMP-frame because L is a P-frame if and only if $\mathcal{R}L$ is Von Neumann regular [22, Remark 3].

(b) A frame is called an *almost P-frame* if the top element is the only dense cozero element. If L is a UMP-frame, then every maximal ideal of $\mathcal{R}L$ consists entirely of zero-divisors. Hence, every ideal of $\mathcal{R}L$ contains only zero-divisors, and therefore L is an almost P-frame by [33, Proposition 4.5].

In the proposition that follows we show that the bounded part of a reduced f-ring with bounded inversion is UMP precisely when the ring is bounded and is a UMP-ring. In the converse direction, we observe that if the ring is UMP, then its bounded part is UMP when and only when the ring is bounded. The application we have in mind is to show that Theorem 1.10 in [7], which is proved rather laboriously in that paper, follows easily (modulo some known topological facts) from a simple f-ring result. Note that a reduced UMP-ring consists of units and zero-divisors only, because a nonunit belongs to some maximal ideal and is therefore a zero-divisor since minimal prime ideals consist entirely of zero-divisors.

Proposition 6.2.1. *Let A be a reduced f-ring with bounded inversion. Then:*

(1) *A^* is a UMP-ring if and only if $A^* = A$ and A is a UMP-ring.*

(2) *If A is a UMP-ring, then A^* is a UMP-ring if and only if $A = A^*$.*

Proof. (1) The implication ($\Leftarrow$) is trivial. Conversely, let A^* be a UMP-ring. Suppose, by way of contradiction, that $A \neq A^*$. Let $a \in A \smallsetminus A^*$. Then $|a| + 1$ is unbounded, and hence $\frac{1}{|a|+1}$ is a member of A^* which is neither a zero-divisor nor a unit in A^*.

(2) This follows immediately from (1). □

Corollary 6.2.1. *βL is a UMP-frame if and only if L is a pseudocompact UMP-frame.*

Proof. This follows from the proposition because $\mathcal{R}(\beta L) \cong \mathcal{R}^* L$. □

Remark 6.2.1. Since pseudocompact realcompact frames are compact [21], it follows that if L is a UMP-frame which is realcompact but not compact, then βL is not a UMP-frame. The "pointed" version of this result is Theorem 1.10 in [7] that we mentioned above.

Given a maximal ideal M of a reduced ring A, denote, as usual, by O_M the intersection of all minimal prime ideals of A contained in M. Then (see, for instance, [29]),

$$O_M = \{a \in A \mid aa' = 0 \text{ for some } a' \notin M\}.$$

Since a prime ideal of a reduced ring is minimal prime if and only if every member of the ideal is annihilated by a non-member, the following lemma is straightforward. It is a generalisation (from $C(X)$ to arbitrary reduced rings) of the equivalence of statements (1) and (3) in [7, Theorem 2.2].

Lemma 6.2.1. *A maximal ideal M of a reduced ring is a UMP-ideal if and only if for every $a \in M \smallsetminus O_M$, there exists $a' \in M \smallsetminus O_M$ such that $aa' = 0$.*

Remark 6.2.2. In view of [28, Proposition 2.2], the result just stated also holds, mutatis mutandis, for bounded distributive lattices.

Let us pause for a moment and consider the following purely algebraic result. In [27] a reduced ring is called an *mp-ring* if every prime ideal contains a unique minimal prime ideal. Such rings were first considered and characterised by Artico and Marconi [2].

Proposition 6.2.2. *A reduced mp-ring is a UMP-ring if and only if it is Von Neumann regular.*

Proof. Since every Von Neumann regular ring is UMP, the one implication is immediate. Conversely, let A be a UMP mp-ring, P be a prime ideal of A, and M be a maximal ideal containing P. Let Q be a minimal prime ideal contained in P. Then Q is the unique minimal prime ideal contained in M. Since A is UMP, $M = Q$, and hence $P = M$. Therefore A is Von Neumann regular. □

In a similar vein we have the following result. It is shown in [60, Proposition 1.4] that the classical ring of quotients of a reduced ring A is Von Neumann regular if and only if every ideal of A which consists only of zero-divisors is contained in a minimal prime ideal. As a consequence we deduce the following.

Proposition 6.2.3. *A reduced ring whose classical ring of quotients is Von Neumann regular is a UMP-ring if and only if it is Von Neumann regular.*

Now, a frame L is an F-frame if and only if $\mathcal{R}L$ is an mp-ring [32, Proposition 3.4]. So, in light of Corollary 6.1.1, it is reasonable to expect that a frame L is UMP precisely when every quotient map out of its Stone-Čech compactification into a frame with at least one point is almost round. We shall see in the main result below that this is indeed the case.

Lemma 6.2.2. *A point I of βL is a UMP-point if and only if the quotient map $\xi\colon \beta L \to \mathbf{2}$, induced by I, is nearly round.*

Proof. The lemma follows easily from the observation that

$$\{\boldsymbol{M}^{\xi_*(p)} \mid p \in \mathrm{Pt}(\mathbf{2})\} = \{\boldsymbol{M}^I\},$$

so that, for any $\alpha \in \mathcal{R}L$, $\alpha \in \bigcap\{\boldsymbol{M}^{\xi_*(p)} \mid p \in \mathrm{Pt}(\mathbf{2})\}$ if and only if $\alpha \in \boldsymbol{M}^I$. □

In the proof that follows, given a frame M, we denote by $\mathrm{Sub}(M)$ the co-frame of its sublocales. For any $a \in M$, we write $\mathfrak{c}_a$ for the closed nucleus induced by a. Recall that $\mathfrak{c}_a(x) = a \vee x$, and recall also that meets of nuclei are computed pointwise. We shall view ${\uparrow}a$ as the closed sublocale $\mathrm{Fix}(\mathfrak{c}_a)$. Let us observe that, for any $I \in \mathrm{Pt}(\beta L)$ and $\alpha \in \mathcal{R}L$,

$$\alpha \notin \boldsymbol{O}^I \quad \Longleftrightarrow \quad r_L((\operatorname{coz}\alpha)^*) \leq I.$$

Theorem 6.2.1. *The following are equivalent for a completely regular frame L.*

(1) *L is a UMP-frame.*

(2) *υL is a UMP-frame.*

(3) *λL is a UMP-frame.*

(4) *Every quotient map $\beta L \to \mathbf{2}$ is nearly round.*

(5) *Every quotient map $\beta L \to M$ into a frame with at least one point is nearly round.*

(6) *For any $I \in Pt(\beta L)$, if $\alpha \in \boldsymbol{M}^I \smallsetminus \boldsymbol{O}^I$, then there exists a $\gamma \in \boldsymbol{M}^I \smallsetminus \boldsymbol{O}^I$ such that $\alpha\gamma = \mathbf{0}$.*

(7) *For every $c \in CozL$, there are cozero elements $\{c_t \mid t \in T\}$ such that*

$$\uparrow r_L(c) = \bigvee_{Sub(\beta L)} \{\uparrow r_L(c_t^*) \mid t \in T\}.$$

Proof. Statements (1), (2) and (3) are equivalent because the rings $\mathcal{R}L, \mathcal{R}(\upsilon L)$ and $\mathcal{R}(\lambda L)$ are isomorphic. The equivalence of (1) and (4) is Lemma 6.2.2, and the equivalence of (1) and (6) follows from Lemma 6.2.1.

(4) $\Leftrightarrow$ (5): That (5) implies (4) is trivial. Assume (4), and let $h\colon \beta L \to M$ be a quotient map with $\mathrm{Pt}(M) \neq \emptyset$. Let $\alpha \in \bigcap\{\boldsymbol{M}^{h_*(p)} \mid p \in \mathrm{Pt}(M)\}$. For any $p \in \mathrm{Pt}(M)$ consider the quotient map $\beta L \xrightarrow{h} M \xrightarrow{\xi^{(p)}} \mathbf{2}$, where $\xi^{(p)}\colon M \to \mathbf{2}$ is the homomorphism induced by p. Observe that

$$\{\boldsymbol{M}^{(\xi^{(p)}\cdot h)_*(q)} \mid q \in \mathrm{Pt}(\mathbf{2})\} = \{\boldsymbol{M}^{h_*(p)}\}.$$

Since $\alpha \in \boldsymbol{M}^{h_*(p)} = \bigcap\{\boldsymbol{M}^{(\xi^{(p)}\cdot h)_*(q)} \mid q \in \mathrm{Pt}(\mathbf{2})\}$, and $\beta L \xrightarrow{h} M \xrightarrow{\xi^{(p)}} \mathbf{2}$ is nearly round, by hypothesis, there is a collection $\mathcal{F}^{(p)} \subseteq \mathrm{Min}(\mathcal{R}L)$ which is adequate for $\xi^{(p)} \cdot h$ such that $\alpha \in \bigcap \mathcal{F}^{(p)}$. Now let

$$\mathcal{F} = \bigcup\{\mathcal{F}^{(p)} \mid p \in \mathrm{Pt}(M)\}.$$

A routine check shows that $\mathcal{F}$ is adequate for h and $\alpha \in \bigcap \mathcal{F}$. Therefore h is nearly round.

(7) $\Rightarrow$ (6): Let I be a point of βL, and take any $\alpha \in \boldsymbol{M}^I \smallsetminus \boldsymbol{O}^I$. By (7), there are cozero elements $\{c_t \mid t \in T\}$ such that, in the language of nuclei, $\mathfrak{c}_{r_L(\operatorname{coz}\alpha)} = \bigwedge_t \mathfrak{c}_{r_L(c_t^*)}$. Because $\alpha \in \boldsymbol{M}^I$, $r_L(\operatorname{coz}\alpha) \leq I$, and therefore

$$I = r_L(\operatorname{coz}\alpha) \vee I = \Big(\bigwedge_t \mathfrak{c}_{r_L(c_t^*)}\Big)(I) = \bigwedge_t \Big(\mathfrak{c}_{r_L(c_t^*)} \vee I\Big) = \bigwedge_t \Big(r_L(c_t^*) \vee I\Big).$$

If we suppose that, for each $t \in T$, $r_L(c_t^*) \nleq I$, then $r_L(c_t^*) \vee I = 1_{\beta L}$ for every t, which leads to the contradiction that $I = 1_{\beta L}$. So there is an index t_0 such that $r_L(c_{t_0}^*) \leq I$. Pick $\gamma \in \mathcal{R}L$ such that $\operatorname{coz}\gamma = c_{t_0}$. Since $r_L((\operatorname{coz}\gamma)^*) \leq I$, it follows that $\gamma \notin \boldsymbol{O^I}$. We claim that $\alpha\gamma = \mathbf{0}$. Evaluating the two equal nuclei at $0_{\beta L}$ yields

$$r_L(\operatorname{coz}\alpha) = \bigwedge_t r_L(c_t^*) = r_L\Big(\bigwedge_t c_t^*\Big) = r_L\Big(\Big(\bigvee_t c_t\Big)^*\Big),$$

which implies $\operatorname{coz}\alpha = \big(\bigvee_t c_t\big)^*$, and hence $\operatorname{coz}\alpha \wedge c_{t_0} = 0$, whence $\alpha\gamma = \mathbf{0}$. Now, since

$$\boldsymbol{O^I} = \{\tau \in \mathcal{R}L \mid \rho\tau = \mathbf{0} \text{ for some } \rho \notin \boldsymbol{M^I}\},$$

we cannot have $\gamma \notin \boldsymbol{M^I}$ because that would imply $\alpha \in \boldsymbol{O^I}$, which is false. Therefore $\gamma \in \boldsymbol{M^I} \smallsetminus \boldsymbol{O^I}$, and so (6) is implied by (7).

(1) $\Rightarrow$ (7): Let $c \in \operatorname{Coz} L$. If $c = 1$, there is nothing to prove as $1 = 0^*$. So assume $c < 1$. Take $\alpha \in \mathcal{R}L$ such that $c = \operatorname{coz}\alpha$. Let $\{I_t \mid t \in T\}$ be the set of all points of βL above $r_L(\operatorname{coz}\alpha)$. This set is not empty. If I is a point of βL above $r_L(\operatorname{coz}\alpha)$, then $\alpha \in \boldsymbol{M^I}$. By (1), there is a minimal prime ideal $P \subseteq \boldsymbol{M^I}$ such that $\alpha \in P$. Since P is minimal prime, there is a $\gamma \notin P$ such that $\alpha\gamma = \mathbf{0}$. Since $\boldsymbol{O^I} \subseteq P$, $\gamma \notin \boldsymbol{O^I}$, and hence $r_L((\operatorname{coz}\gamma)^*) \leq I$. Thus, for each $t \in T$, there is a $\gamma_t \in \mathcal{R}L$ such that

$$\alpha\gamma_t = \mathbf{0} \quad \text{and} \quad r_L((\operatorname{coz}\gamma_t)^*) \leq I_t. \tag{$\dagger$}$$

Put $c_t = \operatorname{coz}(\gamma_t)$. Then each c_t is a cozero element of L. Since $\operatorname{coz}\alpha \wedge c_t = 0$ for every $t \in T$, we have $\operatorname{coz}\alpha \wedge \bigvee_t c_t = 0$, which implies $c \leq \Big(\bigvee_t c_t\Big)^* = \bigwedge_t c_t^*$. On the other hand, the inequality in ($\dagger$) implies

$$r_L(\operatorname{coz}\alpha) = \bigwedge_t I_t \geq \bigwedge_t r_L(c_t^*) = r_L\Big(\bigwedge_t c_t^*\Big),$$

so that $\bigwedge_t c_t^* \leq c$, and hence $c = \bigwedge_t c_t^*$. Thus, $c \leq c_t^*$ for every t, which implies $\mathfrak{c}_{r_L(\operatorname{coz}\alpha)} \leq \mathfrak{c}_{r_L(c_t^*)}$, for every t, and hence

$$\mathfrak{c}_{r_L(\operatorname{coz}\alpha)} \leq \bigwedge_t \mathfrak{c}_{r_L(c_t^*)}.$$

We now reverse this inequality. Let $J \in \beta L$, and consider any $I \in \operatorname{Pt}(\beta L)$ such that $r_L(\operatorname{coz}\alpha) \vee J \leq I$. Then $I = I_{t_0}$ for some index $t_0 \in I$ since I is above $r_L(\operatorname{coz}\alpha)$. Suppose, for contradiction, that $\bigwedge_t \big(r_L(c_t^*) \vee J\big) \nleq I$. Then $\bigwedge_t \big(r_L(c_t^*) \vee J\big) \vee I = 1_{\beta L}$, which implies

$$1_{\beta L} = r_L(c_{t_0}^*) \vee J \vee I_{t_0} = r_L(c_{t_0}^*) \vee I_{t_0},$$

since $J \leq I_{t_0}$. But this contradicts the inequality in (†). Therefore, by spatiality of βL, $\bigwedge_t \mathfrak{c}_{r_L(c_t^*)} \leq \mathfrak{c}_{r_L(\operatorname{coz}\alpha)}$, and hence equality. This shows that

$$\uparrow r_L(c) = \bigvee_{\mathrm{Sub}(\beta L)} \{\uparrow r_L(c_t^*) \mid t \in T\},$$

as required. $\square$

It is not difficult to see that some of the characterisations in this proposition generalise those in [7, Theorem 2.4]. It is however not immediate that our item (7) is a generalisation of item (5) in Theorem 2.4 of [7]. Let us show that it is. Suppose X is a subspace of Y, and let $\varphi\colon \mathfrak{O}Y \to \mathfrak{O}X$ be the frame homomorphism $\mathfrak{O}i$, for the inclusion map $i\colon X \to Y$. We claim that, for any $U \in \mathfrak{O}X$, $Y \smallsetminus \mathrm{cl}_Y(U) = \varphi_*(U^*)$. To see this, denote by $\overline{U}$ the closure of U in X. Now,

$$\mathrm{cl}_Y\,\overline{U} = \mathrm{cl}_Y(X \cap \mathrm{cl}_Y\,U) \subseteq \mathrm{cl}_Y(\mathrm{cl}_Y\,U) = \mathrm{cl}_Y\,U \subseteq \mathrm{cl}_Y\,\overline{U},$$

so that

$$\begin{aligned}
\varphi_*(U^*) &= Y \smallsetminus \mathrm{cl}_Y(X \smallsetminus U^*) \\
&= Y \smallsetminus \mathrm{cl}_Y\left(X \smallsetminus (X \smallsetminus \overline{U})\right) \\
&= Y \smallsetminus \mathrm{cl}_Y\,\overline{U} \\
&= Y \smallsetminus \mathrm{cl}_Y\,U \quad \text{by the calculation above.}
\end{aligned}$$

We therefore have the claimed equality. Thus, if X is Tychonoff space and U a cozero-set of X, then $\beta X \smallsetminus \mathrm{cl}_{\beta X}\,U = r_{\mathfrak{O}X}(U^*)$. Now it should be clear from this that indeed our condition (7) generalises condition (5) in [7, Theorem 2.5] because $\beta(\mathfrak{O}X) \cong \mathfrak{O}(\beta X)$.

In the process of the proof of Theorem 6.2.1 there are two facts that have come to the fore which we now emphasise.

> *If L is a UMP-frame, then every cozero element of L is a pseudocomplement. Conversely, if each cozero element of L is a pseudocomplement of a cozero element, then L is a UMP-frame.*

Of course the first part of this assertion reaffirms (albeit rather heavy-handedly) what we noted in Example 6.2.1(b) because a frame is an almost P-frame precisely if every cozero element of the frame is a pseudocomplement. In spaces, these results say that if a Tychonoff space X is a UMP-space, then every zero-set of X is the closure of some open set, and if every zero-set of X is the closure of some cozero-set, then X is a UMP-space. This latter part is also observed in [7].

We recall from [18] that an *Oz-frame* is a frame in which every pseudocomplement is a cozero element. This class of frames includes perfectly normal frames (i.e. frames L for which $\operatorname{Coz} L = L$), and hence metrisable frames. A Tychonoff space is an Oz-space [26] if and only if $\mathfrak{O}X$ is an Oz-frame.

Proposition 6.2.4. *An Oz-frame is a UMP-frame if and only if it is an almost P-frame. Hence, an Oz-space is a UMP-space if and only if it is an almost P-space.*

Since a frame is Boolean precisely if every element is a pseudocomplement, we also have the following.

Corollary 6.2.2. *A perfectly normal frame is a UMP-frame if and only if it is Boolean.*

Our last proposition is purely algebraic, and it shows that if a $\mathbb{Q}$-algebra A is a UMP-ring, then every ideal of A, when viewed as a ring in its own right, is a UMP-ring. This will then apply to function rings $\mathcal{R}L$, and hence to the rings $C(X)$.

Proposition 6.2.5. *Let A be a UMP-ring which is a $\mathbb{Q}$-algebra. Then every ideal I of A, viewed as a ring, is a UMP-ring.*

Proof. Let Q be a maximal ideal of I. Then, by [64, Corollary 3.6], $Q = I \cap M$ for some maximal ideal M of A which does not contain I. Now, Theorem 2.10 of [42] gives

$$\operatorname{Min}(I) = \{I \cap P \mid P \in \operatorname{Min}(A) \text{ and } P \not\supseteq I\}.$$

We show that

$$Q \subseteq \bigcup \{I \cap P \mid P \in \operatorname{Min}(A), P \not\supseteq I \text{ and } I \cap P \subseteq Q\},$$

which will prove that Q is covered by the minimal prime ideals of I that are contained in Q. Let $x \in Q$. Since A is a UMP-ring and $x \in M$, there is a $P \in \operatorname{Min}(A)$ such that $x \in P \subseteq M$. Thus, $x \in I \cap P$. But $P \not\supseteq I$ because $M \not\supseteq I$; so x is in the displayed union, which establishes the desired containment. $\square$

We close with two examples. The first shows that it is possible for a maximal ideal to be a UMP-ring but fail to be a UMP-ideal. The second is an example of a ring which is not UMP, but whose maximal ideals (in fact it has only one) are UMP-rings, albeit vacuously.

Example 6.2.1. Let $\mathbb{N}^* = \mathbb{N} \cup \{\omega\}$ be the one-point compactification of the discrete space $\mathbb{N}$. By [39, 14G.], $\boldsymbol{M}_n = \boldsymbol{O}_n$ for every $n \in \mathbb{N}$, and $\boldsymbol{M}_\omega$ is not minimal prime. By [7, Observation 1.2], $\mathbb{N}^*$ is not a UMP-space. This implies $\boldsymbol{M}_\omega$ is not a UMP-ideal in $C(\mathbb{N}^*)$. We show that, considered as a ring, $\boldsymbol{M}_\omega$ is a UMP-ring. As recalled above

$$\operatorname{Max}(\boldsymbol{M}_\omega) = \{\boldsymbol{M}_\omega \cap \boldsymbol{M}_n \mid n \in \mathbb{N}\} = \{\boldsymbol{M}_\omega \cap \boldsymbol{O}_n \mid n \in \mathbb{N}\},$$

and

$$\operatorname{Min}(\boldsymbol{M}_\omega) = \{\boldsymbol{M}_\omega \cap \boldsymbol{O}_n \mid n \in \mathbb{N}\} \cup \{P \in \operatorname{Min}(C(\mathbb{N}^*)) \mid P \subseteq \boldsymbol{M}_\omega\}.$$

Now if Q is a maximal ideal of $\boldsymbol{M}_\omega$, then there is an $n \in \mathbb{N}$ such that

$$Q = \boldsymbol{M}_\omega \cap \boldsymbol{M}_n = \boldsymbol{M}_\omega \cap \boldsymbol{O}_n \subseteq \bigcup \{P \in \operatorname{Min}(\boldsymbol{M}_\omega) \mid P \subseteq Q\}.$$

Thus, Q is covered by the minimal prime ideals of $\boldsymbol{M}_\omega$ which are contained in Q. Therefore $\boldsymbol{M}_\omega$ is a UMP-ring.

For the example that follows, note that in an integral domain the zero ideal is the only minimal prime ideal.

Example 6.2.2. Let X be an F-space which is not a P-space. Let p be a non-P-point of $C(X)$. Consider the $\mathbb{Q}$-algebra $A = C(X)/\boldsymbol{O}^p$. This is an integral domain with exactly one maximal ideal, namely $\boldsymbol{M}^p/\boldsymbol{O}^p$. The ring A is not a UMP-ring because the only minimal prime ideal of A (contained in $\boldsymbol{M}^p/\boldsymbol{O}^p$) is the zero ideal, and $\boldsymbol{M}^p/\boldsymbol{O}^p$ is not the zero ideal. However, as a ring, $\boldsymbol{M}^p/\boldsymbol{O}^p$ has no maximal ideals, by [64, Corollary 3.6], and is therefore (vacuously) a UMP-ring.

In view of this latter example, we can state a version of Proposition 6.2.5 as follows:

> *Let A be a $\mathbb{Q}$-algebra. If A is a UMP-ring, then every maximal ideal of A is a UMP-ring. The converse fails.*

Chapter 7

A miscellany of results

In this chapter we establish some other properties of z-ideals and d-ideals of $\mathcal{R}L$. In a number of instances our proofs will "piggyback" on theorems concerning $C(X)$. We also prove that, analogously to spaces, the frame of open sets of the structure space of $\mathcal{R}L$ is isomorphic to the Stone-Čech compactification of L. It should be noted that this can be deduced from results in [25]. The reason we include a different proof is that we wish to highlight the similarities between the classical and the pointfree perspective.

7.1 Existence of n^{th} roots in $\mathcal{R}L$

In $C(X)$ every positive function has an n^{th} root for every integer $n \geq 1$, and every negative function has an n^{th} root for every odd integer $n \geq 1$. In [11], Banaschewski shows that every positive element of $\mathcal{R}L$ has a square root. We extend this result to show that what we have just said about $C(X)$ actually holds in $\mathcal{R}L$. Our proof will not be modelled on that of Banaschewski's for the case $n = 2$, but will rather exploit the fact (established for general reduced f-rings in Chapter 3) that $\mathcal{R}L$ is the ring of fractions of $\mathcal{R}^*L$; and this subring is isomorphic to a $C(X)$.

As has been our practice throughout the thesis, we will actually prove the results just announced for certain f-rings with bounded inversion.

Lemma 7.1.1. *Let A be an f-ring with bounded inversion, and suppose that every element of A^* has an n^{th} root (in A^*) for every odd $n \in \mathbb{N}$, and that every positive element of A^* has an n^{th} root (in A^*) for every $n \in \mathbb{N}$. Then:*

(a) *Every element of A has an n^{th} root for every odd $n \in \mathbb{N}$.*

(b) *Every positive element of A has an n^{th} root in A for every $n \in \mathbb{N}$.*

Proof. We prove only the first part; the second part is proved similarly. Let $a \in A$ and n be an odd positive integer. Now $\frac{a}{1+|a|}$ is an element of A^*, and so, by hypothesis, there is an element $b \in A^*$ such that $b^n = \frac{a}{1+|a|}$. Since $\frac{1}{1+|a|}$ is an element of A^*, there exists a $c \in A^*$ such that $c^n = \frac{1}{1+|a|}$. Then c^n is invertible in A, and hence c is invertible in A, and the equality $ac^n = b^n$ implies $a = (bc^{-1})^n$. So bc^{-1} is an n^{th} root of a. $\square$

Since the bounded part of $\mathcal{R}L$ is a $C(X)$, so that $\mathcal{R}L$ satisfies the hypothesis in the Lemma 7.1.1, we deduce the following.

Corollary 7.1.1. *For any completely regular frame L the following statements hold.*

(a) *Every positive element of $\mathcal{R}L$ has an n^{th} root, for any $n \in \mathbb{N}$.*

(b) *Every element of $\mathcal{R}L$ has an n^{th} root, for any odd $n \in \mathbb{N}$.*

Remark 7.1.1. Professor George Janelidze has shown us that the foregoing result can be proved by a categorical argument.

7.2 Some other properties of z-ideals

In [5], Azarpanah and Mohamadian show that an ideal of $C(X)$ is a z-ideal if and only if its radical is a z-ideal. We remind the reader that the radical of an ideal I of a ring A is the ideal

$$\sqrt{I} = \{a \in A \mid a^n \in I \text{ for some } n \in \mathbb{N}\}.$$

We aim to show that an ideal of $\mathcal{R}L$ is a z-ideal if and only if its radical is a z-ideal. We need a lemma which is itself a "piggyback" on a $C(X)$ result. Observe that if $\alpha \geq \mathbf{0}$ in $\mathcal{R}^*L$ and $n \in \mathbb{N}$, then $\alpha^{\frac{1}{n}} \in \mathcal{R}^*L$. Since the product of bounded elements of an f-ring is bounded, we conclude that if $\alpha \geq \mathbf{0}$ in $\mathcal{R}^*L$ and $q \geq 1$ in $\mathbb{Q}$, then $\alpha^q \in \mathcal{R}^*L$.

Lemma 7.2.1. (cf. [39, 1D.]) Let $\alpha, \beta \in \mathcal{R}L$. If $|\alpha| \leq |\beta|^q$ for some $q > 1$, then α is a multiple of β.

Proof. Multiply by $\frac{1}{\mathbf{1}+|\alpha|} \cdot (\frac{1}{\mathbf{1}+|\beta|})^q$ both sides of the stated inequality to obtain

$$\frac{|\alpha|}{\mathbf{1}+|\alpha|} \cdot \Big(\frac{1}{\mathbf{1}+|\beta|}\Big)^q \leq \frac{1}{\mathbf{1}+|\alpha|} \cdot \Big(\frac{|\beta|}{\mathbf{1}+|\beta|}\Big)^q.$$

Since each of the factors in this inequality is in $\mathcal{R}^*L$, and $\mathcal{R}^*L$ is isomorphic to a $C(X)$ via an f-ring isomorphism, we deduce from [39, 1D.] that $\frac{\alpha}{\mathbf{1}+|\alpha|}$ is a multiple of $\frac{\beta}{\mathbf{1}+|\beta|}$. This implies α is a multiple of β, as desired. □

For use in the lemma that follows, we recall that if (α_n) is a sequence of elements of $\mathcal{R}L$ with $\mathbf{0} \leq \alpha_n \leq \mathbf{1}$ for every n, then the set

$$\Big\{\frac{\alpha_1}{2} + \cdots + \frac{\alpha_n}{2^n} \mid n \in \mathbb{N}\Big\}$$

has a supremum in the poset $\mathcal{R}L$ (see [67, Lemma 4] and [17, §6]). This supremum is denoted by

$$\sum_{n=1}^{\infty} \frac{\alpha_n}{2^n}.$$

Proposition 7.2.1. (cf. [5, Proposition 2.1]) Let Q be an ideal of $\mathcal{R}L$, and let $\alpha \in \mathcal{R}L$. If $\boldsymbol{M}_{\operatorname{coz}\alpha} \subseteq \sqrt{Q}$, then $\boldsymbol{M}_{\operatorname{coz}\alpha} \subseteq Q$.

Proof. Suppose that $\beta \in \boldsymbol{M}_{\operatorname{coz}\alpha} \subseteq \sqrt{Q}$. Without loss of generality, we may assume that $|\beta| \leq \mathbf{1}$. As mentioned above, we can define $\gamma = \sum_{n=1}^{\infty} 2^{-n} \cdot \beta^{\frac{1}{n}}$. Hence

$$\begin{aligned} \operatorname{coz}\gamma &= \bigvee_n \operatorname{coz}\Big(2^{-n} \cdot \beta^{\frac{1}{n}}\Big) \\ &= \bigvee_n \Big(\operatorname{coz} 2^{-n} \wedge \operatorname{coz}\beta^{\frac{1}{n}}\Big) \\ &= \bigvee_n \operatorname{coz}(\beta^{\frac{1}{n}}) \\ &= \operatorname{coz}\beta. \end{aligned}$$

Since $\operatorname{coz}\gamma = \operatorname{coz}\beta$ and $\boldsymbol{M}_{\operatorname{coz}\alpha}$ is a z-ideal, then $\gamma \in \boldsymbol{M}_{\operatorname{coz}\alpha}$. Hence $\gamma \in \sqrt{Q}$ and hence there exists $m \in \mathbb{N}$ such that $\gamma^m \in Q$. Furthermore, since $2^{-n} \cdot \beta^{\frac{1}{n}} \leq \gamma$, for every $n \in \mathbb{N}$, we have $2^{-2m} \cdot \beta^{\frac{1}{2m}} \leq \gamma$ which implies that

$$\Big(2^{-2m} \cdot \beta^{\frac{1}{2m}}\Big)^m \leq \gamma^m$$

and hence

$$2^{-2m^2} \cdot \beta^{\frac{1}{2}} \quad \leq \quad \gamma^m.$$

Therefore, by Lemma 7.2.1, there exists a $\tau \in \mathcal{R}L$ such that

$$\beta \quad = \quad \tau \cdot \gamma^m.$$

This shows that $\beta \in Q$, and hence $\boldsymbol{M}_{\operatorname{coz}\alpha} \subseteq Q$. $\square$

Corollary 7.2.1. *An ideal of $\mathcal{R}L$ is a z-ideal if and only if its radical is a z-ideal.*

Proof. $(\Rightarrow)$: Let Q be a z-ideal of $\mathcal{R}L$. Suppose for $\alpha, \beta \in \mathcal{R}L, \alpha \in \sqrt{Q}$ and $\operatorname{coz}\alpha = \operatorname{coz}\beta$. By definition of a radical of an ideal, $\alpha^n \in Q$ for some $n \in \mathbb{N}$. Since $\operatorname{coz}(\alpha^n) = \operatorname{coz}\alpha = \operatorname{coz}\beta$ and Q is a z-ideal, it follows that $\beta \in Q \subseteq \sqrt{Q}$. Therefore $\sqrt{Q}$ is a z-ideal.

$(\Leftarrow)$: Suppose for $\alpha, \beta \in \mathcal{R}L$, $\alpha \in Q$ and $\operatorname{coz}\alpha = \operatorname{coz}\beta$. Since $\sqrt{Q}$ is a z-ideal, $\beta \in \sqrt{Q}$. By Proposition 7.2.1, $\boldsymbol{M}_{\operatorname{coz}\beta} \subseteq \sqrt{Q}$ and hence $\boldsymbol{M}_{\operatorname{coz}\beta} \subseteq Q$. Since $\beta \in \boldsymbol{M}_{\operatorname{coz}\beta} \subseteq Q$, it follows that $\beta \in Q$. Therefore Q is a z-ideal. $\square$

Corollary 7.2.2. *Let Q be an ideal of $\mathcal{R}L$. Then Q is a z-ideal if and only if every prime ideal minimal over it is a z-ideal.*

Proof. $(\Rightarrow)$: By [58, Theorem 1.1.].

$(\Leftarrow)$: By Corollary 7.2.1, it is enough to show that $\sqrt{Q}$ is a z-ideal. But by [39, 0.18], $\sqrt{Q}$ is an intersection of prime ideals. Therefore $\sqrt{Q}$ is the intersection of prime ideals minimal over $\sqrt{Q}$. Hence $\sqrt{Q}$ is an intersection of z-ideals, therefore it is a z-ideal. $\square$

Recall that an ideal I of an f-ring A is absolutely convex if, for any $a, b \in A$,

$$|a| \leq |b| \quad \text{and} \quad b \in I \quad \Longrightarrow \quad a \in I.$$

In [32, Lemma 3.5] it is shown that every radical ideal of $\mathcal{R}L$ is absolutely convex. The proof (due to B. Banaschewski) given there uses uniform frames. Recall from [39, Theorem 5.5] that in $C(X)$ prime ideals are absolutely convex. The proof given there actually shows that in $C(X)$ radical ideals are absolutely convex. Thus, in $\mathcal{R}^*L$ radical ideals are absolutely convex.

Proposition 7.2.2. *Every radical ideal of $\mathcal{R}L$ is absolutely convex.*

Proof. Let Q be a radical ideal of $\mathcal{R}L$. We show first that Q^c is a radical ideal of $\mathcal{R}^*L$. Consider any $\gamma \in \mathcal{R}^*L$ such that $\gamma^2 \in Q^c$. Then $\gamma^2 \in Q$, and hence $\gamma \in Q$ because Q is a radical ideal. But γ is bounded, so $\gamma \in Q^c$. Thus, Q^c is a radical ideal. Now let $\alpha, \beta \in \mathcal{R}L$ be such that $|\alpha| \leq |\beta|$ and $\beta \in Q$. Then $\frac{\beta}{(\mathbf{1}+|\alpha|)(\mathbf{1}+|\beta|)} \in Q^c$, and since

$$\left|\frac{\alpha}{(\mathbf{1}+|\alpha|)(\mathbf{1}+|\beta|)}\right| \leq \left|\frac{\beta}{(\mathbf{1}+|\alpha|)(\mathbf{1}+|\beta|)}\right|$$

and both these functions are bounded, it follows that $\frac{\alpha}{(\mathbf{1}+|\alpha|)(\mathbf{1}+|\beta|)} \in Q^c \subseteq Q$. Since Q is an ideal of $\mathcal{R}L$, this implies $\alpha \in Q$, and hence Q is absolutely convex. □

In his doctoral thesis [57], Mason shows that if I and J are z-ideals, then IJ is a z-ideal precisely when $IJ = I \cap J$. In $\mathcal{R}L$, just as in $C(X)$, the product of two z-ideals is always a z-ideal, as we show next. We will invoke the f-ring structure of $\mathcal{R}L$. Recall that in any f-ring A, the absolute value of an element a is the element $|a| = a \vee (-a)$, and the elements a^+ and a^- are defined by

$$a^+ = a \vee 0 \quad \text{and} \quad a^- = (-a) \vee 0.$$

Among other properties, they satisfy

$$|a| = a^+ + a^- \quad \text{and} \quad a = a^+ - a^-.$$

Lemma 7.2.2. *If P and Q are z-ideals in $\mathcal{R}L$, then $PQ = P \cap Q$.*

Proof. Since $PQ \subseteq P \cap Q$ always holds, we show the reverse inclusion. Let $\alpha \geq 0$ be in $P \cap Q$. Pick $\beta \in \mathcal{R}L$ such that $\alpha = \beta^2$. Since $\operatorname{coz}\alpha = \operatorname{coz}\beta$ and P and Q are z-ideals, $\beta \in P$ and $\beta \in Q$, and hence $\alpha \in PQ$. Now consider an arbitrary $\alpha \in P \cap Q$, since $|\alpha| = \alpha^+ + \alpha^-$, it follows that

$$\operatorname{coz}\alpha^+ \leq \operatorname{coz}\alpha \quad \text{and} \quad \operatorname{coz}\alpha^- \leq \operatorname{coz}\alpha.$$

Hence we have that $\alpha^+ \in P \cap Q$, and $\alpha^- \in P \cap Q$ since P and Q are z-ideals. By what we showed first, α^+ and α^- are both in PQ, hence $P \cap Q \subseteq PQ$.

□

We observed in Chapter 3 that if I is a d-ideal in a reduced f-ring with bounded inversion, then $I^{ce} = I$. We do not know if this holds for z-ideals in general. However for a special class of z-ideals in $\mathcal{R}L$ we have the following result.

Proposition 7.2.3. *For any $a \in L$, $(\boldsymbol{M}_a)^{ce} = \boldsymbol{M}_a$.*

Proof. Since the inclusion $I^{ce} \subseteq I$ holds in any ring, we need only show that $\boldsymbol{M}_a \subseteq (\boldsymbol{M}_a)^{ce}$. Recall from Corollary 3.2.1 that $\mathcal{R}L = \mathcal{R}^*L[S^{-1}]$ for the set

$$S = \{\alpha \in \mathcal{R}^*L \mid \operatorname{coz} \alpha = 1\}.$$

Thus,

$$(\boldsymbol{M}_a)^{ce} = \{\rho \in \mathcal{R}L \mid \rho = \mu\sigma^{-1} \text{ where } \mu \in (\boldsymbol{M}_a)^c \text{ and } \sigma \in S\}.$$

Now, for any $\gamma \in \boldsymbol{M}_a$,

$$\gamma = \frac{\gamma(\mathbf{1} + |\gamma|)}{\mathbf{1} + |\gamma|} = \frac{\gamma}{\mathbf{1} + |\gamma|} \cdot \left(\frac{1}{\mathbf{1} + |\gamma|}\right)^{-1}.$$

Since $\operatorname{coz}\left(\frac{\gamma}{\mathbf{1}+|\gamma|}\right) = \operatorname{coz} \gamma \leq a$ and $\frac{1}{\mathbf{1}+|\gamma|} \in S$, it follows that $\boldsymbol{M}_a \subseteq (\boldsymbol{M}_a)^{ce}$, whence the result follows. □

7.3 On the Stone-Čech compactification of frames

It is well known that, for any Tychonoff space X, the maximal ideal space, $\operatorname{Max} C(X)$, with the Zariski topology is homeomorphic to βX. As mentioned in Remark 2.3.1, Banaschewski and Sioen prove that the frame of Jacobson radicals of $\mathcal{R}L$ is the compact completely regular coreflection of L. It can be deduced from their results that the frame $\mathfrak{O}(\operatorname{Max} \mathcal{R}L)$ is isomorphic to βL.

In this section we give a direct proof of this fact by actually constructing an isomorphism which witnesses this. Let us recall what the frame $\mathfrak{O}(\operatorname{Max} A)$ looks like for any commutative ring A. For any ideal I of A – the improper ideal A included – the open sets in $\operatorname{Max} A$ are of the form

$$\mathcal{U}(I) = \{M \in \operatorname{Max} A \mid M \not\supseteq I\},$$

so that

$$\mathfrak{O}(\operatorname{Max} A) = \{\mathcal{U}(I) \mid I \text{ is an ideal of } A\}.$$

Next we recall the following Proposition from [65].

Proposition 7.3.1. *For a ring A, a necessary and sufficient condition that* $\operatorname{Max} A$ *be compact Hausdorff is that for every pair M and N of distinct maximal ideals, there exist $a \notin M$ and $b \notin N$ such that $ab \in \operatorname{Jac}(A)$.*

The ring $\mathcal{R}L$ does satisfy this condition, as the following lemma shows.

Lemma 7.3.1. *The frame $\mathfrak{O}(\operatorname{Max}\mathcal{R}L)$ is compact regular.*

Proof. Let I and J be distinct points of βL, so that we have the two distinct maximal ideals $\boldsymbol{M}^I$ and $\boldsymbol{M}^J$ of $\mathcal{R}L$. Since I and J are distinct points in βL, $I \vee J = 1_{\beta L}$, and hence, in view of βL being a normal frame, there exist $U, V \in \beta L$ such that

$$U \wedge V = 0_{\beta L} \quad \text{and} \quad I \vee U = J \vee V = 1_{\beta L}.$$

From this we can find $\tau, \rho \in \mathcal{R}L$ and cozero elements $c \in I$ and $d \in J$, such that

$$\operatorname{coz}\tau \in U,\ \operatorname{coz}\rho \in V \ \text{ and } \ c \vee \operatorname{coz}\tau = 1 = d \vee \operatorname{coz}\rho.$$

Since $r_L(c) \vee r_L(\operatorname{coz}\tau) = 1_{\beta L}$ and $r_L(c) \subseteq I$, it follows that $r_L(\operatorname{coz}\tau) \nsubseteq I$, lest I be the top of βL. Thus, $\tau \notin \boldsymbol{M}^I$. Similarly, $\rho \notin \boldsymbol{M}^J$. Since $U \wedge V = 0_{\beta L}$, $\operatorname{coz}\tau \wedge \operatorname{coz}\rho = 0$, which implies $\tau\rho = \mathbf{0}$, which belongs to every maximal ideal of $\mathcal{R}L$. Therefore, by Proposition 7.3.1, $\operatorname{Max}\mathcal{R}L$ is a compact Hausdorff space, and hence $\mathfrak{O}(\operatorname{Max}\mathcal{R}L)$ is a compact regular frame. $\square$

Observe that, for any $I, J \in \beta L$,

$$\boldsymbol{O}^I \subseteq \boldsymbol{M}^J \quad \Longleftrightarrow \quad I \subseteq J.$$

The right-to-left implication is trivial. Conversely, let $a \in I$, and pick $\gamma \in \mathcal{R}L$ such that $a \prec\!\!\prec \operatorname{coz}\gamma \in I$. Then $\gamma \in \boldsymbol{O}^I \subseteq \boldsymbol{M}^J$, which implies $a \in r_L(\operatorname{coz}\gamma) \subseteq J$. We observed in Remark 2.1.1 that the Jacobson radical of $\mathcal{R}L$ is the zero ideal.

For the next result we recall the following Lemma from [33].

Lemma 7.3.2. *For any $\alpha, \gamma \in \mathcal{R}L$, if $\operatorname{coz}\alpha \prec\!\!\prec \operatorname{coz}\gamma$, then α is a multiple of γ.*

Proposition 7.3.2. *The map* $h\colon \beta L \to \mathfrak{O}(\operatorname{Max}\mathcal{R}L)$ *defined by*

$$h(I) = \mathcal{U}(\boldsymbol{O}^I)$$

is a frame isomorphism.

Proof. It is immediate that h preserves the top and the bottom. Clearly, it also preserves order. Now let $I, J \in \beta L$. If $M \in \mathcal{U}(\boldsymbol{O}^I) \cap \mathcal{U}(\boldsymbol{O}^J)$, then $M \not\supseteq \boldsymbol{O}^I$ and $M \not\supseteq \boldsymbol{O}^J$, and so, in light of M being prime, $M \not\supseteq \boldsymbol{O}^I \cap \boldsymbol{O}^J = \boldsymbol{O}^{I\wedge J}$, which shows that $h(I) \wedge h(J) \subseteq h(I \wedge J)$, and hence equality. Next, let $\{I_\alpha\}$ be a collection of elements of βL. We show that $h(\bigvee_\alpha I_\alpha) \subseteq \bigcup_\alpha h(I_\alpha)$, which will establish that h preserves joins. Consider any $J \in \operatorname{Pt}(\beta L)$ such that $\boldsymbol{M}^J \in h(\bigvee_\alpha I_\alpha) = \mathcal{U}(\boldsymbol{O}^{\bigvee_\alpha I_\alpha})$. Then $\boldsymbol{M}^J \not\supseteq \boldsymbol{O}^{\bigvee_\alpha I_\alpha}$, so that, by what we observed earlier, $\bigvee_\alpha I_\alpha \not\leq J$. Therefore there is an index α_0 such that $I_{\alpha_0} \not\leq J$, whence $\boldsymbol{M}^J \not\supseteq \boldsymbol{O}^{I_{\alpha_0}}$, implying

$$\boldsymbol{M}^J \in \mathcal{U}(\boldsymbol{O}^{I_{\alpha_0}}) \subseteq \bigcup\nolimits_\alpha \mathcal{U}(\boldsymbol{O}^{I_\alpha}) = \bigcup\nolimits_\alpha h(I_\alpha).$$

Therefore h is a frame homomorphism. Now we show that h is one-one. As stated earlier, it suffices to show that it is dense. Consider any $I \in \beta L$ such that

$$h(I) = \mathcal{U}(\boldsymbol{O}^I) = 0_{\mathfrak{O}(\operatorname{Max}\mathcal{R}L)} = \emptyset.$$

This implies that $\boldsymbol{O}^I$ is contained in every maximal ideal of $\mathcal{R}L$, and is therefore the zero ideal. Thus, $I = 0_{\beta L}$, and so h is dense. Finally, we show that h is onto. Consider any ideal Q of $\mathcal{R}L$. We must produce an element I of βL such that $h(I) = \mathcal{U}(Q)$. Define $I \in \beta L$ by

$$I = \bigvee\nolimits_{\beta L} \{r_L(\operatorname{coz}\alpha) \mid \alpha \in Q\} = \bigcup\nolimits_{\alpha\in Q} \{r_L(\operatorname{coz}\alpha) \mid \alpha \in Q\},$$

the latter equality holds because the join is directed. We will show that $\mathcal{U}(\boldsymbol{O}^I) = \mathcal{U}(Q)$. Take any maximal ideal M of $\mathcal{R}L$ such that $M \in \mathcal{U}(\boldsymbol{O}^I)$. Then $M \not\supseteq \boldsymbol{O}^I$, so there is a $\gamma \in \boldsymbol{O}^I$ such that $\gamma \notin M$. Now $\gamma \in \boldsymbol{O}^I$ implies $\operatorname{coz}\gamma \in I$, and hence there is an $\alpha \in Q$ such that $\operatorname{coz}\gamma \in r_L(\operatorname{coz}\alpha)$. Thus, $\operatorname{coz}\gamma \prec\!\!\prec \operatorname{coz}\alpha$, which, by the lemma cited from [33], implies γ is a multiple of α, and is therefore in Q. Consequently, $M \not\supseteq Q$, whence $\mathcal{U}(\boldsymbol{O}^I) \subseteq \mathcal{U}(Q)$. Next, consider any $J \in \operatorname{Pt}(\beta L)$ such that $\boldsymbol{M}^J \not\supseteq Q$. Take an $\alpha \in Q$ with $\alpha \notin \boldsymbol{M}^J$. Then $r_L(\operatorname{coz}\alpha) \not\leq J$, and so $r_L(\operatorname{coz}\alpha) \vee J = 1_{\beta L}$ since J is a maximal element in βL. By compactness of βL, we can find finitely many positive elements $\gamma_1, \dots, \gamma_n$ of $\mathcal{R}L$ such that

$\operatorname{coz}\gamma_i \in I$ for each $i = 1, \dots, n$, and

$$r_L(\operatorname{coz}\gamma_1) \vee \cdots \vee r_L(\operatorname{coz}\gamma_n) \vee J = 1_{\beta L}.$$

Put $\gamma = \gamma_1 + \cdots + \gamma_n$ and observe that $\operatorname{coz}\gamma = \operatorname{coz}\gamma_1 \vee \cdots \vee \operatorname{coz}\gamma_n$. Since r_L preserves finite joins of cozero elements, we have $r_L(\operatorname{coz}\gamma) \vee J = 1_{\beta L}$, whence $r_L(\operatorname{coz}\gamma) \nleq J$, implying $\gamma \notin \boldsymbol{M}^J$. But $\gamma \in \boldsymbol{O}^I$ since $\operatorname{coz}\gamma \in I$, therefore $\boldsymbol{M}^J \not\supseteq \boldsymbol{O}^I$, and hence $\mathcal{U}(Q) \subseteq \mathcal{U}(\boldsymbol{O}^I)$. Thus $\mathcal{U}(Q) = \mathcal{U}(\boldsymbol{O}^I)$, which implies $h(I) = \mathcal{U}(Q)$, showing that h is onto.

$\square$

Next we will describe the inverse of h since it is an isomorphism. Since $\mathcal{U}(Q)$ is not uniquely determined by Q, we need to exercise a bit of care. For any ideal Q in $\mathcal{R}L$, let I_Q be the element of βL given by

$$I_Q = \bigvee\nolimits_{\beta L} \{ r_L(\operatorname{coz}\alpha) \mid \alpha \in Q \}.$$

As observed in the foregoing proof, $\mathcal{U}(Q) = \mathcal{U}(\boldsymbol{O}^{I_Q})$. Now let P and Q be two ideals in $\mathcal{R}L$ with $\mathcal{U}(P) = \mathcal{U}(Q)$. Then

$$h(I_P) = \mathcal{U}(\boldsymbol{O}^{I_P}) = \mathcal{U}(P) = \mathcal{U}(Q) = \mathcal{U}(\boldsymbol{O}^{I_Q}) = h(I_Q),$$

which implies $I_Q = I_P$ since h is one-one. It follows from this that the map

$$\mathfrak{O}(\operatorname{Max}\mathcal{R}L) \to \beta L \quad \text{given by} \quad \mathcal{U}(Q) \mapsto I_Q$$

is well-defined, and is the inverse of the isomorphism h.

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