

**ON SOME RESULTS OF ANALYSIS IN
METRIC SPACES AND FUZZY METRIC
SPACES**

by

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Contents

Table of Contents	ii
Acknowledgements	iii
Summary	iv
Symbols	1
1 Introduction.	2
1.1 Convergence and completeness in metric spaces.	2
1.2 Continuity and uniform continuity in metric spaces.	10
2 Fundamental properties of fuzzy metric spaces.	16
2.1 Basic notions on fuzzy metric spaces.	17
2.2 Topology and fuzzy metric spaces.	28
3 Further properties on fuzzy metric spaces.	39
3.1 Complete fuzzy metric spaces.	39
3.2 Separability and uniform convergence in fuzzy metric spaces.	45
4 Some properties on fuzzy pseudo metric spaces.	49
4.1 Fuzzy pseudo metric spaces and some properties.	49
4.2 Fuzzy pseudo metric spaces and uniformities.	56

4.3	Fuzzy metric identification.	59
4.4	Uniformly continuous maps and extension of t -nonexpansive maps. .	65

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Summary

The notion of a fuzzy metric space due to George and Veeramani has many advantages in analysis since many notions and results from classical metric space theory can be extended and generalized to the setting of fuzzy metric spaces, for instance: the notion of completeness, completion of spaces as well as extension of maps. The layout of the dissertation is as follows:

Chapter 1 provide the necessary background in the context of metric spaces, while chapter 2 presents some concepts and results from classical metric spaces in the setting of fuzzy metric spaces. In chapter 3 we continue with the study of fuzzy metric spaces, among others we show that: the product of two complete fuzzy metric spaces is also a complete fuzzy metric space.

Our main contribution is in chapter 4. We introduce the concept of a standard fuzzy pseudo metric space and present some results on fuzzy metric identification. Furthermore, we discuss some properties of t -nonexpansive maps.

Keywords: Metric space, Cauchy sequence, Compactness, Precompactness, Completeness, Continuity, Uniform Continuity, Isometry, Uniform Convergence, Separable, Nested, Closed sets, Diameter, Pseudo metric space, Fuzzy metric space, Standard fuzzy metric space, Fuzzy pseudo metric space, Metric identification, Fuzzy metric identification, Nonexpansive map, t -nonexpansive map, t -uniformly continuous map, t -isometry map, Quotient map, Quotient topology, Quotient space, Natural map, Topological Space.

Symbols

\forall For all

\in Element of

\exists There exists

\subset Proper subset of

\subseteq Subset of

\mathbb{R} Set of real numbers

\mathbb{Q} Set of rational numbers

\mathbb{N} Set of positive integers $\{1, 2, 3, \dots\}$

$\mathbb{N}_0 = \{0, 1, 2, \dots\}$

\mathbb{R}^n n-Dimensional Euclidean space

$I = \{x \in \mathbb{R} : -1 \leq x \leq 1\}$

\cup Union

\cap Intersection

\Leftrightarrow If and only if

Sup Supremum

Inf Infimum

\emptyset Empty set

f^{-1} Inverse of function f

\therefore Therefore

A^c Complement of the set A

\bar{A} Closure of the set A

$[a, b]$ Closed interval

(a, b) Open interval

$B(x, r, t)$ Open ball with center x and radius $r, 0 < r < 1, t > 0$

$B[x, r, t]$ Closed ball with center x and radius $r, 0 < r < 1, t > 0$

$(X, M, *)$ Fuzzy metric space

Chapter 1

Introduction.

The aim of this chapter is to provide the background information and results that will be useful throughout the dissertation. We provide definitions, propositions, remarks and theorems (mostly without proof) and examples in the context of metric spaces. Some of the results in this chapter will be extended and generalized in the subsequent chapters. Most of the work presented in this chapter is well known and can be found in the literature, see [10],[22],[31],[32],[42] and [49].

1.1 Convergence and completeness in metric spaces.

Definition 1.1.1 A *metric space* (X, d) is a set X together with a function $d : X \times X \rightarrow \mathbb{R}$ such that for all x, y and z in X the following conditions hold:

$$M1. \quad d(x, y) \geq 0$$

$$M2. \quad d(x, y) = 0 \quad \text{if and only if} \quad x = y$$

$$M3. \quad d(x, y) = d(y, x) \quad \text{Symmetric Property}$$

$$M4. \quad d(x, z) \leq d(x, y) + d(y, z) \quad \text{Triangle inequality.}$$

If all these conditions hold but for $M2$ we only have $d(x, x) = 0$, then d is a pseudo metric. We then call (X, d) a **pseudo metric space**.

The following example shows the space (X, d) that satisfies the condition of a metric space.

Example 1.1.1 Let $X = \mathbb{R}^n$, define a function $d_\infty : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$d_\infty(x, y) = \max_{1 \leq i \leq n} |x_i - y_i|.$$

The function d_∞ is usually called the **max metric** on \mathbb{R}^n . Thus (X, d_∞) is a metric space.

The next example provides a space (X, d) which is not a metric space.

Example 1.1.2 Let $X = \mathbb{R}$. Define the function $d : X \times X \rightarrow \mathbb{R}$ by $(x - y)^2$. Then (X, d) is not a metric space.

Definition 1.1.2 Let (X, d) be a metric space and r a real number with $r > 0$. The **open ball** in (X, d) of radius r centered at $x \in X$ is defined by

$$B(x, r) = \{y \in X : d(x, y) < r\}.$$

Proposition 1.1.1 Let $B(x, r_1)$ and $B(x, r_2)$ be open balls with the same center $x \in X$, where $r_1, r_2 > 0$. Then,

$$B(x, r_1) \subseteq B(x, r_2)$$

or

$$B(x, r_2) \subseteq B(x, r_1).$$

Definition 1.1.3 Let (X, d) be a metric space, $\{x_n\}$ be a sequence in X and $x \in X$. The **sequence** $\{x_n\}$ **converges** to a point x in X if for each $\epsilon > 0$ there is a positive integer N such that $d(x_n, x) < \epsilon$ whenever $n \geq N$.

Definition 1.1.4 Let (X, d) be a metric space and A a subset of X . A point $x \in X$ is called a **limit point** of A if each open ball with the center x contains at least one point of A different from x , that is, $\{B(x, r) - \{x\}\} \cap A \neq \emptyset$. The set of all limit points of A is denoted by A' and is called the **derived set** of A .

Remark 1.1.1 If a sequence $\{x_n\}$ in a metric space (X, d) has a limit x we say that the sequence $\{x_n\}$ is convergent and we shall write

$$x_n \rightarrow x$$

or

$$\lim_n d(x_n, x) = 0.$$

If a sequence $\{x_n\}$ in a metric space (X, d) does not converge, it is said to **diverge**.

We provide an example of a convergent sequence in a metric space.

Example 1.1.3 Let $X = \mathbb{R}$ and define a function $d : X \times X \rightarrow [0, \infty)$ by

$$d(x, y) = |x - y|,$$

for all $x, y \in \mathbb{R}$. The function d so defined is called the **usual metric** on \mathbb{R} . Let the sequence $\{x_n\}$ in \mathbb{R} be defined by

$$\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Then

$$\lim_n d(x_n, x) = 0,$$

where $x = 0$.

Definition 1.1.5 Let (X, d) be a metric space. A **neighborhood** of the point $x_0 \in X$ is any open ball in (X, d) with the center x_0 .

Definition 1.1.6 Let (X, d) be a metric space. If $A \subset X$ and $x \in X$, then x is a **cluster point** of A if every neighborhood of x contains a point of A different from x .

Example 1.1.4 Let $X = \mathbb{R}$ be equipped with the usual metric d and define the sequence $\{x_n\}$ in X by

$$\left\{ \frac{1}{n} : n \in \mathbb{N} \right\}.$$

Then 0 is a cluster point of $\{x_n\}$.

Remark 1.1.2 If x is a limit of the sequence $\{x_n\}, n \in \mathbb{N}$ in a metric space (X, d) then x is a cluster point. But the converse is not true.

Example 1.1.5 Let the sequence $\{x_n\}$ in \mathbb{R} with the usual metric d be defined by

$$\{(-1)^n : n \in \mathbb{N}\}.$$

Then -1 and 1 are the cluster points of $\{x_n\}$. But the sequence $\{x_n\}$ does not converge. Note that the sequence $\{(-1)^n\}$ does not have a limit point in (X, d) .

Definition 1.1.7 A sequence $\{x_n\}$ of points in a metric space (X, d) is called a **Cauchy sequence** if for each $\epsilon > 0$ there exists a positive integer N such that $d(x_n, x_m) < \epsilon$ whenever $n, m \geq N$.

Remark 1.1.3 As a matter of notation, if the sequence $\{x_n\}$ in a metric space (X, d) is Cauchy, then we shall write

$$\lim_{n,m} d(x_n, x_m) = 0.$$

Definition 1.1.8 A metric space (X, d) is **complete** if every Cauchy sequence in (X, d) converges.

We provide an example of metric space (X, d) which is complete.

Example 1.1.6 The metric space of Example 1.1.3 is complete.

Proposition 1.1.2 A convergent sequence $\{x_n\}$ in a metric space is a Cauchy sequence.

An example of Cauchy sequence in a metric space (X, d) is given below:

Example 1.1.7 Let $X = \mathbb{R}$ be equipped with the usual metric d and define the sequence $\{x_n\}$ in X by

$$\{1 - \frac{1}{2^n} : n \in \mathbb{N}\}.$$

If $m \geq n$ then

$$d(x_n, x_m) = \frac{1}{2^n} - \frac{1}{2^m}.$$

It follows that

$$\lim_{n,m} d(x_n, x_m) = 0,$$

hence the sequence $\{1 - \frac{1}{2^n} : n \in \mathbb{N}\}$ is a Cauchy sequence.

We provide an example of a sequence in a metric space (X, d) which is not a Cauchy sequence.

Example 1.1.8 Let $X = \mathbb{R}$ be equipped with the usual metric d . Define the sequence $\{x_n\}$ by

$$x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n}$$

for $n \in \mathbb{N}$. If $m > n$, then

$$d(x_m, x_n) = \frac{1}{n+1} + \dots + \frac{1}{m}.$$

It can be shown that the sequence is not Cauchy. Since (X, d) is a complete metric space we conclude that $\{x_n\}$ is not a convergent sequence.

Observe that not every Cauchy sequence in a metric space converges. This is illustrated by an example below:

Example 1.1.9 Let $X = (0, 1)$ be equipped with the usual metric d . Then the sequence $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$ in X is Cauchy but does not converge to a point in X . This shows that a metric space (X, d) is not complete.

Definition 1.1.9 Let (X_1, d_1) and (X_2, d_2) be metric spaces and let $x = (x_1, x_2)$ and $y = (y_1, y_2)$ be arbitrary points in the product $X = X_1 \times X_2$. Define

$$d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}.$$

Then $d(x, y) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}$ is a metric on X and (X, d) called the **product of the metric spaces** (X_1, d_1) and (X_2, d_2) .

Proposition 1.1.3 *Let $(X_n, d_n), n = 1, 2, \dots$ be metric spaces. Then*

$$X = \prod_{n=1}^{\infty} X_n$$

with the metric d defined by

$$d(x, y) = \sum_{n=1}^{\infty} 2^{-n} d_n(x_n, y_n),$$

where $x = \{x_n\}$ and $y = \{y_n\}$ are in X , is a complete metric space if and only if each $(X_n, d_n), n = 1, 2, \dots$ is complete.

Definition 1.1.10 *A subset A of a metric space (X, d) is said to be **open** if given any point $x \in A$, there exists $r > 0$ such that $B(x, r) \subseteq A$.*

Definition 1.1.11 *If A is a subset of X , we define the **complement** of A (relative to X) denoted by A^c as the set of elements that are in X but not in A . Thus*

$$X - A = \{x \in X : x \notin A\}.$$

Definition 1.1.12 *Let A be a subset of a metric space (X, d) . Then A is said to be **closed** if it contains each of its limits points, that is $A' \subseteq A$.*

Closed subsets can be characterized in terms of open subsets as follows: A subset A of a metric space (X, d) is **closed** if and only if its complement A^c is open.

The next example discusses the properties of closed subsets in \mathbb{R} with the usual metric.

Example 1.1.10 *The subset $[a, b]$ of \mathbb{R} equipped with the usual metric is closed since its complement*

$$\mathbb{R} - [a, b] = (-\infty, a) \cup (b, \infty),$$

the union of two open infinite intervals is open. Similarly $[a, \infty)$ is closed, because its complement $(-\infty, a)$ is open.

Proposition 1.1.4 *In any metric space (X, d) each open ball is an open set.*

Proposition 1.1.5 *In any metric space (X, d) each closed ball is a closed set.*

Definition 1.1.13 *Let A be a subset of a metric space (X, d) . The set $A \cup A'$ is called the **closure** of A and is denoted by \bar{A} .*

Definition 1.1.14 *Let (X, d) be a metric space and let A be a nonempty subset of X . We say that A is **bounded** if there exists $N > 0$ such that*

$$d(x, y) \leq N, \quad \text{for all } x, y \in A.$$

*If A is bounded, we define the **diameter** of A as*

$$\text{dia}(A) = \sup\{d(x, y) : x, y \in A\}.$$

*If A is **unbounded**, we write*

$$\text{dia}(A) = \infty.$$

If $A = \emptyset$ we write $\text{dia}(A) = 0$.

Definition 1.1.15 *A subset A of a metric space (X, d) is said to be*

- (i). **Rare** (or **nowhere dense**) in X if its closure \bar{A} has no interior points.*
- (ii). **Meager** (or of **first category**) in X if A is the union of countable many sets each of which is rare in X .*
- (iii). **Non-meager** (or of **second category**) in X if A is not a meager in X .*

The next result characterizes completeness in metric spaces.

Theorem 1.1.1 Baire Category Theorem *(Complete metric space). Let (X, d) be a complete metric space. Then no nonempty open subset of X is of first category, that is, the union of a countable collection of nowhere dense subsets.*

Definition 1.1.16 *Let (X, d) and (Y, ρ) be two metric spaces. A mapping*

$$f : (X, d) \rightarrow (Y, \rho)$$

is an **isometry** if

$$\rho(f(x), f(y)) = d(x, y)$$

for all $x, y \in X$. The metric space (X, d) is said to be **isometric** to the metric space (Y, ρ) when there exists some isometry from (X, d) into (Y, ρ) .

Definition 1.1.17 Let (X, d) be a metric space. A metric space (\tilde{X}, \tilde{d}) is said to be a **completion** of the metric space (X, d) if (\tilde{X}, \tilde{d}) is complete and (X, d) is isometric to a dense subset of (\tilde{X}, \tilde{d}) .

Example 1.1.11 The set \mathbb{R} of real numbers with the usual metric is the completion of the set \mathbb{Q} of rational numbers, since \mathbb{R} is complete and \mathbb{Q} is a dense subset of \mathbb{R} .

Theorem 1.1.2 Every metric space (X, d) has a completion and any two completions of (X, d) are isometric to each other.

Remark 1.1.4 In other words, up to isometry, there exists a unique completion of any metric space. In what follows we shall call such a completion, the completion.

Definition 1.1.18 A **topological space** is a pair (X, τ) consisting of a set X and a collection τ of subsets of X called **open sets**, satisfying the following conditions:

- (i). The union of a family of open sets is open.
- (ii). The intersection of a finite family of closed sets is closed.
- (iii). X and \emptyset are open sets.

Definition 1.1.19 Let (X, d) be a metric space and A_1, A_2, \dots be a sequence of sets. Then A_1, A_2, \dots is said to be **nested** if

$$A_1 \supset A_2 \supset A_3 \supset \dots$$

Theorem 1.1.3 Every nested sequence of nonempty closed sets with metric diameter zero has nonempty intersection.

1.2 Continuity and uniform continuity in metric spaces.

Definition 1.2.1 Let (X, d) be a metric space and $A \subseteq X$. Let \hat{G} be a collection of open sets in X with the property that $A \subseteq \cup\{G : G \in \hat{G}\}$ equivalently, for each $x \in A$, there is a $G \in \hat{G}$ such that $x \in G$. Then \hat{G} is called an **open cover** or an **open covering** of A . A finite sub-collection of \hat{G} which is itself a cover is called a **finite sub-cover** or a **finite sub-covering** of A .

Definition 1.2.2 A metric space (X, d) is said to be **compact** if every open covering \hat{G} of X has a finite sub-covering, that is, there is a finite sub-collection $\{G_1, G_2, G_3, \dots, G_n\} \subseteq \hat{G}$ such that

$$X = \cup_{i=1}^n G_i.$$

Remark 1.2.1 In a compact metric space (X, d) any sequence of closed sets will have a nonempty intersection provided each finite collection of these sets has nonempty intersection.

Proposition 1.2.1 Every compact subset A of a metric space X is bounded.

One of the most important properties of a closed and bounded interval in \mathbb{R} when equipped with the usual metric is given in the next theorem.

Theorem 1.2.1 Heine Borel Theorem. Let $A = [a, b]$ be a closed and bounded interval, and let $\hat{G} = \{G_i : i \in I\}$ be a set of open intervals which covers A , that is, $A \subseteq \cup_i G_i$. Then \hat{G} contains a finite subset, say $\{G_{i_1}, \dots, G_{i_m}\}$ which also covers A , that is,

$$A \subseteq G_{i_1} \cup G_{i_2} \cup \dots \cup G_{i_m}.$$

We provide an example which does not satisfy the properties of Heine Borel Theorem.

Example 1.2.1 Let X be the set of real numbers with the usual metric. Consider the closed infinite interval $A = [0, \infty)$. A is closed but not compact. It is covered by the set of open sets

$$\hat{G} = \{G_n : n \in \mathbb{N}\}$$

where

$$G_n = [0, n),$$

but no finite open subset of sets covers $[0, \infty)$.

Every closed and bounded interval on the real line \mathbb{R} with the usual metric is compact; this follows from the Heine Borel theorem. In particular, we provide the following example:

Example 1.2.2 Let X be the set of real numbers with the usual metric. Then the closed interval $[0, 1]$ is compact by the Heine Borel Theorem, since $[0, 1]$ is bounded.

Next, we provide an example of a metric space which is not compact.

Example 1.2.3 Let $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ and define $d : \mathbb{N}_0 \times \mathbb{N}_0 \rightarrow [0, \infty)$ by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$$

The function d is called **discrete metric**. Therefore the space (\mathbb{N}_0, d) is not a compact metric space, since the sets

$$G = \{G_n\}_{n=1}^{\infty}, G_n = \{x \in \mathbb{N}_0 : 0 \leq x \leq n\},$$

cover \mathbb{N}_0 but no finite collection of such sets can cover \mathbb{N}_0 . However (\mathbb{N}_0, d) is a bounded metric space.

Proposition 1.2.2 Let A be closed subset of a compact metric space (X, d) . Then A is also compact.

Definition 1.2.3 X is **countable infinite**(denumerable) iff $X \simeq \mathbb{N}$,

X is **finite** iff there exists $n \in \mathbb{N}$, $X \simeq n$,

X is **countable** iff X is finite or denumerable.

Example 1.2.4 The set \mathbb{Z} of integers is countably infinite.

Definition 1.2.4 A topological space (X, τ) is called **first countable space** iff it has a countable neighborhood base at each point.

Definition 1.2.5 Let X be a topological space with topology τ . A collection B of subsets of X is called a **base** of τ if:

(i). Each member of B is open in X ,

(ii). Each open subset of X is the union of some collection of sets belonging to B .

Definition 1.2.6 A topological space is said to be **second countable** or is said to satisfy the second axiom of countability if the topology on the space can be generated by countable base.

Example 1.2.5 Let X be the set of real numbers with the usual metric. The collection $\{(x, y) : x, y \in \mathbb{Q}\}$ of all open intervals with rational endpoints form countable base for the open sets of \mathbb{R} .

Theorem 1.2.2 Every metric space satisfies the first condition of countability.

Definition 1.2.7 Let (X, d) be a metric space. If there is a countable dense subset in (X, d) then, (X, d) is said to be **separable**.

We now provide an example of a metric space which is not separable.

Example 1.2.6 Let X denote the infinite set and d be discrete metric. Then the metric space (X, d) is not separable.

Example 1.2.7 *The metric space of Example 1.1.9 is separable.*

Theorem 1.2.3 *Every separable metric space satisfies the second condition of countability.*

Proposition 1.2.3 *Let (X, d) be a metric space and $A \subseteq X$. If X is separable then A with the induced metric is separable, too.*

Definition 1.2.8 *A space X is a T_2 -space (**Hausdorff space**) iff whenever x and y are distinct points of X , there are disjoint open sets U and V in X with $x \in U$ and $y \in V$.*

Proposition 1.2.4 *Every metric space is a Hausdorff space.*

Remark 1.2.2 *Let (X, d) be a metric space. Then the collection $\tau_d = \{A \subset X : x \in A \text{ if and only if there exists } r > 0 \text{ such that } B(x, r) \subset A\}$ is a topology called the **metric topology** induced by d .*

Definition 1.2.9 *Let (X, d) and (Y, ρ) be metric spaces. The function*

$$f : (X, d) \rightarrow (Y, \rho)$$

*is said to be **continuous** at the point $x_0 \in X$ if for each $\epsilon > 0$ there exists a $\delta > 0$ such that*

$$\rho(f(x), f(x_0)) < \epsilon$$

whenever

$$d(x, x_0) < \delta.$$

We shall say that $f : (X, d) \rightarrow (Y, \rho)$ is continuous if it is continuous at every $x \in X$.

The next example provides an example of a continuous function between metric spaces.

Example 1.2.8 Let $X = \mathbb{R}^2$ be equipped with the metric d defined by

$$\begin{aligned}d(x, y) &= d((x_1, x_2), (y_1, y_2)) \\ &= \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}\end{aligned}$$

for all $x = (x_1, x_2), y = (y_1, y_2) \in X$ and $Y = \mathbb{R}$ be equipped with the usual metric ρ . The function $f : (X, d) \rightarrow (Y, \rho)$ defined by

$$f(x, y) = x + y$$

for each $x, y \in \mathbb{R}^2$ is continuous.

In fact, we provide the following characterization of continuity in metric spaces:

Theorem 1.2.4 Let (X, d) and (Y, ρ) be metric spaces. Then the following statements are equivalent:

- (i). $f : (X, d) \rightarrow (Y, \rho)$ is continuous
- (ii). For a sequence $\{x_n\}$ and a point x in (X, d) ,

$$\lim_n \rho(f(x_n), f(x)) = 0$$

whenever

$$\lim_n d(x_n, x) = 0.$$

An example of a discontinuous function is given below:

Example 1.2.9 Let $f : (\mathbb{R}, d) \rightarrow (\mathbb{R}, \rho)$ where $f(x) = x$ for each $x \in \mathbb{R}$ given, let d be the Euclidean metric for \mathbb{R} and let ρ be the discrete metric for the set \mathbb{R} . Then f is not a continuous function. To see this, take a sequence $\{\frac{1}{n} : n \in \mathbb{N}\}$. Then

$$\lim_n d(x_n, 0) = 0$$

but

$$\lim_n \rho(f(x_n), f(0)) = 1.$$

Definition 1.2.10 Let $\{f_n\}$ be a sequence of functions from (X, d) into the metric space (Y, ρ) . Then $\{f_n\}$ is said to **converge uniformly** to a function

$$f : (X, d) \rightarrow (Y, \rho)$$

if for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, and for all $x \in X$, $\rho(f_n(x), f(x)) < \epsilon$.

Definition 1.2.11 Let (X, d) and (Y, ρ) be two metric spaces. The function $f : (X, d) \rightarrow (Y, \rho)$ is **uniformly continuous** on X if and only if for every $\epsilon > 0$ there exists a $\delta > 0$ such that if $x_1 \in X, x_2 \in X$ and $d(x_1, x_2) < \delta$, then, $\rho(f(x_1), f(x_2)) < \epsilon$.

We provide an example of a uniformly continuous function.

Example 1.2.10 Let $X = [0, 1]$ be equipped with the usual metric d and $Y = \mathbb{R}$ be equipped with the usual metric ρ . Consider the function $f : [0, 1] \rightarrow \mathbb{R}$ given by $f(x) = 2x + 1$ for $x \in [0, 1]$. Clearly f is uniformly continuous on $[0, 1]$.

Next we provide an example of a function which is not uniformly continuous.

Example 1.2.11 Let $X = \mathbb{R}, Y = \mathbb{R}$ be equipped with the usual metrics and $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = x^2$. Then f is not uniformly continuous.

Theorem 1.2.5 Uniform limit theorem. Let $f_n : (X, \tau) \rightarrow (Y, \rho)$ be a sequence of continuous function from a topological space (X, τ) to a metric space (Y, ρ) . If the sequence $\{f_n\}$ converges uniformly to $f : (X, \tau) \rightarrow (Y, \rho)$ then f is continuous.

Chapter 2

Fundamental properties of fuzzy metric spaces.

The concept of fuzzy sets and fuzzy logic was introduced by Professor Lofti A Zadeh in 1965. The success of research in fuzzy sets and fuzzy logic has been demonstrated in a variety of fields, such as artificial intelligence, computer science, control engineering, computer applications, robotics and many more. In the dissertation we adopt the notion of fuzzy metric space due to George and Veeramani [14] which is a modification of the notion of fuzzy metric space as studied by Kramosil and Michalek [29]. The notion of fuzzy metric space by George and Veeramani has many advantages in analysis as many notions and results from classical metric spaces can be extended and generalized to the setting of fuzzy metric spaces, for instance: the notion of completeness, completion of spaces as well as extension of maps.

This chapter is based on the work due by A George and P Veeramani [14]. We shall recall the definition of a fuzzy metric space which was modified from [29] to obtain the Hausdorff topology on a fuzzy metric space. We note that just like in the classical metric space case (see chapter 1), every fuzzy metric space induces a topological space. In this chapter we expand on the paper [14] by means of providing detailed examples, propositions, remarks and proofs of some results.

2.1 Basic notions on fuzzy metric spaces.

We start this section with the following well known definition:

Definition 2.1.1 A binary operation $*$: $[0, 1] \times [0, 1] \rightarrow [0, 1]$ is a **continuous triangular norm** if for all $a, b, c, e \in [0, 1]$ the following conditions hold:

$$2.1.1.1 \quad a * b = b * a \quad (\text{commutativity})$$

$$2.1.1.2 \quad a * 1 = a$$

$$2.1.1.3 \quad (a * b) * c = a * (b * c) \quad (\text{associativity})$$

$$2.1.1.4 \quad a * b \leq c * e, \quad \text{whenever} \quad a \leq c \quad \text{and} \quad b \leq e.$$

In the sequel we shall refer to triangular norm as a t -norm.

We provide examples of continuous t -norms.

Example 2.1.1 Define $a * b = ab$, for all $a, b \in [0, 1]$. Note that ab is the usual multiplication in $[0, 1]$ for all $a, b \in [0, 1]$. It follows that $*$ is a continuous t -norm.

Example 2.1.2 Define $a * b = \min(a, b)$, for all $a, b \in [0, 1]$. Then $*$ is a continuous t -norm.

More examples on continuous t -norms can be found in [47].

Remark 2.1.1 Given an arbitrary set X , a **fuzzy set** M on X is a function from X to the unit interval $[0, 1]$. Let $[0, 1]^X = \{f : X \rightarrow [0, 1]\}$. Thus $M \in [0, 1]^X$.

Definition 2.1.2 The 3-tuple $(X, M, *)$ is said to be a **fuzzy metric space**, where X is an arbitrary set, $*$ is continuous t -norm and M is a fuzzy set on $X \times X \times [0, \infty)$ satisfying the following conditions:

$$2.1.2.1 \quad \forall x, y \in X, \quad M(x, y, 0) = 0$$

$$2.1.2.2 \quad \forall x, y \in X, \quad \text{and} \quad \forall t > 0, \quad M(x, y, t) = 1 \quad \text{if and only if} \quad x = y$$

$$2.1.2.3 \quad \forall x, y \in X, \quad \text{and} \quad \forall t > 0, \quad M(x, y, t) = M(y, x, t)$$

$$2.1.2.4 \quad \forall x, y, z \in X, \quad \text{and} \quad \forall s, t > 0, \quad M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$$

$$2.1.2.5 \quad \forall x, y \in X, \quad M(x, y, \bullet) : [0, \infty) \rightarrow [0, 1] \quad \text{is continuous.}$$

Remark 2.1.2 2.1.2.5 means that for each $x, y \in X$ there is a function

$$M_{xy} : [0, \infty) \rightarrow I, t \rightarrow M(x, y, t).$$

Remark 2.1.3 $M(x, y, t)$ can be thought of as the degree of nearness between x and y with respect to $t \geq 0$. If we use the notation

$d(x, y)$ = the distance between x and y , and

$P(S = \alpha)$ if and only if the probability that $S = \alpha$, or

$P(S \geq \alpha)$ probability that $S \geq \alpha$, then

$M(x, y, t) = \alpha$ if and only if $P(d(x, y) \leq t) = \alpha$.

Then, in the case where $*$ = \wedge , 2.1.2.4 reads:

If the probability that $d(x, y) \leq t$ is greater than α and the probability that $d(y, z) \leq s$ is greater than α then the probability that $d(x, z) \leq s + t$ is also greater than α . This is because

$$a \wedge b \geq \alpha \Rightarrow a \geq \alpha \quad \text{and} \quad b \geq \alpha.$$

In other words,

$P(d(x, y) \leq t) \geq \alpha$ and $P(d(y, z) \leq s) \geq \alpha$ implies that $P(d(x, z) \leq s + t) \geq \alpha$.

Or

$$M(x, y, t) \geq \alpha \wedge M(y, z, s) \geq \alpha \Rightarrow M(x, z, t + s) \geq \alpha.$$

We identify $x = y$ if and only if

$$(\forall t > 0, M(x, y, t) = 1),$$

and

$$\lim_{t \rightarrow \infty} M(x, y, t) = 0.$$

The following definition is a modification of Definition 2.1.2. This modification is necessary since the topology induced by the fuzzy metric in Definition 2.1.2 is not Hausdorff.

Definition 2.1.3 *The 3-tuple $(X, M, *)$ is said to be a **fuzzy metric space** where X is an arbitrary set, $*$ is a continuous t -norm and M is a fuzzy set on $X \times X \times (0, \infty)$ satisfying the following conditions:*

$$2.1.3.1 \quad \forall x, y \in X, \quad \text{and} \quad \forall t > 0, \quad M(x, y, t) > 0$$

$$2.1.3.2 \quad \forall x, y \in X, \quad \text{and} \quad \forall t > 0, \quad M(x, y, t) = 1 \quad \text{if and only if} \quad x = y$$

$$2.1.3.3 \quad \forall x, y \in X, \quad \text{and} \quad \forall t > 0, \quad M(x, y, t) = M(y, x, t)$$

$$2.1.3.4 \quad \forall x, y, z \in X, \quad \text{and} \quad \forall s, t > 0, \quad M(x, z, t + s) \geq M(x, y, t) * M(y, z, s)$$

$$2.1.3.5 \quad \forall x, y \in X, \quad M(x, y, \bullet) : (0, \infty) \rightarrow [0, 1] \quad \text{is continuous.}$$

In the sequel the fuzzy set M as in Definition 2.1.3 will be referred to as a **fuzzy metric**. It shall be shown that the topology induced by the fuzzy metric space $(X, M, *)$ is Hausdorff.

Lemma 2.1.1 *$M(x, y, \bullet)$ is nondecreasing for all x, y in X .*

Proof: Suppose that $M(x, y, t) > M(x, y, s)$ for some $0 < t < s$. Then

$$\begin{aligned} M(x, y, t) * M(y, y, s - t) &\leq M(x, y, s) \\ &< M(x, y, t). \end{aligned}$$

By (2.1.3.2) in Definition 2.1.3 we have $M(y, y, s - t) = 1$. Thus

$$\begin{aligned} M(x, y, t) &< M(x, y, s) \\ &< M(x, y, t) \end{aligned}$$

a contradiction.

We recall the following useful remarks.

Remark 2.1.4 (i). Let $(X, M, *)$ be a fuzzy metric space and let $x, y \in X, t > 0, 0 < r < 1$. Then if $M(x, y, t) > 1 - r$ we can find t_0 with $0 < t_0 < t$, such that

$$M(x, y, t_0) > 1 - r.$$

(ii). For any $r_1 > r_2$, we can find an r_3 such that $r_1 * r_3 \geq r_2$ and for any r_4 we can find an r_5 such that $r_5 * r_5 \geq r_4, (r_1, r_2, r_3, r_4, r_5 \in (0, 1))$.

Next we provide an example of a fuzzy metric space.

Example 2.1.3 Let $X = \mathbb{R}$. Define $a * b = ab$ for all $a, b \in [0, 1]$ and

$$M(x, y, t) = \left[\exp\left(\frac{|x - y|}{t}\right) \right]^{-1}$$

for all $x, y \in X$ and $t \in (0, \infty)$. Then $(X, M, *)$ is a fuzzy metric space. We shall show that M is a fuzzy metric.

Proof: 1. $\forall t > 0$. Assume that $x = y$. Then this implies that $|x - y| = 0$. Hence

$$\left[\exp\left(\frac{|x - y|}{t}\right) \right]^{-1} = 1.$$

Therefore

$$M(x, y, t) = 1.$$

Conversely,

assume that $M(x, y, t) = 1$. Therefore

$$\left[\exp\left(\frac{|x - y|}{t}\right) \right]^{-1} = 1$$

implies that

$$e^{\frac{|x - y|}{t}} = e^0.$$

Hence

$$\frac{|x - y|}{t} = 0,$$

it follows that $|x - y| = 0$. Thus $x = y$. Therefore $M(x, y, t) = 1$ if and only if $x = y$.

2. To prove $M(x, y, t) = M(y, x, t)$ we know that

$$|x - y| = |y - x|$$

for all $x, y \in \mathbb{R}$. It follows that for all $x, y \in X$ and for all $t > 0$,

$$M(x, y, t) = M(y, x, t).$$

3. To prove $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$, we know that for all $x, y, z \in X$ and for all $t, s > 0$,

$$|x - z| \leq \left(\frac{t + s}{t}\right) |x - y| + \left(\frac{t + s}{s}\right) |y - z|.$$

That is

$$\frac{|x - z|}{t + s} \leq \frac{|x - y|}{t} + \frac{|y - z|}{s}.$$

Thus

$$e^{\frac{|x-z|}{t+s}} \leq e^{\frac{|x-y|}{t}} e^{\frac{|y-z|}{s}},$$

since e^x is an increasing function for $x > 0$. Therefore

$$\left[\exp\left(\frac{|x - z|}{t + s}\right) \right]^{-1} \geq \left[\exp\left(\frac{|x - y|}{t}\right) \right]^{-1} * \left[\exp\left(\frac{|y - z|}{s}\right) \right]^{-1}.$$

Thus

$$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s).$$

4. Take a sequence $\{t_n\} \in (0, \infty)$ such that the sequence $\{t_n\}$ converges to $t \in (0, \infty)$ where $(0, \infty)$ is equipped with the usual metric. That is,

$$\lim_n |t_n - t| = 0.$$

Without the loss of generality, fix $x, y \in X$. Since the function e^x is continuous on \mathbb{R} we have $e^{\frac{|x-y|}{t_n}}$ converges to $e^{\frac{|x-y|}{t}}$ as t_n converges to t , with respect to the usual metric. Therefore

$$M(x, y, \bullet) : (0, \infty) \rightarrow [0, 1]$$

is continuous. Hence $(X, M, *)$ is a fuzzy metric space.

Remark 2.1.5 Note that in Example 2.1.3 we can replace \mathbb{R} by any non empty set X and the usual metric on \mathbb{R} by any metric d . Also, note that, Example 2.1.3 is a fuzzy metric space with the t -norm defined by $a * b = \min(a, b)$ for all $a, b \in [0, 1]$.

The next example shows that every metric space induces a fuzzy metric space.

Example 2.1.4 Let (X, d) be a metric space. Define $a * b = ab$ for all $a, b \in [0, 1]$ and

$$M(x, y, t) = \frac{kt^n}{kt^n + md(x, y)}, k, m, n \in \mathbb{N}.$$

Then $(X, M, *)$ is a fuzzy metric space.

Remark 2.1.6 Note that Example 2.1.4 holds even with the continuous t -norm $a * b = \min(a, b)$. In particular, taking

$$k = m = n = 1,$$

we get

$$M(x, y, t) = \frac{t}{t + d(x, y)}.$$

We shall call this fuzzy metric induced by a metric d the **standard fuzzy metric**.

In what follows M_d denotes a standard fuzzy metric induced by the metric d .

Example 2.1.5 Let $X = \mathbb{N}$. Define $a * b = ab$ and for all $t > 0$, let

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y \\ \frac{y}{x} & \text{if } y \leq x. \end{cases}$$

We shall show that M is a fuzzy metric.

Proof: 1. $\forall t > 0$. Assume that $x = y$. Then $\frac{x}{y} = \frac{y}{x} = 1$. Hence $M(x, y, t) = 1$.

Conversely,

assume that $M(x, y, t) = 1$. Then $\frac{x}{y} = 1$, and therefore $x = y$. Similarly if $\frac{y}{x} = 1$ then it follows that $y = x$. Thus $M(x, y, t) = 1$ if and only if $x = y$.

2. For all $x, y \in X$ and for all $t > 0$, clearly, $M(x, y, t) = M(y, x, t)$.

3. To prove that $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$. We consider the following cases:

(i). $x = y = z$.

Then

$$M(x, y, t) = 1$$

$$M(y, z, s) = 1$$

$$M(x, z, t + s) = 1.$$

Now

$$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s) = 1.$$

It follows that

$$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$$

holds.

(ii). $x \neq y = z$.

Without loss of generality, we may assume that $x < y$ and $y = z$. Then

$$M(x, y, t) = \frac{x}{y}.$$

Also, we have $M(y, z, t) = 1$ and $M(x, z, t + s) = \frac{x}{z}$. Now

$$\frac{x}{y} * 1 = \frac{x}{y}$$

and

$$\frac{x}{y} = \frac{x}{z}.$$

Therefore

$$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$$

holds.

(iii). $x = y \neq z$.

Without loss of generality, we may assume that $x = y$ and $y < z$. Then $M(x, y, t) = 1$.

Also, we have

$$M(y, z, t) = \frac{y}{z}$$

and

$$M(x, z, t + s) = \frac{x}{z}.$$

Now

$$1 * \frac{y}{z} = \frac{y}{z}$$

and

$$\frac{y}{z} = \frac{x}{z}.$$

Therefore

$$M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$$

holds.

(iv). $x \neq y \neq z$.

Without loss of generality, we may assume that $x < y < z$. Then

$$M(x, y, t) = \frac{x}{y}$$

$$M(y, z, s) = \frac{y}{z}$$

$$M(x, z, t + s) = \frac{x}{z}.$$

Now $z > y$ implies that $z^2 > y^2$.

So

$$\frac{1}{z^2} < \frac{1}{y^2}.$$

Thus

$$\frac{xy}{z^2} < \frac{xy}{y^2}.$$

Therefore

$$\frac{x}{z} * \frac{y}{z} < \frac{x}{y}.$$

Hence $M(x, z, t) * M(z, y, t) < M(x, y, t + s)$.

4. Note that $M(x, y, t)$ is independent of t (that is, $M(x, y, t)$ is a constant in terms of t). For any $s, t > 0$, we have $M(x, y, t) = M(x, y, s)$. Thus $M(x, y, \bullet)$ is continuous.

Therefore $(X, M, *)$ is a fuzzy metric space.

Remark 2.1.7 The fuzzy metric in the above example (Example 2.1.5) is independent of t . Such a fuzzy metric is referred to as a **stationary fuzzy metric**.

Remark 2.1.8 It is interesting to note that there exists no metric d on X satisfying $M(x, y, t) = \frac{t}{t+d(x,y)}$, where $M(x, y, t)$ is as defined in Example 2.1.5. We show that M is not a fuzzy metric on X with the t -norm defined by $a * b = \min(a, b)$.

We start by showing that there is no metric d on X satisfying $M(x, y, t) = \frac{t}{t+d(x,y)}$, where M is defined by

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y \\ \frac{y}{x} & \text{if } y \leq x. \end{cases}$$

Proof: Suppose that there is a metric d on X that induces $M(x, y, t)$. Then, $\forall t > 0$

$$\begin{aligned} M(x, y, t) &= \frac{t}{t + d(x, y)} \\ (t + d(x, y))M(x, y, t) &= t \\ d(x, y) &= \frac{t(1 - M(x, y, t))}{M(x, y, t)}. \end{aligned}$$

1. $\forall t > 0$. Assume that $x = y$. Then this implies that $M(x, y, t) = 1$. Therefore

$$\begin{aligned} d(x, y) &= \frac{t(1 - 1)}{1} \\ &= 0. \end{aligned}$$

Conversely,

assume that $d(x, y) = 0$, then

$$\begin{aligned} \frac{t(1 - M(x, y, t))}{M(x, y, t)} &= 0 \\ t - tM(x, y, t) &= 0 \\ -tM(x, y, t) &= -t \\ M(x, y, t) &= 1. \end{aligned}$$

This implies that $x = y$.

2. To prove that $d(x, y) = d(y, x)$. We note that $M(x, y, t) = M(y, x, t)$ since M is a fuzzy metric on X . Then

$$\begin{aligned} d(x, y) &= \frac{t(1 - M(x, y, t))}{M(x, y, t)} \\ &= \frac{t(1 - M(x, y, t))}{M(x, y, t)} \\ &= d(y, x). \end{aligned}$$

3. We now show that the triangle inequality does not hold: Let $t = 2, x = 1, y = 2$ and $z = 3$. Then

$$\begin{aligned} d(x, y) &= \frac{t(1 - M(x, y, t))}{M(x, y, t)} \\ &= \frac{2(1 - \frac{1}{2})}{\frac{1}{2}} \\ &= 2. \\ d(y, z) &= \frac{2(1 - \frac{2}{3})}{\frac{2}{3}} \\ &= 1. \\ d(x, z) &= \frac{2(1 - \frac{1}{3})}{\frac{1}{3}} \\ &= 4. \end{aligned}$$

Therefore

$$d(x, z) > d(x, y) + d(y, z).$$

Thus (X, d) is not a metric space.

We now prove that M is not a fuzzy metric with the continuous t -norm defined by $a * b = \min(a, b)$.

Proof: 1. $\forall t > 0$. Assume that $x = y$. Then $d(x, y) = 0$. Hence

$$\frac{t}{t + d(x, y)} = 1.$$

Therefore

$$M(x, y, t) = 1.$$

Conversely,

assume that $M(x, y, t) = 1$.

Therefore

$$\begin{aligned}\frac{t}{t + d(x, y)} &= 1 \\ t + d(x, y) &= t \\ d(x, y) &= 0.\end{aligned}$$

Thus $x = y$.

2. To show $M(x, y, t) = M(y, x, t)$ we know that $d(x, y) = d(y, x)$ that is,

$$\begin{aligned}M(x, y, t) &= \frac{t}{t + d(x, y)} \\ &= \frac{t}{t + d(y, x)} \\ &= M(y, x, t).\end{aligned}$$

Thus

$$M(x, y, t) = M(y, x, t).$$

3. To show that the inequality $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$ does not hold, choose $t = 2, x = 1, y = 2$ and $z = 3$. Then we obtain;

$$\begin{aligned}M(x, y, t) * M(y, z, s) &= \frac{2}{3} * \frac{2}{3} \\ &= \min\left(\frac{2}{3}, \frac{2}{3}\right) \\ &= \frac{2}{3}. \\ M(x, z, t + s) &= \frac{2}{4}.\end{aligned}$$

Therefore

$$M(x, z, t + s) < M(x, y, t) * M(y, z, s).$$

Thus M is not a fuzzy metric.

We conclude this section with the following remark.

Remark 2.1.9 *Note that from the discussion above, we conclude that the class of metric spaces is a proper subclass of the class of fuzzy metric spaces.*

2.2 Topology and fuzzy metric spaces.

We continue to present some concepts and results from classical metric spaces theory discussed in chapter 1 in the context of fuzzy metric spaces.

Definition 2.2.1 *Let $(X, M, *)$ be a fuzzy metric space. We define the **open ball** $B(x, r, t)$ with center $x \in X$ and radius $r, 0 < r < 1, t > 0$, as*

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

An extension of Proposition 1.1.1 to fuzzy metric setting is given below:

Proposition 2.2.1 *Let $B(x, r_1, t)$ and $B(x, r_2, t)$ be open balls with the same center $x \in X$ and $t > 0$ with radius $0 < r_1 < 1$ and $0 < r_2 < 1$, respectively. Then we either have*

$$B(x, r_1, t) \subseteq B(x, r_2, t)$$

or

$$B(x, r_2, t) \subseteq B(x, r_1, t).$$

Proof: Let $x \in X$ and $t > 0$. Consider the open balls $B(x, r_1, t)$ and $B(x, r_2, t)$, with

$$0 < r_1 < 1,$$

$$0 < r_2 < 1.$$

If $r_1 = r_2$, then the proposition holds. Next, we assume that $r_1 \neq r_2$. We may assume, without loss of generality, that $0 < r_1 < r_2 < 1$. Then $1 - r_2 < 1 - r_1$. Now, let $a \in B(x, r_1, t)$. It follows that

$$M(a, x, t) > 1 - r_1$$

$$> 1 - r_2.$$

Hence $a \in B(x, r_2, t)$. This shows that $B(x, r_1, t) \subseteq B(x, r_2, t)$. By assuming that $0 < r_2 < r_1 < 1$, we can similarly show $B(x, r_2, t) \subseteq B(x, r_1, t)$.

Definition 2.2.2 A subset A of a fuzzy metric space $(X, M, *)$ is said to be **open** if given any point $a \in A$, there exists $0 < r < 1$, and $t > 0$, such that $B(a, r, t) \subseteq A$.

The next theorem provides a generalization of Proposition 1.1.4 to the setting of fuzzy metric spaces.

Theorem 2.2.1 Every open ball in a fuzzy metric space $(X, M, *)$ is an open set.

Proof: Consider an open ball $B(x, r, t)$. Now $y \in B(x, r, t)$ implies that

$$M(x, y, t) > 1 - r.$$

Since $M(x, y, t) > 1 - r$, by Remark 2.1.4 we can find a $t_0, 0 < t_0 < t$, such that

$$M(x, y, t_0) > 1 - r.$$

Let $r_0 = M(x, y, t_0) > 1 - r$. Since $r_0 > 1 - r$, we can find an $s, 0 < s < 1$, such that

$$r_0 > 1 - s > 1 - r.$$

Now for a given r_0 and s such that $r_0 > 1 - s$ we can find $r_1, 0 < r_1 < 1$, such that

$$r_0 * r_1 \geq 1 - s.$$

Now consider the ball $B(y, 1 - r_1, t - t_0)$. We claim

$$B(y, 1 - r_1, t - t_0) \subset B(x, r, t).$$

Now $z \in B(y, 1 - r_1, t - t_0)$ implies that $M(y, z, t - t_0) > r_1$. Therefore

$$M(x, z, t) \geq M(x, y, t_0) * M(y, z, t - t_0)$$

$$\geq r_0 * r_1$$

$$\geq 1 - s$$

$$> 1 - r.$$

Therefore

$$z \in B(x, r, t)$$

and hence

$$B(y, 1 - r_1, t - t_0) \subset B(x, r, t).$$

Proposition 2.2.2 *Let $(X, M, *)$ be a fuzzy metric space. Define $\tau_M = \{A \subset X : x \in A \text{ if and only if there exists } t > 0, \text{ and } r, 0 < r < 1, \text{ such that } B(x, r, t) \subset A\}$. Then τ_M is a topology on X .*

Proof: (i). Clearly \emptyset and X belong to τ_M .

(ii). Let $A_1, A_2, A_3, \dots, A_i \in \tau_M$, and put

$$U = \cup_{i \in I} A_i.$$

We shall show that $U \in \tau_M$. If $a \in U$, then

$$a \in \cup_{i \in I} A_i$$

which implies that $a \in A_i$ for some $i \in I$. Since $A_i \in \tau_M$, there exists $0 < r < 1, t > 0$, such that $B(a, r, t) \subset A_i$. Hence

$$B(a, r, t) \subset A_i \subset \cup_{i \in I} A_i = U.$$

This shows that $U \in \tau_M$.

(iii). Let $A_1, A_2, A_3, \dots, A_n \in \tau_M$, and $U = \cap_{i=1}^n A_i$. We shall show that $U \in \tau_M$. Let $a \in U$. Then $a \in A_i$ for all $i \in I$. Hence for each $i \in I$, there exists $0 < r_i < 1, t_i > 0$ such that $B(a, r_i, t_i) \subset A_i$. Let

$$r = \min\{r_i, i \in I\}$$

and

$$t = \max\{t_i, i \in I\}.$$

Thus $r \leq r_i$ for all $i \in I$, $1 - r \geq 1 - r_i$ for all $i \in I$. Also, $t > 0$. So, $B(a, r, t) \subseteq A_i$ for all $i \in I$. Therefore

$$B(a, r, t) \subset \bigcap_{i \in I} A_i = U.$$

This shows that $U \in \tau_M$.

The next theorem generalizes Proposition 1.2.4.

Theorem 2.2.2 *Every fuzzy metric space is Hausdorff.*

Proof: Let $(X, M, *)$ be the given fuzzy metric space. Let x, y be two distinct points of X . Then $0 < M(x, y, t) < 1$. Let $M(x, y, t) = r$, for some $r, 0 < r < 1$. For each $r_0, r < r_0 < 1$, we can find an r_1 such that $r_1 * r_1 \geq r_0$. Now consider the open balls $B(x, 1 - r_1, \frac{t}{2})$ and $B(y, 1 - r_1, \frac{t}{2})$. Clearly

$$B(x, 1 - r_1, \frac{t}{2}) \cap B(y, 1 - r_1, \frac{t}{2}) = \emptyset.$$

For if there exists

$$z \in B(x, 1 - r_1, \frac{t}{2}) \cap B(y, 1 - r_1, \frac{t}{2}).$$

Then

$$\begin{aligned} r &= M(x, y, t) \\ &\geq M(x, z, \frac{t}{2}) * M(z, y, \frac{t}{2}) \\ &= r_1 * r_1 \geq r_0 \\ &> r. \end{aligned}$$

which is a contradiction. Therefore $(X, M, *)$ is Hausdorff.

Proposition 2.2.3 *Let (X, d) be a metric space and $M_d(x, y, t) = \frac{t}{t+d(x,y)}$ be the corresponding standard fuzzy metric on X . Then the topology τ_d induced by the metric d and the topology τ_{M_d} induced by the fuzzy metric M_d are the same. That is,*

$$\tau_d = \tau_{M_d}.$$

Proof: Suppose that $A \in \tau_d$. Then there exists $\epsilon > 0$ such that $B(x, \epsilon) \subset A$, for every $x \in A$. For a fixed $t > 0$, we obtain that

$$\begin{aligned} M_d(x, y, t) &= \frac{t}{t + d(x, y)} \\ &> \frac{t}{t + \epsilon}. \end{aligned}$$

Let

$$1 - r = \frac{t}{t + \epsilon}.$$

Then

$$M_d(x, y, t) > 1 - r.$$

It follows that, $B(x, r, t) \subset A$. Hence $A \in \tau_{M_d}$. This shows that $\tau_d \subseteq \tau_{M_d}$.

Conversely,

suppose that $A \in \tau_{M_d}$. Then there exists $0 < r < 1$ and $t > 0$ such that $B(x, r, t) \subset A$ for every $x \in A$. We obtain that

$$\begin{aligned} M(x, y, t) &= \frac{t}{t + d(x, y)} \\ &> 1 - \epsilon \\ t &> (1 - \epsilon)t + (1 - \epsilon)d(x, y) \\ d(x, y) &< \frac{\epsilon t}{1 - \epsilon}. \end{aligned}$$

Let $r = \frac{\epsilon t}{1 - \epsilon}$ where $0 < \epsilon < 1$. Then $d(x, y) < r$, and therefore $B(x, \epsilon) \subset A$. Hence $A \in \tau_d$. This implies that $\tau_{M_d} \subseteq \tau_d$. Therefore $\tau_d = \tau_{M_d}$.

Definition 2.2.3 Let $(X, M, *)$ be a fuzzy metric space. A subset A of X is said to be **F-bounded** if there exists $t > 0$ and $0 < r < 1$ such that

$$M(x, y, t) > 1 - r$$

for all $x, y \in A$.

Remark 2.2.1 Let $(X, M, *)$ be a fuzzy metric space induced by a metric d on X . Then $A \subseteq X$ is F -bounded if and only if it is bounded. This is what R Lowen would call a **good extension** of the notion of boundedness.

Remark 2.2.2 Let (X, d) be a metric space. Define $\tilde{d}(x, y) = \min\{1, d(x, y)\}$ for all $x, y \in X$. Then \tilde{d} is a bounded metric on X , also $\tau_d = \tau_{\tilde{d}}$. Now let $M_{\tilde{d}}$ be the standard fuzzy metric on X induced by \tilde{d} . It follows from Remark 2.2.1 that $(X, M_{\tilde{d}}, *)$ is F -bounded. Hence, we observe that for every metric space (X, d) not necessarily bounded, there exists an F -bounded fuzzy metric space $(X, M, *)$ such that $\tau_d = \tau_M$.

An extension of Proposition 1.2.1 to the context of fuzzy setting is given below:

Theorem 2.2.3 Every compact subset A of a fuzzy metric space X is F -bounded.

Proof: Given A a compact subset of X . Fix $t > 0$ and $0 < r < 1$. Consider an open cover $\{B(x, r, t) : x \in A\}$ of A . Since A is compact, there exists $x_1, x_2, x_3, \dots, x_n \in A$ such that

$$A \subseteq \cup B(x_i, r, t).$$

Let $x, y \in A$. Then $x \in B(x_i, r, t)$ and $y \in B(x_j, r, t)$ for some i, j . Therefore

$$M(x, x_i, t) > 1 - r$$

and

$$M(y, x_j, t) > 1 - r.$$

Now, let

$$\alpha = \min\{M(x_i, x_j, t) : 1 \leq i, j \leq n\}.$$

Then $\alpha > 0$. Now

$$\begin{aligned} M(x, y, 3t) &\geq M(x, x_i, t) * M(x_i, x_j, t) * M(x_j, y, t) \\ &\geq (1 - r) * (1 - r) * \alpha. \end{aligned}$$

Taking $t' = 3t$ and $(1 - r) * (1 - r) * \alpha > 1 - s, 0 < s < 1$, we have

$$M(x, y, t') > 1 - s$$

for all $x, y \in A$. Hence A is F -bounded.

Remark 2.2.3 In a fuzzy metric space every compact subset is closed and bounded.

Theorem 2.2.4 Let $(X, M, *)$ be a fuzzy metric space and τ_M be the topology induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in X , the sequence $\{x_n\}$ converges to x if and only if $M(x_n, x, t)$ converges to 1 as n tends to ∞ .

Proof: Fix $t > 0$. Suppose that the sequence $\{x_n\}$ converges to x . Then for

$$0 < r < 1,$$

there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(x, r, t)$ for all $n \geq n_0$. It follows that

$$M(x_n, x, t) > 1 - r$$

and hence

$$1 - M(x_n, x, t) < r.$$

Therefore

$$M(x_n, x, t)$$

converges to 1 as n tends to ∞ .

Conversely,

if for each $t > 0$, $M(x_n, x, t)$ converges to 1 as n tends to ∞ then for $0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that

$$1 - M(x_n, x, t) < r$$

for all $n \geq n_0$. It follows that

$$M(x_n, x, t) > 1 - r$$

for all $n \geq n_0$. Thus

$$x_n \in B(x, r, t)$$

for all $n \geq n_0$, and hence the sequence $\{x_n\}$ converges to x .

Remark 2.2.4 Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X . Then

$$\lim_n d(x_n, x) = 0$$

if and only if

$$\lim_n M_d(x_n, x, t) = 1$$

for all $t > 0$ and $x \in X$.

Definition 2.2.4 A sequence $\{x_n\}$ in a fuzzy metric space $(X, M, *)$ is a **Cauchy sequence** if for each $\epsilon > 0, t > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_m, t) > 1 - \epsilon$$

for all $n, m \geq n_0$.

Remark 2.2.5 Let (X, d) be a metric space and $\{x_n\}$ be a sequence in X . Then $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if it is Cauchy sequence in $(X, M_d, *)$.

Definition 2.2.5 A sequence $\{x_n\}$ in X is **convergent** to $x \in X$ if

$$\lim_n M(x_n, x, t) = 1$$

for each $t > 0$.

Definition 2.2.6 A fuzzy metric space in which every Cauchy sequence is convergent is called a **complete fuzzy metric space**.

Theorem 2.2.5 Let $(X, M, *)$ be a fuzzy metric space and τ be the topology induced by the fuzzy metric. Then for a sequence $\{x_n\}$ in X, x_n converges to x if and only if $M(x_n, x, t)$ converges to 1 as n converges to ∞ .

Proof: Fix $t > 0$. Suppose that x_n converges to x . Then for $0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that $x_n \in B(x, r, t)$ for all $n \geq n_0$. It follows that $M(x_n, x, t) > 1 - r$ and hence $1 - M(x_n, x, t) < r$. Therefore $M(x_n, x, t)$ converges to 1 as n converges to ∞ .

Conversely,

if for each $t > 0$, $M(x_n, x, t)$ converges to 1 as n converges to ∞ then for $0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that $1 - M(x_n, x, t) < r$ for all $n \geq n_0$. It follows that $M(x_n, x, t) > 1 - r$ for all $n \geq n_0$. Thus $x_n \in B(x, r, t)$ for all $n \geq n_0$, and hence x_n converges to x .

Definition 2.2.7 Let $(X, M, *)$ be a fuzzy metric space. Then we define a **closed ball** with the center $x \in X$ and the radius $r, 0 < r < 1, t > 0$, as

$$B[x, r, t] = \{y \in X : M(x, y, t) \geq 1 - r\}.$$

The following lemma extends Proposition 1.1.5 to the fuzzy setting.

Lemma 2.2.1 Every closed ball in a fuzzy metric space $(X, M, *)$ is a closed set.

Proof: Let $y \in \overline{B[x, r, t]}$. Since X is first countable, there exists a sequence $\{y_n\}$ in $B[x, r, t]$ such that the sequence $\{y_n\}$ converges to y . Therefore $M(y_n, y, t)$ converges to 1 for all t . For a given $\epsilon > 0$,

$$M(x, y, t + \epsilon) \geq M(x, y_n, t) * M(y_n, y, \epsilon).$$

Hence

$$\begin{aligned} M(x, y, t + \epsilon) &\geq \lim_n M(x, y_n, t) * \lim_n M(y_n, y, \epsilon) \\ &\geq (1 - r) * 1 \\ &= 1 - r. \end{aligned}$$

(If $M(x, y_n, t)$ is bounded, the sequence $\{y_n\}$ has a subsequence, which we again denote by $\{y_n\}$ for which $\lim_n M(x, y_n, t)$ exists.) In particular for $n \in \mathbb{N}$, take $\epsilon = \frac{1}{n}$. Then

$$M(x, y, t + \frac{1}{n}) \geq 1 - r.$$

Hence

$$M(x, y, t) = \lim_n M(x, y, t + \frac{1}{n})$$

$$\geq 1 - r.$$

Thus $y \in B[x, r, t]$. Therefore $B[x, r, t]$ is a closed set.

We conclude this section with the Baire category theorem (Theorem 1.1.1) to the setting of fuzzy metric spaces.

Theorem 2.2.6 *Let $(X, M, *)$ be a complete fuzzy metric space. Then the intersection of a countable number of dense open sets is dense.*

Proof: Let X be the given complete fuzzy metric space. Let B_0 be a nonempty open set. Let D_1, D_2, D_3, \dots be dense open sets in X . Since D_1 is dense in X , $B_0 \cap D_1 \neq \emptyset$.

Let

$$x_1 \in B_0 \cap D_1.$$

Since $B_0 \cap D_1$ is open, there exists $0 < r_1 < 1, t > 0$, such that

$$B(x_1, r_1, t_1) \subset B_0 \cap D_1.$$

Choose $r'_1 < r_1$ and $t' = \min\{t_1, 1\}$ such that

$$B(x_1, r'_1, t'_1) \subset B_0 \cap D_1.$$

Let

$$B_1 = B(x_1, r'_1, t'_1).$$

Since D_2 is dense in X , $B_1 \cap D_2 \neq \emptyset$. Let $x_2 \in B_1 \cap D_2$. Since $B_1 \cap D_2$ is open, there exists $0 < r_2 < \frac{1}{2}$ and $t_2 > 0$ such that

$$B(x_2, r_2, t_2) \subset B_1 \cap D_2.$$

Choose $r'_2 < r_2$ and $t'_2 = \min\{t_2, \frac{1}{2}\}$ such that

$$B[x_2, r'_2, t'_2] \subset B_1 \cap D_2.$$

Let $B_2 = B(x_2, r'_2, t'_2)$. Similarly proceeding by induction we can find an

$$x_n \in B_{n-1} \cap D_n.$$

Since $B_{n-1} \cap D_n$ is open, there exists $0 < r_n < \frac{1}{n}$ and $t_n > 0$ such that

$$B(x_n, r_n, t_n) \subset B_{n-1} \cap D_n.$$

Choose $r'_n < r_n, t'_n = \min\{t_n, \frac{1}{n}\}$ such that

$$B[x_n, r'_n, t'_n] \subset B_{n-1} \cap D_n.$$

Let $B_n = B(x_n, r'_n, t'_n)$. Now we claim that $\{x_n\}$ is a Cauchy sequence. For a given $t > 0, \epsilon > 0$ choose n_0 such that $\frac{1}{n_0} < t$ and $\frac{1}{n_0} < \epsilon$. Then for $n \geq n_0, m \geq n$.

$$\begin{aligned} M(x_n, x_m, t) &\geq M(x_n, x_m, \frac{1}{n}) \\ &\geq 1 - (\frac{1}{n}) \\ &\geq 1 - \epsilon. \end{aligned}$$

Therefore $\{x_n\}$ is a Cauchy sequence. Since X is complete, the sequence $\{x_n\}$ converges to x in X . But

$$x_k \in B[x_n, r'_n, t'_n]$$

for all $k \geq n$ and by the previous result $B[x_n, r'_n, t'_n]$ is a closed set. Hence

$$x \in B[x_n, r'_n, t'_n] \subset B_{n-1} \cap D_n$$

for all n . Therefore

$$B_0 \cap (\bigcap_{n=1}^{\infty} D_n) \neq \emptyset.$$

Hence $\bigcap_{n=1}^{\infty} D_n$ is dense in X .

Chapter 3

Further properties on fuzzy metric spaces.

The work presented in this chapter is based on the paper [13]. It is well known that the finite product of metric spaces is metrizable (see Definition 1.1.9 in chapter 1). We present an analogue result in the context of fuzzy metric spaces. Among others we show that a product of two complete fuzzy metric spaces is also a complete fuzzy metric space and subspace of a separable fuzzy metric space is also separable.

3.1 Complete fuzzy metric spaces.

The following two propositions were mentioned as a remark in [13] without proof:

Proposition 3.1.1 *Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be fuzzy metric spaces. For $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2, t > 0$. Let*

$$M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).$$

*Then M is a **fuzzy metric** on $X_1 \times X_2$.*

Proof: 1. Since $M_1(x_1, y_1, t) > 0$ and $M_2(x_2, y_2, t) > 0$ this implies that

$$M_1(x_1, y_1, t) * M_2(x_2, y_2, t) > 0.$$

Therefore

$$M((x_1, x_2), (y_1, y_2), t) > 0.$$

2. Suppose that for all $t > 0$, $(x_1, x_2, t) = (y_1, y_2, t)$. This implies that $x_1 = y_1$ and $x_2 = y_2$, for all $t > 0$. Hence

$$M_1(x_1, y_1, t) = 1$$

and

$$M_2(x_2, y_2, t) = 1.$$

It follows that

$$M(x, y, t) = 1,$$

where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

Conversely,

suppose that $M(x, y, t) = 1$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$. This implies that

$$M_1(x_1, y_1, t) * M_2(x_2, y_2, t) = 1.$$

Since

$$0 < M_1(x_1, y_1, t) \leq 1$$

and

$$0 < M_2(x_2, y_2, t) \leq 1,$$

it follows that

$$M_1(x_1, y_1, t) = 1$$

and

$$M_2(x_2, y_2, t) = 1.$$

Thus $x_1 = y_1$ and $x_2 = y_2$. Therefore $x = y$.

3. To prove that $M(x, y, t) = M(y, x, t)$. We observe that

$$M_1(x_1, y_1, t) = M_1(y_1, x_1, t)$$

and

$$M_2(x_2, y_2, t) = M_2(y_2, x_2, t).$$

It follows that for all $(x_1, x_2), (y_1, y_2) \in X_1 \times X_2$ and $t > 0$,

$$M((x_1, x_2), (y_1, y_2), t) = M((y_1, y_2), (x_1, x_2), t).$$

4. Since $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are fuzzy metric spaces we have that

$$M_1(x_1, z_1, t + s) \geq M_1(x_1, y_1, t) * M_1(y_1, z_1, s)$$

and

$$M_2(x_2, z_2, t + s) \geq M_2(x_2, y_2, t) * M_2(y_2, z_2, s),$$

for all

$$(x_1, x_2), (y_1, y_2), (z_1, z_2) \in X_1 \times X_2$$

and $s, t > 0$. Therefore

$$M((x_1, x_2), (z_1, z_2), t + s) = M_1(x_1, z_1, t + s) * M_2(x_2, z_2, t + s)$$

$$M((x_1, x_2), (z_1, z_2), t + s) \geq M_1(x_1, y_1, t) * M_1(y_1, z_1, s) * M_2(x_2, y_2, t) * M_2(y_2, z_2, s)$$

$$\geq M_1(x_1, y_1, t) * M_2(x_2, y_2, t) * M_1(y_1, z_1, s) * M_2(y_2, z_2, s)$$

$$\geq M((x_1, x_2), (y_1, y_2), t) * M((y_1, y_2), (z_1, z_2), s).$$

5. Note that $M_1(x_1, y_1, t)$ and $M_2(x_2, y_2, t)$ are continuous with respect to t and $*$ is continuous. It follows that

$$M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t),$$

is also continuous.

The next proposition presents Proposition 1.1.3 to the fuzzy metric space setting.

Proposition 3.1.2 *Let $(X_1, M_1, *)$ and $(X_2, M_2, *)$ be fuzzy metric spaces. We define*

$$M((x_1, x_2), (y_1, y_2), t) = M_1(x_1, y_1, t) * M_2(x_2, y_2, t).$$

*Then M is a **complete fuzzy metric** on $X_1 \times X_2$ if and only if $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete.*

Proof: Suppose that $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete fuzzy metric spaces. Let $\{a_n\}$ be a Cauchy sequence in $X_1 \times X_2$. Note that

$$a_n = (x_1^n, x_2^n)$$

and

$$a_m = (x_1^m, x_2^m).$$

Also, $M(a_n, a_m, t)$ converges to 1. This implies that

$$M((x_1^n, x_2^n), (x_1^m, x_2^m), t)$$

converges to 1 for each $t > 0$. It follows that

$$M_1(x_1^n, x_1^m, t) * M_2(x_2^n, x_2^m, t)$$

converges to 1 for each $t > 0$. Thus $M_1(x_1^n, x_1^m, t)$ converges to 1 and also $M_2(x_2^n, x_2^m, t)$ converges to 1. Therefore $\{x_1^n\}$ is a Cauchy sequence in $(X_1, M_1, *)$ and $\{x_2^n\}$ is a Cauchy sequence in $(X_2, M_2, *)$. Since $(X_1, M_1, *)$ and $(X_2, M_2, *)$ are complete fuzzy metric spaces, there exists $x_1 \in X_1$ and $x_2 \in X_2$ such that $M_1(x_1^n, x_1, t)$ converges to 1 and $M_2(x_2^n, x_2, t)$ converges to 1 for each $t > 0$. Let $a = (x_1, x_2)$. Then $a \in X_1 \times X_2$. It follows that $M(a_n, a, t)$ converges to 1 for each $t > 0$. This shows that $(X, M, *)$ is complete.

Conversely,

suppose that $(X, M, *)$ is complete. We shall show that $(X, M_1, *)$ and $(X, M_2, *)$ are complete. Let $\{x_1^n\}$ and $\{x_2^n\}$ be Cauchy sequences in $(X, M_1, *)$ and $(X, M_2, *)$ respectively. Thus $M_1(x_1^n, x_1^m, t)$ converges to 1 and $M_2(x_2^n, x_2^m, t)$ converges to 1 for each $t > 0$. It follows that

$$M(x_1^n, x_2^m, t) = M_1(x_1^n, x_1^m, t) * M_2(x_2^n, x_2^m, t)$$

converges to 1. Let $x^n = (x_1^n, x_2^n)$ in $X_1 \times X_2$ for $n \geq 1$. Then $\{x^n\}$ is a Cauchy sequence in X . Since $(X, M, *)$ is complete, there exists $x \in X_1 \times X_2 = X$ such that $M(x_1^n, x, t)$ converges to 1. Since $x \in X_1 \times X_2$, we may put $x = (x_1, x_2)$, $x_1 \in X_1$ and $x_2 \in X_2$. Clearly, $M_1(x_1^n, x_1, t)$ converges to 1 and $M_2(x_2^n, x_2, t)$ converges to 1. Hence $(X, M_1, *)$ and $(X, M_2, *)$ are complete. This completes the proof.

Definition 3.1.1 Let $(X, M, *)$ be a fuzzy metric space. A collection of sets $\{F_n\}_{n \in I}$ is said to have **fuzzy diameter zero** if for each pair $r, t > 0, 0 < r < 1$, there exists $n \in I$ such that

$$M(x, y, t) > 1 - r$$

for all $x, y \in F_n$.

Remark 3.1.1 A nonempty subset F of a fuzzy metric space X has fuzzy diameter zero if and only if F is a singleton set, where $F = F_n$ for all $n \geq 1$.

We now generalize Theorem 1.1.3:

Theorem 3.1.1 A necessary and sufficient condition that a fuzzy metric space $(X, M, *)$ be complete is that every nested sequence of nonempty closed sets $\{F_n\}_{n=1}^{\infty}$ with fuzzy diameter zero has nonempty intersection.

Proof: First suppose that the given condition is satisfied. We claim that $(X, M, *)$ is complete. Let $\{x_n\}$ be a Cauchy sequence in X . Take

$$A_n = \{x_n, x_{n+1}, x_{n+2}, \dots\}$$

and

$$F_n = \bar{A}_n,$$

then we claim that $\{F_n\}$ has fuzzy diameter zero. For given $s, t > 0, 0 < s < 1$, we can find an $r \in (0, 1)$, such that

$$(1 - r) * (1 - r) * (1 - r) > (1 - s).$$

Since $\{x_n\}$ is a Cauchy sequence, for $r, t > 0, 0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x_n, x_m, \frac{t}{3}) > 1 - r$$

for all $m, n \geq n_0$. Therefore

$$M(x, y, \frac{t}{3}) > 1 - r$$

for all $x, y \in A_{n_0}$. Let $x, y \in F_{n_0}$. Then there exists sequences $\{x'_n\}$ and $\{y'_n\}$ in A_{n_0} such that x'_n converges to x and y'_n converges to y . Hence $x'_n \in B(x, r, \frac{t}{3})$ and $y'_n \in B(y, r, \frac{t}{3})$ for sufficiently large n . Now

$$\begin{aligned} M(x, y, t) &\geq M(x, x'_n, \frac{t}{3}) * M(x'_n, y'_n, \frac{t}{3}) * M(y'_n, y, \frac{t}{3}) \\ &> (1 - r) * (1 - r) * (1 - r) \\ &> 1 - s. \end{aligned}$$

Therefore

$$M(x, y, t) > 1 - s$$

for all $x, y \in F_{n_0}$. Thus $\{F_n\}$ has fuzzy diameter zero. Hence by hypothesis $\bigcap_{n=1}^{\infty} F_n$ is nonempty. Take

$$x \in \bigcap_{n=1}^{\infty} F_n.$$

Then for $r, t > 0, 0 < r < 1$, there exists n_1 such that

$$M(x_m, x, t) > 1 - r$$

for all $n \geq n_1$. Therefore, for each $t > 0$, $M(x_n, x, t)$ converges to 1 as n tends to ∞ . Hence $\{x_n\}$ converges x . Therefore $(X, M, *)$ is a complete fuzzy metric space.

Conversely,

suppose that $(X, M, *)$ is fuzzy complete and $\{F_n\}_{n=1}^{\infty}$ is a nested sequence of nonempty closed sets with fuzzy diameter zero. Let $x_n \in F_n, n = 1, 2, 3, \dots$. Since $\{F_n\}$ has a diameter zero, for $r, t > 0, 0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that

$$M(x, y, t) > 1 - r$$

for all $x, y \in F_{n_0}$. Therefore

$$M(x_n, x_m, \frac{t}{3}) > 1 - r$$

for all $n, m \geq n_0$. Since

$$x_n \in F_n \subset F_{n_0}$$

and

$$x_m \in F_m \subset F_{n_0},$$

$\{x_n\}$ is a Cauchy Sequence. But $(X, M, *)$ is a complete fuzzy metric space and hence $\{x_n\}$ converges to x for some $x \in X$. Now for each fixed n , $x_k \in F_n$ for all $k \geq n$. Therefore

$$x \in \bar{F}_n = F_n$$

for every n , and hence $x \in \bigcap_{n=1}^{\infty} F_n$. This completes our proof.

Remark 3.1.2 The element $x \in \bigcap_{n=1}^{\infty} F_n$ is unique. For if there are two elements

$$x, y \in \bigcap_{n=1}^{\infty} F_n,$$

since $\{F_n\}_{n=1}^{\infty}$ has fuzzy diameter zero, for each fixed

$$t > 0, M(x, y, t) > 1 - \frac{1}{n},$$

for each n . This implies

$$M(x, y, t) = 1$$

and hence

$$x = y.$$

3.2 Separability and uniform convergence in fuzzy metric spaces.

In this section we start by providing an extension of Theorem 1.2.3 to the context of fuzzy metric spaces.

Theorem 3.2.1 Every separable fuzzy metric space is second countable.

Proof: Let $(X, M, *)$ be the given separable fuzzy metric space. Let $A = \{a_n : n \in \mathbb{N}\}$, be a countable dense subset of X . Consider

$$B = \{B(a_j, \frac{1}{k}, \frac{1}{k}) : j, k \in \mathbb{N}\}.$$

Then B is countable. We claim that B is a base for the family of all open sets in X . Let G be an arbitrary open set in X . Let $x \in G$, then there exists $r, t > 0, 0 < r < 1$, such that

$$B(x, r, t) \subset G.$$

Since $r \in (0, 1)$, we can find an $s \in (0, 1)$ such that

$$(1 - s) * (1 - s) > (1 - r).$$

Choose $m \in \mathbb{N}$ such that

$$\frac{1}{m} < \min(s, \frac{t}{2}).$$

Since A is dense in X , there exists $a_j \in A$ such that

$$a_j \in B(x, \frac{1}{m}, \frac{1}{m}).$$

Now if $y \in B(a_j, \frac{1}{m}, \frac{1}{m})$ then,

$$\begin{aligned} M(x, y, t) &\geq M(x, a_j, \frac{t}{2}) * M(y, a_j, \frac{t}{2}) \\ &\geq M(x, a_j, \frac{1}{m}) * M(y, a_j, \frac{1}{m}) \\ &\geq (1 - \frac{1}{m}) * (1 - \frac{1}{m}) \\ &\geq (1 - s) * (1 - s) \\ &> 1 - r. \end{aligned}$$

Thus $y \in B(x, r, t)$ and hence B is a basis. Hence the result.

The next proposition generalizes Proposition 1.2.3.

Proposition 3.2.1 *A subspace of a separable fuzzy metric space is separable.*

Proof: Let X be the given fuzzy metric space and Y be a subspace of X . Let

$$A = \{x_n, n \in \mathbb{N}\}$$

be a countable dense subset of X . For arbitrary but fixed $n, k \in \mathbb{N}$, if there are points $x \in X$ such that

$$M(x_n, x, \frac{1}{k}) > 1 - \frac{1}{k},$$

choose one of them and denote it by x_{nk} . Let

$$B = \{x_{nk}, n, k \in \mathbb{N}\},$$

then B is countable. Now we claim that $Y \subset \bar{B}$. Let $y \in Y$. Given $r, t > 0, 0 < r < 1$, we can find a $k \in \mathbb{N}$ such that

$$(1 - \frac{1}{k}) * (1 - \frac{1}{k}) > 1 - r.$$

Since A is dense in X , there exists an $m \in \mathbb{N}$ such that

$$M(x_m, y, \frac{1}{k}) > 1 - \frac{1}{k}.$$

But by definition of B , there exists $x_{mk} \in A$ such that

$$M(x_{mk}, x_m, \frac{1}{k}) > 1 - \frac{1}{k}.$$

Now

$$\begin{aligned} M(x_{mk}, y, t) &\geq M(x_{mk}, x_m, \frac{t}{2}) * M(x_m, y, \frac{t}{2}) \\ &\geq M(x_{mk}, x_m, \frac{1}{k}) * M(x_m, y, \frac{1}{k}) \\ &\geq (1 - \frac{1}{k}) * (1 - \frac{1}{k}) \\ &= 1 - r. \end{aligned}$$

Thus $y \in \bar{B}$ and hence Y is separable.

Definition 3.2.1 Let X be any nonempty set and $(Y, M, *)$ be a fuzzy metric space. Then a sequence $\{f_n\}$ of functions from X to Y is said to **converge uniformly** to a function f from X to Y if given $r, t > 0, 0 < r < 1$, there exists $n_0 \in \mathbb{N}$ such that

$$M(f_n(x), f(x), t) > 1 - r$$

for all $n \geq n_0$ and for all $x \in X$.

We conclude this section with the uniform limit theorem (Theorem 1.2.5) in the context of fuzzy metric spaces.

Theorem 3.2.2 *Let $f_n : X \rightarrow Y$ be a sequence of continuous functions from a topological space X to a fuzzy metric space Y . If $\{f_n\}$ converges uniformly to f then f is continuous.*

Proof: Let X be the given topological space and $(Y, M, *)$ be the given fuzzy metric space. For any open set V in Y , let $x_0 \in f^{-1}(V)$ and let $y_0 = f(x_0)$. Since V is open, we can find $r, t > 0, 0 < r < 1$, such that

$$B(y_0, r, t) \subset V.$$

Since $r \in (0, 1)$, we can find an $s \in (0, 1)$, such that

$$(1 - s) * (1 - s) * (1 - s) > 1 - r.$$

Since $\{f_n\}$ converges to f , given $s, t > 0, s \in (0, 1)$, there exists $n_0 \in \mathbb{N}$ such that

$$M(f_n(x), f(x), \frac{t}{3}) > 1 - s$$

for all $n \geq n_0$. Since, for all $n \in \mathbb{N}$, f_n is continuous we can find a neighborhood U of x_0 , for a fixed $n \geq n_0$, such that

$$f_n(U) \subset B(f_n(x_0), s, \frac{t}{3}).$$

Hence

$$M(f_n(x), f_n(x_0), \frac{t}{3}) > 1 - s$$

for all x in U . Now

$$\begin{aligned} M(f(x), f(x_0), t) &\geq M(f(x), f_n(x), \frac{t}{3}) * M(f_n(x), f_n(x_0), \frac{t}{3}) * M(f_n(x_0), f(x_0), \frac{t}{3}) \\ &\geq (1 - s) * (1 - s) * (1 - s) \\ &\geq 1 - r. \end{aligned}$$

Thus,

$$f(x) \in B(f(x_0), r, t) \subset V$$

for all x in U . Hence $f(U) \subset V$ and hence f is continuous.

Chapter 4

Some properties on fuzzy pseudo metric spaces.

This chapter introduces the notion of a stationary fuzzy pseudo metric space. In the classical case it is well known that for every pseudo metric space there is a metric identification. We shall extend this to the fuzzy metric case by showing that for every stationary fuzzy pseudo metric space there exists a stationary fuzzy metric identification. We shall also discuss some properties of uniformly continuous maps and extension of t -nonexpansive maps in the context of fuzzy metric spaces. Note that most of the results presented in this chapter are our own contributions.

4.1 Fuzzy pseudo metric spaces and some properties.

In this section we present fundamental results of fuzzy pseudo metric space. We begin with:

Definition 4.1.1 *A 3-tuple $(X, M, *)$ is said to be a **fuzzy pseudo metric space** if X is an arbitrary set, $*$ is a continuous t -norm and M is a fuzzy set on*

$X \times X \times (0, \infty)$ satisfying the following conditions:

$$4.1.1.1 \quad \forall x, y \in X, \quad \text{and} \quad \forall t > 0, \quad M(x, y, t) > 0$$

$$4.1.1.2 \quad \forall x, y \in X, \quad \text{and} \quad \forall t > 0, \quad M(x, y, t) = 1 \quad \text{if} \quad x = y$$

$$4.1.1.3 \quad \forall x, y \in X, \quad \text{and} \quad \forall t > 0, \quad M(x, y, t) = M(y, x, t)$$

$$4.1.1.4 \quad \forall x, y, z \in X, \quad \text{and} \quad \forall s, t > 0, \quad M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)$$

$$4.1.1.5 \quad \forall x, y \in X, \quad M(x, y, \bullet) : (0, \infty) \rightarrow [0, 1] \quad \text{is continuous.}$$

Remark 4.1.1 Clearly every fuzzy metric space is a fuzzy pseudo metric space.

Example 4.1.1 Consider \mathbb{N} with the usual metric d . For any Cauchy sequence $\{x_n\}$ in \mathbb{N} . Let

$$x = \lim_n x_n.$$

It follows that $x \in \mathbb{N}$. Now let $X = \{x_n : x_n \text{ is a Cauchy sequence in } \mathbb{N}\}$. Let $a * b = ab$ for all $a, b \in [0, 1]$ and

$$M(x, y, t) = \begin{cases} \frac{x}{y} & \text{if } x \leq y \\ \frac{y}{x} & \text{if } y \leq x \\ 1 & \text{if } x_n = y_n, \quad \text{for all } n \geq 1, n \in \mathbb{N}. \end{cases}$$

We shall show that M is a fuzzy pseudo metric on X . Note that the outline of the proof is similar to that of Example 2.1.5.

Proof: 1. $\forall t > 0$. Let $x_n, y_n \in X$. If $x_n = y_n$, for all $n \geq 1$, then

$$M(x_n, y_n, t) = 1.$$

2. For all $x_n, y_n \in X$ and for all $t > 0$, clearly $M(x_n, y_n, t) = M(y_n, x_n, t)$.

3. To prove that

$$M(x_n, y_n, t) * M(y_n, z_n, s) \leq M(x_n, z_n, t + s),$$

for all $x_n, y_n, z_n \in X$ and for all $t, s > 0$.

We consider the following cases:

(i). $x_n = y_n = z_n$. Let

$$x = \lim_n x_n, y = \lim_n y_n \quad \text{and} \quad z = \lim_n z_n.$$

$$M(x_n, y_n, t) = 1$$

$$M(y_n, z_n, s) = 1$$

$$M(x_n, z_n, t + s) = 1.$$

Now

$$M(x_n, y_n, t) * M(y_n, z_n, s) = M(x_n, z_n, t + s) = 1.$$

It follows that

$$M(x_n, y_n, t) * M(y_n, z_n, s) \leq M(x_n, z_n, t + s)$$

holds.

(ii). $x_n \neq y_n = z_n$. Let

$$x = \lim_n x_n, y = \lim_n y_n \quad \text{and} \quad z = \lim_n z_n.$$

Without the loss of generality, we may assume that $x < y$ and $y = z$. Then,

$$M(x_n, y_n, t) = \frac{x}{y}$$

and

$$M(y_n, z_n, t) = 1.$$

Also, we have $M(x_n, z_n, t + s) = \frac{x}{z}$. Now $\frac{x}{y} * 1 = \frac{x}{y}$ and $\frac{x}{y} = \frac{x}{z}$. Thus

$$M(x_n, y_n, t) * M(y_n, z_n, s) < M(x_n, z_n, t + s).$$

Therefore

$$M(x_n, y_n, t) * M(y_n, z_n, s) \leq M(x_n, z_n, t + s)$$

holds.

(iii). $x_n = y_n \neq z_n$. Let

$$x = \lim_n x_n, y = \lim_n y_n \quad \text{and} \quad z = \lim_n z_n.$$

Without the loss of generality, we may assume that $x = y$ and $y < z$. Then,

$$M(x_n, y_n, t) = 1.$$

Also, we have

$$M(y_n, z_n, t) = \frac{y}{z}$$

and

$$M(x_n, z_n, t + s) = \frac{x}{z}.$$

Now

$$1 * \frac{y}{z} = \frac{y}{z}$$

and

$$\frac{y}{z} = \frac{x}{z}.$$

Thus

$$M(x_n, y_n, t) * M(y_n, z_n, s) < M(x_n, z_n, t + s).$$

Therefore

$$M(x_n, y_n, t) * M(y_n, z_n, s) \leq M(x_n, z_n, t + s)$$

holds.

(iv). $x_n \neq y_n \neq z_n$. Let

$$x = \lim_n x_n, y = \lim_n y_n \quad \text{and} \quad z = \lim_n z_n.$$

Without the loss of generality, we may assume that $x < y < z$. Then,

$$M(x_n, y_n, t) = \frac{x}{y}$$

$$M(y_n, z_n, s) = \frac{y}{z}$$

$$M(x_n, z_n, t + s) = \frac{x}{z}.$$

Now

$$z > y$$

implies that

$$z^2 > y^2$$

so

$$\frac{1}{z^2} < \frac{1}{y^2}.$$

Thus

$$\frac{xy}{z^2} < \frac{xy}{y^2}.$$

Therefore

$$\frac{x}{z} * \frac{y}{z} < \frac{x}{y}.$$

Hence

$$M(x_n, z_n, t) * M(z_n, y_n, s) < M(x_n, y_n, t + s).$$

So that

$$M(x_n, y_n, t) * M(y_n, z_n, s) \leq M(x_n, z_n, t + s)$$

holds.

4. Note that $M(x_n, y_n, t)$ is independent of t . So for any $s, t > 0$ we have

$$M(x_n, y_n, t) = M(x_n, y_n, s).$$

Thus $M(x_n, y_n, \bullet)$ is continuous. Therefore $(X, M, *)$ is a fuzzy pseudo metric space.

Remark 4.1.2 The topology τ_M induced by the fuzzy pseudo metric in Example 4.1.1 is **discrete**.

In the next example we provide a stationary fuzzy pseudo metric space whose topology is not discrete.

Example 4.1.2 Let $X = \mathbb{R}$ and $a * b = ab$ for all $a, b \in [0, 1]$. Fix $t > 0$. Define

$$M(x, y, t) = \frac{t}{t + |x - y|}.$$

Then $(X, M, *)$ is a stationary fuzzy pseudo metric space on X and the topology τ_M induced by M is the usual topology on X .

Remark 4.1.3 The fuzzy pseudo metrics in Example 4.1.1 and Example 4.1.2 are independent of t , such fuzzy pseudo metrics will be referred to as **stationary fuzzy pseudo metrics**.

Remark 4.1.4 *It is important to note that the notion of fuzzy pseudo metric space also depends on the t -norm. For instance: the fuzzy pseudo metric M of Example 4.1.1 is not a fuzzy pseudo metric with the continuous t -norm $a * b = \min(a, b)$ for all $a, b \in [0, 1]$.*

Remark 4.1.5 *Observe that not every fuzzy pseudo metric space is a fuzzy metric space. Note that Example 4.1.1 provides a fuzzy pseudo metric space which is not a fuzzy metric space. We provide another example:*

Example 4.1.3 *Consider \mathbb{R} with the usual metric. Let $X = \{\{x_n\} : \{x_n\} \text{ is convergent in } \mathbb{R}\}$. Define $a * b = ab$, for all, $a, b \in [0, 1]$, and*

$$M(x_n, y_n, t) = \left[e^{\frac{|\lim(x_n - y_n)|}{t}} \right]^{-1}.$$

*Clearly $(X, M, *)$ is fuzzy pseudo metric space but not fuzzy metric space. To see this, let $\{x_n\} = \frac{1}{n}$ and $\{y_n\} = \frac{3}{n}$. Then $x_n \neq y_n$, for all, $x_n, y_n \in X$, but $M(x_n, y_n, t) = 1$.*

Remark 4.1.6 *Every pseudo metric induces a fuzzy pseudo metric. However, as we have observed by Remark 2.1.8 that not every fuzzy pseudo metric is induced by a pseudo metric.*

We recall:

Definition 4.1.2 ([6], page 215). *A relation R in a set A , that is a subset R of $A \times A$, is termed an **equivalence relation** if satisfies the following condition:*

- (i). *For every $a \in A, (a, a) \in R$ (reflexive).*
- (ii). *If $(a, b) \in R$ then $(b, a) \in R$ (symmetric).*
- (iii). *If $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ (transitive).*

Theorem 4.1.1 *Let $(X, M, *)$ be a fuzzy pseudo metric spaces. The set*

$$R_M = \{(x, y) \in X \times X : M(x, y, t) = 1\}$$

is an equivalence relation on X .

Proof: $(a, a) \in R_M$, since $M(a, a, t) = 1$ for all $a \in X$.

$(a, b) \in R_M$ implies that $(b, a) \in R_M$ as $M(a, b, t) = M(b, a, t)$ for all $a, b \in X$ and $t > 0$.

Suppose that $(a, b) \in R_M$ and $(b, c) \in R_M$.

Then

$$M(a, b, \frac{t}{2}) = 1$$

and

$$M(b, c, \frac{t}{2}) = 1$$

for all $t > 0$. Now for all $t > 0$,

$$\begin{aligned} M(a, c, t) &\geq M(a, b, \frac{t}{2}) * M(b, c, \frac{t}{2}) \\ &\geq 1 * 1 \\ &= 1. \end{aligned}$$

Since $0 \leq M(x, y, t) \leq 1$, for all $x, y \in X, t > 0$, we conclude that,

$$M(a, c, t) = 1.$$

Thus $(a, c) \in R_M$.

Proposition 4.1.1 Let (X, d) be a pseudo metric space and

$$R_d = \{(x, y) \in X \times X : d(x, y) = 0\}.$$

Then

(i). R_d is an equivalence relation.

(ii). $R_d = R_{M_d}$.

Proof: (i). Clear.

(ii). Suppose that $(x, y) \in R_d$. Then $d(x, y) = 0$, this implies that $x = y$. Therefore

$$M_d(x, y, t) = 1.$$

Thus $(x, y) \in R_{M_d}$. Hence $R_d \subseteq R_{M_d}$.

Conversely,

suppose that $(x, y) \in R_{M_d}$, per definition of R_{M_d} , $M_d(x, y, t) = 1$. This implies that $x = y$. Therefore $d(x, y) = 0$. Hence $(x, y) \in R_d$. Thus $R_{M_d} \subseteq R_d$. We conclude that

$$R_d = R_{M_d}.$$

4.2 Fuzzy pseudo metric spaces and uniformities.

Definition 4.2.1 ([54], page 238). If X is any set, we denote Δ the **diagonal** $= \{(x, x) : x \in X \text{ in } X \times X\}$.

Definition 4.2.2 ([54], page 238). Let X be a set and $A \subset X \times X, B \subset X \times X$. We define the set $B \circ A \subset X \times X$ as follows: $(x, y) \in B \circ A$ if there is $z \in X$ such that $(x, z) \in A$ and $(z, y) \in B$.

Remark 4.2.1 If $A_1 = \dots = A_n = A$, we write $A_n \circ \dots \circ A_1 = A^n$.

Definition 4.2.3 ([11], page 217). For any $E \subset X \times X$, we write $E^{-1} = \{(x, y) : (y, x) \in E\}$. A set $E \subset X \times X$ is **symmetric** if $E = E^{-1}$.

We can easily prove the following:

Proposition 4.2.1 Let $A, B, C \subset X \times X$. Then the following properties hold:

(i). $C \circ (B \circ A) = (C \circ B) \circ A$

(ii). $A \circ \Delta = \Delta \circ A = A$

(iii). $(A \circ B)^{-1} = B^{-1} \circ A^{-1}$

(iv). If $A \subset B$, then $A^{-1} \subset B^{-1}$ and $A^n \subset B^n$

(v). $(A^n)^{-1} = (A^{-1})^n, n \in \mathbb{N}$

(vi). If A is symmetric then A^n is also symmetric.

Definition 4.2.4 ([54], page 238). A **uniformity** on a set X is a collection \mathfrak{D} (X), or just \mathfrak{D} , of subsets of $X \times X$, called **surroundings** or **entourages**, which satisfy the following:

(i). $D \in \mathfrak{D}$ implies that $\Delta \subset D$

(ii). $D_1, D_2 \in \mathfrak{D}$ implies that $D_1 \cap D_2 \in \mathfrak{D}$

(iii). $D \in \mathfrak{D}$ implies that $E \circ E \subset D$ for some $E \in \mathfrak{D}$

(iv). $D \in \mathfrak{D}$ implies that $E^{-1} \subset D$ for some $E \in \mathfrak{D}$

(v). $D \in \mathfrak{D} \subset E$ implies that $E \in \mathfrak{D}$.

We shall call the pair (X, \mathfrak{D}) the **uniform space**.

Definition 4.2.5 A **base** for the uniformity \mathfrak{D} is any sub-collection ξ of \mathfrak{D} from which \mathfrak{D} can be recovered by applying condition (v) of Definition 4.2.4

We now provide the following example.

Example 4.2.1 The usual uniformity \mathfrak{D} on \mathbb{R} is the uniformity having for a base the collection of sets $D_\epsilon, \epsilon > 0$, where $D_\epsilon^p = \{(x, y) : |x - y| < \epsilon\}$.

In general every pseudo metric space (X, d) generates a uniform space (X, \mathfrak{D}) , whose base is the collection $D_\epsilon, \epsilon > 0, D_\epsilon^d = \{d(x, y) < \epsilon\}$.

Note that if ϵ runs through \mathbb{Q} then the collection of sets D_ϵ forms a countable base.

Definition 4.2.6 Let (X, \mathfrak{D}) be a uniform space. For $x \in X$, and $D \in \mathfrak{D}$ denote $D[x] = \{y \in X : (x, y) \in D\}$. Now for $A \subseteq X$, say A is **open** if for each $a \in A$ there exists $D \in \mathfrak{D}$ such that $D[a] \subseteq A$. Then the collection of all open subsets of a uniform

space (X, \mathfrak{D}) is a **topology** which we denote by $\tau_{\mathfrak{D}}$. Note that $\tau_{\mathfrak{D}} = \{A \subseteq X : a \in A, \text{ there exists } D \in \mathfrak{D} \text{ such that } D[a] \subseteq A\}$.

Definition 4.2.7 Let (X, d) be a pseudo metric space and (X, \mathfrak{D}_d) be a corresponding uniform space. Then $\tau_d = \tau_{\mathfrak{D}_d}$. In particular, let \mathbb{R} be equipped with the usual pseudo metric d . Then $\tau_{\mathfrak{D}_d}$ is the **usual topology** on \mathbb{R} . Also see Example 4.2.1.

Remark 4.2.2 We shall say that a uniform space (X, ∞) is **pseudo metrizable** if there exists a pseudo metric d on X such that $\tau_d = \tau_{\infty}$.

We recall the following:

Theorem 4.2.1 ([54], page 257). Let (X, \mathfrak{D}) be a uniform space. (X, \mathfrak{D}) is pseudo metrizable if and only if \mathfrak{D} has a countable base.

Definition 4.2.8 A fuzzy pseudo metric space $(X, M, *)$ is **uniformizable** if there exists a uniform space (X, \mathfrak{D}) such that $\tau_{\mathfrak{D}} = \tau_M$.

Proposition 4.2.2 [17]. A T_1 topological space (X, τ) is metrizable if and only if it admits a compatible uniformity with a countable base.

Theorem 4.2.2 Let $(X, M, *)$ be a fuzzy pseudo metric space. Then, (X, τ_M) is a pseudo metrizable topological space.

Proof: For each $n \in \mathbb{N}$ define

$$U_n = \{(x, y) \in X \times X : M(x, y, \frac{1}{n}) > 1 - (\frac{1}{n})\}.$$

We shall prove that $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity \mathfrak{U}_M on X whose induced topology coincides with τ_M . We first note that for each $n \in \mathbb{N}$,

$$\{(x, x) : x \in X\} \subseteq U_n, U_{n+1} \subseteq U_n$$

and $U_n = U_n^{-1}$. On the other hand, for each $n \in \mathbb{N}$, there is, by the continuity of $*$, and $m \in \mathbb{N}$ such that $m > 2n$ and

$$\left(1 - \left(\frac{1}{m}\right)\right) * \left(1 - \left(\frac{1}{m}\right)\right) > 1 - \left(\frac{1}{n}\right).$$

Then, $U_m \circ U_m \subseteq U_n$. Indeed, let $(x, y) \in U_m$ and $(y, z) \in U_m$. Since $M(x, y, \bullet)$ is nondecreasing, $M(x, z, (\frac{1}{n})) \geq M(x, z, \frac{2}{m})$. So

$$\begin{aligned} M(x, z, \frac{1}{n}) &\geq M(x, y, \frac{1}{m}) * M(y, z, \frac{1}{m}) \\ &\geq \left(1 - \left(\frac{1}{m}\right)\right) * \left(1 - \left(\frac{1}{m}\right)\right) \\ &> 1 - \left(\frac{1}{n}\right). \end{aligned}$$

Therefore $(x, z) \in U_n$. Thus $\{U_n : n \in \mathbb{N}\}$ is a base for a uniformity \mathfrak{U}_M on X . Since for each $x \in X$ and each

$$\begin{aligned} n \in \mathbb{N}, U_n(x) &= \{y \in X : M(x, y, \frac{1}{n}) > 1 - \frac{1}{n}\} \\ &= B(x, \frac{1}{n}, \frac{1}{n}), \end{aligned}$$

clearly \mathfrak{U}_M has a countable base hence by Theorem 4.2.1 (X, τ_M) is pseudo metrizable. By Proposition 4.2.1 (X, τ_M) is a pseudo metrizable topological space.

4.3 Fuzzy metric identification.

Definition 4.3.1 ([54], page 59). If X is a topological space, Y is a set and $f : X \rightarrow Y$ is an onto mapping, then the collection τ_f of subsets of Y defined by $\tau_f = \{F \subset Y : f^{-1}(F) \text{ is open in } X\}$ is a topology on Y called the **quotient topology** induced on Y , by f . When Y is given some such quotient topology, it is called a **quotient space** of X and the inducing map f is called a **quotient map**.

Definition 4.3.2 ([54], page 61). Let X be a topological space. A decomposition \mathfrak{D} of X is a collection of disjoint subsets of X whose union is X . If a decomposition \mathfrak{D} is endowed with the topology in which $\mathfrak{F} \subset \mathfrak{D}$ is open if and only if $\cup\{F | F \in \mathfrak{F}\}$ is open in X , then \mathfrak{D} is referred to as a **decomposition space** of X .

Definition 4.3.3 Let $(X, M, *)$ and (Y, N, ∇) be fuzzy pseudo metric spaces. A function $f : (X, M, *) \rightarrow (Y, N, \nabla)$ is a t -**isometry** if and only if

$$M(x, y, t) = N(f(x), f(y), t)$$

for each $x, y \in X$ and $t > 0$.

Theorem 4.3.1 If (X, d) is a pseudo metric space. Then the function

$$f : (X, d) \rightarrow (X, d)$$

is an isometry if and only if $f : (X, M_d, *) \rightarrow (X, M_d, *)$ is a t -isometry.

Proof: Suppose that f is an isometry from $(X, M_d, *)$ to $(X, M_d, *)$. Then

$$M_d(x, y, t) = M_d(f(x), f(y), t)$$

for all $x, y \in X$ and for all $t > 0$. Now from

$$d(x, y) = \frac{t(1 - M(x, y, t))}{M(x, y, t)},$$

we get

$$\begin{aligned} d(f(x), f(y)) &= \frac{t(1 - M(f(x), f(y), t))}{M(f(x), f(y), t)} \\ &= \frac{t(1 - M(x, y, t))}{M(x, y, t)} \\ &= d(x, y). \end{aligned}$$

This implies that $d(x, y) = d(f(x), f(y))$ for all $x, y \in X$. Therefore f is an isometry from (X, d) to (X, d) .

Conversely,

suppose that f is an isometry from (X, d) to (X, d) . Then this implies that

$$d(x, y) = d(f(x), f(y))$$

for all $x, y \in X$. It follows that

$$M_d(f(x), f(y), t) = \frac{t}{t + d(f(x), f(y))}$$

$$\begin{aligned}
&= \frac{t}{t + d(x, y)} \\
&= M_d(x, y, t).
\end{aligned}$$

Thus

$$M_d(x, y, t) = M_d(f(x), f(y), t)$$

for all $x, y \in X$ and for all $t > 0$. Therefore f is an isometry from $(X, M_d, *)$ to $(X, M_d, *)$.

Proposition 4.3.1 *Let $(X, M, *)$ be a stationary fuzzy pseudo metric space and f be a natural map X onto the quotient set $X|_{R_M}$. Then the fuzzy set \bar{M} on*

$$X|_{R_M} \times X|_{R_M} \times (0, \infty)$$

defined by

$$\bar{M}(f(x), f(y), t) = M(x, y, t)$$

is a stationary fuzzy metric on the quotient set.

Proof: Suppose that $f(x) = f(u)$ and $f(y) = f(v)$. We need to show that

$$\bar{M}(f(x), f(y), t) = M(x, y, t) = M(u, v, t) = \bar{M}(f(u), f(v), t).$$

Note that

$$M(x, y, \frac{t}{3}) \geq M(x, u, \frac{t}{3}) * M(u, v, \frac{t}{3}) * M(v, y, \frac{t}{3}).$$

In fact

$$M(x, y, t) \geq M(x, u, t) * M(u, v, t) * M(v, y, t).$$

Since M is stationary fuzzy pseudo metric.

$$M(x, y, t) \geq 1 * M(u, v, t) * 1$$

$$M(x, y, t) \geq M(u, v, t)$$

and

$$M(u, v, t) \geq M(u, x, t) * M(x, y, t) * M(y, v, t)$$

$$M(u, v, t) \geq 1 * M(x, y, t) * 1$$

$$M(u, v, t) \geq M(x, y, t).$$

Therefore

$$M(u, v, t) = M(x, y, t).$$

Since M is a stationary fuzzy pseudo metric. It follows that

$$\bar{M}(f(u), f(v), t) = \bar{M}(f(x), f(y), t).$$

Hence \bar{M} is a stationary fuzzy pseudo metric. Suppose that $(x, y, t) \notin R_M$, then

$$M(x, y, t) > 0, (0 < M(x, y, t) < 1)$$

implies that

$$0 < \bar{M}(f(x), f(y), t) < 1.$$

It follows that \bar{M} is a fuzzy metric.

Definition 4.3.4 Let $(X, M, *)$ be a stationary fuzzy pseudo metric space, we shall refer to the fuzzy metric space $(X|_{R_M}, \bar{M}, *)$ as the **fuzzy metric identification**.

Theorem 4.3.2 Let $(X|_{R_M}, \bar{M}, *)$ be a fuzzy pseudo metric identification of the stationary fuzzy pseudo metric space $(X, M, *)$. Then the topology for the quotient space $X|_{R_M}$ is the topology generated by the fuzzy metric \bar{M} .

Proof: Consider $x \in X, 0 < r < 1$, and $B_M(x, r, t)$ for $t > 0$. Let f be a natural map. To show that

$$f(B_M(x, r, t)) = B_{\bar{M}}(f(x), r, t).$$

Let $a \in f(B_M(x, r, t))$. Then $a \in f(b), b \in B_M(x, r, t)$. Now

$$M(b, x, t) = \bar{M}(f(b), f(x), t)$$

and

$$\bar{M}(f(b), f(x), t) = M(b, x, t)$$

$$> 1 - r.$$

This implies that

$$f(b) \in B_{\bar{M}}(f(x), r, t).$$

Therefore

$$a \in B_{\bar{M}}(f(x), r, t).$$

Thus

$$f(B_M(x, r, t)) \subseteq B_{\bar{M}}(f(x), r, t).$$

Similarly we can show that

$$B_{\bar{M}}(f(x), r, t) \subseteq f(B_M(x, r, t)).$$

Therefore

$$f(B_M(x, r, t)) = B_{\bar{M}}(f(x), r, t),$$

also

$$\begin{aligned} f^{-1}(B_{\bar{M}}(f(x), r, t)) &= f^{-1}f(B_M(x, r, t)) \\ &= B_M(x, r, t). \end{aligned}$$

Hence the collection of all \bar{M} open balls is a base for the quotient topology.

Remark 4.3.1 *If (X, d) is a pseudo metric space then the sequence $\{x_n\}$ in (X, d) is Cauchy if and only if it is Cauchy in $(X, M_d, *)$. Also, a pseudo metric space (X, d) is complete if and only if $(X, M_d, *)$ is complete.*

Let (X, d) be a pseudo metric space we shall denote its metric identification by $(X|_{R_d}, \bar{d})$.

Proposition 4.3.2 *Let $(X, M, *)$ be a complete standard fuzzy pseudo metric space. Then the stationary fuzzy metric identification $(X|_{R_M}, \bar{M}, *)$ is complete.*

Proof: Suppose $(X, M, *)$ is complete. Let $\{x_n\}$ be a Cauchy sequence in $(X|_{R_M}, \bar{M}, *)$, then there exists a sequence $\{a_n\}$ in X such that $x_n = f(a_n), n \geq 1$. Then $\{a_n\}$ is

also a Cauchy sequence in X with respect to M . Since $(X, M, *)$ is complete, there exists a point $x \in X$ such that

$$M(a_n, x, t) \rightarrow 1$$

as n tends to ∞ . It follows that

$$\bar{M}(f(a_n), f(x), t)$$

converges to 1 as n tends to ∞ by continuity of f . Note that $f(a_n) = x_n$. Hence

$$\bar{M}(x_n, f(x), t)$$

converges to 1 as n tends to ∞ . This shows that $(X|_{R_M}, \bar{M}, *)$ is complete. This completes our proof.

Similarly we can proof the following:

Proposition 4.3.3 *Let (X, d) be a complete pseudo metric space. Then $(X|_{R_d}, \bar{d})$ is complete.*

Proposition 4.3.4 *Let (X, d) be a pseudo metric space, $(X|_{R_d}, \bar{d})$ be the metric identification of (X, d) . For the spaces $(X, M_d, *)$, $(X|_{R_d}, M_{\bar{d}}, *)$ and $(X|_{R_{M_d}}, \bar{M}_d, *)$ we have $\tau_{\bar{d}} = \tau_{M_{\bar{d}}} = \tau_{M_d}$.*

Proof: By Proposition 4.1.1 we have observed that $R_d = R_{M_d}$ hence

$$X|_{R_{M_d}} = X|_{R_d}.$$

By Proposition 2.2.3 we have observed that

$$\tau_d = \tau_{M_d}.$$

Similarly we have

$$R_{\bar{d}} = R_{M_{\bar{d}}},$$

hence

$$X|_{R_{\bar{d}}} = X|_{R_{M_{\bar{d}}}}$$

and

$$\tau_{\bar{d}} = \tau_{M_{\bar{d}}}.$$

Finally we have

$$\tau_{\bar{d}} = \tau_{\bar{M}_d}.$$

Hence

$$\tau_{\bar{d}} = \tau_{\bar{M}_d} = \tau_{M_{\bar{d}}}.$$

4.4 Uniformly continuous maps and extension of t -nonexpansive maps.

Definition 4.4.1 Let $(X, M, *)$ be a fuzzy pseudo metric space and $f : (X, M, *) \rightarrow (X, M, *)$ be a function. Then f is t -**uniformly continuous** if for each $0 < \sigma < 1$ there exists $0 < \epsilon < 1$ such that $M(x, y, t) > 1 - \sigma$ implies that $M(f(x), f(y), t) > 1 - \epsilon$, for each $x, y \in X$ and $t > 0$.

Definition 4.4.2 Let $(X, M, *)$ and $(X, N, *)$ be fuzzy pseudo metric spaces, we say that $(X, M, *)$ is **equivalent** to $(X, N, *)$ if and only if for a sequence $\{x_n\}$ and a point x in X we have

$$\lim_n M(x_n, x, t) = 1$$

if and only if

$$\lim_n N(x_n, x, t) = 1.$$

Remark 4.4.1 Recall that for the metric spaces (X, d) and (Y, ρ) , we say that d is equivalent to ρ if and only if for a sequence $\{x_n\}$ and a point x in (X, d) we have

$$\lim_n d(x_n, x) = 0$$

if and only if

$$\lim_n \rho(x_n, x) = 0.$$

Clearly we see that d and ρ are equivalent if and only if the standard fuzzy metric space $(X, M_d, *)$ is equivalent to the standard fuzzy metric space $(Y, M_\rho, *)$.

Theorem 4.4.1 *Let $f : (X, M, *) \rightarrow (X, M, *)$ be a continuous function on a stationary fuzzy pseudo metric space. Then there exists a fuzzy pseudo metric space $(X, \tilde{M}, *)$ such that,*

1. $\tilde{M}(x, y, t)$ is equivalent to $M(x, y, t)$.
2. $f : (X, \tilde{M}, *) \rightarrow (X, M, *)$ is uniformly continuous.

Proof: For all x, y in X and $t > 0$, define

$$\tilde{M}(x, y, t) = M(x, y, t) \wedge M(f(x), f(y), t).$$

1. (i). Since $0 < M(x, y, t) \leq 1$ and $0 < M(f(x), f(y), t) \leq 1$ it follows that

$$0 < \tilde{M}(x, y, t) \leq 1.$$

(ii). Clearly $\tilde{M}(x, y, t) = \tilde{M}(y, x, t)$.

(iii). Since $M(x, y, t) = 1$ and $M(f(x), f(y), t) = 1$ when $x = y$. It follows that $\tilde{M}(x, y, t) = 1$.

(iv). For all $s, t > 0, x, y$ and $z \in X$. We know that

$$M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)$$

and

$$M(f(x), f(y), t + s) \geq M(f(x), f(z), t) * M(f(z), f(y), s).$$

So,

$$\tilde{M}(x, y, t) = M(x, y, t) \wedge M(f(x), f(y), t)$$

implies that

$$\tilde{M}(x, y, t + s) = M(x, y, t) \wedge M(f(x), f(y), t + s)$$

$$\tilde{M}(x, y, t + s) \geq M(x, z, t) * M(z, y, s) \wedge M(f(x), f(y), t) * M(f(z), f(y), s)$$

$$\geq M(x, z, t) * M(f(x), f(z), t) \wedge M(z, y, s) * M(f(z), f(y), s)$$

$$\geq \tilde{M}(x, z, t) * \tilde{M}(z, y, s).$$

Take a sequence $\{x_n\}$ and a point x in X , suppose that for all $t > 0$. $M(x_n, x, t)$ converges to 1. Then by continuity of f , $M(f(x_n), f(x), t)$ converges to 1. So,

$$M(x_n, x, t) \wedge M(f(x_n), f(x), t)$$

converges to 1. It follows that $\tilde{M}(x_n, x, t)$ converges to 1.

Conversely,

suppose that $\tilde{M}(x_n, x, t)$ converges to 1 this means that

$$M(x_n, x, t) \wedge M(f(x_n), f(x), t)$$

converges to 1. Therefore $M(x_n, x, t)$ converges to 1. Thus $\tilde{M}(x_n, x, t)$ and $M(x_n, x, t)$ are equivalent.

2. Observe that

$$M(f(x), f(y), t) \geq \tilde{M}(x, y, t).$$

Now given $0 < \sigma < 1, t > 0$, such that

$$\tilde{M}(x, y, t) \geq 1 - \sigma$$

let $\epsilon = \sigma$. Then

$$M(f(x), f(y), t) > 1 - \epsilon.$$

Therefore

$$f : (X, M, *) \rightarrow (X, M, *)$$

is uniformly continuous.

Definition 4.4.3 Let $(X, M, *)$ be a fuzzy metric. We say that a function is t -**nonexpansive** if

$$\frac{1}{M(x, y, t)} - 1 \leq \frac{1}{M(f(x), f(y), t)} - 1$$

for all $x, y \in X$ and for all $t > 0$.

Remark 4.4.2 Observe that f is t -nonexpansive if and only if

$$M(f(x), f(y), t) \geq M(x, y, t).$$

Remark 4.4.3 Every t -nonexpansive function $f : (X, M, *) \rightarrow (X, M, *)$ is t -uniformly continuous and therefore continuous but not conversely.

Definition 4.4.4 Let $(X, M, *)$ be a fuzzy metric space. We say that $(X, M, *)$ has the **property E** if for a function $f : (X, M, *) \rightarrow (X, M, *)$ and the pair of sequences

$$\{x_i : i \in I\}$$

and

$$\{f(x_i) : i \in I\}$$

in X such that

$$M(f(x_i), f(x_j), t) \geq M(x_i, x_j, t).$$

Then

$$\bigcap_{i \in I} \bar{B}(x_i, r_i, t) \neq \emptyset$$

implies that

$$\bigcap_{i \in I} \bar{B}(f(x_i), r_i, t) \neq \emptyset,$$

where

$$\bar{B}(x, r, t) = \{y \in X : M(x, y, t) \geq 1 - r\},$$

for $0 < r < 1, t > 0$.

Definition 4.4.5 Let S be a subset of fuzzy metric space $(X, M, *)$ and $f : S \rightarrow X$ be a function. We shall say that $F : X \rightarrow X$ is an **extension** of f , if $F|_S = f$.

Remark 4.4.4 Of interest are the extensions which preserve special properties.

Definition 4.4.6 Let $(X, M, *)$ be a fuzzy metric space. We shall say that $(X, M, *)$ has the t -**nonexpansive extension property** if every t -nonexpansive function defined on a subset S of $(X, M, *)$ admits an extension $F : X \rightarrow X$ which is t -nonexpansive.

Theorem 4.4.2 *Let $(X, M, *)$ be a fuzzy metric space and $S \subset X$. Then $(X, M, *)$ has the t -nonexpansive extension property if and only if $(X, M, *)$ has property E.*

Proof: Assume that $(X, M, *)$ has the t -nonexpansive extension property. Consider a map $g : X \rightarrow X$ and a pair of sequences $\{x_i : i \in I\}$ and $\{g(x_i) : i \in I\}$ such that

$$M(g(x_i), g(x_j), t) \geq M(x_i, x_j, t)$$

and

$$\bigcap_{i \in I} \bar{B}(x_i, r_i, t) \neq \emptyset$$

hold. Let $S = \{x_i : i \in I\}$ and define $f = g$ on S . Since $x \in \bigcap_{i \in I} \bar{B}(x_i, r_i, t) \neq \emptyset$, then f is t -nonexpansive and therefore admits an extension $F : X \rightarrow X$ which is t -nonexpansive. In particular, F is defined on x , and therefore there exists $y \in F(X)$ such that

$$y \in \bigcap_{i \in I} \bar{B}(F(x_i), r_i, t).$$

This shows that

$$\bigcap_{i \in I} \bar{B}(F(x_i), r_i, t) \neq \emptyset$$

and so

$$\bigcap_{i \in I} \bar{B}(f(x_i), r_i, t) \neq \emptyset.$$

Conversely, let $S \subset X$ and $f : S \rightarrow X$ be a t -nonexpansive function. For each $x \in X - S$, we can extend f to $S \cup \{x\}$. Consider, the collections of closed balls

$$\{\bar{B}(\omega, M(x, \omega, t), t) : \omega \in S\}$$

and

$$\{\bar{B}(f(\omega), M(x, \omega, t), t) : \omega \in S\}.$$

Then

$$\bigcap_{\omega \in S} \{\bar{B}(\omega, M(x, \omega, t), t) : \omega \in S\} \neq \emptyset$$

Hence, there exists

$$y \in \bigcap_{\omega \in S} \{\bar{B}(f(\omega), M(x, \omega, t), t) : \omega \in S\}.$$

Let $f(x) = y$. Note that this extension is also t -nonexpansive. Now let ζ be the collection of all t -nonexpansive extensions of f to subsets of X that contain S . For f_1 and f_2 in ζ we shall say that $f_1 \leq f_2$ provided that $D(f_1) \subseteq D(f_2)$. Note that every totally ordered subfamily of ζ has a maximal element F and that F belongs to ζ . It follows that F is the required extension of f .

Bibliography

- [1] APOSTOL T. M., *Mathematical Analysis*, Addison Wesley Publishing Co, New York, 1974.
- [2] BRIDGER M., *Real Analysis: a Constructive Approach/Mark Bridger*, Hoboken, N. J.: Wiley-Interscience, 2007.
- [3] BUSHAW D., *Elements of General Topology*, New York: Wiley, 1963.
- [4] CAIRNS S. S., *Introductory Topology*, New York: Ronald Press, 1968.
- [5] CHAE S. B., *Lebesgue Integration*, New York: Springer-Verlag, 1995.
- [6] CONOVER R. A., *A first course in Topology: an introduction to mathematical thinking*, Baltimore: Williams and Wilkins, 1975.
- [7] CROOM F. H., *Principles of Topology*, Philadelphia: Saunders College Pub, 1989.
- [8] CSASZAR A., *General Topology*, Hilger, 1978.
- [9] CULLEN H. F., *Introduction to General Topology*, Boston: Health, 1968.
- [10] EISENBERG M., *Topology*, New York: Holt, Rinehart and Winston, 1974.
- [11] FAIRCHILD W. W., *Topology*, Philadelphia: Saunders, 1971.
- [12] GEMIGNANI M. C., *Introduction to real analysis*, Saunders, 1971.
- [13] GEORGE A., VEERAMANI P., On some results of analysis for fuzzy metrics, *Fuzzy sets and systems*, 90(1997) 365-368.

- [14] GEORGE A., VEERAMANI P., On some results in fuzzy metric spaces, *Fuzzy sets and Systems*, 64(1994) 395-399.
- [15] GEORGE A., VEERAMANI P., Some theorems in fuzzy metric spaces, *The Journal of Fuzzy Mathematics*, 3(1995) 933-940.
- [16] GERVER R. K., *Writing Math Research papers: a guide for students and Instructors/Robert Gerver*, Emeryville, Calif: Key Curriculum Press, 2004.
- [17] GREGORI V., S. ROMAGUERA, Some properties of fuzzy metric spaces, *Fuzzy sets and systems*, 115(2000) 485-489.
- [18] GREGORI V., S. ROMAGUERA, Characterizing completable fuzzy metric spaces, *Fuzzy sets and systems*, 144(2004) 411-420.
- [19] GREGORI V., S. ROMAGUERA, On fixed-point theorems in fuzzy metric spaces, *Fuzzy sets and systems*, 125(2002) 245-252.
- [20] GREGORI V., S. ROMAGUERA, On completion of fuzzy metric spaces, *Fuzzy sets and systems*, 103(2000) 399-404.
- [21] GREGORI V., A. SAPENA, On fixed-point theorems in fuzzy metric spaces, *Fuzzy sets and systems*, 125(2002) 245-252.
- [22] HAASER N.B., *Real Analysis/Norman B. Haaser*, Joseph A. Sullivan, New York: Dover Publication, 1991.
- [23] HIGHMAN N. J., *Handbook of Writing for the Mathematical Sciences*, Philadelphia: Society for Industrial and Applied Mathematics, 1998.
- [24] KALEVA O., *A comment on the completion of fuzzy metric spaces*, *Fuzzy sets and systems*, 159(2008) 2190-2192.
- [25] KAPLANSKY I., *Set Theory and Metric Spaces*, New York: Chelsea Pub Co, 1977.
- [26] KASKRIEL R. H., *Undergraduate Topology*, Philadelphia: Saunders, 1971.

- [27] KELLEY J. L., *General Topology*, New York: Springer-Verlag, 1975.
- [28] KOLMOGOROV A. N., *Introductory Real Analysis*, Prentice-Hall, 1970.
- [29] KRAMOSIL O., MICHALEK J., Fuzzy metric and statistical metric spaces, *Kybernetika*, 11(1975) 326-334
- [30] KRANTZ S. G., *Mathematical publishing: a guidebook/Steven G. Krantz*, Providence R. I.: American Mathematical Society, 1951.
- [31] KREYSIG E., *Introductory functional Analysis with applications*, New York: Wiley, 1978.
- [32] LIPSCHUTZ S., *Schaums outline of theory and problems of general topology*, New York: Schaum, 1965.
- [33] LONG P. E., *An introduction to General Topology*, Columbus, Ohio: Merrill, 1971.
- [34] LYNN A., STEEN J., ARTHUR S. J. R., *Counterexamples in Topology*, New York: Holt Richard and Winston, 1970.
- [35] MANSFIELD M. J., *Introduction to Topology*, Princeton N. J.: Van Nostrand, 1963.
- [36] MARIUSZ G., Fixed points in fuzzy metric spaces, *Fuzzy sets and systems*, 27(1988) 385-389.
- [37] MENDELSON B., *Introduction to Topology*, Boston: Allyn and Bacon, 1968.
- [38] MORGAN F., *Real Analysis and applications: including Fourie series and the calculus of variations/Frank Morgan*, Providence R. I.: American Mathematical Society, 2005.
- [39] MOORE T. O., *Elementary General Topology*, Englewood Cliffs N. J.: Prentice-Hall, 1964.

- [40] MOSHOKOA S. P., *On Hausdorff type distances and their applications to fractals*, MSc Dissertation, UNISA, 2000.
- [41] MOSHOKOA S. P., *Completeness of spaces and extensions of special kinds of maps*, PhD Thesis, UNISA, 2005.
- [42] MUNKRES J. R., *Topology: a first Course*, Englewood Cliffs N. J.: Prentice Hall, 1974.
- [43] MURDESHWAR M. G., *General Topology*, New York: Wiley, 1983.
- [44] PATTY W. C., *Foundations of Topology*, Boston: PWS-KENT Publishing Co, 1993.
- [45] PITTIS C. G. C., *Introduction to Metric Spaces*, Edinburgh: Oliver and Boyd, 1972.
- [46] ROYDEN H. L., *Real Analysis*, New York: Macmillan London: Collier Macmillan, 1988.
- [47] SCHWEIZER B., A. Sklar, *Probabilistic Metric Spaces*, New York: North-Holland, 1983.
- [48] SEGAL I. E., *Integrals and Operators*, New York: McGraw-Hill, 1968.
- [49] SHIRALI S., *Metric Spaces/Satish Shirali and Harkrishan Vasudera L.*, London: Springer, 2006.
- [50] SIERADSKI A. J., *An introduction to Topology and Homotopy/Allan J. Sieradski*, Boston Mass: PWS-KENT, 1992.
- [51] SPRECHER D. A., *Elements of real analysis*, New York: Academic Press, 1970.
- [52] VULIKH B. Z., *Introduction to functional analysis for the scientists and technologists*, Oxford New York: Pergamon Press, 1963.

- [53] WILLIAM J. P., *Foundation of General Topology*, Philadelphia: Saunders, 1964.
- [54] WILLARD S., *General Topology*, New York, Reading Mass: Addison-Wesley, 1970.
- [55] ZIMMERMAN H. J., *Fuzzy set theory and its application third edition*, Boston: Kluwer Academic Publishers, 1991.