LOCAL TIMES OF BROWNIAN MOTION

by

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Abstract

After a review of the notions of Hausdorff and Fourier dimensions from fractal geometry and Fourier analysis and the properties of local times of Brownian motion, we study the Fourier structure of Brownian level sets. We show that if $\delta_a(X)$ is the Dirac measure of one-dimensional Brownian motion X at the level a, that is the measure defined by the Brownian local time L_a at level a, and μ is its restriction to the random interval $[0, L_a^{-1}(1)]$, then the Fourier transform of μ is such that, with positive probability, for all $0 \leq \beta < 1/2$, the function $u \to |u|^\beta |\mu(u)|^2$, $(u \in \mathbf{R})$, is bounded. This growth rate is the best possible. Consequently, each Brownian level set, reduced to a compact interval, is with positive probability, a Salem set of dimension 1/2. We also show that the zero set of X reduced to the interval $[0, L_0^{-1}(1)]$ is, almost surely, a Salem set. Finally, we show that the restriction μ of $\delta_0(X)$ to the deterministic interval $[0, 1]$ is such that its Fourier transform satisfies $E\left(|\hat{\mu}(u)|^2\right) \leq C|u|^{-1/2}$, $u \neq 0$ and $C > 0$.

Key words: Hausdorff dimension, Fourier dimension, Salem sets, Brownian motion, local times, level sets, Fourier transform, inverse local times.

Declaration

"I declare that *Local times of Brownian motion* is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references".

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Contents

Introduction

This thesis is a study of the Fourier structure of level sets of a one-dimensional Brownian motion using the notion of local times. The idea developed from a series of lectures by Prof Fouché (my supervisor) which focused on Fractal geometry, Brownian motion, Fourier analysis and the problem of uniqueness of trigonometric series.

In 1869 Heine proposed to Cantor the problem of determining whether a trigonometric series that converges to 0 at all real numbers must have all its coefficients equal to 0 (that is the series is identical 0). This is equivalent to the following problem: If two trigonometric series converge to the same limit at all real numbers, are they equal (or do they have the same coefficients)? In 1870, using Riemann's ideas, Cantor proved that the answer to the question was "yes" and that was the beginning of various generalizations. The main problem now became the following: Does the answer remain "yes" if exceptional points are allowed. That is, if it is known that the series converge to the same points for all reals, but nothing is known for points in a certain subset E, is it true that the series are identical? Cantor showed that if this exceptional set E is a countable closed set, then again the answer to the question is "yes".

A set $E \subset [0, 1]$ is a set of uniqueness if every trigonometric series which converges to 0 for $x \notin E$ is identically 0, or equivalently any two trigonometric series that converge to the same points for every real $x \notin E$ are identical. Intuitively, this means that the complement E^c of E in $[0, 1]$ is "large" enough that if the two series agree on it, then they are the same on $[0, 1]$ (they have the same coefficients). A set which is not a set of uniqueness is called a set of multiplicity. More clearly, a subset $M \subset [0, 1]$ is a set of multiplicity if there exist different (multiple) trigonometric series that converge to the same points outside M. The convergence to the same limit in M^c is thus not sufficient to guarantee that the series are equal. The problem of uniqueness can thus be formulated as follows: Given a subset $E \subset [0, 1]$, is E a set of uniqueness? Simplest examples of sets of uniqueness are countable closed subsets of $[0, 1]$. It is known that if E is a set of uniqueness and is Lebesgue-measurable, then its Lebesgue measure is 0. It has first been suggested that all Lebesgue null sets should be sets of multiplicity until Menshov constructed an example of a closed set of multiplicity of measure 0. Major progress was made by Salem and Zygmund (see for example [44]) when they completely characterized Cantor type sets of fixed ratio ξ in terms of the number theoretical structure of ξ. However, the characterisation of sets of uniqueness is very far from being complete despite efforts by different mathematicians. (More historical details on the problems of uniqueness can be found in the book by Kechris [28].)

The problem of uniqueness has many interactions with other areas of classical analysis, measure theory, functional analysis, number theory, and set theory. For example, it is well know that if a compact subset E of $[0,1]$ supports a measure μ such that its Fourier transform $\hat{\mu}(u) = \int e^{iux} d\mu(x) \to 0$ as $|u| \to 0$, then E is a set of multiplicity. For example, the Lebesgue measure λ on $[0,1]$ is such that $|\hat{\lambda}(u)|$ converges to 0 (with rate $|u|^{-1}$). It is, therefore, natural to ask for the asymptotic rate at which the Fourier transform of a finite measure tend to zero (if it really tends to 0). It is known that if E is a compact subset of $[0,1]$ with Hausdorff dimension $\alpha \in [0,1]$, then any measure μ supported by E is such that $|u|^\beta |\hat{\mu}(u)|^2$ is unbounded for any $\beta > \alpha$. A compact set $E \subset [0,1]$ is said to have Fourier dimension $\alpha \in (0,1]$ if for any $\beta < \alpha$, there exists a measure μ supported by E such that $|u|^\beta |\hat{\mu}(u)|^2$ is bounded and no such measure exists for $\beta > \alpha$. The Hausdorff dimension is, therefore, an upper-bound of the Fourier dimension. Salem [43] constructed the first example of a compact subsets of $\mathbf{R}$ whose Fourier and Hausdorff dimensions are the same. Such sets were later called Salem sets. The interval $[0,1]$ is itself a trivial example of Salem set. The first non-trivial deterministic example was given by Kaufman [27] when he showed that the set α-well approximable numbers contains a Salem set of dimension $2/(2+\alpha)$. (See also Bluhm [5]).

Kahane, in his seminal book [24], gives a large class of Salem sets by showing, surprisingly that, for any compact $E \subset [0,1]$ of dimension $\alpha < 1/2$, if X denotes the one-dimensional Brownian motion, then the image $X(E)$ is a Salem set (with dimension 2α). There are many interplays between fractal geometry and Brownian motion and stochastic properties in general in such a way that, fractal geometry defines a proper framework to explore deep Brownian motion properties. Conversely, Brownian motion is a source of examples to illustrate fractal properties. For example, knowing that the Hausdorff dimension of the zero set of Brownian motion is 1/2 indicates how "irregular" a Brownian path is. It is now common that in many textbooks on stochastic processes some sections are reserved for fractal properties.

Motivated by Kahane's ideas, we decided to study the Fourier dimension of level sets of Brownian motion. We show that if $\delta_a(X)$ is the Dirac measure of Brownian motion at the level a, that is the measure defined by the Brownian local time L_a at level a, and μ is its restriction to the random interval $[0, L_a^{-1}(1)]$, then the Fourier transform of μ is such that, with positive probability, for all $0 \leq \beta < 1/2$, the function $u \to |u|^\beta |\widehat{\delta_a(X)}(u)|^2$, $(u \in \mathbf{R})$, is bounded. From this we deduce that each Brownian level set, reduced to a compact interval, is with positive probability, a Salem set of dimension 1/2. The proofs are based on the fact that the inverse local time process of Brownian motion has independent and stationary increments.

Using Lévy's original definition of local times, we also show that, almost surely, the restriction of the Dirac measure $\delta_0(X)$ to the deterministic interval $[0,1]$ is such that its

Fourier transform verifies $E\left(|\hat{\mu}(u)|^2\right) \leq C|u|^{-1/2}$, $u \neq 0$ and $C > 0$.
We have extensively used the ideas of Kahane (from his book [24]) and many beautiful results of Lévy on local times and Brownian motion in general (many of them are given in the epoch-making book by Ito and McKean [22] and the book by Peres and Mörters [39]).
If, from the point of view of stochastic processes, Brownian motion can be seen as a simple model because more elaborated models become now available, its fractal properties pose very difficult problems which are far from being fully understood. Its connection with Kolmogorov complexity and descriptive set theory as described in various papers by Fouché ([16], [17], [18]) shows that it has many beautiful and interesting properties that are still to be discovered.
The notion of Salem set (or set of multiplicity in general) is not fully understood and we hope that by exploring different examples provided by nature, it will become clearer how such sets can be characterized.
The thesis is organized as follows. In chapter 1 we introduce the notions of Hausdorff dimension and capacities of compact sets of $\mathbf{R}^n$. Frostman's lemma, which plays a central role in fractal geometry, is discussed in great detail.
Chapter 2 contains a discussion on Fourier transforms of distributions and measures. We provide a detailed proof of the Fourier transform variant of the energy formula. Available proofs in the literature are very sketchy and every effort is made to clarify them. The notions of Fourier dimension and Salem sets are finally presented at the end of the chapter.
General properties of Brownian motion are summarized in chapter 3. We are only interested by those properties which are relevant to the study of fractal and Fourier properties of Brownian motion.
Chapter 4 deals with the notion of local times of Brownian motion. After a brief introduction of the stochastic integration with respect to a Brownian motion, we review some properties of level sets of Brownian motion and present the definition and relevant properties of local times. The notions of Dirac measure and inverse local times of Brownian motion, on which the proofs of the results of this thesis are based, form the last section of the chapter.
In chapter 5 we discuss some fractal properties of Brownian motion. Firstly, we discuss the Hausdorff dimension of level sets and the doubling property of Brownian motion. Secondly, we provide a full proof of Kahane's theorem on the Salemness of Brownian images of compact subsets of Hausdorff dimension $< 1/2$. This chapter is in preparation of the proofs provided in chapter 6.
Finally, in chapter 6 are discussed in depth the asymptotic decays of Fourier transforms of Dirac measures of Brownian motion. We prove in particular that level sets are, with positive probability, Salem sets. The thesis ends with some concluding remarks.

Chapter 1

Hausdorff dimension in Euclidean space

In this chapter we introduce some fundamental notions of fractal geometry that will be used in the sequel. We start with the notion of Hausdorff dimension and we prove Frostman's lemma using ideas of graph theory. Energy and capacities on compact sets and Frostman's theorem which play a central role in this thesis are also discussed in some extent. More details and results of fractal geometry can be found in [13],[24] or [36]) for example. Our exposition is influenced by lectures of Willem Fouché based on Chapter 10 of the book by Kahane [24] as well as the combinatorial treatment of Frostman's lemma by Mörters and Peres [39, Chapter 4].

1.1 Definition of Hausdorff dimension

Given a subset E of the Euclidean space $\mathbf{R}^d$, and real numbers $\alpha \in [0, d]$, $\epsilon > 0$, consider all coverings of E by balls $(B_n : n = 1, 2, \ldots)$ of diameter $\leq \epsilon$ and the corresponding sums

$$\sum_{n \geq 1} |B_n|^\alpha,$$

where $|B| = \sup\{|x - y| : x, y \in B\}$ is the diameter of B. The infimum of these sums over all such coverings by balls of diameter $\leq \epsilon$ is denoted by $H_\alpha^\epsilon(E)$. When ϵ decreases to zero, $H_\alpha^\epsilon(E)$ increases to a limit (which may be infinite). This limit is called the *Hausdorff measure* of E in dimension α and is denoted by $H_\alpha(E)$. In fact one can check that $E \mapsto H_\alpha(E)$ is an outer measure.

If $0 < \alpha < \beta \leq d$, then, for any covering (B_n) of E by balls of diameter $\leq \epsilon$, we have that,

$$\sum_{n \geq 1} |B_n|^\beta \leq \epsilon^{\beta - \alpha} \sum_{n \geq 1} |B_n|^\alpha,$$

from which it follows that

$$H_\beta^\epsilon(E) \leq \epsilon^{\beta - \alpha} H_\alpha^\epsilon(E).$$

Hence, if $H_\alpha(E) < \infty$, then $H_\beta(E) = 0$ and equivalently, if $H_\beta(E) > 0$, then $H_\alpha(E) = \infty$. Therefore,

$$\sup\{\alpha : H_\alpha(E) = \infty\} = \inf\{\beta : H_\beta(E) = 0\}.$$

This common value is called the *Hausdorff dimension* of E and is denoted by $\dim_H(E)$. It has the following elementary properties [36, p 59]:

1. If $E \subset F$, then $\dim_H E \leq \dim_H F$,

2. $\dim_H(\cup_{n=1}^{\infty} E_n) = \sup\{\dim_H(E_n) : n = 1, 2, \ldots\}$,

If $E \subset F$, then $\dim_H E \leq \dim_H F$. Any subset of $\mathbf{R}^d$ of positive Lebesgue measure has Hausdorff dimension d and countable subsets have Hausdorff dimension 0. We will frequently refer to Cantor type sets and we briefly recall their construction. Let $0 < \xi < 1/2$. Starting from the interval $[0, 1]$ we remove an open interval of length $1 - 2\xi$ at the middle of the original interval $[0, 1]$, that is, remove the interval $(\xi, 1-\xi)$. Then from each of the two remaining intervals, remove the interval of length $\xi(1 - 2\xi)$ at the middle of the original intervals. At the nth step, we have 2^n closed intervals of common length ξ^n and each of these generates two subintervals of length ξ^{n+1} by removing an open interval of length $\xi^n(1 - 2\xi)$ at the middle. Let E_n the union of the 2^n intervals of step n and $C_\xi = \cap_{n=1}^{\infty} E_n$. We will call C_ξ the Cantor type set of dissection ratio ξ. We have that $H_\alpha(C_\xi) = 1$ where $\alpha = \log 2/\log(1/\xi)$. (See, for example, the book by Falconer [13]). Then

$$\dim_H C_\xi = \frac{\log 2}{\log(1/\xi)}.$$

The classical ternary Cantor set corresponds to $\xi = 1/3$.
The following proposition provides an upper bound of the Hausdorff dimension of the image of a compact subset by a Hölder-continuous function.

Proposition 1.1 *If E is a compact subset of $\mathbf{R}^n$ and $f : E \to \mathbf{R}^d$ is a function such that*

$$|f(x) - f(y)| \leq C|x - y|^\beta, \ x, \ y \in E$$

where $C > 0$ is a constant and $0 < \beta < 1$, then

$$\dim_H f(E) \leq \min\left\{\frac{\dim_H E}{\beta}, d\right\}.$$

Proof It is clear that $\dim_H f(E) \leq d$ since $f(E)$ is a subset of $\mathbf{R}^d$. We want to show that if $\alpha \geq 0$ is such that $H_\alpha(E) < 1$, then $H_{\alpha/\beta} f(E) < \infty$ and in particular if $\alpha > \dim_H E$, then $\alpha/\beta \geq \dim_H f(E)$. From this it will follow that

$$\dim_H f(E) \leq \frac{\dim_H E}{\beta}.$$

Since for any $\epsilon > 0$, $H_\alpha^\epsilon(E) \leq H_\alpha(E) < 1$, we can cover E by balls (B_n) of diameter $\leq \epsilon$

such that

$$\sum_{n\geq 1} |B_n|^\alpha < 1.$$

Clearly,

$$|f(B_n)| \leq C|B_n|^\beta, \text{ for any } n \geq 0.$$

Let K_n be a ball of diameter $\delta = C|B_n|^\beta$ containing $f(B_n)$. Then

$$H^\delta_{\alpha/\beta} f(E) \leq \sum_{n\geq 1} |K_n|^{\alpha/\beta} \leq C^{\alpha/\beta} \sum_{n\geq 1} |B_n|^\alpha \leq C^{\alpha/\beta}$$

Therefore,

$$H_{\alpha/\beta} f(E) \leq C^{\alpha/\beta} < \infty.$$

∎

1.2 Frostman's lemma

In 1935 Frostman [19] proved the celebrated lemma which provides a relationship between Hausdorff measures and finite measures carried by a compact set. The proof given here is adapted from a proof given in the book by Mörters and Peres [39, pp 83-85]. Other proofs can be found elsewhere [36, pp 112-124].

Theorem 1.2 *If E is a compact subset of $\mathbf{R}^d$ and $0 \leq \alpha \leq d$, then $H_\alpha(E) > 0$ if and only if E carries a probability measure μ such that*

$$\mu(B) \leq c|B|^\alpha \tag{1.1}$$

for all balls B and some constant $c > 0$.

The proof uses ideas from graph theory and the well-known max-flow min-cut theorem of Ford and Fulkerson [14]. Before proving the theorem, we recall the following basic notions of graph theory.

Definition 1.3 *Consider a connected graph $T = (V, F)$ described by a countable set V of vertices including a distinguished vertex ρ, designated as the* root, *and a set $F \subset V \times V$ of ordered edges. T is a* tree *if it has the following properties:*

(1) *for any vertex $v \in V$, $v \neq \rho$, there exists only one $\overline{v} \in V$, called the parent of v such that $(\overline{v}, v) \in F$,*

(2) *for any $v \in V$, there exists a unique path from ρ to v; the number of edges in this path is called the order of v and denoted $|v|$,*

(3) *for every $v \in V$, the set $\{w \in V : (v, w) \in F\}$ of offspring of v is finite.*

For any v, $w \in V$, we denote by $v \wedge w$ the vertex which is the common vertex the paths from the root to v and w with maximal order, that is, the last intersection vertex of the paths $[\rho, v]$ and $[\rho, w]$. The order $|e|$ of an edge $e = (u, v)$ is the order of its end-point v. Every infinite path starting at the root is called a *ray*. The set of rays is denoted ∂T, and is called the boundary of the tree T. For any two rays $\xi, \eta \in \partial T$, we denote by $\xi \wedge \eta$ the vertex in the intersection of these rays of maximum order. The order of $\xi \wedge \eta$ is, therefore, the number of edges that these rays have in common. One can show that

$$d(\xi, \eta) = 2^{-|\xi - \eta|}$$

is a metric on the set ∂T.

Definition 1.4 *Let $T = (V, F)$ be a tree (of root ρ) such that capacities are assigned to its edges by a mapping $k : F \mapsto [0, \infty)$. A* flow *of strength $a > 0$ through T is a function $f : F \mapsto [0, a]$ such that*

(1) $\sum_{w \in V : (\rho, w) \in F} f(\rho, w) = a$,

(2) *(preservation of the flow): $f(\overline{v}, v) = \sum_{w \in V : \overline{w} = v} f(v, w)$ for any $v \in V$, where $\overline{v}$ is the parent of v.*

(3) *$f(e) \leq k(e)$, for any $e \in F$.*

A set Π of edges is called a cutset if every ray contains an edge from Π.

The following proposition is the celebrated max-flow min-cut theorem of Ford and Fulkerson. A proof can be found in Appendix of Mörters and Peres [39].

Proposition 1.5 *The maximum strength of a flow through a tree of capacity k is*

$$\inf \left\{ \sum_{e \in \Pi} k(e) : \Pi \text{ is a cutset of the tree} \right\}.$$

Proof of Theorem 1.2:
Firstly, we prove that the existence of μ implies (1.1). It is clear that $\mu(E) > 0$ since $\mu \neq 0$ and $\mathrm{supp}(\mu) \subset E$. If $(B_n)_{n \geq 1}$ is any covering of E by balls, then

$$0 < \mu(E) \leq \mu(\cup_{n \geq 1} B_n) \leq \sum_{n \geq 1} \mu(B_n) \leq c \sum_{n \geq 1} |B_n|^\alpha$$

and, therefore,

$$\sum_{n \geq 1} |B_n|^\alpha \geq \frac{\mu(E)}{c} > 0.$$

It follows that $H_\alpha^\epsilon(E) \geq \frac{\mu(E)}{c}$ for any $\epsilon > 0$ and hence $H_\alpha(A) \geq \frac{\mu(E)}{c} > 0$.
For the converse of the theorem, we assume without loss of generality that $E \subset [0, 1]^d$ and that $H_\alpha(E) > 0$. The idea is to construct a tree whose boundary contains in some way

the set E and where cutsets of the tree can be interpreted as coverings of E. Let us split the cube $[0,1]^d$ into 2^d new equal dyadic cubes of sidelength $1/2$ and repeat the dissection indefinitely for all the new cubes. Consider the tree $T=(V,F)$ defined as follows:

(1) associate a vertex with each cube; the vertex associated with the original cube $[0,1]^d$ is considered as the root,

(2) for every vertex v, consider 2^d edges emanating from it, corresponding to the 2^d subcubes of the cube associated with v,

(3) remove the edges whose endpoints are subcubes which do not contain elements of E and remove also their endpoints.

It is now clear that a ray of T corresponds to a sequence of nested cubes whose intersection is a point of E. This defines a map $\phi : \partial T \to E$ and it is clear that ϕ is surjective.
Define a capacity k on T by

$$k(e) = (2^{-n}\sqrt{d})^{\alpha} \tag{1.2}$$

where n is the order of the edge e (the dimensional factor $\sqrt{d}$ is omitted in [39]). Note that $k(e) = |A|^{\alpha}$ where A is the subcube associated with the endpoint of e.
For any cutset Π of T, consider the family $\mathcal{C}(\Pi)$ of cubes associated with the vertices of Π. This family is a covering of E. Indeed, any $x \in E$ is of the form $\phi(\xi)$ for some $\xi \in \partial T$. Since Π is a cutset it contains at least one edge e of ξ. The subcube corresponding to the initial point of e contains x. Any cutset can now be seen as a covering of E and therefore (since there are at least many coverings as many cutsets)

$$\inf\left\{\sum_{e\in\Pi} k(e) : \Pi \text{ is a cutset}\right\} \geq \inf\left\{\sum_{n\geq 1} |A_n|^{\alpha} : E \subset \cup_{n\geq 1} A_n\right\}. \tag{1.3}$$

We can now see that the condition $H_{\alpha}(E) > 0$ implies that right hand side of (1.3) is positive. Indeed, suppose that

$$\inf\left\{\sum_{n\geq 1} |A_n|^{\alpha} : E \subset \cup_{n\geq 1} A_n\right\} = 0.$$

This means that $\sum_n |A_n|^{\alpha}$ can be taken arbitrary small; which is possible only if each $|A_n|$ is arbitrary small. Then, after replacing sets A_n by balls of diameter $|A_n|$, this yields $H_{\alpha}^{\epsilon}(E) = 0$, for any $\epsilon > 0$. This contradicts the fact that $H_{\alpha}(E) > 0$. Therefore,

$$\inf\left\{\sum_{e\in\Pi} k(e) : \ \Pi \text{ a cutset}\right\} > 0.$$

By the max-flow min-cut theorem, there exists a flow $f : E \to [0,\infty)$ of positive strength such that $f(e) \leq k(e)$ for any edge $e \in E$.

We want to show how to use the flow f to construct a measure on the boundary ∂T which will induce a measure on E by the map ϕ.
For any edge $e \in F$ we associate the set $T(e) \subset \partial T$ consisting of all rays containing e and denote by $\mathcal{A}$ the class of all such sets. We can add the empty set to this class if it does not contain it. The class $\mathcal{A}$ is a semi-algebra on ∂T in the sense that if $A, B \in \mathcal{A}$ and then $A \cap B \in \mathcal{A}$ and A^c is a finite disjoint union of sets in $\mathcal{A}$. In fact, the first property is obvious, since for any two edges e_1, e_2, $T(e_1) \cap T(e_2) \neq \emptyset$ if and only if one of these sets is a subset of the other. For the second property, note that since $T(e)^c$ is the set of rays that do not contain edge e and the number of offspring of each vertex is finite, there exist $e_1, e_2, \ldots, e_n \in F$ such that $T(e)^c = T(e_1) \cup T(e_2) \cup \ldots \cup T(e_n)$.
We next consider the map $\tilde{\nu} : \mathcal{A} \to [0, \infty)$ defined by

$$\tilde{\nu}(T(e)) = f(e) \text{ and } \tilde{\nu}(\emptyset) = 0.$$

If $T(e_1) \cup T(e_2) \cup \ldots \cup T(e_n) = T(e)$, then because the flow through each vertex is preserved,

$$\tilde{\nu}\left(T(e_1) \cup T(e_2) \cup \ldots \cup T(e_n)\right) = \tilde{\nu}(T(e)) = \tilde{\nu}(T(e_1)) + \ldots + \tilde{\nu}(T(e_n)),$$

which means that ν is finitely additive and hence countably additive, since for each $T(e)$ there is only a finite number of disjoints sets in $\mathcal{A}$ such that $T(e)$ is their union. This is justified by the fact that if $e = (u, v)$ and $v_1, \ldots, v_n$ are the offspring of v, then the only possible decomposition of $T(e)$ as a union of disjoint sets in $\mathcal{A}$ is $T(e) = T(e_1) \cup \ldots \cup T(e_n)$ where $(e_1 = (v, v_1), \ldots, e_n = (v, v_n)$.
By the Caratheodory extension theorem, there exists a measure ν on the σ-algebra $\sigma(\mathcal{A})$ spanned by $\mathcal{A}$ which extends $\tilde{\nu}$. The idea is now to consider the image measure μ of ν by ϕ, which will be a measure on E. Before that, we need to show that ϕ is measurable with respect to $\sigma(\mathcal{A})$ and the Borel σ-algebra $\mathcal{B}(E)$ on E. We first mention the fact that the Borel σ-algebra on $[0, 1]$ is also spanned by the family of intervals of the form $[0, a]$, $(0 \leq a \leq 1)$. If we consider the binary expansion

$$a = \sum_{n=1}^{\infty} a_n 2^{-n}, \;\; a_n = 0 \text{ or } 1,$$

the interval $[0, a]$ can be written as a countable union of dyadic intervals:

$$[0, a] = [0, a_1 2^{-1}] \cup [a_1 2^{-1}, a_1 2^{-1} + a_2 2^{-2}] \cup \ldots \tag{1.4}$$

This indicates that the Borel σ-algebra on $[0, 1]^d$ is also spanned by the family of subcubes obtained from the dissection procedure. It is, therefore, sufficient to show that $\phi^{-1}(S) \in \sigma(\mathcal{A})$, for any subcube S. If $S \cap E = \emptyset$, then $\phi^{-1}(S) = \emptyset$ and there is nothing to prove. Otherwise, we can consider the vertex v associated with S and the edge e pointing to v. Then $\phi^{-1}(S) = T(e)$ and hence it is an element of $\sigma(\mathcal{A})$.

Therefore,

$$\mu(S) = \nu(\phi^{-1}(S)) = \nu(T(e)) = \tilde{\nu}(T(e)) = f(e).$$

The measure μ is nonzero because

$$\mu(E) = \nu(\partial T) = \nu\left(\cup_{j=1}^{n} T(e_j)\right) = \sum_{j=1}^{n} f(e_j) = \text{strength(f)} \ > 0$$

where $e_1, e_2, \dots e_n$ are the edges with common initial point ρ (the source of the tree).
It remains to show that $\mu(B) \leq c|B|^\alpha$ for any ball B of $\mathbf{R}^d$ and some fixed constant c. Consider the integer $n \geq 1$ such that

$$2^{-n} < \frac{|B \cap [0,1]^d|}{\sqrt{d}} \leq 2^{-(n-1)}.$$

Intuitively this means that $B \cap [0,1]^d$ is contained in a cube of side length between 2^{-n} and $2^{-(n-1)}$. In dimension 1, it is clear that at most 3 subcubes (or subintervals) of side length 2^{-n} are needed to cover $B \cap [0,1]$ and one can easily generalise this to 3^d in dimension d. Let us denote these subcubes by S_j, $j = 1, 2, \dots, 3^d$ and consider the edges e_j pointing to the vertices associated with these subcubes. We have that

$$\begin{aligned}
\mu(B) &= \mu(B \cap [0,1]^d \leq \mu\left(\cup_j S_j\right) \\
&\leq \sum_j \mu(S_j) = \sum_j f(e_j) \leq \sum_j k(e_j) \\
&\leq 3^d (2^{-n}\sqrt{d})^\alpha \text{ from relation (1.2))} \\
&\leq 3^d |B \cap [0,1]^d|^\alpha \\
&\leq 3^d |B|^\alpha.
\end{aligned}$$

An obvious scaling of μ by $\mu(E)$ yields a probability measure. ∎

1.3 Energy and capacity

From Frostman's lemma, the Hausdorff dimension of a compact subset of $\mathbf{R}^d$ is closely related to probability measures carried by this set. Another very useful way to characterize a compact subset of $\mathbf{R}^d$ is to consider the energy integrals of non-zero finite measures supported by this set. These energies define another dimension concept which turns out to be identical to Hausdorff dimension by the celebrated theorem of Frostman [19] (see also [24, p 133]).

Definition 1.6 *Consider a compact subset E of $\mathbf{R}^d$ and a real number α such that $0 < \alpha < d$. For a given non-zero finite measure μ supported by E, the energy integral of*

μ with respect to the kernel $k(x) = |x|^{-\alpha}$ is given by

$$I_\alpha(\mu) = \int_{\mathbf{R}^d}\int_{\mathbf{R}^d} \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}.$$

The measure μ is said to have *finite energy* with respect to k if $I_\alpha(\mu) < \infty$. We say that E has *positive capacity* with respect to k and write $\mathrm{Cap}_\alpha(E) > 0$ if E carries a non-zero finite measure of finite energy with respect to k. If there is no such measure we say that E has capacity zero with respect to this kernel and write $\mathrm{Cap}_\alpha(E) = 0$. The following is the Frostman theorem.

Theorem 1.7 *For any compact subset E of $\mathbf{R}^d$ and $0 < \alpha < \beta < d$,*

(1) *if $H_\beta(E) > 0$ then $Cap_\alpha(E) > 0$ and if $Cap_\alpha(E) > 0$ then $H_\alpha(E) > 0$,*

(2) $\sup\{\alpha : Cap_\alpha(E) > 0\} = \inf\{\beta : Cap_\beta(E) = 0\} = \dim_H(E)$.

Proof (1) If $H_\beta(E) > 0$, then, by Frostman's lemma, there is some non-zero finite measure μ carried on E such that $\mu(B) < c|B|^\beta$, for all balls $B \subset \mathbf{R}^d$ and a constant c. In particular, μ is a non-atomic measure. For any fixed $y \in \mathbf{R}^d$, we can partition $\mathbf{R}^d$ into the subsets

$$\begin{aligned} A_j &= \left\{x \in \mathbf{R}^d : \frac{1}{2^{j+1}} \le |x-y| \le \frac{1}{2^j}\right\}, \ j = 1, 2, \ldots \\ A_0 &= \left\{x \in \mathbf{R}^d : |x-y| > \frac{1}{2}\right\}. \end{aligned}$$

Then we have that

$$\int_{\mathbf{R}^d} \frac{d\mu(x)}{|x-y|^\alpha} = \int_{A_0} \frac{d\mu(x)}{|x-y|^\alpha} + \sum_{j=1}^{\infty} \int_{A_j} \frac{d\mu(x)}{|x-y|^\alpha}.$$

The first integral on the right-hand side is bounded by $2^\alpha \mu(E)$ and is, therefore, finite since $\mu(E)$ is finite. For any $j \ge 1$, because $|A_j| \le 1/2^j$, we have that

$$\int_{A_j} \frac{d\mu(x)}{|x-y|^\alpha} \le \int_{A_j} 2^{(j+1)\alpha} d\mu(x) = 2^{(j+1)\alpha}\mu(A_j) \le 2^{(j+1)\alpha} c \left(\frac{1}{2^j}\right)^\beta = \frac{2^\alpha c}{2^{j(\beta-\alpha)}}.$$

It is now clear that

$$\int_{\mathbf{R}^d} \frac{d\mu(x)}{|x-y|^\alpha} < \infty$$

and, therefore, $I_\alpha(E) < \infty$ since $\mu(E) < \infty$. We conclude that $\mathrm{Cap}_\alpha E > 0$.
Let us assume that $\mathrm{Cap}_\alpha E > 0$ and show that $H_\alpha E > 0$. Since $\mathrm{Cap}_\alpha E > 0$, there exists a non-zero finite measure μ carried on E such that $I_\alpha(\mu) < \infty$. For any $t > 0$, define

$$E_t = \left\{y \in E : \int_{\mathbf{R}^d} \frac{d\mu(x)}{|x-y|^\alpha} \le t\right\}.$$

We can choose t so large such that $\mu(E_t) > 0$. Indeed, suppose that for any $t > 0$, $\mu(E_t) = 0$, that is, $\mu(E_t^c) = \mu(E)$. Then for any $t > 0$,

$$I_\alpha(\mu) = \int_{E_t^c} \left(\int_{\mathbf{R}^d} \frac{d\mu(x)}{|x-y|^\alpha} \right) d\mu(y) > \int_{E_t^c} t\mu(y) = t\mu(E).$$

It follows that $I_\alpha(\mu) = \infty$, which is a contradiction. Let us now fix $t > 0$ such that $\mu(E_t) > 0$ and consider a covering $(B_n), n = 1, 2, \ldots$ of E_t by balls of $\mathbf{R}^d$. Our aim to estimate $\sum_n |B_n|^n$ from below. For this purpose, we may assume that $B_n \cap E_t \neq \emptyset$, for all n. Select $y_n \in B_n \cap E_t$, $n = 1, 2, \ldots$ Since $|B_n|^\alpha \geq |x - y_n|^\alpha$ for any $x \in B_n$,

$$\mu(B_n) \leq |B_n|^\alpha \int_{B_n} \frac{d\mu(x)}{|x-y_n|^\alpha} \leq t|B_n|^\alpha \text{ (since } y_n \in E_t).$$

Then

$$\sum |B_n|^\alpha \geq \frac{1}{t}\mu(\cup_n B_n) \geq \frac{1}{t}\mu(E_t) > 0$$

for any covering (B_n). Therefore, $H_\alpha(E_t) \geq \frac{1}{t}\mu(E_t)$ and hence $H_\alpha(E) > 0$.

(2) The first equality is obvious by definition. The second is a consequence of (1). Indeed, denote $\gamma = \sup\{\alpha : \mathrm{Cap}_\alpha(E) > 0\}$. For any $\alpha < \gamma$, we have that $\mathrm{Cap}_\alpha E > 0$ and then from (1), $H_\alpha(E) > 0$. Thus $\dim_H(E) \geq \alpha$ and, therefore, $\dim_H(E) \geq \gamma$. If $\dim_H(E) > \gamma$, then $H_\delta(E) > 0$ for $\gamma < \delta < \dim_H(E)$ and then from (1), $\mathrm{Cap}_\delta(E) > 0$ which is a contradiction. It follows that $\dim_H(E) = \gamma$. ∎

Chapter 2

Fourier transforms and Fourier dimension

The purpose of this chapter is to introduce the notions of Fourier dimension and of Salem sets. These notions are based on the Fourier transforms of measures. In order to give a complete proof of the Fourier transform variant of the capacity formula, we review basic ideas of the Fourier transform of tempered distributions. This formula is one of the most important relations of fractal geometry. A full proof of this formula based on a sketchy proof found in Mattila [36] is provided.

2.1 Fourier transform of integrable functions

Given a function $f \in L^1(\mathbf{R^n})$, its *Fourier transform* is the function defined by

$$\hat{f}(u) = \int e^{iux} f(x)dx, \;\; u \in \mathbf{R}^n. \tag{2.1}$$

The function $\hat{f}$ is continuous.
If $\hat{f}$ is also summable, that is, $\hat{f} \in L^1(\mathbf{R^n})$, then we have the following Fourier inversion formula [42, p 185]:

$$f(x) = \frac{1}{(2\pi)^n} \int e^{-iux} \hat{f}(u)du, \;\; \text{almost everywhere in } \mathbf{R}^n. \tag{2.2}$$

If $f_1,\ f_2 \in L^1(\mathbf{R^n})$ and their Fourier transforms $\hat{f}_1, \hat{f}_2$ belong to $L^2(\mathbf{R^n})$, then we have the formula [42, p 187]:

$$\frac{1}{(2\pi)^n} \int \hat{f}_1(u)\overline{\hat{f}_2(u)}du = \int f_1(x)\overline{f_2(x)}dx. \tag{2.3}$$

In particular,

$$\frac{1}{(2\pi)^n} \int |\hat{f}_1(u)|^2 du = \int |f_1(x)|^2 dx.$$

The Fourier transform of the Gaussian function

$$f(x) = e^{-s|x|^2}, \; s > 0$$

is given by (see, for example [47, p 38-41])

$$\hat{f}(u) = (\pi/s)^{n/2} e^{-|u|^2/4s}. \tag{2.4}$$

2.2 Fourier transform of tempered distributions

We will need Fourier transforms of functions that are not necessarily in $L^1(\mathbf{R^n})$ but are locally integrable. The usual way to define their Fourier transforms is to consider them as tempered distributions.
Let us introduce the following common notations. For any $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbf{N}^n$ and $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$,

$$\begin{aligned} |\alpha| &= \alpha_1 + \ldots + \alpha_n \\ x^\alpha &= x_1^{\alpha_1} \ldots x_n^{\alpha_n} \\ \partial^\alpha &= \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}. \end{aligned}$$

Consider an open subset W of $\mathbf{R}^n$ and the linear space $C_0^\infty(W)$ of C^∞-functions defined on W having compact support. This space can be endowed by the structure of locally convex topological space as follows [1, pp 24-25]:

1. Write
$$C_0^\infty(W) = \cup_{K \in \mathcal{K}} C_K^\infty(W)$$
where $\mathcal{K}$ is the class of all compact subsets of W, and $C_K^\infty(W)$ the subset of $C_0^\infty(W)$ whose elements have support in K.

2. Endow $C_K^\infty(W)$ with the topology defined by the family of norms
$$p_i(\phi) = \sup\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \le i\}, \; i \in \mathbf{N}$$
that is, consider the neighborhood system of zero to be the family of balls
$$B_i(r) = \{\phi \in C_K^\infty(W) : p_i(\phi) < r\}, \; r > 0 \text{ and } i \in \mathbf{N}.$$

3. Endow $C_0^\infty(W)$ with the inductive limit topology of the topologies on the spaces $C_K^\infty(W)$, that is, the neighborhood system of zero is the class of subsets $U \subset C_0^\infty(W)$ such that $U \cap C_K^\infty(W)$ is a neighborhood of zero in $C_K^\infty(W)$ and that U is convex and balanced in the sense that, for any $f \in U$ and $\lambda \in \mathbf{C}$ such that $|\lambda| \le 1$, it follows that $\lambda f \in U$.

The space $C_0^\infty(W)$ endowed with this topology is denoted $\mathcal{D}(W)$.

Definition 2.1 *A distribution on W is a continuous linear functional on $\mathcal{D}(W)$.*

The set of distributions on W is denoted $\mathcal{D}'(W)$. For any locally integrable function f on W, the linear functional

$$\langle f, \phi \rangle = \int_W f(x)\phi(x)dx, \ \ \phi \in \mathcal{D}(R^n),$$

is a distribution on W. Another important example of distribution is the classical Dirac distribution δ_x, $x \in \mathbf{R^n}$ defined by

$$\langle \delta_x, \phi \rangle = \phi(x), \ \phi \in \mathcal{D}(R^n).$$

We will simply denote δ_0 by δ. Any Borel measure μ with compact support defined on $\mathbf{R^n}$ induces a distribution on $\mathbf{R}^n$ as follows:

$$\langle \mu, \phi \rangle = \int \phi(x)d\mu(x), \ \phi \in \mathcal{D}(R^n)$$

Definition 2.2 *A sequence (T_n) in $\mathcal{D}'(W)$ converges (weakly) to the distribution T if, for every $\phi \in \mathcal{D}(W)$, the sequence $\langle T_n, \phi \rangle$ converges to $\langle T, \phi \rangle$ in $\mathbf{C}$.*

Usually, it is useful to consider the Dirac distribution as the limit of a sequence of integrable functions. The following proposition gives such a sequence [1, pp 48-49]:

Proposition 2.3 *For any non-negative integrable function f on $\mathbf{R}^n$ such that $\int f(x)dx = 1$, the family $(f_\epsilon), \epsilon > 0$ defined by*

$$f_\epsilon(x) = \frac{1}{\epsilon^n} f\left(\frac{x}{\epsilon}\right)$$

converges to δ in $\mathcal{D}'(R^n)$ as $\epsilon \to 0$.

An example is given by

$$f_\epsilon(x) = \frac{1}{(2\pi\epsilon)^{-n/2}} e^{-|x|^2/2\epsilon}.$$

We now turn to a specific class of distributions called tempered distributions, to which it is possible to extend Fourier transforms.

Definition 2.4 *A function $\phi \in C^\infty(\mathbf{R})$ is said to be* rapidly decreasing *if*

$$\sup_{x \in \mathbf{R}^n} |x^\alpha \partial^\beta \phi(x)| < \infty$$

for all multi-indices α and β. This is equivalent to

$$\sup_{|\beta| \leq m} \sup_{x \in \mathbf{R}^n} \left(1 + |x|^2\right)^m |\partial^\beta \phi(x)| < \infty,$$

for all integers $m \geq 1$.

The space of rapidly decreasing functions on $\mathbf{R}^n$ is denoted $\mathcal{S}(\mathbf{R}^n)$. It is a linear topological space where the topology is defined by the family of semi-norms $p_{\alpha,\beta}$, α, $\beta \in \mathbf{N}^n$ such that

$$p_{\alpha,\beta}(\phi) = \sup_{x\in\mathbf{R}^n} |x^\alpha \partial^\beta \phi(x)|, \quad \phi \in \mathcal{S}(\mathbf{R}^n).$$

The topological space $\mathcal{D}(\mathbf{R}^n)$ is a dense subspace of $\mathcal{S}(\mathbf{R}^n)$.

Definition 2.5 *A* tempered distribution *is a continuous linear functional on* $\mathcal{S}(\mathbf{R}^n)$.

The set of tempered distributions is denoted $\mathcal{S}'(\mathbf{R}^n)$. Clearly, $\mathcal{S}'(\mathbf{R}^n) \subset \mathcal{D}'(\mathbf{R}^n)$.

Example 2.6

Many properties of fractal geometry are based on the function k defined on $\mathbf{R}^n - \{0\}$ by

$$k(x) = \frac{1}{|x|^\alpha}, \text{ for some } 0 \leq \alpha < n.$$

For any $\phi \in \mathcal{S}(\mathbf{R}^n)$, let

$$\langle k, \phi\rangle = \int k(x)\phi(x)dx.$$

Then $|\langle f, \phi\rangle|$ is finite. Indeed, for some real $A > 0$, we have that

$$\begin{aligned} |\langle k, \phi\rangle| &\leq \int |k(x)\phi(x)|dx \\ &= \int_{|x|\geq A} k(x)|\phi(x)|dx + \int_{|x|<A} k(x)|\phi(x)|dx \end{aligned}$$

For $|x| \geq A$, $k(x) \leq 1/A^\alpha$ and for $|x| < A$, there exists $M > 0$ such that $|\phi(x)| \leq M$ (since ϕ is bounded). Therefore,

$$|\langle k, \phi\rangle| \leq \frac{1}{A^\alpha}\int_{|x|\geq A} |\phi(x)|dx + M\int_{|x|<A} k(x)dx.$$

To show that $\int_{|x|\geq A} |\phi(x)|dx < \infty$, we note since $\phi \in \mathcal{S}(\mathbf{R}^n)$, then

$$\sup_{x\in\mathbf{R}^n} (1 + |x|^2)^n|\phi(x)| < \infty$$

and hence

$$|\phi(x)| = (1 + |x|^2)^{-n}(1 + |x|^2)^n|\phi(x)| \leq H(1 + |x|^2)^{-n}, \text{ for some } H > 0.$$

Therefore,

$$\int |\phi(x)|dx \leq H\int \frac{dx}{(1 + |x|^2)^n} < \infty.$$

The function k defines a tempered distribution (see for example [47] for a more general result).

We have that $\mathcal{S}(\mathbf{R}^n) \subset L^1(\mathbf{R}^n)$ and it is true in general that $\mathcal{S}(\mathbf{R}^n) \subset L^p(\mathbf{R}^n)$ for any $1 \leq p \leq \infty$. Also, $L^p(\mathcal{R}^n) \subset \mathcal{S}'(\mathbf{R}^n)$, ([1, p 122].

The Fourier transform of any $\phi \in \mathcal{S}(\mathbf{R}^n)$ exists. One of the most important properties of the space $\mathcal{S}(\mathbf{R}^n)$ is that the Fourier transform defines a continuous linear operator in $\mathcal{S}(\mathbf{R}^n)$ and the Fourier inversion formula holds, that is: If $\phi \in \mathcal{S}(\mathbf{R}^n)$ then $\hat{\phi} \in \mathcal{S}(\mathbf{R}^n)$ and if ϕ_n converges to ϕ then $\hat{\phi}_n$ converges to $\hat{\phi}$ and, for any $x \in \mathbf{R}^N$,

$$\phi(x) = \frac{1}{(2\pi)^n}\int e^{-iux}\hat{\phi}(u)du,\ \ \phi \in \mathcal{S}(\mathbf{R}^n).$$

(See for example, Theorems 4.3 and 4.4 in [1, p 124-125]).
We are now ready to extend the Fourier transform operator on tempered distributions.

Definition 2.7 *For any $T \in \mathcal{S}'(\mathbf{R}^n)$, the Fourier transform $\hat{T}$ of T is defined by*

$$\langle \hat{T}, \phi\rangle = \langle T, \hat{\phi}\rangle.$$

If $\phi \in \mathcal{S}(\mathbf{R}^n)$, then ϕ defines a tempered distribution T_ϕ by $\langle T_\phi, \psi\rangle = \int \phi(x)\psi(x)dx$ and its Fourier transform is denoted $\hat{T}_\phi$. The Fourier transform $\hat{\phi}$ of ϕ also defines a tempered distribution because it is an element of $\mathcal{S}(\mathbf{R}^n)$. Let us denote it by $T_{\hat{\phi}}$. Then we have that $\hat{T}_\phi = T_{\hat{\phi}}$.
Let us now find the Fourier transform of the tempered distribution defined by the function $k(x) = 1/|x|^\alpha$.

Proposition 2.8 *For any $0 \leq \alpha < n$, the Fourier transform of the tempered distribution defined by the function $k(x) = \frac{1}{|x|^\alpha}$, $x \in \mathbf{R}^n - \{0\}$, is the tempered distribution defined by the function*

$$\hat{k}(u) = \frac{\pi^{n/2}2^{\alpha+n}\Gamma(\alpha/2+n/2)}{\Gamma(\alpha/2)}|u|^{\alpha-n},\ \ u \in \mathbf{R}^n, \tag{2.5}$$

where

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt,\ z > 0$$

is the gamma function.

Proof The following proof is adapted from the book by Strichartz [47, pp 50-51] We start by considering the integral

$$I = \int_0^\infty s^{\frac{\alpha}{2}-1}e^{-s|x|^2}ds$$

With the variable change $h = s|x|^2$, we find, that,

$$I = \frac{1}{|x|^\alpha}\int_0^\infty s^{\frac{\alpha}{2}-1}e^{-s}ds = \frac{\Gamma(\alpha/2)}{|x|^\alpha}.$$

Therefore, we have the identity

$$\frac{1}{|x|^\alpha} = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty s^{\frac{\alpha}{2}-1} e^{-s|x|^2} ds, \ x \neq 0.$$

For any $\phi \in \mathcal{S}(\mathbf{R}^n)$, we have that

$$\begin{aligned}
\langle \hat{k}, \phi \rangle &= \langle k, \hat{\phi} \rangle \\
&= \int_{\mathbf{R}^n} k(x) \hat{\phi}(x) dx \\
&= \frac{1}{\Gamma(\alpha/2)} \int_{\mathbf{R}^n} \left(\int_0^\infty s^{\frac{\alpha}{2}-1} e^{-s|x|^2} ds \right) \hat{\phi}(x) dx \\
&= \frac{1}{\Gamma(\alpha/2)} \int_{\mathbf{R}^n} \left(\int_0^\infty s^{\frac{\alpha}{2}-1} e^{-s|x|^2} ds \right) \int_{\mathbf{R}^n} e^{ixz} \phi(z) dz dx \\
&= \frac{1}{\Gamma(\alpha/2)} \int_{\mathbf{R}^n} \int_0^\infty \left(\int_{\mathbf{R}^n} e^{ixz} e^{-s|x|^2} dx \right) s^{\frac{\alpha}{2}-1} \phi(z) ds dz \text{ (by Fubini's theorem)} \\
&= \frac{1}{\Gamma(\alpha/2)} \int_{\mathbf{R}^n} \int_0^\infty (\pi/s)^{n/2} e^{-|z|^2/4s} s^{\frac{\alpha}{2}-1} \phi(z) ds dz \ \ \text{(from relation (2.4))} \\
&= \frac{\pi^{n/2}}{\Gamma(\alpha/2)} \int_{\mathbf{R}^n} \left(\int_0^\infty s^{\frac{\alpha}{2}-\frac{n}{2}-1} e^{-|z|^2/4s} \phi(z) ds \right) dz \\
&= \frac{\pi^{n/2} 2^{n-\alpha}}{\Gamma(\alpha/2)} \int_{\mathbf{R}^n} \left(\int_0^\infty |z|^{\alpha-n} e^{-h} h^{-\frac{\alpha}{2}+\frac{n}{2}-1} dh \right) \phi(z) dz \text{ (by taking } s = |z|^2/4h) \\
&= \frac{\pi^{n/2} 2^{n-\alpha} \Gamma((n-\alpha)/2)}{\Gamma(\alpha/2)} \int_{\mathbf{R}^n} |z|^{\alpha-n} \phi(z) dz \\
&= c(\alpha, n) \langle g, \phi \rangle
\end{aligned}$$

where $g(z) = |z|^{\alpha-n}$ and

$$c(\alpha, n) \ = \ \frac{\pi^{n/2} 2^{n-\alpha} \Gamma((n-\alpha)/2)}{\Gamma(\alpha/2)}.$$

It follows that $\hat{k}$ is the tempered distribution defined by the function $c(\alpha, n) g$. ∎

2.3 Fourier transform of measures

We will need the notion of Fourier transform of finite measure of compact support.

Definition 2.9 *The Fourier transform of a finite measure μ on $\mathbf{R}^n$ of compact support is the function defined by*

$$\hat{\mu}(u) = \int e^{iux} d\mu(x), \ u \in \mathbf{R}^n.$$

Since $\mu(\mathbf{R}^n) < \infty$, the function $\hat{\mu}$ is bounded uniformly continuous. In the sequel, it will be useful to approximate measures by convolution products. We consider the following definitions.

Definition 2.10 *Let f and g be real functions on $\mathbf{R}^n$ and let μ be a finite measure of compact support on $\mathbf{R}^n$. The convolutions $f * g$ of f and g, and $f * \mu$ of f and μ are* *defined by*

$$\begin{aligned} f * g(x) &= \int f(x-y)g(y)dy \\ f * \mu(x) &= \int f(x-y)d\mu(y) \end{aligned}$$

provided the integrals exist.

Clearly, we have that

$$\widehat{f * g} = \hat{f}\hat{g} \text{ and } \widehat{f * \mu} = \hat{f}\hat{\mu} \tag{2.6}$$

provided the involved integrals exist.

Definition 2.11 *An* approximate identity $(\phi_\epsilon)_{\epsilon>0}$ *is a family of non-negative continuous functions on* $\mathbf{R}^n$ *such that the support of each* ϕ_ϵ *is contained in the ball* $B(\epsilon)$ *of centre* 0 *and radius* ϵ *and* $\int \phi_\epsilon dx = 1$.

Such families are usually constructed by taking a continuous function $f : \mathbf{R}^n \to [0, \infty)$ such that its support is contained in $B(1)$ and $\int f(x)dx = 1$ and then to define

$$\phi_\epsilon(x) = \epsilon^{-n} f(x/\epsilon),\ \epsilon > 0.$$

We have the following proposition. The first part is adapted from [36, p 20] and the second from [42, p 184].

Proposition 2.12 *Let* (ϕ_ϵ) *be an approximate identity.*

(1) *If* μ *is a compactly supported finite measure defined on* $\mathbf{R}^n$*, then the family of functions* $\phi_\epsilon * \mu$ *converges weakly to* μ *as* ϵ *tends to* 0*, in the sense that*

$$\lim_{\epsilon \to 0} \int f(x)(\phi_\epsilon * \mu)(x)dx = \int f(x)d\mu(x)$$

for any uniformly continuous bounded function f *defined on* $\mathbf{R}^n$.

(2) *If* g *is a locally integrable function defined on an open subset of* $\mathbf{R}^n$ *and continuous at* x*, then*

$$\lim_{\epsilon \to 0} g * \phi_\epsilon(x) = g(x).$$

Proof (1) Because $\int \phi_\epsilon dx = 1$, we have that

$$\begin{aligned} & \int f(x)(\phi_\epsilon * \mu)(x)dx - \int f(x)d\mu(x) \\ = & \int f(x) \left(\int \phi_\epsilon(x-y)d\mu(y) \right) dx - \int f(y)d\mu(y) \int \phi_\epsilon(x)dx \\ = & \int \left(\int f(x)\phi_\epsilon(x-y)dx \right) d\mu(y) - \int \left(\int f(y)\phi_\epsilon(x)dx \right) d\mu(y) \ \text{(by Fubini's theorem)} \end{aligned}$$

$$
\begin{aligned}
&= \int \left(\int f(h+y)\phi_\epsilon(h)dh \right) d\mu(y) - \int \left(\int f(y)\phi_\epsilon(x)dx \right) d\mu(y) \ \ \text{(by taking } h = x-y) \\
&= \int \left(\int (f(x+y) - f(y))\phi_\epsilon(x)dx \right) d\mu(y).
\end{aligned}
$$

Since f is uniformly continuous and bounded, then for any $\gamma > 0$, there exists $\delta > 0$ such that for any h, y with $|h - y| < \delta$ we have $|f(h) - f(y)| < \gamma$. Then by taking $\epsilon > 0$ sufficiently small such that $\epsilon < \delta$, we have that for any $x \in B(\epsilon)$, and any $y \in \mathbf{R}^n$, $|(x+y)-y| < \delta$ and hence $|f(x+y)-f(y)| < \gamma$. Using the fact $B(\epsilon)$ contains the support of ϕ_ϵ, one finds that

$$
\begin{aligned}
\left| \int f(x)(\phi_\epsilon * \mu)(x)dx - \int f(x)d\mu(x) \right| &= \left| \int \left(\int_{B(\epsilon)} (f(x+y) - f(y))\phi_\epsilon(x)dx \right) d\mu(y) \right| \\
&\le \int \int \gamma\phi_\epsilon(x)dxd\mu(y) \\
&= \gamma\mu(\mathbf{R}^n).
\end{aligned}
$$

Since $\mu(\mathbf{R}^n) < \infty$, we conclude that

$$
\lim_{\epsilon \to 0} \int f(x)(\phi_\epsilon * \mu)(x)dx - \int f(x)d\mu(x) = 0.
$$

(2) As previously, we write

$$
g * \phi_\epsilon(x) - g(x) = \int_{B(\epsilon)} (g(x-t) - g(x))\phi_\epsilon(t)dt.
$$

Since g is continuous at x, for any $\gamma > 0$, there exists $\delta > 0$ such that $|g(y)-g(x)| < \gamma$ holds for any y such that $|y-x| < \delta$. By taking $y = x-t$ and $\epsilon < \delta$, we find $|g(x-t)-g(x)| < \gamma$, for any $|t| < \epsilon$. It follows that

$$
|g * \phi_\epsilon(x) - g(x)| \le \gamma \int \phi_\epsilon(t)dt = \gamma
$$

and hence $\lim_{\epsilon \to 0} g * \phi_\epsilon(x) = g(x)$. ∎

2.4 Fourier transform and capacities

Consider a compact subset E of $\mathbf{R}^n$. We recall that, by Frostman's theorem (Theorem 1.7), the energy integrals

$$
I_\alpha(\mu) = \int \int k(x-y)d\mu(x)d\mu(y) = \int \int \frac{d\mu(x)d\mu(y)}{|x-y|^\alpha}
$$

are very useful in the calculation of the Hausdorff dimension of E.
The following theorem, which relates $I_\alpha(\mu)$ to $\hat{\mu}$, is an important result of fractal geometry [9, p 22-23], [36, pp 162-163].

Theorem 2.13 *For any finite measure μ on $\mathbf{R}^n$ with compact support, and any $0 \le \alpha < n$,*

$$I_\alpha(\mu) = \frac{1}{(2\pi)^n} \int \hat{k}(u)|\hat{\mu}(u)|^2 du, \tag{2.7}$$

where $\hat{k}$ is given by relation (2.5).

Proof The following proof, which is quite long, is adapted from a sketchy proof given in the book by Mattila [36, pp 162-163].
We approximate the function $k(x) = 1/|x|^\alpha$ by the convolution product of k by an approximate identity. Consider an approximate identity (ϕ_ϵ) defined by $\phi_\epsilon(x) = \epsilon^{-n} f(x/\epsilon)$ where f is a C^∞-function whose support is contained in the ball $B(1/2)$ of radius $1/2$ and centre 0 and such that $\int f(x)dx = 1$. It is clear that $\psi_\epsilon = \phi_\epsilon * \phi_\epsilon$ is also an approximate identity and we have that $\psi(x) = \epsilon^{-n} f * f(x/\epsilon)$. We will also assume that $f(x) = f(-x)$ and that the Fourier transform $\hat{f}$ is a non-negative function.
For any $x \neq 0$, the function k is continuous at x and from Proposition 2.12, it follows that $\lim_{\epsilon\to 0} k * \psi_\epsilon(x) = k(x)$. This is also true for $x = 0$ if we take $k(0) = \infty$. Then, by Fatou's lemma,

$$I_\alpha(\mu) = \int\int k(x-y)d\mu(x)d\mu(y) \quad \le \quad \liminf_{\epsilon\to 0} \int\int k * \psi_\epsilon(x-y)d\mu(x)d\mu(y).$$

(1) Firstly, we want to show that

$$\liminf_{\epsilon\to 0} \int\int k * \psi_\epsilon(x-y)d\mu(x)d\mu(y) \le \frac{1}{(2\pi)^n} \int \hat{k}(u)|\hat{\mu}(u)|^2 du. \tag{2.8}$$

In order to make use of Fubini's theorem to compute these integrals, we need to show that

$$\int\int k * \psi_\epsilon(x-y)d\mu(x)d\mu(y) < \infty.$$

We have that, for any $z \neq 0$ in $\mathbf{R}^n$, if $\frac{|z|}{2} > \epsilon$ then for any $u \in B(\epsilon)$,

$$\frac{1}{|z-u|^\alpha} \le \frac{2^{-\alpha}}{|z|^\alpha}. \tag{2.9}$$

Indeed, the function $u \to |z-u|^\alpha$ attains its minimum at $u = \epsilon z/|z|$ and hence

$$\sup_{u\in B(\epsilon)} \frac{|z|^\alpha}{|z-u|^\alpha} = \frac{1}{(1-\epsilon/|z|)^\alpha} \le 2^{-\alpha}.$$

Now for $x - y \neq 0$, and $|x-y|/2 > \epsilon$, we find that

$$k * \psi_\epsilon(x-y) = \int_{B(\epsilon)} k(x-y-u)\psi_\epsilon(u)du \le 2^{-\alpha}k(x-y) \int_{B(\epsilon)} \psi_\epsilon(u)du = 2^{-\alpha}k(x-y).$$

For $|x-y|/2 \leq \epsilon$, we have that

$$\begin{aligned} k * \psi_\epsilon(x-y) &= \int_{B(\epsilon)} k(x-y-u)\psi_\epsilon(u)du \\ &\leq H \int_{B(\epsilon)} k(x-y-u)du \text{ (where } H = \sup \psi_\epsilon) \\ &\leq H \int_{B(3\epsilon)} \frac{dt}{|t|^\alpha} \text{ (since } |x-y| \leq 2\epsilon) \ (t = x-y-u) \\ &= C(\epsilon). \end{aligned}$$

Therefore, $k * \psi_\epsilon(x-y)$ is bounded by a constant depending only on ϵ. Now it follows that

$$\int\int k * \psi_\epsilon(x-y)d\mu(x)d\mu(y) \leq 2^{-\alpha} \int\int k(x-y)d\mu(x)d\mu(y) + C(\epsilon)A < \infty$$

where $A = (\mu(\mathbf{R}^n))^2$.
(2) We have that,

$$\begin{aligned} k * \psi_\epsilon(x-y) &= (k * \phi_\epsilon) * \phi_\epsilon(x-y) \\ &= \int k * \phi_\epsilon(x-y-h)\phi_\epsilon(h)dh \\ &= \int k * \phi_\epsilon(h-x+y))\phi_\epsilon(h)dh \text{ (by symmetry of } k \text{ and } \phi_\epsilon) \\ &= \int k * \phi_\epsilon(z+y)\phi_\epsilon(z+x)dz \text{ (by taking } z = h-x) \\ &= \int \left(\int k(t)\phi_\epsilon(z+y-t)dt \right) \phi_\epsilon(z+x)dz. \end{aligned}$$

Using Fubini's theorem, we find that

$$\begin{aligned} &\int\int k * \psi_\epsilon(x-y)d\mu(x)d\mu(y) = \\ &\int\int k(t) \left(\int \phi_\epsilon(z+y-t)d\mu(y) \right) \left(\int \phi_\epsilon(z+x)d\mu(x) \right) dtdz. \end{aligned}$$

The first inner integral is

$$\begin{aligned} \int \phi_\epsilon(z+y-t)d\mu(y) &= \int \phi_\epsilon(t-z-y)d\mu(y) \text{ (by symmetry of } \phi_\epsilon) \\ &= \phi_\epsilon * \mu(t-z) \end{aligned}$$

and the second is

$$\int \phi_\epsilon(z+x)d\mu(x) = \int \phi_\epsilon(z-x)d\tilde{\mu}(x) = \phi_\epsilon * \tilde{\mu}(z)$$

where $\tilde{\mu}$ is the measure defined by $\int g(x)d\tilde{\mu}(x) = \int g(-x)d\mu(x)$ for any continuous function g on $\mathbf{R}^n$ with compact support. Therefore,

$$\int\int k * \psi_\epsilon(x-y)d\mu(x)d\mu(y) = \int k(t) \int \phi_\epsilon * \mu(t-z)\phi_\epsilon * \tilde{\mu}(z)dzdt$$

$$= \int k(t)(\phi_\epsilon * \mu) * (\phi_\epsilon * \tilde{\mu})(t)dt.$$

(3) We can now shift to Fourier transforms by using Definition 2.7. By letting

$$(\phi_\epsilon * \mu) * (\phi_\epsilon * \tilde{\mu}) = \hat{H}_\epsilon,\ H_\epsilon \in \mathcal{S}(\mathbf{R}^n)$$

we have, by the inversion formula, that

$$\begin{aligned}
\overline{H_\epsilon} &= \frac{1}{(2\pi)^n}\left(\overline{(\phi_\epsilon * \mu) * (\phi_\epsilon * \tilde{\mu})}\right)^\wedge \\
&= \frac{1}{(2\pi)^n}(\phi_\epsilon * \mu * \phi_\epsilon * \tilde{\mu})^\wedge \text{ (since } \phi \text{ is a real function)} \\
&= \frac{1}{(2\pi)^n}(\hat{\phi}_\epsilon \times \hat{\mu} \times \hat{\phi}_\epsilon \times \hat{\tilde{\mu}}) \text{ (from equation (2.3))} \\
&= \frac{1}{(2\pi)^n}(|\hat{\phi}_\epsilon|^2|\hat{\mu}|^2) \text{ (since } \hat{\tilde{\mu}} = \overline{\hat{\mu}}).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int\int k * \psi_\epsilon(x-y)d\mu(x)d\mu(y) &= \int k(t)\int \phi_\epsilon * \mu(t-z)\phi_\epsilon * \tilde{\mu}(z)dzdt \\
&= \int k(t)\hat{H}_\epsilon(t)dt \\
&= \int \hat{k}(t)H_\epsilon(t)dt \text{ (by Definition 2.7)} \\
&= \frac{1}{(2\pi)^n}\int \hat{k}(t)|\hat{\phi}_\epsilon(t)|^2|\hat{\mu}(t)|^2dt.
\end{aligned}$$

Since for $\epsilon \to 0$, it is the case that $\hat{\phi}_\epsilon(t) = \hat{f}(t\epsilon) \to 1$ (which is the Fourier transform of the Dirac distribution δ), it follows that

$$\liminf_{\epsilon\to 0}\int \hat{k}(t)|\hat{\phi}_\epsilon(t)|^2|\hat{\mu}(t)|^2dt = \frac{1}{(2\pi)^n}\int \hat{k}(t)|\hat{\mu}(t)|^2dt.$$

Therefore

$$I_\alpha(\mu) \le \frac{1}{(2\pi)^n}\int \hat{k}(t)|\hat{\mu}(t)|^2dt.$$

(4) Let us now show that

$$I_\alpha(\mu) \ge \frac{1}{(2\pi)^n}\int \hat{k}(t)|\hat{\mu}(t)|^2dt. \tag{2.10}$$

For this end it is sufficient to show that

$$\limsup_{\epsilon\to 0}\int k * \psi_\epsilon(x-y)d\mu(x)d\mu(y) \le I_\alpha(\mu). \tag{2.11}$$

For $x-y\neq 0$,

$$\begin{aligned} k*\psi_\epsilon(x-y) &= \epsilon^{-n}\int k(x-y-h)(f*f)(h/\epsilon)dh \\ &= \int k(x-y-\epsilon h)(f*f)(h)dh \,(\text{ variable change}). \end{aligned}$$

A useful observation is that for any $z,u\in\mathbf{R}^n$ such that $|z|=1$ and $|u|<1/2$, one has that

$$\frac{1}{|z-u|^\alpha}\leq 1+2|u|. \tag{2.12}$$

Indeed, it is clear that among the u's such that $|u|=\beta<1/2$, the one that maximizes the function $1/|z-u|^\alpha$ is $u=\beta z$ (for fixed z). In that case,

$$\frac{1}{|z-u|^\alpha}=\frac{1}{|z|^\alpha(1-\beta)^\alpha}=\frac{1}{(1-\beta)^\alpha}.$$

The Taylor expansion of

$$\begin{aligned} (1-\beta)^{-\alpha} &= 1+\beta\left(\alpha+\frac{\alpha(\alpha+1)}{2!}\beta+\frac{\alpha(\alpha+1)(\alpha+2)}{3!}\beta^2+\cdots\right) \\ &\leq 1+2\beta \,(\text{ since } \alpha<1,\ \beta<1/2). \end{aligned}$$

Now we write

$$\int\int k*\psi_\epsilon(x-y)d\mu(x)d\mu(y)=J_1+J_2$$

where

$$\begin{aligned} J_1 &= \int\int\int_{\{h\in B(1):\sqrt{\epsilon}|h|\leq|x-y|\}} k(x-y-\epsilon h)(f*f)(h)dhd\mu(x)d\mu(y); \\ J_2 &= \int\int\int_{\{h\in B(1):\sqrt{\epsilon}|h|>|x-y|\}} k(x-y-\epsilon h)(f*f)(h)dhd\mu(x)d\mu(y). \end{aligned}$$

We compute J_1 by noting that if $0<\epsilon<1/4$, $|h|<1$ and $\sqrt{\epsilon}|h|\leq|x-y|$ then

$$\frac{\epsilon|h|}{|x-y|}\leq\sqrt{\epsilon}\leq\frac{1}{2}.$$

Now we have that

$$\begin{aligned} k(x-y-\epsilon h) &= \frac{1}{|x-y|^\alpha}\frac{1}{\left|\frac{x-y}{|x-y|}-\frac{\epsilon h}{|x-y|}\right|^\alpha} \\ &\leq \frac{1}{|x-y|^\alpha}\left(1+\frac{2\epsilon|h|}{|x-y|}\right) \text{ (by relation (2.12))} \\ &\leq \frac{1}{|x-y|^\alpha}(1+2\sqrt{\epsilon}). \end{aligned}$$

Then

$$J_1 \leq (1+2\sqrt{\epsilon})I_\alpha(\mu) \text{ since } \int f * f(h)dh = 1.$$

To compute J_2, we note that for fixed x and y and for $h \in B(1)$ with $\sqrt{\epsilon}|h| > |x-y|\}$, the function $1/|x-y-\epsilon h|^\alpha$ attains its maximum for $h = (x-y)/\sqrt{\epsilon}$. (In general for $u, v \in \mathbf{R}^n$, $|u-v|$ attains its minimum value for $u = Hv$ for some positive constant H.) Therefore, for $h \in B(1)$ and $\sqrt{\epsilon}|h| > |x-y|$,

$$\frac{1}{|x-y-\epsilon h|^\alpha} \leq \frac{1}{(1-\sqrt{\epsilon})^\alpha |x-y|^\alpha} = \frac{c}{|x-y|^\alpha}.$$

Hence

$$\begin{aligned}
J_2 &\leq c \int\int\int_{\{h\in B(1):\sqrt{\epsilon}|h|>|x-y|\}} \frac{1}{|x-y|^\alpha}(f*f)(h)dh d\mu(x)d\mu(y) \\
&\leq c \int_{y\in\mathbf{R}^n}\int_{h\in B(1)}\int_{\{x:\sqrt{\epsilon}>|x-y|\}} \frac{1}{|x-y|^\alpha}(f*f)(h)d\mu(x)dh d\mu(y) \text{ (Fubini's theorem)} \\
&\leq c \int_{y\in\mathbf{R}^n}\int_{\{x:\sqrt{\epsilon}>|x-y|\}} \frac{1}{|x-y|^\alpha} d\mu(x)d\mu(y)
\end{aligned}$$

where the last inequality follows from $\int_{B(1)} f*f(h)dh = 1$.
It follows that $J_1 \to I_\alpha(\mu)$ and $J_2 \to 0$ as $\epsilon \to 0$ and relation (2.11) is proven. ∎

2.5 Fourier dimension

We consider a compact subset E of $\mathbf{R}^n$. The Hausdorff dimension of E and many other notions of dimension give very good indications of the thinness of E. In view of Theorems 1.7(2) and 2.13, the Hausdorff dimension can also be defined as the supremum of the $\alpha \in [0,n)$ for which E carries a nonzero finite measure μ such that

$$\int |u|^{\alpha-n}|\hat{\mu}(u)|^2 du < \infty. \tag{2.13}$$

Roughly speaking [36, p 186], the finiteness of this integral indicates, that for most points u with large norm,

$$|\hat{\mu}(u)| \leq c|u|^{-\alpha/2},\ c>0.$$

One might thing that this is the case for all u such that $|u| \to \infty$ but it is not true in general. For example, there exists a probability measure μ on the ternary Cantor set such that $|\hat{\mu}(u)|$ does not tend to zero for $|u| \to \infty$ even though (2.13) holds for $\alpha = \log 2/\log 3$ (see for example the book by Salem [44, pp 40-41]). This justifies the following definition:

Definition 2.14 *A compact subset E of $\mathbf{R}^n$ is called a M_0-*set *or a* set of multiplicity in the restricted sense *if it carries a non-zero measure μ such that $|\hat{\mu}(u)|^2 \to 0$ as $|u| \to \infty$.*

Such sets are connected with the problem of uniqueness and multiplicity for trigonometric series. If $1/\xi$ is an integer > 2, then the Cantor type set C_ξ of dissection ratio ξ is not a set of multiplicity. This is a particular case of a more general result of Salem and Zygmund (see [44, p 52]) that C_ξ $(0 < \xi < 1/2)$ is a set of multiplicity if and only if $1/\xi$ is not a Pisot number. (A *Pisot number* is a real number > 1 which is a root of a polynomial with integer coefficients and leading coefficient 1 and that all its other roots have norm < 1). For example, all integers > 1 are Pisot numbers and a rational number > 1 is a Pisot number if and only if it is an integer. So, for example, $C_{2/7}$ is a set of multiplicity. If E is a compact set of multiplicity, it is interesting to know at which rate $|\hat{\mu}(u)| \to 0$. This is emphasized by the following definition:

Definition 2.15 *Let E be a compact subset of $\mathbf{R}^n$. The* Fourier dimension *of E is the supremum of the $\alpha \in [0, n]$ for which E carries a non-zero finite measure μ such that*

$$|\hat{\mu}(u)|^2 = O(|u|^{-\alpha}), \quad \textit{for } |u| \to \infty.$$

We will denote the Fourier dimension of E by $\dim_F(E)$. The basic fact concerning Fourier dimension is that it is always less or equal than the Hausdorff dimension. In fact, suppose that $\dim_F(E) = \alpha > 0$. Then for any $0 \leq \beta < \alpha$, and sufficiently small $\epsilon > 0$,

$$|\hat{\mu}(u)|^2 = O(|u|^{-(\beta+\epsilon)}), \quad \text{for } |u| \to \infty.$$

That is, for any $A > 0$ there exists $c > 0$ such that

$$|\hat{\mu}(u)|^2 < c|u|^{-(\beta+\epsilon)}, \quad \text{for } |u| > A.$$

Then

$$\int |u|^{\beta-n}|\hat{\mu}(u)|^2 du = \int_{|u|\leq A} |u|^{\beta-n}|\hat{\mu}(u)|^2 du + \int_{|u|>A} |u|^{\beta-n}|\hat{\mu}(u)|^2 du.$$

The first integral on the right-hand side is bounded by $\int_{|u|\leq A} |u|^{\beta-n} du$ which is finite and the second is bounded by

$$c\int_{|u|>A} |u|^{\beta-n}|u|^{-\beta-\epsilon} du = c\int_{|u|>A} |u|^{-n-\epsilon} < \infty.$$

Therefore, by relation (2.13), $\dim_H E \geq \beta$ and hence $\dim_H E \geq \alpha$, that is $\dim_H E \geq \dim_F E$. However, the Hausdorff and Fourier dimensions may be different. For example the Cantor type set C_ξ, $0 < \xi < 1/2$ where $1/\xi$ is a Pisot number, have Fourier dimension 0 whereas the Hausdorff dimension is nonzero. One should expect that these two notions of dimensions are really different. The Hausdorff dimension is based on the metric structure of the set whereas, as indicated above, the Fourier dimension has something to do with arithmetical properties of the set.

Definition 2.16 *A compact subset of $\mathbf{R}^n$ is called a* Salem set *if its Hausdorff and Fourier dimensions are the same.*

Obvious examples are compact sets of Hausdorff dimension 0 and closed balls of $\mathbf{R}^n$ (with

dimension n). Any sphere of $\mathbf{R}^n$ is also a Salem set of dimension $n-1$. The first example of Salem set defined by Salem [43] was a random set. We will discuss in chapter 5, the beautiful results of Kahane [24] on the existence of other random Salem sets via the theory of Brownian motion. Many other Gaussian stochastic processes generate also Salem sets [46]. Other random Salem sets have been constructed by Bluhm [6]. He showed that a "slight" random perturbation of the Cantor type set C_ξ is a Salem set.
There are, however, very few known examples of non-random Salem sets. Kaufman [27] shows that for any $\alpha > 0$, the set

$$E(\alpha) = \cap_{k=1}^{\infty} \cup_{q=k}^{\infty} \left\{ x \in [0,1] : \|qx\| \leq q^{-1-\alpha} \right\}$$

of "α-well approximable numbers" contains a Salem set of dimension $2/(2+\alpha)$. (Here $\|t\|$ is the distance from t to the nearest integer.) Bluhm [5] showed how, for any $\alpha > 0$, one can construct recursively a sequence of integers M_k such that the following set is a Salem set of dimension $2/(2+\alpha)$:

$$S_\alpha = \cap_{k=1}^{\infty} \cup_{p \in P_{M_k}} \left\{ x \in [0,1] : \|px\| \leq p^{-1-\alpha} \right\},$$

where P_M is the set of prime numbers contained in the interval $[M, 2M]$. Using the notions of "translation sets", Kahane [23] has also shown that there exist Salem sets of dimension one. These are the only deterministic examples of Salem sets known by the author. It is clear that Salem sets are not easy to construct, mainly because the Fourier dimension is very difficult to compute. For example, the author does not know if the Fourier dimension of Cantor type sets C_ξ (for ξ non-Pisot) have been calculated.

Chapter 3

Generalities on Brownian motion

In this chapter we introduce Brownian motion and discuss some results, which are relevant to the study of its fractal properties. Material presented in this chapter is regarded as classical and very clear proofs are available in the literature. We review the Markov properties, the process of passage times and the reflected Brownian motion. Brownian motion is a vast topic and it is impossible to cover everything here.

3.1 Definition and existence of Brownian motion

Definition 3.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A* Brownian motion *on this space is a stochastic process $X = (X_t : t \geq 0)$ defined from $\Omega \times [0, \infty)$ to $\mathbf{R}$ satisfying the following properties:*

1. *Almost surely, $X_0 = 0$, that is, $P\{\omega \in \Omega : X_0(w) = 0\} = 1$.*
2. *For each $\omega \in \Omega$, the path $X(\omega) : [0, \infty) \to \mathbf{R}$, $t \mapsto X_t(\omega)$ is continuous.*
3. *For $0 < t_1 < t_2 < \ldots < t_n$, the random variables*

$$X_{t_1}, X_{t_2} - X_{t_1}, \ldots, X_{t_n} - X_{t_{n-1}}$$

are independent and normally distributed with mean 0 and variance t_1, $t_2 - t_1, \ldots, t_n - t_{n-1}$ respectively.

The last property means that if $A_1, A_2, \ldots, A_n$ are Borel subsets of the reals, and X is a Brownian motion, the probability of the event

$$\Big\{\omega \in \Omega : \Big(X_{t_1}(\omega), X_{t_2}(\omega) - X_{t_1}(w), \ldots, X_{t_n}(\omega) - X_{t_{n-1}}(\omega)\Big) \in A_1 \times A_2 \times \ldots \times A_n\Big\}$$

is given by

$$\prod_{j=1}^{n} \frac{1}{\sqrt{2\pi(t_j - t_{j-1})}} \int_{A_j} \exp\left[\frac{-y^2}{2(t_j - t_{j-1})}\right] dy$$

where $t_0 = 0$ for the sake of convenience.

The canonical model of Brownian motion is constructed as follows (more details can be found in the book by Ito and McKean [22, pp 12-16], also elsewhere). The sample space Ω is $C[0,\infty)$, the set of continuous real functions ω on $[0,\infty)$. A σ-algebra $\mathcal{F}$ and a probability measure on $(C[0,\infty),\mathcal{F})$ are constructed as follows:
For any $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathbf{R}^n$ such that $0 < t_1 < t_2 < \ldots < t_n$, consider the functions

$$X_{\mathbf{t}} : C[0,\infty) \to \mathbf{R}^n$$

defined by

$$X_{\mathbf{t}}(\omega) = (\omega(t_1), \omega(t_2), \ldots \omega(t_n)).$$

Consider the family $\mathcal{C}$ of subsets C of $C[0,\infty)$ of the form

$$C = X_{\mathbf{t}}^{-1}(B) = \{\omega \in C[0,\infty) : X_{\mathbf{t}}(\omega) \in B\} \ B \text{ Borel in } \mathbf{R}^n.$$

The class $\mathcal{C}$ is an algebra on $C[0,\infty)$, (it is non-empty and closed under union and complementary operations). Let us denote the Borel σ-algebra of $\mathbf{R}^n$ by $\mathcal{B}_n$. For any fixed $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ with $0 < t_1 < t_2 < \ldots < t_n$, consider the function

$$P_{\mathbf{t}} : X_{\mathbf{t}}^{-1}(\mathcal{B}_n) \to [0,1]$$

defined by

$$P_{\mathbf{t}}(C) = \int_B g(t_1, 0, b_1) g(t_2 - t_1, b_1, b_2) \ldots g(t_n - t_{n-1}, b_{n-1}, b_n) db_1 db_2 \ldots db_n, \tag{3.1}$$

where $C = X_{\mathbf{t}}^{-1}(B)$ and

$$g(t,a,b) = \frac{1}{\sqrt{2\pi t}} e^{-(b-a)^2/2t}, \ t > 0, \ a, \ b \in \mathbf{R},$$

the density function of the normal distribution with mean a and variance t. Clearly $P_{\mathbf{t}}$ is a probability measure on the σ-algebra $X_{\mathbf{t}}^{-1}(\mathcal{B}_n)$. Using the basic fact that

$$\int g(t-s,a,c) g(s,c,b) dc = g(t,a,b)$$

one can show if $C = X_{\mathbf{t}}^{-1}(B) = X_{\mathbf{s}}^{-1}(D)$ with B Borel in $\mathbf{R}^n$ and D Borel in $\mathbf{R}^m$, then $P_{\mathbf{t}}(C) = P_{\mathbf{s}}(C)$. Then we can consider the function $P : \mathcal{C} \to [0,1]$ defined by

$$P(C) = P_{\mathbf{t}}(C) \text{ if } C \in X_{\mathbf{t}}^{-1}(\mathcal{B}_n) \text{ for some } \mathbf{t} \text{ and some } n.$$

It is proven in [22, pp 17-18] that P can be (uniquely) extended to a probability measure on the σ-algebra $\mathcal{F}$ on $C[0,\infty)$ spanned by $\mathcal{C}$. This probability measure is called the Wiener measure and we will denote it simply by P. Clearly, the σ-algebra $\mathcal{F}$ is also spanned by the family of sets of the form $X_t^{-1}(B)$, $t \geq 0$ and B a Borel subset of $\mathbf{R}$. It is now clear that the process

$$X = (X_t : t \geq 0) \text{ where } X_t(\omega) = \omega(t),\ \omega \in C[0,\infty)$$

is a Brownian motion on the space $(C[0,\infty), \mathcal{F}, P)$. This is the model that we shall assume when we refer to Brownian motion.

Associated with P is the family of probability measures $(P_a : a \in \mathbf{R})$ on $(C[0,\infty), \mathcal{F})$ such that

$$P_a(C) = \int_B g(t_1, a, b_1)g(t_2 - t_1, b_1, b_2)\ldots g(t_n - t_{n-1}, b_{n-1}, b_n)db_1 db_2 \ldots db_n,$$

for $C = X_{\mathbf{t}}^{-1}(B)$, $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ with $0 < t_1 < t_2 < \ldots < t_n$. Obviously, $P_0 = P$. One can show that for any $A \in \mathcal{F}$,

$$P_a(A) = P\{\omega \in C[0,\infty) : \omega + a \in A\}$$

where $\omega + a$ is the translated path defined by $(\omega + a)(t) = \omega(t) + a$, $t \geq 0$. For example, if $A = X_t^{-1}(B)$ for $t > 0$ and B is a Borel subset of $\mathbf{R}$, then

$$P_a(A) = P\{X(t) + a \in B\}.$$

Also, for $t, s \geq 0$,

$$P\{X_{t+s} \in B | X_s = a\} = P_a\{X_t \in B\}.$$

Then for $C \in \mathcal{F}$, $P_a(C)$ is the probability that the event C occurs given that the Brownian path starts at level a. In the sequel, unless otherwise indicated, P will be the probability in use, as defined above.

It is easy to verify that the following processes are also Brownian motions.

$$\begin{aligned} Y_t &= X_{t+s} - X_s \quad \text{for fixed } s \geq 0. \\ Y_t &= \frac{1}{\sqrt{a}} X_{at} \quad \text{for fixed } a > 0. \\ Y_t &= -X_t. \end{aligned}$$

3.2 Some properties of Brownian motion

3.2.1 Markov properties

Let $\mathcal{F}_t$, for $t > 0$, be the sub σ-algebra of $\mathcal{F}$ spanned by the family $(X_s : 0 \leq s \leq t)$ of random variables, that is, the smallest σ-algebra containing the events of the form

$$\{\omega \in C[0,\infty) : a \leq \omega(s) \leq b\},\ 0 \leq s \leq t,\ a, b \in \mathbf{R}.$$

$\mathcal{F}_t$ describes the "past" (and the present) with respect to time t. Similarly, the sub σ-algebra spanned by $(X_u : u > t)$ describes the "future" with respect to t. We shall associate to any $\omega \in C[0, \infty)$ the shifted path

$$\omega_t^+ : [0, \infty) \to \mathbf{R} \text{ defined by } \omega_t^+(s) = \omega(s+t)$$

obtained by cutting off the path corresponding to [0, t] and shifting it back to the point $(0, \omega(t))$. The event $\{\omega \in C[0, \infty) : \omega_t^+ \in A\}$ will be simply denoted $\{\omega_t^+ \in A\}$.

Let us recall that given a sub σ-algebra $\mathcal{G}$ of $\mathcal{F}$ and a non-negative or integrable random variable Y on $(C[0, \infty), \mathcal{F}, P)$, the conditional expectation $E[Y|\mathcal{G}]$ is the Radon-Nikodym derivative of the measure μ, defined on $(C[0, \infty), \mathcal{F})$ by $\mu(A) = \int_A Y dP$, with respect to the restriction of P to $\mathcal{G}$. In that sense, $E[Y|\mathcal{G}]$ is the equivalent class of $\mathcal{G}$-measurable random variables Z such that

$$\int_A Z dP = \int_A Y dP, \ A \in \mathcal{G}. \tag{3.2}$$

The notation $E[Y|\mathcal{G}] = Z$ means that Z is $\mathcal{G}$-measurable and satisfies (3.2).

If H is a random variable, by definition $E[Y|H] = E[Y|\sigma(H)]$ where $\sigma(H)$ is the sub σ-algebra spanned by H. By definition, the conditional probability $P\{A|\mathcal{G}\}$ is $E[1_A|\mathcal{G}]$ for any $A \in \mathcal{F}$.

The following basic properties of conditional expectation can be found elsewhere (see, for example, [12, pp 219-231]):

1. If Y is $\mathcal{G}$-measurable, then $E[Y|\mathcal{G}] = Y$.
2. Assume that Y is independent of $\mathcal{G}$, that is,

   $$P(A \cap B) = P(A)P(B) \text{ for any } A \in \mathcal{G} \text{ and } B \in \sigma(Y),$$

   then $E[Y|\mathcal{G}] = E[Y]$.
3. If Y is $\mathcal{G}$-measurable, then $E[YZ|\mathcal{G}] = YE[Z|\mathcal{G}]$
4. $E[E[Y|\mathcal{G}]] = E[Y]$ provided that the two conditional expectations are defined.

The following theorem is the weak Markov property of Brownian motion. For a proof of a more general statement, see the book by Durrett [12, p 381].

Theorem 3.2 *For any Borel subset A of $\mathbf{R}$ and $s, t \geq 0$, $a \in \mathbf{R}$,*

$$P_a\{X_{t+s} \in A|\mathcal{F}_s\} = P_{X_s}\{X_t \in A\}$$

where $P_{X_s}\{X_t \in A\}$ is the function $C[0, \infty) \to \mathbf{R}$ defined by

$$P_{X_s}\{X_t \in A\}(\omega) = P_{X_s(\omega)}\{X_t \in A\} = \int_A g(t, X_s(\omega), b) db.$$

From this property, we can deduce that for fixed $t \geq 0$, the process Y defined by $Y_s = X_{t+s} - X_t, s \geq 0$ is also a Brownian motion and is independent of $\mathcal{F}_t$.
The strong Markov property is an extension of the weak Markov property to some random functions called stopping times.

Definition 3.3 *A random variable* $\Gamma : C[0, \infty) \to [0, \infty]$ *is called a* Markov time *(stopping time or optional time) if* $\{\Gamma < t\} \in \mathcal{F}_t$ *for any* $t > 0$

Some authors replace the condition $\{\Gamma < t\} \in \mathcal{F}_t$ by $\{\Gamma \leq t\} \in \mathcal{F}_t$ in the definition of stopping times but in this thesis we use the terminology of Ito and McKean [22]. Every constant function $C[0, \infty) \to [0, \infty)$, is a stopping time. Any stopping time Γ is associated with the sub σ-algebra

$$\mathcal{F}_{\Gamma+} = \{A \in \mathcal{F} : A \cap \{\Gamma < t\} \in \mathcal{F}_t, \, \forall t > 0\}.$$

In the particular case of a constant stopping time $\Gamma(\omega) = t \geq 0$,

$$\mathcal{F}_{\Gamma+} = \mathcal{F}_{t+} = \cap_{\epsilon>0} \mathcal{F}_{t+\epsilon}.$$

The weak Markov property remains valid if the sub σ-algebra $\mathcal{F}_t$ is replaced by $\mathcal{F}_{t+}$.
More details on stopping times can be found in the book by Bauer [3, p 435].
Let us recall the Blumenthal 0-1 law as it will prove to be useful in the next chapter.

Theorem 3.4 *Let* $x \in \mathbf{R}$ *and* $A \in \mathcal{F}_{0+}$, *then* $P_x(A) \in \{0, 1\}$.

A proof is given in the book by Mörters and Peres [39, p 35].
The following theorem due to Hunt [21] is called the strong Markov property of Brownian motion.

Theorem 3.5 *For any stopping time* Γ, *under the condition* $\Gamma < \infty$, *the process* $(X_{t+\Gamma} - X_\Gamma : t \geq 0)$ *defined on* $C[0, \infty)$ *by*

$$\omega \to X_{t+\Gamma(\omega)}(\omega) - X_{\Gamma(\omega)}(\omega) = \omega(t + \Gamma(\omega)) - \omega(\Gamma(\omega))$$

is a Brownian motion and it is independent of $\mathcal{F}_{\Gamma+}$.

A detailed proof of this theorem is given in [39, p 38] and an equivalent version of this result is that [22, p 23], for any stopping time Γ, under the condition $\Gamma < \infty$,

$$P_a\{\omega_\Gamma^+ \in C | \mathcal{F}_{\Gamma+}\} = P_{X_\Gamma}(C), \; C \in \mathcal{F}, \, (a \in \mathbf{R})$$

where

$$P_{X_\Gamma}(C)(\omega) = P_b(C) \text{ with } b = X_{\Gamma(\omega)}(\omega), \, \omega \in C[0, \infty)$$

and ω_Γ^+ is the function defined by $\omega_\Gamma^+(t) = \omega(t + \Gamma(\omega))$.

It is also equivalent to the following: For any measurable function $f : C[0, \infty) \to [0, 1]$

$$E[f(\theta_\Gamma)|\mathcal{F}_{\Gamma+}] = \int f(v)dP_{X_\Gamma}(v), \text{ for } \Gamma < \infty, \tag{3.3}$$

where $\theta_\Gamma(\omega) = \omega_\Gamma^+$. This means that for any $A \in \mathcal{F}_{\Gamma+}$ and $A \subset \{\Gamma < \infty\}$,

$$\int_A f(\omega_\Gamma^+)dP(\omega) = \int_A \int f(v)dP_{X_{\Gamma(\omega)}}(v)dP(\omega),$$

the inner integral is taken on the whole space $C[0, \infty)$. A proof can be found in the book by Durrett [12, pp 390-392].

The following result, called the reflection principle of André, is a consequence of the strong Markov property of Brownian motion as shown in [39, p 39]

Theorem 3.6 *If Γ is a stopping time, then the process $(Y_t : t \geq 0)$ defined by*

$$Y_t(\omega) = \begin{cases} X_t(\omega) \text{ if } t \leq \Gamma(\omega) \\ 2X_\Gamma(\omega) - X_t(\omega) \text{ otherwise} \end{cases} \tag{3.4}$$

is a Brownian motion.

Intuitively, if we assume that $X_0(\omega) = 0$, the path $Y_t(\omega)$ is obtained by "stopping" the path ω at time $\Gamma(\omega)$ and considering the symmetry with respect to the line $y = \omega(\Gamma(\omega))$ of the path $\omega(s) : s \geq \Gamma$.

3.2.2 Modulus of continuity of Brownian motion

We have assumed that Brownian paths are continuous and if restricted to the interval $[0, 1]$ (or any other closed interval), they become uniformly continuous. We recall that the modulus of continuity of a uniformly continuous function f is the function K_f defined by

$$K_f(h) = \sup_{|t-s|<h} |f(t) - f(s)|.$$

Lévy [32] showed that the modulus of continuity of Brownian motion K_X is such that, almost surely,

$$K_X(h) \leq \sqrt{2h \log(1/h)}. \tag{3.5}$$

More precisely, he showed that, almost surely

$$\limsup_{h\downarrow 0} \sup_{0\leq t\leq 1-h} \frac{|X_{t+h} - X_t|}{\sqrt{2h \log(1/h)}} = 1.$$

A proof can also be found in the book by Ito and McKean [22, pp 36-38]. In view of Proposition 1.1, as will be discussed later, this result yields an upper bound to the Hausdorff dimension of the image of a compact subset of $[0, 1]$ by a Brownian motion.

3.3 Passage times of Brownian motion

Definition 3.7 *Let* $a \in \mathbf{R}$*. The* passage time *of Brownian motion at level* a *is the function* $\Gamma_a : C[0,\infty) \to [0,\infty]$ *defined by:*

$$\Gamma_a(\omega) = \inf\{t \geq 0 : X_t(\omega) = a\}$$

where $\inf \emptyset = \infty$.

In fact, the name "first passage time" is often used and is obviously appropriate. The following well-known result is the starting point in the analysis of passage times. It is true in general for the hitting time of any closed subset of $\mathbf{R}$ (see the book by Bauer [3, p 439]).

Proposition 3.8 *For any* $a \geq 0$*, the passage time* Γ_a *is a stopping time.*

Proof We want to show that $\{\Gamma_a < t\} \in \mathcal{F}_t$, where $\mathcal{F}_t$ is the σ-algebra spanned by the $(X_s : 0 \leq s \leq t)$. Let

$$T_n = \inf\left\{t \geq 0 : X_t \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right)\right\}, \quad n = 1, 2, \ldots$$

For any $s > 0$,

$$\omega \in \{T_n < s\} \Leftrightarrow \text{ there exists } t < s \text{ such that } X_t \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right).$$

Then

$$\{T_n < s\} = \cup_{0 \leq t < s}\left\{X_t \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right)\right\}.$$

Using the continuity of Brownian paths, this union can be restricted to the rational numbers, that is

$$\{T_n < s\} = \bigcup_{0 \leq t < s,\ t \in \mathbf{Q}} \left\{X_t \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right)\right\}.$$

Now since $\{T_n < s\}$ is a countable union of elements of $\mathcal{F}_s$, it follows that $\{T_n < s\} \in \mathcal{F}_s$ and hence T_n is a stopping time. The next step is to show that $\sup_n T_n = \Gamma_a$. Clearly, for any n, $T_n \leq T_{n+1} \leq \Gamma_a$ and hence $\sup_n T_n \leq \Gamma_a$. To show that $\Gamma_a \leq \sup_n T_n$, we consider 3 mutually exclusive cases.

1. If $\Gamma_a(\omega) = 0$, then $T_n(\omega) = 0$ for all n. Conversely, $T_n(\omega) = 0$ is equivalent to $0 = \inf\left\{\omega^{-1}\left(a - \frac{1}{n}, a + \frac{1}{n}\right)\right\}$. By continuity of Brownian paths, there exists a sequence t_k converging to 0 such that $X_{t_k}(\omega) = \omega(t_k) \in \left(a - \frac{1}{n}, a + \frac{1}{n}\right)$ for all k. Then by continuity, $\omega(t_k) \to \omega(0) \in \left[a - \frac{1}{n}, a + \frac{1}{n}\right]$. Therefore, $T_n(w) = 0$ for all n implies that

$$\omega(0) \in \cap_{n=1}^{\infty}\left[a - \frac{1}{n}, a + \frac{1}{n}\right] = a.$$

Hence $\Gamma_a(\omega) = 0$. Then for the first case, the equality $T = \sup_n T_n = \Gamma_a$ holds.
2. Suppose that $\Gamma_a(\omega) > 0$ and $T = \sup_n T_n < \infty$. Since $\Gamma_a(\omega) > 0$ is equivalent to $w(0) \neq a$ (the path does not start at a), then there exists $n_0 \geq 1$ such that $T_{n_0}(\omega) > 0$. Because $T < \infty$, we have that

$$T_{n_0}(\omega) < T_{n_0+1}(\omega) < T_{n_0+2} < \ldots < \sup_n T_n.$$

Since $T_n(\omega) \to T(\omega)$ as $n \to \infty$, by continuity,

$$X_{T_n}(\omega) = \omega(T_n(\omega)) \to X_T(\omega) = \omega(T(\omega)).$$

But clearly, for $n \geq n_0$, the fact that $T_n(\omega) > 0$ yields $X_{T_n}(\omega) \in \{a - 1/n, a + 1/n\}$, the end points of interval $\left(a - \frac{1}{n}, a + \frac{1}{n}\right)$. Then,

$$\lim X_{T_n}(\omega) = a = X_T(\omega).$$

From $X_T(\omega) = a$ we find $\Gamma_a(\omega) \leq T(\omega)$. It follows that $\Gamma_a = T = \sup_n T_n$.
3. Finally, for $\Gamma_a(\omega) > 0$ and $\sup_n T_n = \infty$, we have that $\Gamma_a(\omega) = \infty$.
Therefore, in all cases, $\Gamma_a(\omega) = \sup T_n(\omega)$.
It is now clear that

$$\{\Gamma_a \leq t\} = \cap_{n=1}^{\infty} \{T_n < t\}$$

from which it follows that $\{\Gamma_a \leq t\} \in \mathcal{F}_t$ and $\{\Gamma_a < t\} = \cup_{n=1}^{\infty} \{\Gamma_a \leq t - 1/n\} \in \mathcal{F}_t$. ∎

The maximum function of Brownian motion is defined by

$$M_t = \sup\{X_s : 0 \leq s \leq t\},\ t \geq 0.$$

For any $a \geq 0$, by the reflection principle, the process

$$Y_t(\omega) = \begin{cases} X_t(\omega) \text{ if } t \leq \Gamma_a(\omega) \\ 2a - X_t(\omega) \text{ otherwise} \end{cases}$$

is a Brownian motion. As almost all paths start at the origin, we have that

$$\begin{aligned} P\{\Gamma_a \leq t\} &= P\{M_t \geq a\} \\ &= P\{M_t \geq a, X_t \geq a\} + P\{M_t \geq a, X_t < a\} \\ &= P\{X_t \geq a\} + P\{\Gamma_a \leq t, X_t < a\} \\ &= P\{X_t \geq a\} + P\{Y_t \geq a\} \\ &= 2P\{X_t \geq a\} \text{ (by the reflection principle)} \\ &= P\{|X_t| \geq a\}. \end{aligned}$$

The probability distribution of the passage time Γ_a follows immediately:

$$P\{\Gamma_a \leq t\} = \frac{2}{\sqrt{2\pi t}} \int_a^\infty e^{-x^2/2t} dx$$

and using the variable change $x = a\sqrt{t/s}$, it follows that

$$P\{\Gamma_a \leq t\} = \int_0^t \frac{a}{\sqrt{2\pi s^3}} e^{-a^2/2s} ds, \tag{3.6}$$

a result discovered by Lévy [33].
From this result we deduce that

$$P\{\Gamma_a = \infty\} = P\{\Gamma_a > t, \text{ for all } t\} = 0. \tag{3.7}$$

In the analysis of passage times that will be done in the sequel, we will need the Fourier transform of their distribution.
For any $a > 0$,

$$E\left[e^{iu\Gamma_a}\right] = e^{-a\sqrt{|u|}(1-i \text{ sign}(u))}. \tag{3.8}$$

As proof, one can show that for $\alpha > 0$,

$$L(\alpha) = E(e^{-\alpha\Gamma(a)}) = e^{-a\sqrt{2\alpha}}$$

by direct calculation and using the integral

$$\int_0^\infty \frac{e^{-\alpha t} e^{-a^2/2t}}{\sqrt{2\pi t}} dt = \frac{e^{-a\sqrt{2\alpha}}}{\sqrt{2\alpha}},$$

(see [22, p 26]). Another way to prove this result is to use the optional sampling theorem (see, for example, the book by Medvegyev [38, p 82]). The formula can now be extended to yield the Fourier transform and the calculations are done on pages 85–86 of the same book by Medvegyev [38].
The stochastic process $(\Gamma_a : a \geq 0)$ of passage times has interesting properties. The following result can be ascribed to Lévy [33] (see also [12, p 393]):

Theorem 3.9 *The process* $(\Gamma_a : a \geq 0)$ *of passage times has independent and stationary increments and its paths are left-continuous.*

Proof We can assume, without loss of generality, that all paths start at the origin. For $0 < a < b$, and $s \geq 0$, we have that $\Gamma_b(\omega) \leq s + \Gamma_a(\omega)$ means that the path ω has reached level b before time $s + \Gamma_a(\omega)$. Then, since $(X_{t+\Gamma_a} - X_{\Gamma_a} : t \geq 0)$ is also a Brownian motion (by the strong Markov property),

$$P\{\Gamma_b - \Gamma_a \leq s\} \quad = \quad P\{\Gamma_b \leq s + \Gamma_a\}$$

$$
\begin{aligned}
&= P\left\{\sup_{0\le u\le s} X_{u+\Gamma_a} \ge b\right\} \qquad (\text{since } X_{\Gamma_a}=a)\\
&= P\left\{\sup_{0\le u\le s} (X_{u+\Gamma_a} - X_{\Gamma_a}) \ge b-a\right\}\\
&= P\left\{\sup_{0\le u\le s} X_u \ge b-a\right\}\\
&= P\left\{\Gamma_{b-a} \le s\right\}.
\end{aligned}
$$

Therefore, $\Gamma_b - \Gamma_a$ and Γ_{b-a} have the same distribution.

To prove the independence, let $0 = a_0 < a_1 < \ldots < a_n$ be real numbers. It is sufficient to show that for any measurable and bounded functions $f_i : \mathbf{R} \to [0,1]$, $(i = 1,2,\ldots,n)$,

$$
E\left[\prod_{i=1}^{n} f_i(\Gamma_{a_i} - \Gamma_{a_{i-1}})\right] = \prod_{i=1}^{n} E(f_i(\Gamma_{a_i} - \Gamma_{a_{i-1}})). \tag{3.9}
$$

Indeed, for fixed Borel subsets A_i, $(i = 1,2,\ldots,n)$ of $\mathbf{R}$, one may consider $f_i = 1_{A_i}$ for any i and obtain that

$$
P\{\Gamma_{a_i} - \Gamma_{a_{i-1}} \in A_i, \forall i = 1,2,\ldots,n\} = \prod_{i=1}^{n} P\{\Gamma_{a_i} - \Gamma_{a_{i-1}} \in A_i\}.
$$

It is clear that for $0 \le a < b$, $\Gamma_b(\omega) - \Gamma_a(\omega) = \Gamma_b(\omega^+_{\Gamma_a})$ for all paths ω. Then $\Gamma_b - \Gamma_a = \Gamma_b \circ \theta_{\Gamma_a}$ where $\theta_{\Gamma_a}(\omega)$ is the path $w^+_{\Gamma_a}$.
From the strong Markov property (relation (3.3)), we have that for any bounded and measurable function $f : \mathbf{R} \to [0,1]$, and $0 \le a < b$,

$$
\begin{aligned}
E[f(\Gamma_b - \Gamma_a)|\mathcal{F}_{\Gamma_a+}] &= E[f(\Gamma_b)\circ\theta_{\Gamma_a}|\mathcal{F}_{\Gamma_a+}]\\
&= \int f(\Gamma_b)(v)dP_{X_{\Gamma_a}}(v)\\
&= \int f(\Gamma_b)(v)dP_a(v) \qquad (\text{because } X_{\Gamma_a}=a)\\
&= E[f(\Gamma_{b-a})]
\end{aligned}
$$

(because $P_a\{\Gamma_b < t\} = P\{\Gamma_b < t|X_0 = a\} = P\{\Gamma_b - a < t\}$.) Therefore

$$
E[f(\Gamma_b - \Gamma_a)] = E[E[f(\Gamma_b - \Gamma_a)|\mathcal{F}_{\Gamma_a+}]] = E[f(\Gamma_{b-a})].
$$

Now relation (3.9) can be proven by induction by conditioning on $\mathcal{F}_{\Gamma_{a_{n-1}}}$.We use the fact that for $0 < a < b$, $\Gamma_a < \Gamma_b$ (almost surely), $\mathcal{F}_{\Gamma_a+} \subset \mathcal{F}_{\Gamma_b+}$ and Γ_a is measurable with respect to $\mathcal{F}_{\Gamma_a+}$. Indeed, if $A = \{\Gamma_a \le t\}$, and $t = s$, then $A \cap \{\Gamma_a < s\} = \{\Gamma_a < s\} \in \mathcal{F}_s$. If $t < s$, then $A \cap \{\Gamma_a < s\} = A \in \mathcal{F}_{s-\epsilon} \subset \mathcal{F}_s$ for some $\epsilon > 0$. Finally, if $t > s$, then $A \cap \{\Gamma_a < s\} = \{\Gamma_a < s\} \in \mathcal{F}_s$.

Then for any $i = 1, 2, \ldots, n-1$, the random variable $f_i(\Gamma_{a_i} - \Gamma_{a_{i-1}})$ is measurable with respect to $\mathcal{F}_{\Gamma_{a_{n-1}}+}$. Therefore,

$$\begin{aligned} E\left[\prod_{i=1}^{n} f_i(\Gamma_{a_i} - \Gamma_{a_{i-1}}) | \mathcal{F}_{\Gamma_{a_{n-1}}+}\right] &= \prod_{i=1}^{n-1} f_i(\Gamma_{a_i} - \Gamma_{a_{i-1}}) E[f_n(\Gamma_{a_n} - \Gamma_{a_{n-1}})) | \mathcal{F}_{\Gamma_{a_{n-1}}+}] \\ &= \prod_{i=1}^{n-1} f_i(\Gamma_{a_i} - \Gamma_{a_{i-1}}) E[f_n(\Gamma_{a_n - a_{n-1}})]. \end{aligned}$$

Relation (3.9) follows by induction (after taking the expectation of both sides) ∎

Remark 3.10

The passage time Γ_a can be seen as the left-continuous inverse of the maximum function of Brownian motion because $\Gamma_a = \inf\{t \geq 0 : M_t \geq a\}$. In fact, Γ_a is the left-end point of the interval where $M_t = a$. It is, therefore, natural to consider the right-hand endpoint of this interval and define the process

$$\rho_a = \inf\{t \geq 0 : M_t > a\},$$

the first time the Brownian motion becomes greater than a. By the same arguments used to study the properties of the process Γ of passage times, one can prove that ρ_a is a stopping time (for any a) and the process $\rho = (\rho_a : a \geq 0)$ is right-continuous, has independent and stationary increments and has the same distribution as Γ. The process ρ will be called the right-continuous inverse of the maximum function of Brownian motion.

3.4 Reflected Brownian motion

In this section, we suppose that all paths start at the origin. Lévy [34] proved that the process $(|X_t| : t \geq 0)$, called the reflected Brownian motion, is a Markov process and is statistically equivalent to the process $(M_t - X_t : t \geq 0)$ where $M_t = \sup_{0 \leq s \leq t} X_s$. Let us recall the following definition:

Definition 3.11 *A function* $p : [0, \infty) \times \mathbf{R} \times \mathcal{B} \to [0, 1]$ (*where* $\mathcal{B}$ *is the Borel* σ*-algebra on* $\mathbf{R}$) *is called a* Markov transition kernel *provided*

(1) $p(., ., A)$ *is measurable on* $[0, \infty) \times \mathbf{R}$ *for any fixed* A,

(2) $p(t, x, .)$ *is a probability measure on* $\mathbf{R}$ *for all* t *and* x,

(3) *for all* $A \in \mathcal{B}$, $x \in \mathbf{R}$ *and* $t, s > 0$,

$$p(t + s, x, A) = \int_{\mathbf{R}} p(t, y, A) p(s, x, dy).$$

Consider a stochastic process $(Y_t : t \geq 0)$ defined on a probability space $(\Omega, \mathcal{H}, Q)$, taking values in $\mathbf{R}$ and a filtration $\{\mathcal{H}_t : t \geq 0\}$ of $\mathcal{H}$ (that is, a family of sub σ-algebras of

$\mathcal{H}$ such that $\mathcal{H}_s \subseteq \mathcal{H}_t$, for $s \leq t$). The process $(Y_t : t \geq 0)$ is a Markov process with transition kernel p if Y_t is $\mathcal{H}_t$-measurable and

$$P\{Y_t \in A|\mathcal{H}_s\} = p(t-s, Y_s, A),$$

for all $t \geq s, A \in \mathcal{B}$, where $p(t-s, Y_s, A)$ is defined on Ω by

$$p(t-s, Y_s, A)(\omega) = p(t-s, Y_s(\omega), A).$$

The reflected Brownian motion is a Markov process with transition kernel

$$p(t, x, A) = P_x\{|X_t| \in A\} = \int_A g^+(t, x, b)db, \ x \geq 0, \ t > 0 \text{ and } A \subseteq [0, \infty),$$

where $g^+(t, x, b) = g(t, -x, b) + g(t, x, b)$ and

$$g(t, a, b) = \frac{e^{-(b-a)^2/2t}}{\sqrt{2\pi t}} \quad \text{is the Gaussian kernel.}$$

The following result is due to Lévy.

Theorem 3.12 *The process $(Y_t = M_t - X_t : t \geq 0)$, where $M_t = \sup_{0 \leq s \leq t} X_s$, is a Markov process and is identical in law with the reflected Brownian motion.*

Proof (Adapted from [39, pp 44-45]). We want to show that the two processes have the same finite dimensional distributions, that is, for any $0 \leq t_1 < t_2 < \ldots < t_n$ and any Borel subsets $A_1, A_2, \ldots, A_n$ of $\mathbf{R}$

$$P\{|X_{t_1}| \in A_1, \ldots, |X_{t_n}| \in A_n\} = P\{Y_{t_1} \in A_1, \ldots, Y_{t_n} \in A_n\}.$$

Because finite dimensional distributions of a Markov process are fully determined by its transition kernel, in fact, the probability $P\{|X_{t_1}| \in A_1, \ldots, |X_{t_n}| \in A_n\}$ is given by

$$\int_{A_1} p(t_1, 0, x_1)dx_1 \int_{A_2} p(t_2 - t_1, x_1, x_2)dx_2 \ldots \int_{A_n} p(t_n - t_{n-1}, x_{n-1}, x_n)dx_n,$$

it is sufficient to show that $(Y_t : t \geq 0)$ is also a Markov process and has the same transition kernel as $(|X_t| : t \geq 0)$. We denote, as previously, the transition kernel of the reflecting Brownian motion by p. The question is to show that, for any $t, s \geq 0$ and any Borel subset A of $\mathbf{R}$,

$$P\{Y_{t+s} \in A|\mathcal{F}_s\} = p(t, Y_s, A) = P_{Y_s}\{|X_t| \in A\}. \tag{3.10}$$

Let us fix $s \geq 0$ and consider the Brownian motion

$$\tilde{X}_t = X_{t+s} - X_s, \ t \geq 0.$$

Let

$$\begin{aligned} M_t &= \sup_{0\le u\le t} X_u \\ \tilde{M}_t &= \sup_{0\le u\le t} \tilde{X}_u \end{aligned}$$

Clearly, $M_{s+t} = M_s \vee (X_s + \tilde{M}_t)$ where $a \vee b = \max(a, b)$. Then since $Y_{s+t} = M_{s+t} - X_{s+t}$, we find $Y_{s+t} = [M_s \vee (X_s + \tilde{M}_t)] - (X_s + \tilde{X}_t)$. From the identity $(a \vee b) - c = (a-c) \vee (b-c)$, we have that $Y_{s+t} = (Y_s \vee \tilde{M}_t) - \tilde{X}_t$. Since both $\tilde{M}_t$ and $\tilde{X}_t$ are independent of $\mathcal{F}_s$, it is sufficient to show that for any $y \ge 0$,

$$P\{(y \vee \tilde{M}_t) - \tilde{X}_t \in A\} = P_y\{|X_t| \in A\} = P\{|y + X_t| \in A\}.$$

Let us fix $a \ge 0$. From $(y \vee \tilde{M}_t) - \tilde{X}_t = (y - \tilde{X}_t) \vee (\tilde{M}_t - \tilde{X}_t)$ we have that

$$P\{(y \vee \tilde{M}_t) - \tilde{X}_t \ge a\} = P\{y - \tilde{X}_t > a\} + P\{y - \tilde{X}_t \le a,\, \tilde{M}_t - \tilde{X}_t) > a\}.$$

By symmetry of Brownian motion, $P\{y - \tilde{X}_t > a\} = P\{y + \tilde{X}_t > a\}$. For the second term, consider the process $(H_u : 0 \le u \le t)$ (for $t > 0$ fixed) defined by $H_u = \tilde{X}_{t-u} - \tilde{X}_t$. One can easily show that this process is also a Brownian motion (it is called the time reversed Brownian motion). The corresponding maximum process is

$$M_l^H = \sup_{0\le u\le l} H_u, \text{ for } (0 \le l \le t).$$

Clearly, $M_t^H = \tilde{M}_t - \tilde{X}_t$ and because $H_t = -\tilde{X}_t$, we have that

$$P\{y - \tilde{X}_t \le a,\, \tilde{M}_t - \tilde{X}_t) > a\} = P\{H_t \le a - y,\, M_t^H > a\}.$$

Consider the Brownian motion $(H_u^* : 0 \le u \le t)$ defined by (see relation (3.4))

$$H_u^*(\omega) = \begin{cases} H_u(\omega) \text{ if } u \le T(\omega) \\ 2H_T(\omega) - H_u(\omega) \text{ otherwise} \end{cases}$$

$(0 \le u \le t)$ where $T(\omega) = \inf\{u \ge 0 : H_u(\omega) = a\}$. Since $M_t^H > a \Leftrightarrow T < a$ and $T < a \Leftrightarrow H_t^* = 2a - H_t$,
$P\{y - \tilde{X}_t \le a,\, \tilde{M}_t - \tilde{X}_t) > a\} = P\{H_t^* \ge a + y\} = P\{H_t \ge a + y\} = P\{-\tilde{X}_t \ge a + y\}$.
Therefore,

$$\begin{aligned} P\{(y \vee \tilde{M}_t) - \tilde{X}_t > a\} &= P\{y + \tilde{X}_t > a\} + P\{y + \tilde{X}_t \le -a\} \\ &= P\{|y + \tilde{X}_t| \ge a\} \\ &= P\{|y + \tilde{X}_t| > a\} = P\{|y + X_t| > a\}. \end{aligned}$$

∎

Chapter 4

Local times of Brownian motion

In this chapter, we discuss the local times of Brownian motion. The local time at a level a is a measure of the time that a Brownian traveler spends at that level. It is a key concept in the study of the properties of Brownian motion. The modern approach to local times is based on the theory of stochastic integration. After an introduction to stochastic integration with respect to Brownian motion in section 1, some classical properties of Brownian level sets are given in section 2. We then introduce local times in section 3, the corresponding measure (called Dirac measure of Brownian motion) in section 4 and the inverse local times in the final section. We keep our exposition to the minimum necessary for the exploration of the Fourier structure of the Dirac measures of Brownian motion.

4.1 Introduction to stochastic integration

In this section, we summarise the construction of stochastic integration with respect to Brownian motion, adapted from the book by Chung and Williams [10, p 28-40]. Proofs of results and extra detail may be found there. For the general theory of stochastic integration and different applications, the reader is referred to the book by Medvegyev [38]. As in the previous chapter, we consider the canonical model of Brownian motion $X = (X_t : t \geq 0)$ on $(C[0,\infty), \mathcal{F}, P)$. For simplification purposes, we denote $C[0,\infty)$ by Ω. Recall that $X_t(\omega) = \omega(t)$ (for $\omega \in C[0,\infty)$) and $\mathcal{F}$ is the σ-algebra spanned by all the variables X_t $(t \geq 0)$. We also consider the filtration $(\mathcal{F}_t : t \geq 0)$ of $\mathcal{F}$ where $\mathcal{F}_t$ is the smallest σ-algebra containing all the P-null sets of $\mathcal{F}$ and with respect to which all X_s , $0 \leq s \leq t$ are measurable. In fact, if we denote by $\mathcal{G}_t$ the σ-algebra spanned by the family $(X_s : 0 \leq s \leq t)$ and $\mathcal{A}$ the collection of P-null sets of $\mathcal{F}$, then $\mathcal{F}_t$ is spanned by $\mathcal{G}_t \cup \mathcal{A}$.

Unless otherwise indicated, all stochastic processes $Y = (Y_t : t \geq 0)$ will be considered as functions $Y : [0,\infty) \times \Omega \mapsto \mathbf{R}$. In order to define integrals of such functions, we first need to define a σ-algebra on the product space $[0,\infty) \times \Omega$.

Let $\mathcal{R}$ be the set of all rectangles of the form $\{0\} \times F_0$ and $(s,t] \times F$, where $0 \leq s < t$, $F_0 \in \mathcal{F}_0$ and $F \in \mathcal{F}_s$. We will consider in the sequel the ring, the algebra and the σ-algebra on $[0, \infty) \times \Omega$ spanned by $\mathcal{R}$.

Recall that a *ring* on a set S is a non-empty class of subsets of S closed under union and difference of sets. As a consequence, it is also closed under intersection. An *algebra* on S is a ring on S which contains S, or equivalently, is closed under complements, while a σ*-algebra* is an algebra which is closed under countable unions.

The ring $\mathcal{G}$ spanned by $\mathcal{R}$ consists of the empty set and all finite unions of disjoint rectangles in $\mathcal{R}$. The algebra $\mathcal{A}$ generated by $\mathcal{R}$ is $\mathcal{G} \cup \{A^c : A \in \mathcal{G}\}$, where A^c denotes the complement of A.

The σ-algebra on $[0, \infty) \times \Omega$ spanned by $\mathcal{R}$ is called the *predictable* σ*-algebra* and is denoted by $\mathcal{P}$. A function $Y : [0, \infty) \times \Omega \mapsto \mathbf{R}$ is called *predictable* if it is $\mathcal{P}$-measurable. If $A \in \mathcal{R}$ then for any $t \geq 0$, the function $1_A(t, .)$ is $\mathcal{F}_t$-measurable. Indeed, if $A = (s, h] \times F$ with $F \in \mathcal{F}_s$ and B is a Borel subset of $\mathbf{R}$, then $\{\omega : 1_A(t, \omega) \in B\} \in \{F, F^c, \Omega, \emptyset\}$ if $t \in (s, h]$ and $\{w : 1_A(t, w) \in B\} \in \{\Omega, \emptyset\}$ otherwise. So, in all cases, $\{1_A(t, \omega) \in B\} \in \mathcal{F}_t$ because $\mathcal{F}_s \subset \mathcal{F}_t$. This means that the process 1_A is adapted to the filtration. The same property holds for A^c. Since for $A, B \in \mathcal{R}$, $A \cap B$ is also in $\mathcal{R}$ if it is nonempty, then $1_{A \cap B}$ is also adapted. Using elementary properties of the indicator function, one can show that if A is in the ring spanned by $\mathcal{R}$, then 1_A is also adapted. The generalization to the algebra spanned by $\mathcal{R}$ follows immediately. The idea is to extend the property to the whole σ-algebra $\mathcal{P}$. This can be achieved by using the following variant of the monotone class theorem (see, for example, the book by Dellacherie and Meyer [11, 14-I]: Let V be a vector space of real-valued bounded functions defined on a set W. Assume that V contains all the constant functions and is such that: for any uniformly bounded increasing sequence of positive functions $f_n \in V$, the function $f = \lim_n f_n$ belongs to V. Let C be a subset of V which is closed under multiplication. Then V contains all bounded functions measurable with respect to the σ-algebra $\sigma(C)$ spanned by C on V.

We take V to be the space of all functions $f : [0, \infty) \times \Omega \to \mathbf{R}$ that are adapted and $C = \{1_A : A \in \mathcal{A}\}$, where $\mathcal{A}$ is the algebra on $[0, \infty) \times \Omega$ spanned by $\mathcal{R}$. Clearly all the hypotheses of the monotone class theorem are verified and $\sigma(C) = \mathcal{P}$. Therefore, all bounded $\mathcal{P}$-measurable functions are adapted. In particular, for $A \in \mathcal{P}$, 1_A is adapted. From the fact that any $\mathcal{P}$-measurable function is a pointwise limit of a sequence of finite linear combinations of indicator functions of sets in $\mathcal{P}$, such a function is an adapted process.

The next step is to define a measure on the space $([0, \infty) \times \Omega, \mathcal{P})$. It is useful to regard the σ-algebra $\mathcal{P}$ as a subset of the product σ-algebra $\mathcal{B} \otimes \mathcal{F}$ where $\mathcal{B}$ is the Borel σ-algebra on $[0, \infty)$ and $\mathcal{F}$ is the fixed σ-algebra on $\Omega = C[0, \infty)$. A canonical measure on the product space $([0, \infty) \times \Omega, \mathcal{B} \otimes \mathcal{F})$ is the product measure $\lambda \times P$ where λ is the Lebesgue measure on $[0, \infty)$ and P is the Wiener measure on Ω (this is the probability measure that we have fixed on Ω). Then $\lambda \times P$ induces a measure on $([0, \infty) \times \Omega)$ that will be denoted by μ for the sake of simplicity. This measure is such that for any $A = (s, t] \times F \in \mathcal{R}$,

$$\mu(A) = E\left[1_F (X_t - X_s)^2\right]$$

where E denotes the expectation (with respect to probability measure P). Indeed, since $F \in \mathcal{F}_s$ and $X_t - X_s$ is independent of $\mathcal{F}_s$, by conditioning on $\mathcal{F}_s$, we find that

$$\begin{aligned} E\left[1_F(X_t - X_s)^2\right] &= E\left[E\left[1_F(X_t - X_s)^2|\mathcal{F}_s\right]\right] \\ &= E[1_F E\left[(X_t - X_s)^2\right] \\ &= E[1_F](t-s) \\ &= (\lambda \times P)[(s,t] \times F]. \end{aligned}$$

And the property is obviously true for $A = \{0\} \times F_0$, $F_0 \in \mathcal{F}_0$.
We denote by $\mathcal{L}^2(\mu)$ the space $L^2([0,\infty) \times \Omega, \mathcal{P}, \mu)$ and $L^2(P)$ the space $L^2(\Omega, \mathcal{F}, P)$. Then a predictable process $Y : [0,\infty) \times \Omega \mapsto \mathbf{R}$ is in $\mathcal{L}^2(\mu)$ if

$$\int_{[0,\infty)\times\mathbf{R}} Y^2(s,\omega)d\mu(s,\omega) = E\left(\int_0^\infty Y^2(s,\omega)ds\right) < \infty.$$

We define the *stochastic integral* with respect to a Brownian motion X of a predictable processes Y as follows:
1. If Y is a $\mathcal{R}$-step function, that is $Y = 1_{(s,t]\times F}$, the indicator function of the rectangle $(s,t] \times F$ in $\mathcal{R}$, then

$$\int Y_s dX_s = 1_F(X_t - X_s),$$

and if $Y = \{0\} \times F_0$, for $F_0 \in \mathcal{F}_0$, then

$$\int Y_s dX_s = 0.$$

2. Suppose now that Y is a finite linear combination of $\mathcal{R}$-step functions (such functions are called $\mathcal{R}$-simple functions), that is,

$$Y = \sum_{j=1}^n c_j 1_{(s_j,t_j]\times F_j} + c_0 1_{\{0\}\times F_0}, \tag{4.1}$$

where $c_j \in \mathbf{R}$, $F_j \in \mathcal{F}_{s_j}$, $s_j < t_j$ for all j and $F_0 \in \mathcal{F}_0$. Any such representation can be taken such that the rectangles $(s_j, t_j] \times F_j$ are disjoint. Then

$$\int Y_s dX_s = \sum_{j=1}^n c_j 1_{F_j}(X_{t_j} - X_{s_j})$$

and one can verify that this integral does not depend on the representation of Y.
3. The stochastic integral can be generalised to all predictable processes $Y \in \mathcal{L}^2(\mu)$. Firstly, we note that if Y is an $\mathcal{R}$-step function, then $Y \in \mathcal{L}^2(\mu)$ and by direct calculation this extends to $\mathcal{R}$-simple functions. Then the space $\mathcal{E}$ of all $\mathcal{R}$-simple functions is a subspace of $\mathcal{L}^2(\mu)$. Moreover (see [10, pp 37-38]), we have that for $Y \in \mathcal{E}$,

$$E\left[\left(\int Y_s dX_s\right)^2\right] = \int_{[0,\infty)\times\Omega} Y^2 d\mu. \tag{4.2}$$

Therefore, the map $\mathcal{E} \mapsto L^2(P)$ defined by $Y \to \int Y_s dX_s$ is an isometry. To finish the construction, note that $\mathcal{E}$ is a dense subspace of the Hilbert space $\mathcal{L}^2(\mu)$ (see [10], p 38, Lemma 2.4) and, therefore, the isometry can be extended uniquely to the whole space $\mathcal{L}^2(\mu)$. By definition $\int Y_s dX_s$ is the image of Y by this isometry. This means that the integral $\int Y_s dX_s$ is obtained by approximating Y (in the $\mathcal{L}^2(\mu)$-norm) by a sequence of $\mathcal{R}$-simple functions and taking the $L^2(P)$-limit of the sequence of integrals of these functions.
4. For any predictable process $Y \in \mathcal{L}^2(\mu)$ and $t \geq 0$, the process $1_{[0,t]}Y$ is also predictable and belongs to $\mathcal{L}^2(\mu)$. By definition,

$$\int_0^t Y_s dX_s = \int 1_{[0,t]}(s) Y_s dX_s.$$

If $Y \in \mathcal{E}$ and (4.1) is a representation for Y, then for $t \geq 0$, $1_{[0,t]}Y \in \mathcal{E}$ and

$$\int_0^t Y_s dX_s = \sum_{j=1}^{n} c_j 1_{F_j} (X_{t_j \wedge t} - X_{s_j \wedge t}),$$

($a \wedge b = \min(a, b)$). By continuity of Brownian paths, we see that the process $H_t = \int_0^t Y_s dX_s$ for $\mathcal{R}$-simple processes Y also has continuous paths. This fact is generalized as follows (see [10, p 40] for a proof):

Theorem 4.1 *For any predictable process $Y \in \mathcal{L}^2(\mu)$, the process $(H_t : t \geq 0)$ defined by $H_t = \int_0^t Y_s dX_s$ has a continuous modification, in the sense that there exists a process $(K_t : t \geq 0)$ with continuous paths such that for each $t \geq 0$, $H_t = K_t$, almost surely.*

We conclude the section with the famous Ito formula (see [10, pp 88-90] for the proof).

Theorem 4.2 (*Ito formula*) *Let $f : \mathbf{R} \mapsto \mathbf{R}$ be twice continuously differentiable such that $1_{[0,t]} f'(X_s) \in \mathcal{L}^2(\mu)$ for some $t > 0$. Then, almost surely, for all $0 \leq s \leq t$,*

$$f(X_s) - f(X_0) = \int_0^s f'(X_u) dX_u + \frac{1}{2} \int_0^s f''(X_u) du.$$

4.2 Level sets of Brownian motion

As already mentioned, local times provide a natural measure of the time that the Brownian motion spends at a given level. For any $a \in \mathbf{R}$, the *level set* of Brownian motion is defined as $Z_a = \{t \geq 0 : X_t = a\}$. We use Z instead of Z_0. For every $\omega \in \Omega$, $Z_a(\omega) = \{t \geq 0 : \omega(t) = a\}$ is a closed set, because Brownian paths are continuous. Clearly for any $t > 0$, $P\{t \in Z_a\} = P\{X_t = a\} = 0$. This means that $\int 1_{t \in Z_a(\omega)} dP(\omega) = 0$ for any $t > 0$. Then if we denote by $\lambda(Z_a)$ the Lebesgue measure of Z_a, by Fubini's theorem we find that

$$E\lambda(Z_a) = \int \int_0^\infty 1_{\{t \in Z_a(\omega)\}} dt dP(\omega) = 0. \tag{4.3}$$

Therefore, almost surely, $\lambda(Z_a) = 0$. That is, almost surely, for λ-almost every $s \geq 0$, we have that $s \notin Z_a$. So Brownian level sets are almost surely of null Lebesgue measure.

One surprising property of level sets of Brownian motion is that they are infinite and have non isolated points. This can be justified as follows [39, p 42]:
Let $T = \inf\{t > 0 : X_t > 0\}$ and $R = \inf\{t > 0 : X_t = 0\}$. We want to show that $P\{R = 0\} = 1$, which means that for any $t > 0$, there exists $0 < s < t$ such that $X_s = 0$ and hence Z is infinite. It is clear that, $T(\omega) = 0$ is equivalent to say that for any $n \in \mathbf{N}$, there exists $\epsilon \in (0, 1/n)$ such that $X_\epsilon > 0$. Then $\{T = 0\} = \cap_{n=1}^{\infty} E_n$ where $E_n = \{\omega :$ there exists $0 < \epsilon < 1/n : X_\epsilon > 0\}$. From this it is now clear that $\{T = 0\} \in \mathcal{F}_{0+}$. By the Blumenthal $0-1$ law (Theorem 3.4), $P\{T = 0\} \in \{0, 1\}$. Let $t > 0$. If $X_t > 0$, then $T \leq t$.
Hence, $P\{T \leq t\} \geq P\{X_t > 0\} = 1/2$. It follows that $P\{T = 0\} \geq P\{T \geq t\} \geq 1/2$ and therefore, $P\{T = 0\} = 1$. Similarly, if $S = \inf\{t > 0 : X_t < 0\}$, then $P\{S = 0\} = 1$. Then we have that almost surely, for any $t > 0$, there exist $s_1, s_2 \in (0, t)$ such that $X_{s_1} > 0$ and $X_{s_2} < 0$. By continuity of Brownian paths we conclude that, almost surely, for any $t > 0$, there exists $0 < s < t$ such that $X_s = 0$. This is equivalent to $P\{R = 0\} = 1$.
To show that Z has no isolated point, we fix $0 < t_1 < t_2$ and define

$$W = \{\omega \in \Omega : Z(\omega) \text{ has only one element in } (t_1, t_2)\}.$$

We want to show that $P(W) = 0$. Consider $R_{t_1} = \inf\{s > t_1 : X_s = 0\}$, which is a stopping time (in fact, $R_t = t + \Gamma_0(\omega_t^+)$, where Γ_0 is the first passage time at 0). By the strong Markov property, $Y_s = X_{s+R_{t_1}} - X_{R_{t_1}} = X_{s+R_{t_1}}$ is a Brownian motion. For any $\omega \in W$, $t_1 < R_{t_1} < t_2$ and the only element of $Z(\omega)$ in the interval (t_1, t_2) is R_{t_1}. From the construction above, $P\{\inf\{s > 0 : Y_s = 0\} = 0\} = 1$. This means that $\inf\{s > 0 : X_{s+R_{t_1}} = 0\} = 0$ almost surely. Therefore, almost surely, Z has infinitely many elements in the interval $[R_{t_1}, t_2)$ and hence W has probability zero.
To extend these properties to other Brownian level sets, it is sufficient to note that, for any $a \in \mathbf{R}$, the level set $Z_a(X) = \{t \geq 0 : X_t = a\}$ is such that $Z_a(X) = \Gamma_a + Z(Y)$, where $Z(Y)$ is the zero set of the Brownian motion $Y_s = X_{s+\Gamma_a} - a$ and Γ_a is the first passage time at level a (for the Brownian motion X).
The following proposition, known as Lévy's arcsine law, gives the distribution of the last element of Z before a specified $t > 0$.

Proposition 4.3 *For any $t > 0$, let $K_t = \sup\{s < t : X_s = 0\}$. Then*

$$P\{K_t \leq h\} = \frac{2}{\pi} \arcsin \sqrt{h/t}, \; 0 \leq h < t.$$

A detailed proof is given in [39, p 113].

4.3 Brownian local times

The following construction is adapted from the book by Chung and Williams ([10], pp 127–142). For other constructions of local times (including Lévy's original ideas) we refer to the book by Ito and McKean [22]. More results on the subject can be found in [4] and [8].

To introduce local time of Brownian motion at level $a \in \mathbf{R}$, consider the function $f_a(x) = (x-a)^+ = \max\{x-a, 0\}$. The idea is to apply the Ito formula but this function is not differentiable. Firstly, we approximate f_a by the family of functions $(f_{a,\epsilon} : \epsilon > 0)$ defined as follows:

$$f_{a,\epsilon}(x) = \begin{cases} 0, & \text{for } x \leq a-\epsilon \\ (x-a+\epsilon)^2/4\epsilon, & \text{for } a-\epsilon \leq x \leq a-\epsilon \\ x-a, & \text{for } x \geq a+\epsilon. \end{cases}$$

Clearly,

$$f'_{a,\epsilon}(x) = \begin{cases} 0, & \text{for } x \leq a-\epsilon \\ (x-a+\epsilon)/2\epsilon, & \text{for } a-\epsilon \leq x \leq a-\epsilon \\ 1, & \text{for } x \geq a+\epsilon \end{cases}$$

and

$$f''_{a,\epsilon}(x) = \begin{cases} 0, & \text{for } x < a-\epsilon \\ 1/2\epsilon, & \text{for } a-\epsilon < x < a-\epsilon \\ 0, & \text{for } x > a+\epsilon \end{cases}$$

and we set $f''_{a,\epsilon}(x \pm \epsilon) = 0$. The function $f''_{a,\epsilon}$ is not continuous at $a \pm \epsilon$, so Ito's formula is not applicable here. We fix $\epsilon > 0$ and approximate $f_{a,\epsilon}$ by a sequence of convolution products $g_n = \phi_n * f_{a,\epsilon}$ where (ϕ_n) is a sequence of $C^\infty-$functions defined on $\mathbf{R}$ such that the support of each ϕ_n is contained in the interval $[-1/n, 1/n]$ and $\int \phi_n(x)dx = 1$. (In fact, $(\varphi_{1/n})$ defined by $\varphi_{1/n} = \phi_n$ is an approximate of identity (see Definition 2.11)). We recall that g_n is C^∞ and

$$g_n(x) = \int_{\mathbf{R}} f_{a,\epsilon}(x-z)\phi_n(z)dz, \quad \text{for all } x$$
$$g'_n(x) = \int_{\mathbf{R}} f'_{a,\epsilon}(x-z)\phi_n(z)dz, \quad \text{for all } x.$$

Clearly, $g_n \to f_{a,\epsilon}$ and $g'_n \to f'_{a,\epsilon}$ uniformly on $\mathbf{R}$ and $g''_n \to f''_{a,\epsilon}$ pointwise except at $a \pm \epsilon$. An application of Ito's formula to g_n yields that, almost surely, for all $t \geq 0$,

$$g_n(X_t) - g_n(X_0) = \int_0^t g'_n(X_s)dX_s + \frac{1}{2}\int_0^t g''_n(X_s)ds. \tag{4.4}$$

We have that

$$\sup_{(s,\omega)\in[0,t]\times\Omega} |1_{[0,t]}(s)g'_n(X_s(\omega) - 1_{[0,t]}(s)f'_{a,\epsilon}(X_s(\omega))| \to 0 \text{ as } n \to \infty.$$

Indeed, since $\text{supp}(\phi_n) \subseteq [-1/n, 1/n]$ and

$$|f'_{a,\epsilon}(y) - f'_{a,\epsilon}(x)| \leq |x-y|/2\epsilon, \quad \text{for all } x, y \in \mathbf{R},$$

then (using the properties of (ϕ_n), we find that

$$\begin{aligned}
|1_{[0,t]}(s)[g'_n(X_s(\omega) - f'_{a,\epsilon}(X_s(\omega))]| &= \left| \int_0^t f'_{a,\epsilon}(\omega(s) - z)\phi_n(z)dz - \int_0^t f'_{a,\epsilon}(\omega(s))\phi_n(z)dz \right| \\
&\leq \int_0^t \left| f'_{a,\epsilon}(\omega(s) - z) - f'_{a,\epsilon}(\omega(s)) \right| \phi_n(z)dz \\
&\leq \int_{-1/n}^{1/n} |f'_{a,\epsilon}(\omega(s) - z) - f'_{a,\epsilon}(\omega(s))| \phi_n(z)dz \\
&\leq \int_{-1/n}^{1/n} \frac{|z|}{2\epsilon}\phi_n(z)dz \\
&\leq \frac{1}{2n\epsilon} \int_{-1/n}^{1/n} \phi_n(z)dz \\
&= \frac{1}{2n\epsilon}.
\end{aligned}$$

From this uniform convergence and the fact that $1_[0,t]g'_n(X)$ is dominated by $1_[0,t]$ which is in $\mathcal{L}^2(\mu)$, we deduce that the process $1_{[0,t]}(s)g'_n(X)$ converges to the process $1_{[0,t]}(s)f'_{a,\epsilon}(X)$ in the $\mathcal{L}^2(\mu)-$norm. (See for example the book by Bartle [2, p 75] for more details on convergence results). Therefore, by the isometry between $\mathcal{L}^2(\mu)$ and $L^2(P)$,

$$\int_0^t g'_n(X_s)dX_s \to \int_0^t f'_{a,\epsilon}(X_s)dX_s$$

in the $L^2(P)$-norm.

For the last term of (4.4), we also have that $g''_n(x)$ converges to $f''(x)$ pointwise except for $x = a \pm \epsilon$. Then $g''_n(X_s)(\omega) \to f''(X_s)(\omega)$ except for $\omega(s) = a \pm \epsilon$ ($\omega \in \Omega$). From (4.3), we know that for any fixed real b, almost surely, for λ-almost every $s \geq 0$, it is the case that $s \notin Z_b$. Therefore, almost surely, for $\lambda-$almost every $s \in [0,t]$, we have that $g''_n(X_s) \to f''(X_s)$. Since clearly, $|g''_n| \leq 1/2\epsilon$, we can use the bounded convergence theorem to find that, almost surely,

$$\int_0^t g''_n(X_s)ds \to \int_0^t f''_{a,\epsilon}(X_s)ds$$

and this convergence also holds in $L^2(P)$-norm.

Therefore, by letting $n \to \infty$, we find that, for each a and t, almost surely,

$$\begin{aligned}
f_{a,\epsilon}(X_t) - f_{a,\epsilon}(X_0) &= \int_0^t f'_{a,\epsilon}(X_s)dX_s + \frac{1}{2}\int_0^t f''_{a,\epsilon}(X_s)ds \\
&= \int_0^t f'_{a,\epsilon}(X_s)dX_s + \frac{1}{2}\int_0^t \frac{1}{2\epsilon}1_{(a-\epsilon,a+\epsilon)}(X_s)ds. \qquad (4.5)
\end{aligned}$$

One can check that for any $x \in \mathbf{R}$, $|f_{a,\epsilon}(x) - (x-a)^+| \leq \epsilon/4$ (the maximum being attained at $x = a$). Then

$$|(f_{a,\epsilon}(x) - f_{a,\epsilon}(y)) - ((x-a)^+ - (y-a)^+)| \leq \epsilon/2, \quad \text{for all } x, y \in \mathbf{R}$$

because $|a - b| \leq |a| + |b|$. In particular, for any $t \geq 0$,

$$|(f_{a,\epsilon}(X_t) - f_{a,\epsilon}(X_0)) - ((X_t - a)^+ - (X_0 - a)^+)| \leq \epsilon/2.$$

Hence for fixed $t \geq 0$,

$$f_{a,\epsilon}(X_t) - f_{a,\epsilon}(X_0) \longrightarrow (X_t - a)^+ - (X_0 - a)^+ \text{ as } \epsilon \longrightarrow 0,$$

both almost surely and in $L^2(P)$-norm.
Also, for every $x \in \mathbf{R}$,

$$|f'_{a,\epsilon}(x) - 1_{[a,\infty)}(x)| \leq 1_{(a-\epsilon,a+\epsilon)}(x), \text{ for all } x \in \mathbf{R}.$$

Therefore,

$$\begin{aligned} E\left[\int_0^t |f'_{a,\epsilon}(X_s) - 1_{[a,\infty)}(X_s)|^2 ds\right] &\leq E\left[\int_0^t 1_{(a-\epsilon,a+\epsilon)}(X_s) ds\right] \\ &= \int_0^t P\{X_s \in (a-\epsilon, a+\epsilon)\} ds \\ &= \int_0^t \frac{2}{\sqrt{2\pi s}} \int_a^{a+\epsilon} e^{-h^2/2s} dh ds \\ &\leq \int_0^t \frac{2\epsilon}{\sqrt{2\pi s}} ds \text{ (because } e^{-h^2/2s} \leq 1) \\ &\longrightarrow 0 \text{ (as } \epsilon \longrightarrow 0). \end{aligned}$$

This means that, for $\epsilon \longrightarrow 0$, $f'_{a,\epsilon}(X_s)$ converges to $1_{[a,\infty)}(X_s)$ in $\mathcal{L}^2(\mu)$. It follows, by the isometry $\mathcal{L}^2(\mu) \longrightarrow L^2(P)$, that $\int_0^t f'_{a,\epsilon}(X_s) dX_s$ converges to $\int_0^t 1_{[a,\infty)}(X_s) dX_s$ in $L^2(P)$.
We conclude that, for every $t \geq 0$ and $a \in \mathbf{R}$, almost surely,

$$(X_t - a)^+ - (X_0 - a)^+ = \int_0^t 1_{[a,\infty)}(X_s) dX_s + \frac{1}{2} L_a(t), \tag{4.6}$$

where

$$L_a(t) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{(a-\epsilon,a+\epsilon)}(X_s) ds, \tag{4.7}$$

the limit being considered in the $L^2(P)$-norm.

Definition 4.4 *The process* $(L_a(t) : t \geq 0)$ *is called the* local time *at level* a *of the Brownian motion* X*.*

Since by Theorem 4.1, the process $(\int_0^t 1_{[a,\infty)}(X_s) dX_s : t \geq 0)$ has a continuous version, then by (4.6), for any $a \in \mathbf{R}$, the local time process $(L_a(t) : t \geq 0)$ has a continuous version. More than that is true: there exists a family of random variables $J = (J_a(t) : (t,a) \in [0,\infty) \times \mathbf{R})$ and a subset Ω_0 of Ω of probability 1 such that the function $(t,a) \to J_a(t)(\omega)$ is continuous for all $\omega \in \Omega_0$ and for each fixed (t,a):

$$P\left\{\int_0^t 1_{[a,\infty)}(X_s)dX_s = J_a(t)\right\} = 1. \tag{4.8}$$

(this is Lemma 7.2 of [10]). From this, we can now redefine the local time by replacing the integral $\int_0^t 1_{[a,\infty)}(X_s)dX_s$ by $J_a(t)$ in (4.6). That is,

$$\frac{1}{2}L_a(t) = (X_t - a)^+ - (X_0 - a)^+ - J_a(t).$$

Now we have a version of local time, which is continuous in (a,t). This is the version that we will be considering unless otherwise indicated. This fact was first proved by Trotter [48]. For this new version, we still have relations (4.6) and (4.7).
So far, relation (4.7) is true when the limit is taken in the L^2 sense. It is an important fact that it is also true almost surely.

Theorem 4.5 *For any $(t,a) \in [0,\infty) \times \mathbf{R}$, almost surely,*

$$L_a(t) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{(a-\epsilon,a+\epsilon)}(X_s)ds.$$

Proof (Adapted from [10, pp 134-136] We can easily verify that

$$f'_{a,\epsilon}(x) = \frac{1}{2\epsilon} \int_{a-\epsilon}^{a+\epsilon} 1_{[x,\infty)}(x)dx.$$

The idea is to replace in relation (4.5) the stochastic integral by a deterministic integral depending on J. We have that

$$\int_0^t f'_{a,\epsilon}(X_s)dX_s = \frac{1}{2\epsilon} \int_{a-\epsilon}^{a+\epsilon} J_x(t)dx.$$

Indeed, for fixed t, by continuity of $J_x(t)$, we can approximate the corresponding integral by Riemann sums:

$$\int_{a-\epsilon}^{a+\epsilon} J_x(t)dx = \lim_{n\to\infty} \frac{1}{2^n} \sum_{k \in A_n} J_{k/2^n}(t), \text{ where } A_n = \{k \in \mathbf{Z} : k/2^n \in (a-\epsilon, a+\epsilon)\}.$$

From (4.8),

$$J_{k/2^n}(t) = \int_0^t 1_{[k/2^n,\infty)}(X_s)dX_s \text{ almost surely.} \tag{4.9}$$

Then, almost surely,

$$\int_{a-\epsilon}^{a+\epsilon} J_x(t)dx = \lim_{n\to\infty} \frac{1}{2^n} \sum_{k\in A_n} \int_0^t 1_{[k/2^n,\infty)}(X_s)dX_s$$

$$= \lim_{n\to\infty} \frac{1}{2^n} \int_0^t \phi_n(X_s) dX_s$$

where

$$\phi_n = \frac{1}{2^n} \sum_{k\in A_n} 1_{[k/2^n,\infty)}.$$

The sequence (ϕ_n) converges uniformly to $2\epsilon f_{a,\epsilon}$. Then the process $(\phi_n(X_s) : 0 \le s \le t)$ converges to $(2\epsilon f'_{a,\epsilon}(X_s) : 0 \le s \le t)$ uniformly and hence in $\mathcal{L}^2(\mu)$ because $\phi_n(x)$ is dominated by $1_{[a-\epsilon,a+\epsilon]}$(see [2, p 75]). By the isometry $\mathcal{L}^2(\mu) \to L^2(P)$, we deduce that

$$\lim_{n\to\infty} \int_0^t \phi_n(X_s) dX_s = 2\epsilon f'_{a,\epsilon}(X_s) dX_s$$

and, therefore,

$$\int_{a-\epsilon}^{a+\epsilon} J_x(t) dx = 2\epsilon \int_0^t f'_{a,\epsilon}(X_s) dX_s.$$

From relation (4.5), we find that, for each fixed $\epsilon > 0$, almost surely,

$$f_{a,\epsilon}(X_t) - f_{a,\epsilon}(X_0) - \frac{1}{2\epsilon} \int_{a-\epsilon}^{a+\epsilon} J_x(t) dx = \frac{1}{2} \int_0^t \frac{1}{2\epsilon} 1_{(a-\epsilon,a+\epsilon)}(X_s) ds. \tag{4.10}$$

We know fix a subset Ω_0 of probability 1 such that (4.10) holds for all $\omega \in \Omega$. To consider the limit in (4.10) for $\epsilon \to 0$, it is first required that this relation holds almost surely simultaneously for all $\epsilon > 0$. For this end, we consider the left hand side of this relation as a function of $\epsilon > 0$ and note that it is continuous (for any $\omega \in \Omega$). Similarly, the right hand side, is also continuous (on $(0, \infty)$). Indeed, we can write the integral on the right hand side as

$$h_\epsilon = \lambda \left\{ [0, t] \cap X^{-1}(a - \epsilon, a + \epsilon) \right\}$$

where λ is the Lebesgue measure. Then, for $\alpha > 0$,

$$\begin{aligned}
\lim_{\epsilon\downarrow\alpha} h_\epsilon &= \lambda \left\{ [0, t] \cap \left(\cap_{\epsilon>\alpha} X^{-1}(a - \epsilon, a + \epsilon) \right) \right\} \\
&= \lambda \left\{ [0, t] \cap X^{-1}[a - \alpha, a + \alpha] \right\} \\
&= \lambda \left\{ [0, t] \cap X^{-1}(a - \alpha, a + \alpha) \right\} + \lambda\{X^{-1}(a - \alpha)\} + \lambda\{X^{-1}(a + \alpha)\} \\
&= h_\alpha \text{ almost surely,}
\end{aligned}$$

where the last equality follows from the fact that the level sets of Brownian motion are of Lebesgue measure zero. Also, $\lim_{\epsilon\uparrow\alpha} h_\epsilon = h_\alpha$.

Relation (4.10) holds simultaneously for all rationals. For any positive real number r consider a sequence (r_n) of rationals converging to r. Then by continuity of both sides of relation (4.10), we deduce that, on Ω_0,

$$f_{a,r}(X_t) - f_{a,r}(X_0) - \frac{1}{2r} \int_{a-r}^{a+r} J_x(t) dx = \frac{1}{2} \int_0^t \frac{1}{2r} 1_{(a-r,a+r)}(X_s) ds$$

Now we can consider the limit to find that

$$\lim_{\epsilon \to 0} f_{a,\epsilon}(X_t) - f_{a,\epsilon}(X_0) - \frac{1}{2\epsilon}\int_{a-\epsilon}^{a+\epsilon} J_x(t)dx = \lim_{\epsilon \to 0} \frac{1}{2}\int_0^t \frac{1}{2\epsilon} 1_{(a-\epsilon,a+\epsilon)}(X_s)ds.$$

Then

$$(X_t - a)^+ - (X_0 - a)^+ = J_a(t) + \lim_{\epsilon \to 0} \frac{1}{2}\int_0^t \frac{1}{2\epsilon} 1_{(a-\epsilon,a+\epsilon)}(X_s)ds$$

and therefore, almost surely,

$$L_a(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon}\int_0^t 1_{(a-\epsilon,a+\epsilon)}(X_s)ds.$$

∎

Remark 4.6 : ***Tanaka's formula.***

By replacing function $f_{a,\epsilon}$ in the construction above by the following

$$f_{a,\epsilon}(x) = \begin{cases} a - x, & \text{for } x \leq a - \epsilon \\ (a - x + \epsilon)^2/4\epsilon, & \text{for } a - \epsilon \leq x \leq a + \epsilon \\ 0, & \text{for } x \geq a + \epsilon, \end{cases}$$

we obtain that

$$(X_t - a)^- - (X_0 - a)^- = -\int_0^t 1_{(-\infty,a]}(X_s)dX_s + \frac{1}{2}L_a(t), \tag{4.11}$$

where $(x - a)^- = \max(0, -(x - a))$.
Adding this equation to (4.6) yields

$$L_a(t) = |X_t - a| - |X_0 - a| - \int_0^t \text{sign}(X_s - a)dX_s. \tag{4.12}$$

This relation is known as the Tanaka formula. It is an important fact that the process

$$\left(\int_0^t \text{sign}(X_s - a)dX_s : \ t \geq 0\right)$$

is another version of Brownian motion. The simplest proof uses the notion of quadratic variation and can be found in [10, pp 138-139] while a direct proof is given [39, pp 235-236]. We conclude this section with the occupation measure formula. A proof can be found in [10, pp 136-137]. See also [41, p 215] and [38, p 435] for a general result. Let f be a Borel measurable and locally integrable on $\mathbf{R}$. Then for each $t \geq 0$, almost surely,

$$\int L_a(t)f(a)da = \int_0^t f(X_s)ds. \tag{4.13}$$

4.4 The Dirac measure of Brownian motion

We have seen that for any fixed $a \in \mathbf{R}$, the local time process $(L_a(t) : t \geq 0)$ at level a is almost surely continuous. Furthermore, for any fixed $t \geq 0$, almost surely,

$$L_a(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{(a-\epsilon,a+\epsilon)}(X_s)ds.$$

Therefore, for any $0 \leq s \leq t$, almost surely,

$$L_a(t) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^s 1_{(a-\epsilon,a+\epsilon)}(X_h)dh + \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_s^t 1_{(a-\epsilon,a+\epsilon)}(X_h)dh. \tag{4.14}$$

Hence $L_a(s) \leq L_a(t)$ almost surely. This means that the process $(L_a(t) : t \geq 0)$ is almost surely increasing.
We can, therefore, consider the (random) measure whose distribution is $L_a(t)$. We call it the Dirac measure of Brownian motion at level a and denote it by $\delta_a(X)$. This name is justified by the fact that in (4.14),

$$\frac{1}{2\epsilon} 1_{(a-\epsilon,a+\epsilon)} \to \delta_a, \quad \text{as } \epsilon \to 0$$

in the distributional sense. So we have that

$$\delta_a(X)_\omega[0,t] = L_a(t)(\omega).$$

Theorem 4.7 *The support of $\delta_a(X)$ is contained in the level set $Z_a = \{t \geq 0 : X_t = a\}$.*

Proof Assume that $t \notin Z_a(\omega)$ for some $t \geq 0$. Then in the case where $\omega(t) > a$, by continuity of ω, there exists $s_1 < s_2$ such that $\omega(s) > a + \gamma$ for all s in the interval (s_1, s_2) for some fixed rational $\gamma > 0$. Then for all sufficiently small rationals $\epsilon < \gamma$,

$$\lambda \{s \in (s_1, s_2) : X_s \in (a - \epsilon, a + \epsilon)\} = 0.$$

Since, almost surely,

$$\delta_a(X)(s_1, s_2) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \lambda \{s \in (s_1, s_2) : X_s \in (a - \epsilon, a + \epsilon)\},$$

then we find that, on a subspace Ω_0 of probability 1, $\delta_a(X)(s_1, s_2)(\omega) = 1$. The same argument applies for $\omega(t) < a$. Therefore, $t \notin \text{supp}\delta_a(X)(\omega)$ on Ω_0.

An immediate consequence of this theorem is that local time L_a is constant on the complement of the level set Z_a.

4.5 Inverse local times

To further explore the properties of Brownian local times we need to use the following classical result of Lévy.

Theorem 4.8 *The processes* $(|X_t|, L_0(t)) : t \geq 0)$ *and* $(M_t - X_t, M_t)$ *are identical in law.*

Proof (Adapted from [39, p 138]). From Tanaka's formula (4.12), we have that (since $X_0 = 0$ almost surely)

$$|X_t| = \int_0^t \text{sign}(X_s) dX_s + L_0(t) = W_t + L_0(t)$$

where W_t is another version of Brownian motion. Consider now the version of Brownian motion $\tilde{W} = (\tilde{W}_t = -W_t)$ and the corresponding maximum process $\tilde{M}_t$. We want to show that $\tilde{M}_t = L_0(t)$, from which it will follow that the process $((|X_t|, L_0(t)) : t \geq 0)$ agrees pointwise with the process $((\tilde{M}_t - \tilde{W}_t, \tilde{M}_t) : t \geq 0)$, which has the same distribution with $((M_t - X_t, M_t) : t \geq 0)$. Clearly, for any $s \geq 0$,

$$\tilde{W}_s = L_0(s) - |X_s| \leq L_0(s).$$

Then, since $(L_0(s))$ is increasing, we find $\tilde{M}_t \leq L_0(t)$. Also since the function $L_0(t)$ is constant on the complement of the zero set $Z_0 = \{t \geq 0 : X_t = 0\}$ and on the set, $L_0(t) = \tilde{W}_t \leq \tilde{M}_t$, we conclude that $L_0(t) = \tilde{M}_t$ and the equality follows. ∎

From the proof above, the local time of Brownian motion (at the origin) is the maximum function of another version of Brownian motion. We can, therefore, transfer all results concerning probabilistic properties of the maximum function of Brownian motion to Brownian local time. The distribution of $L_0(t)$ is, therefore, equal to the distribution of M_t. In particular, for any $t > 0$, almost surely, $L_0(t) > 0$ from which we deduce that the measure $\delta(X)$ is non zero, almost surely.

Definition 4.9 *The process* $L_0^{-1} = (L_0^{-1}(t) : t \geq 0)$ *defined by*

$$L_0^{-1}(t) = \inf\{s \geq 0 : L_0(s) > t\} \tag{4.15}$$

is called the inverse local time process at 0.

Because $L_0(t)$ is the maximum function of another version of Brownian motion, the process L_0^{-1} can be seen as the right-continuous inverse of the maximum function of a Brownian motion. Therefore, from results of section 3.3, the process L_0^{-1} is right-continuous, has independent and stationary increments and has a distribution given by

$$P\{L_0^{-1}(t) \leq h\} = \int_0^h \frac{t}{\sqrt{2\pi s^3}} e^{-t^2/2s} ds, \; h > 0. \tag{4.16}$$

Its Fourier transform is

$$E\left[e^{iuL_0^{-1}(t)}\right] = e^{-t\sqrt{|u|}(1-i\,\mathrm{sign}(u))}. \tag{4.17}$$

These results will be useful in the sequel. In general, the inverse local time of Brownian motion at level a is defined by

$$L_a^{-1}(t) = \inf\{s \geq 0 : L_a(s) > t\}.$$

Chapter 5

Some fractal properties of Brownian motion

In this chapter we discuss some known fractal properties of Brownian motion on which the results of chapter 6 will be based. We start with the Hausdorff dimension of Brownian level sets and show that they are of dimension 1/2. We also discuss, in section 2, the dimension doubling property of Brownian images of compact subsets. The last section is devoted to the beautiful construction of Kahane where he showed that Brownian images of compact subsets of Hausdorff dimension $< 1/2$ are Salem sets. Kahane's proof is not easy to follow and we made every effort to clarify the construction by filling in many gaps between the main steps of the proof. The main references for this chapter are [22], [24] and [39].

5.1 Hausdorff dimension of Brownian level sets

Theorem 5.1

(1) *Almost surely, for all reals $\kappa > 0$, the zero set of Brownian motion*

$$Z = \{t \in [0, \kappa] : X_t = 0\}$$

has Hausdorff dimension 1/2.

(2) *For any fixed $a \in \mathbf{R}$, the level set $Z_a = \{t \in [0, 1] : X_t = a\}$ is non-empty with positive probability and in this case, its Hausdorff dimension is also* 1/2.

Proof (Adapted from [39] and [31, pp 181-182].) Firstly, let us fix $\kappa > 0$ and Consider the Dirac measure $\delta(X)$ of Brownian motion as discussed in section 4.4. For any interval $I = [a, b) \subset [0, \kappa]$, $\delta(X)(I) = L_0(b) - L_0(a)$. As discussed in section 4.5, the local times $L_0(t)$ is the maximum function of another version of Brownian motion $(\widetilde{W}(t) : t \geq 0)$. Then $\delta(X)(I) = \widetilde{M}_b - \widetilde{M}_a$ where $\widetilde{M}$ is the maximum function of $\widetilde{W}$. We have, by the modulus of continuity of Brownian motion (see relation (3.5)), that almost surely,

$$\begin{aligned} 0 \le \widetilde{M}_b - \widetilde{M}_a &\le \sup_{0 \le h \le b-a} |\widetilde{W}_{a+h} - \widetilde{W}_a| \le \sqrt{2(b-a)\log(1/(b-a))} \\ &\le \sqrt{2}(b-a)^{\frac{1}{2}-\epsilon} \text{ for any } \epsilon > 0. \end{aligned}$$

Therefore, almost surely,

$$\delta(X)(I) \le C|I|^{\frac{1}{2}-\epsilon} \text{ for any } \epsilon > 0.$$

Since the support of $\delta(X)$ is contained in Z (Theorem 4.7), Frostman's lemma implies that $\dim_H Z \ge \frac{1}{2} - \epsilon$, for any $\epsilon > 0$. It follows that $\dim_H Z \ge \frac{1}{2}$. To prove that $\dim_H Z \le 1/2$, we use Lévy's arcsine law of Brownian motion (see Proposition 4.3). We subdivide the interval $(0, 1]$ into n subintervals

$$I_k = \left(\frac{k-1}{n}, \frac{k}{n}\right] \quad k = 1, 2, \ldots, n.$$

We want to determine how many of these subintervals are needed to cover $Z \cap (0, \kappa]$ in order to find an upper bound on the Hausdorff measure $H_{1/2}(Z)$ of Z. For this purpose, consider the random variables $T_k, (k = 1, 2, \ldots, n)$ defined on Ω by $T_k = 1$ if $I_k \cap Z \neq \emptyset$ and 0 otherwise. The required number is $N = T_1 + T_2 + \ldots + T_n$ and we want to estimate its expectation $E[N]$. Let $K_t = \sup\{s < t : X_s = 0\}$. Clearly, $I_k \cap Z \neq \emptyset$ is equivalent to $K_{k/n} > (k-1)/n$. Then, using Lévy's arcsine law (Proposition 4.3),

$$\begin{aligned} P\{T_k = 1\} &= P\{I_k \cap Z \neq \emptyset\} \\ &= 1 - P\{K_{k/n} \le (k-1)/n\} \\ &= 1 - \frac{2}{\pi} \arcsin \sqrt{\frac{k-1}{k}}. \end{aligned}$$

Clearly, if $y = \arcsin \sqrt{\frac{k-1}{k}}$, then $\sin y = \sqrt{\frac{k-1}{k}}$, $\cos y = \sqrt{\frac{1}{k}}$ and $\tan y = \sqrt{k-1}$. Therefore

$$P\{T_k = 1\} = 1 - \frac{2}{\pi} \arctan \sqrt{k-1}.$$

Hence

$$E(N) = \sum_{k=1}^{n} \left(1 - \frac{2}{\pi} \arctan \sqrt{k-1}\right) = 1 + \sum_{k=1}^{n-1} \left(1 - \frac{2}{\pi} \arctan \sqrt{k}\right).$$

Let $f(x) = 1 - \frac{2}{\pi} \arctan \sqrt{x}$. Then the sum

$$f(1)(1-0) + f(2)(2-1) + \ldots + f(n-1)(n-1-(n-2))$$

can be seen as a Riemann sum of the function f corresponding to the partition $[0, 1], (1, 2], \ldots, (n-2, n-1]$ of $[0, n-1]$. Therefore, since f is a decreasing function, then

$f(k+1)(k+1-k) \leq \int_k^{k+1} f(x)dx$. Then

$$\begin{aligned} E[N] &\leq 1 + \int_0^{n-1} f(x)dx \\ &\leq 1 + \int_0^n f(x)dx \\ &= 1 + n - \left[\frac{2}{\pi}\left((n+1)\arctan\sqrt{n} - \sqrt{n}\right)\right] \end{aligned}$$

because an antiderivative of $\arctan\sqrt{x}$ is $(x+1)\arctan\sqrt{x} - \sqrt{x}$.
Since

$$\arctan(\sqrt{n}) = \frac{\pi}{2} + O(n^{-1/2}) \quad \text{for } n \to \infty,$$

we find that

$$1 + n - \left[\frac{2}{\pi}\left((n+1)\arctan\sqrt{n} - \sqrt{n}\right)\right] = -\frac{\pi}{2}\left[(n+1)O(n^{-1/2}) + n^{1/2}\right] = O(n^{1/2})$$

for $n \to \infty$.
Then

$$E[N] = O(n^{1/2}) \text{ for } n \to \infty.$$

It follows that

$$\lim_{n\to\infty} E[n^{-1/2}N] < \infty.$$

Hence, using Fatou's lemma,

$$\lim_{n\to\infty} n^{-1/2}N < \infty \text{ almost surely.}$$

By considering the covering

$$\mathcal{C}_n = (I_k : I_k \cap Z \neq \emptyset,\, k \in \{1, \ldots, n\})$$

of $Z \cap (0, \kappa]$, we find that

$$H_{1/2}^{1/n}(Z) \leq \sum_{I \in \mathcal{C}_n} |I|^{1/2} = Nn^{-1/2} \text{ (because } |I| = 1/n).$$

Hence

$$H_{1/2}(Z) = \lim_{n\to\infty} H_{1/2}^{1/n}(Z) \leq \lim_{n\to\infty} Nn^{-1/2} < \infty \quad \text{almost surely,}$$

from which it follows that $\dim_H Z \leq 1/2$. Therefore, for any fixed $\kappa > 0$, almost surely, $\dim_H\{t \in [0,\kappa] : X_t = 0\} = 1/2$. Then, almost surely, simultaneously, for all rationals $r > 0$, $\dim_H\{t \in [0,r] : X_t = 0\} = 1/2$. This means that there exists a subset Ω_0 of Ω of probability 1 such that for all $\omega \in \Omega_0$ and for all rationals $r > 0$,

$$\dim_H\{t \in [0,r] : X_t(\omega) = 0\} = 1/2.$$

Let us show that for all $\omega \in \Omega_0$ and a real $\kappa > 0$, $\dim_H\{t \in [0,\kappa] : X_t(\omega) = 0\} = 1/2$. Consider an increasing sequence of positive rationals (r_n) converging to κ. Clearly,

$$\{t \in [0,\kappa] : X_t(\omega) = 0\} = \cup_{n=1}^{\infty}\{t \in [0,r_n] : X_t(\omega) = 0\}$$

and hence

$$\dim_H\{t \in [0,\kappa] : X_t(\omega) = 0\} = \sup_n \dim_H\{t \in [0,r_n] : X_t(\omega) = 0\} = 1/2.$$

Therefore, on Ω_0, we have that $\dim_H\{t \in [0,\kappa] : X_t(\omega) = 0\} = 1/2$ simultaneously for all reals $\kappa > 0$.
For the general level set $Z_a = \{t \in [0,1] : X_t = 0\}$, it is clear that, Z_a is empty if and only if $\Gamma_a > 1$. (Recall that $\Gamma_a = \inf\{t \in [0,1] : X_t = a\}$). We now want to show that if $\Gamma_a < 1$, then $\dim_H Z_a = 1/2$. By the strong Markov property, the process $Y = (Y_t = X_{t+\Gamma_a} - a : t \geq 0)$ is a Brownian motion and we have that $Z_a = \Gamma_a + Z(Y)$ where $Z(Y) = \{t \in [0, 1-\Gamma_a] : Y_t = 0\}$. Under the condition $\Gamma_a < 1$, Theorem 5.1 (i) shows that, $Z(Y)$ is non-empty and has Hausdorff dimension 1/2. Therefore, Z_a also has Hausdorff dimension 1/2 because obviously $\dim_H(x + E) = \dim_H E$ (for $x \in \mathbf{R}$ and $E \subset \mathbf{R}$. We conclude that Z_a is non-empty with positive $p = P\{\Gamma_a < 1\} > 0$ and has Hausdorff dimension 1/2. ∎

5.2 Brownian images: Hausdorff dimension

Theorem 5.2 *For any compact subset of $E \subset [0,1]$ of Hausdorff dimension $\alpha < 1/2$, almost surely, its Brownian image $X(E) = \{X(t) : t \in E\}$ has Hausdorff dimension 2α.*

Proof (Following Fouché's lectures based on Kahane [24, chapter 14]). Because the modulus of continuity K_X of the Brownian motion satisfies

$$K_X(h) \leq \sqrt{2|h|\log(1/|h|)} \leq \sqrt{2}|h|^{\gamma}$$

for any $0 < \gamma < 1/2$, Proposition 1.1 yields that, $\dim_H X(E) \leq (1/\gamma)\dim_H E$. Since this holds for any $0 < \gamma < 1/2$, then $\dim_H X(E) \leq 2\dim_H E$.
To prove that $\dim_H X(E) \geq 2\dim_H E$, we first choose $\beta \in (0,1)$ such that $\beta/2 < \dim_H E$. By Frostman's theorem (Theorem 1.7), E carries a probability measure θ such that

$$I_{\beta/2}(\theta) = \int_{\mathbf{R}}\int_{\mathbf{R}} \frac{d\theta(x)d\theta(y)}{|x-y|^{\frac{\beta}{2}}} < \infty.$$

Let μ be the image measure of θ under the Brownian motion X, that is, $\mu(A) = \theta(X^{-1}(A))$ for any Borel subset A of $\mathbf{R}$. Note that μ is a non-zero random measure carried by $X(E)$. We want to show that $I_\beta(\mu) < \infty$, almost surely, which implies that $\dim_H X(E) \geq \beta$ and hence $\dim_H X(E) \geq 2 \dim_H E$. We recall (by Theorem 2.13, $n = 1$) that

$$I_\beta(\mu) = c \int |u|^{\beta-1} |\hat{\mu}(u)|^2 du$$

for some constant $c = c_\beta > 0$. It is sufficient to show that $J = E[I_\beta(\mu)] < \infty$. By Fubini's theorem,

$$J = c \int |u|^{\beta-1} E[|\hat{\mu}(u)|^2] du.$$

We have that (after using several applications of Fubini's theorem),

$$\begin{aligned} E(|\hat{\mu}|^2) &= \int_\Omega \int_{\mathbf{R}} \int_{\mathbf{R}} e^{iu(x-y)} d\mu(x) d\mu(y) dP \\ &= \int_\Omega \int_{\mathbf{R}} \int_{\mathbf{R}} e^{iu(X_t - X_s)} d\theta(t) d\theta(s) dP \\ &= \int_{\mathbf{R}} \int_{\mathbf{R}} \int_\Omega e^{iu(X_t - X_s)} dP d\theta(t) d\theta(s) \end{aligned}$$

Because X is a Brownian motion, $X(t) - X(s)$ has the same distribution as $X(t-s)$ for $t \geq s$. Because $-X$ is also a Brownian motion, $X(t) - X(s)$ also has the same distribution as $X(s-t)$ for $t < s$. So in both cases, $X(t) - X(s)$ is normally distributed with mean 0 and variance $|t-s|$. Therefore,

$$\begin{aligned} E(e^{iu(X_t - X_s)}) &= \frac{1}{\sqrt{2\pi |t-s|}} \int_{\mathbf{R}} e^{-x^2/2|t-s|} e^{iux} dx \\ &= e^{-|t-s|u^2/2}, \end{aligned} \tag{5.1}$$

where the last equality is just the Fourier transform of the normal distribution (relation (2.4). Hence, by another application of Fubini's theorem,

$$E(|\hat{\mu}(u)|^2 = \int_{\mathbf{R}} \int_{\mathbf{R}} e^{-|t-s|u^2/2} d\theta(t) d\theta(s)$$

Therefore, by another application of Fubini's theorem,

$$J = c \int_{\mathbf{R}} \int_{\mathbf{R}} \int_{\mathbf{R}} |u|^{\beta-1} e^{-|t-s|u^2/2} du d\theta(t) d\theta(s).$$

The substitution $v = u\sqrt{|t-s|}$ yields

$$J = C \int_{\mathbf{R}} \int_{\mathbf{R}} \frac{1}{|t-s|^{\beta/2}} d\theta(t) d\theta(s),$$

where

$$C = c \int_{\mathbf{R}} |v|^{\beta-1} e^{-v^2/2} dv < \infty \text{ (see Example 2.6).}$$

It follows that $J < \infty$. ∎

Remark 5.3

A simple proof of this doubling dimension property using nonstandard analysis is given in [40]. There are many other interesting fractal properties of Brownian motion. Theorem 5.2 is a particular case of the following result ascribed to McKean [37]: for any fixed compact subset $E \subset \mathbf{R}_+$, $\dim_H X(E) = \min(2\dim_H E, 1)$. But this property does not hold simultaneously for all compact sets $E \subset \mathbf{R}_+$ because this should imply that it also holds for random compact subsets. For example, it should imply that the Hausdorff dimension of the image of the zero set of Brownian motion has dimension 1, which is a contradiction.
For $\dim_H E > 1/2$, Kaufman [26] has shown that $X(E)$ has an interior point. Kaufman [25] has shown that if X is a Brownian motion in dimension $d \geq 2$ (that is $X = (X^1, X^2, \ldots, X^d)$ and the X^i's are independent one dimensional Brownian motions), then almost surely, for all $E \subset [0, 1]$, $\dim_H X(E) = 2\dim_H E$.
There are also some results for inverse images. For example, for any compact $E \subset \mathbf{R}$, we have $\dim_H X^{-1}(E) = (1 + \dim_H E)/2$, almost surely (a more general result is given in [20]). Serlet [45] proved that this holds simultaneously for all compact subsets E of $\mathbf{R}$, that is, there exists a subset Ω_0 of Ω of probability 1 such that for all compact subsets E of $\mathbf{R}$ and for all $\omega \in \Omega_0$, $\dim_H X^{-1}(\omega)(E) = (1 + \dim_H E)/2$. Many researchers are interested in extending Brownian fractal properties to more general processes like Lévy processes [29], [35].

5.3 Brownian images: Fourier dimension

In this section, we discuss Kahane's construction of Salem sets via Brownian motion [24, pp 251-255].

Theorem 5.4 *Let E be a compact subset of $[0, 1]$ of Hausdorff dimension $0 < \alpha < 1/2$. Then $X(E)$ is almost surely a Salem set of dimension 2α.*

What we have to show is that the Fourier dimension of $X(E)$ is 2α. The proof is based on the following lemma which is a special case of Lemma 1 of Kahane's book [24, p 252].

Lemma 5.5 *Let μ be a positive finite measure carried by a compact set interior to the closed interval $[-1, 1]$ such that $|\hat{\mu}(u)| \leq 1$ for all $u \in \mathbf{R}$. Let $\alpha \in (0, 1)$ and $0 < \kappa < 1$. If there exists a constant $C > 0$ such that*

$$|\hat{\mu}(n)| \leq C\sqrt{|\kappa n|^{-\alpha} \log |\kappa n|} \ \text{ for all large } |n|, \text{ with } n \text{ and integer }, \tag{5.2}$$

then there exists a positive constant C' depending only on C and α such that

$$|\hat{\mu}(u)| \leq C'\sqrt{|\kappa u|^{-\alpha} \log |\kappa u|} \ \text{ for all large } |u|, \text{ with } u \text{ a real.}$$

Proof Let f be a C^∞-function carried by a compact set interior to $[-1, 1]$ and equal to

1 on the support of μ. For any $a \in [-1,1]$, consider the function

$$f_a(x) = e^{iax} f(x),\ x \in \mathbf{R}.$$

Because f is differentiable, then we have the equality

$$f_a(x) \quad = \quad \sum_{n\in\mathbf{Z}} \hat{f}_a(n) e^{inx}, \tag{5.3}$$

where

$$\hat{f}_a(n) = \int_{\mathbf{R}} e^{inx} f_a(x) dx$$

are the Fourier coefficients. Because f is C^∞ and has bounded support, then each of its derivatives is bounded. Using the fact that $|a| \leq 1$, one can easily verify that for any $q \in \mathbf{N}$, there exists a constant $K > 0$ (independent of a) such that the derivative $f_a^{(q)}$ of order q verifies $|f_a^{(q)}(x)| \leq K$. For example, for $q = 1$, we have that $|f_a'(x)| \leq |a||f(x)| + |f'(x)| \leq \|f\| + \|f'\|$ where $\|f\| = \sup_{x\in\mathbf{R}} |f(x)|$.
This is also true for Fourier transforms $\widehat{f_a^{(q)}}$ because $|\hat{f}_a^{(q)}(u)| \leq \int |f_a^{(q)}(x)| dx \leq 2\|f_a^{(q)}\|$. Then for any positive integer q, there exists a constant $K_1 > 0$ independent of a, such that

$$|\widehat{f_a^{(q)}}(u)| \leq K_1,\ \text{ for all } u.$$

Since

$$\widehat{f_a^{(q)}}(u) = (-iu)^q \widehat{f_a}(u)$$

it follows that

$$|\widehat{f_a}(u)| \leq K_1 |u|^{-q} \tag{5.4}$$

for arbitrary q (and K_1 depending only on q). We can now choose $q \geq 2$ and conclude that the series in (5.3) is absolutely convergent.
Let us now fix $m \in \mathbf{Z}$ and $a \in [-1,1]$ and estimate $\hat{\mu}(m+a)$. Using the fact that $f = 1$ on the support of μ and relation (5.3) we find that

$$\begin{aligned}
\hat{\mu}(m+a) &= \int e^{imx} e^{iax} d\mu(x) \\
&= \int e^{imx} e^{iax} f(x) d\mu(x) \\
&= \int e^{imx} f_a(x) d\mu(x) \\
&= \int e^{imx} \sum_{n\in\mathbf{Z}} \hat{f}_a(n) e^{inx} d\mu(x) \text{ (by relation (5.3))} \\
&= \sum_{n\in\mathbf{Z}} \hat{f}_a(n) \int e^{i(n+m)x} d\mu(x) \\
&= \sum_{n\in\mathbf{Z}} \hat{f}_a(n) \hat{\mu}(n+m).
\end{aligned}$$

Therefore, we write

$$|\hat{\mu}(m+a)| \le \sum_{|n|\le |m|/2} |\hat{f}_a(n)\hat{\mu}(n+m)| + \sum_{|n|>|m|/2} |\hat{f}_a(n)\hat{\mu}(n+m)|.$$

Now from condition (5.2) we have that

$$|\hat{\mu}(n+m)| \le C\sqrt{|\kappa(n+m)|^{-\alpha}\log(|\kappa(n+m)|)} \text{ for large } |m+n|.$$

If $|n| \le |m|/2$, then $|m|/2 \le |n+m| \le |2m|$. Therefore,

$$\begin{aligned} |\kappa(n+m)|^{-\alpha}\log(|\kappa(n+m)|) &\le |\kappa m/2|^{-\alpha}\log(|2\kappa m|) \le 2^{\alpha}|\kappa m|^{-\alpha}\log|2\kappa m| \\ &\le 2^{\alpha}|\kappa m|^{-\alpha}\, 2\log|\kappa m| \quad \text{(for large } m\text{)} \end{aligned}$$

It is now clear that for $|n| \le |m|/2$ and m large,

$$|\hat{\mu}(n+m)| \le C_1\sqrt{|\kappa m|^{-\alpha}\log(|\kappa m|)}$$

where $C_1 = 2^{(1+\alpha)/2}C$.
In the case $n > |m|/2$, we use the fact that $|\hat{\mu}(u)| \le 1$ for all u. Therefore,

$$|\hat{\mu}(m+a)| \le C_1\sqrt{|m|^{-\alpha}\log(|m|)} \sum_{|n|\le|m|/2} |\hat{f}_a(n)| + \sum_{|n|>|m|/2} |\hat{f}_a(n)|.$$

From relation (5.4), there exists $K > 0$ (independent of a) such that

$$|\widehat{f_a}(n)| \le K|n|^{-3} \le K|n|^{-2-\alpha/2}, \text{ for all } n.$$

Then

$$\sum_{|n|\le|m|/2} |\widehat{f_a}(n)| \le K \sum_{|n|\in\mathbf{Z}-\{0\}} |n|^{-3} < K\pi^2/3.$$

On the other hand,

$$\begin{aligned} \sum_{|n|>|m|/2} |\hat{f}_a(n)| &\le K \sum_{|n|>|m|/2} |n|^{-\alpha/2}|n|^{-2} \\ &\le K|m/2|^{-\alpha/2} \sum_{|n|>|m|/2} |n|^{-2} \\ &\le 2^{\alpha/2}K(\pi^2/3)|m|^{-\alpha/2} \\ &\le 2^{\alpha/2}K(\pi^2/3)|\kappa m|^{-\alpha/2} \end{aligned}$$

It follows that, for large m,

$$\sum_{|n|>|m|/2} |\hat{f}_a(n)| \le 2^{\alpha/2}(\pi^2/3)K|\kappa m|^{-\alpha/2}\left(\log|\kappa m|\right)^{1/2}.$$

Therefore,

$$|\hat{\mu}(m+a)| \leq (C_1 K\pi^2/3 + 2^{\alpha/2}K(\pi^2/3))\sqrt{|\kappa m|^{-\alpha}\log(|\kappa m|)} \text{ for large } |m|,$$

that is,

$$|\hat{\mu}(m+a)| \leq C_2\sqrt{|\kappa m|^{-\alpha}\log(|\kappa m|)} \text{ for large } |m|$$

where $C_2 = C_1K(\pi^2/3) + 2^{\alpha/2}K(\pi^2/3)$ and is independent of a. Now we consider a large $|u|$ for u real and write $u = m + a$ for m integer and $a \in (-1, 1)$. Then obviously, as $|m|$ and $|u|$ are large enough, $|\kappa m| \geq |\kappa u/2|$ and $\log|\kappa m| \leq 2\log|\kappa u|$. Then we find, for large $|m|$,

$$\begin{aligned} |\hat{\mu}(u)| &\leq C_2\sqrt{|\kappa m|^{-\alpha}\log(|\kappa m|)} \\ &\leq C_2\sqrt{2|\kappa u/2|^{-\alpha}\log(|\kappa u|)}. \end{aligned}$$

Therefore,

$$|\hat{\mu}(u)| \leq C'\sqrt{|\kappa u|^{-\alpha}\log(|\kappa u|)}$$

where $C' = 2^{(1+\alpha)/2}C_2$. ∎

The fact that the constant C' depends only on C will be helpful in the final part of the proof of Theorem 5.4. We are now ready to prove Theorem 5.4.

Proof (of Theorem 5.4)

Since $\dim_H E = \alpha$, then $\dim_H E > \alpha - \gamma$, for any $\gamma > 0$, and hence $H_{\alpha-\gamma}(E) = \infty$. From Frostman's lemma, E carries a probability measure θ such that

$$\theta(I) \leq C|I|^{\alpha-\gamma}, \text{ for any interval } I. \tag{5.5}$$

Let μ be the image measure of θ by the Brownian motion X, that is, $\mu(A) = \theta(X^{-1}(A))$ for any Borel subset A of $\mathbf{R}$. It is a random measure carried by $X(E)$. We want to show that there exists a positive constant C' depending only on C and $\alpha - \gamma$ such that, almost surely, for any $\epsilon > 0$,

$$|\hat{\mu}(u)| \leq C'|u|^{-\alpha+\gamma+\epsilon}, \text{ for } |u| \to \infty.$$

This will imply that the Fourier dimension of $X(E)$ is $\geq 2\alpha$ almost surely. The result will then follow because the $\dim_F X(E) \leq \dim_H X(E)$ and $\dim_H X(E) = 2\alpha$ (Theorem 5.2). We have that

$$\hat{\mu}(u) = \int e^{ixu}d\mu(x) = \int e^{iX_s u}d\theta(s).$$

Then

$$|\hat{\mu}(u)|^2 = \int_{\mathbf{R}^2} e^{i(X_s - X_t)u}d\theta(s)d\theta(t)$$

and for any integer $q \geq 1$, this generalizes to

$$|\hat{\mu}(u)|^{2q} = \int_{\mathbf{R}^{2q}} \exp iu[(X_{s_1} + \ldots + X_{s_q}) - (X_{s_1'} + \ldots + X_{s_q'})] \\ d\theta(s_1) \ldots d\theta(s_q) d\theta(s_1') \ldots d\theta(s_q'). \quad (5.6)$$

To simplify the notation, let

$$f(u, s_1, \ldots, s_q, s_1', \ldots, s_q') = \exp iu[(X_{s_1} + \ldots + X_{s_q}) - (X_{s_1'} + \ldots + X_{s_q'})].$$

Since obviously, f does not change under any permutation of $\{s_1, s_2, \ldots, s_n\}$ and also for any permutation of $\{s_1', s_2', \ldots, s_n'\}$, the integral in (5.6) is equal to

$$(q!)^2 \int_{0 \leq s_1 \ldots \leq s_q} \int_{0 \leq s_1' \ldots \leq s_q'} f(u, s_1, \ldots, s_q, s_1', \ldots, s_q') d\theta(s_1) \ldots d\theta(s_q) d\theta(s_1') \ldots d\theta(s_q').$$

By rearranging the numbers $s_1, s_2, \ldots, s_q, s_1', s_2', \ldots, s_q'$ as an increasing sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_{2q}$, we obtain that

$$|\hat{\mu}(u)|^{2q} = (q!)^2 \sum_{(\epsilon_1, \ldots, \epsilon_{2q}) \in T} \int_{0 \leq t_1 \ldots \leq t_{2q}} \exp[iu(\epsilon_1 X_{t_1} + \epsilon_2 X_{t_2} + \ldots + \epsilon_{2q} X_{t_{2q}})] \\ d\theta(t_1) d\theta(t_2) \ldots d\theta(t_{2q})$$

where T is the set of all sequences $(\epsilon_1, \ldots, \epsilon_{2q})$ such that $\epsilon_j \in \{-1, 1\}$ for each j and $\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{2q} = 0$. Clearly, T is the set all sequences of $2q$ objects, for which q of tehm are equal to 1 and the other remaining q are equal to -1. Then T has $(2q)!/(q!q!)$ elements.

For any sequence $0 \leq t_1 \leq t_2 \leq \ldots \leq t_2 q$, the random variables X_{t_1}, $X_{t_2} - X_{t_1}$, ..., $X_{t_{2q}} - X_{t_{2q-1}}$ are independent. Then

$$E[\exp(iu(\epsilon_1 X_{t_1} + \epsilon_2 X_{t_2} + \ldots + \epsilon_{2q} X_{t_{2q}}))] \\ = E[\exp(iu(\epsilon_1 + \ldots + \epsilon_{2q}) X_{t_1}] \times E[\exp(iu(\epsilon_2 + \ldots + \epsilon_{2q})(X_{t_2} - X_{t_1})] \\ \times E[\exp iu\epsilon_{2q}(X_{t_{2q}} - X_{t_{2q-1}})].$$

Because these random variables (X_{t_1}, $X_{t_2} - X_{t_1}$, ..., $X_{t_{2q}} - X_{t_{2q-1}}$) are normally distributed with mean 0 and variances $t_1, t_2 - t_1, \ldots, t_{2q} - t_{2q-1}$ (respectively), we find, using relation (5.1), that

$$E[\exp(iu(\epsilon_1 X_{t_1} + \epsilon_2 X_{t_2} + \ldots + \epsilon_{2q} X_{t_{2q}}))] \\ = \exp[-u^2 t_1 (\epsilon_1 + \ldots + \epsilon_{2q})^2/2) \times \exp[-u^2 (t_2 - t_1)(\epsilon_2 + \ldots + \epsilon_{2q})^2/2] \\ \times \exp[-u^2 (t_{2q} - t_{2q-1}) \epsilon_{2q}^2/2].$$

Therefore,

$$\begin{aligned} |\hat{\mu}(u)|^{2q} = \; & (q!)^2 \textstyle\sum_{(\epsilon_1,\ldots,\epsilon_{2q})\in T} \int_{0\le t_1 \ldots \le t_{2q}} \exp[-u^2 t_1(\epsilon_1+\ldots+\epsilon_{2q})^2/2) \\ & \times \exp[-u^2(t_2-t_1)(\epsilon_2+\ldots+\epsilon_{2q})^2/2] \\ & \times \exp[-u^2(t_{2q}-t_{2q-1})\epsilon_{2q}^2/2] \\ & d\theta(t_1)d\theta(t_2)\ldots d\theta(t_{2q}). \end{aligned}$$

Let

$$\psi_j = u^2(\epsilon_j+\ldots+\epsilon_{2q})^2/2,\; j=1,2,\ldots,2q.$$

It is clear that for each j, $\psi_j \ge 0$ and for even j's, $\psi_j \ge u^2/2$ since $|\epsilon_j+\ldots+\epsilon_{2q}| \ge 1$ since $\epsilon_j \in \{-1,1\}$. Therefore by dropping all the factors corresponding to odd j's the inequality is reinforced, that is,

$$\begin{aligned} E(|\hat{\mu}(u)|^{2q}) \;\le\; & (q!)^2 \sum_{(\epsilon_1,\ldots,\epsilon_{2q})\in T} \int_{0\le t_1 \ldots \le t_{2q}} \exp(-(t_2-t_1)\psi_2) \times \exp(-(t_4-t_3)\psi_4)\ldots \\ & \times \exp(-(t_{2q}-t_{2q-1})\psi_{2q}) d\theta(t_1)d\theta(t_2)\ldots d\theta(t_{2q}). \end{aligned}$$

We can now integrate with respect to t_j for even j's. For example, for $j=2$, we find, using $\psi_j \ge u^2/2$, that

$$\int_{t_1}^{t_3} \exp[-(t_2-t_1)\psi_2]d\theta(t_2) = \int_0^{t_3-t_1} e^{-t\psi_2} d\theta(t+t_1) \le \int_0^\infty e^{-tu^2/2} d\theta(t+t_1). \tag{5.7}$$

This integral can be calculated by parts as follows: take

$$\begin{aligned} U &= e^{-tu^2/2},\; dU = (-u^2/2)e^{-tu^2/2}dt; \\ dV &= d\theta(t+t_1);\; V(t) = \int_0^t d\theta(s+t_1) = \theta[t_1, t_1+t] \\ I := \int_0^\infty e^{-tu^2/2}\, d\theta(t+t_1) &= (u^2/2)\int_0^\infty e^{-tu^2/2}\theta[t_1,t_1+t]dt \\ &\le C(u^2/2)\int_0^\infty e^{-tu^2/2} t^{\alpha-\gamma} dt \text{ from (5.5))} \\ &= C\,\Gamma(\alpha-\gamma+1)\,2^{\alpha-\gamma}u^{-2(\alpha-\gamma)}. \end{aligned}$$

By repeating the same calculations for $t_4, t_6, \ldots, t_{2q}$, we obtain that

$$\begin{aligned} E(|\hat{\mu}(u)|^{2q}) \;\le\; & (q!)^2 \sum_{(\epsilon_1,\ldots,\epsilon_{2q})\in T} \left[C\,\Gamma(\alpha-\gamma+1)\,2^{\alpha-\gamma}u^{-2(\alpha-\gamma)}\right]^q \times \\ & \int_{0\le t_1\le t_3 \ldots \le t_{2q-1}} d\theta(t_1)d\theta(t_3)\ldots d\theta(t_{2q-1}). \end{aligned}$$

Because θ is a probability measure, we have by symmetry that,

$$\int_{0\le t_1\le t_3 \ldots \le t_{2q-1}} d\theta(t_1)d\theta(t_3)\ldots d\theta(t_{2q-1}) = 1/q!.$$

Hence, since T has $(2q)!/(q!q!)$ elements, we find that

$$E(|\hat{\mu}(u)|^{2q}) \leq \frac{(2q)!}{q!}(C_1)^q(u^{-2(\alpha-\gamma)})^q, \tag{5.8}$$

where $C_1 = C\,\Gamma(\alpha-\gamma+1)\,2^{\alpha-\gamma}$. Because $(2q)!/(q!) \leq (2q)^q$, relation (5.8) yields,

$$E(|\hat{\mu}(u)|^{2q}) \leq \left(C_2 q u^{-2(\alpha-\gamma)}\right)^q, \tag{5.9}$$

where $C_2 = 2C_1$.

Now we write inequality (5.9) for all $u = n \in \mathbf{Z}$, $n \neq 0$, by taking $q = q_n = [\log|n|]$, the integer such that $q_n \leq \log|n| < q_n + 1$. Then

$$E(|\hat{\mu}(n)|^{2q_n}) \leq \left(C_2 q_n n^{-2(\alpha-\gamma)}\right)^{q_n},$$

and, therefore,

$$E\left[\left(\frac{|\hat{\mu}(n)|^2}{C_2 q_n n^{-2(\alpha-\gamma)}}\right)^{q_n}\right] \leq 1.$$

This implies that

$$E\left[|n|^{-2}\left(\frac{|\hat{\mu}(n)|^2}{C_2 q_n n^{-2(\alpha-\gamma)}}\right)^{q_n}\right] \leq |n|^{-2}.$$

By summing, we find:

$$E\left(\sum_{n\in\mathbf{Z},n\neq 0}\left[|n|^{-2}\left(\frac{|\hat{\mu}(n)|^2}{C_2 q_n n^{-2(\alpha-\gamma)}}\right)^{q_n}\right]\right) < \infty.$$

Therefore, by Fatou's lemma, the series

$$\sum_{n\in\mathbf{Z},n\neq 0}\left[|n|^{-2}\left(\frac{|\hat{\mu}(n)|^2}{C_2 q_n |n|^{-2(\alpha-\gamma)}}\right)^{q_n}\right] \tag{5.10}$$

converges almost surely. Then, almost surely, its general term tends to zero. That is, almost surely,

$$|n|^{-2}\left(\frac{|\hat{\mu}(n)|^2}{C_2 q_n |n|^{-2(\alpha-\gamma)}}\right)^{q_n} \to 0 \text{ as } |n| \to \infty.$$

Let

$$a_n = \frac{|\hat{\mu}(n)|^2}{C_2 q_n |n|^{-2(\alpha-\gamma)}}.$$

Then $|n|^{-2}a_n^{q_n} \to 0$ for $|n| \to \infty$. Since $q_n + 1 > \log|n|$, then

$$\log(|n|^{-2}a_n^{q_n}) = q_n \log(a_n) - 2\log|n| \geq q_n \log(a_n) - 2(q_n+1) = q_n(\log(a_n) - 2) - 2.$$

Then $q_n(\log(a_n) - 2) - 2 \to -\infty$ and in particular as $q_n \to +\infty$, $\log(a_n) < 2$ for large $|n|$. Therefore,

$$\frac{|\hat{\mu}(n)|^2}{C_2 q_n |n|^{-2(\alpha-\gamma)}} < e^2$$

and hence

$$|\hat{\mu}(n)|^2 < C_3 q_n |n|^{-2(\alpha-\gamma)} \leq C_3 \log|n||n|^{-2(\alpha-\gamma)} \text{ for large } |n|$$

($C_3 = e^2 C_2$) and this holds almost surely.

For any $\kappa > 0$, we can repeat the procedure above by taking for all $u = \kappa n$, $n \in \mathbf{Z}$, $n \neq 0$, $q_n = [\log|\kappa n|]$ instead of $q_n = [\log|n|]$ in relation (5.9). Then the series (5.10) becomes

$$\sum_{n \in \mathbf{Z}, n \neq 0} \left[|\kappa n|^{-2} \left(\frac{|\hat{\mu}(\kappa n)|^2}{C_2 q_n |\kappa n|^{-2(\alpha-\gamma)}} \right)^{q_n} \right].$$

Since this series must converge almost surely, we deduce, as above that, almost surely,

$$|\hat{\mu}(\kappa n)|^2 < C_3 q_n |\kappa n|^{-2(\alpha-\gamma)} \leq C_3 \log|\kappa n||\kappa n|^{-2(\alpha-\gamma)} \text{ for large } |n| \tag{5.11}$$

($C_3 = e^2 C_2$).

We now fix a subset Ω_0 of Ω (of probability 1) such that inequality (5.11) holds on Ω_0 simultaneously for all rational numbers $\kappa > 0$. Let us now fix $\omega \in \Omega_0$. Because Brownian paths are continuous and E is compact, then $\omega(E)$ is compact. Suppose now that $\omega(E)$ is contained in $(-1/\kappa, 1/\kappa)$, for some rational $\kappa > 0$. In the same way, we find

$$|\hat{\mu}_\omega(\kappa n)|^2 < C_3 q_n |\kappa n|^{-2(\alpha-\gamma)} \leq C_3 \log|\kappa n||\kappa n|^{-2(\alpha-\gamma)} \text{ for large } |n|,$$

where μ_ω is the value of the random measure μ at ω.

Consider now the measure ν_ω defined by $\nu_\omega(A) = \mu_\omega(A/\kappa)$ for any Borel set A of the reals. Clearly the support of ν_ω is contained in $(-1, 1)$ and $\hat{\nu}_\omega(u) = \hat{\mu}_\omega(\kappa u)$ for any $u \in \mathbf{R}$. Indeed, for all Borel-measurable function f,

$$\int f(s) d\nu_\omega(s) = \int f(\kappa s) d\mu_\omega(s).$$

In particular,

$$\hat{\nu}_\omega(u) = \int e^{ius} d\nu_\omega(s) = \int e^{iu\kappa s} d\mu_\omega(s) = \hat{\mu}_\omega(\kappa s).$$

Furthermore, $|\hat{\nu}_\omega(u)| = |\hat{\mu}_\omega(\kappa u)| \leq \theta(E) = 1$. Then ν_ω fulfills all the hypotheses of Lemma 5.5. Therefore, there exists a constant $C' > 0$ (depending only on C_3 and $\alpha - \gamma$) such that

$$|\hat{\mu}_\omega(\kappa u)|^2 = |\hat{\nu}_\omega(u)|^2 \leq C' \log|\kappa u||\kappa u|^{-2(\alpha-\gamma)} \text{ for large real } |u|.$$

Then, by the variable change $u \to \kappa u$, we find that

$$|\hat{\mu}_\omega(u)|^2 \leq C' \log|u||u|^{-2(\alpha-\gamma)} \text{ for large real } |u|$$

and this holds for all $\omega \in \Omega_0$. Hence, almost surely, for any $\epsilon > 0$,

$$|\hat{\mu}(u)|^2 \leq C'|u|^{-2(\alpha-\gamma)+\epsilon}, \text{ as } |u| \to \infty,$$

since $\log |u| \leq |u|^\epsilon$ for large values of $|u|$.
Summarizing, we have that

for any $\gamma > 0$, almost surely, for any $\epsilon > 0$,

$$|\hat{\mu}(u)|^2 \leq C'|u|^{-2(\alpha-\gamma)+\epsilon}, \text{ as } |u| \to \infty.$$

Therefore, for any $\gamma > 0$, almost surely, $\dim_F X(E) \geq 2\alpha - \gamma$. By considering a sequence (γ_n) of rational numbers converging to 0, this implies that, almost surely,

$$\dim_F X(E) \geq 2\alpha.$$

∎

This result has been extended by Kahane to fractional Brownian motion [24, pp 265-267].

Chapter 6

Fourier analysis on Brownian level sets

In this chapter, we first apply Kahane's method to study the asymptotic decays of Fourier transform of Dirac measures of Brownian motion. We show that if $\delta_a(X)$ is the Dirac measure of Brownian motion at the level a, that is the measure defined by the Brownian local time L_a at level a, and μ is its restriction to the random interval $[0, L_a^{-1}(1)]$, then the Fourier transform of μ is such that, with positive probability, for all $0 \leq \beta < 1/2$, the function $u \to |u|^\beta |\widehat{\delta_a(X)}(u)|^2$, $(u \in \mathbf{R})$ is bounded. From this result we deduce that each Brownian level set, reduced to a compact interval, is with positive probability, a Salem set of dimension $1/2$. After using Lévy's formula of local times, we show that the restriction μ of $\delta_0(X)$ to the deterministic interval $[0, 1]$ is such that its Fourier transform satisfies $E\left(|\hat{\mu}(u)|^2\right) \leq C|u|^{-1/2}$, $u \neq 0$ and $C > 0$.
We consider as previously the canonical model of Brownian motion $X = (X_t : t \geq 0)$ defined on the space $\Omega = C[0, \infty)$ and we assume that all paths start at the origin.

6.1 Fourier analysis on passage times

Let $(\Gamma_a : a \geq 0)$ be the passage times process of Brownian motion. We recall that

$$\Gamma_a = \inf\{t \geq 0 : X_t = a\} \quad \text{and} \quad \inf \emptyset = \infty.$$

It is shown in [7] that, for any fixed compact subset E of $[0, \infty)$ of dimension α, its image by a stable process with index $\gamma \in (0, 2]$ in $\mathbf{R}$ has, almost surely, Hausdorff dimension $\min\{\alpha\gamma, 1\}$. (See also [29] for recent generalisations to Lévy processes.) We, therefore, have in particular that $\dim_H \Gamma(E) = \frac{\dim_H E}{2}$, because $(\Gamma_a : a \geq 0)$ is a stable process of index $1/2$. The same result applies for the process $(\rho_a : a \geq 0)$, the right-continuous inverse of the maximum function of Brownian motion. Clearly, the closure of the image $\Gamma(E)$ is such that

$$\overline{\Gamma(E)} \subset \Gamma(E) \cup \rho(E).$$

Therefore,

$$\dim_H \overline{\Gamma(E)} = \dim_H \Gamma(E) = \frac{\dim_H E}{2}.$$

To show the claim $\overline{\Gamma}(E) \subseteq \Gamma(E) \cup \rho(E)$, we argue as follows: let $x \in \overline{\Gamma}(E)$. Then there is a sequence (x_n) of elements of $\Gamma(E)$ that converges to x. Then we assume that $x_n = \Gamma(t_n)$ for $t_n \in E$ for each n. If we can extract from (x_n) an increasing subsequence $(y_n = \Gamma(s_n))$ that converges also to x, then because Γ is injective and increasing, the sequence s_n is also increasing (in E) and hence converges to some point $s \in E$ because E is compact. Then since Γ is left-continuous, we find that $y_n = \Gamma(s_n) \to \Gamma(s)$. Then $x = \Gamma(s)$.
If such an increasing subsequence (y_n) does not exist, then we can consider a decreasing subsequence (z_n) of (x_n) that converges to x and assume $z_n = \gamma(h_n)$ where $h_n \in E$. The sequence (h_n) is therefore decreasing (because Γ is increasing) and hence converges to a limit $t \in E$. In that case, the sequence $\rho(h_n)$ also converges to x. Indeed from the definition of Γ and ρ, we have that for any $n \in \mathbf{N}$,

$$\Gamma(h_n) \leq \rho(h_n) \leq \Gamma(h_{n+1}).$$

Then

$$|\rho(h_n) - \Gamma(h_n)| \leq |\Gamma(h_n) - \Gamma(h_{n+1})| \to 0$$

and hence $\Gamma(h_n)$ and $\rho(h_n)$ converge to the same limit x.

Because ρ is right-continuous, we conclude that $\rho(h_n) \to \rho(t)$. Therefore, $\rho(t) = x$. So, in all cases, $x \in \Gamma(E) \cup \rho(E)$.

Theorem 6.1 *For any compact subset $E \subset [0,1]$ of Hausdorff dimension α, there exists a subset Ω_1 of Ω of probability 1 such that, for each $\omega \in \Omega_1$, the image $\Gamma_\omega(E) = \{\Gamma_a(\omega) : a \in E\}$ is bounded and its closure is a Salem set of dimension $\alpha/2$.*

Proof The proof is based on the proof of Theorem 5.4. From relation (3.7), we have that $\Gamma_1 < \infty$ almost surely. Then we consider a subset Ω_0 of Ω of probability 1 such that Γ_1 is finite on Ω_0 in the sense that for each $\omega \in \Omega_0$, there exists a positive real number h_ω such that $\Gamma_1(\omega) < h_\omega$. This implies that $\Gamma_\omega(E) \subset [0, h_\omega]$. We now restrict ourselves to Ω_0. Since $\dim_H E = \alpha$, then for any $\gamma > 0$, $\dim_H E > \alpha - \gamma$. We have, by Frostman's lemma, a probability measure θ carried by E such that

$$\theta(I) \leq C|I|^{\alpha-\gamma}, \text{ for any interval } I.$$

Let μ be the image measure of θ by the process $(\Gamma_a : a \geq 0)$. This means that $\mu(A) = \theta(\Gamma_a^{-1}(A))$ for any Borel subset A of $\mathbf{R}$. It is a random measure carried by the closure $\overline{\Gamma(E)}$ of $\Gamma(E)$.
We want to show that there exists a positive constant $C' > 0$, depending only on C and $\alpha - \gamma$, such that, almost surely, for any $\epsilon > 0$,

$$|\hat{\mu}(u)| \leq C'|u|^{-\frac{\alpha-\gamma}{4}+\epsilon}, \text{ for large } |u|.$$

This will imply that

$$\dim_F \overline{\Gamma(E)} \geq \alpha/2.$$

From the fact that $\dim_H \overline{\Gamma(E)} = \alpha/2$, it will follow that the Fourier and Hausdorff dimensions are equal to $\alpha/2$.
We have that

$$\hat{\mu}(u) = \int e^{ixu} d\mu(x) = \int e^{i\Gamma_s u} d\theta(s).$$

Then

$$|\hat{\mu}(u)|^2 = \int \int e^{i(\Gamma_s - \Gamma_t)u} d\theta(s) d\theta(t).$$

For any integer $q \geq 1$,

$$|\hat{\mu}(u)|^{2q} = \int_{\mathbf{R}^{2q}} \exp iu[(\Gamma_{s_1} + \ldots + \Gamma_{s_q}) - (\Gamma_{s_1^{'}} + \ldots + \Gamma_{s_q^{'}})] \\ d\theta(s_1) \ldots d\theta(s_q) d\theta(s_1^{'}) \ldots d\theta(s_q^{'}).$$

By symmetry this integral is equal to

$$(q!)^2 \int_{0 \leq s_1 \ldots \leq s_q} \int_{0 \leq s_1^{'} \ldots \leq s_q^{'}} f(u, s_1, \ldots, s_q, s_1^{'}, \ldots, s_q^{'}) d\theta(s_1) \ldots d\theta(s_q) d\theta(s_1^{'}) \ldots d\theta(s_q^{'})$$

where

$$f(u, s_1, \ldots, s_q, s_1^{'}, \ldots, s_q^{'}) = \exp iu[(\Gamma_{s_1} + \ldots + \Gamma_{s_q}) - (\Gamma_{s_1^{'}} + \ldots + \Gamma_{s_q^{'}})].$$

By rearranging $s_1, s_2, \ldots, s_q, s_1^{'}, s_2^{'}, \ldots, s_q^{'}$ as an increasing sequence $t_1 \leq t_2 \leq \ldots \leq t_{2q}$, we obtain that

$$|\hat{\mu}(u)|^{2q} = (q!)^2 \quad \sum_{(\epsilon_1, \ldots, \epsilon_{2q}) \in T} \int_{0 \leq t_1 \ldots \leq t_{2q}} \exp[iu(\epsilon_1 \Gamma_{t_1} + \epsilon_2 \Gamma_{t_2} + \ldots + \epsilon_{2q} \Gamma_{t_{2q}})] \\ d\theta(t_1) d\theta(t_2) \ldots d\theta(t_{2q})$$

where, as previously, T is the set of all sequences $(\epsilon_1, \ldots, \epsilon_{2q})$ such that $\epsilon_j \in \{-1, 1\}$ for each j and $\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{2q} = 0$.
The random variables $\Gamma(t_1)$, $\Gamma(t_2) - \Gamma(t_1)$, ..., $\Gamma(t_{2q}) - \Gamma(t_{2q-1})$ are independent. Then

$$E[\exp(iu(\epsilon_1 \Gamma_{t_1} + \epsilon_2 \Gamma_{t_2} + \ldots + \epsilon_{2q} \Gamma_{t_{2q}}))] \\ = E[\exp(iu(\epsilon_1 + \ldots + \epsilon_{2q}) \Gamma_{t_1})] \times E[\exp(iu(\epsilon_2 + \ldots + \epsilon_{2q})(\Gamma_{t_2} - \Gamma_{t_1}))] \\ \times E[\exp(iu\epsilon_{2q}(\Gamma_{t_{2q}} - \Gamma_{t_{2q-1}}))].$$

We recall that $\Gamma_x - \Gamma_y$ has the same distribution as Γ_{y-x}, $x \leq y$ and by relation (3.8)

$$E(\exp(iu\Gamma_a) = \exp[-a\sqrt{|u|}(1 - i\,\mathrm{sgn}(u))], \ a > 0.$$

Then

$$\begin{aligned}
E(|\hat{\mu}(u)|^{2q}) &= (q!)^2 \sum_{(\epsilon_1,\ldots,\epsilon_{2q})\in T} \int_{0\le t_1\ldots\le t_{2q}} \exp[-t_1\sqrt{|u||\epsilon_1+\ldots+\epsilon_{2q}|}(1-i\,\mathrm{sgn}(u(\epsilon_1+\ldots+\epsilon_{2q})))] \\
&\quad \times \exp[-(t_2-t_1)\sqrt{|u||\epsilon_2+\ldots+\epsilon_{2q}|}(1-i\,\mathrm{sgn}(u(\epsilon_2+\ldots+\epsilon_{2q})))] \\
&\quad \times \exp(-(t_3-t_2)\sqrt{|u||\epsilon_3+\ldots+\epsilon_{2q}|}(1-i\,\mathrm{sgn}(u(\epsilon_3+\ldots+\epsilon_{2q})))) \\
&\quad \times \ldots \\
&\quad \times \exp[-(t_{2q}-t_{2q-1})\sqrt{|u||\epsilon_{2q}|}(1-i\,\mathrm{sgn}(u\epsilon_{2q}))] \\
&\quad \times d\theta(t_1)d\theta(t_2)\ldots d\theta(t_{2q}).
\end{aligned}$$

There are $(2q)!/(q!q!)$ terms in the sum and each term is there with its conjugate. Since $z+\bar{z}\le|z|+|\bar{z}|$, ($z$ complex), we find that

$$\begin{aligned}
E(|\hat{\mu}(u)|^{2q}) &\le (q!)^2 \sum_{\{\epsilon_1,\ldots,\epsilon_{2q}\}\in T} \int_{0\le t_1\ldots\le t_{2q}} \exp(-t_1\sqrt{|u||\epsilon_1+\ldots+\epsilon_{2q}|}) \\
&\quad \times \exp(-(t_2-t_1)\sqrt{|u||\epsilon_2+\ldots+\epsilon_{2q}|}) \\
&\quad \times \exp(-(t_3-t_2)\sqrt{|u||\epsilon_3+\ldots+\epsilon_{2q}|}) \\
&\quad \times \ldots \\
&\quad \times \exp(-(t_{2q}-t_{2q-1})\sqrt{|u||\epsilon_{2q}|}) \\
&\quad d\theta(t_1)d\theta(t_2)\ldots d\theta(t_{2q}).
\end{aligned}$$

Let $\psi_j=\sqrt{|u||\epsilon_j+\ldots+\epsilon_{2q}|}$. It is clear that for each j, $\psi_j\ge 0$ and for even j's, $\psi_j\ge\sqrt{|u|}$ because $|\epsilon_j+\ldots+\epsilon_{2q}|\ge 1$. Therefore, by dropping all the factors corresponding to odd j's the inequality remains valid, that is,

$$\begin{aligned}
E(|\hat{\mu}(u)|^{2q}) &\le (q!)^2 \sum_{\{\epsilon_1,\ldots,\epsilon_{2q}\}\in T} \int_{0\le t_1\ldots\le t_{2q}} \exp[-(t_2-t_1)\psi_2]\times\exp[-(t_4-t_3)\psi_4]\ldots \\
&\quad \times \exp[-(t_{2q}-t_{2q-1})\psi_{2q}]d\theta(t_1)d\theta(t_2)\ldots d\theta(t_{2q}).
\end{aligned}$$

We can now integrate with respect to t_j for j even and obtain for example for $j=2$,

$$\int_{t_1}^{t_3}\exp(-(t_2-t_1)\psi_2)d\theta(t_2)=\int_0^{t_3-t_1}e^{-t\psi_2}d\theta(t+t_1)\le\int_0^\infty e^{-t\sqrt{|u|}}\,d\theta(t+t_1). \tag{6.1}$$

This integral can be calculated by parts as follows: take

$$\begin{aligned}
U &= e^{-t\sqrt{|u|}},\; dU=\sqrt{|u|}e^{-t\sqrt{|u|}}dt; \\
dV &= d\theta(t+t_1);\; V(t)=\int_0^t d\theta(s+t_1)=\theta[t_1,t_1+t]\le Ct^{\alpha-\gamma} \\
I:=\int_0^\infty e^{-t\sqrt{|u|}}\,d\theta(t+t_1) &= \sqrt{|u|}\int_0^\infty e^{-t\sqrt{|u|}}\theta[t_1,t_1+t]dt
\end{aligned}$$

$$\leq C\sqrt{|u|}\int_0^\infty e^{-t\sqrt{|u|}}t^{\alpha-\gamma}dt.$$

Using the variable change $t \to -t\sqrt{|u|}$, we find that

$$I \leq C\,\Gamma(\alpha-\gamma+1)|u|^{\frac{-\alpha+\gamma}{2}},$$

(here Γ is the gamma function and not the passage time process).
By repeating the same calculation for $t_4, t_6, \dots, t_{2q}$, we obtain that

$$E(|\hat{\mu}(u)|^{2q}) \leq \frac{(2q)!}{q!}(C_1)^q(|u|^{\frac{-\alpha+\gamma}{2}})^q,$$

where $C_1 = C\,\Gamma(\alpha-\gamma+1)$. Therefore,

$$E(|\hat{\mu}(u)|^{2q}) \leq \left(C_2 q|u|^{\frac{-\alpha+\gamma}{2}}\right)^q, \tag{6.2}$$

where $C_2 = 2C_1$. Now we use this inequality in the similar way we did for inequality (5.9) in the proof of Theorem 5.4. We find that, almost surely, for large integer $|n|$,

$$|\hat{\mu}(n)|^2 \leq C_3|n|^{\frac{-\alpha+\gamma}{2}}\log|n|, \quad \text{where } C_3 = e^2C_2. \tag{6.3}$$

As we discussed in the proof of Theorem 5.4, there exists a subset Ω_1 of Ω_0 such that, simultaneously for all rationals $k > 0$,

$$|\hat{\mu}(\kappa n)|^2 < C_3 q_n|\kappa n|^{\frac{-(\alpha-\gamma)}{2}} \leq C_3\log|\kappa n||\kappa n|^{-2(\alpha-\gamma)} \text{ for large } |n| \tag{6.4}$$

For an arbitrary $\omega \in \Omega_1$, there exists a rational number $\kappa > 0$ such that $\Gamma_1(\omega) \leq 1/\kappa$. By the same argument as in the proof of Theorem 5.4, Lemma 5.5 implies that there exists a constant C' depending only on C_3 and $\alpha - \gamma$ such that, for all $w \in \Omega_1$,

$$|\hat{\mu}_\omega(u)|^2 \leq C'|u|^{\frac{-\alpha+\gamma}{2}}\log|u| \text{ for large reals} |u|.$$

∎

Remark 6.2 *Kahane's method can be generalized to other stochastic processes for which an equivalent of relation (6.2) can be explicitly calculated.*

Corollary 6.3 *For any compact subset $E \subset [0,1]$ of Hausdorff dimension α, the closure of $\Gamma(E)$ is, with positive probability contained in $[0,1]$ and in that case, its closure is a Salem set with dimension $\alpha/2$.*

Proof Using the notations of Theorem 6.1, consider $A = \{\omega \in \Omega_1 : \Gamma_1(\omega) < 1\}$. We have that $P(A) = P\{\Gamma_1 < 1\} > 0$ and the closure of $\Gamma_\omega(E)$ is a Salem set with dimension $\alpha/2$ for every $w \in A$.

Remark 6.4

If we replace the process $(\Gamma_a : a \geq 0)$ by the process $(\rho_a : a \geq 0)$ where

$$\rho_a = \inf\{t \geq 0 : M_t > a, \},\ M_t = \sup_{s\in[0,t]} X_s,$$

(ρ is the right continuous inverse of the maximum process of Brownian motion), then using the same ideas as in the proof of Theorem 6.1, we have that, for any compact subset $E \subset [0,1]$ of dimension α, almost surely, the closure $\overline{\rho(E)}$ is bounded and is a Salem set of dimension $\alpha/2$.

6.2 Fourier dimension of zero set

Now we consider the process $L_0^{-1} = (\Lambda_t : t \geq 0)$ of inverse local times of Brownian motion. We recall that

$$L_0^{-1}(t) = \inf\{s \geq 0 : L_0(s) > t\},$$

where L_0 is the local time process at 0. As discussed in Section 4.5, L_0 can be seen as the maximum function of another version of Brownian motion. Then L_0^{-1} is the right-continuous inverse of the maximum process of another version of Brownian motion. Therefore, by Remark 6.4, we have the following theorem.

Theorem 6.5 *For any compact subset $E \subset [0,1]$ of Hausdorff dimension α, there exists a subset Ω_1 of Ω of probability 1 such that, for each $\omega \in \Omega_1$, the image $L_0^{-1}(\omega)(E) = \{L_0^{-1}(a)(\omega) : a \in E\}$ is bounded and its closure is a Salem set of dimension $\alpha/2$.*

We will now consider the case where E is the interval $[0,1]$.

Theorem 6.6 *Let Z be the zero set of Brownian motion. Then,*

(*i*) *almost surely, $Z \cap [0, L_0^{-1}(1)]$ is bounded and is a Salem set of dimension 1/2.*

(*ii*) *with positive probability, the intersection $Z \cap [0,1]$ is a Salem set with dimension 1/2.*

Proof It is clear that (*ii*) is a direct consequence of (*i*) because $L_0^{-1}(1) \leq 1$ with positive probability. If $L_0^{-1}(1) \leq 1$, then $Z \cap [0, L_0^{-1}(1)] \subset Z \cap [0,1]$ and hence $Z \cap [0,1]$ is a Salem set of dimension 1/2 because, obviously, $\dim_F Z \cap [0, L_0^{-1}(1) \leq \dim_F Z \cap [0,1]$.
The proof of (*i*) is based on Theorem 6.5. Consider the case where $E = [0,1]$ and θ is the Lebesgue measure on $[0,1]$. Clearly, $\theta(I) = |I|$. Let us now take the image of E by the process L_0^{-1}. We have that $L_0^{-1}(E) \subset [0, L_0^{-1}(1)]$, because the process L_0^{-1} is increasing. Since, almost surely, $L_0^{-1}(1) < \infty$, we consider again a subset Ω_0 of Ω such that $L_0^{-1}(1) < \infty$ everywhere in Ω_0. Then, $L_0^{-1}(E) \subset Z$. The image measure of θ by the process L_0^{-1} is clearly the Dirac measure $\delta(X)$ restricted to the interval $[0, L_0^{-1}(1)]$. We

denote this restriction by μ. By Theorem 6.5, there exists a constant $C' > 0$ and a subset Ω_1 of Ω_0 such that everywhere in Ω_1,

$$|\widehat{\mu}(u)|^2 \le C'|u|^{-\frac{1}{2}} \log |u| \text{ for large reals} |u|.$$

Because the support of the measure μ is contained in $Z \cap [0, L_0^{-1}(1)]$, we conclude that $Z \cap [0, L_0^{-1}(1)]$ is a Salem set everywhere in Ω_1. ∎

Remark 6.7

What we have shown in Theorem 6.6 (ii) is that, with positive probability, the restriction μ of the Dirac measure on $[0, L_0^{-1}(1)]$ has its support included in the interval $[0, 1]$ and verifies relation

$$|\widehat{\mu}(u)|^2 \le C'|u|^{-\frac{1}{2}} \log |u| \text{ for large reals} |u|.$$

Nothing is said about the restriction of the Dirac measure on a deterministic interval like $[0, 1]$. We will discuss this issue in the last section of this chapter.

6.3 Fourier dimension of level sets

Theorem 6.8 *For any $a \in \mathbf{R}$, the Brownian level set $Z_a = \{t \ge 0 : X_t = a\}$ is, such that, $Z_a \cap [0, 1]$ is non-empty with positive probability and in this case, is a Salem set.*

Proof The idea is to reduce to zero set by applying the strong Markov property of Brownian motion. We consider the Dirac measure $\delta_a(X)$ of Brownian motion at level a, the measure defined by the local time process $L_a(t)$ at level a. Consider now the Brownian motion $Y_t = X_{t+\Gamma_a} - a$, where Γ_a is the passage time process at a. The zero set Z^Y of Y is equal to the level set Z_a of X. Let us denote by $L_0^Y(t)$ the local time of the Brownian motion Y at zero. We have that

$$\begin{aligned} L^Y(t) &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{|Y| \le \epsilon}(s) ds \\ &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{|X-a| \le \epsilon}(s + \Gamma) ds \\ &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^t 1_{|X-a| \le \epsilon}(s + \Gamma_a) ds \\ &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{\Gamma_a}^{t+\Gamma_a} 1_{|X-a| \le \epsilon}(h) dh \\ &= \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_0^{t+\Gamma_a} 1_{|X-a| \le \epsilon}(h) dh \\ &= L_a(t + \Gamma_a). \end{aligned}$$

We conclude that $L_a(t + \Gamma_a)$ is the local time at zero of another version of Brownian motion (independent of a) and in particular, it has the same distribution with $L_0(t)$ (the local time of Brownian motion X at 0). Let L_a^{-1} be the inverse of the local time L_a (of X) defined by

$$L_a^{-1}(t) = \inf\{s \geq 0 : L_a(s) > t\}.$$

Then, we have that

$$L_a^{-1}(t) = \Gamma_a + \Lambda^Y(t)$$

where $\Lambda^Y(t)$ is the inverse local time at zero of the Brownian motion Y. The measure defined by $L_a^{-1}(t)$ is the Dirac measure $\delta_a(X)$ of Brownian motion at level a. We have that, almost surely, $L_a^{-1}(1) < \infty$. Let us now fix a subset Ω_0 of Ω of probability 1 for which $L_a^{-1}(1) < \infty$ everywhere. Let ν_a be the image measure of the Lebesgue measure on $[0,1]$ by the process L_a^{-1}. Then ν_a is the restriction of $\delta_a(X)$ to the interval $[0, L_a^{-1}(1)]$. We have that

$$\begin{aligned}
\widehat{\nu_a}(u) &= \int e^{iuL_a^{-1}(t)}dt \\
&= e^{iu\Gamma_a}\int e^{iu(L_a^{-1}(t)-\Gamma_a)}dt \\
&= e^{iu\Gamma_a}\int e^{iu\Lambda^Y(t)}dt
\end{aligned}$$

Then

$$|\widehat{\nu_a}(u)| = |\int e^{iu\Lambda^Y(t)}dt| = |\hat{\eta}(u)|$$

where η is the image measure of the Lebesgue measure on $[0,1]$ by the process $(\Lambda^Y(t) : t \geq 0)$. We already know, from the proof of Theorem 6.6 that, there exists a constant $C' > 0$, and a subset Ω_1 of Ω_0 of probability 1 such that, everywhere on Ω_1,

$$|\widehat{\eta}(u)|^2 \leq C'|u|^{-\frac{1}{2}} \log|u| \text{ for large reals} |u|.$$

Then we have that on Ω_1,

$$|\widehat{\nu_a}(u)| \leq C'|u|^{-\frac{1}{2}} \log|u| \ \text{ for } |u| \to \infty.$$

It follows that $Z_a \cap [0, L_a^{-1}(1)] = Z_a \cap [\Gamma_a, L_a^{-1}(1)]$ is a Salem set.
We now consider the subset A of Ω_1 defined by $A = \{w \in \Omega_1 : L_a^{-1}(1) < 1\}$. Clearly $P(A) > 0$ and then on A, we have that $Z_a \cap [0, L_a^{-1}(1)] \subset Z_a \cap [0,1]$ and is a Salem set. It follows that, with positive probability, $Z_a \cap [0,1]$ is non-empty and is a Salem set with dimension 1/2. ∎

6.4 The Fourier transform of the Dirac measure of Brownian motion on the unit interval

In the previous sections, we analysed the restrictions of the Dirac measure $\delta(X)$ of Brownian motion only on random intervals like $[0, L^{-1}(1)]$ where L^{-1} is the inverse local time at 0. We have shown that, if μ denotes the restriction of $\delta(X)$ on $[0, L^{-1}(1)]$, then, almost surely,

$$|\hat{\mu}(u)|^2 = O(|u|^{-\frac{1}{2}} \log |u|) \text{ for large reals } |u|.$$

We essentially used the fact that the process of inverse local times of Brownian motion (at the origin) has independent and stationary increments and considered the measure μ as the image of the Lebesgue measure λ on $[0, 1]$ by the process $(L^{-1}(t) : t \geq 0)$.
The restriction of the Dirac measure of Brownian motion on a deterministic interval like $[0, 1]$ can also be seen as the image measure of the Lebesgue measure on the random interval $[0, L(1)]$ by the same process L^{-1} (L is the Brownian local time process zero). However, it is difficult to analyse its Fourier transform by the method developed in the proof of Theorem 6.1. Indeed, we should have to calculate the expectation of the integral

$$\int_0^{L(1)} e^{iuL_0^{-1}(t)} dt$$

which seems to be very difficult to handle. There is also a problem of generalization to other stochastic processes. The notion of local times has been extended to other more general processes and it is an open problem to study the Fourier structure of their zero sets [46]. No one can expect that the inverse local times of general processes will have independent stationary increments as it is the case for Brownian motion. In this section we develop a different approach to study the Fourier structure of the zero set of Brownian motion, which may be extended to some general processes.
We consider the restriction of the Dirac measure of Brownian motion $\delta(X)$ on the interval $[0, 1]$ and prove the following result:

Theorem 6.9 *For some constant* $C > 0$,

$$E\left(|\widehat{\delta(X)}(u)|^2\right) \leq C|u|^{-1/2},\ u \neq 0. \tag{6.5}$$

The proof of this theorem is based on the following propositions. The first is due to Fouché [15].

Proposition 6.10 *Let* $\delta(X)$ *be the Dirac measure of Brownian motion restricted on the interval* $[0, 1]$. *Then, almost surely,*

$$\widehat{\delta(X)}(u) = \lim_{n\to\infty} \frac{n}{2} \int_0^1 1_{|X|\leq 1/n}(s) e^{ius} ds. \tag{6.6}$$

Proof We recall that the measure $\delta(X)$ is defined by the local time $L(t) : 0 \leq t \leq 1$ at zero and

$$L(t) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t 1_{|X|\leq\epsilon}(s) ds.$$

There exists a subspace $\Omega_0 \subset \Omega$ of probability 1 such that for any $0 \leq t \leq 1$, $L(t)$ exists (and is finite) on Ω_0. So, we restrict ourselves on Ω_0. Consider the sequence (L_n) of functions defined on $[0,1]$ by

$$L_n(t) = \frac{n}{2}\int_0^t 1_{|X|\leq 1/n}(s)ds, n = 1, 2, \dots$$

Because L_n is an non-decreasing function, it defines a measure μ_n on $[0,1]$ by $\mu_n[a,b) = L_n(b) - L_n(a)$. Clearly, (L_n) converges pointwise to L and, therefore, by the portmanteau theorem, the sequence (μ_n) converges weakly to the Dirac measure $\delta(X)$ of Brownian motion. As a consequence, by the same theorem, the Fourier transform of μ_n converges pointwise to the Fourier transform of $\delta(X)$, that is

$$\widehat{\delta(X)}(u) = \lim_{n\to\infty} \hat{\mu}_n(u) = \lim_{n\to\infty} \int_0^1 e^{ius} d\mu_n(s), \; u \in \mathbf{R}.$$

The distribution function of μ_n is L_n and is differentiable with derivative

$$L_n'(s) = \frac{n}{2} 1_{|X|\leq 1/n}(s).$$

Therefore,

$$\hat{\mu}_n(u) = \frac{n}{2}\int_0^1 e^{ius} 1_{|X|\leq 1/n}(s)ds$$

from which we deduce that

$$\widehat{\delta(X)}(u) = \lim_{n\to\infty} \frac{n}{2}\int_0^1 1_{|X|\leq 1/n}(s) e^{ius} ds.$$

∎

Proposition 6.11 *With the notations in the proof of Proposition* 6.10*, we have that*

$$E\left(|\widehat{\delta(X)}(u)|^2\right) \leq \lim_{n\to\infty} E\left(|\hat{\mu}_n(u)|^2\right)$$

where E denotes the expectation.

Proof This is a consequence of Fatou's lemma. We already know that $|\hat{\mu}_n(u)|^2$ converges to $|\widehat{\delta(X)}(u)|^2$ pointwise on Ω_0. Then

$$\begin{aligned} E\left(|\widehat{\delta(X)}(u)|^2\right) &= \int_{\Omega_0} |\widehat{\delta(X)}(u)|^2 dP \\ &= \int_{\Omega_0} \lim_{n\to\infty} |\hat{\mu}_n(u)|^2 dP \\ &\leq \liminf_{n\to\infty} \int_{\Omega_0} |\hat{\mu}_n(u)|^2 dP \\ &= \liminf_{n\to\infty} E\left(|\hat{\mu}_n(u)|^2\right). \end{aligned}$$

Let us show that the limit $\lim_{n\to\infty} E\left(|\hat{\mu}_n(u)|^2\right)$ exists. By Fubini's theorem, we have that

$$E\left(|\hat{\mu}_n(u)|^2\right) = \left(\frac{n}{2}\right)^2 \int_0^1\int_0^1 E\left(1_{|X|\leq 1/n}(s) 1_{|X|\leq 1/n}(t)\right) e^{ius} e^{-iut} ds dt$$

$$\begin{aligned}
&= \left(\frac{n}{2}\right)^2 \int_0^1 \int_0^1 P\left(|X(s)| \le 1/n, |X(t)| \le 1/n\right) e^{ius} e^{-iut} ds dt \\
&= \left(\frac{n}{2}\right)^2 \int_{0\le s\le t\le 1} P\left(|X(s)| \le 1/n, |X(t)| \le 1/n\right) e^{ius} e^{-iut} ds dt \\
&\quad + \left(\frac{n}{2}\right)^2 \int_{0\le t\le s\le 1} P\left(|X(s)| \le 1/n, |X(t)| \le 1/n\right) e^{ius} e^{-iut} ds dt.
\end{aligned}$$

Then

$$E\left(|\hat{\mu}_n(u)|^2\right) = \left(\frac{n}{2}\right)^2 (A + \bar{A})$$

where

$$A = \int_{0\le s\le t\le 1} P\left(|X(s)| \le 1/n, |X(t)| \le 1/n\right) e^{ius} e^{-iut} ds dt.$$

Using relation (3.1), we have that

$$A = \int_{0\le s\le t\le 1} \int_{-1/n}^{-1/n} \int_{-1/n}^{-1/n} \frac{e^{-x^2/2s}}{\sqrt{2\pi s}} \frac{e^{-(y-x)^2/2(t-s)}}{\sqrt{2\pi(t-s)}} e^{iu(t-s)} dx dy ds dt.$$

Because

$$\begin{aligned}
f_n(s,t) &= \int_{-1/n}^{-1/n} \int_{-1/n}^{-1/n} \frac{e^{-x^2/2s}}{\sqrt{2\pi s}} \frac{e^{-(y-x)^2/2(t-s)}}{\sqrt{2\pi(t-s)}} dx dy \\
&\le \left(\frac{2}{n}\right)^2 \frac{1}{2\pi\sqrt{s(t-s)}},
\end{aligned}$$

we can now apply the dominated convergence theorem. We find that,

$$\lim_{n\to\infty} \left(\frac{n}{2}\right)^2 A = \int_{0\le s\le t\le 1} \left[\lim_{n\to\infty} \left(\frac{n}{2}\right)^2 f_n(s,t)\right] e^{iu(t-s)} ds dt. \tag{6.7}$$

Now we have that

$$\lim_{n\to\infty} \left(\frac{n}{2}\right)^2 f_n(s,t) = \lim_{n\to\infty} \frac{1}{\lambda(B_n)} \int_{B_n} \frac{e^{-x^2/2s}}{\sqrt{2\pi s}} \frac{e^{-(y-x)^2/2(t-s)}}{\sqrt{2\pi(t-s)}} dx dy$$

where λ is the Lebesgue measure and B_n is the square $-1/n \le x, y \le 1/n$. Since the function

$$h(x,y) = \frac{e^{-x^2/2s}}{\sqrt{2\pi s}} \frac{e^{-(y-x)^2/2(t-s)}}{\sqrt{2\pi(t-s)}}$$

is continuous at the origin, we find that

$$\lim_{n\to\infty} \left(\frac{n}{2}\right)^2 f_n(s,t) = \lim_{n\to\infty} \frac{1}{\lambda(B_n)} \int_{B_n} h(x,y) dx dy = h(0,0) = \frac{1}{2\pi\sqrt{s(t-s)}}.$$

(We have used the fact that if $g \in L^1(\mathbf{R}^n)$ and g is continuous at x, then x is a Lebesgue point of g, see for example, [42, p 138].) Then, relation (6.7) yields

$$\begin{aligned}
\lim_{n\to\infty}\left(\frac{n}{2}\right)^2 A &= \int_{0\le s\le t\le 1}\frac{1}{2\pi\sqrt{s(t-s)}}e^{iu(t-s)}dsdt \\
&= (2\pi)^{-1}\int_0^1\int_s^1\frac{e^{iu(t-s)}}{\sqrt{s(t-s)}}dsdt \\
&= (2\pi)^{-1}\int_0^1\frac{1}{\sqrt{s}}\left(\int_0^{1-s}\frac{e^{iuz}}{\sqrt{z}}dz\right)ds \\
&= \pi^{-1}\int_0^1\frac{1}{\sqrt{s}}\left(\int_0^{\sqrt{1-s}}e^{iuz^2}dz\right)ds \\
&= \pi^{-1}|u|^{-1/2}\int_0^1\frac{1}{\sqrt{s}}\left(\int_0^{\sqrt{(1-s)|u|}}e^{iz^2\ \mathrm{sgn}(u)}dz\right)ds.
\end{aligned}$$

Therefore, the limit $\lim_{n\to\infty} E\left(|\hat{\mu}_n(u)|^2\right)$ exists and is given by

$$\lim_{n\to\infty} E\left(|\hat{\mu}_n(u)|^2\right) = 2\pi^{-1}|u|^{-1/2}\int_0^1\frac{1}{\sqrt{s}}\left(\int_0^{\sqrt{(1-s)|u|}}\cos(z^2)dz\right)ds.$$

It is well-known (see, for example, [30]) that the Fresnel integral

$$C(x) = \int_0^x \cos(z^2)dz, (x\ge 0)$$

is such that $0\le C(x)\le C(\sqrt{\pi/2}) < 1$. Therefore,

$$E|\widehat{\delta(X)}(u)|^2 \le \lim_{n\to\infty} E\left(|\hat{\mu}_n(u)|^2\right) \le 4\pi^{-1}|u|^{-1/2}.$$

∎

6.5 Some open problems

The following problems still require further investigations.

Problem 1 We have shown that, for every $a\in[0,1]$, the level set $Z_a\cap[0,1]$ is a Salem set with positive probability. It is not known whether this property holds simultaneously for all $a\in[0,1]$.

Problem 2 Given a compact subset $E\subset[0,1]$, it is not known whether its Brownian inverse image $X^{-1}(E)\cap[0,1]$ is a Salem set.

Problem 3 We have proven that the Dirac measure of Brownian motion $\delta(X)$ (restricted to $[0,1]$) is such that its Fourier transform satisfies $E\left(|\widehat{\delta(X)}(u)|^2\right)\le C|u|^{-1/2}$, $u\ne 0$ and $C>0$. An improvement of this result should be to find the exact asymptotic decay of

$\widehat{\delta(X)}(u)$ and to generalize the procedure to other stochastic processes, for example the fractional Brownian motion and the Brownian bridge.

Concluding remarks

In this thesis, we have been able to study, partially, the Fourier structure of level sets of Brownian motion using the notion of local times.
We showed that, if $\delta_a(X)$ is the Dirac measure of Brownian motion at the level a, that is the measure defined by the Brownian local time L_a at level a, and μ is its restriction to the random interval $[0, L_a^{-1}(1)]$, then the Fourier transform of μ is such that, with positive probability, for all $0 \leq \beta < 1/2$, the function $u \to |u|^\beta |\widehat{\delta_a(X)}(u)|^2$, $(u \in \mathbf{R})$, is bounded. From this we deduced that each Brownian level set, reduced to a compact interval, is with positive probability, a Salem set of dimension 1/2. The proofs are based on the fact that the inverse local time process of Brownian motion has independent and stationary increments. Using Levy's original definition of local times, we showed that the restriction μ of $\delta_0(X)$ to the deterministic interval $[0, 1]$ is such that its Fourier transform satisfies $E\left(|\hat{\mu}(u)|^2\right) \leq C|u|^{-1/2}$, $u \neq 0$ and $C > 0$. This result can be improved by estimating the exact decay of this measure at infinity. The procedure may possibly be generalized to study the Fourier structure of level sets of other stochastic processes.

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