SATISFICING SOLUTIONS FOR MULTIOBJECTIVE STOCHASTIC LINEAR PROGRAMMING PROBLEMS

by

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Summary
Multiobjective Stochastic Linear Programming is a relevant topic. As a matter of fact, many real life problems ranging from portfolio selection to water resource management may be cast into this framework.

There are severe limitations in objectivity in this field due to the simultaneous presence of randomness and conflicting goals. In such a turbulent environment, the mainstay of rational choice does not hold and it is virtually impossible to provide a truly scientific foundation for an optimal decision.

In this thesis, we resort to the bounded rationality and chance-constrained principles to define satisficing solutions for Multiobjective Stochastic Linear Programming problems. These solutions are then characterized for the cases of normal, exponential, chi-squared and gamma distributions.

Ways for singling out such solutions are discussed and numerical examples provided for the sake of illustration.

Extension to the case of fuzzy random coefficients is also carried out.

Keywords
Satisficing solution, chance constrained, Linear programming, multiobjective programming, stochastic Programming, expected value optimality/efficiency, variance optimality/efficiency, expected value and standard deviation optimality/efficiency, minimum risk optimality/efficiency, optimality /efficiency with given probabilities, fuzzy random variables, random closed sets, Embedding Theorem.

Declaration

Student number: 3655-598-3

I declare that

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is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references.

. . . .

SIGNATURE (Mr A.S Adeyefa) DATE

Dedication

This thesis is dedicated to

Professor John Omoniyi Abiri,
The Obapero of Abiri

a retired professor of Educational Psychology, now a crowned paramount ruler and king of Abiri town through whose advice I became a mathematician.

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Introduction

Many concrete real-life situations, including engineering, industrial and economic problems, involve the challenging task of incorporating conflicting goals as well as random data into an optimization setting, giving rise to mathematical programs with complicated structure. Among those complex optimization problems, linear models are the most popular, mainly because they are simple to understand and easy to represent [11], [63], [92], [114].

Competing objective functions cannot be arbitrarily squeezed within the narrow framework of a single utility function without running the risk of invalidating all implications that are supposed to be drawn from the analysis. Simple examples (see e.g., [52], [72], [145]) are in line with the endorsed paradox [127] and the Arrow's impossibility Theorem [14], where there are no good ways of aggregating conflicting objective functions into a single one. This has given rise to the field of Multiobjective Programming (MOP) [74], [121], [123], [146].
Moreover, the above mentioned problems may include some level of uncertainty about the values to be assigned to various parameters. In this connection the noted philosopher Nietzche was quoted as saying,

"No one is gifted with immaculate perception".

False certainty is bad science and it could be dangerous if it stunts articulation of critical choices. Interested readers are referred to [90], [120], [135], [148] for problems where uncertainty should be accommodated in an optimization framework.

Uncertainty presents unique difficulties in constrained optimization problems, because decision makers are faced with doubtful situations, requiring an analysis of multiple outcomes in different state of nature. When the uncertainty in question is stochastic in nature, then we are in the realm of Multiobjective Stochastic Programming (MOSP).

The problem under consideration in this thesis is that of solving the following multiobjective program:

$$(P1) \left\{ \min_{x \in D(\omega)} \left(c^1 x, \ldots, c^K x \right) \right.$$

where $c^1, \ldots, c^K$ are random vectors defined on a Probability space $(\Omega, \mathcal{F}, \mathbb{P})$. That is, for all $k \in \{1, \ldots, K\}$

$$\begin{aligned} c^k : \Omega &\mapsto \mathbb{R}^n \\ \omega &\mapsto c^k(\omega) \end{aligned}$$

is a measurable function and

$$D(\omega) = \{x \in \mathbb{R}^n \mid A(\omega)x - b(\omega) \leq 0,\ x \geq 0\}$$

where $A(\omega),\ b(\omega)$ are respectively $m \times n$ and $m \times 1$ random matrices.

As examples of concrete problems that may be put into the form of $(P1)$, we mention:

- The automated manufacturing system in a production planning situation [1], [34], [57], [147].
- Water use planning [29], [36], [41], [45]
- Financial and banking planning [9], [16], [20], [30], [38]
- Distributed energy resources and power systems planning [10], [131], [132], [133], [136].

Owing to the presence of conflicting goals and the randomness surrounding data, the mathematical program described in $(P1)$ is an ill-stated problem. Therefore, neither the notion of feasibility nor that of optimality is clearly defined for this problem.
A look at the literature reveals that most existing solution concepts for MOSP problems rely heavily on the expected, pessimistic and optimistic values of involved random variables [31], [33].
Findings from several research works [90], [100], [116] leave no doubt about the fact that these values may be useful in providing the range of possible outcomes. Nevertheless, they ignore such important factors as the size and the probability of deviations outside the likely range. They also neglect other aspects concerning the dispersion of involved random variables. As a result, these values often, offer a short-sighted view of the under consideration.

This has motivated a search for more flexible formulations of optimization problems, that, although remains rigorous, bridge the gap between mathematical programming models and real decision-making situations, through introduction of uncertainty and conflicting objective functions.

In this thesis, we resort to the bounded rationality principle [125], to present satisficing solution for $(P1)$. These solutions concepts, are based on the chance constrained paradigm [43]. Chance constrained applies for the purpose of limiting the probability that a constraint will be violated. In this form, it adds considerably to both the flexibility and the reality of the stochastic model under consideration.

Mathematical characterization of the above mentioned solution concepts are provided for the case of normal, exponential, chi-squared and gamma distributions and ways for singling out these satisficing solutions are also addressed.

In some applications, input data of a Multiobjective Programming problem may be af-

fected by a relatively large amount of both randomness and fuzziness [144], [148]. In portfolio selection, for example, the returns on security may be expressed as a fuzzy random variable in a way to couple stock experts' judgements and investors' conflicting opinions. Some methodological approaches, for dealing with situations where randomness and fuzziness are under one roof in a multiobjective optimization setting maybe found in [21], [76], [79], [81], [89], [91], [108], [142] and in references therein. In these papers, the spotlight is on Multiobjective Programming problems with flexible relationships and random data containing fuzzy parameters [21].
In the last part of this thesis, we add to the spectrum of methods on Multiobjective Programming problems under randomness and fuzziness, a complementary method of coping with situations where fuzzy random variables are to be incorporated into a Multiobjective Programming framework.

The thesis is organized as follows. In Chapter 1 we provide some preliminary materials. In Chapter 2, we present an overview of what has been done in the field of Multiobjective Stochastic Linear Programming, emphasizing the limitations of previous studies. In Chapter 3 we endeavor to overcome these limitations by introducing satisficing solution concepts for Multiobjective Stochastic Linear Programming problems. An extension to Multiobjective Programming problems with fuzzy random coefficients is discussed in Chapter 4. We end up with some concluding remarks along with lines for further developments in this field.

Chapter 1

Preliminaries

In this chapter, we present some preliminary notions on Probability Theory and Mathematical Programming that will be needed in the sequel.

1.1 Elements of Probability Theory

1.1.1 Probability space

1.1.1 Definition. Let Ω be a set. A family $\mathcal{F} = (A_i)_{i \in I}$ of subset of Ω is called a $\sigma-$algebra on Ω if the following three properties hold:

1. $\Omega \in \mathcal{F}$
2. for any sequence $A_1, A_2, \ldots$ of elements of $\mathcal{F}$, $\bigcup_{i=1}^{\infty} A_n \in \mathcal{F}$
3. for any $A \in \Omega$, the complement $\bar{A} = \Omega - A \in \mathcal{F}$.

If $\mathcal{F}$ is a $\sigma-$algebra on Ω, then $(\Omega, \mathcal{F})$ is called a measurable space. The elements of $\mathcal{F}$ are termed $\mathcal{F}-$measurable sets or simply measurable sets.

If $\mathcal{H}$ is a family of subsets of Ω then there exists at least one $\sigma-$algebra on Ω containing $\mathcal{H}$. The smallest $\sigma-$algebra on Ω containing $\mathcal{H}$ is called the $\sigma-$algebra generated by $\mathcal{H}$.
1.1.2 Definition. Let $(\Omega, \mathcal{F})$ be a measurable space. A measure on $(\Omega, \mathcal{F})$ is a function μ from $\mathcal{F}$ into the set $[0, \infty]$ such that the following properties hold:

1. for any sequence $A_1, A_2, \ldots, A_n, \ldots$ of mutually exclusive elements of $\mathcal{F}$,

$$\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

2. $\mu(\phi) = 0$.

If μ is a measure on a measurable space $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \mu)$ is called a measure space.
1.1.3 Definition. A probability on a measurable space $(\Omega, \mathcal{F})$ is a measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ verifying the condition $\mathbb{P}(\Omega) = 1$.

If $\mathbb{P}$ is a probability on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space.
1.1.4 Definition. A topological space is a set E endowed with a family $\mathcal{T}$ of subsets of E such that the following three properties hold:

1. the empty subset and E belong to $\mathcal{T}$
2. if $(O_i)_{i\in I}$ is a family of elements of $\mathcal{T}$ then $\bigcup_{i\in I} O_i \in \mathcal{T}$
3. if $O_1, \ldots, O_n$ are elements of $\mathcal{T}$ then $\bigcap_{i=1}^n O_i \in \mathcal{T}$.

Elements of $\mathcal{T}$ are called the open subsets of E.
A complement of an open subset is called a closed subset.
A neighborhood of a point $x \in E$ is any subset of E that contains an open set containing the point x.

The closure of a subset A of E is the smallest closed subset of E containing A.
If E and F are topological spaces, a map $f : E \to F$ is continuous if for any open subset O of F, f^{-1} is also continuous, f is called a homeomorphism.

The space $\mathbb{R}^n$ is an example of a separable locally compact space. A subset of this space is compact if and only if it is bounded and closed.
1.1.5 Definition. Let E be a topological space and $\mathcal{T}$ be the set of open subsets of E. The $\sigma-$algebra on E generated by $\mathcal{T}$ is called the Borel $\sigma-$algebra on E.

When a topological space is considered as a measurable space, measurable sets are members of the Borel $\sigma-$algebra.
1.1.6 Definition. Let $\mathbb{B}$ be the Borel $\sigma-$algebra on $\mathbb{R}^n$. There exists a measure μ on $(\mathbb{R}^n, \mathbb{B})$ such that for any intervals $A_i = (a_i, b_i)$ of $\mathbb{R}$ with $a_i \leq b_i$

$$\mu(A_1 \times A_2 \times \ldots, \times A_n) = (b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n).$$

This measure is called the Lebesgue measure on $\mathbb{R}^n$.

The $\sigma-$algebra generated by a family $(X_i)_{i\in I}$ of functions $X_i : \Omega \to (E, \mathcal{E})$ is the smallest $\sigma-$algebra on Ω for which all the functions X_i are measurable. It is equal to the $\sigma-$algebra on Ω generated by the sets of the form $X^{-1}(A_i)$, where for any $i \in I, A_i \in \mathcal{E}$.

1.1.2 Integration

1.1.7 Definition. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A measurable function $f : \Omega \to \mathbb{R}$ is said to be a step function if there exists a partition $A_1, \ldots, A_n$ of Ω such that the subsets A_i are measurable and f is constant on each of them.

If $f = a_i$ on A_i, then the integral of the function f is defined by

$$\int f d\mu = \sum_{i=1}^{n} a_i \mu(A_i).$$

For any $A \in \mathcal{F}$ and any step function f,

$$\int_A f d\mu = \int f.1_A d\mu$$

where 1_A is the characteristic function of A.

1.1.8 Definition. Let $f : \Omega \to \mathbb{R}$ be a nonnegative measurable function. It can be shown that there exists a nondecreasing sequence (f_n) of nonnegative measurable step functions that converges to f. The integral of the function f is

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

1.1.9 Definition. Let $f : \Omega \to \mathbb{R}$ be a measurable function. If

$$\int |f| d\mu < \infty,$$

then the function f is said to be $\mu-$integrable.

The integral of the function f is

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

where

$$f^+ = \sup(f, 0), \;\; f^- = \inf(-f, 0).$$

The integral $\int f d\mu$ is also denoted by $f(x)d\mu(x)$.

1.1.3 Random variables, Random vectors

A random variable is a function, which maps events or outcomes of a random experiment to real numbers. A random variable's possible values might represent the possible outcomes of a yet-to-be-performed experiment, or the potential values of a quantity whose already-existing value is uncertain (e.g. as a result of incomplete information or imprecise measurements). A more rigorous definition of a random variable is given below.

1.1.10 Definition. A random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a function

$$\begin{aligned} Y : \Omega &\mapsto \mathbb{R} \\ \omega &\mapsto Y(\omega) \end{aligned}$$

such that for any Borel subset A of $\mathbb{R}$, the inverse image

$$Y^{-1}(A) = \{\omega \in \Omega : Y(\omega) \in A\} \in \mathcal{F}.$$

In the sequel $Y^{-1}(A)$ is merely denoted by $\{y \in A\}$.

It is an obvious fact that $Y : \Omega \mapsto \mathbb{R}$ is a random variable if and only if for any $a \in \mathbb{R}^n$, the set

$$Y^{-1}[a, \infty) = \{\omega \in \Omega : Y(\omega) \geq a\} \in \mathcal{F}.$$

1.1.11 Definition. An $n-$dimensional random vector on $(\Omega, \mathcal{F}, \mathbb{P})$ is a function

$$\begin{aligned} Y : \Omega &\mapsto \mathbb{R}^n \\ \omega &\mapsto Y(\omega) \end{aligned}$$

such that $Y^{-1}(A) \in \mathcal{F}$, for any Borel subset A of $\mathbb{R}^n$.

Observe that every component of a random vector is itself a random variable. It is worth making a distinction between discrete and continuous random variables.
A discrete random variable is one whose set of assumed values is countable. A continuous random variable is one whose set of assumed values is uncountable.

1.1.4 Probability distribution of a random variable

1.1.12 Definition. Let Y be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The Cumulative Distribution Function of Y is the function:

$$F_Y : \mathbb{R} \to \mathbb{R}$$

defined by:

$$F_Y(y) = \mathbb{P}\{\omega \in \Omega \mid Y(\omega) \leq y\}.$$

If $\bar{\mathbb{P}}$ is absolutely continuous with respect to the Lebesgue measure $\mathbb{P}$ [118], by the Radon-Nikodym Theorem [77], there is a Probability Density Function $f_Y(\tau)$ defined on $\mathbb{R}$ such that

$$\bar{\mathbb{P}}(B) = \int_B f_Y(\tau) d\mathbb{P}$$

for all $B \in \mathcal{B}$.

1.1.13 Definition. A Cumulative Distribution Function F_Y is absolutely continuous if there exists a non-negative, Lebesgue integrable function f [118] such that

$$F_Y(b) - F_Y(a) = \int_a^b f(y)dy \quad \textit{for all } a < b.$$

The function f is called the Probability Density Function of F. The area under the curve between any two ordinates $y = a$ and $y = b$ is the probability that $Y(\omega)$ lies between a and b.

$$\int_a^b f(y)dy = \mathbb{P}(\{\omega \in \Omega \mid a \leq Y(\omega) \leq b\})$$

For a discrete probability distribution, we have a Probability Mass Function, which is a function that gives the probability that a discrete random variable is exactly equal to some value.

1.1.14 Definition. Let Y be a random variable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Then the Probability Mass Function

$$\mathbb{P}_Y : R \mapsto [0, 1]$$

for Y is defined as

$$\mathbb{P}_Y(y) = \mathbb{P}(Y = y) = \mathbb{P}(\{\omega \in \Omega : Y(\omega) = y\}).$$

The total probability for all Y must be equal to 1.

$$\sum \mathbb{P}_Y(y) = 1$$
$$y = Y(\omega)$$
$$\omega \in \Omega.$$

The discontinuity of Probability Mass Functions reflects the fact that the Cumulative Distribution Function of a discrete random variable is also discontinuous. Where it is differentiable, the derivative is zero, just as the Probability Mass Function is zero at all such points.

To any given probability measure $\mathbb{P}$ on $(\mathbb{R}, \mathbb{R})$ we can associate a Cumulative Distribution Function F_Y via the relation:

$$F_Y(b) - F_Y(a) = \mathbb{P}(\{\omega \in \Omega \mid a < \omega \leq b\}) \quad \textit{for all a,b}, \quad -\infty < a \leq b < \infty,$$

and since $\mathbb{P}$ as well as F_Y are uniquely defined by their values on the rectangles.

The joint distribution function of an n-dimensional random vector Y with components $Y_1, \ldots, Y_n$ is the function $F_Y : \mathbb{R}^n \to \mathbb{R}$ defined by

$$F_Y(y_1, \ldots, y_n) = P\left\{\omega \in \Omega \mid Y_1(\omega) \le y_1, \ldots, Y_n(\omega) \le y_n\right\}.$$

In this thesis, we adopt the notation $Y \sim H$ for random variables $Y_i;\ i = 1, \ldots, n$ that follow the distribution H with Probability Density Function $f_Y(y)$.

One of the central concepts in Probability Theory is independence, which we consider in the next subsection.

1.1.5 Independence

Events or random variables are independent if they do not affect each other's probabilities. Random variables Y and V are independent if for all Borel sets S and $\bar{S}$, the events $Y^{-1}(S)$ and $V^{-1}(\bar{S})$ are independent, i.e.

$$P(Y \in S,\ V \in \bar{S}) = P(Y \in S)P(V \in \bar{S}).$$

More formally, we have the following definitions.
1.1.15 Definition. The random variables

$$Y_1, \ldots, Y_n$$

are independent if and only if for arbitrary Borel sets

$$A_1, \ldots, A_n \text{ and for } \omega \in \Omega$$

$$P\left(\bigcap_{j=1}^{n} \{\omega \mid Y_j(\omega) \in A_j\}\right) = \prod_{j=1}^{n} P(\{\omega \mid Y_j(\omega) \in A_j\}).$$

1.1.16 Definition. The random variables $Y_1, \ldots, Y_n$ are independent if and only if

$$F_{Y(\omega)}(y) = \prod_{j=1}^{n} F_{Y_j(\omega)}(y_j), \quad \textit{for all } y \in \mathbb{R}^n,\ \omega \in \Omega.$$

Definitions (1.1.15) and (1.1.16) are equivalent. The second definition is implied by the first, since the half-open infinite sets are a subclass of all measurable sets.
Moreover, functions of independent random variables are independent and independence is preserved under deterministic transformation as stipulated in the next two results whose proofs can be found in [65].
1.1.1 Theorem. *1. If Y and V are discrete random variables, then Y and V are*

independent if and only if

$$\mathbb{P}_{Y(\omega),V(\omega)}(y,v) = \mathbb{P}_{Y(\omega)}(y) \cdot \mathbb{P}_{V(\omega)}(v) \quad \textit{for all } \omega \in \Omega,\ Y(\omega), V(\omega) \in \mathbb{R}.$$

2. *If Y and V are absolutely continuous, then Y and V are independent if and only if*

$$f_{Y(\omega),V(\omega)}(y,v) = f_{Y(\omega)}(y) \cdot f_{V(\omega)}(v) \quad \textit{for all } \omega \in \Omega,\ Y(\omega), V(\omega) \in \mathbb{R}.$$

1.1.2 Proposition. *Let*

$$Y_1, \ldots, Y_n \tag{1.1}$$

be random variables and

$$f_1, \ldots, f_n$$

be measurable functions. If $Y_1, \ldots, Y_n$ are independent, then so are

$$f \circ Y_1, \ldots, f \circ Y_n. \tag{1.2}$$

Other important notions on Probability Theory that will be needed in the sequel include: expected value, variance and moment generating function. We briefly discuss them in what follows.

1.1.6 Expected value

1.1.17 Definition. Let $y_1, \ldots, y_n$ be real numbers, $A_1, \ldots, A_n \in \mathcal{F}$ such that

$$\bigcup_{i=1}^{n} A_i = \Omega.$$

The expected value of a random variable

$$Y = \sum_{j=1}^{n} y_j 1_{A_j}$$

is defined as follows:

$$E\left(\sum_{j=1}^{n} y_j 1_{A_j}\right) = \sum_{j=1}^{n} y_j P(A_j)$$

where $\{y_j;\ 1 \leq j \leq n\}$ are real numbers, and $\{A_j;\ 1 \leq j \leq n\}$ is a finite partition of Ω, and 1_{A_j} denotes the characteristic (indication) function of A_j.

The expected value of a non-negative random variable is defined below.

1.1.18 Definition. Suppose Y is a non-negative random variable. The expected value of Y is defined as

$$E(Y) = lim_{n\to\infty} \sum_{j=1}^{n2^n} \frac{j-1}{2^n} P\left(\{\omega \in \Omega \mid \frac{j-1}{2^n} \leq Y(\omega) < \frac{j}{2^n}\} \right).$$

We consider arbitrary random variables which may be neither simple nor non-negative. For such a random variables we may write

$$Y = Y^+ - Y^-$$

where

$$Y^+ = \max(Y, 0)$$

and

$$Y^- = \max(-Y, 0).$$

Both Y^+ and Y^- are non-negative.

1.1.19 Definition. For an arbitrary random variable Y, the expected value of Y is

$$E(Y) = E(Y^+) - E(Y^-)$$

provided at least one of $E(Y^+)$ and $E(Y^-)$ is finite.
We write

$$E(Y) = \int_\Omega Y(\omega)dP(\omega) or\ simply \int Y(\omega)dP(\omega)$$

If $E(Y) < \infty$, *we say that* Y *is integrable.*

The following lemma, the proof of which may be found in [119], highlights properties of expectations of random variables.

1.1.3 Lemma. *Let* V *and* Y *be random variables, then*

1. $E(V + Y) = E(V) + E(Y)$
2. $E(cV) = cE(V)$ *where* c *is a constant*
3. *if* $V = 0$ *then* $E(V) = 0$
4. *if* $V \leq Y$ *then* $E(V) \leq E(Y)$
5. $|E(V)| \leq E(|V|)$
6. *if* V *and* Y *are independent, then* $E(XY) = E(X)E(Y)$.

Let $(\Omega, \mathcal{F})$ be a measurable space and $\mathbb{P}, \bar{\mathbb{P}}$ be two measures defined on $\mathcal{F}$. If $\bar{\mathbb{P}}$ is absolutely continuous with respect to the Lebesgue measure $\mathbb{P}$, by the Radon-Nikodym Theorem, the expectation $E(Y)$ of a random vector Y is the vector of the integrals of the components of Y. We write:

$$E(Y) = (\int_\Omega Y_1 d\mathbb{P}, \ldots, \int_\Omega Y_n d\mathbb{P})^T = \int_\Omega Y d\mathbb{P}.$$

Hence we have

$$E(Y) = \int_\Omega Y d\mathbb{P} = \int_{R^n} \omega d\bar{\mathbb{P}} = \int_{R^n} \omega dF_Y(\omega)$$

where the last expression is the so-called Lebesgue-Stieltjes integral.

If Y has a Probability Density Function, we may also write

$$E(Y) = \int_{R^n} \omega f_Y(\omega) d\mathbb{P}_n$$

where $d\mathbb{P}_n$ refers to the Lebesgue measure on $\mathcal{B}_n$.

1.1.7 Variance and standard deviation

1.1.20 Definition. Let Y be a random variable, the variance of Y denoted by $Var(Y)$ or σ^2 is defined as:

$$Var(Y) = E[Y - E(Y)]^2.$$

Using the notation μ for $E(Y)$, we can also write:

$$Var(Y) = E[Y - \mu]^2.$$

It is an easy matter to see that:

$$Var(Y) = E(Y^2) - [E(Y)]^2.$$

The square root of the variance of Y, denoted by $\sqrt{Var(Y)}$, is called the standard deviation of the probability distribution. Some properties of variance, are found in [119] and are given below as follows:

1.1.4 Lemma. *Let Y and V be independent random variables with finite variances, and $a, b \in \mathbb{R}$. Then*

1. $Var(aY) = a^2 Var(Y)$
2. $Var(Y + V) = Var(Y) + Var(V)$
3. $Var(aY + bV) = a^2 Var(Y) + b^2 Var(V)$

1.1.8 Moment generating function

1.1.21 Definition. Let Y be a random variable. The moment generating function of a random variable Y is the function

$$M_Y(s) = E(e^{sY}) = \int_{-\infty}^{\infty} e^{sy} dF_Y(y)$$

provided the expectation exists and $s \in \mathbb{R}$.

A lot of information about the distribution of Y can be obtained from $M_Y(s)$.
1.1.22 Definition. Let Y be a random variable, then

1. $E(Y^n);\ n = 1, \ldots;$ are called moments of Y
2. $E(Y - E(Y))^n;\ n = 1, \ldots;$ are called central moments of Y
3. $E|Y|^n;\ n = 1, \ldots;$ are called absolute moments of Y
4. $E|Y - E(Y)|^n;\ n = 1, \ldots;$ are called absolute central moments of Y.

It is clear that moment of order 1 is the mean and the second central moment is the variance. The third and the fourth moments are the skewness and the kutorsis respectively.

Some properties of moment generating function of random variables, which can be found in [118], are given below.

1.1.5 Lemma. *Let a and c be constants, and let $M_Y(s)$ be the moment generating function of a random variable Y. Then the moment generating function of the random variable $V = a + cY$ is the following:*

$$M_V(s) = E[e^{sV}] = E[e^{s(a+cY)}] = e^{as} M_Y(cs).$$

1. *Let V and Y be independent random variables having the respective moment generating functions $M_V(s)$ and $M_Y(s)$. The moment generating function of the sum $U = V + Y$ random variable is*

$$M_U(s) = E[e^{sU}] = E[e^{s(V+Y)}] = E[e^{sV} e^{sY}] = E[e^{sV}].[e^{sY}] = M_V(s).M_Y(s).$$

2. *By differentiating $M_Y(s)$ r times, we obtain*

$$M_Y^{(r)}(s) = \frac{d^r}{ds^r} e^{sY} = E[\frac{d^r}{ds^r} e^{sY}] = E[Y^r e^{sY}].$$

3. *When $s = 0$, $M_Y^{(r)}(s)$ generates the r^{th} moment of Y. As a matter of fact*

$$M_Y^{(r)}(0) = E[Y^r];\ r = 1, 2, \ldots.$$

Figure 1.1: Normal Probability Density Function

In the following subsections, we look into more detail at the distributions we shall consider in this study.

1.1.9 Some usual distributions

1. **The normal distribution**

   A normal or Gaussian random variable is a continuous random variable with a symmetric, bell-shaped Probability Density Function. The Probability Density Function of a normal random variable Y with mean μ and standard deviation σ is given by:

$$f(y) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(y-\mu)^2}{2\sigma^2}}; \quad y \in \mathbb{R},\ \mu \in \mathbb{R},\ \sigma > 0.$$

   This distribution is commonly denoted by $N(\mu, \sigma^2)$.
   Its Cumulative Density Function is given by

$$F(y) = \int_{-\infty}^{y} f(y)dt; \quad y \in \mathbb{R}.$$

   The normal random variable with mean $\mu = 0$ and standard deviation $\sigma = 1$ is called the standard normal random variable, and its Cumulative Distribution Function is denoted by $\Phi(z)$.
   If Y is a normal random variable with mean μ and standard deviation σ, and Z is the standard normal random variable, then

$$\begin{aligned} \mathbb{P}(Y \leq y) &= \mathbb{P}\left(Z \leq \frac{y-\mu}{\sigma}\right) \\ &= \int_{-\infty}^{(y-\mu)/\sigma} e^{(-t^2/2)}dt \\ &= \Phi\left(\frac{y-\mu}{\sigma}\right). \end{aligned} \tag{1.3}$$

   The mean μ is the location parameter and the standard deviation σ is the scale parameter. A normal distribution is often a rough substitute for any distribution. The normal distribution governs many aspects of human performance [85]. For example a diversified portfolio will typically have returns that fall in a normal distribution [20].

2. **The chi-squared distribution**

   Let $Y_1, \ldots, Y_n$ be independent standard normal random variables. The distribution

Figure 1.2: Chi-squared Probability Density Function

of

$$Y = \sum_{j=1}^{n} Y_j^2$$

is called the chi-squared distribution with degree of freedom (df) n, and is denoted by χ_n^2. Its Probability Density Function is given by

$$f(n) = \frac{e^{-y/2} y^{(n/2)-1}}{2^{n/2}\Gamma(n/2)}; \qquad y \geq 0, \; n > 0$$

where Γ stands for the gamma function. The Cumulative Distribution Function of the chi-squared distribution is given by Krishnamoorthy [85] and Sleeper [126] as follows:

$$F(n) = \frac{1}{\Gamma(n/2)} \sum_{j=0}^{\infty} \frac{(-1)^j (y/2)^{n/2+j}}{j!\Gamma(n/2+j)} = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^y e^{-t/2} t^{(n/2)-1} dt.$$

The Inverse Cumulative Distribution Function of χ_n^2 has no easy formula, and must be calculated iteratively [126]. One way of doing that is by using the Wilson-Hilferty transformation [140].
Let $Y_1, \ldots, Y_n$ be independent standard normal variables and $Y = \sum_{j=1}^{n} Y_j^2$ be the chi-squared distribution with n degree of freedom, then the Wilson-Hilferty transformation of Y into an appropriate normal distribution is given below:

$$\sqrt{\frac{9n}{2}} \left\{ \left(\frac{Y^2}{\sigma^2 n} \right)^{\frac{1}{3}} - 1 + \frac{2}{9n} \right\} \sim N(0,1).$$

The inverse of this formula can be used to approximate quantiles of the χ_n^2 distribution from $\Phi^{-1}(p)$, the standard normal Inverse Cumulative Distribution Function

$$F^{-1}(p) = \sigma \sqrt{n \left(\sqrt{\frac{9n}{2}} \Phi^{-1}(p) + 1 - \frac{2}{9n} \right)^3}. \tag{1.4}$$

If $Y_1, \ldots, Y_n$ are independent chi-squared random variables with degrees of freedom $n_1, \ldots, n_k$ respectively, then

$$\sum_{j=1}^{k} Y_j \sim \chi_m^2 \text{ with } m = \sum_{j=1}^{k} n_j$$

which implies that $\sum_{j=1}^{k} Y_j$ is also a chi-squared random variable with degree of

Figure 1.3: Exponential Probability Density Function

freedom $\sum_{j=1}^{k} n_j$. The mean and variance of χ_n^2 are n and $2n$ respectively. Its characteristic function is given by [126]

$$\phi(y) = \frac{1}{(1 - i2\sigma^2 y)^{\frac{1}{2}}} e^{\frac{it\mu^2}{(1-i2\sigma^2 y)}}$$

where $i = \pm\sqrt{-1}$, μ and σ^2 are mean and variance of independent normal distribution.

The chi-squared is useful in many testing procedures related to the variation of normally distributed data [85].

3. **The exponential distribution**

The Probability Density Function of a random variable following an exponential distribution of rate parameter λ is given by

$$f_{Exp(\lambda)}(y) = \lambda e^{-y\lambda};\ y > 0,\ \lambda > 0 \qquad y \geq 0.$$

This Probability Density Function may be defined in terms of mean parameter μ as follows:

$$f_{Exp(\mu)}(y) = \frac{1}{\mu} e^{-y/\mu}; \qquad y \geq 0,\ \mu > 0.$$

The corresponding Cumulative Distribution Functions are respectively:

$$F_{Exp(\lambda)}(y)) = 1 - e^{-y\lambda};\ y \geq 0,\ \lambda > 0$$

and

$$F_{Exp(\mu)}(y)) = 1 - e^{-y/\mu}; \quad y \geq 0,\ \mu > 0.$$

Exponential distribution has the following properties [126]:

(a) The mean of exponential distribution in term of rate and mean parameters are respectively $E[Exp(\lambda)] = \frac{1}{\lambda}$ and $E[Exp(\mu)] = \mu$.

(b) The variance of exponential distribution in term of rate and mean parameters are respectively $Var[Exp(\lambda)] = \frac{1}{\lambda^2}$ and $Var[Exp(\mu)] = \mu^2$.

(c) A standard exponential random variable has $\lambda = \mu = 1$, if $Y = Exp(1)$, then $\mu Y \sim Exp(\mu)$ and $\frac{Y}{\lambda} \sim Exp(\lambda)$.

(d) The minimum of a set of independent observations of an exponential random variable is also an exponential random variable.

The following result, ith proof found in [27] is given below:

1.1.6 Lemma. *Assume that* $\rho_1, \ldots, \rho_n$ *are independent random variables, exponentially distributed with respective parameters* $\mu_1, \ldots, \mu_n$. *Then for any real numbers* $x_1, \ldots, x_n$ *the random variable* $Y = x_1\rho_1 + \ldots + x_n\rho_n$ *has the following density distribution:*

$$f(y) = \prod_{j=1}^{n} \mu_j \left(\sum_{q=1}^{n} \frac{x_q^{n-2} e^{-\frac{\mu_q y}{x_q}}}{\prod_{\{l=1, l\neq q\}}^{n} (x_q\mu_l - x_l\mu_q)} \right) \qquad \text{if } y \geq 0$$

with mean and variance $E(Y) = \sum_{j=1}^{n} \frac{x_j}{\mu_j}$ *and* $Var(Y) = \sum_{j=1}^{n} \frac{x_j^2}{\mu_j^2}$ *respectively.*

Exponential distribution has tremendous applications in modeling life time [122]. In queuing theory, the exponential distribution is commonly used to model the probability distribution of a random variable that represents service time. In reliability applications, the exponential distribution is commonly used to model the life time of components which are subject to wear out [119].

4. **The gamma distribution**

The gamma distribution can be viewed as a generalization of the exponential distribution with mean parameter $\mu = 1/\lambda,\ \lambda > 0$.

Let $X_1, \ldots, X_n$ be independent exponential random variables with mean $1/\lambda$,

$$Y = \sum_{i=1}^{\alpha} X_i$$

is a gamma random variable.

The chi-squared family of distributions is a subset of gamma family of distributions. Specifically a χ_n^2 is also a gamma random variable with shape parameter $\alpha = n/2$ and scale parameter $\beta = 2$.

Because of this relationship, tools intended for gamma distributions work well for both chi-squared distributions and exponential distributions, and vice versa [122].

A random variable Y is a gamma random variable with shape parameter α and scale parameter β, if its Probability Density Function is a

$$f_{\gamma(\alpha,\beta)}(y) = \frac{y^{\alpha-1} e^{-y\beta}}{\beta^{\alpha}\Gamma(\alpha)}; \qquad y \geq 0$$

where Γ stands for the gamma function. In this case we note $Y \sim \gamma(\alpha, \beta)$.
The corresponding Cumulative Distribution Function is

$$F_{\gamma(\alpha,\beta)}(y) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} \int_0^y u^{\alpha-1} e^{-u/\beta} du; \qquad y \geq 0.$$

Figure 1.4: Gamma Probability Density Function

When the shape parameter α is an integer, gamma random variables are called Erlang random variables, with Probability Density Function and Cumulative Distribution Function are respectively given as follows:

$$f_{\gamma(\alpha,\beta)}(y) = \frac{y^{\alpha-1}e^{-y\beta}}{\beta^{\alpha}(\alpha-1)!}\ ; \qquad y \geq 0$$

and

$$F_{\gamma(\alpha,\beta)}(y) = 1 - e^{y/\beta}\sum_{i=0}^{\alpha-1}\frac{y/\beta}{i!}; \quad y \geq 0.$$

The weighted sum of independent gamma random variable is a gamma random variable [138].

Some notable properties of gamma random variable are highlighted below [126]:

(a) The mean and variance of gamma distribution are $E[\gamma(\alpha,\beta)] = \alpha\beta$ and $Var[\gamma(\alpha,\beta)] = \alpha\beta^2$.

(b) The sum of n mutually independent gamma random variables with different shape parameters α_j and the same scale parameter β is a gamma random variable with shape parameter equal to the sum of the component shape parameters. If $Y_j \sim \gamma(\alpha_j, \beta)$ and the Y_j are mutually independent, then $\sum_{j=1}^{n} Y_j(\omega) \sim \gamma(\sum_{j=1}^{n}\alpha_j, \beta)$.

(c) The Cumulative Distribution function of gamma random variables have a variety of shapes.

(d) $Y_j \sim \gamma(\beta_j, t_j) \Leftrightarrow c_jY_j \sim \gamma\left(\frac{\beta_j}{c_j,t_j}\right)$.

(e) As the shape parameter α increases, gamma random variables become very similar in shape to normal random variables.

The following lemma due to Vellaisamy & Upadhye [138] is useful in Chapter 3 Subsection (3.6).

1.1.7 Lemma. *Let $Y_1, \ldots, Y_n$ be independent random variables, where $\gamma(\beta_j, \alpha_j)$, the gamma distribution with scale parameter β_j^{-1} and shape parameter $\alpha_j > 0$. For $c_j > 0$ and $i \in \mathbb{Z}_+ \backslash \{0\}$, let $T_n = \sum_{j=1}^{n} c_jY_j$, $\beta = \max_{1\leq j\leq n}\frac{\beta_j}{(c_j+\beta_j)}$, $d_n = \prod_{j=1}^{n}((1-\beta)\beta_j)/(c_j\beta))_j^{\alpha}$ and $a_i = (1/i)\sum_{j=1}^{n}\alpha_j(1-(1-\beta)\beta_j/c_j\beta))^i$. Then $T_n \sim \gamma(\frac{\beta}{1-\beta}), L_n + \alpha)$, where L_n is a random variable with $\mathbb{P}(L_n = k) = d_nb_k,\ k \in \mathbb{R}$, and $\alpha = \sum_{j=1}^{n}\alpha_j$. Here, $b_0 = 1$ and $b_k = (\frac{1}{k}\sum_{i=1}^{k} ia_ib_{k-i})$ for $k \in \mathbb{Z}_+ \backslash \{0\}$.*

5. **Relationships between the normal, chi-squared, exponential and gamma distributions**

In what follows we mention some relationships between normal distribution and the other distributions, used in this thesis.

(a) The normal random variable is the limiting distribution for the standardized form

$$\frac{Y - E(Y)}{\sqrt{Y}}$$

of many other random variables including chi-squared, exponential and gamma distributions [85].

This idea is clearly reflected in the central limit Theorem.

1.1.8 Theorem. *If $Y_1, \ldots, Y_n$ are mutually independent, identically distributed random variables with finite mean μ and standard deviation σ, then the distribution of the standardized sum*

$$\frac{\left\{\left(\sum_{j=n}^{n} Y_j\right) - n\mu\right\}}{\sigma\sqrt{n}}$$

tends to a standard normal distribution as n goes to infinity.

(b) Any continuous random variable Y with Cumulative Distribution Function $F_Y(y)$ can be transformed into a standard exponential random variable by the function -ln$(1 - F_Y(y)) \sim Exp(1)$ [126].

1.2 Mathematical Programming

1.2.1 Problem formulation

Mathematical Programming is a branch of optimization theory in which a single-valued objective function f of n real variables $x_1, \ldots, x_n$ is minimized (or maximized), subject to a finite number of constraints, which are written as inequalities or equations. A mathematical program is therefore a problem of the form

$$(P2)\begin{cases} \min \quad f(x) \\ g_i(x) \leq 0;\ i = 1, \ldots, m \\ h_j(x) = 0;\ j = 1, \ldots, p \\ x \geq 0 \end{cases}$$

where $x \in \mathbb{R}^n$ and $f,\ g_i;\ (i = 1, \ldots, m)$ and $h_j\ (j = 1, \ldots, p)$ are real valued functions of $\mathbb{R}^n$.

If one or more of the functions appearing in $(P2)$ are nonlinear in x, we call it a nonlinear program, in contrast to a linear program, where all these functions must be linear.

Mathematical Programming problems arise in such various disciplines as engineering, economics, business administration, physical sciences, and mathematics, or in any other area where decisions (in a broad sense) must be taken in some complex (or conflicting) situations that can be represented by a mathematical model [63], [92], [115]. In order to illustrate some types of mathematical programs, an example is presented below.

1.2.1 Example. *Suppose that in some scientific research, say in biology or physics, a certain phenomenon f is measured in the laboratory as a function of time. Also suppose that we are given a mathematical model of the phenomenon, and from the model we know that the value of f is assumed to vary with time t as*

$$f(t) = x_1 + x_2 e^{-x_3 t}. \tag{1.5}$$

The purpose of the laboratory experiments is to find the unknown parameters x_1, x_2 and x_3 by measuring values of f at times $t_1, \ldots, t_m$. The decision-making process involves assigning values to the parameters, and it is reasonable to ask those values of x_1, x_2 and x_3 that are optimal in some sense. For instance, we can seek optimal values of the parameters in the least-squares sense, that is, those values for which the sum of squares of the experimental deviations from the theoretical curve is minimized.

Formally, we have the following optimization problem:

$$(P2)' \qquad \min \left\{ \sum_{i=1}^{m} [f(t_i) - x_1 - x_2 e^{-x_3 t_i}]^2 \right\}$$

Note that this is an unconstrained program that, if solved, may yield unacceptable values of the parameters. To avoid such a situation, we can impose restrictions in the form of constraints. For example, the parameters x_3 can be restricted to have a nonnegative value-that is,

$$x_3 \geq 0. \tag{1.6}$$

Also suppose that, for the particular phenomenon under consideration, the mathematical model proposed can be acceptable only if the parameters are so chosen that at $t = 0$ we have $f(0) = 1$. Hence we must add a constraint

$$x_1 + x_2 = 1. \tag{1.7}$$

Solving $(P2)'$, subject to (1.6) *and* (1.7), *is then a constrained Mathematical Programming problem having a nonlinear objective function with constraints.*

1.2.2 Equality Constrained extremum and the method of Lagrange

In this subsection we consider the problem :

$$(P2)'' \qquad \begin{cases} \min f(x) \\ g_i(x) = 0; \quad i = 1, \ldots, m \end{cases} \tag{1.8}$$

where $g_i;\ i = 1, \ldots, m$ and f are real-valued function defined on $D \subset \mathbb{R}^n$. The assumption that $m < n$ will simplify subsequent discussions. The problem is, therefore, to find an extremum of f in the region determined by the equations in (1.8). The first and most intuitive method of solution of such problem involves the elimination of m variables from the problem by using equations in (1.8). The conditions for such an elimination will be stated later in the Implicit Function Theorem, and the proof can be found in most advanced calculus textbooks (see, e.g., [17], [102]). This theorem assumes the differentiability of the functions g_i and the fact that the $n \times m$ Jacobian matrix $[\partial g_i / \partial x_j]$ has rank m. The actual solution of equations (1.8) for m variables in terms of the remaining $n - m$ can often prove difficult, if not impossible task. For this reason, we do not pursue this method further.

Another method, also based on the idea of transforming a constrained problem into unconstrained one, was proposed by Lagrange [92]. Many results in mathematical programming are actually a direct extension and generalization of Lagrange's method, mainly to problems with inequality constraints.

Before presenting a result that will provide a direction in which we must proceed in order to transform an equality constrained problem into an equivalent unconstrained problem, we state the well-known Implicit Function Theorem.

1.2.1 Theorem. *(Implicit Function Theorem [15])*
Suppose that ϕ_i are real-valued functions defined on D and continuously differentiable on an open set $D^1 \subset D \subset \mathbb{R}^{m+p}$, where $p > 0$ and $\phi_i(x^0, y^0) = 0$ for $i = 1\ldots, m$ and $(x^0, y^0) \in D^1$. Assume that the Jacobian matrix $[\partial\phi_i(x^0, y^0)/\partial x_j]$ has the rank m. Then there exists a neighborhood $N_\delta(x^0, y^0) \subset D^1$, an open set $D^2 \subset \mathbb{R}^p$ containing y^0 and real-valued functions ψ_k; $k = 1, \ldots, m$, continuously differentiable on D^2, such that the following conditions are satisfied:

$$x_k^0 = \psi_k(y^0); \quad k = 1, \ldots, m$$

for every $y \in D^2$, we have

$$\phi_i(\psi(y), y) = 0; \quad i = 1, \ldots, m$$

where $\psi(y) = (\psi_1(y), \ldots, \psi_m(y))$ and for all $(x, y) \in N_\delta(x^0, y^0)$ the Jacobian matrix $[\partial\phi_i(x, y)/\partial x_j]$ has rank m. Furthermore, for $y \in D^2$, the partial derivatives of $\psi_k(y)$ are the solutions of the set of linear equations

$$\sum_{k=1}^{m} \frac{\partial\phi_i(\psi(y), y)}{\partial x_k}\frac{\partial\psi_k(y)}{\partial y_j} = -\frac{\partial\phi_i(\psi(y), y)}{\partial y_j}, \quad i = 1, \ldots, m. \tag{1.9}$$

Before introducing the method of Lagrange, we discuss the following result due to Beltrami [19].

1.2.2 Theorem. *Let f and g_i; $i = 1, \ldots, m$ be real-valued functions on $D \subset \mathbb{R}^n$ and continuously differentiable on a neighborhood $N_\epsilon(x^*) \subset D$. Suppose that x^* is a local minimum of f for all points x in $N_\epsilon(x^*)$ that also satisfy*

$$g_i(x) = 0; \quad i = 1, \ldots, m. \tag{1.10}$$

Also assume that the Jacobian matrix of $g_i(x^*)$ has rank m. Under these hypotheses the gradient of f at x^* is a linear combination of the gradients of g_i at x^*; that is, there exist real numbers λ_i^* such that:

$$\nabla f(x^*) = \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*).$$

Proof. By suitable re-arrangement and relabeling of rows, we can always assume that the $m \times m$ matrix, formed by taking the first m rows of the Jacobian $[\partial g_i(x^*)/\partial x_j]$, is nonsingular. The set of linear equations

$$\sum_{i=1}^{m} \frac{\partial g_i(x^*)}{\partial x_j}\lambda_i = \frac{\partial f(x^*)}{\partial x_j}; \quad j = 1, \ldots, m \tag{1.11}$$

has a unique solution for the λ_i denoted by λ_i^*.
Let $\hat{x} = (x_{m+1}, \dots, x_n)$. Then by applying Theorem 1.2.1 to (1.10) at x^*, there exist real functions $h_j(\hat{x})$ and an open set $\hat{D} \subset \mathbb{R}^{n-m}$ containing x^*, such that

$$x_j^* = h_j(\hat{x}^*); \quad j = 1, \dots, m$$

and

$$f(x^*) = f(h_1(\hat{x}^*), \dots, h_m(\hat{x}^*), x_{m+1}^*, \dots, x_n^*).$$

As a result of the last relation, it follows that the first partial derivatives of f with respect to $x_{m+1}, \dots, x_n$ must vanish at x^* [15]. Thus

$$\frac{\partial f(x^*)}{\partial x_j} = \sum_{k=1}^{m} \frac{\partial f(x^*)}{\partial x_k} \frac{\partial h_k(\hat{x}^*)}{\partial x_j} + \frac{\partial f(x^*)}{\partial x_j} = 0; \quad j = m+1, \dots, n. \tag{1.12}$$

From (1.9) we have for every $j = m+1, \dots, n$

$$\sum_{k=1}^{m} \frac{\partial g_i(x^*)}{\partial x_k} \frac{\partial h_k(\hat{x}^*)}{\partial x_j} = -\frac{\partial g_i(x^*)}{\partial x_j}; \quad i = 1, \dots, m. \tag{1.13}$$

Multiplying each of the equations in (1.13) by λ_i^* and adding up, we get

$$\sum_{i=1}^{m} \sum_{k=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_k} \frac{\partial h_k(\hat{x}^*)}{\partial x_j} + \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_j} = 0; \quad j = 1, \dots, n. \tag{1.14}$$

Subtracting (1.14) from (1.12) and rearranging yield

$$\sum_{k=1}^{m} \left[\frac{\partial f(x^*)}{\partial x_k} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_k} \right] \frac{\partial h_k(\hat{x}^*)}{\partial x_j} + \frac{\partial f(x^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_j} = 0; \quad j = m+1, \dots, n. \tag{1.15}$$

But the expression in brackets is zero by (1.11), and so

$$\frac{\partial f(x^*)}{\partial x_j} - \sum_{i=1}^{m} \lambda_i^* \frac{\partial g_i(x^*)}{\partial x_j} = 0, \quad j = m+1, \dots, n.$$

The last expression, together with (1.11), yields the desired result. □

The relation between the gradient of the function to be minimized and the gradients of the constraint functions at a local extremum, as expressed in the last theorem, leads to the formulation of the Lagrangian, $L(x, \lambda)$ of $(P2)''$ as follows:

$$L(x, \lambda) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) \tag{1.16}$$

where the $\lambda_i; \ \ i = 1, \ldots, m$ are called Lagrange Multipliers.

Lagrange's method consists of transforming an equality constrained extremum problem into a problem of finding a stationary point of the Lagrangian. This can be seen by the following result [102].

1.2.3 Theorem. *Suppose that $f, \ g_i; \ \ i = 1, \ldots, m$, satisfy the hypotheses of Theorem* (1.2.2)*. Then there exists a vector of multipliers $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)^T$ such that*

$$\nabla L(x^*, \lambda^*) = 0.$$

Proof. Follows directly from Theorem (1.2.2) and the definition of L as given by (1.16). □

Several different proofs of the last two theorems exist in the literature. See, for example [19], [102]. We have chosen the one based on the Implicit Function Theorem, since it requires no additional background material. Theorem (1.2.3) provides necessary conditions for an extremum of f with equality constraints.

In the next subsection we turn to a discussion of sufficient conditions for such an extremum, see for example [15] [19] for details.

1.2.4 Theorem. *Let $f, \ g_1, \ldots, g_m$ be twice continuously differentiable real-valued functions of $\mathbb{R}^n$. If there exist vectors $x^* \in \mathbb{R}^n, \ \lambda^* \in \mathbb{R}^m$ such that*

$$\nabla L(x^*, \lambda^*) = 0 \tag{1.17}$$

and for every nonzero vector $z \in \mathbb{R}^n$ satisfying

$$z^T \nabla g_i(x^*) = 0; \quad i = 1, \ldots, m \tag{1.18}$$

it follows that

$$z^T \nabla_x^2 L(x^*, \lambda^*) z > 0 \tag{1.19}$$

*and hence f has a strict local minimum at x^*, subject to $g_i(x) = 0; \ \ i = 1, \ldots, m$. If the sense of the inequality in (*1.19*) is reversed, then f has a strict local minimum at x^*.*

Proof. Assume that x^* is not a strict local minimum. Then there exists a neighborhood $N_\delta(x^*)$ and a sequence $\{z^k\}, \ z^k \in N_\delta(x^*), \ z^k \neq x^*$, converging to x^* such that for every

$z^k \in \{z^k\}$

$$g_i(z^k) = 0; \quad i = 1, \ldots, m \tag{1.20}$$

and

$$f(x^*) \geq f(z^k). \tag{1.21}$$

Let $z^k = x^* + \theta^k y^k$, where $\theta^k > 0$ and $||y^k|| = 1$. The sequence $\{\theta^k, y^k\}$ has a subsequence that converges to $(0, \hat{y})$, where $||\hat{y}|| = 1$. By the Mean Value Theorem [17], we get for each k in this subsequence

$$g_i(z^k) - g_i(x^*) = \theta^k (y^k)^T \nabla g_i(x^* + \eta_i^k \theta^k y^k) = 0; \quad i = 1, \ldots, m \tag{1.22}$$

where η_i^k is a number between 0 and 1 and

$$f(z^k) - f(x^*) = \theta^k (y^k)^T \nabla f(x^* + \xi^k \theta^k y^k) \leq 0 \tag{1.23}$$

where ξ^k is, again, a number between 0 and 1.

Dividing (1.22) and (1.23) by θ^k and taking limits as $k \longrightarrow \infty$, we get

$$(\hat{y})^T \nabla g_i(x^*) = 0; \quad i = 1, \ldots, m$$

and

$$(\hat{y})^T \nabla f(x^*) \leq 0.$$

From Taylor's theorem we have

$$L(x^k, \lambda^*) = L(x^*, \lambda^*) + \theta^k (y^k)^T \nabla_x L(x^*, \lambda^*) + \frac{1}{2}(\theta^k)^2 (y^k)^T \nabla_x^2 L(x^* + \eta^k \theta^k y^k, \lambda^*) y^k$$

where $1 > \eta^k > 0$.

By (1.16), (1.17), (1.20), and (1.21), and dividing (1.24) by $\frac{1}{2}(\theta^k)^2$ we obtain

$$(y^k)^T \nabla_x^2 L(x^* + \eta^k \theta^k y^k, \lambda^*) y^k \leq 0.$$

Letting $k \longrightarrow \infty$, we obtain from the last expression

$$(\hat{y})^T \nabla_x^2 L(x^*, \lambda^*) \hat{y} \leq 0.$$

This completes the proof, since $\hat{y} \neq 0$, and it satisfies (1.18). $\square$

The sufficient conditions stated in the last theorem involve determining the sign of a quadratic form, subject to linear constraints. This task can be accomplished by a result, due to Mann [99].

Let $A = [\alpha_{ij}]$ be an $n \times n$ symmetric real matrix and $B = [\beta ij]$ an $n \times m$ real matrix. Denote by M_{pq} the matrix obtained from a matrix M by keeping only the elements in the first p rows and q columns.

Consider the next result whose proof can be found in [47] or [99]:

1.2.5 Theorem. *Suppose that det* $[B_{mm}] \neq 0$. *Then the quadratic form*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_{ij} \xi_i \xi_j \tag{1.24}$$

is positive for all nonzero ξ_i; $i = (\xi_1, \ldots, \xi_n)$ *satisfying*

$$\sum_{i=1}^{n} \beta_{ij} \xi_i = 0 \tag{1.25}$$

if and only if

$$(-1)^m \det \begin{bmatrix} A_{pp} \; \mathrm{B}_{pm} \\ B_{pm}^T \; 0 \end{bmatrix} > 0$$

for $p = m + 1, \ldots, n$.

Similarly, (1.24) *is negative for all nonzero vector* ξ *satisfying (*1.25*) if and only if*

$$(-1)^p \det \begin{bmatrix} A_{pp} \; \mathrm{B}_{pm} \\ B_{pm}^T \; 0 \end{bmatrix} > 0$$

for $p = m + 1, \ldots, n$.

Suppose now that the $n \times m$ Jacobian matrix $[\frac{\partial g_i(x^*)}{\partial x_j}]$ has rank m and the variables are indexed in such a way that

$$\det \begin{bmatrix} \frac{\partial g_1(x^*)}{\partial x_1}, \ldots, \frac{\partial g_m(x^*)}{\partial x_1} \\ \frac{\partial g_1(x^*)}{\partial x_m}, \ldots, \frac{\partial g_m(x^*)}{\partial x_m} \end{bmatrix} \neq 0.$$

Then we have the following result due to Avriel [15].

Let $f, g_1, \ldots, g_m$ be twice continuously differentiable real-valued functions. If there exist vector $x^* \in \mathbb{R}^n$, $\lambda^* \in \mathbb{R}^m$, such that

$$\nabla L(x^*, \lambda^*) = 0$$

and if

$$(-1)^m \det \begin{bmatrix} \frac{\partial^2 L(x^*,\lambda^*)}{\partial x_1 \partial x_1}, & \dots, & \frac{\partial^2 L(x^*,\lambda^*)}{\partial x_1 \partial x_p} & \frac{\partial g_1(x^*)}{\partial x_1}, & \dots, & \frac{\partial g_m(x^*)}{\partial x_1} \\ \cdot & , \dots, & \cdot & \cdot & , \dots, & \cdot \\ \cdot & , \dots, & \cdot & \cdot & , \dots, & \cdot \\ \cdot & , \dots, & \cdot & \cdot & , \dots, & \cdot \\ \frac{\partial^2 L(x^*,\lambda^*)}{\partial x_p \partial x_1}, & \dots, & \frac{\partial^2 L(x^*,\lambda^*)}{\partial x_p \partial x_p} & \frac{\partial g_1(x^*)}{\partial x_p}, & \dots, & \frac{\partial g_m(x^*)}{\partial x_p} \\ \frac{\partial g_1(x^*)}{\partial x_1} & , \dots, & \frac{\partial g_1(x^*)}{\partial x_p} & 0, & \dots, & 0 \\ \cdot & , \dots, & \cdot & \cdot & , \dots, & \cdot \\ \cdot & , \dots, & \cdot & \cdot & , \dots, & \cdot \\ \cdot & , \dots, & \cdot & \cdot & , \dots, & \cdot \\ \frac{\partial g_m(x^*)}{\partial x_1}, & \dots, & \frac{\partial g_m(x^*)}{\partial x_p} & 0, & \dots, & 0 \end{bmatrix} > 0$$

for $p = m+1, \dots, n$, then f has a strict local minimum at x^*, such that

$$g_i(x^*) = 0, \quad i = 1, \dots, m.$$

Proof. Follows directly from Theorems (1.2.4) and (1.2.5). □

The similar result for strict maxima is obtained by changing $(-1)^m$ to $(-1)^p$ in the matrix above.

1.2.3 First-order necessary conditions for Inequality Constrained Extremum

We begin deriving first-order necessary conditions for inequality and equality constrained extremum problems involving only first derivatives by considering the most general mathematical program $(P2)$ defined earlier. Here the functions $f,\ g_1, \dots, g_m,\ h_1, \dots, h_p$ are assumed to be defined and differentiable on some open set $D \subset \mathbb{R}^n$.
Let $X \subset D$ denote the feasible set for problem $(P2)$.
1.2.1 Definition. A point $x^* \in X$ is said to be a local solution of problem $(P2)$, if there exists a positive number $\delta > 0$ such that:

$$f(x) \geq f(x^*) \tag{1.26}$$

for all $x \in X \cap N_\delta(x^*)$.
If (1.26) holds for all $x \in X$, then x^* is said to be a global minimum (global solution) of problem $(P2)$.

Every point x in a neighborhood of x^* can be written as $x^* + z$, where z is a nonzero vector if and only if $x \neq x^*$.

1.2.2 Definition. A vector $z \neq 0$ is called a feasible direction vector from x^* if there exists a number $\delta_1 > 0$ such that $(x^* + \theta z) \in X \cap N_{\delta_1}(x^*)$ for all $0 \leq \theta \leq \delta_1/||z||$.

Feasible direction vectors are important in many numerical optimization algorithms. Momentarily, we are interested in them for the simple reason that if x^* is a local solution of problem $(P2)$ and z is a feasible direction vector, then we must have $f(x^* + \theta z) \geq f(x^*)$ for sufficiently small positive θ.

Let us characterize the feasible direction vectors in terms of the constraints $g_i;\ i = 1, \ldots, m$ and equations $h_j;\ j = 1, \ldots, p$.

Define

$$I(x^*) = \{i \ : \ g_i(x^*) = 0\}$$

and suppose that $z^T \nabla g_k(x^*) < 0$ for some $k \in I(x^*)$ z being a feasible direction vector from x^*. By the differentiability assumption, we can write:

$$g_k(x^* + \theta z) = g_k(x^*) + \theta z^T \nabla g_k(x^*) + \theta \epsilon_k(\theta)$$

where $\epsilon_k(\theta)$ tends to zero as $\theta \to 0$. If θ is small enough, then $z^T \nabla g_k(x^*) + \epsilon_k(\theta) < 0$; and since $g_k(x^*) = 0$, we obtain $g_k(x^* + \theta z) < 0$ for all sufficiently small $\theta > 0$, contradicting the fact that z is a feasible direction vector from x^*. Hence we must have $z^T \nabla g_k(x^*) \geq 0$ for all $i \in I(x^*)$.

Similar reasoning can be applied to show that, for a feasible direction vector z, we must also have $z^T \nabla h_j(x^*) = 0$ for $j = 1, \ldots, p$.

Define

$$Z^1(x^*) = \{z \mid z^T \nabla g_k(x^*) \geq 0,\ i \in I(x^*),\ z^T \nabla h_j(x^*) = 0;\ j = 1, \ldots, p\}.$$

From the foregoing discussion it follows that if z is a feasible direction vector from x^*, then $z \in Z^1(x^*)$.

1.2.3 Definition. A set $K \subset \mathbb{R}^n$ is called a cone if $x \in K$ implies $\alpha x \in K$ for every nonnegative number α.

The set $Z^1(x^*)$ is clearly a cone. It is also called a linearizing cone of X at x^* [64], since it is generated by linearizing the constraint functions at x^*.

Let us define $Z^2(x^*)$, another "'linearizing"' set that will be needed later:

$$Z^2(x^*) = \{z \ : \ z^T \nabla f(x^*) < 0\}.$$

If $z \in Z^2(x^*)$, it can be shown that there exists a point $x = x^* + \theta z$, sufficiently close to x^*, such that $f(x^*) > f(x)$.

Figure 1.5: Interpretation of the Farkas lemma for a 3×2 matrix

The following lemma, due to Minkowski and Farkas [56] is needed in the sequel.

1.2.6 Lemma. *Let A be a given $m \times n$ real matrix and b a given n vector. The inequality $b^T y \geq 0$ holds for all the vectors y satisfying $Ay \geq 0$ if and only if there exists an m vector $\rho \geq 0$ such that $A^T \rho = b$.*

An illustration of the Farkas Lemma is given in Figure 1.5 for a 3×2 matrix A. The vectors $A_1,\ A_2,\ A_3$ are the row vectors of the matrix A. Consider the set Y consisting of all vectors y that make an acute angle with every row vector of A. The Farkas lemma then states that b makes an acute angle with every $y \in Y$ if and only if b can be expressed as a nonnegative linear combination of the row vectors of A. In Figure 1.5, b^1 is a vector that satisfies these conditions, whereas b^2 is a vector that does not.

The Lagrangian associated with problem $(P2)$ is defined as follows:

$$L(x, \lambda, \mu) = f(x) - \sum_{i=1}^{m} \lambda_i g_i(x) - \sum_{j=1}^{p} \mu_j h_j(x)$$

and give the next three results with proofs can be found in [15].

1.2.7 Theorem. *Suppose that $x^0 \in X$. Then $Z^1(x^0) \cap Z^2(x^0) = \emptyset$ if and only if there exist vectors λ^0, μ^0 such that*

$$\nabla_x L(x^0, \lambda^0, \mu^0) = \nabla f(x^0) - \sum_{i=1}^{m} \lambda_i^0 \nabla g_i(x^0) - \sum_{j=1}^{p} \mu_j^0 \nabla h_j(x^0) \tag{1.27}$$

$$\lambda_i^0 g_i(x^0) = 0; \quad i = 1, \ldots, m \tag{1.28}$$

$$\lambda^0 \geq 0. \tag{1.29}$$

Proof. The set $Z^1(x^0)$ is never empty, since the origin always belongs to it; and $Z^1(x^0) \cap Z^2(x^0)$ is empty if and only if for every z satisfying

$$z^T \nabla g_i(x^0) \geq 0; \quad i \in I(x^0) \tag{1.30}$$

$$z^T \nabla h_j(x^0) = 0; \quad j = 1, \ldots, p \tag{1.31}$$

we have

$$z^T \nabla f(x^0) \geq 0. \tag{1.32}$$

We can write (1.31) as two inequalities

$$z^T \nabla h_j(x^0) \geq 0; \quad j = 1, \dots, p \tag{1.33}$$

$$z^T[-\nabla h_j(x^0)] \geq 0; \quad j = 1, \dots, p \tag{1.34}$$

It follows that (1.32) holds for all vectors z satisfying (1.30),(1.33), and (1.34) if and only if there exist vectors $\lambda^0 \geq 0, \mu^1 \geq 0, \mu^2 \geq 0$ such that

$$\nabla f(x^0) = \sum_{i \in I(x^0)}^{m} \lambda_i^0 \nabla g_i(x^0) - \sum_{j=1}^{p} (\mu_j^1 - \mu_j^2) \nabla h_j(x^0).$$

Letting $\lambda_i^0 = 0$ for $i \notin I(x^0)$, $\mu^0 = \mu^1 - \mu^2$, we conclude that $Z^1(x^0) \cap Z^2(x^0)$ is empty if and only if (1.27) to (1.29) hold. □

1.2.8 Theorem. *Suppose that f, $g_1, \dots, g_m, \dots, h_p$ are continuously differentiable on an open set containing X. If x^* is a solution of problem $(P2)$, then there exist vectors $\lambda^* = (\lambda_0^*, \dots, \lambda_m^*)^T$ and $\mu^* = (\mu_1^*, \dots, \mu_p^*)^T$ such that*

$$\nabla_x \bar{L}(x^*, \lambda^*, \mu^*) = \lambda_0^* \nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) - \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*) = 0 \tag{1.35}$$

$$\lambda_i^* g_i(x^*) = 0; \quad i = 1, \dots, m \tag{1.36}$$

$$(\lambda^*, \mu^*) \neq 0, \quad \lambda^* \geq 0. \tag{1.37}$$

Proof. We shall consider necessary conditions for the solution x^* of the problem

$$\begin{cases} \min \ f(x) \\ g_i(x) \geq 0; \quad i = 1, \dots, m \end{cases}$$

The conditions are the existence of a vector λ^* such that

$$\lambda_0^* \nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) \tag{1.38}$$

$$\lambda_i^* g_i(x^*) = 0; \quad i = 1, \dots, m \tag{1.39}$$

$$\lambda^* \neq 0, \quad \lambda^* \geq 0. \tag{1.40}$$

If $g_i(x^*) > 0, \ \forall \ i$, then $I(x^*) = 0$. Choose $\lambda_0^* = 1$, $\lambda_1^* = \lambda_2^* =, \dots, \lambda_m^* = 0$, and (1.38) to (1.40) hold with $\nabla f(x^*) = 0$.

Suppose now that $I(x^*) \neq \emptyset$. Then for every z satisfying

$$z^T \nabla g_i(x^*) > 0; \quad i \in I(x^*) \tag{1.41}$$

we cannot have

$$z^T \nabla f(x^*) < 0. \tag{1.42}$$

This result follows from the previously seen fact that if there exist a z satisfying (1.41), then we can find a sufficiently small δ such that the point $x = x^* + \theta z$ satisfies

$$g_i(x) > 0; \quad i = 1, \dots, m$$

for all $0 < \theta < \delta$; that is, x is feasible.
If (1.42) also holds, then

$$f(x) < f(x^*),$$

contradicting the fact that x^* is a minimum. Thus the system of inequalities (1.41) and (1.42) has no solution. By Theorem (1.2.7), there exists a nonzero vector $\lambda^* \geq 0$ such that

$$\lambda_0^* \nabla f(x^*) + \sum_{i \in I(x^*)}^{m} \lambda_i^* [-\nabla g_i(x^*)] = 0 \tag{1.43}$$

Letting $\lambda_i^* = 0$ for $i \in I(x^*)$, we get from (1.43), after rearrangement,

$$\lambda_0^* \nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) = 0 \tag{1.44}$$

and clearly

$$(\lambda_i^*, g_i(x^*)) = 0; \quad i = 1, \dots, m. \tag{1.45}$$

□

Conditions (1.35) to (1.37) of Theorem (1.2.8) become conditions (1.27) to (1.29) of Theorem (1.2.7), if λ_0^* is positive. Conversely, the conditions of Theorem (1.2.7) trivially imply those of Theorem (1.2.8) with $\lambda_0^* = 1$. The lemma below with proof found in [15] characterizes $S(A, x)$.

1.2.9 Lemma. *A vector z is contained in $S(A, x)$ if and only if there exists a sequence of vectors $\{x^k\} \subset A$ converging to x and a sequence of nonnegative numbers $\{\alpha^k\}$ such that the sequence $\{\alpha^k(x^k - x)\}$ converges to z.*

1.2.10 Lemma. *Suppose that $x^0 \in X$. The set $Z^1(x^0) \cap Z^2(x^0)$ is empty if and only if*

$$\nabla f(x^0) \in (Z^1(x^0))^T.$$

Proof. The set $Z^1(x^0) \cap Z^2(x^0)$ is empty if and only if for every $z \in Z^1(x^0)$ we have $z^T \nabla f(x^0) \geq 0$. It follows, then that $\nabla f(x^0)$ is contained in the positively normal cone of $Z^1(x^0)$. □

1.2.11 Lemma. *Suppose that x^0 is a solution of problem $(P2)$. Then*

$$\nabla f(x^0) \in (S(X, x^0))^T.$$

Proof. We must show that $z^T \nabla f(x^0) \geq 0$ for every $z \in S(X, x^0)$. Suppose that $z \in S(X, x^0)$. Then, by Lemma 1.2.9, there exist a sequence $\{x^k\} \in X$, converging to x^0 and a sequence of nonnegative numbers $\{\alpha\}$ such that $\{\alpha^k(x^k - x^0)\}$ converges to z. If f is differentiable at x^0, we can write

$$f(x^k) = f(x^0) + (x^k - x^0)^T \nabla f(x^0) + \epsilon ||x^k - x^0||,$$

where ϵ is a function that tends to zero as $k \to \infty$. Hence

$$\alpha^k[f(x^k) - f(x^0)] = (\alpha^k(x^k - x^0))^T \nabla f(x^0) + \epsilon ||x^k - x^0||.$$

Since $x^k \in X$ and x^0 is a local minimum, it follows that by letting $k \to \infty$ the term $\epsilon ||x^k - x^0|| \to 0$, and the expression $\alpha^k[f(x^k) - f(x^0)]$ converges to a nonnegative limit. Thus

$$\lim_{k \to \infty} (\alpha^k(x^k - x^0))^T \nabla f(x^0) = z^T \nabla f(x^0) \geq 0$$

where $\nabla f(x^0) \in (S(X, x^0))'$. □

Now we can state and prove the main result of this subsection, a set of necessary conditions, stronger than those presented in Theorem (1.2.4). The conditions stated below can be viewed as a direct extension of the Kuhn-Tucker necessary conditions [86] for optimality.

1.2.12 Theorem. *(Generalized Kuhn-Tucker Necessary Conditions)*
Let x^* be a solution of problem $(P2)$ and suppose that

$$(Z^1(x^*))' = (S(X, x^*))'.$$

Then there exist vectors $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)^T$, $\lambda^* \geq 0$ and $\mu^* = (\mu_1^*, \ldots, \mu_p^*)^T$ such that

$$\nabla f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) - \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*) = 0 \tag{1.46}$$

$$\lambda_i^* g_i(x^*) = 0; \quad i = 1, \ldots, m. \tag{1.47}$$

Proof. Suppose that x^* is a solution of $(P2)$. By Lemma (1.2.11), $\nabla f(x^*) \in (S(X, x^*))'$. If $Z^1(x^*) = (S(X, x^*))'$, then

$$\nabla f(x^*) \in (Z^1(x^*))'.$$

By Lemma (1.2.10), the set $Z^1(x^*) \cap Z^2(x^*)$ is empty; and by Theorem (1.2.7), conditions (1.46) and (1.47) hold. □

1.2.4 Second-order optimality conditions

In this subsection optimality conditions for problem $(P2)$ that involve second derivatives are discussed.

In the following discussion we assume that the functions $f,\ g_1, \ldots, g_m, h_1 \ldots, h_p$ appearing in problem $(P2)$ are twice continuously differentiable. A second-order constraint qualification will be stated first. Let $x \in X$ and define

$$\hat{Z}^1(x) = \{z \ : \ z^T \nabla g_i(x) = 0;\ i \in I(x),\ z^T \nabla h_j(x) = 0; \quad j = 1, \ldots, p\}.$$

1.2.4 Definition. The second-order constraint qualification is said to hold at $x^0 \in X$ if every nonzero $z \in \hat{Z}^1(x^0)$ is tangent to a twice differentiable arc contained in the boundary of X; that is, for each $z \in \hat{Z}^1(x^0)$ there exist a twice differentiable function α defined on $[0, \epsilon]$ with range in $\mathbb{R}^n$ such that $\alpha(0) = x^0$,

$$g_i(\alpha(\theta)) = 0;\ i \in I(x^0),\ h_j(\alpha(\theta)) = 0;\ j = 1, \ldots, p \tag{1.48}$$

for $0 \leq \theta \leq \epsilon$ and

$$\frac{d\alpha(0)}{d\theta} = \lambda z$$

for some positive number λ.

We have then the second-order conditions [15].

1.2.13 Theorem. *Let x^* be a solution of problem $(P2)$ and suppose that there exist vectors $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*)$, $\lambda^* \geq 0$, $\mu^* = (\mu_1^*, \ldots, \mu_m^*)$ satisfying* (1.46) *and* (1.47)*. Further suppose that the second-order constraint qualification holds at x^*. Then for $z \neq 0$ such that $z \in \hat{Z}^1(x^*)$, we have*

$$z^T[\nabla^2 f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla^2 g_i(x^*) - \sum_{j=1}^{p} \mu_j^* \nabla^2 h_j(x^*)]z \geq 0. \tag{1.49}$$

Proof. Let $z \neq 0$, and $z \in \hat{Z}^1(x^*)$, and let $\alpha(\theta)$ be the vector-valued function assumed in the second-order constraint qualification; that is, $\alpha(0) = x^*,\ d\alpha(0)/d\theta = z$ (since $\hat{Z}^1(x^*)$ is a cone, we assumed, without loss of generality, that $\lambda = 1$). Let $d^2\alpha(0)/d\theta^2 = w$. From

(1.48) and the chain rule it follows that

$$\frac{d^2 g_i(\alpha(0))}{d\theta^2} = z^T \nabla^2 g_i(x^*) z + w^T \nabla g_i(x^*) = 0; \quad i \in I(x^*) \tag{1.50}$$

$$\frac{d^2 h_j(\alpha(0))}{d\theta^2} = z^T \nabla^2 h_j(x^*) z + w^T \nabla h_j(x^*) = 0; \quad j = 1, \ldots, p. \tag{1.51}$$

From (1.46), (1.47) and the definition of $\hat{Z}^1(x^*)$, we have

$$\frac{df(\alpha(0))}{d\theta} = z^T \nabla f(x^*) = z^T [\sum_{i=1}^{m} \lambda_i^* \nabla g_i(x^*) + \sum_{j=1}^{p} \mu_j^* \nabla h_j(x^*)] = 0. \tag{1.52}$$

Since x^* is a local minimum and

$$\frac{df(\alpha(0))}{d\theta} = 0$$

it follows that

$$\frac{d^2 h_j(\alpha(0))}{d\theta^2} \geq 0$$

that is,

$$\frac{d^2 f(\alpha(0))}{d\theta^2} = z^T \nabla^2 f(x^*) z + w^T \nabla f(x^*) \geq 0; \quad i \in I(x^*). \tag{1.53}$$

Multiplying (1.50) and (1.51) by the corresponding multipliers, subtracting from (1.53) and by (1.46) and (1.47), we obtain

$$z^T [\nabla^2 f(x^*) - \sum_{i=1}^{m} \lambda_i^* \nabla^2 g_i(x^*) - \sum_{j=1}^{p} \mu_j^* \nabla^2 h_j(x^*)] z \geq 0. \tag{1.54}$$

□

1.2.5 Finding optimal solutions of Mathematical Programming problems

Mathematical Programming problems can be solved by using a variety of methods such as penalty- and barrier- function methods, gradient projection methods and Sequel Quadratic Programming (SQP) algorithms [11].

Among these methods, SQP algorithms have proved highly effective for solving general constrained problems with smooth objective and constraint functions [58]. A more recent development in constrained optimization is the extension of the modern interior-point approaches to the general class of Mathematical Programs.

In this thesis we discuss the algorithm of SQP method for a Mathematical Program with equality constraints, adapted from [11].

Consider the optimization problem

$$(P2)''' \qquad \begin{cases} \min \ f(x) \\ x \in D \end{cases}$$

where

$$D = \{x \in \mathbb{R}^n \mid h_j(x) = 0; \ j = 1, \ldots, p; \ x \geq 0\}$$

and $f(x), \ h_j(x); \ j = 1, \ldots, p$ are continuous functions which have continuous second partial derivatives. We assume that the feasible region D is non empty and $p \leq n$.

From Subsection (1.2.2), we know that the first-order necessary conditions for x^* to be a local minimizer of the problem $(P2)'''$ is that there exists a $\lambda^* \in \mathbb{R}^p$, $\lambda^* \geq 0$ such that:

$$\nabla L_{(P2)'''}(x^*, \lambda^*) = 0$$

where $L_{(P2)'''}(x, \lambda)$ is the Lagrangian of $(P2)'''$ defined by

$$L_{(P2)'''}(x, \lambda) = f(x) - \sum_{j=1}^{p} \lambda_j h_j(x).$$

If $\{x_k, \lambda_k\}$ is the k^{th} iterate, which is assumed to be $\{x^*, \lambda^*\}$ i.e., $x_k \simeq x^*$ and $\lambda_k \simeq \lambda^*$, we need to find an increment $\{\delta_x, \delta_\lambda\}$ such that the next iterate

$$\{x_{k+1}, \lambda_{k+1}\} = \{x_k + \delta_x, \lambda_k + \delta_{\lambda\}}$$

is closer to $\{x^*, \lambda^*\}$.

If we approximate $\nabla L_{(P2)'''}(x_{k+1}, \lambda_{k+1})$ by using the first two terms of the Taylor series of $\nabla L_{(P2)'''}$ for $\{x_k, \lambda_k\}$, i.e.,

$$\nabla L_{(P2)'''lem:}(x_{k+1}, \lambda_{k+1}) \simeq \nabla L_{(P2)'''}(x_k, \lambda_k) + \nabla^2 L_{(P2)'''}(x_k, \lambda_k) \left(\delta_x, \delta_\lambda\right)^T$$

then $\{x_{k+1}, \lambda_{k+1}\}$ is an approximation of $\{x^*, \lambda^*\}$ if the increment $\{\delta_x, \delta_\lambda\}$ satisfies the equality

$$\nabla^2 L_{(P2)'''}(x_k, \lambda_k) \left(\delta_k \delta_\lambda\right)^T = -\nabla L_{(P2)''}(x_k, \lambda_k). \tag{1.55}$$

More specifically we can write (1.55) as follows [11]:

$$\begin{pmatrix} W_k & -A_k^T \\ -A_k & 0 \end{pmatrix} \begin{pmatrix} \delta_x \\ \delta_\lambda \end{pmatrix} = \begin{pmatrix} A_k^T \lambda_k - g_k \\ a_k \end{pmatrix} \tag{SQ1}$$

where

$$W_k = \nabla_x^2 f(x_k) - \sum_{j=1}^{p} (\lambda_k)_j \nabla_x^2 a_j(x_k)$$

$$A_k = \begin{pmatrix} \nabla_x^T a_1(x_k) \\ \vdots \\ \nabla_x^T a_p(x_k) \end{pmatrix}$$

$$g_k(x) = \nabla_x f(x_k)$$

$$\big(a_k =)a_1, \ldots, a_p(x_k)\big)^T.$$

If W_k is positive definite and A_k has a full rank, then the matrix at the left-hand side of relation $SQ1$ is nonsingular and symmetric and the system of relation $SQ1$ can be solved efficiently for $\{\delta_x, \delta_\lambda\}$.

Relation in ($SQ1$) can also be written as

$$W_k\delta_x + g_k = A_k^T \lambda_{k+1} A_k \delta_x = -a_k$$

and these relations may be interpreted as the first-order necessary conditions for δ_x to be a local minimizer of the QP problem

$$(P2)^{iv} \qquad \begin{cases} \min \ \left(\frac{1}{2}\delta^T W_k \delta + \delta^T g_k\right) \\ A_k\delta = -a_k \end{cases}$$

If W_k is positive definite and A_k has full rank, the minimizer of the problem $(P2)^{iv}$ can be found. Once the minimizer, δ_x, is found, the next iterate is set to $x_{k+1} = x_k + \delta_x$ and the Lagrangian multiplier vector λ_{k+1} is determined as

$$\lambda_{k+1} = (A_k A_k^T)^{-1} A_k (W_k \delta_x + g_k) \tag{1.56}$$

by using $(P2)^{iv}$.

With x_{k+1} and λ_{k+1} known, W_{k+1}, g_{k+1}, A_{k+1} and a_{k+1} can be evaluated. The iterations should continue until $||\delta_x||$ is sufficiently small to terminate the algorithm.

We can see that the entire solution procedure consists of solving a series of QP sub-problems in a sequential manner, and as a consequence, the method is referred to as the *Sequencial Quadratic Programming* (SQP) *method*

A stepwise description of the method is as follows:

1.2.1 Algorithm.

Step 1. Set $\{x, \lambda\} = \{x_0, \lambda_0\}$, $k = 0$ *and initialize the tolerance* ϵ.

Step 2. Evaluate W_k, A_k, g_k *and* a_k.

Step 3. Solve the QP problem $(P2)^{iv}$ *for* δ_x *and compute the Lagrangian multiplier* λ_{k+1} *using relation* (1.56).

Step 4. Set $x_{k+1} = x_k + \delta_x$.
If $||\delta_x|| \leq \epsilon$, *print* $x^* = x_{k+1}$ *and stop;*
otherwise, set $k = k + 1$ *and repeat from step 2.*

The following results by [58] show that the SQP algorithms have good local properties.
1.2.14 Lemma. *For* $\{x_0, \lambda_0\}$ *the QP problem has a unique solution* δ_x, *and Algorithm* (1.2.1) *converges quadratically.*

The main disadvantage of Algorithm (1.2.1)*yusinglevel* is that they may fail to converge when the initial point is not sufficiently close to x^* [11].

1.3 Multiobjective Programming

1.3.1 Problem formulation

A Multiobjective Program is a problem of the form:

$$(P3) \quad \min_{x \in D} \left(f^1(x), \ldots, f^K(x)\right)$$

where

$$D = \{x \in \mathbb{R}^n \mid g_i(x) \leq 0;\ i = 1, \ldots, m\}$$

is the feasible set, $K \geq 2, f^k(x);\ k = 1, \ldots, K$ and $g_i(x);\ i = 1, \ldots, m$ are real-valued functions of $\mathbb{R}^n$.

The feasible criterion space Z is defined as

$$Z = \{f(x) \ : \ x \in D\}.$$

The image of D under

$$f = (f^1 \ldots, f^K)$$

is denoted by

$$Z_D = f(D) = \{z \in \mathbb{R}^K \ : \ z = f(x) \text{ for some } x \in D\}$$

is referred to as the **feasible set** in criterion space.

A Multiobjective Programming (MOP) problem may be considered as the selection of a *best compromise criterion* vector z from Z_D, which may then be stated as:

$$(P3)' \quad \min_{z \in Z_D} z = \left(z^1(x), \ldots z^K(x)\right)$$

where

$$Z_D = f(D) = \{z \in \mathbb{R}^K \ : \ z = f(x) \text{ for some } x \in D\}$$

and $z^j;\ \ k = 1, \ldots, K$ are components of criterion vector z.
For illustration purposes, let us have a look at the following example from [50].
1.3.1 Example. *Consider the mathematical program*

$$\min_{x \geq 0} \ (f^1(x), f^2(x)),$$

where $f^1(x) = \sqrt{x+1}$ *and* $f^2(x) = x^2 - 4x + 5$.

The feasible set is

$$D = \{x \in \mathbb{R} \ : \ x \geq 0\}$$

and the objective functions are

$$f^1(x) = \sqrt{x+1} \text{ and } f^2(x) = x^2 - 4x + 5.$$

The decision space is $\mathbb{R}$ *because* $D \subset \mathbb{R}$. *The criterion space is* $\mathbb{R}^2$, *to obtain the image of the feasible set in criterion space we substitute* z_1 *for* $f^1(x)$ *and* z_2 *for* $f^2(x)$ *to get* $x = (z_1)^2 - 1$ *(solving* $\sqrt{1+x}$ *for* x*). Therefore, we obtain* $z_2 = ((z_1)^2-1)^2+4-4(z_1)^2+5 = (z_1)^4 - 6(z_1)^2 + 10$. *Note that* $x \geq 0$ *translate to* $z_1 \geq 1$ *so that* $Z_D = f(D)$.

1.3.2 Efficient solution and nondominated point

1.3.1 Definition. Consider $(P3)$, a point $x^1 \in D$ is dominated by another point $x^2 \in D$ if and only if

$$f^k(x^2) \geq f^k(x^1); \; k = 1, \ldots, K$$

and $f^l(x) > f^{*l}(x)$ for at least one l.

We are now in a position to introduce the notion of efficiency for a Multiobjective Programming problem.
1.3.2 Definition. Consider $(P3)'$, a criterion vector $z^* \in Z$ is nondominated for $(P3)'$, if the decision vector corresponding to it is efficient for $(P3)$.

In other words, a nondominated criterion vector $z^* \in Z_D$ is such that any other points in Z_D, which increase the value of one criterion also decrease the value of at least one other criterion.
Intuitively, when we have a set of solutions such that we cannot improve any objective further without at the same time worsening another then we have the *Pareto-optimal set* or *Pareto front* (of objective vectors). In such a case all the other lesser solutions are said to be *dominated* by these better ones and we can discard them for decision making purposes.
1.3.3 Definition. The set of all efficient solutions $x^* \in D$ is denoted by D_E. The set of all nondominated points $z^* = f(x^*) \in Z_D$, where $x^* \in D_E$ is denoted by Z_N.

In the following definitions, we present variants of the notion of efficiency.
1.3.4 Definition. Consider $(P3)$, $x^* \in D$ is a local efficient solution for $(P3)$, if there exists $\delta > 0$ such that x^* is efficient for $(P3)$ in $D \cap B(x^*, \delta)$.
1.3.5 Definition. Consider $(P3)'$, a criterion vector $z^* \in Z$ is locally nondominated for $(P3)'$, if the decision vector corresponding to it is locally efficient for $(P3)$.

Often, multiobjective programming algorithms provide solutions that may not be efficient but may satisfy other criteria that make them significant for practical applications. As an example, we present the notion of weak efficiency.
1.3.6 Definition. A solution $x^* \in D$ is called weakly efficient solution for $(P3)$, if there

is no $x \in D$ such that $f^k(x) < f^k(x^*)$ for all $k = 1, \dots, K$.
The point $z^* = f(x^*)$ is then called weakly non-dominated for $(P3)'$.
1.3.7 Definition. A solution $x^* \in D$ is called strictly efficient for $(P3)$ if there is no $x \in D,\ x \neq x^*$ such that $f^k(x) \leq f^k(x^*)$ for all $k = 1, \dots, K$.
The point $z^* = f(x^*)$ is then called strictly non-dominated for $(P3)'$.

The weakly(strictly) efficient and nondominated sets are denoted by $D_{WE}(D_{SE})$ and Z_{WE} respectively. Efficient solutions are weakly efficient, but weakly efficient solutions are not necessarily efficient [137].

For characterization, we introduce level sets and level curves of functions.
1.3.8 Definition. Let $D \subset \mathbb{R}^n;\quad f\ :\ D \to \mathbb{R}$, and $x^* \in D$

$$\mathcal{L}_{\leq}(f(x^*)) = \{x \in D\ :\ f(x) \leq f(x^*)\} \tag{1.57}$$

is called the level set of f at x^*.

$$\mathcal{L}_{=}(f(x^*)) = \{x \in D\ :\ f(x) = f(x^*)\} \tag{1.58}$$

is called the level curve of f at x^*.

$$\begin{aligned} \mathcal{L}_{<}(f(x^*)) &= \mathcal{L}_{\leq}(f(x^*)) \backslash \mathcal{L}_{=}(f(x^*)) \\ &= \{x \in D\ :\ f(x) < f(x^*)\} \end{aligned} \tag{1.59}$$

is called the strict level set of f at x^*.

Obviously $\mathcal{L}_{=}(f(x^*)) \subset \mathcal{L}_{\leq}(f(x^*))$. We can now formulate the characterization of (strict, weak) efficiency using level sets, due to Ehrgott [50].
1.3.1 Theorem. *Let $x^* \in D$ be a feasible solution and define $z_k^* = f^k(x^*);\ k = 1, \dots, K$, then*

1. *x^* is strictly efficient if and only if*

$$\bigcap_{k=1}^{K} \mathcal{L}_{\leq}(z^*) = \{x^*\}.$$

2. *x^* is efficient if and only if*

$$\bigcap_{k=1}^{K} \mathcal{L}_{\leq}(z^*) = \bigcap_{k=1}^{K} \mathcal{L}_{=}(z^*).$$

3. x^* is weakly efficient if and only if

$$\bigcap_{k=1}^{K} \mathcal{L}_{<}(z^*) = \emptyset.$$

Proof. 1. x^* is strictly efficient

$\Leftrightarrow$ there is no $x \in D$, $x \neq x^*$ such that $f(x) \leq f(x^*)$

$\Leftrightarrow$ there is no $x \in D$, $x \neq x^*$ such that $f^k(x) \leq f^k(x^*)$ for all $k = 1, \dots, K$

$\Leftrightarrow$ there is no $x \in D$, $x \neq x^*$ such that $x \in \cap_{k=1}^{K} \mathcal{L}_{\leq}(z_k^*)$

$\Leftrightarrow \cap_{k=1}^{K} \mathcal{L}_{\leq}(z_k^*) = \{x^*\}$.

2. x^* is efficient

$\Leftrightarrow$ there is no $x \in D$, such that both $f^k(x) \leq f^k x^*)$ for all $k = 1, \dots, K$ and $f^l(x) \leq f^l(x^*)$ for some l

$\Leftrightarrow$ there is no $x \in D$, such that both $x \in \cap_{k=1}^{K} \mathcal{L}_{\leq}(z_k^*)$ and $x \in \mathcal{L}_{<}(z_l^*)$ for some l

$\Leftrightarrow \cap_{k=1}^{K} \mathcal{L}_{\leq}(z_k^*) = \cap_{k=1}^{K} \mathcal{L}_{=}(z_k^*)$.

3. x^* is weakly efficient

$\Leftrightarrow$ there is no $x \in D$, such that $f^k(x) < f^k(x^*)$ for all $k = 1, \dots, K$

$\Leftrightarrow$ there is no $x \in D$, such that $x \in \cap_{k=1}^{K} \mathcal{L}_{<}(z_k^*)$

$\Leftrightarrow \cap_{k=1}^{K} \mathcal{L}_{\leq}(z_k^*) = \emptyset$.

$\square$

In order to eliminate undesirable nondominated solutions with unbounded trade-offs between the various objectives, the concept of proper efficiency and proper nondominance have been introduced by Geoffrion [61] below.

1.3.3 Proper efficiency and proper nondominance

1.3.9 Definition. $z^* \in Z_D$ is *properly non-dominated* if and only if it is non dominated and there exist a scalar $M > 0$ such that for each $k = \{1, \dots, K\}$ and each $z \in Z_D$ with $z^k > z^{*k}$, there exists at least one $j \in \{1, \dots, K\}$ with $z^{*j} > z^j$ and $\left(\frac{z^k - z^{*k}}{z^{*j} - z^j}\right) \leq M$.

If this ratio is not bounded from above, an extremely large improvement in the numerator would result from an extremely small decrement in the denominator. Thus, one would always choose a solution which is properly efficient.
Geoffrion [61] suggested that the analyst unsure about the proper efficiency of a solution should test numerous values of the ratio about the selected point.

To characterize proper efficiency in Geoffrion's sense consider the following weighted sum optimization problem and the subsequent propositions and theorem which can be found in [50].

1.3.10 Definition. Let $\{\lambda_k\};\ 1 \leq k \leq K$ be such that $\lambda_k \geq 0; \forall\ k$ and $\sum_{k=1}^{K} \lambda_k = 1$. We call the single objective optimization problem

$$(P3)'' \quad \min_{x\in D} \sum_{k=1}^{K} \lambda_k f^k(x)$$

a weighted sum optimization problem of the MOP $(P3)$.

1.3.2 Proposition. *Suppose that x^* is an optimal solution of the weighted sum optimization problem $(P3)''$, then the following statement holds:*

1. $x^* \in D_{WE}$
2. $x^* \in D_E$
3. *if x^* is a unique optimal solution of $(P3)''$ then $x^* \in D_{SE}$.*

1.3.3 Proposition. *Let D be a convex set, and let $f^k;\ k = 1,\dots,K$ be convex functions. If $x^* \in D_{WE}$, then x^* is an optimal solution of $(P3)''$.*

1.3.4 Theorem. *Let $D \subset \mathbb{R}^n$ be convex and assume $f^k : D \to \mathbb{R}$ are convex for $k = 1,\dots,K$. Then $x \in D$ is properly efficient if and only if x^* is an optimal solution of $(P3)''$, with strict positive weights $\lambda_k;\ k = 1,\dots,K$.*

Let the properly efficient set be denoted by D_{pE}.

Other definitions of proper efficiency have been proposed by Borwen [28], Benson [22] and Kuhn-Tucker [86].
We give Kuhn-Tucker definition of proper efficiency in the sequel.

1.3.11 Definition. (Kuhn-Tucker (KT) proper efficiency [86])
A solution $x^* \in D$ is called properly efficient (in Kuhn-Tucker's sense) if it is efficient and if there is no $d \in \mathbb{R}^n$ satisfying

$$\begin{aligned}
&\nabla f^k(x^*)^T d \le 0 \ for\ all\ k = 1,\dots,K,\\
&\nabla f^j(x^*)^T d < 0 \ for\ some\ j \in \{1,\dots,K\},\\
&\nabla g_i(x^*)^T d \le 0;\ \forall i \in \mathcal{I}(x^*) = \{i = 1,\dots,m,\ g_i(x^*) = 0\}.
\end{aligned}$$

The set $\mathcal{I}(x^*)$ is called the set of active indices.

1.3.12 Definition. (Kuhn-Tucker (KT) constraint qualification [86])
A differentiable MOP $(P3)$ satisfies the KT constraint qualification at $x^* \in D$ if for any $d \in \mathbb{R}^n$ with $\nabla f^k(x^*)^T d \le 0, \forall\ j \in \mathcal{I}(x^*)$, there is a real number $t > 0$, a function $\theta : [0,t] \to \mathbb{R}^n$, and $\alpha > 0$ such that $\theta(0) = x^*, g(\theta(\bar{t})) \le 0$ for all $\bar{t} \in [0,t]$ and $\theta'(0) = \alpha d$.

This constraint qualification means that, the feasible set D has a local description as a differentiable curve (at x^*): Every feasible direction d can be written as the gradient of a feasible curve starting at x^*.

1.3.5 Theorem. *(Geoffrion [61])*
If a differentiable MOP $(P3)$ satisfies the KT constraint qualification at x^* and x^* is

properly efficient in Geoffrion sense, then it is properly efficient in Kuhn-Tucker sense.

Proof. Suppose x^* is efficient, but not properly efficient according to Definition (1.3.11). Then there is some $d \in \mathbb{R}^n$ such that

$$\begin{aligned}
&\nabla f^1(x^*)^T d < 0 \\
&\nabla f^k(x^*)^T d \leq 0 \ for \ all \ k = 2, \ldots, K, \\
&\nabla g_i(x^*)^T d \leq 0, \ \forall i \in \mathcal{I}(x^*)\}.
\end{aligned}$$

Using the function θ from the constraint qualification, we take a sequence $\bar{t_k} \to 0$, and if necessary a subsequence such that

$$Q = \{l \ : \ f^l(\theta(\bar{t_k})) > f^l(x^*)\}$$

is the same for all k. Since for $l \in Q$ by Taylor expansion of f^l at $\theta(t_k)$

$$f^l(\theta(t_k)) - f^l(x^*) = t_k \left\langle \nabla f^l(x^*), \alpha d \right\rangle + 0(t_k) > 0$$

and $\left\langle \nabla f^l(x^*), d \right\rangle \leq 0$ it must be that

$$\left\langle \nabla f^l(x^*), \alpha d \right\rangle = 0 \ \ \forall \ l \in Q.$$

But since $\langle \nabla f^1(x^*), d \rangle < 0$ the latter implies

$$\Rightarrow \frac{f^1(x^*) - f^1(\theta(t_k))}{f^j(\theta(t_k)) - f^j(x^*)} = \frac{-\langle \nabla f^1(x^*), \alpha d \rangle + \frac{0(t_k)}{t_k}}{\langle \nabla(x^*), \alpha d \rangle + \frac{0(t_k)}{t_k}} \to \infty$$

whenever $i \in Q$. Hence x^* is not properly efficient according to Geoffrion's definition. □

The converse of Theorem (1.3.5) holds without the constraint qualification [50].

1.3.4 Ideal point and compromise function

One way of obtaining initial information about a multiobjective decision problem is by finding *the ideal point* [145] also called utopia point" [143].
1.3.13 Definition. A point $z_* = (z_{*1}, \ldots, z_{*K})$ where

$$z_{*k} = \min_{x \in D} f^k(x) = \min_{z \in Z_D} \ z_k$$

is called the ideal point of the multiobjective problem $(P3)$.

One of the most frequently used measures of distance in decision making is the family of

L_p metrics, which can be described by the following relation:

$$L_p = \left\{ \sum_{k=1}^{K} \left(f^k(x) - z_{*k} \right)^p \right\}^{1/p} \tag{1.60}$$

where $p \in [1, \infty]$. The ideal point and the concept of compromise function are the basic idea of compromise programming [50], [117].

1.3.5 Efficiency conditions for Multiobjective Programming problems

In this section we prove necessary and sufficient conditions for weak and proper efficiency of the solution of MOP problem $(P3)$. The results follow along the lines of Karush-Kuhn-Tucker optimality conditions, known from single objective Mathematical Programming discussed in subsection (1.2.3).

We state the following theorem of the alternatives that can be found in [98].

1.3.6 Theorem. *(Motzkin's theorem of the alternative [98])*
Let B, C, D be $k \times n$, $l \times n$ and $o \times n$ matrices, respectively. Then either

$$Bx < 0, \ Cx \leq 0, \ Dx = 0$$

has a solution $x \in \mathbb{R}^n$ or

$$B^T y^1 + C^T y^2 + D^T y^3 = 0, \ y^1 \geq 0, \ y^2 \geq 0$$

has a solution $y^1 \in \mathbb{R}^k$, $y^2 \in \mathbb{R}^l$, $y^3 \in \mathbb{R}^o$, but never both.

We start with conditions for weak efficiency of Multiobjective Program $(P3)$.

1.3.7 Theorem. *(Ehrgott [50])*
The Lagrangian of $(P3)$ is defined as follows:

$$L_{B1}(x, \lambda, \mu) = \sum_{k=1}^{K} \lambda_k \nabla f^k(x) + \sum_{i=1}^{m} \mu_i \nabla g_i(x)$$

where $(x, \lambda, \mu) \in \mathbb{R}^n \times \mathbb{R}^K \times \mathbb{R}^m$ and $(\lambda, \mu) \neq (0, 0)$. Suppose that the Kuhn-Tucker constraint qualification given in Definition (1.3.11) is satisfied at $x^* \in D$.

If x^* is a weakly efficient solution of $(P3)$ there exist $(x^*, \lambda^*, \ \mu^*)$ that satisfy the Fritz

John system

$$\left.\begin{array}{c}\nabla_x L_{B1}(x,\lambda,\mu)=0\\ \nabla_\mu L_{B1}(x,\lambda,\mu)\le 0\\ \mu\nabla_\mu L_{B1}(x,\lambda,\mu)=0\\ \lambda\ge 0.\end{array}\right\}\quad KT1$$

Proof. Let $x^* \in D_{WE}$. We first show that there can be no $d \in \mathbb{R}^n$ such that

$$\nabla f^k(x^*)^T d < 0, \quad \forall\, k \tag{1.61}$$

$$\nabla g_i(x^*)^T d < 0, \quad \forall\, i \in \mathcal{I}(x^*) = \{i\ : g_i(x^*) = 0\}. \tag{1.62}$$

We then apply Motzkin's Theorem of the alternative (1.3.6) to obtain the multipliers $\lambda_k^*;\ k = 1, \dots, K$ and $\mu_i^*;\ i = 1, \dots, m$.

Suppose that such a $d \in \mathbb{R}^n$ exists. From the KT constraint qualification there is a continuously differentiable function $\theta\ :\ [0, \bar{t}] \to \mathbb{R}^n$ such that $\theta(0) = x^*,\ g(\theta(t)) \le 0$ for all $t \in [0, \bar{t}]$ and $\theta'(0) = \alpha d$ with $\alpha > 0$. Thus

$$f^k(\theta(t)) = f^k(x^*) + t\left\langle \nabla f^k(x^*), \alpha d\right\rangle + 0(t) \tag{1.63}$$

and using

$$\left\langle \nabla f^k(x^*), d\right\rangle < 0$$

it follows that $f^k(\theta(t)) < f^k(x^*), \quad k = 1, \dots, K$ for t sufficiently small, which contradicts $x^* \in D_{WE}$.

It remains to show that (1.62) and (1.63) imply conditions of $(KT1)$. This is achieved by using matrices

$$B = (\nabla f^k(x^*))_{k=1,\dots,K},\ C = (\nabla g_i(x^*))_{i\in\mathcal{I}(x^*)},\ D = 0\ \mathit{with}\ l = |\mathcal{I}(x^*)|$$

in Theorem (1.3.6). Then since (1.61) and (1.62) have no solution $d \in \mathbb{R}^n$, according to Theorem (1.3.6) there must be $y^1 = \lambda^*,\ y^2 = \mu^*$ and y^3 such that

$$B^T y^1 + C^T y^2 = 0,\ \mathit{with}\ y^1 \ge 0\ \mathit{and}\ y^2 \ge 0$$

.i.e.

$$\sum_{k=1}^{K} \lambda_k^* \nabla f^k(x^*) + \sum_{i\in\mathcal{I}(x^*)}^{m} \mu_i^* \nabla g_i(x^*) = 0.$$

We complete the proof by setting $\mu_i^* = 0$ for $i \in \{1, \dots, m\}\backslash\mathcal{I}(x^*)$. □

If involved objective functions are convex and the feasible set is convex, then the Karush-Kuhn-Tucker conditions for Multiobjective Programming problems are also sufficient.

1.3.8 Theorem. *(Miettinen [103])*

Assume $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^K$, $\mu \in \mathbb{R}^m$ are nonnegative. Let $x^* \in \mathbb{R}^n$ and suppose $f^1(x), \ldots, f^K(x)$, $g_1(x), \ldots, g_m(x)$ are continuously differentiable at x^* and convex. If $(x^*,\ \lambda^*,\ \mu^*)$ satisfies system $KT1$ then x^* is weakly efficient for $(P3)$.

It is well known in Mathematical Programming that global optimality implies local optimality. So it is in Multiobjective Programming: global efficiency solution implies local efficiency. The converse is valid only for convex Multiobjective Programming problems, see e.g. [40].

It is also well known that if all the objective functions are quasi-convex on a convex feasible region with at least one being strictly quasi-convex, then every locally efficient solution for $(P3)$ is also globally efficient. Interested readers may consult [103] for more details.

Next we prove similar conditions for properly efficient solution of $(P3)$ in Kuhn-Tucker's sense and in Geoffrion's sense. We consider the following Tucker's theorem of the alternative, which we use in the sequel.

1.3.9 Theorem. *(Tucker's theorem of the alternative)*

Let B, C, and D be $k \times n$, $l \times n$ and $o \times n$ matrices. Then either

$$Bx \leq 0,\ Cx \leq 0,\ Dx = 0$$

has a solution $x \in \mathbb{R}^n$ or

$$B^T y^1 + C^T y^2 + D^T y^3 = 0,\ y^1 > 0,\ y^2 \geq 0$$

has a solution $y^1 \in \mathbb{R}^k$, $y^2 \in \mathbb{R}^l$, $y^3 \in \mathbb{R}^o$, but never both.

A proof of (1.3.9) can be found in [98].

Kuhn-Tucker definition of proper efficiency (Definition (1.3.11)) is based on the system of inequalities

$$\nabla f^k(x^*)^T d \leq 0 \ for\ all\ k = 1, \ldots, K, \tag{1.64}$$

$$\nabla f^j(x^*)^T d < 0 \ for\ some\ j \in \{1, \ldots, K\}, \tag{1.65}$$

$$\nabla g_i(x^*)^T d \leq 0,\ \forall i \in \mathcal{I}(x^*) = \{i = 1, \ldots, m :\ g_i(x^*) = 0\} \tag{1.66}$$

having no solution. We apply Tucker's theorem of the alternative, to show that a dual system of inequalities has a solution. This system yields a necessary condition for proper efficiency in Kuhn-Tucker's sense.

1.3.10 Theorem. *If x^* is properly efficient for $(P3)$ in Kuhn-Tucker's sense, there exist $\lambda^* \in \mathbb{R}^n,\ \mu^* \in \mathbb{R}^m$ such that*

$$\left.\begin{array}{c} \nabla_x L_{B1}(x^*, \lambda^*, \mu^*) = 0 \\ \nabla_\mu L_{B1}(x^*, \lambda^*, \mu^*) \leq 0 \\ \mu \nabla_\mu L_{B1}(x^*, \lambda^*, \mu^*) = 0 \\ \lambda > 0, \mu \geq 0. \end{array}\right\} \qquad (KT2)$$

Proof. Because x^* is properly efficient in Kuhn-Tucker's sense, there are no $d \in \mathbb{R}^n$ satisfying conditions $(KT2)$. We apply Theorem (1.3.9) to matrices

$$\begin{aligned} B &= (\nabla f^k(x^*))_{k=1,\ldots,K} \\ C &= (\nabla g_i(x^*))_{i \in \mathcal{I}(x^*)} \\ D &= 0 \end{aligned}$$

with $l \in |\mathcal{I}(x^*)|$. Since (1.64) to (1.66) do not have a solution $d \in \mathbb{R}^n$, we obtain $y^1 = \lambda^*,\ y^2 = \mu^*$ and y^3 with $\lambda_k^* > 0$ for $k = 1, \ldots, K;\ \mu_i^* \geq 0$ for $i \in \mathcal{I}(x^*)$ satisfying

$$\sum_{k=1}^{K} \lambda_k^* \nabla f^k(x^*) + \sum_{i \in \mathcal{I}(x^*)}^{m} \mu_i^* \nabla g_i(x^*) = 0.$$

Letting $\mu_i^* = 0$ for all $i \in \{1, \ldots, m\} \backslash \mathcal{I}(x^*)$, the proof is complete. □

With Theorem (1.3.10) providing necessary conditions for Kuhn-Tucker proper efficiency and Theorem (1.3.5), which shows that Geoffrion's proper efficiency implies Kuhn-Tucker's under the constraint qualification we obtain the following corollary as an immediate consequence. If x^* is properly efficient in Geoffrion's sense and the KT constraint qualification is satisfied at x^*, then there are $\lambda^* \in \mathbb{R}^n$, $\mu^* \in \mathbb{R}^m$ such that $(KT2)$ holds.

For the missing link in the relationships of proper efficiency definitions, we use the single objective Karush-Kuhn-Tucker sufficient conditions of Theorem (1.2.12) and apply them to the weighted sum problem $(P3)''$. We obtain the following result.

1.3.11 Theorem. *Assume that $f^k, g_i : \mathbb{R}^n \to \mathbb{R}$ are convex, continuously differentiable functions for all k and i. Suppose that there are $x^* \in D$, $\lambda^* \in \mathbb{R}^K$ and $\mu^* \in \mathbb{R}^m$ satisfying system $(KT2)$. Then x^* is properly efficient in the sense of Geoffrion.*

Proof. Let

$$f(x) = \sum_{k=1}^{K} \lambda_k^* \nabla f^k(x),$$

be a convex function. By adding convexity assumption to Theorem (1.2.12), we have that

x^* is an optimal solution of $(P3)''$. Since $\lambda_k^* > 0$ for $k = 1, \ldots, K$, Theorem 1.3.4 yields that x^* is properly efficient in the sense of Geoffrion. □

We can derive two corollaries, the first one shows that for convex problems proper efficiency in Kuhn-Tucker's sense implies efficiency in Geoffrion's sense. Let $f^k, g_i : \mathbb{R}^n \to \mathbb{R}$ be convex, continuously differentiable functions for all k, for all i, and suppose x^* is properly efficient in the Kuhn-Tucker sense. Then x^* is properly efficient in Geoffrion's sense.

Proof. The result follows from Theorems (1.3.10) and (1.3.11) □

The second corollary provides sufficient conditions for proper efficiency in Kuhn-Tucker's sense. It follows immediately from Theorems (1.3.11) and (1.3.5). If in addition to the assumptions of Theorem (1.3.11) the KT constraint qualification is satisfied at x^*, the system $(KT2)$ are sufficient for x^* to be properly efficient solution for $(P3)$ in Kuhn-Tucker's sense.

1.3.6 Solving Multiobjective Programming problems

In what follows we briefly discuss a few approaches for solving nonlinear Multiobjective programming problems.
The main approaches are highlighted as follows:

1. **a prior−articulation of DM preference methods** *A priori* knowledge requires a correct articulation of the target concept [39]. Therefore, these methods allow DM to specify preferences for, or relative importance of, objective functions. Examples of these methods include Weighted Sum, Weighted global criterion, Goal programming Method [51].

2. ***posteriori*** **articulation of DM preference methods** These methods generate representative Pareto optimal set and then the DM selects from these solutions based on his preference.

   Examples of these methods include Normal boundary intersection, Normal constraint, Min-max and the Benson method to mention a few [50].

3. **No preference articulation methods** No preference articulation methods are methods that do not require any input from the DM. As an example of no preference articulation method, we may mention multiobjective simplex method.

4. **Progressive articulation of DM preference methods** If any of the methods in the above named categories are programmed to interact with the DM, we obtain a Progressive articulation method. Only part of the efficient solutions has to be generated and evaluated, and the DM can specify and correct his references and selections as the solution process continues Examples of these methods may be

found in [103]. They include interactive weighted Tchebycheff, interactive surrogate worth trade-off, Geoffrion-Dyer-Feinberg, sequencial proxy optimization and STEP methods.

We consider in more details the methods used explicitly in this thesis.

1.3.7 Weighted sum approach

The weighted sum approach consists of aggregating the different objectives into one and then solving the resulting single objective optimization problem. For instance, consider the case where the aggregation function is given by

$$V(x) = \sum_{k=1}^{K} \lambda_k f^k(x) \tag{1.67}$$

where $\lambda_k \geq 0, \sum_{k=1}^{K} \lambda_k = 1$. The resulting single objective is in this case is

$$(P3)'' \quad \min_{x \in D} \sum_{k=1}^{K} \lambda_k f^k(x)$$

We have seen that the method enables computation of the properly efficient and weakly efficient solutions for convex problems by varying $\lambda_k;\ k = 1, \ldots, K$. The following results are important in this regard.

1.3.12 Theorem.

Let $x^* \in D$ be an optimal solution of $(P3)''$. The following statements hold

1. - *if $\lambda > 0$, then $x^* \in D_{pE}$*
   - *if $\lambda \geq 0$, then $x^* \in D_{WE}$*
   - *if $\lambda \geq 0$ and x^* is a unique optimal solution of $(P3)''$ then $x^* \in D_{sE}$.*

2. *Let D be convex set and $f^k,\ k = 1, \ldots, K$ be convex functions. Then the following statements hold*

   - *If $x^* \in D_{pE}$, then there are $\lambda_k > 0$ for all k such that x^* is an optimal solution of $(P3)''$,*
   - *If $x^* \in D_{WE}$, then there are $\lambda_k \geq 0$ for all k such that x^* is an optimal solution of $(P3)''$,*

that if $\lambda_k > 0,\ \forall k$ in $(P3)''$ then the solution obtained is a proper efficient solution for $(P3)$.

1.3.8 Compromise programming

The basic idea of the method of compromise programming is to minimize the distance between a feasible point in the objective space and the ideal point [117].

This may be done using L_p metrics. The resulting single objective program is, in this case,

$$(P3)''' \quad \min_{x\in D} \left\{ \sum_{k=1}^{K} \lambda_k \left(f^k(x) - z_{*k}\right)^p \right\}^{\frac{1}{p}}$$

for $p < \infty$, and

$$(P3)^{iv} \quad \min_{x\in D} \max_{k=1,\ldots,K} \lambda_k \left(f^k(x) - z_{*k}\right)$$

for $p = \infty$, where z_{*k} is the k^{th} component of the ideal vector z_{*k}.

It is of interest to note that for $p = 1$, program $(P3)'''$ can be written as

$$\min_{x\in D} \sum_{k=1}^{K} (\lambda_k f^k(x) - z_{*k}) = \min_{x\in D} \left\{ \sum_{k=1}^{K} \left(\lambda_k f^k(x)\right) - \sum_{k=1}^{K} (\lambda_k z_{*k}) \right\}$$

Hence weighted sum scalarization can be seen as a special case of weighted compromise programming.

The emphasis on the distinction between $1 < p < \infty$ and $p = \infty$ is justified by two reasons: The former is the most interesting case [50] and the most widely used, and the results are often different from those for $p = \infty$.

1.3.14 Definition. Given a weight vector λ, x^* is a compromise solution if and only if it minimizes L_p.

One may wonder whether a solution of $(P3)'''$ is efficient for $(P3)$. This is the subject matter of the next theorem.

1.3.13 Theorem. *[50]*
An optimal solution x^* of $(P3)'''$ with $p < \infty$ is efficient if one of the following conditions holds:

1. *x^* is a unique optimal solution of $(P3)'''$.*
2. *$\lambda_k > 0$ for all $k = 1, \ldots, K$.*

Goal programming

The Goal programming approach consists of minimizing deviations to some fixed targets by the DM.

There are different types of Goal programming approaches [72]. The most used ones are weighted goal programming (WGP) and the lexicographic goal programming (LGP). The WGP variant, considers all goals simultaneously within a composite objective function comprising the sum of all the respective deviations of the goals from their aspiration levels. The deviations are weighted according to the relative importance of each goal. So the WGP approaches for solving $(P3)$ consists of tackling the following single objective program.

For illustration, consider the following program:

$$(P3)^v \begin{cases} \min \left\{ \sum_{k=1}^{K} \lambda_k (\delta_k^+ + \delta_k^-) \right\} \\ \text{subject to} \\ f^1(x) - \delta_1^+ + \delta_1^- = s_1 \\ \vdots \\ f^K(x) - \delta_K^+ + \delta_K^- = s_K \\ x \in D, x \geq 0, \delta_k^+ \geq 0, \delta_k^- \geq 0 \end{cases}$$

where $s_k,\ k = 1, \ldots, K$ are the aspiration levels for objective functions $f^k\ k = 1, \ldots, K$, $\delta_k^+\ k = 1, \ldots, K$ and $\delta_k^-\ k = 1, \ldots, K$ are positive and negative deviations from $s_k\ k = 1, \ldots, K$ respectively while $\lambda_k\ k = 1, \ldots, K$ are weights for rating the importance of objective functions.

The following result, the proof of which may be found in [50], tells us something about the efficiency of the solution obtained by the Goal programming approach.

1.3.14 Theorem. *The solution of a WGP problem $(P3)^v$ is efficient if either the aspiration levels form an efficient reference point or all the deviational variables δ_k^+ for functions to be minimized and δ_k^- for functions to be maximized have positive values at the optimum.*

LGP on the other hand, accomplishes the minimization process by attaching pre-emptive or absolute weights to the sets of goals situated in different priorities, that is, the fulfillment of a set of goals situated in a certain priority is immeasurably preferable to the achievement of any other set placed in a lower priority [52]. Hence in LGP the higher priority goals are satisfied first, and it is only then that the lower priorities are considered.

In short, in WGP the relative importance of the goals is dealt with using their relative weights, while in LGP the absolute goals are handled by their rankings.

We write the lexicographic optimization problem with *lex min* operator as follows:

$$(P3)^{vi} \quad \operatorname{lex} \min_{x \in D} \ (f^1(x), \ldots, f^K(x))$$

In the sequel we define the order relation $<_{lex}$ as follows:

$z^1 <_{lex} z^2$ if $z_q^1 < z_q^2$ where $q = \min\ \{k\ :\ z_k^1 \neq z_k^2\}$.
The above defined order is total.

1.3.15 Definition. A feasible solution $x^* \in D$ is lexicographically optimal if there is no $x \in D$ such that $f(x) <_{lex} f(x^*)$.

We can then state that $x^* \in D$ is lexicographically optimal, if

$$f(x^*) \leq_{lex} f(x) \; for \; all \; x \in D.$$

The following lemma establishes the relationship between lexocographically optimal solutions and efficient solutions.

1.3.15 Lemma. *Let $x^* \in D$ be such that $f(x^*) \leq_{lex} f(x)$ for all $x \in D$. Then $x^* \in D_E$.*

Proof. Suppose that x^* is not efficient. Then there is an $x \in D$ such that $f(x) \leq f(x^*)$. So for some $k \in \{1, \ldots, K\}$ we have $f^k(x) \leq f^k(x^*)$. Defining $q = \min\{k \; : \; f^k(x) < f^k(x^*)\}$ we get that $f^k(x) = f^k(x^*)$ for $k = 1, \ldots, q-1$ and $f^q(x) < f^q(x^*)$. Therefore, $f(x) \leq_{lex} f(x^*)$ contradicting lexicographic optimality of x^*. □

Apart from the above mentioned approaches, there are other emerging techniques that are used for solving MOP problems. Here are some of them metaheuristics [48], evolutionary computing [53] and particle swarm optimization [141].

Chapter 2

Multiobjective Stochastic Linear Programming: An overview

2.1 Preamble

In this chapter we discuss key ideas developed in the field of Multiobjective Stochastic Linear Programming (MSLP) [7]. This will help us to motivate our own contribution in this field.

To avoid complications unrelated to our subject, we assume that involved random variables have known distributions with finite means and variances.

Consider the problem $(P1)$ described in the Introduction and let us attempt to provide some meaning to problem $(P1)$.

2.2 Transformation of the feasible set of (P1)

One generally transform $D(\omega)$ to a deterministic set, say D, according to the rules used in Stochastic Programming (see e.g. [26], [78], [112]).

Some commonly used deterministic counterparts of $D(\omega)$ are listed below.

1. $D' = \{x \in \mathbb{R}^n \ : \ E\left(A(\omega)\right) x \leq \mathrm{E}\left(b(\omega)\right), \ x \geq 0\}$

   where E stands for the expected value.

2. $D''(\alpha) = \{x \in \mathbb{R}^n \ : \ P(A(\omega)x \leq b(\omega)) \geq \alpha, \ x \geq 0\}$

   where α is a probability level pre-defined by the Decision maker.

3. $D'''\left(\alpha_1, \ldots, \alpha_m\right) = \bigcap_{i=1}^{m} D_i\left(\alpha_i\right)$

where for $i = \{1, \ldots, m\}$,

$$D_i(\alpha_i) = \{x \in \mathbb{R}^n \ : \ P\left(A_i(\omega)x \leq b_i(\omega)\right) \geq \alpha_i, \ x \geq 0\}$$

here $\alpha_i; \ i = 1, \ldots, m$ are probability levels *a priori* fixed by the Decision maker and

4. $D^{iv} = \{x \in \mathbb{R}^n \ : \ Q(x, \omega) < +\infty, \text{ with probability } 1\}$

   where

$$Q(x, \omega) = \begin{cases} \inf \ \mathrm{q}(\omega)\mathrm{y}; \ \mathrm{y} \in \Upsilon & \text{if} \quad \Upsilon \neq \emptyset \\ \\ +\infty & \text{if} \quad \Upsilon = \emptyset \end{cases}$$

   where $q(\omega)$ is a penalty cost, $W(\omega)$ is a recourse matrix and

$$\Upsilon = \{y \in \mathbb{R}^m \ : \ W(\omega)y = b(\omega) - A(\omega)x, \ y \geq 0\}.$$

In the next subsection, we discuss some existing solution concepts for MSLP problems.

2.3 Solution concepts

2.3.1 Expected value and variance optimalities

Consider now the following deterministic mathematical programs:

$$(P4) \qquad \min_{x \in D} E\left(c(\omega)\right) x$$

$$(P4)' \qquad \min_{x \in D} V\left(c(\omega)\right) x$$

$$(P4)'' \qquad \min_{x \in D} \{E(c(\omega))x + \sigma(c(\omega))x\}$$

with V and σ denoting the variance and the standard deviation respectively and $c(\omega)$ stands for an aggregation of $c^1(\omega), \ldots, c^K(\omega)$ based on techniques of multiattribute utility theory [139].

2.3.1 Definition. If x^* is an optimal solution for Program $(P4)$, $(P4)'$ or $(P4)''$, then x^* is called an expected value, a variance or an expected value/standard deviation optimal solution for problem $(P1)$ when D is a transformation of $D(\omega)$ obtained through techniques of Stochastic Optimization.

Henceforth ψ_E, ψ_V stand respectively for the set of expected value and variance optimal solutions for problem $(P1)$.

A shortcoming of the above defined solution concepts is that, the expected value and the variance do not exhaust the information contained in the distributions of involved random variables. To overcome this drawback, other solution concepts have been proposed. We discuss some of them in the next three subsections.

2.3.2 Tammer and minimum risk optimalities and optimality in probability

2.3.2 Definition. x^* is a Tammer α-optimal solution for Problem $(P1)$, if there is no $x \in D$ such that

$$P\left(\omega : c(\omega)x \leq c(\omega)x^*\right) \geq 1-\alpha \text{ and } P\left(\omega : c(\omega)x < c(\omega)x^*\right) > 0$$

when D is a transformation of $D(\omega)$ obtained through techniques of Stochastic Optimization and α is a probability level pre-defined by the Decision maker.

For details on this solution concept, we invite the reader to consult [130].

2.3.3 Definition. x^* is an α-minimum risk solution for Problem $(P1)$ if x^* is an optimal solution for the following program:

$$\max_{x \in D} P\left(c(\omega)x \leq \alpha\right)$$

when D is a transformation of $D(\omega)$ obtained through techniques of Stochastic Optimization. Where α is an aspiration level a priori fixed by the Decision maker.

An interested reader is referred to [23] for key facts about the minimum risk solution concept.

2.3.4 Definition. x^* is a ρ-optimal solution in probability for Problem $(P1)$ if there is $\beta^* \in \mathbb{R}$ such that (x^*, β^*) is optimal for the program

$$\begin{aligned} &\min_{(x,\beta)\in D\times\mathbf{R}} \beta \\ &\text{subject to} \\ &P\big(c\big(\omega\big)x \leq \beta\big) = \rho \end{aligned}$$

where D is a transformation of $D(\omega)$ obtained through techniques of Stochastic Optimization and ρ is a probability level pre-defined by the Decision maker.

A reader interested to know more about this solution concept is referred to [80].

2.3.3 Expected value and variance efficiencies

Consider the following deterministic multiobjective programs:

$$(P4)''' \qquad \min_{x\in D} \left\{E(c^1(\omega))x, \dots, E(c^K(\omega))x\right\}$$

$$(P4)^{iv} \qquad \min_{x\in D} \left\{V(c^1(\omega))x, \ldots, V(c^K(\omega))x\right\}$$

$$(P4)^{v} \qquad \min_{x\in D} \left\{E(c^1(\omega))x, \ldots, E(c^K(\omega))x, \sigma(c^1(\omega))x, \ldots, \sigma(c^K(\omega))x\right\}$$

where σ stands for the standard deviation.

2.3.5 Definition. x^* is called an expected value, a variance or an expected value/standard deviation efficient solution for problem $(P1)$ if it is efficient for Programs $(P4)'''$, $(P4)^{iv}$ or $(P4)^{v}$ respectively, when D is a transformation of $D(\omega)$ obtained through techniques of Stochastic Optimization.

The sets of expected value, variance and expected value/standard deviation efficient solutions for Program $(P1)$ are denoted by ϕ_E, ϕ_V and $\phi_{E/\sigma}$ respectively.

The concept of expected value weak efficiency, variance weak efficiency and expected value/standard deviation weak efficiency and those of expected value proper efficiency, variance proper efficiency and expected value/standard deviation proper efficiency are obtained by replacing "efficiency" by "weak efficiency" and by "proper efficiency" respectively.

In the sequel ϕ_E^w (ϕ_E^p), ϕ_V^w (ϕ_V^p) and $\phi_{E/\sigma}^w$ $(\phi_{E/\sigma}^p)$ denote the sets of expected value weakly (properly) efficient solutions, variance weakly (properly) efficient solutions and expected value/standard deviation weakly (properly) efficient solutions for program $(P1)$ respectively.

2.3.4 Minimum risk efficiency and efficiency with given probabilities

2.3.6 Definition. x^* is an $(\alpha_1, \ldots, \alpha_K)$-minimum risk efficient solution for MOP problem $(P1)$ if it is efficient for the multiobjective program

$$\max_{x\in D} \left\{P\left(c^1(\omega)x \le \alpha_1\right), \ldots, P\left(c^K(\omega)x \le \alpha_K\right)\right\}$$

when D is a transformation of $D(\omega)$ obtained through techniques of Stochastic Optimization. Here $\alpha_1, \ldots, \alpha_K$ are aspiration levels a priori fixed by the Decision maker.

Characterizations of minimum risk efficiency with aspiration levels, may be found elsewhere [128].

As in the case of expected value efficiency, the concepts of $(\alpha_1, \ldots, \alpha_K)$-minimum risk weak efficiency and $(\alpha_1, \ldots, \alpha_K)$-minimum risk proper efficiency may be obtained by respectively replacing "efficiency" by "weak efficiency" or "proper efficiency" in the above definition.

In what follows $\psi_{MR}(\alpha_1, \ldots, \alpha_K)$, $\psi_{MR}^w(\alpha_1, \ldots, \alpha_K)$ and $\psi_{MR}^p(\alpha_1, \ldots, \alpha_K)$ denote the sets of $(\alpha_1, \ldots, \alpha_K)$-minimum risk efficient solutions, $(\alpha_1, \ldots, \alpha_K)$-minimum risk weakly efficient solutions and $(\alpha_1, \ldots, \alpha_K)$-minimum risk properly efficient solutions for Program $(P1)$ respectively.

2.3.7 Definition. x^* is a $(\rho_1, \ldots, \rho_K)$-efficient solution for MOP Problem $(P1)$ if there is $\beta^* = (\beta_1^*, \ldots, \beta_K^*)$ such that (x^*, β^*) is efficient for the mathematical program:

$$\begin{aligned} &\min_{(x,\beta)\in D\times\mathbf{R}^K} \Big(\beta_1, \ldots, \beta_K\Big) \\ &\text{subject to} \\ &P(c^k\big(\omega\big)x \le \beta_k) \ge \rho_k;\ k = 1, \ldots, K \end{aligned}$$

when D is a transformation of $D(\omega)$ obtained through techniques of Stochastic Optimization. Where $\rho_1, \ldots, \rho_K$ are probability levels *a priori* fixed by the Decision maker.

An interested reader may consult [127] for a thorough discussion on this efficiency concept.

Concepts of $(\rho_1, \ldots, \rho_K)$-weak efficiency and $(\rho_1, \ldots, \rho_K)$-proper efficiency may also be obtained in a way similar to the one in which minimum risk weak and proper efficiencies were obtained.

From now on $\psi_{KT}(\rho_1, \ldots, \rho_K)$, $\psi_{KT}^w(\rho_1, \ldots, \rho_K)$ and $\psi_{KT}^P(\rho_1, \ldots, \rho_K)$ denote the sets of $(\rho_1, \ldots, \rho_K)$-efficient solutions, $(\rho_1, \ldots, \rho_K)$-weak efficient solutions and $(\rho_1, \ldots, \rho_K)$-proper efficient solutions for Program $(P1)$ respectively.

In the next subsection we present some theoretical results related to problem $(P1)$.

2.4 Related mathematical results

Most stochastic constraint transformations yield nonconvexity on resulting deterministic feasible sets. This precludes the application of existing powerful convex optimization algorithms (see e.g [18], [24]). It is therefore, relevant to know when a deterministic counterpart of $D(\omega)$ is convex. The following four propositions; the proofs of which may be found in [77], provide some insights to this issue.

2.4.1 Proposition. *$D''(0)$, $D''(1)$, $D_i(0)$; $i = 1, \ldots, m$, $D_i(1)$; $i = 1, \ldots, m$ and D^{iv} are convex sets.*

2.4.2 Proposition. *Consider Problem $(P1)$ and suppose that $A(\omega)$ is a fixed matrix with maximal rank. Then*

$$D_i(\alpha_i) = \{x \in \mathbb{R}^n : F_i(A_i x) \ge \alpha_i;\ i = 1, \ldots, m\}$$

are convex for every probability distribution F_i of $b_i(\omega)$.

2.4.3 Proposition. *Assume that the probability space under consideration is discrete,* *that is,* $\Omega = \{\omega_1, \dots, \omega_L\}$ *and* $P(\omega_l) = p_l > 0;\ l = 1, \dots, L$. *Let*

$$\alpha_l^* = \max\left(1 - p_l : 1 \le l \le L\right)$$

then the set $D_l(\alpha_l)$ *is convex for any* $\alpha_l > \alpha_l^*$ *and* $D''(\alpha)$ *is convex for any* $\alpha > \alpha^*$ *where* α^* *and* α_l^* *are real numbers.*

2.4.4 Proposition. *Suppose that the probability space under consideration is* $\Omega = \{\omega_1, \dots, \omega_L\}$ *and suppose that* $p_l = P(\omega_l) > 0$ *if and only if* $l \in N = \{1, \dots, r\}$. *Assume also that only one element* $l_o \in N$ *exists such that*

$$p_{l_o} = \min_{l \in N} p_l,$$

then the sets $D''(\alpha)$ *and* $D''_l(\alpha)$ *are convex for every* $\alpha > 1 - p_{l_1}$, *where*

$$p_{l_1} = \min_{l \in (N \setminus \{l_o\})} p_l.$$

The next two results established in [32], [33], bridge the gap between solution concepts based on the first two moments (Proposition (2.4.5)) and establish a connection between a minimum risk efficient solution with aspiration levels and an efficient solution with given probabilities (Proposition (2.4.6)).

2.4.5 Proposition. *We have the following:*

$$1. \phi_E \cap \phi_V \subset \phi_{E/\sigma},$$

$$2. \phi_E \cup \phi_V \subset \phi_{E/\sigma}^w,$$

$$3. \phi_E^w \cup \phi_V^w \subset \phi_{E/\sigma}^w,$$

where ϕ_E, ϕ_V, $\phi_{E/\sigma}$, ϕ_E^w (ϕ_E^p), ϕ_V^w (ϕ_V^p) *and* $\phi_{E/\sigma}^w$ $(\phi_{E/\sigma}^p)$ *are the sets of expected value, variance, expected value/standard deviation, expected value weakly (properly) efficient solutions, variance weakly (properly) efficient solutions and expected value/standard deviation weakly (properly) efficient solutions for program* $(P1)$ *respectively.*

2.4.6 Proposition. *Assume that the probability distributions of the random vectors* $c^1(\omega), \dots, c^K(\omega)$ *are continuous and strictly increasing. Then for any* $(\alpha_1, \dots, \alpha_K) \in \mathbb{R}^K$, $x^* \in \psi_{MR}(\alpha_1, \dots, \alpha_K)$ *if and only if* $x^* \in \psi_{KT}(\rho_1, \dots, \rho_K)$, *where*

$$\rho_k = P\left(c^k(\omega)x \le \alpha_k\right);\ k \in \{1, \dots, K\}.$$

Moreover, we have the following proposition:

2.4.7 Proposition.

$$\bigcup_{(\alpha_1, \dots, \alpha_K) \in R^K} \psi_{MR}(\alpha_1, \dots, \alpha_K) = \bigcup_{(\rho_1, \dots, \rho_K) \in B} \psi_{KT}(\rho_1, \dots, \rho_K)$$

where

$$B = \{(\rho_1, \ldots, \rho_K) : \rho_k \in (0,1];\ k = 1, \ldots, K\}.$$

Well-known characterizations of proper efficiency (see Theorem (1.3.4)), have been explored to relate optimality of $(P1)$ in the sense of Definition (2.3.1) and the efficiency of the same program in the sense of Definition (2.3.5). This is the subject matter of the next two propositions.

2.4.8 Proposition. *If x^* is an expected value optimal solution for problem $(P14)$, then x^* is an expected value properly efficient solution for program $(P4)''$. That is,*

$$\psi_E \subseteq \phi_E^P.$$

2.4.9 Proposition. *If D is a convex set and $E\left(c^k(\omega)\right)x,\ k = 1, \ldots, K$ are convex functions, then x^* is an expected value proper efficient solution for the multiobjective program $(P4)''$, if and only if, x^* is an expected value optimal solution for the problem $(P4)$. That is,*

$$\phi_E^P = \psi_E.$$

Details on this matter are contained in subsection (1.3.3).

2.5 Methodological approaches for solving Multiobjective Stochastic Linear Programs

Methods for singling out satisficing solutions in MSLP problems have been developed in the literature, leading to three main trends, namely: the hard, the soft and the metaheuristic. The hard trend consists of methodologies with proven anayltical justifications. Examples of approaches within this trend may be found in [6], [84], [96].
The soft and metaheuristic trends based on soft Operation Research methodology, are structured and rigorous but non-mathematical. Prime examples of these methods ranging from over these trends may be found in [2], [90], [113] and on [4],[59] and [71] respectively.

Within each group, the original problem may be either reduced to a single objective stochastic program and then solved with a technique used in Stochastic Programming (stochastic approach) or converted to a deterministic multiobjective program before solving with a well known method of Multiobjective Programming (multiobjective approach). A third alternative is to combine in an appropriate manner a technique of single objective Stochastic Programming with a technique of Multiobjective Programming (hybrid approach).

The ideas discussed in the previous subsections have served as guidelines in implementing efficient techniques for solving Multiobjective Stochastic Linear Programming problems.

In what follows we outline a method within each of the three existing approaches namely, the stochastic approach, the multiobjective approach and the hybrid one.

2.5.1 Stochastic approach

In this subsection we present a method described in [70] for solving problem $(P1)$, using the stochastic approach. For this method the following assumptions should be met: $A_i(\omega)$; $i = 1, \ldots, m$; $b(\omega)$ and $c^k(\omega)$, $k = 1, \ldots, K$ are normally distributed random vectors. λ_k; $k = 1, \ldots, K$ are strictly positive real numbers in the interval $(0, 1]$ such that $\sum_{k=1}^{K} \lambda_k = 1$.

Moreover, the following notations are used:

1. $h_i(\omega, x) = A_i(\omega)x - b_i(\omega)$; $i = 1, \ldots, m$.
2. The weights associated with the expected value and the standard deviation of $c(\omega)$, are given and denoted respectively by q_1 and q_2.
3. Probability levels prescribed by the Decision maker for constraints satisfaction are denoted by $\alpha = (\alpha_1, \ldots, \alpha_m)$.

2.5.1 Algorithm. *A stepwise description of the method is given below.*

Step 1. Read λ_k*;* $k = 1, \ldots, K$, $c^k(\omega)$*;* $k = 1, \ldots, K$, $h_i(\omega, x)$*;* $i = 1, \ldots, m$, α_i*;* $i = 1, \ldots, m$.

Step 2. Find

$$c(\omega) = \sum_{k=1}^{K} \lambda_k c^k(\omega).$$

Step 3. Replace $D(\omega)$ *by*

$$D^v = \left\{ x \in R^n : E\left(h_i(\omega, x)\right) + \Phi^{-1}(\alpha_i)\sigma\left(h_i(\omega, x)\right) \leq 0;\; i = 1, \ldots, m \; x \geq 0 \right\}.$$

Step 4. Solve the mathematical program

$$\min_{x \in D^v} \left(q_1 E\left(c(\omega)x\right) + q_2 \sigma\left(c(\omega)x\right)\right). \tag{2.1}$$

Let x^* *be a solution of* (2.1).

Step 5. Print "x^* *is a satisficing solution of* $(P1)$*".*

Step 6. Stop.

As can be seen, this algorithm transforms the original problem into a single objective problem, with the following deterministic objective function

$$q_1 E\left(c(\omega)x\right) + q_2 \sigma\left(c(\omega)x\right).$$

The solution x^* obtained is an expected value/standard deviation optimal solution in the sense of Definition (2.3.1) for problem $(P1)$ as defined in subsection (2.3.1).

Other techniques closely related to the stochastic approach for solving problem $(P1)$ include, decomposition method [35], [46], [69], chance-constrained method [42], [43], simulation based techniques [88], [110], [111], two stage method [147] and multistage method [38].

2.5.2 Multiobjective approach

Here we outline a method within the multiobjective approach. For this method, we need λ_k; $k = 1, \ldots, K$; such that $\lambda_k > 0$, $\sum_{k=1}^{K} \lambda_k = 1$ as in subsection (2.5.1).
2.5.2 Algorithm. *The steps of the method are as follow:*
Step 1. Read λ_k*;* $k = 1, \ldots, K$*,* $c^k(\omega)$*;* $k = 1, \ldots, K$*,* $A_i(\omega)$*;* $i = 1, \ldots, m$*,* $b_i(\omega)$*;* $i = 1, \ldots, m$*.*

Step 2. Replace $D(\omega)$ *by*

$$D' = \{x \in R^n : E\left(A(\omega)\right)x - E\left(b(\omega)\right) \leq 0;\ x \geq 0\}. \tag{2.2}$$

Step 3. Find

$$E\left(c^1(\omega)\right), \ldots, E\left(c^K(\omega)\right).$$

Step 4. Solve the mathematical program

$$\min_{x \in D'} \left(\sum_{k=1}^{K} \lambda_k E\left(c^k(\omega)\right) x \right) \tag{2.3}$$

Let x^* *be a solution of* (2.3)*.*

Step 5. Print "x^* *a satisficing solution of* $(P1)$*".*

Step 6. Stop.

In this method, Steps 2 and 3 tackle randomness, while Step 4 deals with multiplicity of objective functions. The solution x^* obtained is an expected value efficient solution for MSLP problem $(P1)$ as defined in Subsection (**??**.

For a more thorough discussion of other methods for solving $(P1)$ based on the multiobjective approach, the reader is referred to [13], [16], [49], [106] and [124]

2.5.3 Hybrid approach

In this section, we describe a hybrid method due to Mortazavi [105], for solving MSLP problem ($P1$). This method is based on the assumptions given in subsection (2.5.1). The following notations are used in the sequel.

1. δ_s^+; $s = 1, \ldots, S$, δ_t^-; $t = 1, \ldots, T$, δ_u^+, δ_u^-, $u = 1, \ldots, U$ denote positive, negative and two sided deviations from targets g_s; $s = 1, \ldots, S$, g_t; $t = 1, \ldots, T$, g_u; $u = 1, \ldots, U$ respectively. S, T and U are respectively the total number of positive, negative and two-sided deviations from targets g_s, g_t and g_u.
2. α_s; $s = 1, \ldots, S$, α_t; $t = 1, \ldots, T$, α_u; $u = 1, \ldots, U$ are probability levels *a priori* fixed by the Decision maker.

2.5.3 Algorithm. *A stepwise description of the method is as follows:*

Step 1. Read S, T, U, g_s,α_s; $s = 1, \ldots, S$, g_t, α_t; $t = 1, \ldots, T$, g_u, α_u,

$$u = 1, \ldots, U,\ c^k(\omega);\ k = 1, \ldots K,\ h_i(\omega, x);\ i = 1, \ldots, m.$$

Step 2. Put $D(\omega)$ *in the following form :*

$$D^{vi} = \big\{x \in \mathbb{R}^n : E\left(c^s(\omega)\right) x + \Phi^{-1}(\alpha_s)\sigma\left(c^s(\omega)\right) x - g_s - \delta_s^+ \le 0;$$

$$s = 1, \ldots, S,\ E\left(c^t(\omega)\right) x + \Phi^{-1}\left(1 - \alpha_t\right) \sigma\left(c^t(\omega)\right) x - g_t + \delta_t^+ \le 0;$$

$$t = 1, \ldots, T,\ E\left(c^u(\omega)\right) x + \Phi^{-1}\left(\frac{1 - \alpha_u}{2}\right) \sigma\left(c^k(\omega)\right) x - g_u \le 0;$$

$$u = 1, \ldots, U,\ E\left(h_i(\omega, x)\right) + \Phi^{-1}(\alpha_i)\sigma\left(h_i(\omega, x)\right) \le 0;$$

$$i = 1, \ldots, m,\ \delta_s^+ \ge 0, \delta_t^+ \ge 0,\ x \ge 0\big\}.$$

Step 3. Solve the mathematical program

$$\min_{x \in D^{vi}} \left(\sum_{u=1}^{U}(\delta_u^+ + \delta_u^-) + \sum_{s=1}^{S}\delta_s^+ + \sum_{t=1}^{T}\delta_t^-\right) \tag{2.4}$$

Let x^* *be a solution of* (2.4).

Step 4. Print "x^* *is a satisficing solution of* ($P1$)*".*

Step 5. Stop.

It is clear that this method combines the goal programming technique for solving a multi-objective program with the chance-constrained method for solving a stochastic optimization problem.

Other methods pertaining to the hybrid approach may be found in [12], [54] and [84].

2.5.4 Comparison of different approaches

The main lessons that can be drawn while comparing the above described approaches are as follow:

1. The stochastic approach takes into account dependencies between objective functions, whereas the multiobjective approach does not (see for example [31]). This makes the stochastic approach closer to reality. Therefore, the stochastic approach is more effective for finding solutions to a MSLP problem than the multiobjective approach.
2. The multiobjective approach is more efficient than the stochastic approach, in the sense that it requires fewer computations. These computations are easier to handle than those required by the stochastic approach. (see e.g., [101], [132], [134]).
3. The hybrid approach combines the strengths of the stochastic and the multiobjective approaches. Consequently, the hybrid approach could perform better than either of the other two approaches for a given problem. Interested readers may consult [20] for a substantiation of this claim.
4. Methods pertaining to the hybrid approach create more flexibility in allowing the Decision maker to specify his preferences (see e.g., [12]).

Nevertheless, it is the nature and the structure of the problem that determines which approach to use. In what follows, we briefly discuss some applications of Multiobjective Stochastic Linear Programming to concrete real-life problems.

2.6 Applications

2.6.1 Applications of the stochastic approach

Production planning problems, lend themselves better to the use of the stochastic approach. As a matter of fact, the structure of these problems dictates that one starts dealing with the multiplicity of objective functions and later tackles the randomness in data [55].

Some other applications of the stochastic approach to MSLP problems include power system security problem [8], power plant preventive maintenance scheduling [35], capacity planning [60], hydro-thermal electricity generation [107], deployment of roadway incident response vehicles [109] and multi-product batch plant design [147].

2.6.2 Applications along the multiobjective approach

Water resource planning and management problems [104], are most appropriately dealt with using the multiobjective approach. Random parameters are first transformed into appropriate fixed data, before the conflicting goals are sorted out. The literature is rich in models using the multiobjective approach. We list a few of them.

Water use planning [29], workforce scheduling model [37], transportation network design problem [44] and nuclear generation of electricity problem [87], [131].

2.6.3 Applications within the hybrid approach

To significantly bridge the dangerous gap between the problems of designing reliable portfolio assets and the mathematical programming models used to solve them, the Decision maker should be able to consider different objective functions and incorporate imprecision into the model. Owing to the complexity of such problems, it is best to couple different techniques in an appropriate way to solve them.

There are several good papers using this approach, to which the reader may refer. The papers, Ben Abdelaziz [20], [25], [41], [75], [82], [83] and [129] are some of them.

Chapter 3

Satisficing Solutions for Multiobjective Stochastic Linear Programming Problems

3.1 Preamble

In this chapter, we introduce new solution concepts for MSLP problems. The solution concepts are based on the chance constrained approach for tackling randomness (see e.g., Kall [77], Liu [89]).
We also briefly discuss their advantages over other optimality/efficiency notions discussed in Chapter 3.

These solution concepts are then characterized under the assumptions that involved random variables have normal, chi-squared, exponential and gamma distributions.

Recall that the optimization problem at hand is of the forM

$$(P1)\begin{cases}\min\limits_{x\in D(\omega)}\left(c^1x,\ldots,c^Kx\right)\end{cases}$$

where $c^1,\ldots,c^K$ are random vectors defined on a Probability space $(\Omega,\mathcal{F},\mathbb{P})$. That is for all $k\ \in\{k=1,\ldots,K\}$

$$\begin{aligned}c^k:\Omega&\mapsto\mathbb{R}^n\\ \omega&\mapsto c^k(\omega)\end{aligned}$$

is a measurable function and

$$D(\omega)=\{x\in\mathbb{R}^n\mid A(\omega)x-b(\omega)\leq 0;\ \ i=1,\ldots,m;\ x\geq 0\}$$

where $A(\omega),\ b(\omega)$ are respectively $m \times n$ and $m \times 1$ random matrices.

3.2 Satisficing solutions for $(P1)$

3.2.1 Definitions of satisficing solutions for $(P1)$

3.2.1 Definition. (*(α, β) − satisficing solution of type 1*)
We say that $x^* \in \mathbb{R}^n$ is an $(\alpha, \beta)-$satisficing solution of type 1 for $(P1)$ if (x^*, s^*) is optimal for the following optimization problem:

$$(P5)\begin{cases} \min\ s \\ \text{subject to} \\ \mathbb{P}\left(cx \leq s\right) \geq \beta \\ \mathbb{P}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m \\ x \geq 0,\ s,\ u.r.s. \end{cases}$$

Where cx is an aggregation of $c_1x, \ldots, c^Kx$, $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\alpha_i \in (0, 1];\ i = 1, \ldots, m$ and $\beta \in (0, 1]$. $\alpha_i(i = 1, \ldots, m)$ and β are probability levels *a priori* fixed by the decision maker.
3.2.2 Definition. ($(\alpha, \beta)-$satisficing solution of type 2)
We say that $x^* \in \mathbb{R}^n$ is an $(\alpha, \beta)-$satisficing solution of type 2 for $(P1)$ if (x^*, s^*), where $s^* \in \mathbb{R}^K$, is efficient for the following multiobjective optimization problem:

$$(P5)'\begin{cases} \min\ (s_1, \ldots, s_K) \\ \text{subject to} \\ \mathbb{P}\left(c^1(\omega)x \leq s_1\right) \geq \beta_1 \\ \vdots \\ \mathbb{P}\left(c^Kx \leq s_K\right) \geq \beta_K \\ \{\mathbb{P}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m \\ x \geq 0,\ s_k,\ u.r.s;\ k = 1, \ldots, K \end{cases}$$

Here $\alpha_i \in (0, 1];\ i = 1, \ldots, m$ and $\beta_k \in (0, 1];\ k = 1, \ldots, K$ are thresholds fixed by the decision maker.
3.2.3 Definition. ($(\alpha, \beta)-$satisficing solution of type 3)
We say that $x^* \in \mathbb{R}^n$ is an $(\alpha, \beta)-$satisficing solution of type 3 for $(P1)$ if (x^*, δ^*), where

$\delta^* \in \mathbb{R}^K$, is optimal for the following mathematical program:

$$(P5)'' \begin{cases} \min \left(\sum_{k=1}^{K} \nu_k \delta_k\right) \\ \text{subject to} \\ \mathbb{P}\left(c^1 x - \delta_1 \leq t_1\right) \geq \beta_1 \\ \vdots \\ \mathbb{P}\left(c^K x - \delta_K \leq t_K\right) \geq \beta_K \\ \mathbb{P}\left(\sum_{j=1}^{n} a_{ij} x_j - b_i \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m \\ x \geq 0,\ \delta_k \geq 0;\ k = 1, \ldots, K \end{cases}$$

Here δ_k, ν_k and t_k are respectively the upper deviation of the k^{th} goal from the k^{th} target, the weight of the k^{th} goal and the target associated with the k^{th} objective function.
We note that $\alpha_i \in (0, 1];\ i = 1, \ldots, m$ and $\beta_k \in (0, 1];\ k = 1, \ldots, K$ are as in Definition (3.2.2).

3.2.2 Interpretation of satisficing solutions for $(P1)$

Solution concepts discussed here are based on the individual chance-constrained approach [4]. One may equally well, consider the joint chance-constrained approach [112].
The Chance constrained approach allows partial violation of the constraints, and it may be viewed as a technique for providing an appropriate safety margin [122].
The Chance constrained technique in general has been applied to various industrial and economic problems [135].

Intuitive ways of thinking about the above solution concepts are given below.

1. An $(\alpha, \beta)-$satisficing solution of type 1 for $(P1)$ is an alternative that keeps the probability of the aggregate cx of the objective functions $c^1x, \ldots, c^Kx$ of this problem, lower than a variable s to a desired extent β, while minimizing this variable.
Moreover the chance of meeting the i^{th} constraint of the problem is requested to be higher than a fixed thresholds α_i. It is clear that for α_i and β close to 1, such an alternative is a an interesting one.
As a matter of fact, it maintains the probability of keeping the aggregation of objective functions low, while keeping the probabilities of realization of constraints to desired extents.

2. An $(\alpha, \beta)-$satisficing solution of type 2 for $(P1)$ is an action that keeps the probability of the k^{th} objective c^kx, for all k less than some variable s_k to a level β_k while the vector $(s_1, \ldots, s_K)$ is minimized in the Pareto sense.
In the meantime, the chance of having the i^{th} constraint, for all i, be satisfied is greater than α_i.

Again for $\beta_k;\ k = 1, \ldots, K$ and $\alpha_i;\ i = 1, \ldots, m$ close to one, such a solution is interesting from the standpoints of feasibility and efficiency.

3. Finally, an $(\alpha, \beta)-$satisficing solution of type 3 minimize the weighted sum of deviations from given targets while maintaining the probability of each objective to be close to the corresponding target to some desired levels. It also meets the requirement of being feasible with high probability levels.

In what follows, we consider the case when the involved random variables of $(P1)$ are normally distributed.

3.3 Characterization of satisficing solutions for the Normal case

We recall that a random vector

$$\begin{aligned} Y\ :\ \Omega &\to \mathbb{R}^n; \\ \omega &\mapsto Y(\omega) \end{aligned}$$

is said to have the multivariate normal distribution if any linear combination of its components is normally distributed. The following lemma due to Kataoka [80] will be needed in the sequel.

3.3.1 Lemma.

(i) *Assume that $Y_1, \ldots, Y_n$ are independent and normally distributed random variables with means $E(Y_1), \ldots, E(Y_n)$ respectively. Then for any real numbers $\lambda_1, \ldots, \lambda_n$, the aggregation $Y = \lambda_1 Y_1 + \ldots + \lambda_n Y_n$ is normally distributed with mean $E(Y) = \lambda_1 E(Y_1) + \ldots + \lambda_n E(Y_n)$ and variance $Var(Y) = \sum_{j=1}^{n} Var(Y_j)\lambda_j^2$.*

(ii) *Suppose $Y = (Y_1, \ldots, Y_n)$ has a multivariate normal distribution with mean $E(Y) = (E(Y_1), \ldots, E(Y_n))$ and let covariance matrix $V = (v_{ij})_{i,j}$ where $v_{ij} = cov(Y_i, Y_j)$ be the covariance of Y_i and Y_j. Then for any $x \in \mathbb{R}^n$, the inner product $Yx = Y_1x_1 + \ldots + Y_nx_n$ is normally distributed with mean $E(Y)x = E(Y_1)x_1 + \ldots + E(Y_n)x_n$ and variance $x'Var(Y)x = \sum_{i,j=1}^{n} x_i cov(Y_i, Y_j) x_j$.*

We are now in a position to characterize an $(\alpha, \beta)-$satisficing solution of type 1.

3.3.1 Satisficing solution of type 1

3.3.2 Proposition. *Assume that objective functions of* $(P1)$ *are aggregated as follows:*

$$cx = \lambda_1 c^1 x + \cdots + \lambda_K c^K x$$

where $\lambda_1, \ldots, \lambda_K \geq 0,\ \sum_{k=1}^K \lambda_k = 1$. *Suppose that the random variables*

$$c_j^k, k = 1, \ldots, K;\ j = 1, \ldots, n$$

are independent and normally distributed. Suppose also that for any $i = 1, \ldots, m$, *the random variables* $a_{ij}; j = 1, \ldots, n$ *and* b_i *are normally distributed and independent. Then* x^* *is an* (α, β)*-satisficing solution of type 1 for* $(P1)$ *if and only if* x^* *is optimal for the following optimization problem:*

$$(P6)\begin{cases} \min\ \left(E(c)x + \phi^{-1}(\beta)\sqrt{x'Var(c)x}\right) \\ \textit{subject to} \\ E(A_i)x - E(b_i) + \phi^{-1}(\alpha_i)\sqrt{Var(b_i) + x'Var(A_i)x} \leq 0; \quad i = 1, \ldots, m \\ x \geq 0 \end{cases}$$

where $\alpha_i,\ i = 1, \ldots, m$ *and* β *are as defined in Definition* (3.2.1)*, and*

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.$$

Proof. Assume that x^* is a satisficing solution of type 1 for $(P1)$, then (x^*, s^*) is a solution of the program:

$$(P5)\begin{cases} \min\ s \\ \text{subject to} \\ \mathbb{P}\left(cx \leq s\right) \geq \beta \\ \mathbb{P}\left(\sum_{j=1}^n a_{ij}x_j - b_i \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m \\ x \geq 0;\ s,\ u.s.r. \end{cases}$$

Let's show that the first constraint of $(P5)$ is equivalent to

$$E(c)x + \phi^{-1}(\beta)\sqrt{x'Var(c)x} \leq s. \tag{3.1}$$

Indeed we have by Lemma (3.3.1) that cx is normally distributed with mean $E(c)x$ and variance $x'Var(c)x$. Then

$$
\begin{aligned}
\mathbb{P}(cx \leq s) &= \mathbb{P}\left(\frac{cx - E(c)x}{\sqrt{x'Var(c)x}} \leq \frac{s - E(c)x}{\sqrt{x'Var(c)x}}\right) \\
&= \phi\left(\frac{s - E(c)x}{\sqrt{x'Var(c)x}}\right)
\end{aligned}
$$

because

$$\frac{cx - E(c)x}{\sqrt{x'Var(c)x}}$$

is normally distributed with mean 0 and variance 1. Therefore the first constraint of $(P5)$ becomes

$$\phi\left(\frac{s - E(c)x}{\sqrt{x'Var(c)x}}\right) \geq \beta$$

that is

$$\frac{s - E(c)x}{\sqrt{x'Var(c)x}} \geq \phi^{-1}(\beta)$$

which is equivalent to

$$E(c)x + \phi^{-1}(\beta)\sqrt{x'Var(c)x} \leq s,$$

and we have established that the first constraint of $(P5)$ is equivalent to relation (3.1).
Define now the random variable

$$l_i = \sum_{j=1}^{n} a_{ij}x_j - b_i;\ i = 1, \ldots, m$$

$l_i;\ i = 1, \ldots, m$ are independent and normally distributed ([70]).
The expected values of $l_i;\ i = 1, \ldots, m$ are

$$E(l_i) = \sum_{j=1}^{n} E(a_{ij})x_j - E(b_i);\ i = 1, \ldots, m \tag{3.2}$$

and their variances are

$$Var(l_i) = Var(b_i) + x'Var(A_i)x;\ i = 1, \ldots, m. \tag{3.3}$$

The chance constraints

$$\mathbb{P}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m \tag{3.4}$$

can therefore, be written as

$$\mathbb{P}(l_i \leq 0) \geq \alpha_i;\ i = 1, \ldots, m.$$

We can also write

$$\mathbb{P}(l_i \leq 0) = \mathbb{P}\left(\frac{l_i - E(l_i)}{\sqrt{Var(l_i)}} \leq \frac{-E(l_i)}{\sqrt{Var(l_i)}}\right);\ i = 1, \ldots, m$$

or

$$\mathbb{P}(l_i \leq 0) = \phi\left(\frac{-E(l_i)}{\sqrt{Var(l_i)}}\right);\ i = 1, \ldots, m$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}.$$

Thus the chance constraints (3.4) can be merely written as follows:

$$\phi\left(\frac{-E(l_i)}{\sqrt{Var(l_i)}}\right) \geq \alpha_i;\ i = 1, \ldots, m$$

that is

$$\frac{-E(l_i)}{\sqrt{Var(l_i)}} \geq \phi^{-1}(\alpha_i);\ i = 1, \ldots, m$$

which leads to

$$E(l_i) + \phi^{-1}(\alpha_i)\sqrt{Var(l_i)} \leq 0;\ i = 1, \ldots, m. \tag{3.5}$$

Replacing $E(l_i)$ and $\sqrt{Var(l_i)}$ by their values given by (3.2) and (3.3) respectively in (3.5), we obtain

$$\begin{aligned} &E(A_i)x - E(b_i) \\ &+\phi^{-1}(\alpha_i)\sqrt{Var(b_i) + x' Var(A_i)x} \leq 0;\ i = 1, \ldots, m. \end{aligned} \tag{3.6}$$

Combining relations (3.1) and (3.6) we obtain that (x^*, s^*) is optimal to the following program:

$$
(P6)' \begin{cases} \min\ s \\ \text{subject to} \\ E(c)x + \phi^{-1}(\beta)\sqrt{x'Var(c)x} \leq s \\ E(A_i)x - E(b_i) + \phi^{-1}(\alpha_i)\sqrt{Var(b_i) + x'Var(A_i)x} \leq 0;\ i = 1, \ldots, m \\ x \geq 0;\ s, u.s.r. \end{cases}
$$

This problem is equivalent to $(P6)$ as desired.

□

3.3.1 Example. *Consider the following Multiobjective Stochastic Linear Program:*

$$
(P6)'' \begin{cases} \min f_1(x) = \ (c_1^1 x_1 + c_2^1 x_2) \\ \max f_2(x) = \ (\mathrm{c}_1^2 \mathrm{x}_1 + \mathrm{c}_2^2 \mathrm{x}_2) \\ \textit{subject to} \\ a_{11}x_1 + a_{12}x_2 \leq b_1 \\ a_{21}x_1 + a_{22}x_2 \leq b_2 \\ a_{31}x_1 + a_{32}x_2 \leq b_3 \\ x \geq 0 \end{cases}
$$

where

$$
c_j^k;\ j = 1, 2;\ j = k;\ a_{ij};\ i = 1, 2, 3,\ j = 1, 2
$$

are independent normally distributed random variables, with the following parameters:
$c_1^1 \sim N(20, 5),\ c_2^1 \sim N(40, 10)$
$c_1^2 \sim N(50, 20),\ c_2^2 \sim N(100, 50)$
$a_{11} \sim N(10, 6),\ a_{12} \sim N(5, 4)$
$a_{21} \sim N(4, 4),\ a_{22} \sim N(10, 7)$
$a_{31} \sim N(1, 2),\ a_{32} \sim N(1.5, 3)$
$b_1 \sim N(2500, 500),\ b_2 \sim N(2000, 400),\ b_3 \sim N(450, 50).$

If the Decision maker is interested in a satisficing solution of type 1, he should use the weighted sum method described in Subsection (1.3.7)*. For the following weights:* $\lambda_1 = 0.4$ *and* $\lambda_2 = 0.6$ *and for* $\alpha_i = 0.99$ *for all* i*, and* $\beta_k = 0.99;\ k = 1, 2.$
We have by virtue of Proposition (3.3.2) *to solve the following problem, in a way to obtain*

an $(\alpha, \beta)-$satisficing solution of type 1; with $\alpha = \beta = (0.99, 0.99)$.

$$(P6)''' \begin{cases} \min \Big\{ \Big(20x_1 + 40x_2 + 2.33\sqrt{25x_1^2 + 100x_2^2}\Big)\, 0.4 \\ -\Big(50x_1 + 100x_2 + 2.33\sqrt{400x_1^2 + 2500x_2^2}\Big)\, 0.6 \Big\} \\ \textit{subject to} \\ 10x_1 + 5x_2 + 2.33\sqrt{36x_1^2 + 16x_2^2 + 250000} - 2500 \leq 0 \\ 4x_1 + 10x_2 + 2.33\sqrt{16x_1^2 + 49x_2^2 + 160000} - 2000 \leq 0 \\ x_1 + 1.5x_2 + 2.33\sqrt{4x_1^2 + 9x_2^2 + 2500} - 450 \leq 0 \\ x \geq 0 \end{cases}$$

We use LINGO 13.1 to solve this optimization problem. Local optimal solution was found at iteration: 38, and the optimal solution: $z^* = -5383.792$ *at the points* $x_1^* = 14.96307$, $x_2^* = 48.13617$.

As far as $(\alpha, \beta)-$satisficing solution of type 2 is concerned we have the following result.

3.3.2 Satisficing solution of type 2

3.3.3 Proposition. *Consider* $(P1)$ *and assume that the random variables*

$$c_j^k; \quad k = 1, \ldots, K, \; j = 1, \ldots, n,$$

are independent and normally distributed. Assume further that for any $i = 1, \ldots, m$, *the random variables* $a_{ij}; j = 1, \ldots, n$, *and* b_i *are also independent and normally distributed. Then* x^* *is an* (α, β)*-satisficing solution of type 2 for* $(P1)$ *if and only if* x^* *is efficient for the following mathematical program:*

$$(P6)^{iv} \begin{cases} \min \Big\{ E(c^1)x + \phi^{-1}(\beta_1)\sqrt{x' Var(c^1)\, x} \Big\} \\ \vdots \\ \min \Big\{ E(c^K)x + \phi^{-1}(\beta_K)\sqrt{x' Var(c^K)\, x} \Big\} \\ \textit{subject to} \\ E(A_i)x - E(b_i) + \phi^{-1}(\alpha_i)\sqrt{Var(b_i) + x' Var(A_i)x} \leq 0; \; i = 1, \ldots, m \\ x \geq 0 \end{cases}$$

where $\alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_K \in (0, 1]$ *are as in Definition* (3.2.2), $\phi(x)$ *is as in Proposition* (3.3.2).

Proof. As shown in the proof of Proposition (3.3.2), the chance constraints

$$\mathbb{P}(c^k x \leq s_k) \geq \beta_k;\ k = 1, \ldots, K,$$

in $(P5)'$ are equivalent to

$$E(c^k)x + \phi^{-1}(\beta_k)\sqrt{x'Var(c^k)x} \leq s_k;\ \ k = 1, \ldots, K.$$

The chance constraints:

$$\mathbb{P}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i \leq 0\right) \geq \alpha_i;\ \ i = 1, \ldots, m$$

are equivalent to

$$E(A_i)x - E(b_i) + \phi^{-1}(\alpha_i)\sqrt{Var(b_i) + x'Var(A_i)x} \leq 0;\ \ i = 1, \ldots, m.$$

Then $(P6)^{iv}$ is equivalent to the following program:

$$\begin{cases} \min\ (s_1, \ldots, s_K) \\ \text{subject to} \\ E(c^k)x + \phi^{-1}(\beta_k)\sqrt{x'Var(c^k)x} \leq s_k;\ k = 1 \ldots K \\ E(A_i)x - E(b_i) + \phi^{-1}(\alpha_i)\sqrt{Var(b_i) + x'Var(A_i)x} \leq 0;\ \ i = 1, \ldots, m \\ x \geq 0;\ \ s_k, u.s.r.\ k = 1, \ldots, K \end{cases}$$

And clearly, this problem is equivalent to $(P5)^{iv}$ in the sense that they have the same efficient frontier. □

3.3.2 Example. *Consider Example* (3.3.1) *and assume now, that, a satisficing solution of type 2 is sought with* $\alpha_i = 0.99\ \forall\ i$, *and* $\beta_k = 0.99\ \forall\ k$.
Then by Proposition (3.3.3), *one has to solve the following problem:*

$$(P6)^v \begin{cases} \min\ \mathrm{f}_1 = \left\{20\mathrm{x}_1 + 40\mathrm{x}_2 + 2.33\sqrt{25\mathrm{x}_1^2 + 100\mathrm{x}_2^2}\right\} \\ \min\ \mathrm{f}_2 = \left\{-\left(50\mathrm{x}_1 + 100\mathrm{x}_2 + 2.33\sqrt{400\mathrm{x}_1^2 + 2600\mathrm{x}_2^2}\right)\right\} \\ \textit{subject to} \\ 10x_1 + 5x_2 + 2.33\sqrt{36x_1^2 + 16x_2^2 + 250000} - 2500 \leq 0 \\ 4x_1 + 10x_2 + 2.33\sqrt{16x_1^2 + 49x_2^2 + 160000} - 2000 \leq 0 \\ x_1 + 1.5x_2 + 2.33\sqrt{4x_1^2 + 9x_2^2 + 2500} - 450 \leq 0 \\ x \geq 0 \end{cases}$$

Using compromise programming discussed in Subsection (1.3.8), *we shall solve the follow-*

	f_1^*=0	f_2^*= -11220.36	z^*=-9538.066
x_1^*	0	18.25769	12.57766
x_2^*	0	47.30804	48.67925

Table 3.1: Compromise solution for Program $(P6)^v$

ing subproblems:

$$\begin{cases} \min\ \mathrm{f_1} = \left\{20\mathrm{x_1} + 40\mathrm{x_2} + 2.33\sqrt{25\mathrm{x_1^2} + 100\mathrm{x_2^2}}\right\} \\ \textit{subject to} \\ 10x_1 + 5x_2 + 2.33\sqrt{36x_1^2 + 16x_2^2 + 250000} - 2500 \leq 0 \\ 4x_1 + 10x_2 + 2.33\sqrt{16x_1^2 + 49x_2^2 + 160000} - 2000 \leq 0 \\ x_1 + 1.5x_2 + 2.33\sqrt{4x_1^2 + 9x_2^2 + 2500} - 450 \leq 0 \\ x \geq 0 \end{cases}$$

$$\begin{cases} \min\ \mathrm{f_2} = -\left(50\mathrm{x_1} + 100\mathrm{x_2} + 2.33\sqrt{400\mathrm{x_1^2} + 2600\mathrm{x_2^2}}\right) \\ \textit{subject to} \\ 10x_1 + 5x_2 + 2.33\sqrt{36x_1^2 + 16x_2^2 + 250000} - 2500 \leq 0 \\ 4x_1 + 10x_2 + 2.33\sqrt{16x_1^2 + 49x_2^2 + 160000} - 2000 \leq 0 \\ x_1 + 1.5x_2 + 2.33\sqrt{4x_1^2 + 9x_2^2 + 2500} - 450 \leq 0 \\ x \geq 0 \end{cases}$$

and

$$\begin{cases} \min\ \mathrm{z} = \left\{20\mathrm{x_1} + 40\mathrm{x_2} + 2.33\sqrt{25\mathrm{x_1^2} + 100\mathrm{x_2^2}}\right\} - \mathrm{f_{*1}} \\ -\left(50\mathrm{x_1} + 100\mathrm{x_2} + 2.33\sqrt{400\mathrm{x_1^2} + 2600\mathrm{x_2^2}}\right) - \mathrm{f_{*2}} \\ \textit{subject to} \\ 10x_1 + 5x_2 + 2.33\sqrt{36x_1^2 + 16x_2^2 + 250000} - 2500 \leq 0 \\ 4x_1 + 10x_2 + 2.33\sqrt{16x_1^2 + 49x_2^2 + 160000} - 2000 \leq 0 \\ x_1 + 1.5x_2 + 2.33\sqrt{4x_1^2 + 9x_2^2 + 2500} - 450 \leq 0 \\ x \geq 0 \end{cases}$$

Using LINGO 13.0 software we obtain the solutions in the table above.

3.3.3 Satisficing solution of type 3

3.3.4 Proposition. *Consider* $(P1)$ *and assume that the random variables*

$$c_j^k; \quad k = 1, \ldots, K;\ j = 1, \ldots, n$$

are independent and normally distributed. Suppose also that for any $i = 1, \ldots, m$, the random variables $a_{ij}; j = 1, \ldots, n$ and b_i are normally distributed and independent.
Then x^* is an (α, β)-satisficing solution of type 3 for problem $(P1)$ if and only if x^* is optimal for the following mathematical program:

$$(P6)^{vi}\begin{cases} \min \left(\sum_{k=1}^{K} \nu_k \delta_k\right) \\ \textit{subject to} \\ E(c^1)x + \phi^{-1}(\beta_1)\sqrt{x'Var(c^1)x} \leq t_1 + \delta_1 \\ \vdots \\ E(c^K)x + \phi^{-1}(\beta_K)\sqrt{x'Var(c^K)x} \leq t_K + \delta_K \\ E(A_i)x - E(b_i) + \phi^{-1}(\alpha_i)\sqrt{Var(b_i) + x'Var(A_i)x} \leq 0;\ i = 1, \ldots, m \\ x \geq 0;\ \delta_k \geq 0;\ k = 1 \ldots, K \end{cases}$$

where δ_k; $(k = 1, \ldots, K)$, ν_k; $(k = 1, \ldots, K)$ and t_k; $(k = 1, \ldots, K)$ are as in Definition (3.2.3)*; $\phi(x)$; $\alpha_i \in (0, 1]$; $(i = 1, \ldots, m)$; $\beta_k \in (0, 1]$; $(k = 1, \ldots, K)$ are as defined in Proposition* (3.3.3)*.*

Proof. Consider the mathematical program

$$(P5)''\begin{cases} \min \left(\sum_{k=1}^{K} \nu_k \delta_k\right) \\ \text{subject to} \\ \mathbb{P}\left(c^1(\omega)x - \delta_1 \leq t_1\right) \geq \beta_1 \\ \vdots \\ \mathbb{P}\left(c^K(\omega)x - \delta_K \leq t_K\right) \geq \beta_K \\ \mathbb{P}\left(\sum_{j=1}^{n} a_{ij}(\omega)x_j - b_i(\omega) \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m \\ x \geq 0;\ \delta_k \geq 0;\ k = 1, \ldots, K \end{cases}$$

As discussed in the proof of Proposition (3.3.3), the chance constraints

$$\mathbb{P}\big(c^k x \leq t_k + \delta_k\big) \geq \beta_k;\ \ k = 1, \ldots, K,$$

are equivalent to

$$E(c^k)x + \phi^{-1}(\beta_k)\sqrt{x'Var(c^k)x} \leq t_k + \delta_k;\ k = 1, \ldots, K.$$

As shown in the proof of Proposition (3.3.2), we have that

$$\mathbb{P}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m$$

can be written as

$$E(A_i)x - E(b_i) + \phi^{-1}(\alpha_i)\sqrt{Var(b_i) + x'Var(A_i)x} \leq 0,\ i = 1, \ldots, m.$$

Therefore, program $(P5)''$ reads the following:

$$\begin{cases} \min\ \left(\sum_{k=1}^{K} \nu_k \delta_k\right) \\ \text{subject to} \\ E(c^1)x + \phi^{-1}(\beta_1)\sqrt{x'Var(c^1)x} \leq t_1 + \delta_1 \\ \vdots \\ E(c^K)x + \phi^{-1}(\beta_K)\sqrt{x'Var(c^K)x} \leq t_K + \delta_K \\ E(A_i)x - E(b_i) + \phi^{-1}(\alpha_i)\sqrt{Var(b_i) + x'Var(A_i)x} \leq 0;\ \ i = 1, \ldots, m \\ x \geq 0; \delta_k \geq 0;\ \ k = 1, \ldots, K \end{cases}$$

which is exactly $(P6)^{vi}$.

□

3.3.3 Example. *Consider Example* (3.3.1) *and suppose that, a satisficing solution of type 3 is desirable for* $\alpha_i = 0.99\ \forall\ i,$ *and* $\beta_k = 0.99\ \forall\ k.$
Suppose further that the decision maker goals are 2,000 and 15,000 for objectives 1 and 2 respectively. Using equal weights (i.e $\nu_1 = \nu_2 = 0.5$) *for the two objectives, we have to solve the following problem:*

$$(P6)^{vii} \begin{cases} \min\ (0.5\delta_1 + 0.5\delta_2) \\ \textit{subject to} \\ 20x_1 + 40x_2 + 2.33\sqrt{25x_1^2 + 100x_2^2} \leq 2000 + \delta_1 \\ 50x_1 + 100x_2 + 2.33\sqrt{400x_1^2 + 2600x_2^2} \geq 15000 - \delta_2 \\ 10x_1 + 5x_2 + 2.33\sqrt{36x_1^2 + 16x_2^2 + 250000} - 2500 \leq 0 \\ 4x_1 + 10x_2 + 2.33\sqrt{16x_1^2 + 49x_2^2 + 160000} - 2000 \leq 0 \\ x_1 + 1.5x_2 + 2.33\sqrt{4x_1^2 + 9x_2^2 + 2500} - 450 \leq 0 \\ x \geq 0 \end{cases}$$

Using LINGO 13.0 software to solve this problem, we obtain the following solution:
$\delta_1^* = 1342.376,\ \delta_2^* = 3801.852,\ x_1^* = 12.57766,\ x_2^* = 48.67925$ *with optimal value* $z^* =$ 2572.114.

The assumption of normality may be unrealistic for a wide class of problem, for example, resource allocation and investment for capacity expansion where the cost, input coefficient, demand or resources has to be non-negative [122]. This calls for distributions with non-negative range such as the chi-squared, exponential or gamma distribution. The basic

motivation is the fact that these distributions may retain the convexity of the set of feasible solution [122], which is of great advantage in applying any algorithm of nonlinear programming.
In what follows, we characterize solution concepts introduced, in the case involved random variables follow the chi-squared, the exponential and the gamma distributions.

3.4 Characterization of satisficing solutions for the Chi-squared case

The major motivating factors for choosing chi-squared distribution to estimate imprecision in Multiobjective Stochastic Programming include the following [122].

1. The reproductive property by which the sum of N variates, each having a fixed chi-squared distribution, reproduces the same distribution in the same form of chi-squared distribution.
2. The reproductive property of a chi-squared distribution relates very closely to a normal distribution.
3. A chi-squared distribution is related very closely to a normal distribution
4. Any chi-squared distribution with a degree of freedom larger than 30 can be approximated by a normal distribution.
5. The error involved in approximating a central or noncentral chi-squared distribution by a normal distribution can in principle be evaluated in terms of the optimal solution of a chance constrained programming model.

The following lemma will be needed in the sequel.
3.4.1 Lemma. *Let*

$$r^2 = \sum_{j=1}^{n} r_j^2 \ \textit{where} \ \sum_{j=1}^{n} r_j^2 = \sum_{j=1}^{n} (\eta_j \upsilon_j)^2; \ \eta_j \sim N(0, \sigma^2); \ j = 1, \ldots, n$$

and $\upsilon_j = \sqrt{x_j}; \ j = 1, \ldots, n$ *be some scalar quantity.*
Then r^2 *has a chi-squared distribution with degree of freedom* n $(df = n)$ *. The expected value and variance of* r^2 *are respectively given by*

$$E(r^2) = \sum_{j=1}^{n} Var(r_j) + Var(r_j)\kappa_{r_j}^2 \tag{3.7}$$

$$Var(r^2) = 2\sum_{j=1}^{n} Var(r_j)^2 + 2Var(r_j)^2\kappa_{r_j}^2 \tag{3.8}$$

where

$$\kappa_r = \frac{E(\eta)x^{\frac{1}{2}}}{Var(r)^{\frac{1}{2}}} \tag{3.9}$$

and

$$Var(r) = xE(\eta - E(\eta))^2. \tag{3.10}$$

Proof. Suppose $r^2 = \sum_{j=1}^{n} r_j^2$ where $\sum_{j=1}^{n} r_j^2 = \sum_{j=1}^{n}(\eta_j \upsilon_j)^2$; $\eta_j \sim N(0,\sigma^2)$ and $\upsilon_j = \sqrt{x_j}$ be some scalar quantity.

Since

$$r = \sum_{j=1}^{n} \eta_j \sqrt{x_j}$$

given that $\eta_j \sim N(0,\sigma^2)$, then r_j; $j = 1, \ldots, n$ are independent standard normal random variables with finite expectations and variances.

$$m_r = \sum_{j=1}^{n} E(r_j)$$

and

$$Var(r) = \sum_{j=1}^{n} E(r_j - m_{r_j})^2$$

respectively.

The distribution function of r^2 is obtained as follows:

$$\begin{aligned} F_{r^2}(y) &= P(r^2 \leq y) \\ &= \mathbb{P}\left(|r| \leq \sqrt{y}\right) \\ &= F_r(\sqrt{y}) - F_r(-\sqrt{y}). \end{aligned}$$

Taking the derivative we obtain the density function

$$\begin{aligned} \frac{d}{dy}F_{r^2}(y) &= \frac{d}{dy}(F_r(\sqrt{y}) - F_r(-\sqrt{y}) \\ &= \frac{1}{2\sqrt{y}}P_r(\sqrt{y}) + \frac{1}{2\sqrt{y}}P_r(-\sqrt{y}). \end{aligned}$$

As the density function of r is even, we have

$$\frac{P_r(\sqrt{y})}{\sqrt{y}} = \frac{1}{\sigma\sqrt{2\pi y}}e^{\frac{y}{2\sigma^2}}, \quad \text{with } y \geq 0.$$

r^2 can be expressed (see e.g. [122]) as follows:

$$r^2 = \sum_{j=1}^{n} Var(r_j)(q_{r_j} + \hat{m}_{r_j})^2$$

where

$$\hat{m}_r = \sum_{j=1}^{n} \frac{m_{r_j}}{\sqrt{Var(r_j)}}$$

and

$$q_r = \sum_{j=1}^{n} \frac{(r - m_{r_j})}{\sqrt{Var(r_j)}}$$

is normally distributed with mean 0 and variance 1. We find the characteristic function of r^2 through the moment generating function

$$M_{r^2}(it) = E[e^{itr^2}]$$

$$= \frac{1}{\sigma\sqrt{2\pi y}} \int_0^{\infty} \frac{1}{\sqrt{y}} e^{\frac{y}{2\sigma^2}} e^{itr^2} dy.$$

That is, the characteristic function of r^2 [126] is

$$\phi(t) = \prod_{j=1}^{n} \left[(2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{\{itVar(r_j)(q_{r_j}+\hat{m}_{r_j})^2 - \frac{1}{2}q_{r_j}^2\}} dq_{r_j} \right] \tag{3.11}$$

where $i = +(-1)^{\frac{1}{2}}$ and $j = 1, \ldots, n$.
Since the integral in (3.11), see for example [122], is equal to

$$\frac{2\pi}{(1 - 2itVar(r_j))} e^{\left(\frac{itVar(r_j)\hat{m}_{r_j}^2}{1 - 2itVar(r_j)} \right)}$$

the characteristic function $\phi(t)$ can be merely written as

$$\phi(t) = \prod_{j=1}^{n} (1 - 2itVar(r_j))^{-\frac{1}{2}} e^{\left(\frac{\sum_{j=1}^{n} (itVar(r_j)\hat{m}_{r_j}^2)}{1 - 2itVar(r_j)} \right)}. \tag{3.12}$$

We can obtain the first two moments from (3.12) as follows:

$$E(r^2) = \sum_{j=1}^{n} Var(r_j) + Var(r_j)\kappa_{r_j}$$

$$Var(r^2) = 2\sum_{j=1}^{n} Var(r_j)^2 + 2Var(r_j)^2\kappa_{r_j}^2$$

where κ_r and $Var(r)$ are given in relations (3.9) and (3.10) respectively.

$\square$

In the sequel, we characterize satisficing solution concepts introduced earlier in this chapter.

3.4.1 Satisficing solution of type 1

3.4.2 Proposition. *Consider* $(P1)$*, suppose*

$$cx = \lambda_1 c^1 x + \cdots + \lambda_K c^K x,$$

and assume that

$$c^2 = \sum_{j=1}^{n} \bar{c}_j^2 \text{ where } \sum_{j=1}^{n} \bar{c}_j^2 = \sum_{j=1}^{n} (c_j \upsilon_j)^2 \text{ and } \upsilon_j = \sqrt{x_j};\ j = 1, \ldots, n.$$

Suppose also that

$$h_i^2 = \sum_{i=1}^{n} h_{ij}^2;\ i = 1, \ldots, m \text{ where } \sum_{i=1}^{n} h_{ij}^2 = \sum_{j=1}^{n} \left\{ (a_{ij}\upsilon_j)^2 - (b_i \bar{w}_{ij})^2 \right\};\ i = 1, \ldots, m$$

where c^2 *and* h_i^2 *are mutually independent following a chi-squared distribution.*
Then a point $x^* \in \mathbb{R}^n$ *is an* (α, β)*-satisficing solution of type 1 for* $(P1)$ *if and only if* x^* *is optimal for the following optimization problem:*

$$(P7) \begin{cases} \min \Big\{ \sum_{j=1}^{n} Var(c_j) + Var(c_j)\kappa_{c_j}^2 \\ +F_{\chi_n^2}^{-1}(\beta)\sqrt{2\sum_{j=1}^{n} Var(c_j)^2 + 2Var(c_j)^2\kappa_{c_j}^2} \Big\} \\ \text{subject to} \\ \sum_{j=1}^{n} Var(h_{ij}) + Var(h_{ij})\kappa_{h_{ij}}^2 \\ +F_{\chi_n^2}^{-1}(\alpha_i)\sqrt{2\sum_{j=1}^{n} Var(h_{ij})^2 + 2Var(h_{ij})^2\kappa_{h_{ij}}^2} \leq 0;\ i = 1, \ldots, m \\ x_j \geq 0;\ j = 1, \ldots, n \end{cases}$$

where
$\sum_{j=1}^{n} \lambda_j = 1$; κ_c *and* $Var(c)$ *are as defined in relations* (3.9) *and* (3.10) *respectively,* $\alpha_i \in (0, 1]$; $(i = 1, \ldots, m)$*;* $\beta \in (0, 1]$ *as in Definition* (3.2.1),

$$\bar{w}_{ij} = \sqrt{w_{ij}};\ j = 1, \ldots, n,\ i = 1, \ldots, m \text{ where } \bar{w}_{ij} \geq 0;\quad j = 1, \ldots, n,\ i = 1, \ldots, m$$

and $\sum_{j=1}^{n} \bar{w}_{ij} = 1;\ i = 1, \ldots, m$ *are weights,*

$$\kappa_{h_i} = \frac{E(h_i)^{\frac{1}{2}}}{Var(h_i)^{\frac{1}{2}}};\ i = 1, \ldots, m; \quad Var(h_i) = 2\sum_{j=1}^{n} Var(h_{ij})^2 + 2Var(h_{ij})^2\kappa_{rj}^2;\ i = 1, \ldots, m$$

$$F_{\chi_n^2} = \frac{1}{2^{n/2}\Gamma(n/2)} \int_0^y e^{y/2} t^{(n/2)-1} dt \tag{3.13}$$

$c_j^k;\ k = 1, \ldots, K,\ j = 1, \ldots, n;\ a_{ij};\ j = 1, \ldots, n,\ i = 1, \ldots, m;$ *and* $b_i;\ i = 1, \ldots, m$ *are independent standard normal variates.*

Proof. We know that $x^* \in \mathbb{R}^n$ is an (α, β)-satisficing solution of type 1 for (P1), if there exists $s^* \in \mathbb{R}^n$, such that (x^*, s^*) is optimal for the following problem:

$$(P5) \begin{cases} \min\ s \\ \text{subject to} \\ \mathbb{P}\left(cx \leq s\right) \geq \beta \\ \mathbb{P}\left(\sum_{j=1}^n a_{ij}x_j - b_i \leq 0\right) \geq \alpha_i, i = 1; \ldots, m \\ x \geq 0;\ s,\ u.s.r. \end{cases}$$

We need to put the following relation in a simpler form

$$P(cx \leq s) \geq \beta. \tag{3.14}$$

As

$$c^2 = \sum_{j=1}^n \bar{c}_j^2 \text{ where } \sum_{j=1}^n \bar{c}_j^2 = \sum_{j=1}^n (c_j \upsilon_j)^2 \text{ and } \upsilon_j = \sqrt{x_j};\ j = 1, \ldots, n$$

we have that

$$c^2 \sim \chi_n^2.$$

By Lemma (3.4.1), the expected value and variance of c^2 are

$$E(c^2) = \sum_{j=1}^n Var(c_j) + Var(c_j)\kappa_{c_j}^2 \tag{3.15}$$

and

$$Var(c^2) = 2\sum_{j=1}^n Var(c_j)^2 + 2Var(c_j)^2\kappa_{c_j}^2 \tag{3.16}$$

respectively, where $c_j;\ j = 1, \ldots, n$ are independent standard normal random variables. Then (3.14) can be written as

$$\mathbb{P}\left(\frac{c^2 - E(c^2)}{\sqrt{Var(c^2)}} \leq \frac{s - E(c^2)}{\sqrt{Var(c^2)}}\right) \geq \beta. \tag{3.17}$$

But we know that

$$\mathbb{P}(c^2 \leq s) = F_{\chi_n^2}\left(\frac{s - E(c^2)}{\sqrt{Var(c^2)}}\right)$$

where $F_{\chi_n^2}$ is as defined by relation (3.13).

Hence (3.17) becomes

$$F_{\chi_n^2}\left(\frac{s - E(c^2)}{\sqrt{Var(c^2)}}\right) \geq \beta$$

that is

$$\frac{s - E(c^2)}{\sqrt{Var(c^2)}} \geq F_{\chi_n^2}^{-1}(\beta).$$

Therefore, we have that

$$E(c^2) + F_{\chi_n^2}^{-1}(\beta)\sqrt{Var(c^2)} \leq s. \tag{3.18}$$

Substituting relations (3.15) and (3.16) in (3.18) we obtain

$$\begin{aligned} &\sum_{j=1}^{n} Var(c_j) + Var(c_j)\kappa_{c_j} \\ &+F_{\chi_n^2}^{-1}(\beta_j)\sqrt{2\sum_{j=1}^{n} Var(c_j)^2 + 2Var(c_j)^2\kappa_{c_j}^2} \leq \sum_{j=1}^{n} s_j. \end{aligned} \tag{3.19}$$

Now define

$$h_i^2 = \sum_{j=1}^{n} h_{ij}^2;\ i = 1,\ldots,m \text{ where } \sum_{i=1}^{n} h_{ij}^2 = \sum_{j=1}^{n}\left\{(a_{ij}\upsilon_j)^2 - (b_i\bar{w}_{ij})^2\right\};\ i = 1,\ldots,m$$

and h_i^2 is chi-squared distributed,

$$\bar{w}_{ij} = \sqrt{w_{ij}};\ j = 1,\ldots,n,\ i = 1,\ldots,m \text{ where } \bar{w}_{ij} \geq 0;\quad j = 1,\ldots,n,\ i = 1,\ldots,m$$

and $\sum_{j=1}^{n} \bar{w}_{ij} = 1;\ i = 1,\ldots,m.$

For all i

$$a_{ij};\ j = 1,\ldots,n,\ i = 1,\ldots,m \text{ and } b_i;\ i = 1,\ldots,m$$

are independent and normally distributed.

Clearly $h_i;\ i = 1,\ldots,m$ are independent normal variates with the following respective finite expected values and variances

$$m_{h_i} = \sum_{j=1}^{n} E(h_{ij});\quad i = 1,\ldots,m$$

$$Var(h_i) = \sum_{j=1}^{n} E(h_{ij} - m_{h_{ij}})^2;\ i = 1, \ldots, m.$$

By following the procedure used in the Lemma (3.4.1), the characteristic function $\phi(t)$ of $h_i^2;\ \ i = 1, \ldots, m$ can be written as

$$\phi(t) = \prod_{j=1}^{n} \{1 - 2itVar(h_{ij})\}^{-\frac{1}{2}}\, e^{\left[\frac{\sum_{j=1}^{n}(itVar(h_{ij})\hat{m}_{h_{ij}}^2)}{1-2itVarh_{ij}}\right]};\ \ i = 1, \ldots, m \tag{3.20}$$

where

$$\hat{m}_{h_i} = \sum_{j=1}^{n} \frac{M_{h_{ij}}}{\sqrt{Varh_{ij}}};\ i = 1, \ldots, m.$$

We can obtain the first two moments of h_i^2 from (3.20) as follows

$$E(h_i^2) = \sum_{j=1}^{n} Var(h_{ij}) + Var(h_i)\kappa_{h_{ij}}^2;\ i = 1, \ldots, m \tag{3.21}$$

$$Var(h_i^2) = 2\sum_{j=1}^{n} Var(h_{ij})^2 + 2Var(h_{ij})^2\kappa_{h_{ij}}^2;\ i = 1, \ldots, m. \tag{3.22}$$

Now

$$\mathbb{P}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m$$

can be written as

$$\mathbb{P}(h_i^2 \leq 0) \geq \alpha_i;\ \ i = 1, \ldots, m$$

which can be expressed as

$$\mathbb{P}\left(\frac{h_i^2 - E(h_i^2)}{\sqrt{Var(h_i^2)}} \leq \frac{-E(h_i^2)}{\sqrt{Var(h_i^2)}}\right) \geq \alpha_i;\ \ i = 1, \ldots, m.$$

Then

$$\mathbb{P}(h_i^2 \leq 0) = F_{\chi_n^2}\left(\frac{-E(h_i^2)}{\sqrt{Var(h_i^2)}}\right);\ i = 1, \ldots, m$$

where $F_{\chi_n^2}$ is defined as in (3.13).

We therefore have

$$F_{\chi_n^2}\left(\frac{-E(h_i^2)}{\sqrt{Var(h_i^2)}}\right) \geq \alpha_i;\ i = 1, \ldots, m$$

that is

$$\frac{-E(h_i^2)}{\sqrt{Var(h_i^2)}} \geq F^{-1}_{\chi^2_n}(\alpha_i); \quad i = 1, \ldots, m$$

which leads to

$$E(h_i^2) + F^{-1}_{\chi^2_n}(\alpha_i)\sqrt{Var(h_i^2)} \leq 0; \ i = 1, \ldots, m. \tag{3.23}$$

Substitute relations (3.21) and (3.22) in (3.23) to get

$$\begin{aligned} &\sum_{j=1}^{n} Var(h_{ij}) + Var(h_{ij})\kappa^2_{h_{ij}} \\ &+F^{-1}_{\chi^2_n}(\alpha)\sqrt{2\sum_{j=1}^{n} Var(h_{ij})^2 + 2Var(h_{ij})^2\kappa^2_{h_{ij}}} \leq 0. \end{aligned} \tag{3.24}$$

Combine (3.19) and (3.24) in $(P6)$ to obtain

$$\begin{cases} \min \ \sum_{j=1}^{n} s_j \\ \text{subject to} \\ \sum_{j=1}^{n} Var(c_j) + Var(c_j)\kappa^2_{c_j} \\ +F^{-1}_{\chi^2_n}(\beta)\sqrt{2\sum_{j=1}^{n} Var(c_j)^2 + 2Var(c_j)^2\kappa^2_{c_j}} \leq \sum_{j=1}^{n} s_j \\ . \sum_{j=1}^{n} Var(h_{ij}) + Var(h_{ij})\kappa^2_{h_{ij}} \\ +F^{-1}_{\chi^2_n}(\alpha_i)\sqrt{2\sum_{j=1}^{n} Var(h_{ij})^2 + 2Var(h_{ij})^2\kappa^2_{h_{ij}}} \leq 0; \ i = 1, \ldots, m \\ x_j \geq 0; \ j = 1, \ldots, n \end{cases}$$

which is equivalent to $(P7)$.

□

We know that

$$Var(c) = \sum_{j=1}^{n} x_j E(c_j - E(c_j))^2,$$

$$\text{let } \sigma_c^2 = \sum_{j=1}^{n} E(c_j - E(c_j))^2$$

$$\text{then } Var(c) = \sum_{j=1}^{n} x_j \sigma^2_{c_j}.$$

Let

$$\kappa_c = \sum_{j=1}^{n} \frac{E(c_j)x_j^{\frac{1}{2}}}{Var(c_j)^{\frac{1}{2}}} \text{ then } \kappa_c^2 = \sum_{j=1}^{n} \frac{E(c_j)^2 x_j}{x_j \sigma^2_{c_j}}.$$

The objective function of $(P7)$ can be expanded to give

$$\begin{aligned}
&\sum_{j=1}^{n} x_j\sigma_{c_j}^2 + x_j\sigma_{c_j}^2\frac{E(c_j)^2x_j}{x_j\sigma_{c_j}^2} \\
&+F_{\chi_n^2}^{-1}(\beta)\sqrt{2\sum_{j=1}^{n} x_j^2(\sigma_{c_j}^2)^2 + 2x_j^2(\sigma_{c_j}^2)^2\frac{E(c_j)^2x_j}{x_j(\sigma_{c_j}^2)}} \\
&= \sum_{j=1}^{n} x_j\sigma_{c_j}^2\left\{1+\frac{E(c_j)^2}{\sigma_{c_j}^2}\right\} \\
&+F_{\chi_n^2}^{-1}(\beta)\sqrt{2\sum_{j=1}^{n} x_j^2(\sigma_{c_j}^2)^2\left\{1+\frac{E(c_j)^2}{\sigma_{c_j}^2}\right\}}.
\end{aligned} \tag{3.25}$$

Recall

$$h_i = \sum_{j=1}^{n}\{a_{ij}x_j - \bar{w}_{ij}b_i\};\ i = 1,\ldots,m$$

where

$$\sum_{j=1}^{n}\{a_{ij}x_j - \bar{w}_{ij}b_i\} = \sum_{j=1}^{n} a_{ij}x_j - \sum_{j=1}^{n}\bar{w}_{ij}b_i;\ i = 1,\ldots,m$$

then

$$Var(h_i) = \sum_{j=1}^{n} Var(a_{ij}x_j) - \sum_{j=1}^{n} Var(\bar{w}_{ij}b_i)\}$$

or

$$\begin{aligned}
Var(h_i) &= \sum_{j=1}^{n} x_j\{a_{ij} - E(a_{ij}\}^2 - \bar{w}_{ij}\{b_i - E(b_i)\}^2 \\
&= \sum_{j=1}^{n} x_j\sigma_{a_{ij}}^2 - \sum_{j=1}^{n}\bar{w}_{ij}\sigma_{b_i}^2.
\end{aligned} \tag{3.26}$$

Substitute (3.26) in the constraints of $(P6)$

$$\begin{aligned}
&\sum_{j=1}^{n}\{x_j\sigma_{a_{ij}}^2 - \bar{w}_{ij}\sigma_{b_i}^2\}\left\{1+\frac{E(a_{ij})^2x_j}{\{x_j\sigma_{a_{ij}}^2 - \bar{w}_{ij}\sigma_{b_i}^2\}}\right\} \\
&+F_{\chi_n^2}^{-1}(\alpha_i)\sqrt{2\sum_{j=1}^{n}\{x_j\sigma_{a_{ij}}^2 - \bar{w}_{ij}\sigma_{b_i}^2\}^2\left\{1+2\frac{E(a_{ij})^2x_j}{\{x_j\sigma_{a_{ij}}^2 - \bar{w}_{ij}\sigma_{b_i}^2\}}\right\}} \le 0.
\end{aligned}$$

Combine (3.25) and (3.27) in $(P5)$ to obtain

$$(P7)' \begin{cases} \min\left(\sum_{j=1}^{n} x_j\sigma^2_{c_j}\left\{1+\frac{E(c_j)^2}{\sigma^2_{c_j}}\right\}+F^{-1}_{\chi^2_n}(\beta)\sqrt{2\sum_{j=1}^{n} x_j^2(\sigma^2_{c_j})^2\left\{1+\frac{E(c_j)^2}{\sigma^2_{c_j}}\right\}}\right) \\ \text{subject to} \\ \sum_{j=1}^{n}\{x_j\sigma^2_{a_{ij}}-\bar{w}_{ij}\sigma^2_{b_i}\}\left\{1+\frac{E(a_{ij})^2x_j}{\{x_j\sigma^2_{a_{ij}}-\bar{w}_{ij}\sigma^2_{b_i}\}}\right\} \\ +F^{-1}_{\chi^2_n}(\alpha_i)\sqrt{2\sum_{j=1}^{n}\{x_j\sigma^2_{a_{ij}}-\bar{w}_{ij}\sigma^2_{b_i}\}^2}\left\{1+2\frac{E(a_{ij})^2x_j}{\{x_j\sigma^2_{a_{ij}}-\bar{w}_{ij}\sigma^2_{b_i}\}}\right\}\leq 0;\ i=1,\ldots,m \\ x_j\geq 0;\ j=1,\ldots,n \end{cases}$$

3.4.1 Example. *Consider the following Multiobjective Stochastic Linear Program:*

$$(P7)'' \begin{cases} \min\ \{(\bar{c}^1)^2\} \\ \min\ \{(\bar{c}^2)^2\} \\ \textit{subject to} \\ \sum_{j=1}^{n} h^2_{1j}\leq 0 \\ \sum_{j=1}^{n} h^2_{2j}\leq 0 \\ \sum_{j=1}^{n} h^2_{3j}\leq 0 \\ x\geq 0 \end{cases}$$

where

$$(\bar{c}^k)^2=\sum_{j=1}^{2}(c_j^k\upsilon_j)^2;\ k=1,2;\ \sum_{i=1}^{2}h^2_{ij}=\sum_{j=1}^{2}\left\{(a_{ij}\upsilon_j)^2-(b_i\bar{w}_{ij})^2\right\};\ i=1,\ldots,3$$

$$x_j=\sqrt{\upsilon_j};\ j=1,2;\bar{w}_{ij}\geq 0;\ j=1,2,\ i=1,\ldots,3\ \ \textit{and}\quad \sum_{j=1}^{n}\bar{w}_{ij}=1;\ i=1,\ldots,m$$

$$c_j^k;\ j=1,2;\ j=k;\ a_{ij};\ i=1,2,3;\ j=1,2,\ b_i;\ i=1,2,3$$

are independent normally distributed random variables, with the following parameters:

$c_1^1\sim N(20,5),\ c_2^1\sim N(40,10)$
$c_1^2\sim N(50,20),\ c_2^2\sim N(100,50)$
$a_{11}\sim N(10,6),\ a_{12}\sim N(5,4)$
$a_{21}\sim N(4,4),\ a_{22}\sim N(10,7)$
$a_{31}\sim N(1,2),\ a_{32}\sim N(1.5,3)$
$b_1\sim N(2500,500),\ b_2\sim N(2000,400),\ b_3\sim N(450,50).$

With these data, $(P7)''$ reads

$$
(P7)''' \left\{
\begin{array}{l}
\min\ f_1 = \ \Big(25x_1 \left\{1+\frac{400}{25}\right\} + 9.2103\sqrt{1250x_1^2\left\{1+\frac{400}{25}\right\}} \\
+100x_2\left\{1+\frac{1600}{100}\right\} + 9.2103\sqrt{20000x_2^2\left\{1+\frac{1600}{100}\right\}}\Big) \\
\min\ f_2 = \Big(400x_1\left\{1+\frac{2500}{400}\right\} + 9.2103\sqrt{320000x_1^2\left\{1+\frac{2500}{400}\right\}} \\
+2500x_2\left\{1+\frac{10000}{2500}\right\} + 9.2103\sqrt{12500000x_2^2\left\{1+\frac{10000}{2500}\right\}}\Big) \\
\textit{subject to} \\
16x_1 - 0.5(25000) + 100x_1\{16x_1 - 0.5(25000)\} \\
+9.2103\sqrt{2\{16x_1 - 0.5(25000)\}^2 + 216x_1\{16x_1 - 0.5(25000)\}} \\
+36x_2 - 0.5(25000) + 25x_2\{36x_2 - 0.5(25000)\} \\
+9.2103\sqrt{2\{36x_2 - 0.5(25000)\}^2 + 50x_2\{36x_2 - 0.5(25000)\}} \leq 0 \\
16x_1 - 0.5(160000) + 16x_1\{16x_1 - 0.5(160000)\} \\
+9.2103\sqrt{2\{16x_1 - 0.5(160000)\}^2 + 32x_1\{16x_1 - 0.5(160000)\}} \\
+49x_2 - 0.5(160000) + 100x_2\{49x_2 - 0.5(160000)\} \\
+9.2103\sqrt{2\{49x_2 - 0.5(160000)\}^2 + 200x_2\{49x_2 - .5(160000)\}} \leq 0 \\
4x_1 - 0.5(2500) + x_1\{4x_1 - 0.5(2500)\} \\
+9.2103\sqrt{2\{4x_1 - 0.5(2500)\}^2 + 2x_1\{4x_1 - 0.5(2500)\}} \\
+9x_2 - 0.5(2500) + 2.25x_2\{9x_2 - 0.5(2500)\} \\
+9.2103\sqrt{2\{9x_2 - 0.5(2500)\}^2 + 4.5x_2\{9x_2 - 0.5(2500)\}} \leq 0 \\
x_1, x_2 \geq 0
\end{array}
\right.
$$

If the Decision maker is interested in a satisficing solution of type 1, and if the objective functions are aggregated with weights $\lambda_1 = 0.4$ and $\lambda_2 = 0.6$ for instance, then by virtue of Proposition (3.3.2) the counterpart of $(P7)'''$ is given below

$$
(P7)^{iv}\begin{cases}
\min\ z = \ 0.4\left(25x_1\left\{1+\frac{400}{25}\right\}+9.2103\sqrt{1250x_1^2\left\{1+\frac{400}{25}\right\}}\right.\\
\left.+100x_2\left\{1+\frac{1600}{100}\right\}+9.2103\sqrt{20000x_2^2\left\{1+\frac{1600}{100}\right\}}\right)\\
+0.6\left(400x_1\left\{1+\frac{2500}{400}\right\}+9.2103\sqrt{320000x_1^2\left\{1+\frac{2500}{400}\right\}}\right.\\
\left.+2500x_2\left\{1+\frac{10000}{2500}\right\}+9.2103\sqrt{12500000x_2^2\left\{1+\frac{10000}{2500}\right\}}\right)\\
\textit{subject to}\\
16x_1-0.5(25000)+100x_1\{16x_1-0.5(25000)\}\\
+9.2103\sqrt{2\{16x_1-0.5(25000)\}^2+216x_1\{16x_1-0.5(25000)\}}\\
+36x_2-0.5(25000)+25x_2\{36x_2-0.5(25000)\}\\
+9.2103\sqrt{2\{36x_2-0.5(25000)\}^2+50x_2\{36x_2-0.5(25000)\}}\le 0\\
16x_1-0.5(160000)+16x_1\{16x_1-0.5(160000)\}\\
+9.2103\sqrt{2\{16x_1-0.5(160000)\}^2+32x_1\{16x_1-0.5(160000)\}}\\
+49x_2-0.5(160000)+100x_2\{49x_2-0.5(160000)\}\\
+9.2103\sqrt{2\{49x_2-0.5(160000)\}^2+200x_2\{49x_2-.5(160000)\}}\le 0\\
4x_1-0.5(2500)+x_1\{4x_1-0.5(2500)\}\\
+9.2103\sqrt{2\{4x_1-0.5(2500)\}^2+2x_1\{4x_1-0.5(2500)\}}\\
+9x_2-0.5(2500)+2.25x_2\{9x_2-0.5(2500)\}\\
+9.2103\sqrt{2\{9x_2-0.5(2500)\}^2+4.5x_2\{9x_2-0.5(2500)\}}\le 0\\
x_1,x_2\ge 0
\end{cases}
$$

Solving this optimization problem using LINGO 13.0 software we obtain optimal solution $z^* = 599033.3$ *at* $x_1^* = 0,\ x_2^* = 11.08983$ *after 324 iterations.*

3.4.2 Satisficing solution of type 2

3.4.3 Proposition. *Consider* $(P1)$*, suppose that*

$$c^k\ (k = 1,\ldots,K)$$

are such that

$$(c^k)^2 = \sum_{j=1}^{n}(\bar{c}_j{}^k)^2;\quad \textit{where}\ \sum_{j=1}^{n}(\bar{c}_j{}^k)^2 = \sum_{j=1}^{n}\left\{(c_j\upsilon_j)^k\right\}^2$$

$\upsilon_j = \sqrt{x_j};\quad j = 1,\ldots,n.$

Suppose also that

$$h_i^2 = \sum_{j=1}^{n} h_{ij}^2;\ i = 1,\ldots,m\ \textit{where}\ \ \sum_{j=1}^{n} h_{ij}^2 = \sum_{j=1}^{n}\left\{(a_{ij}\upsilon_j)^2-(b_i\bar{w}_{ij})^2\right\};\ i = 1,\ldots,m$$

where $(c^k)^2;\ k = 1,\dots,K$ *and* $h_i^2;\ i = 1,\dots,m$ *are mutually independent following a chi-squared distribution.*
Then a point $x^* \in \mathbb{R}^n$ *is an* (α,β)*-satisficing solution of type 2 for* $(P1)$ *if and only if* x^* *is efficient for the following optimization problem:*

$$(P7)^v \begin{cases} \min \Big(\sum_{j=1}^n Var(c_j^1) + Var(c_j^1)\kappa^2_{c_j^1} \\ +F^{-1}_{\chi^2_n}(\beta_1)\sqrt{2\sum_{j=1}^n Var(c_j^1)^2 + 2Var(c_j^1)^2\kappa^2_{c_j^1}}\Big) \\ \vdots \\ \min \ \Big(\sum_{j=1}^n Var(c_j^K) + Var(c_j^K)\kappa^2_{c_j^K} \\ +F^{-1}_{\chi^2_n}(\beta_K)\sqrt{2\sum_{j=1}^n Var(c_j^K)^2 + 2Var(c_j^K)^2\kappa^2_{c_j^K}}\Big) \\ \textit{subject to} \\ \sum_{j=1}^n Var(h_{ij}) + Var(h_{ij})\kappa^2_{h_{ij}} \\ +F^{-1}_{\chi^2_n}(\alpha_i)\sqrt{2\sum_{j=1}^n Var(h_{ij})^2 + 2Var(h_{ij})^2\kappa^2_{h_{ij}}} \le 0;\ i = 1,\dots,m \\ x_j \ge 0;\ j = 1,\dots,n \end{cases}$$

where
$\kappa_{c^k};\ k = 1,\dots,K;$ *and* $Var(c^k);\ k = 1,\dots,K$ *are as defined in relations* (3.9) *and* (3.10) *respectively,* $\alpha_i \in (0,1];\ (i = 1,\dots,m)$*;* $\beta_k \in (0,1];\ (k = 1,\dots,K)$ *are as in Definition* (3.2.2)*,* $\bar{w}_i;\ i = 1,\dots,m;\ \kappa_{h_i};\ i = 1,\dots,m;\ Var(h_i);\ i = 1,\dots,m;\ F_{\chi^2_n}$ $c_j^k; k = 1,\dots,K,\ j = 1,\dots,n;$ $a_{ij};\ j = 1,\dots,n,\ i = 1,\dots,m;$ *and* $b_i;\ i = 1,\dots,m$ *are as given in Proposition* (3.4.2)*.*

Proof. As shown in the proof of Proposition (3.4.2), the chance constraints

$$P(c^1x \le s_1) \ge \beta_1,\dots,P(c^Kx \le s_K) \ge \beta_K \tag{3.27}$$

in $(P5)'$ are equivalent to

$$\begin{cases} \sum_{j=1}^n Var(c_j^1) + Var(c_j^1)\kappa^2_{c_j^1} \\ +F^{-1}_{\chi^2_n}(\beta_1)\sqrt{2\sum_{j=1}^n Var(c_j^1)^2 + 2Var(c_j^1)^2\kappa^2_{c_j^1}} \le s_1 \\ \vdots \\ \sum_{j=1}^n Var(c_j^K) + Var(c_j^K)\kappa^2_{c_j^K} \\ +F^{-1}_{\chi^2_n}(\beta_K)\sqrt{2\sum_{j=1}^n Var(c_j^K)^2 + 2Var(c_j^K)^2\kappa^2_{c_j^K}} \le s_K \end{cases} \tag{3.28}$$

From the proof of Proposition (3.4.2)

$$h_i^2 = \sum_{j=1}^n \left\{(a_{ij}\upsilon_j)^2 - (b_i\bar{w}_{ij})^2\right\},\ i = 1,\dots,m$$

is equivalent to

$$\sum_{j=1}^{n} Var(h_{ij}) + Var(h_{ij})\kappa_{h_{ij}}^2 + F_{\chi_n^2}^{-1}(\alpha_i)\sqrt{2\sum_{j=1}^{n}(Var(h_{ij})^2 + 2Var(h_{ij})^2\kappa_{h_{ij}}^2} \le 0.$$

Therefore, $(P5)'$ can be written as

$$\begin{cases} \min\,(s_1,\ldots,s_K) \\ \text{subject to} \\ \sum_{j=1}^{n} Var(c_j^1) + Var(c_j^1)\kappa_{c_j^1}^2 \\ +F_{\chi_n^2}^{-1}(\beta_1)\sqrt{2\sum_{j=1}^{n} Var(c_j^1)^2 + 2Var(c_j^1)^2\kappa_{c_j^1}^2} \le s_1 \\ \vdots \\ \sum_{j=1}^{n} Var(c_j^K) + Var(c_j^K)\kappa_{c_j^K}^2 \\ +F_{\chi_n^2}^{-1}(\beta_K)\sqrt{2\sum_{j=1}^{n} Var(c_j^K)^2 + 2Var(c_j^K)^2\kappa_{c_j^K}^2} \le s_K \\ \sum_{j=1}^{n} Var(h_{ij}) + Var(h_{ij})\kappa_{h_{ij}}^2 \\ +F_{\chi_n^2}^{-1}(\alpha_i)\sqrt{2\sum_{j=1}^{n} Var(h_{ij})^2 + 2Var(h_{ij})^2\kappa_{h_{ij}}^2} \le 0 \\ x_j \ge 0;\ j=1,\ldots,n;\ s_k,\ u.s.r;\ k=1,\ldots,K \end{cases}$$

which is equivalent to $(P7)^v$. $\square$

To solve $(P7)^v$ we have to put it in a more tractable form

$$(P7)^{vi}\begin{cases} \min\left(\sum_{j=1}^{n} x_j\sigma_{c_j^1}^2\left\{1+\frac{E(c_j^1)^2}{\sigma_{c_j^1}^2}\right\} + F_{\chi_n^2}^{-1}(\beta_1)\sqrt{2\sum_{j=1}^{n} x_j^2(\sigma_{c_j^1}^2)^2\left\{1+\frac{E(c_j^1)^2}{\sigma_{c_j^1}^2}\right\}}\right) \\ \vdots \\ \min\left(\sum_{j=1}^{n} x_j\sigma_{c_j^K}^2\left\{1+\frac{E(c_j^K)^2}{\sigma_{c_j^K}^2}\right\} + F_{\chi_n^2}^{-1}(\beta_K)\sqrt{2\sum_{j=1}^{n} x_j^2(\sigma_{c_j^K}^2)^2\left\{1+\frac{E(c_j^K)^2}{\sigma_{c_j^K}^2}\right\}}\right) \\ \textit{subject to} \\ \sum_{j=1}^{n}\{x_j\sigma_{a_{ij}}^2 - \bar{w}_{ij}\sigma_{b_i}^2\}\left\{1+\frac{E(a_{ij})^2x_j}{\{x_j\sigma_{a_{ij}}^2-\bar{w}_{ij}\sigma_{b_i}^2\}}\right\} \\ +F_{\chi_n^2}^{-1}(\alpha_i)\sqrt{2\sum_{j=1}^{n}\{x_j\sigma_{a_{ij}}^2 - \bar{w}_{ij}\sigma_{b_i}^2\}^2\left\{1+2\frac{E(a_{ij})^2x_j}{\{x_j\sigma_{a_{ij}}^2-\bar{w}_{ij}\sigma_{b_i}^2\}}\right\}} \le 0, \quad i=1,\ldots,m, \\ x_j \ge 0,\ j=1,\ldots,n \end{cases}$$

3.4.2 Example. *Assume now that a satisficing solution of type 2 is sought. Then by Proposition* (3.3.3)*, one has to solve problem $(P7)'''$ of Example* (3.4.1)*. Using the compromise programming approach (see Subsection* (1.3.8)*), we have to solve the following subproblems:*

$$
\begin{cases}
\min\ f_1 = \ \Big(25x_1\left\{1+\frac{400}{25}\right\} + 9.2103\sqrt{1250x_1^2\left\{1+\frac{400}{25}\right\}} \\
+100x_2\left\{1+\frac{1600}{100}\right\} + 9.2103\sqrt{20000x_2^2\left\{1+\frac{1600}{100}\right\}}\Big) \\
\textit{subject to} \\
16x_1 - 0.5(25000) + 100x_1\{16x_1 - 0.5(25000)\} \\
+9.2103\sqrt{2\{16x_1 - 0.5(25000)\}^2 + 216x_1\{16x_1 - 0.5(25000)\}} \\
+36x_2 - 0.5(25000) + 25x_2\{36x_2 - 0.5(25000)\} \\
+9.2103\sqrt{2\{36x_2 - 0.5(25000)\}^2 + 50x_2\{36x_2 - 0.5(25000)\}} \le 0 \\
16x_1 - 0.5(160000) + 16x_1\{16x_1 - 0.5(160000)\} \\
+9.2103\sqrt{2\{16x_1 - 0.5(160000)\}^2 + 32x_1\{16x_1 - 0.5(160000)\}} \\
+49x_2 - 0.5(160000) + 100x_2\{49x_2 - 0.5(160000)\} \\
+9.2103\sqrt{2\{49x_2 - 0.5(160000)\}^2 + 200x_2\{49x_2 - .5(160000)\}} \le 0 \\
4x_1 - 0.5(2500) + x_1\{4x_1 - 0.5(2500)\} \\
+9.2103\sqrt{2\{4x_1 - 0.5(2500)\}^2 + 2x_1\{4x_1 - 0.5(2500)\}} \\
+9x_2 - 0.5(2500) + 2.25x_2\{9x_2 - 0.5(2500)\} \\
+9.2103\sqrt{2\{9x_2 - 0.5(2500)\}^2 + 4.5x_2\{9x_2 - 0.5(2500)\}} \le 0 \\
x_1, x_2 \ge 0
\end{cases}
$$

$$
\begin{cases}
\min\ f_2 = \Big(400x_1\left\{1+\frac{2500}{400}\right\} + 9.2103\sqrt{320000x_1^2\left\{1+\frac{2500}{400}\right\}} \\
+2500x_2\left\{1+\frac{10000}{2500}\right\} + 9.2103\sqrt{12500000x_2^2\left\{1+\frac{10000}{2500}\right\}}\Big) \\
\textit{subject to} \\
16x_1 - 0.5(25000) + 100x_1\{16x_1 - 0.5(25000)\} \\
+9.2103\sqrt{2\{16x_1 - 0.5(25000)\}^2 + 216x_1\{16x_1 - 0.5(25000)\}} \\
+36x_2 - 0.5(25000) + 25x_2\{36x_2 - 0.5(25000)\} \\
+9.2103\sqrt{2\{36x_2 - 0.5(25000)\}^2 + 50x_2\{36x_2 - 0.5(25000)\}} \le 0 \\
16x_1 - 0.5(160000) + 16x_1\{16x_1 - 0.5(160000)\} \\
+9.2103\sqrt{2\{16x_1 - 0.5(160000)\}^2 + 32x_1\{16x_1 - 0.5(160000)\}} \\
+49x_2 - 0.5(160000) + 100x_2\{49x_2 - 0.5(160000)\} \\
+9.2103\sqrt{2\{49x_2 - 0.5(160000)\}^2 + 200x_2\{49x_2 - .5(160000)\}} \le 0 \\
4x_1 - 0.5(2500) + x_1\{4x_1 - 0.5(2500)\} \\
+9.2103\sqrt{2\{4x_1 - 0.5(2500)\}^2 + 2x_1\{4x_1 - 0.5(2500)\}} \\
+9x_2 - 0.5(2500) + 2.25x_2\{9x_2 - 0.5(2500)\} \\
+9.2103\sqrt{2\{9x_2 - 0.5(2500)\}^2 + 4.5x_2\{9x_2 - 0.5(2500)\}} \le 0 \\
x_1, x_2 \ge 0
\end{cases}
$$

	f_1^*= 78410.41	f_2^*= 946115.3	z^*= -0.6822355E-02
x_1^*	0	0	0
x_2^*	11.08983	11.08983	11.08983

Table 3.2: Compromise solution for Program $(P7)'''$

$$
(P7)^{vii}\left\{\begin{array}{l}
\min\ z = 0.5\Big(25x_1\left\{1+\frac{400}{25}\right\}+9.2103\sqrt{1250x_1^2\left\{1+\frac{400}{25}\right\}}\\
+100x_2\left\{1+\frac{1600}{100}\right\}+9.2103\sqrt{20000x_2^2\left\{1+\frac{1600}{100}\right\}}-f^*1\Big)\\
+0.5\Big(400x_1\left\{1+\frac{2500}{400}\right\}+9.2103\sqrt{320000x_1^2\left\{1+\frac{2500}{400}\right\}}\\
+2500x_2\left\{1+\frac{10000}{2500}\right\}+9.2103\sqrt{12500000x_2^2\left\{1+\frac{10000}{2500}\right\}}-f_2^*\Big)\\
\textit{subject to}\\
16x_1-0.5(25000)+100x_1\{16x_1-0.5(25000)\}\\
+9.2103\sqrt{2\{16x_1-0.5(25000)\}^2+216x_1\{16x_1-0.5(25000)\}}\\
+36x_2-0.5(25000)+25x_2\{36x_2-0.5(25000)\}\\
+9.2103\sqrt{2\{36x_2-0.5(25000)\}^2+50x_2\{36x_2-0.5(25000)\}}\le 0\\
16x_1-0.5(160000)+16x_1\{16x_1-0.5(160000)\}\\
+9.2103\sqrt{2\{16x_1-0.5(160000)\}^2+32x_1\{16x_1-0.5(160000)\}}\\
+49x_2-0.5(160000)+100x_2\{49x_2-0.5(160000)\}\\
+9.2103\sqrt{2\{49x_2-0.5(160000)\}^2+200x_2\{49x_2-.5(160000)\}}\le 0\\
4x_1-0.5(2500)+x_1\{4x_1-0.5(2500)\}\\
+9.2103\sqrt{2\{4x_1-0.5(2500)\}^2+2x_1\{4x_1-0.5(2500)\}}\\
+9x_2-0.5(2500)+2.25x_2\{9x_2-0.5(2500)\}\\
+9.2103\sqrt{2\{9x_2-0.5(2500)\}^2+4.5x_2\{9x_2-0.5(2500)\}}\le 0\\
x_1,x_2\ge 0
\end{array}\right.
$$

using LINGO 13.0 software to solve the the above problems, we obtain the solutions in the table above.

Finally, a characterization of an $(\alpha,\beta)-$satisficing solution of type 3 is given below.

3.4.3 Satisficing solution of type 3

3.4.4 Proposition. *Consider $(P1)$, suppose that for $k=1,\ldots,K,\ \ c^k$ is such that*

$$(c^k)^2=\sum_{j=1}^{n}(\bar{c_j}^k)^2 \quad \textit{where} \quad \sum_{j=1}^{n}(\bar{c_j}^k)^2=\sum_{j=1}^{n}\left\{(c_j\upsilon_j)^k\right\}^2$$

$\upsilon_j = \sqrt{x_j}; \quad j = 1, \ldots, n$

Suppose also that for $i = 1, \ldots, m$

$$h_i^2 = \sum_{j=1}^{n} h_{ij}^2 \text{ where } \sum_{j=1}^{n} h_{ij}^2 = \sum_{j=1}^{n} \left\{ (a_{ij}\upsilon_j)^2 - (b_i \bar{w}_{ij})^2 \right\}$$

suppose further that $(c^k)^2;\ k = 1, \ldots, K$ *and* $h_i^2;\ i = 1, \ldots, m$ *are mutually independent and follow chi-squared distributions.*
Then a point $x^* \in \mathbb{R}^n$ *is an* (α, β)*-satisficing solution of type 3 for* $(P1)$ *if and only if there exists* $\delta^* \in \mathbb{R}^K$ *such that* (x^*, δ^*) *is optimal for the following mathematical programming problem*

$$(P7)^{viii} \begin{cases} \min \left(\sum_{k=1}^{K} \nu_k \delta_k \right) \\ \textit{subject to} \\ \sum_{j=1}^{n} Var(c_j^1) + Var(c_j^1)\kappa_{c_j^1}^2 + F_{\chi_n^2}^{-1}(\beta_1)\sqrt{2\sum_{j=1}^{n} Var(c_j^1)^2} \\ +2Var(c_j^1)^2\kappa_{c_j^1}^2 \leq t_1 + \delta_1, \\ \vdots \\ \sum_{j=1}^{n} Var(c_j^K) + Var(c_j^K)\kappa_{c_j^K}^2 + F_{\chi_n^2}^{-1}(\beta_K)\sqrt{2\sum_{j=1}^{n} Var(c_j^K)^2} \\ +2Var(c_j^K)^2\kappa_{c_j^K}^2 \leq t_K + \delta_K \\ \sum_{j=1}^{n} Var(h_{ij}) + Var(h_{ij})\kappa_{h_{ij}}^2 \\ +F_{\chi_n^2}^{-1}(\alpha_i)\sqrt{2\sum_{j=1}^{n} Var(h_{ij})^2 + 2Var(h_{ij})^2\kappa_{h_{ij}}^2} \leq 0 \\ x_j \geq 0;\ j = 1, \ldots, n;\ \delta_k \geq 0;\ k = 1, \ldots, K \end{cases}$$

where $\delta_k;\ (k = 1, \ldots, K),\ \ \nu_k;\ (k = 1, \ldots, K),\ \ t_k;\ (k = 1, \ldots, K);$
$\kappa_{c^k};\ k = 1, \ldots, K,$ *and* $Var(c^k);\ k = 1, \ldots, K$ *are as defined in relations* (3.9) *and* (3.10) *of Lemma* (3.4.1) *respectively,* $\alpha_i \in (0, 1];\ (i = 1, \ldots, m)$, $\beta_k \in (0, 1];\ (k = 1, \ldots, K)$ *are as in Definition* (3.2.2), $\bar{w}_i;\ i = 1, \ldots, m,\ \kappa_{h_i};\ i = 1, \ldots, m,\ Var(h_i);\ i = 1, \ldots, m,\ F_{\chi_n^2}$ $c_j^k; k = 1, \ldots, K,\ j = 1, \ldots, n,\ a_{ij};\ j = 1, \ldots, n,\ i = 1, \ldots, m$ *and* $b_i;\ i = 1, \ldots, m$ *are as defined in Proposition* (3.4.3).

Proof. Consider

$$(P5)'' \begin{cases} \min \left(\sum_{k=1}^{K} \nu_k \delta_k \right) \\ \text{subject to} \\ \mathbb{P}\left(c^k x \leq t_k + \delta_k \right) \geq \beta_k;\ k = 1, \ldots, K \\ \mathbb{P}\left(\sum_{j=1}^{n} a_{ij} x_j - b_i \leq 0 \right) \geq \alpha_i; i = 1, \ldots, m \\ x \geq 0, \delta_k \geq 0;\ k = 1, \ldots, K \end{cases}$$

As discussed in the proof of Proposition (3.4.3), the chance constraints

$$\mathbb{P}\big(c^k x \leq t_k + \delta_k\big) \geq \beta_k;\ k = 1, \ldots, K \tag{3.29}$$

are equivalent to

$$\begin{aligned}
&\sum_{j=1}^{n} Var(c_j^k) + Var(c_j^k)\kappa^2_{c_j^k} + \\
&F^{-1}_{\chi^2_n}(\beta_k)\sqrt{2\sum_{j=1}^{n} Var(c_j^k)^2 + 2Var(c_j^k)^2\kappa^2_{c_j^k}} \leq t_k + \delta_k;\ \ k = 1, \ldots, K.
\end{aligned} \tag{3.30}$$

From the proof of Proposition (3.4.2), we know that

$$h_i^2 = \sum_{j=1}^{n} \left\{(a_{ij}\upsilon_j)^2 - (b_i\bar{w}_{ij})^2\right\},\ i = 1, \ldots, m$$

can be transformed as

$$\begin{aligned}
&\sum_{j=1}^{n} Var(h_{ij}) + Var(h_{ij})\kappa^2_{h_{ij}} \\
&+F^{-1}_{\chi^2_n}(\alpha_i)\sqrt{2\sum_{j=1}^{n} Var{h_{ij}}^2 + 2Var{h_{ij}}^2\kappa^2_{h_{ij}}} \leq 0;\ \ i = 1, \ldots, m
\end{aligned} \tag{3.31}$$

Substitute (3.30) and (3.31) in $(P5)''$ we obtain $(P7)^{viii}$.

$\square$

To enable us solve $(P7)^{viii}$ we need to put it in a more tractable form

$$(P7)^{ix}\left\{\begin{array}{l}
\min\ \left(\sum_{1=k}^{K} \nu_k\delta_k\right) \\
\textit{subject to} \\
\sum_{j=1}^{n} x_j\sigma^2_{c_j^1}\left\{1 + \frac{E(c_j^1)^2}{\sigma^2_{c_j^1}}\right\} + F^{-1}_{\chi^2_n}(\beta_1)\sqrt{2\sum_{j=1}^{n} x_j^2(\sigma^2_{c_j^1})^2\left\{1 + \frac{E(c_j^1)^2}{\sigma^2_{c_j^1}}\right\}} \leq t_1 + \delta_1 \\
\vdots \\
\sum_{j=1}^{n} x_j\sigma^2_{c_j^K}\left\{1 + \frac{E(c_j^K)^2}{\sigma^2_{c_j^K}}\right\} + F^{-1}_{\chi^2_n}(\beta_K)\sqrt{2\sum_{j=1}^{n} x_j^2(\sigma^2_{c_j^K})^2\left\{1 + \frac{E(c_j^K)^2}{\sigma^2_{c_j^K}}\right\}} \leq t_K + \delta_K \\
\sum_{j=1}^{n}\{x_j\sigma^2_{a_{ij}} - \bar{w}_{ij}\sigma^2_{b_i}\}\left\{1 + \frac{E(a_{ij})^2 x_j}{\{x_j\sigma^2_{a_{ij}} - \bar{w}_{ij}\sigma^2_{b_i}\}}\right\} \\
+F^{-1}_{\chi^2_n}(\alpha_i)\sqrt{2\sum_{j=1}^{n}\{x_j\sigma^2_{a_{ij}} - \bar{w}_{ij}\sigma^2_{b_i}\}^2\left\{1 + 2\frac{E(a_{ij})^2 x_j}{\{x_j\sigma^2_{a_{ij}} - \bar{w}_{ij}\sigma^2_{b_i}\}}\right\}} \leq 0, \quad i = 1, \ldots, m, \\
x_j \geq 0,\ \ j = 1, \ldots, n, \delta_k \geq 0; k = 1, \ldots, K
\end{array}\right.$$

.

3.4.3 Example. *Suppose that a satisficing solution of type 3 of Program $(P7)'''$ is desirable by the DM whose goals are 2,000 and 15,000 for objectives 1 and 2 respectively. Using equal weights (i.e $\nu_1 = \nu_2 = 0.5$) for the two objectives, we have to solve the following problem:*

$$
(P7)^x \left\{
\begin{array}{l}
\min \ (0.5\delta_1 + 0.5\delta_2) \\
\textit{subject to} \\
25x_1 \left\{1 + \frac{400}{25}\right\} + 9.2103\sqrt{1250x_1^2 \left\{1 + \frac{400}{25}\right\}} \\
+100x_2 \left\{1 + \frac{1600}{100}\right\} + 9.2103\sqrt{20000x_2^2 \left\{1 + \frac{1600}{100}\right\}} \leq 2000 + \delta_1 \\
400x_1 \left\{1 + \frac{2500}{400}\right\} + 9.2103\sqrt{320000x_1^2 \left\{1 + \frac{2500}{400}\right\}} \\
+2500x_2 \left\{1 + \frac{10000}{2500}\right\} + 9.2103\sqrt{12500000x_2^2 \left\{1 + \frac{10000}{2500}\right\}} \geq 15000 - \delta_2 \\
16x_1 - 0.5(25000) + 100x_1\{16x_1 - 0.5(25000)\} \\
+9.2103\sqrt{2\{16x_1 - 0.5(25000)\}^2 + 216x_1\{16x_1 - 0.5(25000)\}} \leq 0 \\
+36x_2 - 0.5(25000) + 25x_2\{36x_2 - 0.5(25000)\} \\
+9.2103\sqrt{2\{36x_2 - 0.5(25000)\}^2 + 50x_2\{36x_2 - 0.5(25000)\}} \leq 0 \\
\{16x_1 - 0.5(160000) + 16x_1\{16x_1 - 0.5(160000)\} \\
+9.2103\sqrt{2\{16x_1 - 0.5(160000)\}^2 + 32x_1\{16x_1 - 0.5(160000)\}} \\
+\{49x_2 - 0.5(160000) + 100x_2\{49x_2 - 0.5(160000)\} \\
+9.2103\sqrt{2\{49x_2 - 0.5(160000)\}^2 + 200x_2\{49x_2 - .5(160000)\}} \leq 0 \\
4x_1 - 0.5(2500) + x_1\{4x_1 - 0.5(2500)\} \\
+9.2103\sqrt{2\{4x_1 - 0.5(2500)\}^2 + 2x_1\{4x_1 - 0.5(2500)\}} \\
+9x_2 - 0.5(2500) + 2.25x_2\{9x_2 - 0.5(2500)\} \\
+9.2103\sqrt{2\{9x_2 - 0.5(2500)\}^2 + 4.5x_2\{9x_2 - 0.5(2500)\}} \leq 0 \\
x_1, x_2 \geq 0, \delta_1, \ \delta_2 \geq 0
\end{array}
\right.
$$

Using LINGO 13.0 software to solve $(P7)^x$, we obtain the following solution after 355 iterations: optimal solution $z^* = 38205.20$ at $x_1 = 0.000$, $x_2 = 11.08983$, $\delta_1 = 76410.41$, $\delta_2 = 0$.

3.5 Characterization of satisficing solutions for the Exponential case

The exponential distribution provides another suitable continuous distribution defined in a nonnegative domain, which could be used for a wide class of economic models involving nonnegative prices, input coefficients and resource vectors.

Major advantages of exponential distribution are listed below.

1. It is relatively easy to analyze because it does not deteriorate with time and is often a good approximation to the actual distribution.
2. It is a memoryless distribution and the unique distribution that possess this interesting property.
3. The convolution of exponential random variables produce an exponential variate.
4. With exponential distribution it is possible to determine the probability that one exponential variable is smaller than another.

3.5.1 Lemma. *Assume Y is an exponential random variable, b is a constant and $\alpha \in (0,1]$ then*

$$\mathbb{P}\{Y \leq b\} \geq \alpha$$

is equivalent to

$$\prod_{j=1}^{n} \mu_j \left(\sum_{q=1}^{n} \frac{x_q^{n-1} e^{-\frac{\mu_q b}{x_q}}}{\mu_j \prod_{\{l=1, l\neq q\}}^{n} (x_q \mu_l - x_l \mu_q)} \right) \leq 1-\alpha \qquad \text{if } \; b \geq 0.$$

This result was proved for 2 variables by Goicoechea *et al* [62] and Biswal *et al* [27] generalized it.

3.5.1 Satisficing solution of type 1

3.5.2 Proposition. *Suppose that*

$$c_j^k; \;\; k = 1, \ldots, K; \;\; j = 1, \ldots, n$$

are independent exponentially distributed random variables with parameters μ_j^k. Assume that the objective functions of $(P1)$ can be aggregated as the following weighted sum:

$$cx = \lambda_1 c^1 x + \cdots + \lambda_K c^K x$$

where $\lambda_1, \ldots, \lambda_K \geq 0$, $\sum_{k=1}^{K} \lambda_k = 1$. *Suppose also that for any* $i = 1, \ldots, m$; $a_{ij}; j = 1, \ldots, n$ *are independent exponentially distributed with known parameters* μ_{ij} *and assume that* b_i; $i = 1, \ldots, m$ *are positive constants. Then a point* $x^* \in \mathbb{R}^n$ *is an* (α, β)*-satisficing solution of type 1 for* $(P1)$ *if and only if* x^* *is optimal for the following optimization problem:*

$$(P8)\begin{cases} \min\ s \\ \textit{subject to} \\ \sum_{k=1}^{K} \left\{ \prod_{j=1}^{n} \mu_{c_j^k} \left(\sum_{p=1}^{n} \frac{\lambda_k x_p^{n-1} e^{-\frac{\mu_{c_p^k} s_k}{\lambda_k x_p}}}{\mu_{c_j^k} \prod_{\{l=1, l\neq p\}}^{n} (\lambda_k x_p \mu_{c_l^k} - \lambda_k x_l \mu_{c_p^k})} \right) \right\} \leq 1-\beta \\ \prod_{j=1}^{n} \mu_{a_{1j}} \left(\sum_{p=1}^{n} \frac{x_p^{n-1} e^{-\frac{\mu_{a_{1p}} b_1}{x_p}}}{\mu_{a_{1j}} \prod_{\{l=1, l\neq p\}}^{n} (x_p \mu_{a_{1l}} - x_l \mu_{a_{1p}})} \right) \leq 1-\alpha_1 \\ \vdots \\ \prod_{j=1}^{n} \mu_{a_{mj}} \left(\sum_{p=1}^{n} \frac{x_p^{n-1} e^{-\frac{\mu_{a_{mp}} b_m}{x_p}}}{\mu_{a_{mj}} \prod_{\{l=1, l\neq p\}}^{n} (x_p \mu_{a_{ml}} - x_l \mu_{a_{mp}})} \right) \leq 1-\alpha_m \\ x \geq 0,\ s;\ u.s.r \end{cases}$$

where $\alpha_1, \ldots, \alpha_m \in (0,1]$, $\beta \in (0,1]$.

Proof. Recall that if (x^*, s^*) is optimal for the following optimization problem:

$$(P5)\begin{cases} \min\ s \\ \text{subject to} \\ \mathbb{P}\left(cx \leq s\right) \geq \beta \\ \mathbb{P}\left(\sum_{j=1}^{n} a_{ij} x_j - b_i \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m \\ x \geq 0;\ s,\ u.r.s \end{cases}$$

then x^* is the type 1 solution for $(P1)$.
As

$$cx = \lambda_1 c^1 x + \ldots + \lambda_K c^K x$$

we have by Lemma (3.5.1)

$$\mathbb{P}(cx \leq s) \geq \beta$$

is equivalent to

$$\sum_{k=1}^{K} \left\{ \prod_{j=1}^{n} \mu_{c_j^k} \left(\sum_{p=1}^{n} \frac{\lambda_k x_p^{n-1} e^{-\frac{\mu_{c_p^k} s_k}{\lambda_k x_p}}}{\mu_{c_j^k} \prod_{\{l=1, l\neq p\}}^{n} (\lambda_k x_p \mu_{c_l^k} - \lambda_k x_l \mu_{c_p^k})} \right) \right\} \leq 1-\beta. \qquad (3.32)$$

We have also that

$$\mathbb{P}\left(\sum_{j=1}^{n} a_{ij}x_j \leq b_i\right) \geq \alpha_i;\ \ i = 1, \ldots, m$$

are equivalent to

$$\prod_{j=1}^{n} \mu_{a_{1j}} \left(\sum_{p=1}^{n} \frac{x_p^{n-1} e^{-\frac{\mu_{a_{1p}} b_1}{x_p}}}{\mu_{a_{1j}} \prod_{\{l=1,l\neq p\}}^{n} (x_p\mu_{a_{1l}} - x_l\mu_{a_{1p}})}\right) \leq 1 - \alpha_1$$

to

(3.34)

$$\vdots$$

$$\prod_{j=1}^{n} \mu_{a_{mj}} \left(\sum_{p=1}^{n} \frac{x_p^{n-1} e^{-\frac{\mu_{a_{mp}} b_m}{x_p}}}{\mu_{a_{mj}} \prod_{\{l=1,l\neq p\}}^{n} (x_p\mu_{a_{ml}} - x_l\mu_{a_{mp}})}\right) \leq 1 - \alpha_m \tag{3.35}$$

Substitute (3.32) and (3.33) in $(P5)$ gives $(P8)$. □

3.5.1 Example. *Consider the following optimization problem, for illustration purposes:*

$$(P8)'' \begin{cases} \min\ (c_1^1x_1 + c_2^1x_2 + c_3^1x_3) \\ \min\ (c_1^2x_1 + c_2^2x_2 + c_3^2x_3) \\ \textit{subject to} \\ a_{11}x_1 + a_{12}x_2 + a_{13}x_3 \leq b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 \leq b_2 \\ x_1, x_2, x_3 \geq 0. \end{cases}$$

where

$$c_i^k, i = 1, 2; i = k;\ \ a_{ij}, i = 1, 2; j = 1, 2, 3$$

are independent exponentially distributed random variables, with the following means:
$E(c_1^1(\omega)) = 5,\ E(c_2^1(\omega)) = 6,\ E(c_3^1(\omega)) = 3$
$E(c_1^2(\omega)) = 10,\ E(c_2^2(\omega)) = 4,\ E(c_3^2(\omega)) = 7$
$E(a_{11}(\omega)) = 5,\ E(a_{12}(\omega)) = 4,\ E(a_{12}(\omega)) = 8$
$E(a_{21}(\omega)) = 10\ E(a_{22}(\omega)) = 2,\ E(a_{23}(\omega)) = 20$
$b_1 = 10,\ b_2 = 20.$

If the Decision maker is interested in a satisficing solution of type 1 with
$\alpha_1 = \frac{95}{100}, \alpha_2 = \frac{90}{100},$ *and* $\beta = \frac{90}{100},\ \lambda_1 = \frac{4}{10}$ *and* $\lambda_2 = \frac{6}{10}.$

Then the following program should be solved:

$$\begin{cases} \min\ s \\ \textit{subject to} \\ \Big(\frac{1}{5}\frac{1}{6}\frac{1}{3}\Big\{\frac{0.4x_1^2e^{-5\frac{s}{0.4x_1}}}{(0.4x_1\frac{1}{6}-0.4x_2\frac{1}{5})(0.4x_1\frac{1}{3}-0.4x_3\frac{1}{5})}+\frac{0.4x_2^2e^{-6\frac{s}{0.4x_2}}}{\frac{1}{6}(0.4x_2\frac{1}{5}-0.4x_1\frac{1}{6})(0.4x_2\frac{1}{3}-0.4x_3\frac{1}{6})} \\ +\frac{0.4x_3^2e^{-3\frac{s}{0.4x_3}}}{\frac{1}{3}(0.4x_3\frac{1}{5}-0.4x_1\frac{1}{3})(0.4x_3\frac{1}{6}-0.4x_2\frac{1}{3})}\Big\}+ \\ \frac{1}{10}\frac{1}{4}\frac{1}{2}\Big\{\frac{0.6x_1^2e^{-10\frac{s}{0.6x_1}}}{\frac{1}{10}(0.6x_1\frac{1}{4}-0.6x_2\frac{1}{10})(0.6x_1\frac{1}{2}-0.6x_3\frac{1}{10})}+\frac{0.6x_2^2e^{-4\frac{s}{0.6x_2}}}{\frac{1}{4}(0.6x_2\frac{1}{10}-0.6x_1\frac{1}{4})(0.6x_2\frac{1}{2}-0.6x_3\frac{1}{4})} \\ +\ \frac{0.6x_3^2e^{-2\frac{s}{0.6x_3}}}{\frac{1}{2}(0.6x_3\frac{1}{10}-0.6x_1\frac{1}{2})(0.6x_3\frac{1}{4}-0.6x_2\frac{1}{2})}\Big\}\Big)\leq\frac{10}{100} \\ \frac{1}{5}\frac{1}{4}\frac{1}{8}\Big\{\frac{x_1^2e^{-\frac{10}{5}x_1}}{\frac{1}{5}(x_1\frac{1}{4}-x_2\frac{1}{5})(x_1\frac{1}{8}-x_3\frac{1}{5})}+\frac{x_2^2e^{-\frac{10}{4}x_2}}{\frac{1}{4}(x_2\frac{1}{5}-x_1\frac{1}{4})(x_2\frac{1}{8}-x_3\frac{1}{4})}+\frac{x_3^2e^{-\frac{10}{8}x_3}}{\frac{1}{8}(x_3\frac{1}{5}-x_1\frac{1}{8})(x_3\frac{1}{4}-x_2\frac{1}{8})}\Big\}\leq\frac{5}{100} \\ \frac{1}{10}\frac{1}{2}\frac{1}{20}\Big\{\frac{x_1^2e^{-\frac{20}{10}x_1}}{\frac{1}{10}(x_1\frac{1}{2}-x_2\frac{1}{10})(x_1\frac{1}{20}-x_3\frac{1}{10})}+\frac{x_2^2e^{-\frac{20}{2}x_2}}{\frac{1}{2}(x_2\frac{1}{10}-x_1\frac{1}{2})(x_2\frac{1}{20}-x_3\frac{1}{2})}+\frac{x_3^2e^{-\frac{20}{20}x_3}}{\frac{1}{20}(x_3\frac{1}{10}-x_1\frac{1}{20})(x_3\frac{1}{2}-x_2\frac{1}{20})}\Big\}\leq\frac{10}{100} \\ x\geq 0,\ \ s,\ \ u.s.r. \end{cases}$$

This program can be written as

$$(P8)'''\begin{cases} \min\ s \\ \textit{subject to} \\ \Big(\frac{1}{90}\Big\{\frac{5(0.4)x_1^2e^{-\frac{5s}{(0.4)x_1}}}{(\frac{(0.4)x_1}{6}-\frac{(0.4)x_2}{5})(\frac{(0.4)x_1}{3}-\frac{(0.4)x_3}{5})}+\frac{6(0.4)x_2^2e^{-\frac{6s}{(0.4)x_2}}}{(\frac{(0.4)x_2}{5}-\frac{(0.4)x_1}{6})(\frac{(0.4)x_2}{3}-\frac{(0.4)x_3}{6})} \\ +\frac{3(0.4)x_3^2e^{-\frac{3s}{(0.4)x_3}}}{(\frac{(0.4)x_3}{5}-\frac{(0.4)x_1}{3})(\frac{(0.4)x_3}{6}-\frac{(0.4)x_2}{3})}\Big\}+ \\ \frac{1}{80}\Big\{\frac{10(0.6)x_1^2e^{-\frac{10s}{(0.6)x_1}}}{(\frac{(0.6)x_1}{4}-\frac{(0.6)x_2}{10})(\frac{(0.6)x_1}{7}-\frac{(0.6)x_3}{10})}+\frac{4(0.6)x_2^2e^{-\frac{4s}{(0.6)x_2}}}{(\frac{(0.6)x_2}{10}-\frac{(0.6)x_1}{4})(\frac{(0.6)x_2}{7}-\frac{(0.6)x_3}{4})}+ \\ +\ \frac{7(0.6)x_3^2e^{-\frac{7s}{(0.6)x_3}}}{(\frac{(0.6)x_3}{10}-\frac{(0.6)x_1}{7})(\frac{(0.6)x_3}{4}-\frac{(0.6)x_2}{7})}\Big\}\Big)\leq\frac{10}{100} \\ \frac{1}{160}\Big\{\frac{5x_1^2e^{\frac{-2}{x_1}}}{(\frac{x_1}{4}-\frac{x_2}{5})(\frac{x_1}{8}-\frac{x_3}{5})}+\frac{4x_2^2e^{-\frac{5}{2x_2}}}{(\frac{x_2}{5}-\frac{x_1}{4})(\frac{x_2}{8}-\frac{x_3}{4})}+\frac{8x_3^2e^{-\frac{5}{4x_3}}}{(\frac{x_3}{5}-\frac{x_1}{8})(\frac{x_3}{4}-\frac{x_2}{8})}\Big\}\leq\frac{5}{100} \\ \frac{1}{140}\Big\{\frac{10x_1^2e^{-\frac{2}{x_1}}}{(\frac{x_1}{2}-\frac{x_2}{10})(\frac{x_1}{20}-\frac{x_3}{10})}+\frac{2x_2^2e^{-\frac{10}{x_2}}}{(\frac{x_2}{10}-\frac{x_1}{2})(\frac{x_2}{20}-\frac{x_3}{2})}+\frac{20x_3^2e^{-\frac{1}{x_3}}}{(\frac{x_3}{10}-\frac{x_1}{20})(\frac{x_3}{2}-\frac{x_2}{20})}\Big\}\leq\frac{10}{100} \\ x\geq 0,\ \ s,\ \ u.s.r. \end{cases}$$

Solving this optimization problem using LINGO 13.0 software, at the 61^{th} iteration, we obtain the following solution $s^* = 0, x_1^* = 0.2185930,\ x_2^* = 0.2085628,\ x_3^* = 0.2486902$.

3.5.2 Satisficing solution of type 2

3.5.3 Proposition. *Consider $(P1)$, suppose that*

$$c_j^k;\ k = 1, \ldots, K,\ j = 1, \ldots, n$$

are independent exponentially distributed with parameters μ_j^k. Suppose also that for any $i = 1, \ldots, m$; $a_{ij}, j = 1, \ldots, n$ are independent exponentially distributed random variables with known parameters μ_{ij} and assume that b_i; $i = 1, \ldots, m$ are positive constants.

Then a point $x^* \in \mathbb{R}^n$ is an (α, β)-satisficing solution of type 2 for $(P1)$ if and only if x^* is efficient for the following optimization problem:

$$(P8)^{iv} \begin{cases} \min\ (s_1, \ldots, s_K) \\ \textit{subject to} \\ \prod_{j=1}^n \mu_{c_j^1} \left(\sum_{p=1}^n \frac{x_p^{n-1} e^{-\frac{\mu_{c_p^1} s_1}{x_p}}}{\mu_{c_j^1} \prod_{\{l=1, l\neq p\}}^n (x_p \mu_{c_l^1} - x_l \mu_{c_p^1})} \right) \leq 1 - \beta_1 \\ \vdots \\ \prod_{j=1}^n \mu_{c_j^K} \left(\sum_{p=1}^n \frac{x_p^{n-1} e^{-\frac{\mu_{c_p^K} s_k}{x_p}}}{\mu_{c_j^K} \prod_{\{l=1, l\neq p\}}^n (x_p \mu_{c_l^k} - x_l \mu_{c_p^K})} \right) \leq 1 - \beta_K \\ \prod_{j=1}^n \mu_{a_{1j}} \left(\sum_{p=1}^n \frac{x_p^{n-1} e^{-\frac{\mu_{a_{1p}} b_1}{x_p}}}{\mu_{a_{1j}} \prod_{\{l=1, l\neq p\}}^n (x_p \mu_{a_{1l}} - x_l \mu_{a_{1p}})} \right) \leq 1 - \alpha_1 \\ \vdots \\ \prod_{j=1}^n \mu_{a_{mj}} \left(\sum_{p=1}^n \frac{x_p^{n-1} e^{-\frac{\mu_{a_{mp}} b_m}{x_p}}}{\mu_{a_{mj}} \prod_{\{l=1, l\neq p\}}^n (x_p \mu_{a_{ml}} - x_l \mu_{a_{mp}})} \right) \leq 1 - \alpha_m \\ x \geq 0,\ s_k,\ u.s.r.;\quad k = 1, \ldots, K \end{cases}$$

where $\alpha_1, \ldots, \alpha_m \in (0, 1]$, $\beta_1, \ldots, \beta_K \in (0, 1]$.

Proof. We know that if (x^*, s^*), is efficient for the following multiobjective optimization problem:

$$(P5)' \begin{cases} \min\ (s_1, \ldots, s_K) \\ \text{subject to} \\ \mathbb{P}\left(c^1(\omega)x \leq s_1\right) \geq \beta_1 \\ \vdots \\ \mathbb{P}\left(c^K x \leq s_K\right) \geq \beta_K \\ \mathbb{P}\left(\sum_{j=1}^n a_{ij}x_j - b_i \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m \\ x \geq 0;\ s_k,\ u.r.s.;\ k = 1, \ldots, K \end{cases}$$

then $x^* \in \mathbb{R}^n$ is an $(\alpha, \beta)-$satisficing solution of type 2 for $(P1)$.

By Lemma (3.5.1) the following constraints:

$$\mathbb{P}\left(c^1(\omega)x \leq s_1\right) \geq \beta_1$$
$$\vdots$$
$$\mathbb{P}\left(c^K x \leq s_K\right) \geq \beta_K$$

are equivalent to

$$\begin{aligned}
\prod_{j=1}^{n}\mu_{c_j^1}\left(\sum_{p=1}^{n}\frac{x_p^{n-1}e^{-\frac{\mu_{c_p^1}s_1}{x_p}}}{\mu_{c_j^1}\prod_{\{l=1,l\neq p\}}^{n}(x_p\mu_{c_l^1}-x_l\mu_{c_p^1})}\right) &\leq 1-\beta_1 \\
&\vdots \\
\prod_{j=1}^{n}\mu_{c_j^K}\left(\sum_{p=1}^{n}\frac{x_p^{n-1}e^{-\frac{\mu_{c_p^K}s_k}{x_p}}}{\mu_{c_j^K}\prod_{\{l=1,l\neq p\}}^{n}(x_p\mu_{c_l^k}-x_l\mu_{c_p^K})}\right) &\leq 1-\beta_K
\end{aligned} \tag{3.36}$$

For constraints

$$\mathbb{P}\left(\sum_{j=1}^{n}a_{ij}x_j-b_i\leq 0\right)\geq\alpha_i;\ \ i=1,\ldots,m$$

is equivalent to the last technological constraints of $(P8)^{iv}$ and we are done. □

We finally consider the type 3 satisficing solution.

3.5.3 Satisficing solution of type 3

3.5.4 Proposition. *Consider $(P1)$, suppose that the random variables*

$$c_j^k;\ \ k=1,\ldots,K;\ \ j=1,\ldots,n$$

are independent exponentially distributed with parameters μ_j^k. Suppose also that for any $i=1,\ldots,m$; a_{ij}, $j=1,\ldots,n$ are independent exponentially distributed random variables with known parameters μ_{ij} and b_i is a positive constant. Then a point $x^*\in\mathbb{R}^n$ is an

(α, β)-satisficing solution of type 3 for $(P1)$ if and only if x^* is optimal for the problem

$$(P8)^v \begin{cases} \min \ \left(\sum_{k=1}^{K} \nu_k \delta_k\right) \\ \textit{subject to} \\ \prod_{j=1}^{n} \mu_{c_j^1} \left(\sum_{p=1}^{n} \frac{x_p^{n-1} e^{-\frac{\mu_{c_p^1} t_1 + \delta_1}{x_p}}}{\mu_{c_j^1} \prod_{\{l=1, l \neq p\}}^{n} (x_p \mu_{c_l^1} - x_l \mu_{c_p^1})}\right) \leq 1 - \beta_1 \\ \vdots \\ \prod_{j=1}^{n} \mu_{c_j^K} \left(\sum_{p=1}^{n} \frac{x_p^{n-1} e^{-\frac{\mu_{c_p^K} t_K + \delta_K}{x_p}}}{\mu_{c_j^K} \prod_{\{l=1, l \neq p\}}^{n} (x_p \mu_{c_l^K} - x_l \mu_{c_p^K})}\right) \leq 1 - \beta_K. \\ \prod_{j=1}^{n} \mu_{a_{1j}} \left(\sum_{p=1}^{n} \frac{x_p^{n-1} e^{-\frac{\mu_{a_{1p}} b_1}{x_p}}}{\mu_{a_{1j}} \prod_{\{l=1, l \neq p\}}^{n} (x_p \mu_{a_{1l}} - x_l \mu_{a_{1p}})}\right) \leq 1 - \alpha_1 \\ \vdots \\ \prod_{j=1}^{n} \mu_{a_{mj}} \left(\sum_{p=1}^{n} \frac{x_p^{n-1} e^{-\frac{\mu_{a_{mp}} b_m}{x_p}}}{\mu_{a_{mj}} \prod_{\{l=1, l \neq p\}}^{n} (x_p \mu_{a_{ml}} - x_l \mu_{a_{mp}})}\right) \leq 1 - \alpha_m \\ x \geq 0, \ \delta_k \geq 0; \ k = 1, \ldots, K \end{cases}$$

where δ_k; $(k = 1, \ldots, K)$, ν_k; $(k = 1, \ldots, K)$, t_k; $(k = 1, \ldots, K)$, $\alpha_i \in (0, 1]$; $i = 1, \ldots, m$; $\beta_i \in (0, 1]$; $k = 1, \ldots, K$ are as in Definition (3.2.3)

The proof is similar to that of Proposition (3.5.4).

3.6 Characterization of satisficing solutions for the Gamma case

It is well know that a quantile function of a probability distribution is the inverse of its Cumulative Distribution Function.

3.6.1 Definition. Let F_v be the Cumulative Distribution Function of V. Then the quantile function of

$$F_v(\beta) = \mathbb{P}(V \leq v) \geq \beta \text{ for } \beta \in (0, 1]$$

is defined as the inverse function

$$Q_\beta = F_v^{-1}(\beta).$$

Consequently, we state the following lemma which proof can be found in [68].

3.6.1 Lemma. *The chance constraints*

$$P(g_i(x) \leq \omega) \geq \alpha_i,\ i = 1, \ldots, m$$

can be written as

$$g(x_i) \geq Q_{\alpha_i},\ i = 1, \ldots, m$$

where $Q_{\alpha_i} = F_\omega^{-1}(\alpha_i),\ i = 1, \ldots, m.$

In what follows we state a result for the case of random coefficients [127].

3.6.2 Lemma. *Suppose that*

$$g_i(x, \omega) = b_i(x) - \langle A(\omega_i), g_i(x) \rangle, i = 1, \ldots, m \tag{3.37}$$

where $A(\omega_i);\ i = 1, \ldots, m$ *are random coefficients of* $g_i(x)$*, then the deterministic equivalent of*

$$\mathbb{P}\big(b_i(x) \leq \langle A(\omega_i), g_i(x) \rangle\big) \geq \alpha_i;\ i = 1, \ldots, m \tag{3.38}$$

is the following:

$$b_i(x) \geq \sum_{i=1}^{m} A_{i0} g_i(x) \tag{3.39}$$

here

$$A_{i0} = F_i^{-1}(\alpha_i), i = 1, 2, \ldots, m.$$

In other words individual chance constraints with random right hand side inherit their structure from the underlying stochastic constraints [68].

In the sequel we use the concept of quantile function and Lemma (1.1.7) given in Subsection (1.1.9).

3.6.1 Satisficing solution of type 1

3.6.3 Proposition. *Consider* $(P1)$*, assume that objective functions of* $(P1)$ *are aggregated as follows:*

$$X = \sum_{k=1}^{K} \sum_{j=1}^{n} \lambda_k c_j^k x_j$$

where $\lambda_1, \ldots, \lambda_K \geq 0,\ \sum_{k=1}^{K} \lambda_k = 1$*. Suppose that the random variables*

$$c_j^k;\ k = 1, \ldots, K,\ j = 1, \ldots, n$$

are independent Gamma distributed with η_{jk} as the inverse scale parameter and θ_{jk} as the shape parameter. For any $\lambda_k x_j > 0, \ \forall \ j,k$, the variable $X \sim \gamma(\frac{\eta}{1-\eta}, L+\theta)$ where

$$\eta = \max_{i \le j \le n} \frac{\eta_j}{x_j + \eta_j}, \ \ \theta = \sum_{j=1}^{n} \theta_j.$$

L is a discrete random variable such that $\mathbb{P}[L = r] = dh_r$, $(r \in \mathbf{N})$ where

$$d = \prod_{j=1}^{n} \Big(\frac{(1-\eta_j)\eta_j}{x_j \eta_j}\Big)^{\theta_j}$$

$$h_o = 1, \ \ h_r = \frac{1}{r} \sum_{l=1}^{r} l q_l b_r - l, \ r \ge 1$$

$$q_l = \frac{1}{l} \sum_{j=1}^{n} \theta_j \Big(\frac{1-(1-\eta_j)\eta_j}{c_j \eta_j}\Big)^{l}$$

Assume further that for any $i = 1, \dots, m$ the random variables b_i are independent Gamma distributed with ζ_i as the inverse scale parameter and ξ_i as the shape parameter, while a_{ij}; $j = 1, \dots, n$ are deterministic. Then x^* is a $(\alpha, \beta)-$satisficing solution of type 1 for (P1) if x^* is optimal for

$$(P9) \begin{cases} \min \ s \\ \textit{subject to} \\ \sum_{r=0}^{\infty} F(s, \frac{\eta}{1-\eta}, r+\theta) P[L=r] \ge \beta \\ \sum_{j=1}^{n} a_{ij} x_j \ge Q_b(1-\alpha i), \ \ i = 1, \dots, m; \\ x_j \ge 0; \ = 1, \dots, n \ s, \ \ u.s.r. \end{cases}$$

where $\alpha_1, \dots, \alpha_m \in (0,1], \beta \in (0,1]$ are a priori fixed by the decision maker, Q_{α_i}; $i = 1, \dots, m$ are the quantiles of the distribution function of Gamma random variable with inverse scale $\frac{\eta}{1-\eta}$ and shape parameter $L+\theta$, and $F(.)$ is the distribution function of random variable that is $\gamma(\frac{\eta}{1-\eta}, L+\theta)$.

Proof. We know that the density function of $\gamma(\tau, u)$ is given by

$$f(x, \tau, u) = \frac{\tau^u}{\Gamma(u)} x^{u-1} e^{-\tau x}, \ x > 0$$

where τ is the inverse scale parameter and u is the shape parameter. By Lemma (1.1.7) we have

$$f_c(x) = \sum_{r=0}^{\infty} f(s, \frac{\eta}{1-\eta}, r+\theta) P[L=r].$$

Let us denote the distribution function of $\gamma(\tau, u)$ by

$$F(x, \tau, u) = \int_{-\infty}^{x} f(y, \tau, u)dy.$$

The distribution function of c is a Gamma distribution with inverse scale parameter $\frac{\eta}{1-\eta}$ and shape parameter $L+\theta)$. We have then

$$\begin{aligned} F_c(x) &= \int_{-\infty}^{x} f_c(y)dy \\ &= \sum_{r=0}^{\infty} \int_{-\infty}^{x} f(y, \frac{\eta}{1-\eta}, r+\theta)P[L=r]dy \\ &= \sum_{r=0}^{\infty} F(y, \frac{\eta}{1-\eta}, r+\theta)P[L=r]. \end{aligned}$$

This means $\forall s \in \mathbb{R}$,

$$\mathbb{P}\left(cx \le s\right) = \sum_{r=0}^{\infty} F(s, \frac{\eta}{1-\eta}, r+\theta)P[L=r]$$

then the deterministic counterpart of

$$\mathbb{P}\left(cx \le s\right) \ge \beta \tag{3.40}$$

is given by

$$\sum_{r=0}^{\infty} F(s, \frac{\eta}{1-\eta}, r+\theta)P[L=r] \ge \beta. \tag{3.41}$$

Consider the chance constraint

$$\mathbb{P}\left(\sum_{j=1}^{n} a_{ij}x_j \le b_i\right) \ge \alpha_i,\ i=1,\dots,m \tag{3.42}$$

where only $b_i,\ i=1,\dots,m$ are independent Gamma distributed with ζ_i as the inverse scale parameters and ξ_i as the shape parameters, while $a_{ij}, j=1,\dots,n,\ i=1,\dots,m$ are deterministic. For each confidence level $\alpha_i \in [0,1],\ (i=1,\dots,m)$,

$$\sum_{j=1}^{n} a_{ij}x_j \ge Q_{bi}(1-\alpha_i);\ i=1,\dots,m \tag{3.43}$$

where $Q_{bi}(1-\alpha i);\ i=1,\dots,m$ be the $(1-\alpha_i)-$quantile functions of random variables $b_i;\ \ i=1,\dots,m.$

$$Q_{1-\alpha_i} = F_{bi}^{-1}(1-\alpha_i);\ i=1,\dots,m$$

where F_{bi} is Gamma cumulative distribution function. By replacing (3.40) with (3.41) and (3.42) with (3.43) in $(P5)$, we obtain the desired mathematical program $(P9)$.

$\square$

The following result is a generalization of Proposition (3.6.3).

3.6.2 Satisficing solution of type 2

3.6.4 Proposition. *Suppose that the random variables*

$$c_j^k;\ k=1,\ldots,K,\ j=1,\ldots,n$$

are independent Gamma distributed with η_{jk} as the inverse scale parameter and θ_{jk} as the shape parameter. For any $x_j > 0,\ \forall\ j$, the variables $c^1x,\ldots,c^kx$, are $\gamma(\frac{\eta_k}{1-\eta_k}, L_k+\theta_k)$ where

$$\eta_k = \max_{i\leq j\leq n,\ 1\leq k\leq K} \frac{\eta_{jk}}{x_{jk}+\eta_{jk}},\ \ \theta_k=\sum_{j=1}^{n}\theta_{jk},\ \ k=1,\ldots,K.$$

L_k are discrete random variables such that $\mathbb{P}[L_k=r]=d_kh_{rk}$, $k=1,\ldots,K,\ (r\in\mathbf{N})$ where

$$d_k=\prod_{j=1}^{n}\Big(\frac{(1-\eta_{jk})\eta_{jk}}{x_j\eta_{jk}}\Big)^{\theta_{jk}};\ \ k=1,\ldots,K$$

$$h_{ok}=1,\ \ h_{rk}=\frac{1}{r}\sum_{l=1}^{r} l_k q_{l_k} b_{r_k-l_k},\ r\geq 1;\ \ k=1,\ldots,K$$

$$q_{lk}=\frac{1}{l_k}\sum_{j=1}^{n}\theta_{jk}\Big(\frac{1-(1-\eta_{jk})\eta_{jk}}{c_{jk}\eta_{jk}}\Big)^{l_k};\ \ k=1,\ldots,K.$$

Assume further that for any $i=1,\ldots,m$ the random variables $b_i\sim\gamma(\zeta_i,\xi_i)$ while $a_{ij};\ j=1,\ldots,n$ are deterministic. Then x^* is a $(\alpha,\beta)-$satisficing solution of type 2 for (P1) if x^* is efficient for

$$(P9)'\begin{cases}\min\ (s_1,\ldots,s_K)\\ \textit{subject to}\\ \sum_{r=0}^{\infty}F(s_k,\frac{\eta_k}{1-\eta_k},r+\theta_k)P[L_k=r]\geq\beta_k;\ \ k=1,\ldots,K,\\ \sum_{j=1}^{n}a_{ij}x_j\geq Q_b(1-\alpha_i);\ \ i=1,\ldots,m\\ x,\geq 0,\ \ s_k,\ \ u.s.r.;\ \ k=1,\ldots,K\end{cases}$$

where $\alpha_1,\ldots,\alpha_m\in(0,1],\beta_1,\ldots,\beta_K\in(0,1]$ are as in $(P5)'$, $Q_{\alpha i};\ i=1,\ldots,m$ and $F(.)$ are as in Proposition (3.6.3).

3.6.3 Satisficing solution of type 3

3.6.5 Proposition. *Consider $(P1)$ and suppose that the random variables*

$$c_j^k;\ k = 1, \ldots, K,\ j = 1, \ldots, n$$

are independent Gamma distributed with η_{j^k} as the inverse scale parameter and θ_{j^k} as the shape parameter. Assume further that the random variables $b_i \sim \gamma(\zeta_i, \xi_i);\ i = 1, \ldots, m$ while $a_{ij};\ j = 1, \ldots, n,\ i = 1, \ldots, m$ are deterministic. Then x^* is a $(\alpha, \beta)-$satisficing solution of type 3 for (P1) if x^* is optimal for the following optimization problem:

$$(P9)'' \begin{cases} \min \sum_{k=1}^{K} \nu_k \delta_k \\ \textit{subject to} \\ c_{10}x - s_1 \leq t_1 + \delta_1 \\ \vdots \\ c_{K0}x - s_K \leq t_K + \delta_K \\ \sum_{j=1}^{n} a_{ij}x_j \geq Q_b(1 - \alpha_i);\ i = 1, \ldots, m \\ x \geq 0,\ \delta_k \geq 0;\quad k = 1, \ldots, K \end{cases}$$

where $\alpha_1, \ldots, \alpha_m \in (0, 1], \beta_1, \ldots, \beta_K \in (0, 1]$ are as in $(P5)'$, $Q_{\alpha i};\ i = 1, \ldots, m$ as in Proposition (3.6.3) and $c_{k0} = F_k^{-1}(1 - \beta_k), k = 1, \ldots, K$ are as in Lemma (3.6.2).

Proof. We know that $x^* \in \mathbb{R}^n$ is an $(\alpha, \beta)-$satisficing solution of type 3 for $(P1)$ if (x^*, δ^*), where $\delta^* \in \mathbb{R}^K$, is optimal for the following mathematical program:

$$(P5)'' \begin{cases} \min \left(\sum_{k=1}^{K} \nu_k \delta_k \right) \\ \text{subject to} \\ \mathbb{P}\left(c^1 x - \delta_1 \leq t_1\right) \geq \beta_1 \\ \vdots \\ \mathbb{P}\left(c^K x - \delta_K \leq t_K\right) \geq \beta_K \\ \mathbb{P}\left(\sum_{j=1}^{n} a_{ij}x_j - b_i \leq 0\right) \geq \alpha_i;\ i = 1, \ldots, m \\ x \geq 0,\ \delta_k \geq 0;\ k = 1, \ldots, K \end{cases}$$

By lemma (3.6.2) the chance constraint

$$\mathbb{P}\left(c^k x \leq s_k\right) \geq \beta_k;\ k = 1, \ldots, K \tag{3.44}$$

are equivalent to

$$c_{k0}x \leq s_k + t_k + \delta_k;\ k = 1, \ldots, K \tag{3.45}$$

where $c_{k0} = F_k^{-1}(1 - \beta_k), k = 1, \ldots, K$ and the chance constraints

$$\mathbb{P}\left(\sum_{j=1}^{m} a_{ij}x_j - b_i\right) \geq \alpha_i;\ \ i = 1, \ldots, m \tag{3.46}$$

are equivalent to

$$\sum_{j=1}^{n} a_{ij}x_j \geq Q_b(1 - \alpha_i);\ \ i = 1, \ldots, m. \tag{3.47}$$

By replacing (3.44) with (3.45) and (3.46) with (3.47) in Program $(P5)''$ we obtain mathematical program $(P9)''$. $\square$

Chapter 4

Extension

4.1 Multiobjective Programming problems with fuzzy random coefficients

4.1.1 A connection between fuzzy random variables and random closed sets

A) **Auxiliary mappings**

The maps π, f_ω, σ_ω described below are needed to state the Embedding Theorem for fuzzy random variables proved in [94].

1) π: $\Im_{cc}(\mathbb{R}) \to \tilde{C}[0,1] \times \tilde{C}[0,1]$

$$\tilde{a} \quad \rightsquigarrow (\tilde{a}^L(\alpha), \tilde{a}^U(\alpha))$$

where $\tilde{a}^L(\alpha) = \tilde{a}^L_\alpha$ and $\tilde{a}^U(\alpha) = \tilde{a}^U_\alpha$, $\tilde{a}^L_\alpha$ and $\tilde{a}^U_\alpha$ standing for the lower and upper endpoints of the α-level of $\tilde{a}$.

Here $\Im_{cc}(\mathbb{R})$ denotes the space of fuzzy numbers with compact supports and $\tilde{C}[0,1]$ is the set of real-valued bounded functions f on [0,1] such that

- f is left continuous for any $t \in [0,1)$ and right continuous at 0.
- f has a right limit for any $t \in [0,1)$.

2) $f_\omega : F(\Omega) \to \Im_{cc}(\mathbb{R})$

$$X \rightsquigarrow X(\Omega) = X_\omega.$$

Here $F(\Omega)$ stands for the set of fuzzy random variables (FRV) in the sense of Puri & Ralescu [113].

3) $\sigma_\omega : F(\Omega) \to \tilde{C}[0,1] \times \tilde{C}[0,1]$

$$X \rightsquigarrow \pi(X_\omega).$$

It is clear that the three maps make the following diagram commutative.

Figure 4.1: Commutative Diagram 1

As a matter of fact, we have for $X \in F(\Omega)$,
$\pi f_\omega(X) = \pi(X_\omega) = \sigma_\omega(X)$.

B) **Main mapping**
Let $E = \tilde{C}[0,1] \times \tilde{C}[0,1]$ and let $\Im_E$ be the family of closed subset of E. Consider now the probability space (Ω, Σ, P) where Σ is the σ-algebra of subsets of E.
The main mapping is defined as follows:

$$\sigma : F(\Omega) \to RACS(E)$$

$$X \to \sigma X$$

where $RACS(E)$ stands for the set of random closed sets E and σX is defined as follows:

$$\sigma X : \Omega \to \Im_E$$

$$\omega \rightsquigarrow (X^L_\omega(\alpha), X^U_\omega(\alpha))$$

where $\alpha \in [0,1]$,
$X^L_\omega(\alpha) = X^L_{\omega\alpha}$ and $X^U_\omega(\alpha) = X^U_{\omega\alpha}$.

It is clear that the mapping σ thus defined makes the following diagram commutative.

Figure 4.2: Commutative Diagram 2

As a matter of fact, for $\omega \in \Omega$ we have $\sigma X(\omega) = \pi(X_\omega) = (\pi \circ X)(\omega)$ i.e
$\sigma X = \pi \circ X$.
In the sequel we consider the distance and the partial order relation defined on the set of random closed sets on E (RACS(E)) and on the set of fuzzy random variables $F(\Omega)$ as follows:

1) For X_1, $X_2 \in RACS(E)$,
$d_{RACS(E)}(X_1, X_2) = \sup_\omega d_H(X_1(\omega), X_2(\omega))$
$X_1 \leq_{RACS(E)} X_2$ if and only if $\sup_\omega(X_1(\omega) \leq \inf_\omega X_2(\omega))$, $\forall \omega \in \Omega$.

2) For Y_1, $Y_2 \in F(\Omega)$
$d_{F(\Omega)}(Y_1, Y_2) = \sup_\omega d_{Fcc(\mathbb{R})}(Y_1(\omega), Y_2(\omega))$
$Y_1 \leq Y_2$ if and only if $Y_{1\omega} \leq_{F_{cc}(\mathbb{R})} Y_{2\omega}$; $\forall \omega \in \Omega$.

4.1.1 Theorem.

(i) σ *is injective*

(ii) *For* $\lambda,\ \mu \in \mathbb{R}$, *we have*
$\sigma[(1_{\{\lambda\}} \boxdot X) \boxplus (1_{\{\lambda\}} \boxdot Y)] = \lambda\sigma X + \mu\sigma Y$

(iii) $d_{RACS(E)}(\sigma X, \sigma Y) = d_{F(\Omega)}(X, Y)$

(iv) $X \leq_{F(\Omega)} Y \Longleftrightarrow \sigma X \leq_{RACS(E)} \sigma Y$
where $\boxdot$, $\boxplus$ *indicate that the operations are in space* $F_{(\Omega)}$ *and* $\leq_{F(\Omega)}$ *means the inequality* $\leq$ *is between elements of* $F_{(\Omega)}$.

This Theorem tells us that the set of fuzzy random variables in the sense of Puri, Ralescu can be embedded on the set of closed random set isomorphically and isometrically. The proof of this theorem may be found elsewhere [94].

In the next subsection, we shall use this result in an essential manner in describing an approach for solving multiobjective programming problems with fuzzy random coefficients.

4.1.2 Solving a multiobjective programming problem with fuzzy random objective functions

A) **Case of deterministic feasible set**

1) **Problem formulation**
Consider the mappings

$$\tilde{f}_i : \mathbb{R}^n \to F(\Omega);\ i = 1, \ldots, p.$$

We are interested in solving the following optimization problem:

$$(P10) \quad \min_{x \in X} \quad (\tilde{f}_1(x), \ldots, \tilde{f}_p(x))$$

where $X = \{x \in \mathbb{R}^n / g_j(x) \leq 0;\ j = 1, \ldots, m\}$ is a convex and bounded subset of $\mathbb{R}^n$.

2) **Analysis**
Consider now the following surrogate of (P10):

$$(P10)' \quad \min_{x \in X} \quad \left(\sigma\tilde{f}_1(x), \ldots, \sigma\tilde{f}_p(x)\right)$$

The following result relates $(P10)$ and $(P10)'$ in an expected manner.

4.1.2 Proposition. x^* *is an efficient solution for* $(P10)$ *if and only if* x^* *is* *efficient for* $(P10)'$.

Proof. Assume x^* is efficient for $(P10)$ and not efficient for $(P10)'$. As x^* is efficient for $(P10)$, there is no $x \in X$ such that

$$\begin{aligned}&\tilde{f}_i(x) \leq_{F(\Omega)} \tilde{f}_i(x^*),\ \forall i \in \{1, \ldots, p\}\\&\text{and}\\&\tilde{f}_l(x) <_{F(\Omega)} \tilde{f}_l(x^*) \text{ for some } l \in \{1, \ldots, p\}.\end{aligned} \tag{4.1}$$

As x^* is not efficient for $(P10)'$, we may find $x \in X$ such that

$$\begin{aligned}&\sigma\tilde{f}_i(x) \leq \sigma\tilde{f}_i(x^*),\ \forall i \in \{1, \ldots, p\}\\&\text{and}\\&\sigma\tilde{f}_l(x) < \sigma\tilde{f}_l(x^*) \text{ for some } l \in \{1, \ldots, p\}.\end{aligned} \tag{4.2}$$

This means for $\omega \in \Omega$ we have

$$\begin{aligned}&(\sigma\tilde{f}_i(x))(\omega) \leq \sigma(\tilde{f}_i(x^*))(\omega),\ \forall i \in \{1, \ldots, p\}\\&\text{and}\\&(\sigma\tilde{f}_l(x))(\omega) < \sigma(\tilde{f}_l(x^*))(\omega) \text{ for some } l \in \{1, \ldots, p\}.\end{aligned} \tag{4.3}$$

That is

$$\begin{aligned}&(\tilde{f}^L_{i\omega}(x)(\alpha), \tilde{f}^U_{i\omega}(x))(\alpha) \leq (\tilde{f}^L_{i\omega}(x^*)(\alpha), \tilde{f}^U_{i\omega}(x^*))(\alpha),\ \forall \alpha \in (0, 1]\\&\text{and}\\&(\tilde{f}^L_{l\omega}(x)(\alpha), \tilde{f}^U_{l\omega}(x))(\alpha) < (\tilde{f}^L_{l\omega}(x^*)(\alpha), \tilde{f}^U_{l\omega}(x^*))(\alpha) \text{ for some } \alpha \in (0, 1].\end{aligned} \tag{4.4}$$

(4.4) is equivalent to

$$\begin{aligned}&\tilde{f}_{i\omega}(x) \leq_{F(\Omega)} \tilde{f}_{i\omega}(x^*),\ \forall i \in \{1, \ldots, p\}\\&\text{and}\\&\tilde{f}_{l\omega}(x) \leq_{F(\Omega)} \tilde{f}_{l\omega}(x^*), \text{ for some } l \in \{1, \ldots, p\}.\end{aligned} \tag{4.5}$$

(4.5) is in contradiction with (4.1) and x^* is efficient for $(P10)$. □

The other implication can be proved in a similar way.

Using the Expectation model approach to convert $(P10)'$ into deterministic terms yields the following program:

$$(P10)'' \begin{cases} \min \quad \{(E\tilde{f}^L_\omega(x)(\alpha), E\tilde{f}^U_\omega(x)(\alpha));\ \alpha \in (0,1]\} \\ x \in X \end{cases}$$

Unfortunately $(P10)''$ is a mathematical program with infinitely many objective functions about which, to the best of our knowledge, there is no available solution technique.
Nevertheless, making use of nested property of α-level sets of fuzzy sets i.e, for $\alpha \geq \beta$ we have $\tilde{f}^\alpha_\omega(x) \subset \tilde{f}^\beta_\omega(x)$ for ω, it is clear that $E\tilde{f}^L_\omega(x)(\alpha)$ and $E\tilde{f}^U_\omega(x)(\alpha)$ will be kept at minimal extent when $E\tilde{f}^L_\omega(x)(0)$ and $E\tilde{f}^U_\omega(x)(0)$ are minimal. Apart from the above intuitive argument, another justification for restriction on $\alpha = 0$ while solving $(P10)''$ can be formulated as follows:

4.1.3 Proposition. *If $x^* \in X$ is weakly efficient for the following bi-objective program:*

$$(P10)''' \begin{cases} \min \left\{E\tilde{f}^L_\omega(x)(0), E\tilde{f}^U_\omega(x)(0)\right\} \\ x \in X \end{cases}$$

then x^* is weakly efficient for $(P10)''$.

Proof. Proving this proposition amounts to prove its contrapositive. That is to prove that if $x^* \in X$ is not weakly efficient for $(P10)''$ then it is not weakly efficient for $(P10)'''$.

Assume that $x^* \in X$ is not weakly efficient for $(P10)''$, Then we may find $x \in X$ such that

$$E\tilde{f}^L_\omega(x)(\alpha) < E\tilde{f}^L_\omega(x^*)(\alpha),\ \forall \alpha \in (0,1]$$

and

$$E\tilde{f}^U_\omega(x)(\alpha) < E\tilde{f}^U_\omega(x^*)(\alpha),\ \forall \alpha \in (0,1].$$

Taking the limit for $\alpha \to 0$ we have that there is $x \in X$ such that

$$\lim_{\alpha \to 0} E\tilde{f}^L_\omega(x)(0) < \lim_{\alpha \to 0} E\tilde{f}^L_\omega(x^*)(0)$$

and

$$\lim_{\alpha \to 0} E\tilde{f}^U_\omega(x)(0) < \lim_{\alpha \to 0} E\tilde{f}^U_\omega(x^*)(0).$$

This means, there is $x \in X$ such that

$$E\tilde{f}^L_\omega(x)(0) < E\tilde{f}^L_\omega(x^*)(0)$$

and

$$E\tilde{f}^U_\omega(x)(0) < E\tilde{f}^U_\omega(x^*)(0).$$

But this is tantamount to saying that x^* is not weakly efficient for $(P10)'''$. □

The foregoing discussion inspires us the procedure described below for solving (P10).

3) **Algorithm for solving (P10)**

4.1.1 Algorithm. ‘

Step 1: Read objectives and constraints of the problem.

Step 2: Construct the following multiobjective program:

$$(P10)^{iv} \begin{cases} \min \left\{ (\tilde{f}_{1\omega}(x))^L(0), (\tilde{f}_{1\omega}(x))^U(0), \ldots, (\tilde{f}_{p\omega}(x))^L(0), (\tilde{f}_{p\omega}(x))^U(0) \right\} \\ x \in X \end{cases}$$

Step 3: Find a solution of $(P10)^{iv}$ using techniques of Multiobjecive Stochastic Optimization [2].

Step 4: Print the obtained solution.

Step 5 Stop.

B) **Case of fuzzy random feasible set**

1) Problem formulation Consider the fuzzy random-valued mappings:

$$\tilde{f}_i : \mathbb{R}^n \to F(\Omega);\ i = 1, \ldots, p$$

and

$$\tilde{g}_j : \mathbb{R}^n \to F(\Omega);\ j = 1, \ldots, m.$$

The optimization problem we ponder here is

$$(P10)^v \begin{cases} \min\left\{\tilde{f}_1(x), \dots, \tilde{f}_p(x)\right\} \\ X = \{x \in \mathbb{R}^n \mid \tilde{g}_j(x) \le \tilde{b}_j;\ j = 1, \dots, m\} \end{cases}$$

where $\tilde{b}_j (j = 1, \dots, m)$ are fuzzy random variables on (Ω, Σ, P).

2) **Analysis**
Consider the following optimization problem:

$$(P10)^{vi} \begin{cases} \min\left\{\sigma\tilde{f}_1(x), \dots, \sigma\tilde{f}_p(x)\right\} \\ x \in Z = \{x \in \mathbb{R}^n | \sigma\tilde{g}_j(x) \le_{F(\Omega)} \tilde{b}_j;\ j = 1, \dots, m \end{cases}$$

Before stating a result that bridges the gap between $(P10)^v$ and $(P10)^{vi}$, let us first introduce the following respective surrogates to these two programs.

$$(P10)^{vii} \begin{cases} \min\left\{\tilde{f}_1(x), \dots, \tilde{f}_p(x)\right\} \\ x \in Y' \end{cases}$$

and

$$(P10)^{viii} \begin{cases} \min\left\{\sigma\tilde{f}_1(x), \dots, \sigma\tilde{f}_p(x)\right\} \\ x \in Z' \end{cases}$$

where Y' and Z' are deterministic counterparts of Y and Z respectively. Then the following results holds.

4.1.4 Proposition. *x^* is a Pareto optimal solution for $(P10)^{vii}$ if and only if x^* is Pareto optimal for $(P10)^{viii}$.*

Proof. By Theorem (4.1.1 (iv)), we have that $Y' = Z'$. By Proposition (1), $(P10)^{vii}$ and $(P10)^{viii}$ are equivalent in terms of Pareto optimality and we are done. □

This discussion suggests the following method for finding a solution of $(P10)^v$.

3) **Method for solving (P10)^v**

4.1.2 Algorithm.

Step 1 Read objective functions and constraints of $(P10)^v$.

Step 2 Construct the mathematical program

$$(P10)^{viii} \begin{cases} \min\left\{(\tilde{f}_{1\omega}(x))^L(0), (\tilde{f}_{1\omega}(x))^U(0), \dots, (\tilde{f}_{p\omega}(x))^L(0), (\tilde{f}_{p\omega}(x))^U(0)\right\} \\ x \in Y' \end{cases}$$

FRV	Fuzzy values	Probabilities
$\tilde{c}_1^1$	$c_{11}^1 = \triangle(-1, 2, 2)$	$p_{11}^1 = 1/4$
	$c_{12}^1 = \triangle(0, 1, 1)$	$p_{12}^1 = 3/4$
$\tilde{c}_2^1$	$c_{21}^1 = \triangle(-2, 2, 1)$	$p_{21}^1 = 1/4$
	$c_{22}^1 = \triangle(1, 1, 1)$	$p_{22}^1 = 3/4$
$\tilde{c}_1^2$	$c_{11}^2 = \triangle(3, 1, 1)$	$p_{11}^2 = 1/4$
	$c_{12}^2 = \triangle(1, 1, 2)$	$p_{12}^2 = 3/4$
$\tilde{c}_2^2$	$c_{21}^2 = \triangle(2, 2, 1)$	$p_{21}^2 = 1/4$
	$c_{22}^2 = \triangle(-3, 1, 1)$	$p_{22}^2 = 3/4$

Table 4.1: Details of FRV $\tilde{c}_j^i$

Step 3 Solve $(P10)^{ix}$ *using techniques of Multiobjective Stochastic Optimization [2].*

Step 4 Print the obtained solution.

Step 5 Stop.

4.1.3 Numerical example

4.1.1 Example. *Consider the multiobjective program*

$$(P10)^x \begin{cases} \min \quad (\tilde{c}_1^1 x_1 \boxplus \tilde{c}_2^1 x_2, \tilde{c}_1^2 x_1 \boxplus \tilde{c}_2^2 x_2) \\ \textit{subject to} \\ -2x_1 + x_2 \leq 2 \\ -x_1 + x_2 \leq 3 \\ x_1 \leq 3 \\ x_1 \geq 0; x_2 \geq 0 \end{cases}$$

where $\tilde{c}_j^i$; $j = 1, 2$; $i = 1, 2$ *are fuzzy random variables defined on* $\Omega = \{\omega_1, \omega_2\}$ *with probabilities* $p_{j1}^i = 1/4$ *and* $p_{j2}^i = 3/4$; $j = 1, 2$; $i = 1, 2$.

Details of the 4 fuzzy random variables is given in Table (4.1.1).

Here $\triangle(a, b, c)$ *denotes a triangular fuzzy number, the membership of which is defined as follows:*

Figure 4.3: Triangular fuzzy number

According to Theorem (4.1.1) *and Proposition* (4.1.2)*, solving* $(P10)^x$ *is tantamount to* *solving*

$$(P10)^{xi}\begin{cases}\min \quad \{(\sigma\tilde{c}_1^1)x_1+(\sigma\tilde{c}_2^1)x_2,(\sigma\tilde{c}_1^2)x_1+(\sigma\tilde{c}_2^2)x_2\}\\ \textit{subject to}\\ -2x_1+x_2\leq 2\\ -x_1+x_2\leq 3\\ x_1\leq 3\\ x_1\geq 0; x_2\geq 0.\end{cases}$$

As explained in Subsection (4.1.2)*, solving* $(P10)^{xi}$ *amounts to solving the following program:*

$$(P10)^{xii}\begin{cases}\min\Big\{(c_{1\omega}^{1L}(0),(c_{2\omega}^{1L}(0))(x_1,x_2)^T,\\ (c_{1\omega}^{1U}(0),(c_{2\omega}^{1U}(0))(x_1,x_2)^T,\\ (c_{1\omega}^{2L}(0),(c_{2\omega}^{2L}(0))(x_1,x_2)^T,\\ (c_{1\omega}^{2U}(0),(c_{2\omega}^{1U}(0))(x_1,x_2)^T\Big\}\\ \textit{subject to}\\ -2x_1+x_2\leq 2\\ -x_1+x_2\leq 3\\ x_1\leq 3\\ x_1\geq 0,x_2\geq 0\end{cases}$$

where $c_{j\omega}^{iL}(0)$ *and* $c_{j\omega}^{iU}(0)$ *stand respectively for the left endpoint and right endpoint of the support of* $\sigma\tilde{c}_j^i$*. Using the expectation model approach to solve* $(P10)^{xii}$ *yields the following deterministic multiobjective program:*

$$\begin{cases}\min\ \Big\{-\frac{6}{4}x_1-\frac{3}{4}x_2,x_1+\frac{6}{4}x_2,-\frac{1}{4}x_1-\frac{11}{4}x_2,\frac{10}{4}x_1-\frac{2}{4}x_2\Big\}\\ \textit{subject to}\\ -2x_1+x_2\leq 2\\ -x_1+x_2\leq 3\\ x_1\leq 3\\ x_1\geq 0,x_2\geq 0\end{cases}$$

A Pareto optimal solution for this program may be obtained by solving the following weighted problem. For $\omega_1=4$*,* $\omega_2=8$*,* $\omega_3=4$*,* $\omega_4=12$ *we obtain the program*

$$\begin{cases} \min \quad (25x_1 - 8x_2) \\ \textit{subject to} \\ -2x_1 + x_2 \le 2 \\ -x_1 + x_2 \le 3 \\ x_1 \le 3 \\ x_1 \ge 0; x_2 \ge 0 \end{cases}$$

The solution of this program, obtained by using the software LINGO is (0,2). This is a satisficing solution for $(P10)^v$.

Conclusion

This thesis is on Stochastic Multiobjective Programming problems. In such turbulent environments involving both randomness and conflicting objective functions, neither the notion of feasibility nor that of optimality is clearly defined. We then resorted to the Simon's bounded rationality principle [125] to seek for satisficing rather than optimal solutions. Three satisficing solutions based on the chance constrained paradigm [77] were introduced. Chance constrained applies for the purpose of limiting the probability that constraints will be violated. In this form, it adds considerably to both the flexibility and the reality of the stochastic model under consideration.

The solution of type 1 (see Definition 3.2.1) may be of great help when the Decision Maker can aggregate his preferences through an appropriate utility function. When the Decision Maker has some targets for his objectives, the solution of type 3 (see Definition 3.2.3) is the most suitable for his purpose. When neither utility functions nor targets for objectives are available, the Decision Maker should resort to the solution of type 2 (see Definition 3.2.3).

The above mentioned solution concepts for Stochastic Multiobjective Programming problems have a clear advantage over those based on the first two moments of involved random variables [7], [127].
As a matter of fact, the expectation values and the variances of random variables at hand often offer short-sighted views of uncertainty surrounding random data under consideration.

Mathematical characterizations of these solution concepts have been obtained for the case of normal, exponential, chi-squared and gamma distributions.

Extension to the case of fuzzy random variables has also been carried out. We find it interesting to include numerical examples for the sake of illustration. It might be pointed out that a general methodology for solving Fuzzy Stochastic Optimization problems has been outlined in [93]. The quintessential of that methodology is to perform a couple of transformations, either sequentially or in parallel in a way to put the original problem into deterministic terms. In this thesis the transformation is done by the connections between fuzzy random variables and random closed sets.

This approach contrasts markedly with those where approximation of fuzzy values by real ones is followed by approximation of random variables by their moments [91]. It also differs form approaches based on fuzzy-stochastic simulation [73]. Moreover, our approach can handle both linear and non linear optimization problems. It is also less demanding in terms of information that the decision maker should provide before having his problem solved. This departs strongly with extant methods (see for example [66] and [67])

Lines for further developments in this field include the following:

- Find ways based on multiparametric optimization for fixing threshold $\alpha_i,\ \beta_k$ optimally instead of requesting the Decision Maker to provide them.
- In the approach for incorporating fuzzy random data in an optimization setting, consider another interpretation for interval optimization rather than the one consisting of sticking on mid-points.
- Characterize the introduced solution concepts in the case of other distributions that are not considered here.

Some results from this thesis have been published [7], [95], [97]. Others have been submitted for publication or presented at conferences [2], [3], [4], [5], [6], [96].

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