LATENT RELATIONSHIPS BETWEEN MARKOV PROCESSES, SEMIGROUPS AND PARTIAL DIFFERENTIAL EQUATIONS

by

SAFARI MUKERU KAJAMA

submitted in part fulfilment of the requirements
for the degree of

MASTER OF SCIENCE

in the subject

OPERATIONS RESEARCH

at the

UNIVERSITY OF SOUTH AFRICA

SUPERVISOR: PROF MK LUHANDJULA

June 2008

Declaration

"I declare that Latent Relationships between Markov processes, Semigroups and Partial Differential Equations is my own work and that all the sources that I have used or quoted have been indicated and acknowledged by means of complete references".

MR S MUKERU DATE: June 27, 2008.

Dedication

To my parents Ferdinand MUKERU KAJAMA
and Concilie CIREZI M'RUTEGA

Acknowledgements

I would like first to express my gratitude to Prof. LUHANDJULA who kindly proposed me the topic and guided me during the period of research. I also thank him for having interested me in the field of Operations research.

My gratitude is also due to the staff of the Department of Decision Sciences for the training I have been receiving from them since 2004. The Head of Department, Prof. SWANEPOEL encouraged me to take admission to the MSC programme and I thank him.

I am very grateful to the Financial Aid Bureau (FAB) of UNISA for having granted me with tuition fees since 2006.

I also thank my spouse NAMULISA for her support. My brother BAHATI has supported me financially and I express to him my deepest gratitude. I also thank my parents, COKORO, KALOLO, MAVUDIKO and NAMULAMBI families and all people who contributed to the completion of this work.

Finally, I thank my children Rachel, Concilie, Esther and Grace for having accepted my long absence from home.

S. Mukeru

Abstract

This research investigates existing relationships between the three apparently unrelated subjects: Markov process, Semigroups and Partial differential equations.

Markov processes define semigroups through their transition functions. Conversely particular semigroups determine transition functions and can be regarded as Markov processes. We have exploited these relationships to study some Markov chains.

The infinitesimal generator of a Feller semigroup on the closure of a bounded domain of $\mathbf{R}^n, (n \geq 2)$, is an integro-differential operator in the interior of the domain and verifies a boundary condition.

The existence of a Feller semigroup defined by a differential operator and a boundary condition is due to the existence of solution of a bounded value problem. From this result other existence sufficient conditions on the existence of Feller semigroups have been obtained and we have applied some of them to construct Feller semigroups on the unity disk of $\mathbf{R}^2$.

Key-words

Markov process, semigroup, partial differential equation, Feller semigroup, infinitesimal generator, birth and death process.

Contents

Declaration i
Dedication ii
Acknowledgements iii
Abstract iv
Key words iv
Contents v

Introduction **1**

1 Basics of Markov Processes, Semigroups and Partial Differential Equations **3**

1.1 Basic notions on probability and measure theory 3
1.1.1 Probability space 3
1.1.2 Notions from Topology 4
1.1.3 Measurable functions, Random variables 5
1.1.4 Integration 7
1.1.5 Conditional expectation, Independency 8
1.1.6 Kernels 11
1.2 Stochastic Processes 13
1.2.1 Definitions 13
1.2.2 Equivalent stochastic processes 14
1.2.3 Kolmogorov extension Theorem 15
1.2.4 Adapted stochastic processes 17
1.3 Markov Processes 17
1.3.1 Markov property 17
1.3.2 Transition function of a Markov process 22
1.3.3 Realisation of a transition function 23
1.4 Semigroups on Banach spaces 25
1.4.1 Notions from Functional Analysis 25
1.4.2 Definition of Semigroups 28
1.4.3 The Hille-Yosida Theorem 30
1.4.4 Feller semigroups 33
1.5 Notions on Differential Manifolds 34

1.5.1 Definitions and Examples of Maninfolds 35
1.5.2 Tangent space 36
1.6 Partial differential equations 39
1.6.1 Definitions 39
1.6.2 Linear partial differential equation 39
1.6.3 Existence Theorem for elliptic bounded value problems 40

2 Relationships between Markov processes, Semigroups and Partial differential equations 43
2.1 Markov Processes and Semigroups 43
2.1.1 Semigroups associated to Markov Processes 43
2.1.2 Markov Processes associated to Semigroups 45
2.2 Infinitesimal generators of Feller semigroups on a Bounded Domain 49
2.2.1 Form of infinitesimal generators in the interior of the Domain . . . 50
2.2.2 Form of the infinitesimal generator on the boundary of the Domain 59
2.3 Feller Semigroups and Bounded value problems 68
2.3.1 Statement of the Problem 69
2.3.2 Existence of Feller semigroups 70

3 Applications 80
3.1 Semigroups associated to Markov Chains 80
3.1.1 Discrete-Time Markov Chains 80
3.1.2 Continuous-time Markov chains 85
3.1.3 Infinitesimal generators of birth and death processes 93
3.1.4 Particular case: Poisson process 94
3.1.5 Illustrative Example 95
3.2 Feller semigroups on a closed disk 97
3.2.1 Atlas on the unity circle 97
3.2.2 Euclidean and non-Euclidean balls on the unity circle 99
3.2.3 Examples of Feller semigroups 105

Concluding Remarks 107

Bibliography 109

Introduction

It is often convenient to describe a system by a family $(X_t)_{t\geq 0}$ of random variables in order to handle imprecision related to its evolution. In this representation, the random variable X_t is interpreted as the state of the system at time t [13]. It can be for instance the inventory of some commodity at time t, a position of a particle that moves on a given surface at time t, etc.... Such a family is called a stochastic process. A stochastic process is said to be Markovian if it has the so-called Markov property: the future state of the process given the current state at time t is independent on observed states during the time interval $[0, t)$ [17]. Any Markov process is mainly described by its initial distribution (which gives the initial state of the process) and its transition probabilities (which define the probabilities for the process to move from a state to another).

A semigroup on a Banach space E is a family of bounded linear operators $(T_t)_{t\geq 0}$ which verifies the so-called "semigroup propert": the composite of T_s and T_t is equal to T_{s+t} and such that T_0 is the identity map. If these operators are contractions and strongly continuous, the semigroup is completely described by a linear operator, called its infinitesimal generator [11], [12], [17].

Roughly speaking, a Partial differential equation is an equation that depends on a function $u(x, y, \ldots)$, some of its partial derivatives and the independent variables $(x, y, \ldots)$. If the unknown function is restricted to some initial values or boundary conditions, the term "Boundary value problem" is used instead of Partial differential equation.

These three subjects are from different branches of Mathematics, Probability theory for the first, Functional Analysis for the second and its own area for the third. Yet there are many interplays between them. The purpose of this dissertation is to highlight some of these relationships and to show how they can be used with good reasons to the study of particular Markov chains and to the construction of Markov processes.

Markov processes are related to Semigroups through their transition functions. The Chapman-Kolmogorov equation is equivalent to the semigroup property. Conversely, one can associate transition functions (and hence Markov processes) to some semigroups, and Feller semigroups in particular. The construction uses mainly the "Riesz representation

Theorem"[4], [19].

Semigroups and Partial differential equations are related through the infinitesimal generator of the semigroup. There are methods of solving partial differential equations using semigroup theory based on the Hille-Yosida theory. For details on these matters, an interested reader may consult [12].

The infinitesimal generator of a Feller semigroup on the closure of a bounded domain of $\mathbf{R}^n$, $(n \geq 2)$, is defined by an integro-differential operator in the interior of the domain. A boundary condition which turns out to be an integro-differential operator is also verified. In the particular case where no integral terms appear in these operators, they become differential operators [3], [17].

This observation has raised the problem of knowing under which conditions a differential operator A and a boundary condition L determine a Feller semigroup on a bounded domain. It has been established that the existence of such a Feller semigroup is due to the existence of solutions of a bounded value problem defined from A and L [3],[17]. From this, other more efficient conditions on the existence of Feller semigroups have been obtained.

Our dissertation is organized as follows. In Chapter 1, the basic definitions and properties of Markov processes, semigroups and partial differential equations are provided. Useful results from Probability and Measure theory, Stochastic processes, Topology, Linear operators theory, Semigroups, Differential geometry and Partial differential equations have been also included.

In Chapter 2, relationships between Markov processes, Semigroups and Partial differential equations are discussed. It has been shown that Semigroups are related to Markov processes. The form of the infinitesimal generator of a Feller semigroup in the closure of a bounded domain of $\mathbf{R}^n$ has also been presented. The last section of the chapter discusses some sufficient conditions under which a differential operator and a boundary condition define a Feller semigroup on the closure of a bounded domain of $\mathbf{R}^n$.

In Chapter 3, the Semigroups associated with Markov chains and the "birth and death processes", in particular, are investigated. We have applied some existence Theorems of Chapter 2 to construct Feller semigroups on the unity disk of $\mathbf{R}^2$. The dissertation ends with some concluding remarks.

Chapter 1

Basics of Markov Processes, Semigroups and Partial Differential Equations

1.1 Basic notions on probability and measure theory

1.1.1 Probability space

Definition 1.1 *Let Ω be a set. A family $\mathcal{F} = (A_i)_{i\in I}$ of subsets of Ω is called a $\sigma-$algebra on Ω if the following three properties hold:*

1. $\Omega \in \mathcal{F}$,
2. *for any sequence $A_1, A_2, \ldots$ of elements of $\mathcal{F}$, $\cup_{n=1}^{\infty} A_n \in \mathcal{F}$,*
3. *for any $A \in \mathcal{F}$, the complement $\overline{A} = \Omega - A \in \mathcal{F}$.*

If $\mathcal{F}$ is a $\sigma-$algebra on Ω, then the pair $(\Omega, \mathcal{F})$ is called a measurable space. The elements of $\mathcal{F}$ are termed $\mathcal{F}$-measurable sets or simply measurable sets.

If $\mathcal{H}$ is a family of subsets of Ω then there exists at least one $\sigma-$algebra on Ω containing $\mathcal{H}$. The smallest $\sigma-$algebra on Ω containing $\mathcal{H}$ is called the $\sigma-$algebra generated by $\mathcal{H}$.

Definition 1.2 *Let $(\Omega, \mathcal{F})$ be a measurable space. A measure on $(\Omega, \mathcal{F})$ is a function μ from $\mathcal{F}$ into the set $[0, \infty]$ such that the following properties hold:*

1. *for any sequence $A_1, A_2, \ldots, A_n, \ldots$ of mutually exclusive elements of $\mathcal{F}$,*

$$\mu(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n).$$

2. $\mu(\phi) = 0$.

If μ is a measure on a measurable space $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, \mu)$ is called a measure space.

Definition 1.3 *A probability on a measurable space $(\Omega, \mathcal{F})$ is a measure P on $(\Omega, \mathcal{F})$ verifying the condition $P(\Omega) = 1$.*

If P is a probability on $(\Omega, \mathcal{F})$, then $(\Omega, \mathcal{F}, P)$ is called a probability space.

1.1.2 Notions from Topology

Definition 1.4 *A topological space is a set E endowed with a family $\mathcal{T}$ of subsets of E such that the following three properties hold:*

1. *the empty subset and E belong to $\mathcal{T}$,*
2. *if $(O_i)_{i\in I}$ is a family of elements of $\mathcal{T}$ then $\cup_{i\in I} O_i \in \mathcal{T}$.*
3. *if $O_1, O_2, \ldots, O_n$ are elements of $\mathcal{T}$ then $\cap_{i=1}^{n} O_i \in \mathcal{T}$.*

Elements of $\mathcal{T}$ are called the open subsets of E.
A complement of an open subset is called a closed subset.
A neighborhood of a point $x \in E$ is any subset of E that contains an open set containing the point x.
The closure of a subset A of E is the smallest closed subset of E containing A.
If E and F are topological spaces, a map $f : E \to F$ is continuous if for any open subset O of F, $f^{-1}(O)$ is an open subset of E. If in addition f is bijective and f^{-1} is also continuous, f is called an homeomorphism.

Definition 1.5 *A topological space E is separated if for any $x, y \in E$, there exist neighborhoods V_x and V_y of x and y respectively such that $V_x \cap V_y$ is empty.*
A separated topological space E is compact if for any family $(O_i)_{i\in I}$ of open subsets of E such that

$$\cup_{i\in I} O_i = E,$$

there exists a finite subset J of I such that

$$\cup_{i\in J} O_i = E.$$

A subset K of E is compact if it is compact as a subspace of E.
A separated topological space E is locally compact if each point x of E has a compact neighborhood in E.
A topological space is connected if it cannot be written as a union of two disjoint open sets. A topological space E is separable if it contains a countable and dense subset.

The space $\mathbf{R}^n$ is an example of a separable locally compact space. A subset of this space is comapct if and only if it is bounded and closed.

Definition 1.6 *Let E be a topological space and $f : E \to \mathbf{R}$ be a continuous function. The support of f is the closure of the set*

$$\{x \in E : f(x) \neq 0\}.$$

Definition 1.7 *Let E be a topological space and $\mathcal{T}$ be the set of open subsets of E. The $\sigma-$algebra on E generated by $\mathcal{T}$ is called the Borel $\sigma-$algebra on E.*

When a topological space is considered as a measurable space, measurable sets are members of the Borel $\sigma-$algebra.

Definition 1.8 *Let $\mathcal{B}$ be the Borel $\sigma-$algebra on $\mathbf{R}^n$. There exists a measure μ on $(\mathbf{R}^n, \mathcal{B})$ such that for any intervals $A_i = (a_i, b_i)$ of $\mathbf{R}$ with $a_i \leq b_i$,*

$$\mu(A_1 \times A_2 \times \ldots \times A_n) = (b_1 - a_1)(b_2 - a_2)\ldots(b_n - a_n).$$

This measure is called the Lebesgue measure on $\mathbf{R}^n$.

1.1.3 Measurable functions, Random variables

Definition 1.9 *Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be measurable spaces. A function f from Ω into E is termed a measurable function if*

$$\forall A \in \mathcal{E}, \quad f^{-1}(A) \in \mathcal{F}.$$

Definition 1.10 *Let $(\Omega, \mathcal{F}, P)$ be a probability measure. A function X from Ω into $\mathbf{R}$ is called a random variable if it is measurable, that is, for any Borelian A of $\mathbf{R}$,*

$$X^{-1}(A) \in \mathcal{F}.$$

In the sequel we will use the term *random variable* in the genral sense of being a measurable function from a probability space into a measurable space. We will always have to specify the second space.

Let X be a random variable from $(\Omega, \mathcal{F}, P)$ into $\mathbf{R}$. The probability distribution of X is the function $P^X : \mathbf{R} \to [0, 1]$ defined by

$$P^X(x) = P\{w \in \Omega : X(w) < x\}.$$

Definition 1.11 *Let Ω be a nonempty set and X be a function from Ω into a measurable space $(E, \mathcal{E})$. The set*

$$\sigma(X) = \{X^{-1}(A) : A \in \mathcal{E}\}$$

is the smallest $\sigma-$algebra on Ω such that X is measurable. It is called the $\sigma-$algebra generated by X.

The $\sigma-$algebra generated by a family $(X_i)_{i\in I}$ of functions $X_i : \Omega \to (E, \mathcal{E})$ is the smallest $\sigma-$algebra on Ω for which all the functions X_i are measurable. It is equal to the $\sigma-$algebra on Ω generated by the sets of the form $X_i^{-1}(A_i)$, where for any $i \in I, A_i \in \mathcal{E}$.

Definition 1.12 *Let $(E_i, \mathcal{E}_i)_{i\in I}$ be a family of measurable spaces. Consider the coordinate maps*

$$\pi_i : \prod_{i\in I} E_i \to (E_i, \mathcal{E}_i)$$

defined by:

$$\forall e = (e_i)_{i\in I}, \quad \pi_i(e) = e_i, \quad \forall i \in I. \tag{1.1}$$

The $\sigma-$algebra on $\prod_{i\in I} E_i$ generated by these maps is called the product of the $\sigma-$ algebras $(\mathcal{E}_i)_{i\in I}$ and is denoted $\prod_{i\in I} \mathcal{E}_i$.

This $\sigma-$algebra is also generated by the sets of the form

$$\prod_{i\in I} A_i$$

where $\forall i \in I, A_i \in \mathcal{E}_i$ and $A_i = E$ except for a finite number of indices.
If

$$(E_i, \mathcal{E}_i) = (E, \mathcal{E}), \quad \forall i \in I,$$

then the product space $(\prod_{i\in I} E_i, \prod_{i\in I} \mathcal{E}_i)$ is simply denoted $(E^I, \mathcal{E}^I)$. In this case, E^I represents the set of all maps from I into E.

The following theorem will be used in the sequel.

Theorem 1.1 (The monotone class theorem) *Let V be a vector space of real-valued bounded functions defined on Ω verifying the following properties:*

1. *the constant function $1 \in V$,*
2. *if $(f_n)_{n\geq 0}$ is a sequence of functions from V to $\mathbf{R}$ that converges uniformly to f, then $f \in V$,*
3. *if (f_n) is a nondecreasing sequence of elements of V such that*

$$\sup_n \left(\sup_{x\in\Omega} |f_n(x)| \right) < +\infty,$$

then

$$\lim f_n \in V.$$

Let C be a subset of V, such that for any $f, g \in C$, the product $fg \in C$. Then V contains all bounded measurable functions with respect to the $\sigma-$algebra $\sigma(C)$ generated by C on V.

A proof of this theorem along with some more details may be found elsewhere [4], [9]. The following result also from [4] is a consequence of Theorem 1.1 and will be used in the sequel.

Theorem 1.2 *Let $\mathcal{C}$ be a set of subsets of Ω such that for any $A, B \in \mathcal{C}$, $A \cup B \in \mathcal{C}$ and the complement $\bar{A} = \Omega - A \in \mathcal{C}$. Let $\mathcal{F}$ be the $\sigma-$algebra on Ω generated by $\mathcal{C}$. If P and P' are two probability measures on $(\Omega, \mathcal{F})$ such that $P = P'$ on $\mathcal{C}$, then $P = P'$*

1.1.4 Integration

Definition 1.13 *Let $(\Omega, \mathcal{F}, \mu)$ be a measure space. A measurable function $f : \Omega \to \mathbf{R}$ is said to be a step function if there exists a partition $A_1, A_2, \cdots, A_n$ of Ω such that the subsets A_i are measurable and f is constant on each of them.*
If $f = a_i$ on A_i, then the integral of the function f is defined by

$$\int f d\mu = \sum_{i=1}^{n} a_i \mu(A_i). \tag{1.2}$$

For any $A \in \mathcal{F}$ and any step function f,

$$\int_A f d\mu = \int f.1_A d\mu$$

where 1_A is the characteristic function of A.

Definition 1.14 *Let $f : \Omega \to \mathbf{R}$ be a nonnegative measurable function. It can be shown that there exists an nondecreasing sequence (f_n) of nonnegative measurable step functions that converges to f. The integral of the function f is*

$$\int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

Definition 1.15 *Let $f : \Omega \to \mathbf{R}$ be a measurable function. If*

$$\int |f| d\mu < \infty,$$

then the function f is said to be $\mu-$integrable.

The integral of the function f is

$$\int f d\mu = \int f^+ d\mu - \int f^- d\mu$$

where

$$f^+ = \sup(f, 0), \quad f^- = \inf(-f, 0).$$

The integral $\int f d\mu$ is also denoted by $\int f(x) d\mu(x)$.

Definition 1.16 *Let E be a compact topological space and $\mathcal{E}$ be its Borel $\sigma-$algebra. A sequence $(\mu_n)_{n\geq 0}$ of measures on $(E,\mathcal{E})$ converges weakly to a measure μ on the same space if for any continuous function $f : E \to \mathbf{R}$ the following property holds:*

$$\lim_{n\to\infty} \int_E f(x)d\mu_n(x) = \int_E f(x)d\mu(x). \tag{1.3}$$

Theorem 1.3 *If $(\mu_n)_{n\geq 0}$ is a sequence of measures on the space $(E,\mathcal{E})$ where E is a compact space and $\mathcal{E}$ is the Borel $\sigma-$algebra on E and if*

$$\sup_{n\geq 0} \mu_n(E) < \infty \tag{1.4}$$

then the sequence $(\mu_n)_{n\geq 0}$ contains a subsequence which converges weakly to a measure μ on $(E,\mathcal{E})$.

An interested reader may consult [17] for proof of this theorem.

Theorem 1.4 (The Riesz representation Theorem) *Let E be a locally compact space. Let $C_c(E)$ be the set of continuous functions $f : E \to \mathbf{R}$ with compact support. Let*

$$L : C_c(E) \to \mathbf{R}$$

be a nonnegative linear map. Then there exists a $\sigma-$algebra $\mathcal{E}$ on E containing the Borel $\sigma-$algebra of E, and there exists a unique measure μ on $(E,\mathcal{E})$ such that the following properties hold:

1. $L(f) = \int_E f(x)d\mu(x), \qquad \forall f \in C_c(E)$
2. *for any compact subset K of E, $\mu(K) < \infty$*
3. *for any $A \in \mathcal{E}$, $\mu(A) = \inf\{\mu(V) : A \subseteq V, \quad \text{and } V \text{ open in } E\}$*
4. *for any $A \in \mathcal{E}$ such that $\mu(A) < +\infty$,*

$$\mu(A) = \sup\{\mu(K) : K \subseteq A, K \text{ compact}\}$$

A complete and very comprehensive proof of this theorem is given in [19]. Other equivalent forms of this theorem may be found in [4].

1.1.5 Conditional expectation, Independency

Definition 1.17 *Let $(\Omega,\mathcal{F},P)$ be a probability space and $X : \Omega \to \mathbf{R}$ be a random variable. If X is $P-$integrable, then the expectation of X is defined by*

$$E(X) = \int X(w)dP(w).$$

Definition 1.18 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $X : \Omega \rightarrow \mathbf{R}$ a random variable. Let $\mathcal{H}$ be a sub $\sigma-$algebra of $\mathcal{F}$. If X is $P-$integrable then there exists a random variable $Y : \Omega \rightarrow \mathbf{R}$, $\mathcal{H}-$ measurable, such that*

$$\int_B X dP = \int_B Y dP, \qquad \forall B \in \mathcal{H}. \tag{1.5}$$

The class of such random variables is called the conditional expectation of the random variable X with respect to the sub $\sigma-$algebra $\mathcal{H}$ and is denoted $E(X|\mathcal{H})$. If Z and Y are two members of this class then, Y and Z are equal almost surely (a.s.), that is,

$$P\{w \in \Omega : Y(w) = Z(w)\} = 0. \tag{1.6}$$

Formally, we write

$$\int_B X dP = \int_B E(X|\mathcal{H}) dP, \qquad \forall B \in \mathcal{H} \tag{1.7}$$

Here are some properties of the conditional expectation [4], [10].

1. If $X \geq 0$ then
$$E(X|\mathcal{H}) \geq 0. \tag{1.8}$$

2. If X is integrable, then $E(X|\mathcal{H})$ is also integrable and
$$E(E(X|\mathcal{H})) = E(X). \tag{1.9}$$

3. If $\mathcal{H} = \{\emptyset, \Omega\}$, then
$$E(X|\mathcal{H}) = E(X). \tag{1.10}$$

4. If X is $\mathcal{H}-$measurable then
$$E(X|\mathcal{H}) = X \quad a.s. \tag{1.11}$$

5. If $\mathcal{G}$ and $\mathcal{H}$ are two sub $\sigma-$algebra of $\mathcal{F}$ such that $\mathcal{G} \subseteq \mathcal{H}$ then
$$E(E(X|\mathcal{H})|\mathcal{G}) = E(X|\mathcal{G}) \quad a.s. \tag{1.12}$$

6. If X and Y are integrable random variables such that Y is $\mathcal{H}$ - measurable, and XY is integrable then
$$E(XY|\mathcal{H}) = YE(X|\mathcal{H}) \quad a.s. \tag{1.13}$$

Definition 1.19 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $(E, \mathcal{E})$ be a measurable space. Let $X_1 : (\Omega, \mathcal{F}, P) \rightarrow (E, \mathcal{E})$ and $X_2 : (\Omega, \mathcal{F}, P) \rightarrow \mathbf{R}$ be random variables. The conditional expectation of the random variable X_2 given X_1 is defined as follows:*

$$E(X_2|X_1) = E(X_2|\mathcal{H}) \tag{1.14}$$

where $\mathcal{H} = \sigma(X_1)$ *is the sub* $\sigma-$*algebra of* $\mathcal{F}$ *generated by* X_1.

The following result, the detail of which may be found elsewhere [4], is usually used to show that the conditional expectation $E(X_2|X_1)$ can be seen as a function of X_1.

Theorem 1.5 *Let* X *be a random variable defined on a probability space* $(\Omega, \mathcal{F}, P)$ *and taking value in a measurable space* $(E, \mathcal{E})$ *and let* $f : \Omega \to \mathbf{R}$ *be a function. Let* $\mathcal{H}$ *be the* $\sigma-$*algebra generated by* X. *Then* f *is* $\mathcal{H}-$*measurable if and only if there exists a measurable function* $g : (E, \mathcal{E}) \to \mathbf{R}$ *such that*

$$f = g \circ X.$$

Now returning to the notations above, we have that the function

$$X_1 : (\Omega, \mathcal{F}, P) \to (E, \mathcal{E})$$

is measurable and since

$$E(X_2|X_1) : (\Omega, \mathcal{F}, P) \to \mathbf{R}$$

is $\sigma(X_1)-$ measurable then from Theorem 1.5, there exists a measurable function

$$g : (E, \mathcal{E}) \to \mathbf{R}$$

such that

$$E(X_2|X_1) = g \circ X_1 \quad a.s. \tag{1.15}$$

Definition 1.20 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space. Let* $\mathcal{H}$ *be a sub* $\sigma-$ *algebra of* $\mathcal{F}$ *and let* $A \in \mathcal{F}$. *The conditional probability* $P(A|\mathcal{H})$ *of* A *given* $\mathcal{H}$ *is the conditional expectation*

$$E(1_A|\mathcal{H})$$

where 1_A *is the characteristic function of* A *in* Ω.

We have that $P(A|\mathcal{H})$ is a $\mathcal{H}$-measurable random variable such that

$$\int_B P(A|\mathcal{H})dP = P(A \cap B), \qquad \forall B \in \mathcal{H}. \tag{1.16}$$

Definition 1.21 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and* $A_1, A_2, \ldots A_n$ *be elements of* $\mathcal{F}$. *Then* $A_1, A_2, \ldots A_n$ *are said to be independent if for any integer* k *such that* $1 \leq k \leq n$ *and any subset* $\{i_1, i_2, \ldots i_k\}$ *of* $\{1, 2, \cdots, n\}$ *of* k *elements, one has:*

$$P(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \ldots P(A_{i_k}). \tag{1.17}$$

Definition 1.22 *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space and let* $(\mathcal{F}_t)_{t \in T}$ *be a family of sub* $\sigma-$*algebras of* $\mathcal{F}$. *These sub* $\sigma-$ *algebras are said to be independent if for any finite subset*

$\{t_1, t_2, \ldots t_k\}$ of T and distinct elements $A_{t_i} \in \mathcal{F}_{t_i}$, $(i = 1, 2, \ldots k)$, one has:

$$P(A_{t_1} \cap A_{t_2} \cap \ldots \cap A_{t_k}) = P(A_{t_1})P(A_{t_2}) \ldots P(A_{t_k}). \tag{1.18}$$

Definition 1.23 *A family $(X_t)_{t\in T}$ of random variables defined from a probability space $(\Omega, \mathcal{F}, P)$ to a measurable space $(E, \mathcal{E})$ is said to be independent if for any finite subset $\{t_1, t_2, \ldots t_k\}$ of T and distinct elements $A_{t_1}, A_{t_2}, \ldots, A_{t_k}$ of $\mathcal{E}$, one has:*

$$P\{X_{t_1} \in A_{t_1}, X_{t_2} \in A_{t_2}, \ldots, X_{t_k} \in A_{t_k}\} = \prod_{i=1}^{k} P\{X_{t_i} \in A_{t_i}\}. \tag{1.19}$$

The following property obtained from [4] is very useful.

Theorem 1.6 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_1, \mathcal{F}_2, \cdots, \mathcal{F}_n$ be independent sub $\sigma-$algebras of $\mathcal{F}$. Let $X_1, X_2, \ldots, X_n$ be independent and integrable random variables from $(\Omega, \mathcal{F}, P)$ into $\mathbf{R}$ and measurable with respect to $\mathcal{F}_1, \mathcal{F}_2, \ldots, \mathcal{F}_n$ respectively. Then the product $X_1.X_2.\ldots.X_n$ is integrable and*

$$E[X_1.X_2.\ldots.X_n] = E[X_1].E[X_2].\ldots.E[X_n]. \tag{1.20}$$

Definition 1.24 *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be three sub $\sigma-$ algebras of $\mathcal{F}$. The $\sigma-$ algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be conditionally independent given $\mathcal{F}_3$ if for all nonnegative random variables $X_1, X_2 : (\Omega, \mathcal{F}, P) \to \mathbf{R}$, measurable with respect to $\mathcal{F}_1, \mathcal{F}_2$ respectively one has:*

$$E[X_1.X_2|\mathcal{F}_3] = E[X_1|\mathcal{F}_3].E[X_2|\mathcal{F}_3] \quad a.s. \tag{1.21}$$

The following result [4] can be used to check the conditional independency.

Theorem 1.7 *Let $\mathcal{F}_{1,3}$ be the sub $\sigma-$algebra generated by $\mathcal{F}_1$ and $\mathcal{F}_3$. Then the sub $\sigma-$algebras $\mathcal{F}_1, \mathcal{F}_2$ are conditionally independent given $\mathcal{F}_3$ if and only if for any integrable random variable $Y : (\Omega, \mathcal{F}, P) \to \mathbf{R}$ and measurable with respect to $\mathcal{F}_2$ one has*

$$E[Y|\mathcal{F}_{1,3}] = E[Y|\mathcal{F}_3] \quad a.s. \tag{1.22}$$

1.1.6 Kernels

Another key concept that will be helpful in the study of Markov processes and semigroups is that of *kernel.*

Definition 1.25 *Let $(\Omega, \mathcal{F})$ and $(E, \mathcal{E})$ be measurable spaces. A kernel from Ω to E is a function*

$$N : \Omega \times \mathcal{E} \to [0, +\infty]$$

satisfying the following properties.

1. *For any* $x \in \Omega$ *the function* $N(x,.)$ *defined by*

$$N(x,.)(A) = N(x,A), \qquad \forall A \in \mathcal{E}$$

is a measure on $(E,\mathcal{E})$.

2. *For any* $A \in \mathcal{E}$, *the function* $N(.,A)$ *defined by*

$$N(.,A)(x) = N(x,A) \quad \forall x \in \Omega$$

is $\mathcal{F}-$ *measurable.*

The measure $N(x,.)$ will be denoted $N(x,dy)$.
The kernel N is said to be a Markov kernel if

$$N(x,E) = 1, \forall x \in \Omega. \tag{1.23}$$

It is said to be a sub-Markov kernel if

$$N(x,E) \leq 1, \forall x \in \Omega. \tag{1.24}$$

It is said to be bounded if

$$N(x,E) < +\infty, \forall x \in \Omega. \tag{1.25}$$

A sub-Markov kernel can be transformed into a Markov kernel by the following procedure [6],[11].
Let N be a sub-Markov kernel from a measurable space $(E,\mathcal{E})$ into itself. Let $E_\partial = E \cup \{\partial\}$ where ∂ is an element not in E and let $\mathcal{E}_\partial$ be the $\sigma-$algebra on E_∂ generated by

$$\mathcal{E} \cup \{\partial\}.$$

Then the kernel $\tilde{N}$ defined on $(E_\partial, \mathcal{E}_\partial)$ as follows:

for any $x \in E, A \in \mathcal{E}$,

$$\begin{aligned} \tilde{N}(x,A) &= N(x,A) \\ \tilde{N}(x,\{\partial\}) &= 1 - N(x,E) \end{aligned}$$

and for any $A \in \mathcal{E}_\partial$,

$$\tilde{N}(\partial, A) = \begin{cases} 1 & \text{if } \partial \in A \\ 0 & \text{otherwise} \end{cases}$$

is a Markov kernel and is an extension of N. We have the following important facts.

Theorem 1.8 *Let N be a kernel from $(\Omega, \mathcal{F})$ to $(E, \mathcal{E})$ and let $f : E \to [0, +\infty]$ be a nonnegative measurable function. Then the function*

$$Nf : \Omega \to [0, +\infty]$$

defined as follows

$$Nf(x) = \int_E f(y)N(x, dy) \tag{1.26}$$

is $\mathcal{F}$-measurable.
Let μ be a measure on the measurable space $(\Omega, \mathcal{F})$. Then the function

$$\mu N : \mathcal{E} \to [0, +\infty]$$

defined as follows

$$\mu N(A) = \int_\Omega N(x, A)d\mu(x) \tag{1.27}$$

is a measure on the space $(E, \mathcal{E})$.

For the proof of this theorem and more details on kernels, the reader may consult [5], [17].

1.2 Stochastic Processes

1.2.1 Definitions

Definition 1.26 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space. A stochastic process defined on Ω with state space E is a family $(X_t)_{t\in T}$ of random variables defined from $(\Omega, \mathcal{F}, P)$ to $(E, \mathcal{E})$ and indexed by a set T.*

The space $(\Omega, \mathcal{F}, P)$ is called the base space of the process and the space $(E, \mathcal{E})$ is called the state space of the process. The index set T will generally be a subset of $\mathbf{R}_+$. For any $t \in T$, the random variable X_t is called the state of the process at time t.

Definition 1.27 *Let $(X_t)_{t\in T}$ be a stochastic process with base space $(\Omega, \mathcal{F}, P)$ and state space $(E, \mathcal{E})$. Let $J = \{t_1, t_2, \ldots, t_n\}$ be a finite subset of T. There exists a unique measure μ_J on the product space $(E^J, \mathcal{E}^J)$ such that, for any $A_1, A_2, \ldots, A_n \in \mathcal{E}$,*

$$\mu_J\left(\prod_{i=1}^n A_i\right) = P\left(\cap_{i=1}^n X_{t_i}^{-1}(A_i)\right). \tag{1.28}$$

The measure μ_J is called the finite-dimensional distribution of the process $(X_t)_{t\in T}$ corresponding to the subset J. It is the joint distribution of the random variables $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$.

Definition 1.28 *Let $X = (X_t)_{t\in T}$ be a stochastic process. For any $w \in \Omega$, the function $X(w) : T \to E$ such that*

$$X(w)(t) = X_t(w), \quad \forall t \in T$$

is called the path of the process X corresponding to w.

If $T = \mathbf{R}_+$ or any interval of $\mathbf{R}$, E is a topological space and $\mathcal{E}$ its Borel $\sigma-$algebra, we can consider continuity of the paths of the process X since they are maps between two topological spaces. We have the following the following definition.

Definition 1.29 *The process X is continuous if all its paths are continuous almost surely, that is:*

$$P\{w \in \Omega : X(w) \text{ is continuous }\} = 1.$$

1.2.2 Equivalent stochastic processes

Definition 1.30 *Let $(X_t)_{t\in T}$ and $(X_t^{'})_{t\in T}$ be stochastic processes with base spaces $(\Omega, \mathcal{F}, P)$ and $(\Omega^{'}, \mathcal{F}^{'}, P^{'})$ respectively and the same state space $(E, \mathcal{E})$. These stochastic processes are said to be equivalent if they have the same finite-dimensional distributions.*

That is, for any finite sequence $(t_1, t_2, \ldots, t_n)$ of elements of T and any finite sequence $A_1, A_2, \ldots, A_n$ of elements of $\mathcal{E}$,

$$P\{\cap_{i=1}^{n} X_{t_i}^{-1}(A_i)\} = P'\{\cap_{i=1}^{n} X_{t_i}^{'-1}(A_i)\}. \tag{1.29}$$

An example of equivalent stochastic process is as follows [4].
Let $(X_t)_{t\in T}$ be a stochastic process with base space $(\Omega, \mathcal{F}, P)$ and state space $(E, \mathcal{E})$. Consider the product measurable space $(E^T, \mathcal{E}^T)$ where E^T is the set of functions from T into E and $\mathcal{E}^T$ is the $\sigma-$ algebra generated by the coordinate maps

$$\pi_t : E^T \to (E, \mathcal{E})$$

defined by:

$$\pi_t(e) = e_t, \qquad \forall e = (e_s)_{s\in T} \in E. \tag{1.30}$$

Then the map

$$\tau : (\Omega, \mathcal{F}) \to (E^T, \mathcal{E}^T)$$

that maps any element $w \in \Omega$ to its path $(X_t(w))_{t\in T}$ is measurable. Indeed, for any $B \in \mathcal{E}$, we have:

$$\begin{aligned}
\tau^{-1}(\pi_t^{-1}(B)) &= \{w \in \Omega : \tau(w) \in \pi_t^{-1}(B)\} \\
&= \{w \in \Omega : X_s(w)_{s\in T} \in \pi_t^{-1}(B)\} \\
&= \{w \in \Omega : X_t(w) \in B\} \\
&= X_t^{-1}(B).
\end{aligned}$$

Then $\tau^{-1}(\pi_t^{-1}(B) \in \mathcal{F}$ and the property follows since $\mathcal{E}^T$ is generated by the sets of the form $\pi_t^{-1}(B)$ with $B \in \mathcal{E}$.

The map τ defines therefore a probability measure μ on $(E^T, \mathcal{E}^T)$as follows:

$$\mu(M) = P\{w \in \Omega : \tau(w) \in M\} = P\{\tau^{-1}(M)\}. \tag{1.31}$$

Then the coordinate maps $(\pi_t)_{t\in T}$ form a stochastic process with base space $(E^T, \mathcal{E}^T, \mu)$ and state space $(E, \mathcal{E})$.
This process is equivalent to the original process $(X_t)_{t\in T}$. The process $(\pi_t)_{t\in T}$ is called the canonical stochastic process associated with the process $(X_t)_{t\in T}$. It is an easy matter to show that two equivalent stochastic processes have the same canonical process.

1.2.3 Kolmogorov extension Theorem

Let $(X_t)_{t\in T}$ be a stochastic process taking values in $(E, \mathcal{E})$ and given in its canonical form. This means the random variables X_t are considered as the canonical coordinate maps defined from $(E^T, \mathcal{E}^T, \mu)$ to $(E, \mathcal{E})$ where μ is a probability measure on $(E^T, \mathcal{E}^T)$.
Let $A_1, A_2, \ldots A_n$ be elements of $\mathcal{E}$ and let $J = \{t_1, t_2, \ldots, t_n\}$ be a finite subset of T. Let μ_J be the finite-dimensional distribution corresponding to J. Let $t_{n+1}, t_{n+2}, \ldots, t_m, (m \geq n)$ be other elements of T and let

$$K = \{t_1, t_2, \ldots, t_n, t_{n+1}, t_{n+2}, \ldots, t_m\}.$$

Let us take

$$A_{n+1} = A_{n+2} = \cdots = A_m = E.$$

Then

$$\begin{aligned} \mu\left(\cap_{i=1}^m X_{t_i}^{-1}(A_i)\right) &= \mu\left(\cap_{i=1}^n X_{t_i}^{-1}(A_i)\right) \\ &= \mu_J\left(\prod_{i=1}^n A_i\right). \end{aligned}$$

This means that

$$\mu_K\left(A_1 \times A_2 \times \ldots \times A_n \times E \ldots \times E\right) = \mu_J\left(A_1 \times A_2 \times \ldots \times A_n\right). \tag{1.32}$$

Let $\pi_J : E^T \to E^J$ be the canonical projection map defined as follows:

$$\forall e = (e_s)_{s\in T} \in E^T, \pi_J(e) = (e_s)_{s\in J} = (X_s(e))_{s\in J}.$$

It is clear that this map is measurable (with respect to the product $\sigma-$ algebras). If π_J^{-1} is the map from $\mathcal{E}^J$ to $\mathcal{E}^T$ defined as follows:

$$\pi_J^{-1}(A) = \left\{e \in E^T : \pi_J(e) = (e_s)_{s\in J} \in A\right\} \tag{1.33}$$

then

$$\mu \circ \pi_J^{-1} = \mu_J. \tag{1.34}$$

Indeed, let $\mathcal{C}$ be the set of finite unions of subsets of E^J of the form $A_1 \times A_2 \times \ldots \times A_n$ where $A_i \in \mathcal{E}, \forall i$. It is an easy matter to show that $\mathcal{C}$ satisfies the conditions of Theorem 1.2. We have that

$$\begin{aligned}
\mu \circ \pi_J^{-1}(A_1 \times \ldots \times A_n) &= \mu\left(\left\{e \in E^T : (e_s)_{s\in J} \in A_1 \times \ldots \times A_n\right\}\right. \\
&= \mu\left(\left\{e \in E^T : X_{t_i}(e) \in A_i, \quad \forall i \in J\right\}\right) \\
&= \mu\left(\cap_{i=1}^n X_{t_i}^{-1}(A_i)\right) \\
&= \mu_J(A_1 \times \ldots \times A_n)
\end{aligned}$$

and relation (1.34) follows from Theorem 1.2 since $\mathcal{C}$ generates $\mathcal{E}^T$.

Definition 1.31 *Let $(E, \mathcal{E})$ be a measurable space and T be a nonempty set. Let $Fin(T)$ denote the set of all finite nonempty subsets of T. For any $J \in Fin(T)$, let μ_J be a probability measure on the $\sigma-$ algebra $(E^J, \mathcal{E}^J)$. For any $J, K \in Fin(T)$ such that$J \subseteq K$, let*

$$\pi_J : E^T \to E^J, \qquad \pi_J^K : E^K \to E^J$$

be the canonical projection maps.
The family $(\mu_J)_{J\in Fin(T)}$ is said to be a projective family of measures if the equality

$$\mu_J = \mu_K \circ (\pi_J^K)^{-1} \tag{1.35}$$

holds for all $J \subseteq K \in Fin(T)$.

Theorem 1.9 *The finite-dimensional distributions of a stochastic process form a projective family.*

Proof. This is so since $\mu_J = \mu \circ \pi_J^{-1}$ and obviously $\pi_J = \pi_J^K \circ \pi_K$ •

As mentioned in [14], an immediate consequence of this proposition is that a necessary condition for a family of finite-dimensional distributions to be finite-dimensional distributions of a stochastic process is to form a projective family. A necessary and sufficient condition for such a family to be extended to the full distribution of a stochastic process is given by the well known Kolmogorov Theorem. Here, one version of this important theorem is given.

Theorem 1.10 (Kolmogorov extension Theorem) *Let $(E, \mathcal{E})$ be a measurable space where E is a separable locally compact metric space and $\mathcal{E}$ its Borel $\sigma-$algebra. Let T be an index set. For any finite nonnempty subset J of T, let μ_J be a probability measure on the measurable space $\left(E^J, \mathcal{E}^J\right)$. Then there exists a unique probability measure μ on $\left(E^T, \mathcal{E}^T\right)$ such that*

$$\mu_J = \mu \circ \pi_J^{-1}$$

if and only if the measures μ_J form a projective family.

A proof of this theorem as well as other versions of this theorem can be found elsewhere [4], [14].

Another key notion in the theory of stochastic processes is that of adapted stochastic processes. This notion will be briefly discussed in the next section.

1.2.4 Adapted stochastic processes

Let $(\Omega, \mathcal{F})$ be a measurable space and let $(\mathcal{F}_t)_{t\in T}$ be a family of sub $\sigma-$algebras of $\mathcal{F}$. This family is said to form a filtration of $(\Omega, \mathcal{F})$ if

$$\mathcal{F}_s \subseteq \mathcal{F}_t, \quad \forall s \leq t \in T.$$

The sub $\sigma-$algebra $\mathcal{F}_t$ is usually interpreted as the set of events prior to time t.

Definition 1.32 *Let $(X_t)_{t\in T}$ be a stochastic process on a measurable space $(\Omega, \mathcal{F})$ with state space $(E, \mathcal{E})$ and let $(\mathcal{F}_t)_{t\in T}$ be a filtration of $(\Omega, \mathcal{F})$. The process $(X_t)_{t\in T}$ is said to be adapted to the filtration $(\mathcal{F}_t)_{t\in T}$ if for any $t \in T$, the random variable X_t is $\mathcal{F}_t-$measurable.*

An obvious example is given by any stochastic process $(X_t)_{t\in T}$ with base space $(\Omega, \mathcal{F})$ and state space $(E, \mathcal{E})$ where the sub $\sigma-$ algebra $\mathcal{F}_t$ is the $\sigma-$algebra

$$\sigma\left(X_s : s \leq t\right)$$

generated by the family $(X_s)_{s\leq t}$. This filtration is called the natural filtration of the process $(X_t)_{t\in T}$.

We are now ready to define a Markov process.

1.3 Markov Processes

1.3.1 Markov property

Roughly speaking, the Markov property of a stochastic process means that the past and the future are conditionally independent given the present. This section is intended to formalize this idea.

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space. Let $(X_t)_{t\in T\subseteq \mathbf{R}_+}$ be a stochastic process with base space $(\Omega, \mathcal{F}, P)$ and state space $(E, \mathcal{E})$. Assume $(X_t)_t$ is adapted to a filtration $(\mathcal{F}_t)_{t\in T}$ of $\mathcal{F}$.

Definition 1.33 *The process* $(X_t)_{t\in T}$ *is said to be a Markov process with respect to the filtration* $(\mathcal{F}_t)_{t\in T}$ *if it verifies the following property called the Markov Property: for any* $t > s$ *in* T*, and for any bounded measurable function* $f : E \to \mathbf{R}$*,*

$$E[f \circ X_t|\mathcal{F}_s] = E[f \circ X_t|X_s] \quad a.s. \tag{1.36}$$

Since the conditional expectation $E[X|Y]$ of a random variable X given another random variable Y is a function of Y, the Markov property can also be stated as follows:
for any $t > s$ *in* T*, and for any bounded measurable function* $f : E \to \mathbf{R}$*, there exists a measurable function* $g : E \to \mathbf{R}$ *such that:*

$$E[f \circ X_t|\mathcal{F}_s] = g \circ X_s \quad a.s. \tag{1.37}$$

This means that:

$$\int_A f \circ X_t dP = \int_A g \circ X_s dP, \quad \forall A \in \mathcal{F}_s. \tag{1.38}$$

In the case where no filtration is explicitly given, the natural filtration is the one to be considered.
In what follows, some equivalent statements of the Markov property are given in order to have a full understanding of its probability meaning [6], [11]. But before giving them, we need to state some assumptions.

Let $(X_t)_{t\in T}$ be a stochastic process on a probability space $(\Omega, \mathcal{F}, P)$, taking values in a measurable space $(E, \mathcal{E})$ and let $(\mathcal{F}_t)_{t\in T}$ be its natural filtration, that is

$$\mathcal{F}_s = \sigma(X_t : t \leq s), \forall s \in T.$$

Let

$$\mathcal{G}_s = \sigma(X_s : t \geq s), s \in T.$$

The $\sigma-$ algebras $\mathcal{F}_s$ and $\mathcal{G}_s$ are said to represent respectively the past and the future of the process since their events are determined by the process up to time s and after time s respectively. The $\sigma-$algebra generated by X_s represents the present (of course at time s). Under these assumptions, we have:

Theorem 1.11 *The following statements are equivalent:*

1. *The process* $(X_t)_{t\in T}$ *is Markovian.*

2. *The past* $\mathcal{F}_s$ *and the future* $\mathcal{G}_s$ *are independent conditional on the present.*

3. *For any* $s \geq 0, A \in \mathcal{F}_s, B \in \mathcal{G}_s$

$$P(A \cap B|X_s) = P(A|X_s).P(B|X_s).$$

4. For any $s \geq 0, B \in \mathcal{G}_s$

$$P(B|\mathcal{F}_s) = P(B|X_s).$$

5. For any $s \geq 0, A \in \mathcal{F}_s$

$$P(A|\mathcal{G}_s) = P(A|X_s).$$

Proof. It is important to note from Theorem 1.7, that Property (2) in Theorem 1.11 is equivalent to the following: for any function $h : \Omega \to \mathbf{R}$, $\mathcal{G}_s-$measurable and integrable, one has

$$E[h|\mathcal{F}_s] = E[h|X_s] \quad a.s. \tag{1.39}$$

since the $\sigma-$ algebra generated by the past $\mathcal{F}_s$ and the present $\sigma(X_s)$ is clearly equal to $\mathcal{F}_s$.

1. (2) $\Rightarrow$ (1). Suppose (2) is true. For any bounded and measurable function $f : (E, \mathcal{E}) \to \mathbf{R}$ and any $t \geq s$, the function $f \circ X_t$ is bounded and $\mathcal{G}_s-$measurable since X_t is one of the variables generating $\mathcal{G}_s$. Hence it is integrable since each measurable and bounded function is obviously integrable. Then from equation (1.39), one has

$$E[f \circ X_t|\mathcal{F}_s] = E[f \circ X_t|X_s] \quad a.s.$$

and hence (1) is also true.

2. (1) $\Rightarrow$ (2). Let V be the set of bounded measurable and integrable functions $h : (\Omega, \mathcal{F}) \to \mathbf{R}$ such that

$$E[h|\mathcal{F}_s] = E[h|X_s] \quad a.s. \tag{1.40}$$

and let C be the set of functions of the form

$$(f_1 \circ X_{t_1}).(f_2 \circ X_{t_2}) \ldots (f_n \circ X_{t_n}) \tag{1.41}$$

where n is a positive integer, $t_1, t_2, \ldots, t_n$ are real numbers such that

$$s \leq t_1 \leq t_2 \leq \ldots \leq t_n$$

and $f_1, f_2, \ldots, f_n$ are bounded and measurable functions from $(E, \mathcal{E})$ to $\mathbf{R}$. It is clear that V is a vector space and verifies all the properties of Theorem 1.1. The only no obvious property is that C is a subset of V and this is proved by induction as follows. Let

$$h = (f_1 \circ X_{t_1}).(f_2 \circ X_{t_2}) \ldots (f_n \circ X_{t_n}) \in C.$$

(i) If $n = 1$, and $t_1 = s$, then

$$E[f_1 \circ X_s|\mathcal{F}_s] = f_1 \circ X_s = E[f_1 \circ X_s|X_s]$$

since $f_1 \circ X_s$ is $\mathcal{F}_s$ - measurable. Otherwise if $t_1 > s$, then

$$E[f_1 \circ X_{t_1} | \mathcal{F}_s] = E[f_1 \circ X_{t_1} | X_s]$$

by the Markov property. Then for $n = 1$, we have that $h \in V$.
(ii) Suppose that the property holds for $n-1$. Let us show that it holds for n. Define

$$h' = E[h | \mathcal{F}_{t_{n-1}}].$$

Then

$$\begin{aligned} E[h|\mathcal{F}_s] &= E\left[E\left[h|\mathcal{F}_{t_{n-1}}\right]|\mathcal{F}_s\right] \quad a.s. \quad \text{since } \mathcal{F}_s \subseteq \mathcal{F}_{t_{n-1}} \text{ (see equation (1.12))} \\ &= E[h'|\mathcal{F}_s] \qquad a.s. \end{aligned}$$

By the same argument,

$$E[h|X_s] = E[h'|X_s] \qquad a.s. \tag{1.42}$$

Since all the functions $(f_1 \circ X_{t_1}), (f_2 \circ X_{t_2}), \ldots, (f_{n-1} \circ X_{t_{n-1}})$ are measurable with respect to $\mathcal{F}_{t_{n-1}}$ then

$$\begin{aligned} h' &= E\left[(f_1 \circ X_{t_1}).(f_2 \circ X_{t_2}) \ldots (f_n \circ X_{t_n})|\mathcal{F}_{t_{n-1}}\right] \\ &= (f_1 \circ X_{t_1}).(f_2 \circ X_{t_2}) \ldots (f_{n-1} \circ X_{t_{n-1}}) E\left[f_n|\mathcal{F}_{t_{n-1}}\right] \\ &= (f_1 \circ X_{t_1}).(f_2 \circ X_{t_2}) \ldots (f_{n-1} \circ X_{t_{n-1}}) E\left[f_n|X_{t_{n-1}}\right] \end{aligned}$$

where the last equality follows from the Markov property.
Since $E\left[f_n|X_{t_{n-1}}\right]$ can be written as $g \circ X_{t_{n-1}}$ where $g : E \to \mathbf{R}$ is a bounded and $\mathcal{E}-$measurable function, it follows that

$$h' = (f_1 \circ X_{t_1}).(f_2 \circ X_{t_2}) \ldots ((f_{n-1}.g) \circ X_{t_{n-1}}).$$

Therefore $h' \in V$ since there are only $n-1$ factors. Thus,

$$E[h|\mathcal{F}_s] = E[h'|\mathcal{F}_s] = E[h'|X_s] \qquad (\text{since } h' \in V)$$

and hence from equation (1.42), we have

$$E[h|\mathcal{F}_s] = E[h|X_s]$$

and therefore $h \in V$.
By Theorem 1.1, the space V contains all bounded integrable $\sigma(C)-$measurable functions $h : \Omega \to \mathbf{R}$. It is clear that $\sigma(C) \subseteq \mathcal{G}_s$ since X_t is measurable with respect to $\mathcal{G}_s$ for any

$t \geq s$. Also, $\mathcal{G}_s \subseteq \sigma(C)$ since for any $A \in \mathcal{E}$

$$X_t^{-1}(A) = (1_A \circ X_t)^{-1}(\{1\}) \in \sigma(C)$$

meaning that X_t is $\sigma(C)-$measurable, for any $t \geq s$. It follows that for any function $h : \Omega \to \mathbf{R}$ that is $\mathcal{G}_s-$ measurable and integrable, one has

$$E[h|\mathcal{F}_s] = E[h|X_s] \quad a.s.$$

3. (2) $\Rightarrow$ (3) is straightforward. For any $A \in \mathcal{F}_s, B \in \mathcal{G}_s$,we have

$$\begin{aligned} P(A \cap B|X_s) &= E[1_{A\cap B}|X_s] \\ &= E[1_A.1_B|X_s] \\ &= E[1_A|X_s].E[1_B|X_s] \\ &= P(A|X_s).P(B|X_s) \end{aligned}$$

4. (3) $\Rightarrow$ (4). Indeed, let $A \in \mathcal{F}_s, B \in \mathcal{G}_s$. Then

$$E[1_A.P(B|X_s)|X_s] = P(B|X_s).E[1_A|X_s] \quad a.s.$$

since $P(B|X_s)$ is measurable with respect to $\sigma(X_s)$. Then

$$\begin{aligned} E[1_A.P(B|X_s)|X_s] &= P(B|X_s).P(A|X_s) \\ &= P(A \cap B|X_s) \quad a.s. \\ &= E[1_{A\cap B}|X_s] \quad a.s. \end{aligned}$$

By taking expectation of both sides, we obtain that:

$$E[1_A.P(B|X_s)] = E[1_A.1_B] \qquad \text{since } 1_{A\cap B} = 1_A.1_B$$

and then, from the definition of conditional expectation, it follows that

$$E[P(B|X_s)|\mathcal{F}_s] = E[1_B|\mathcal{F}_s] \quad a.s.$$

Since the random variable $P(B|X_s)$ is measurable with respect to $\mathcal{F}_s$, this equality implies that

$$P(B|X_s) = P(B|\mathcal{F}_s) \quad a.s.$$

as desired.

5. (3) $\Rightarrow$ (5). Here we have just to exchange the role of A and B in step 4.

6. (4) $\Rightarrow$ (2). Let $h : \Omega \to \mathbf{R}$ be a bounded and measurable function. If h is a nonnegative

step function, say $h = \sum_{i=1}^n a_i 1_{A_i}$ (where $a_i \in \mathbf{R}, A_i \in \mathcal{G}_s, \forall i$), then

$$\begin{aligned} E[h|\mathcal{F}_s] &= \sum_{i=1}^n a_i E[1_{A_i}|\mathcal{F}_s] \\ &= \sum_{i=1}^n a_i P(A_i|\mathcal{F}_s) \\ &= \sum_{i=1}^n a_i P(A_i|X_s) \\ &= E[h|X_s]. \end{aligned}$$

If h is nonnegative then it is known that h can be represented as a limit of a nondecreasing sequence (h_n) of simple functions [9] and then

$$E[h|\mathcal{F}_s] = E[\lim h_n|\mathcal{F}_s] = \lim E[f_n|\mathcal{F}_s] = E[h|X_s].$$

If h is not nonnegative, then h is a sum of two nonnegative measurable functions and the property follows immediately.
7. The same argument can be used to show that (5) $\Rightarrow$ (2) and we are done •

Another important characteristic of Markov processes is their transition functions. We discuss these functions in the next section.

1.3.2 Transition function of a Markov process

Definition 1.34 *A transition function on a measurable space $(E, \mathcal{E})$ is a family $(P_{s,t})_{0\le s\le t}$ of sub-Markovian kernels on $(E, \mathcal{E})$ such that for any $A \in \mathcal{E}$ and any $x \in E$, the following properties hold.*
1. For any $t \ge 0$,

$$P_{t,t}(x, A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases} \tag{1.43}$$

2. For any nonnegative real numbers $s \le t \le u$

$$P_{s,u}(x, A) = \int_E P_{s,t}(x, dy) P_{t,u}(y, A). \tag{1.44}$$

Definition 1.35 *Let $(X_t)_{t\ge 0}$ be a Markov process taking value in a measurable space $(E, \mathcal{E})$. A transition function $(P_{s,t})_{0\le s\le t}$ on $(E, \mathcal{E})$ is said to be a transition function of the Markov Process $(X_t)_{t\ge 0}$ if for any bounded function $f : E \to \mathbf{R}$ and for any nonnegative real numbers s, t such that $s < t$ one has:*

$$(P_{s,t} f) \circ X_s = E[f \circ X_t | X_s] \quad a.s. \tag{1.45}$$

A transition function $(P_{s,t})_{0\leq s\leq t}$ is said to be homogeneous if $P_{s,t}$ depends only on the length $t-s$ of the interval $[s,t]$. This means that if

$$t-s=t^{'}-s^{'},$$

then

$$P_{s,t}=P_{s^{'},t^{'}}.$$

In this case, $P_{s,t}$ is simply denoted by P_{t-s}. In what follows only homogeneous transition functions will be considered.
A Markov transition function with homogeneous transition function is termed a homogeneous Markov process.
It is pointed out in [6], that it is hard to construct a transition function from a Markov process and that in practice, Markov processes are usually given with corresponding transition functions. However, given a transition function on particular measurable spaces, it is possible to construct a Markov process that admits it as its transition function. The process is summarized in the following section.

1.3.3 Realisation of a transition function

Let $(X_t)_{t\geq 0}$ be a homogeneous Markov process on a probability space $(\Omega,\mathcal{F},P)$ taking value in a measurable space $(E,\mathcal{E})$ with respect to a filtration $(\mathcal{F}_t)$ with transition function (P_t). Let μ be distribution (or probability law) of X_0. It is called the initial distribution of the process. Then the distribution μ_t of X_t is given by

$$\mu_t=\mu P_t \tag{1.46}$$

This means that

$$\mu_t(A)=P\{X_t\in A\}=\int_E P_t(x,A)d\mu(x),\quad \forall A\in\mathcal{E}. \tag{1.47}$$

Indeed, for any $A\in\mathcal{E}$, we have from equation (1.9) that:

$$E[1_A\circ X_t]=E[E[1_A\circ X_t]|X_0]. \tag{1.48}$$

But clearly

$$E[1_A\circ X_t]=E[1_{X_t^{-1}(A)}]=P\{X_t\in A\}.$$

Since (P_t) is the transition function of the Markov process (X_t), from equation (1.45) we have that

$$E[1_A\circ X_t|X_0]=(P_t1_A)\circ X_0\quad a.s.$$

Then

$$P\{X_t\in A\}=E[(P_t1_A)\circ X_0].$$

For any Borel set B of $\mathbf{R}$, we have that:

$$\begin{aligned} P\{w \in \Omega : P_t.1_A \circ X_0(w) \in B\} &= P\{X_0^{-1}((P_t.1_A)^{-1}(B))\} \\ &= \mu((P_t.1_A)^{-1}(B)) \\ &= \mu\{x \in E : P_t.1_A(x) \in B\}. \end{aligned}$$

Then

$$E[(P_t 1_A) \circ X_0] = \int_E P_t.1_A(x) d\mu(x)$$

and since

$$P_t.1_A(x) = \int_E P_t(x, dy) 1_A(y) = P_t(x, A)$$

it follows that

$$E[(P_t 1_A) \circ X_0] = \int_E P_t(x, A) d\mu(x).$$

Therefore

$$P\{X_t \in A\} = \int_E P_t(x, A) d\mu(x)$$

as desired.
This means that if the transition function of a Markov process is given along with its initial distribution, then we can determine the distributions of the variables $X_t, t > 0$.

The finite-dimensional distributions of the process are also expressed in terms of the transition function and the initial distribution in the following way.
Let $0 = t_0 < t_1 < \ldots < t_n$ be real numbers and $X_{t_0}, X_{t_1}, \ldots, X_{t_n}$ be the corresponding random variables. The distribution $\mu_{t_0, t_1, \ldots, t_n}$ of $(X_{t_0}, X_{t_1}, \ldots, X_{t_n})$ on the measurable space $(E^{n+1}, \mathcal{E}^{n+1})$ is such that, for any $A_0, A_1, \ldots A_n \in \mathcal{E}$,

$$\begin{aligned} \mu_{t_0, t_1, \ldots, t_n}(A_0 \times A_1 \times \ldots \times A_n) = \int_{A_0} \mu(dx_0) \cdot \int_{A_1} P_{t_1}(x_0, dx_1) \ldots \\ \int_{A_n} P_{t_n - t_{n-1}}(x_{n-1}, dx_n). \end{aligned} \qquad (1.49)$$

These finite-dimensional distributions are consistent in the sense that for any real numbers $0 = t_0 < t_1 < \ldots < t_n < t_{n+1} < \ldots < t_m$ and any $A_0, A_1, \ldots A_n \in \mathcal{E}$, we have

$$P\{\cap_{i=0}^{n} X_{t_i}^{-1}(A_i)\} = P\{X_{t_0} \in A_0, X_{t_1} \in A_1, \ldots, X_{t_n} \in A_n, X_{t_{n+1}} \in E, \ldots, X_{t_m} \in E\}.$$

Conversely let E be a separable locally compact metric space and $\mathcal{E}$ its Borel $\sigma-$algebra. Let $(P_t)_{t \geq 0}$ be a transition function on $(E, \mathcal{E})$ and μ be a probability measure on E. Consider now the product measurable space $(E^T, \mathcal{E}^T)$ where T stands for the set of nonnegative real numbers. The coordinate maps $(X_t)_{t \geq 0}$ defined from E^T into E by $X_t(w) = w(t)$ for any $w \in E^T$ form a stochastic process.
Then for any subset $J = \{t_0, t_1, \ldots, t_n\}$ of T with $0 = t_0 < t_1 < \ldots < t_n$, let $\pi_J : E^T \to E^J$ be the projection map and let μ_J be the measure defined on $(E^J, \mathcal{E}^J)$ such that:

$$\mu_J(A_0 \times A_1 \times \ldots \times A_n) = \int_{A_0} \mu(dx_0) \cdot \int_{A_1} P_{t_1}(x_0, dx_1) \ldots$$
$$\int_{A_n} P_{t_n - t_{n-1}}(x_{n-1}, dx_n), \quad A_i \in \mathcal{E}, \forall i.$$

These finite-dimensional distributions form a projective system. from the Kolmogorov extension theorem (Theorem 1.10), the projective limit measure P exists and is such that is such that

$$\mu_J = P \circ \pi_J^{-1}. \tag{1.50}$$

This means that:

$$\mu_J(A) = P(\pi_J^{-1}(A)), \forall A \in \mathcal{E}^J.$$

The measure P is the unique probability measure on the measurable space $(E^T, \mathcal{E}^T)$ such that $(X_t)_{t\geq 0}$ is a Markov process on $(E^T, \mathcal{E}^T, P)$, with transition function (P_t) and initial measure μ [6], [11].
The Markov process (X_t) is called a realization of the transition function (P_t).

A particular case is to consider that the initial distribution is centred at a particular point x of E, that is

$$\mu_0(A) = \delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

The process $(X_t)_{t\geq 0}$ is said to start at the point x. We have that for any $t > 0$ and any $A \in \mathcal{E}$,

$$P\{X_t \in A\} = \int_A P_t(x, dy) = P_t(x, A). \tag{1.51}$$

Then $P_t(x, A)$ has the intuitive meaning to be the probability that the state of the process will be in A at time t given that it starts at position x.
For more details on transition functions of Markov processes, the reader may consult [6], [7], [11].

1.4 Semigroups on Banach spaces

1.4.1 Notions from Functional Analysis

Definition 1.36 *A Banach space is a normed vector space E defined on* **R** *or* **C** *in which each Cauchy sequence converges.*

Definition 1.37 *Let E and F be Banach spaces. A linear operator from E into F is a pair $(\mathcal{D}(T), T)$ where $\mathcal{D}(T)$ is a subspace of E and T is a linear map from $\mathcal{D}(T)$ to F that* *is,*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y), \forall \alpha, \beta \in \mathbf{C}, x, y \in \mathcal{D}(T). \tag{1.52}$$

The subspace $\mathcal{D}(T)$ is called the domain of the operator and the subspace

$$R(T) = \{T(x) : x \in \mathcal{D}(T)\}$$

is called the range of the operator.

Definition 1.38 *A linear operator $(\mathcal{D}(T), T)$ from E into F is bounded if there exists a constant C such that $\|T(x)\| \leq C\|x\|$ for any $x \in \mathcal{D}(T)$.*

If $(\mathcal{D}(T), T)$ is a bounded linear operator then its norm is defined as follows:

$$\|T\| = \sup\{\|T(x)\| \ : x \in \mathcal{D}(T) \text{ and } \|x\| = 1\}. \tag{1.53}$$

It is important to notice that

$$\|T(x)\| \leq \|T\|\|x\|, \qquad \forall x \in \mathcal{D}(T). \tag{1.54}$$

Definition 1.39 *A linear operator $(\mathcal{D}(T), T)$ from E into F is said to be closed if its graph*

$$\Gamma(T) = \{(x, T(x)) : x \in \mathcal{D}(T)\} \tag{1.55}$$

is a closed subset of the space $E \times F$ endowed with the topology defined by the norm

$$\|(x, y)\| = \|x\| + \|y\|. \tag{1.56}$$

This operator is said to be closable if it can be extended to a closed operator. Any closable operator has a minimum closed extension [12].

Theorem 1.12 *(i) A linear operator $(\mathcal{D}(T), T)$ is closed if and only if for any sequence (x_n) of $\mathcal{D}(T)$ such that (x_n) converges to x and $T(x_n)$ converges to y, $x \in \mathcal{D}(T)$ and $T(x) = y$.*
(ii) A linear operator $(\mathcal{D}(T), T)$ is closable if and only if for any sequence (x_n) of $\mathcal{D}(T)$ such that (x_n) converges to 0 we have that $T(x_n)$ converges to 0 or does not converge.

Proof. Part (i) follows from the fact that the product $E \times F$ of two Banach spaces E and F is also a Banach space by the norm (1.56). Part (ii) is now a consequence of (i). Indeed, define the operator

$$(\mathcal{D}(\bar{T}), \bar{T})$$

as follows.
For any sequence (x_n) of $\mathcal{D}(T)$ such that (x_n) converges to x and $T(x_n)$ converges to y we put

$$x \in \mathcal{D}(\bar{T}) \text{ and } \bar{T}(x) = y.$$

Then it is clear that the linear operator

$$(\mathcal{D}(\bar{T}), \bar{T})$$

is the minimal closed extension of $(\mathcal{D}(T), T)$.

Conversely let $\bar{T}$ be a closed extension of T and let (x_n) be a sequence in $\mathcal{D}(T)$ that converges to zero. If $\bar{T}(x_n) = T(x_n)$ converges to y, then we have $\bar{T}(0) = y$ since $\bar{T}$ is closed. And by linearity, we get $y = 0$ •

Corollary 1.1 *If an operator $(\mathcal{D}(T), T)$ is closable and $(\mathcal{D}(T), T)$ is its minimal closed extension, then*

$$\begin{aligned} \Gamma(\bar{T}) &= \overline{\Gamma(T)} \\ R(\bar{T}) &\subseteq \overline{R(T)} \end{aligned}$$

Proof. Let $(x, y) \in \Gamma(\bar{T})$, then $x \in \mathcal{D}(\bar{T})$ and $\bar{T}(x) = y$. Hence there exists a sequence (x_n) in $\mathcal{D}(T)$ that converges to x and such that $T(x_n) \to y$. Then we have that

$$(x_n, T(x_n)) \in \Gamma(T)$$

and $(x, y) = \lim(x_n, T(x_n))$. It follows that

$$(x, y) \in \overline{\Gamma(T)}.$$

Conversely, let

$$(x, y) \in \overline{\Gamma(T)}.$$

There exists a sequence $(x_n, T(x_n))$ in $\Gamma(T)$ that converges to (x, y). This means that x_n converges to x and $T(x_n)$ converges to y. Then $\bar{T}(x) = y$ and hence $(x, y) \in \Gamma(\bar{T})$ and we have proven (i).
(ii) can be proven in a similar way •

Definition 1.40 *Let E be a Banach space on the field $\mathbf{C}$ of complex numbers. The resolvent of a linear operator $(\mathcal{D}(T), T)$ on E is the set of linear operators*

$$R_\lambda(T) = (T - \lambda I)^{-1}, \quad \lambda \in \mathbf{C}$$

where the operator $(T - \lambda I)$ is one-to-one.
The resolvent set of the operator $(\mathcal{D}(T), T)$ is the set of complex numbers λ such that

$R_\lambda(T)$ exists (that is $(T-\lambda I)$ is one-to-one), is bounded and its domain is a dense subset of E.
The spectrum of the operator $(\mathcal{D}(T),T)$ is the complement of its resolvent set.

Remark 1.1 (particular case: bounded linear maps) *If the domain of a linear operator $(\mathcal{D}(T),T)$ from E into F is known to be equal to E, the term linear map will be used instead of linear operator.*

Let E and F be Banach spaces. The set $\mathcal{L}(E,F)$ of bounded linear maps from E into F is a Banach space with norm

$$\|T\| = \sup\{\|T(x)\| : x \in E \text{ and } \|x\| = 1\}. \tag{1.57}$$

A linear map T from E ito F is said to be continuous if for any sequence (x_n) of E that converges to x, the sequence $(T(x_n)$ converges to $T(x)$ in F. A linear map is continuous if and only it is bounded.

Definition 1.41 *Let $(\mathcal{D}(T),T)$ be a bounded linear operator on a Banach space E. The exponential operator e^T is defined as follows*

$$e^T = \sum_{k=0}^{\infty} \frac{1}{k!} T^k \tag{1.58}$$

It is also a bounded linear operator and verifies the following property

$$\|e^{\mathcal{A}}\| \leq e^{\|\mathcal{A}\|}. \tag{1.59}$$

Furthermore if T_1 and T_2 are two bounded linear operators such that

$$T_1T_2 = T_2T_1.$$

then

$$e^{T_1+T_2} = e^{T_1}e^{T_2} \tag{1.60}$$

For more details on these matters the reader is referred to [11], [12].

1.4.2 Definition of Semigroups

Definition 1.42 *Let E be a Banach space. A semigroup on E is a family $(T_t)_{t\geq 0}$ of bounded linear maps on E such that the following two properties hold.*

1. *For any t, $s \geq 0$, $T_{t+s} = T_t.T_s$.*

2. *T_0 is the identity map*

A semigroup $(T_t)_{t\geq 0}$ is said to be strongly continuous if

$$\lim_{t\downarrow 0}\|T_t(x)-x\|=0, \quad \forall x\in E. \tag{1.61}$$

If in addition, operators T_t are contractions, that is,

$$\|T_t\|\leq 1, \qquad \forall t\geq 0, \tag{1.62}$$

then $(T_t)_{t\geq 0}$ is said to be a strongly continuous contraction semigroup.

Definition 1.43 *Let $(T_t)_{t\geq 0}$ be a strongly continuous semigroup on a Banach space E. The infinitesimal generator of the semigroup $(T_t)_{t\geq 0}$ is the linear operator $(\mathcal{D}(\mathcal{U}),\mathcal{U})$ defined on E as follows:*

$$\mathcal{D}(\mathcal{U}) = \left\{x\in E:\lim_{t\downarrow 0}\frac{1}{t}(T_t(x)-x) \text{ exists in } E\right\} \text{ and} \tag{1.63}$$

$$\mathcal{U}(x) = \lim_{t\downarrow 0}\frac{1}{t}\left(T_t(x)-x\right). \tag{1.64}$$

As we will see later on, the infinitesimal generator is an important characteristic of a semigroup on a Banach space. Here some properties of the infinitesimal generator of a semigroup are given[11], [12].

Theorem 1.13 *For any strongly continuous semigroup $(T_t)_{t\geq 0}$ on a Banach space E with infinitesimal generator $(\mathcal{D}(\mathcal{U}),\mathcal{U})$ and for all $t>s, x\in E$, the following properties hold.*

1. $\lim_{t\downarrow s}\|T_t(x)-T_s(x)\|=0$.

2. $$\lim_{h\downarrow 0}\frac{1}{h}\int_t^{t+h}T_s(x)ds=T_t(x). \tag{1.65}$$

3. $\int_0^t T_s(x)ds\in\mathcal{D}(\mathcal{U})$ *and*

$$\mathcal{U}\left(\int_0^t T_s(x)ds\right)=T_t(x)-x. \tag{1.66}$$

4. *If $x\in\mathcal{D}(\mathcal{U})$, then $T_t(x)\in\mathcal{D}(\mathcal{U})$and*

$$\frac{d}{dt}T_t(x) = \mathcal{U}(T_t(x))=T_t(\mathcal{U}(x)) \tag{1.67}$$

$$T_t(x)-T_s(x) = \mathcal{U}\left(\int_s^t T_h(x)dh\right)=\int_s^t T_h(\mathcal{U}(x))dh. \tag{1.68}$$

Corollary 1.2 *The infinitesimal generator of a strongly continuous contraction semigroup $(T_t)_{t\geq 0}$ defined on a Banach space E is a closed linear operator and its domain is a dense subset of E.*

Proof. From Theorem 1.13, we have that for any $x \in E$,

$$\frac{1}{t}\int_0^t T_s(x)ds \in \mathcal{D}(\mathcal{U})$$

and

$$\lim_{t\downarrow 0}\frac{1}{t}\int_0^t T_s(x)ds = T_0(x) = x.$$

Now let (t_n) be a sequence of positive real numbers that converges to 0 and let

$$y_n = \frac{1}{t_n}\int_0^{t_n} T_s(x)ds.$$

The sequence (y_n) of elements of $\mathcal{D}(\mathcal{U})$ converges to x. This means that $\mathcal{D}(\mathcal{U})$ is dense in E. The closeness follows from the fact that if (x_n) is a sequence of $\mathcal{D}(\mathcal{U})$ that converges to x and if $\mathcal{U}(x_n)$ converges to y then

$$\begin{aligned}
\frac{1}{t}(T_t(x) - x) &= \frac{1}{t}(T_t(\lim_{n\to\infty} x_n) - \lim_{n\to\infty} x_n) \\
&= \lim_{n\to\infty}\frac{1}{t}(T_t(x_n) - x_n) \\
&= \frac{1}{t}\lim_{n\to\infty}\int_0^t T_s\mathcal{U}(x_n)ds \\
&= \frac{1}{t}\int_0^t T_s(y)ds \quad \text{since } \mathcal{U}(x_n)\to y.
\end{aligned}$$

Then

$$\mathcal{U}(x) = \lim_{t\downarrow 0}\frac{1}{t}(T_t x - x) = \lim_{t\downarrow 0}\frac{1}{t}\int_0^t T_s(y)ds = y \quad \bullet$$

1.4.3 The Hille-Yosida Theorem

The Hille-Yosida Theorem provides necessary and sufficient conditions for a linear operator to be an infinitesimal generator of a strongly continuous contraction semigroup.

Let $(T_t)_{t\geq 0}$ be a contraction strongly continuous semigroup on a Banach space E. It is shown in [11] that any real $\lambda > 0$, for any $x \in E$, the integral

$$\int_0^\infty e^{-\lambda t}T_t(x)dt \tag{1.69}$$

exists in E since

$$\int_0^\infty \|e^{-\lambda t}T_t(x)\|dt \leq \frac{\|x\|}{\lambda}$$

In addition, the map R_λ defined on E by

$$R_\lambda(x) = \int_0^\infty e^{-\lambda t}T_t(x)dt \tag{1.70}$$

is linear and is bounded as we have

$$\|R_\lambda\| \leq \frac{1}{\lambda}. \tag{1.71}$$

Hence R_λ is bounded.

Definition 1.44 *The set of the linear operators $R_\lambda, \lambda \geq 0$ is called the resolvent of the semigroup $(T_t)_{t\geq 0}$.*

The following property justifies the name resolvent.

Theorem 1.14 *Let $(\mathcal{D}(\mathcal{U}), \mathcal{U})$ be the infinitesimal generator of a strongly continuous semigroup $(T_t)_{t\geq 0}$ on a Banach space E. Then for any real number $\lambda > 0$, the operator*

$$\lambda I - \mathcal{U}$$

is a bijection from $\mathcal{D}(\mathcal{U})$ onto E and

$$(\lambda I - \mathcal{U})^{-1} = R_\lambda. \tag{1.72}$$

Proof. For any $x \in E$ and any $h \geq 0$,

$$\begin{aligned} T_h(R_\lambda(x)) - R_\lambda(x) &= T_h \int_0^\infty e^{-\lambda t} T_t(x)dt - \int_0^\infty e^{-\lambda t} T_t(x)dt \\ &= \int_0^\infty e^{-\lambda t} T_{t+h}(x)dt - \int_0^\infty e^{-\lambda t} T_t(x)dt \\ &= (e^{\lambda h} - 1)\int_h^\infty e^{-\lambda t} T_t(x)dt - \int_0^h e^{-\lambda t} T_t(x)dt \end{aligned}$$

Dividing by h and taking the limit when $h \to 0$, it follows that

$$\mathcal{U}(R_\lambda(x)) = \lambda R_\lambda(x) - x.$$

Therefore

$$R_\lambda(x) \in \mathcal{D}(\mathcal{U})$$

and

$$(\lambda I - \mathcal{U})R_\lambda = I.$$

Moreover, for any $x \in \mathcal{D}(\mathcal{U})$, one has that

$$R_\lambda(\mathcal{U}(x)) = \int_o^\infty e^{-\lambda t} d(T_t(x))$$

since

$$T_t(\mathcal{U}(x)) = \mathcal{U}(T_t(x)) = \frac{d(T_t(x))}{dt}.$$

This gives, after integrating by parts,

$$R_\lambda \mathcal{U}(x) - \lambda R_\lambda(x) = -x.$$

And hence

$$R_\lambda(\lambda I - \mathcal{U}) = I,$$

that is

$$R_\lambda = (\lambda I - \mathcal{U})^{-1} \quad \bullet$$

Theorem 1.15 *Let $(\mathcal{D}(\mathcal{U}), \mathcal{U})$ be a linear operator on a Banach space E. Suppose that the following two conditions hold.*

1. *$\mathcal{U}$ is a closed operator and its domain $\mathcal{D}(\mathcal{U})$ is dense in E.*
2. *For any real number $\lambda > 0$, λ belongs to the resolvent set of the operator $\mathcal{U}$ and verifies*
$$\|(\lambda I - \mathcal{U})^{-1}\| \leq \frac{1}{\lambda}.$$

Let $R_\lambda = \lambda I - \mathcal{U}$. Then $\mathcal{U}$ has the following properties.

1. *For any $x \in E$,*
$$\lim_{\lambda \to \infty} \lambda R_\lambda(x) = x. \tag{1.73}$$
2. *For any real number $\lambda > 0$ the operator*
$$\mathcal{U}_\lambda = \lambda \mathcal{U} R_\lambda \tag{1.74}$$
is bounded and verifies
$$\lim_{\lambda \to \infty} \mathcal{U}_\lambda(x) = \mathcal{U}(x), \quad \forall x \in \mathcal{D}(\mathcal{U}). \tag{1.75}$$
3. *For any $\lambda > 0$, the family $(T_t^\lambda)_{t \geq 0}$ defined by*
$$T_t^\lambda = e^{t \cdot \mathcal{U}_\lambda} \tag{1.76}$$
is a strongly continuous contraction semigroup on E and $\mathcal{U}_\lambda$ is its infinitesimal generator.
4. *For any $t \geq 0$,*
$$\lim_{\lambda \to \infty} T_t^\lambda(x) \in E, \quad x \in E.$$
5. *The map $T_t, t \geq 0$ defined on E as follows*
$$T_t(x) = \lim_{\lambda \to \infty} T_t^\lambda(x) = \lim_{\lambda \to \infty} e^{t \cdot \mathcal{U}_\lambda}(x)$$

is linear and bounded. Furthermore, the the family $(T_t)_{t\geq 0}$ is a strongly continuous contraction semigroup and $\mathcal{U}$ is its infinitesimal generator.

The operator T_t of this theorem is usually denoted $e^{\mathcal{U}t}$.
A proof of this Theorem can be found elsewhere [11]. We can now state the Hille-Yosida Theorem:

Theorem 1.16 (Hille-Yosida Theorem) *A linear operator $(\mathcal{D}(\mathcal{U}),\mathcal{U})$ on a Banach space E is an infinitesimal generator of a strongly continuous semigroup of contractions on E if and only if the following two conditions hold.*

1. *$\mathcal{U}$ is a closed operator and its domain is dense in E.*

2. *Any positive real number λ belongs to the resolvent set of the operator $\mathcal{U}$ and verifies:*

$$\|(\lambda I - \mathcal{U})^{-1}\| \leq 1/\lambda.$$

The fact that the conditions are necessary is clear from Corollary 1.2 and Theorem 1.14 and the fact that the conditions are sufficient as well as the procedure to construct the semigroup is given by Theorem 1.15.
Other versions of the Hille-Yosida theorem are given elsewhere [6], [12].

In the next subsection, we introduce particular semigroups which will be used in the sequel.

1.4.4 Feller semigroups

Definition 1.45 *Let E be a locally compact metric space. Let ∂ be a point not in E and*

$$E_\partial = E \cup \{\partial\}.$$

Define a topology $\mathcal{T}$ on E_∂ by

$$\mathcal{T} = \{O, O \cup \{\partial\} : O \textit{ is an open subset of } E\}$$

Endowed with this topology, the space E_∂ is compact and is called the one-point Alexandrov compactification of E.

Let E be a locally compact metric space and let E_∂ be its one-point Alexandrov compactification of E. Let $C_b(E)$ be the Banach space of bounded continuous functions $f : E \to \mathbf{R}$. Let $C_0(E)$ denote the set of functions $f \in C_b(E)$ such that

$$\lim_{x\to\partial} f(x) = 0$$

in the sense that, for any $\epsilon > 0$, there exists a compact subset $K \subseteq E$ such that

$$|f(x)| < \epsilon, \quad \forall x \in E - K.$$

It is clear that $C_0(E)$ is a closed subspace of $C_b(E)$ and therefore a Banach space.

Definition 1.46 *A semigroup $(T_t)_{t\geq 0}$ on the space $C_0(E)$ is called a Feller semigroup on E if it is strongly continuous, non-negative and contractive, that is: for any $f \in C_0(E)$ one has:*

1. $\lim_{t\downarrow 0} \|T_t f - f\| = 0$,
2. *if $f \geq 0$, then $T_t f \geq 0, \quad \forall t \geq 0$,*
3. $\|T_t\| \leq 1$.

If the space E is compact, then $C_0(E)$ is identified with $C_b(E)$ [17]. We will denote simply $C(E)$ since each continuous function on a compact space is bounded. It is this particular case that will be considered in the sequel.

Theorem 1.17 (The Hille-Yosida-Ray theorem) *Let E be a locally compact and separable metric space and A be a linear operator on $C_b(E)$.*

1. *If*

   the domain $\mathcal{D}(A)$ of A is dense in $C_b(E)$ and there exists an open and dense subset F of E such that for any $u \in \mathcal{D}(A)$, if u attains its positive maximum at a point $x_0 \in F$ then $Au(x_0) \leq 0$

   then the operator A has a closed extention in $C_b(E)$.

2. *If*

   for any $u \in \mathcal{D}(A)$ that takes its positive maximum at a point $x' \in E$ one has $Au(x') \leq 0$ and there exists a constant $\alpha_0 \geq 0$ such that the range of the operator $\alpha_0 I - A$ is dense in $C_b(E)$,

   then the minimum closed extension of A is the infinitesimal generator of a Feller semigroup on E.

For the proof of this theorem and more details on these matters, the reader is referred to [6],[17].

1.5 Notions on Differential Manifolds

Notions and results on differential manifolds contained into $\mathbf{R}^n$ that will be used in the sequel are discussed in this section.

1.5.1 Definitions and Examples of Maninfolds

Definition 1.47 *Let U and V be open subsets of $\mathbf{R}^n$. A map $f : U \to V$ is called a diffeomorphism of class C^k if it is bijective, $k-$continuously differentiable and its inverse f^{-1} is also $k-$continuously differentiable.*

Definition 1.48 *Let M be a topological space and let n, k be nonnegative integers with $k \geq 1$. An $n-$dimensional atlas of class C^k on M is a family of pairs $\{(U_i, \varphi_i)\}_{i\in I}$ such that the following properties hold.*

1. *For any $i \in I, U_i$ is an open subset of M and $\cup_{i\in I} U_i = M$.*
2. *For any $i \in I, \varphi_i$ is an homeomorphism from U_i onto an open subset of $\mathbf{R}^n$.*
3. *For any $i, j \in I$, the map*

$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

   is a diffeomorphism of class C^k.

Any pair (U_i, φ_i) of an atlas is called a *chart*. If $x \in U_i$, then (U_i, φ_i) is said to be a chart at x. If

$$\varphi_i(x) = (x_1(x), x_2(x), \ldots, x_n(x)), \forall x \in U_i,$$

the functions $x_1, x_2, \ldots, x_n$ are called the local coordinates associated with this chart. Two $n-$dimensional atlases of same class on M are said to be compatible if their union is also another n-dimensional atlas of the same class on M. The compatibility relation in the set of all atlases on M is an equivalence relation [2].

Definition 1.49 *A differential manifold of dimension n of class C^k is a pair $(M, \mathcal{A})$ of a topological space M and an equivalent class $\mathcal{A}$ of $n-$dimensional atlases of class C^k on M.*

In practice, a differential manifold is defined by taking just one atlas and considering the structure defined by its equivalence class. A chart of a differential manifold is a chart of an atlas equivalent to the fixed atlas.

Example 1.1 *For any positive integer n, $\mathbf{R}^n$ is a $C^\infty-$ differential manifold of dimension n with the obvious atlas $(\mathbf{R}^n, id)$ where id is the identity map in $\mathbf{R}^n$.*

In general any open subset U of $\mathbf{R}^n$ is a $C^\infty-$ differential manifold of dimension n with the obvious atlas (U, id_U) (called canonical atlas on U). We will always assume that any open subset of $\mathbf{R}^n$ is endowed with its canonical differential structure.
Let S^n be the unit sphere of $\mathbf{R}^{n+1}$, that is

$$S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbf{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1\}.$$

Define $U_1 = S^n - \{(0,0,\ldots,0,1)\}$ and $U_2 = S^n - \{(0,0,\ldots,0,-1)\}$ and the maps $\varphi_1 : U_1 \to \mathbf{R}^n$ and $\varphi_2 : U_2 \to \mathbf{R}^n$ as follows:

$$\begin{aligned}
\varphi_1(x_1, x_2, \ldots, x_{n+1}) &= \frac{1}{1 - x_{n+1}}(x_1, x_2, \ldots, x_n), \\
\varphi_2(x_1, x_2, \ldots, x_{n+1}) &= \frac{1}{1 + x_{n+1}}(x_1, x_2, \ldots, x_n).
\end{aligned}$$

It is an easy matter to show that $\{(U_1, \varphi_1), (U_2, \varphi_2)\}$ is a $C^\infty-$atlas of dimension n on S^n and hence S^n is a $C^\infty-$differential manifold of dimension n. This atlas is called the canonical atlas of S^n.

Definition 1.50 *Let M be a differential manifold of class C^k and dimension n and let p be a nonnegative integer such that $p \leq k$. A map $f : M \to \mathbf{R}$ is said to be a differentiable function of class C^p (or a C^p-function) if for any x in M, there exists a chart (U, φ) of M at x such that the function*

$$f \circ \varphi^{-1} : \varphi(U) \to \mathbf{R}$$

is differentiable of class C^p.

The set of C^p-differentiable functions on M is denoted $C^p(M)$.
If $p = 0$, $C^0(M)$ is simply denoted $C(M)$ and it is the set of continuous functions on M.

One should note that for a function defined on an open subset U of $\mathbf{R}^n$, differentiability in the sense of differential manifold is equivalent to differentiability in the common sense.

1.5.2 Tangent space

Definition 1.51 *Let M be a $n-$dimensional differential manifold of class C^k. Let $x \in M$ and (U, φ) be a chart at x with coordinate system $(x_1, x_2, \ldots, x_n)$. A curve of M at the point x is a map $\gamma : I \to M$ where I is an open interval of $\mathbf{R}$ containing 0 such that the function $\varphi \circ \gamma$ is C^k-differentiable at 0. Let $C_x(M)$ be the set of curves of M at x. In this set, the relation $\sim$ defined as follows:*

$$\gamma \sim \beta \iff \frac{d(\varphi \circ \gamma)}{dt}(0) = \frac{d(\varphi \circ \beta)}{dt}(0) \tag{1.77}$$

is an equivalence relation and it is independent of the choice of the chart. The equivalence class of the curve γ is denoted $\gamma'(0)$. The quotient space is called the tangent space of M at the point x and denoted T_xM. Its elements are called tangent vectors of M at x. If $V \in T_xM$ and $V = \gamma'(0)$, then the vector V is said to be tangent to the curve γ at the point x.

This construction does not change if we consider any interval I that does not contain 0 and take derivatives at t such that $\gamma(t) = x$. But we will use the first approach for the sake of simplification.

The choice of the chart (U, φ) induces a map from $h : T_xM \to \mathbf{R}^n$. Indeed, let $V \in T_x(M)$ and suppose that V is tangent to a curve γ. Let

$$h(V) = \frac{d(\varphi \circ \gamma)}{dt}(0) = \left(\frac{d(x_1 \circ \gamma)}{dt}(0), \dots, \frac{d(x_n \circ \gamma)}{dt}(0) \right). \tag{1.78}$$

The map h is well defined and it is bijective [16]. We can now transfer the vector structure of $\mathbf{R}^n$ to T_xM by taking

$$V + V' = h^{-1}(h(V) + h(V'); \quad \alpha V = h^{-1}(\alpha h(V)), \quad V, V' \in T_xM, \quad \alpha \in \mathbf{R}.$$

Furthermore this structure is independent of the choice of the chart. The vector space T_xM is of dimension n.
Elements of T_xM can also be seen as derivatives [16].
Suppose that M is of class C^∞. A derivative on M at x is a function $D : C^\infty(M) \to \mathbf{R}$ such that

$$D(f.g) = f(x).D(g) + g(x).D(f), \qquad \forall f, g, \in C^\infty(M).$$

Let us denote $\mathcal{D}_x$ the set of all derivatives on M at x. Consider the map $L : T_xM \to \mathcal{D}_x$ defined as follows: for any $V \in T_xM$ (tangent to the curve γ at x), $L(V)$ is the derivative such that

$$L(V)(f) = \frac{d(f \circ \gamma)}{dt}(0), \quad f \in C^\infty(M).$$

This map is well defined, bijective and linear. More precisely, for any $D \in \mathcal{D}_x$, if (U, φ) is such that $\varphi(x) = (0, 0, \dots, 0)$, then

$$D(f) = \frac{d(f \circ \gamma)}{dt}(0), \quad f \in C^\infty(M)$$

where γ is a curve on M at x such that

$$\varphi \circ \gamma(t) = (tD(x_1), tD(x_2), \dots, tD(x_n)). \tag{1.79}$$

This means that $L(V) = D$ where V is a tangent vector to the curve γ at x. For this reason, T_xM is identified with $(D)_x$ by identifying $L(V)$ and V. Consider now the curves $\gamma_i \quad (i = 1, 2, \dots, n)$ defined as follows:

$$x_j \circ \gamma_i(t) = \delta_i^j.t, \quad \forall j = 1, 2, \dots, n. \tag{1.80}$$

Then the tangent vector V defined by γ_i is such that

$$\begin{aligned} L(V)(f) &= \frac{d(f \circ \gamma)}{dt}(0) \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x) \frac{dx_i}{dt}(0) \\ &= \frac{\partial f}{\partial x_i}(x). \end{aligned}$$

This means that

$$L(V) = \left(\frac{\partial}{\partial x_i} \right)_x$$

where the indice x indicates that the derivative is taken at the point x.
Furthermore, the system

$$B = \left\{ \left(\frac{\partial}{\partial x_1} \right)_x, \left(\frac{\partial}{\partial x_2} \right)_x, \ldots, \left(\frac{\partial}{\partial x_n} \right)_x \right\} \tag{1.81}$$

forms a basis of $T_x M$.
Any vector $V \in T_x M$ tangent to γ can be expressed as:

$$V = \sum_{i=1}^{n} \frac{dx_i}{dt}(0) \left(\frac{\partial}{\partial x_i} \right)_x . \tag{1.82}$$

The dual of $T_x M$ is called the cotangent space and is denoted $T_x^* M$. The dual basis of B is simply denoted

$$B^* = \{dx_1, d_2, dx_n\}.$$

Example 1.2 *Consider the $C^\infty-$differential structure defined on the sphere S^2 in Example 1.1 and the curve*

$$\gamma : (-\pi/6, \pi/6) \to S^2, \qquad \textit{given by } \gamma(t) = (\cos 2t \sin 3t, \sin 2t \sin 3t, \cos 3t).$$

This curve passes through the point $x = (0, 0, 1)$.

We have that in the chart (U_2, φ_2),

$$\varphi_2 \circ \gamma(t) = \frac{1}{1 + \cos 3t}(\cos 2t \sin 3t, \sin 2t \sin 3t).$$

Then

$$\begin{aligned} \frac{dx_1}{dt}(0) &= 3, \\ \frac{dx_2}{dt}(0) &= 0. \end{aligned}$$

The tangent vector to γ at this point is

$$V = \gamma^{'}(0) = 3\left(\frac{\partial}{\partial x_1}\right)_x.$$

For further discussion on differential manifolds, the reader may refer to [2], [16].

1.6 Partial differential equations

1.6.1 Definitions

Notations For any $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbf{N}^n$ (called a multi-index), $|\alpha|, D_j^{\alpha_j}$ and D^α are defined as follows:

$$\begin{aligned} |\alpha| &= \alpha_1 + \alpha_2 + \ldots + \alpha_n, \\ D_j^{\alpha_j} &= \frac{\partial^{\alpha_j}}{\partial x_j^{\alpha_j}}, \\ D^\alpha &= D_1^{\alpha_1} D_2^{\alpha_2} \ldots D_n^{\alpha_n}. \end{aligned}$$

Definition 1.52 *Let Ω be an open subset of $\mathbf{R}^n$. Let u and f be functions from Ω to $\mathbf{R}$. A partial differential equation in u is an equation of the form*

$$F(x_1, \ldots, x_n, u, D^\alpha u, D^\beta u, \ldots) = f(x_1, x_2, \ldots, x_n) \tag{1.83}$$

where F is a function taking value in $\mathbf{R}$ and $\alpha, \beta, \ldots,$ are multi-index and u is the unknown function.

The order of this equation is the highest derivative that appears in it. A solution of this equation is any function u that makes it an identity.

In general for practical reasons the unknown function u is assumed to verify some boundary or initial conditions and the problem of finding such a function is called a boundary value problem.
We will be interested by linear equations of order two which are briefly discussed in the next section.

1.6.2 Linear partial differential equation

Definition 1.53 *A second order linear partial differential equation is an equation of the form:*

$$\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u = f \tag{1.84}$$

where a_{ij}, b_i, c, f are functions defined on an open subset Ω of $\mathbf{R}^n$ and $a_{ij} = a_{ji}$.

Let us consider the differential expression

$$L(x,.) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x) \tag{1.85}$$

$L(x,.)$ is a linear operator on the space $C(\Omega)$ of continuous functions $f : \Omega \to \mathbf{R}$. It is called a differential operator of order two [17]. We have that

$$L(x, i\xi) = -\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j + i\sum_{i=1}^{n} b_i(x)\xi_i + c(x) \tag{1.86}$$

where $\xi \in \mathbf{R}^n$.
The principal part of $L(x, i\xi)$ is

$$Q(\xi) = -\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j, \forall \xi \in \mathbf{R}^n \tag{1.87}$$

and it is a quadratic form on $\mathbf{R}^n$.
The equation (1.84) is said to be *elliptic* if the quadratic form Q is strictly definite, that is

$$\forall \xi \in \mathbf{R}^n - \{0\}, Q(\xi) > 0 \quad \text{ or } \forall \xi \in \mathbf{R}^n - \{0\}, Q(\xi) < 0$$

It is said to be *hyperbolic* if Q is indefinite, that is, it vanishes only for $\xi = 0$ and changes the sign in $\mathbf{R}^n$. It is *parabolic* if the quadratic form Q is degenerate, that is, there exists a nonzero vector at which Q vanishes.
The equation is elliptic if and only if all eigenvalues of the matrix (a_{ij}) are different from zero and have the same sign, it is hyperbolic if some of them are negative and other are positive but no one is null, and is parabolic if and only if at least one eigenvalue is null. An interested reader is invited to consult [12] for more details on Linear partial differential equations.

1.6.3 Existence Theorem for elliptic bounded value problems

Definition 1.54 *Let Ω be a subset of $\mathbf{R}^n$ and let θ be a real number such that $0 < \theta < 1$. A function $u : \Omega \to \mathbf{R}$ is said to be Hölder continuous on Ω with exponent θ if*

$$\sup_{x,y\in\Omega, x\neq y} \frac{|u(x) - u(y)|}{|x - y|^\theta}$$

is finite. A function u is locally Hölder continuous on Ω with exponent θ if it is Hölder continuous with exponent θ on any compact subset of Ω.

The following definitions and notations will be used in the sequel.

1. $C^k(\Omega)$ is the set of functions $u : \Omega \to \mathbf{R}$ such that $D^\alpha u$ is continuous in Ω for any $\alpha \in \mathbf{N}^n$ such that $|\alpha| \leq k$,

2. $C^{\theta}(\Omega)$ is the set of locally Hölder continuous functions on Ω with exponent θ.

3. $C^{k+\theta}(\Omega)$ is the set of functions $u \in C^k(\Omega)$ such that their k-th derivatives are in $C^{\theta}(\Omega)$.

4. $C^k(\bar{\Omega})$ is the set of functions $u \in C^k(\Omega)$ such that $D^{\alpha}u$ can be continuously extended to $\bar{\Omega}$, for any $\alpha \in \mathbf{N}^n$ such that $|\alpha| \leq k$.

5. $C^{k+\theta}(\bar{\Omega})$ is the set of functions $u \in C^k(\overline{\Omega})$ such that for any $\alpha \in \mathbf{N}^n$ such that $|\alpha| = k$, $D^{\alpha}u \in C^{\theta}(\bar{\Omega})$.

If Ω is bounded then, $C^{k+\theta}(\bar{\Omega})$ is a Banach space with the norm

$$\|u\| = \sup_{x\in\bar{\Omega}, |\alpha|\leq k} |D^{\alpha}u(x)| + \sup_{|\alpha|=k} [D^{\alpha}u].$$

where

$$[D^{\alpha}u] = \sup_{x,y\in\overline{\Omega}, x\neq y} \frac{|u(x) - u(y)|}{|x-y|^{\theta}}.$$

Definition 1.55 *A subset Ω of $\mathbf{R}^n$ is a domain if it is open and connected. A bounded domain Ω of $\mathbf{R}^n$ is said to be of class C^k $(0 \leq k \leq \infty)$ if for any $x \in \partial\Omega$ there exists an open neighborhood U of x in $\mathbf{R}^n$ such that $U \cap \partial\Omega$ is the graph of a C^k- differentiable function of $n-1$ variables from the canonical variables $x_1, x_2, \ldots, x_n$ of U.*

Any open ball of $\mathbf{R}^n$ is a $C^{\infty}-$domain. If D is a $C^{\infty}-$domain its boundary ∂D is a $C^{\infty}-$differential manifold.

The following theorem [15], [17] gives conditions for existence and uniqueness of a solution for an elliptic linear bounded problem.

Theorem 1.18 (Existence and uniqueness of solution) *Let Ω be a bounded domain of $\mathbf{R}^n$ and let $\partial\Omega$ be its boundary. Let*

$$L(x,.) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)$$

be a differential operator of order two with real coefficients such that there exists $\theta \in (0,1)$ with the following properties:

1. $a_{ij} \in C^{\theta}(\Omega)$,

2. $a_{ij} = a_{ji}$,

3. *there exists* $a_0 > 0$ *such that*

$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a_0\|\xi\|^2, \quad \forall x \in \Omega, \xi \in \mathbf{R}^n,$$

4. $b_i \in C^\theta(\Omega)$,

5. $c \in C^\theta(\Omega)$ *and* $c(x) \leq 0, \forall x \in \Omega$.

Let f *and* ϕ *be functions defined in* Ω *and* $\partial\Omega$ *respectively. Then the so-called Dirichlet problem of finding a function* u *defined on the closure* $\overline{\Omega}$ *such that*

$$\begin{cases} L(x,u) = f & \text{in } \Omega \\ u = \phi & \text{on } \partial\Omega \end{cases} \tag{1.88}$$

is such that:

> *if the domain* Ω *is of class* C^2 *and if* $f \in C^\theta(\Omega)$ *and* $\phi \in C(\partial\Omega)$*, then the it has a unique solution*
>
> $$u \in C(\overline{\Omega}) \cap C^{2+\theta}(\Omega).$$

Chapter 2

Relationships between Markov processes, Semigroups and Partial differential equations

2.1 Markov Processes and Semigroups

In this section relationships between Markov processes and Semigroups are investigated. Throughout this section, any Markov process is defined by its transition function. It will be shown later that the Chapman-Kolmogorov relation expresses in some sense the semigroup property and that Markov Processes can be constructed from some special semigroups. All Markov processes are supposed to be homogeneous and indexed by the set of nonnegative real numbers.

2.1.1 Semigroups associated to Markov Processes

Theorem 2.1 *Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(E, \mathcal{E})$ be a measurable space. Let $(X_t)_{t\geq 0}$ be a Markov process on $(\Omega, \mathcal{F}, P)$, taking value in $(E, \mathcal{E})$ with transition function $(P_t)_{t\geq 0}$. Let $b\mathcal{E}$ be the Banach space of bounded and measurable functions $f : (E, \mathcal{F}) \to \mathbf{R}$. For any $t \geq 0$, let $T_t : b\mathcal{E} \to b\mathcal{E}$ be the map defined as follows:*

$$T_t(f)(x) = \int_E f(y) P_t(x, dy), \quad \forall f \in b\mathcal{E}, x \in E. \tag{2.1}$$

Then T_t is a linear map and the family $(T_t)_{t\geq 0}$ is a contraction semigroup on the Banach space $b\mathcal{E}$.

Proof. The linearity of T_t is an obvious property of the integral and the measurability follows from the fact that P_t is a kernel (see Theorem 1.8). It is also clear that T_0 is the identity map. Let $s, t \geq 0, f \in b\mathcal{E}, x \in E$. Then

$$T_{s+t} f(x) = \int_E f(y).P_{t+s}(x, dy).$$

Since

$$\forall A \in \mathcal{E}, \quad P_{t+s}(x, A) = \int_E P_t(x, dz).P_s(z, A),$$

then

$$\begin{aligned} T_{s+t}f(x) &= \int_E f(y). \int_E P_s(x, dz).P_t(z, dy) \\ &= \int_E P_s(x, dz) \int_E f(y)P_t(z, dy) \\ &= \int_E P_s(x, dz)T_t f(z) \\ &= T_s(T_t f)(x). \end{aligned}$$

Therefore

$$T_{s+t} = T_s.T_t.$$

We have also that:

$$\begin{aligned} |T_t f(x)| &= \left| \int_E f(y)P_t(x, dy) \right| \\ &\leq \int_E |f(y)|P_t(x, dy) \\ &\leq \|f\| \int_E P_t(x, dy) \\ &\leq \|f\|P_t(x, E) \\ &\leq \|f\| \qquad \text{since } P_t(x, E) \leq 1 \end{aligned}$$

It follows that

$$\|T_t f\| = \sup_{x \in E} |T_t f(x)| \leq \|f\|$$

and hence

$$\|T_t\| = \sup_{\|f\|=1} \|T_t f\| \leq 1$$

which means that T_t is a contraction •

As particular case, we consider the so-called *Feller transition functions* described in the section below.

Definition 2.1 *Let E be a locally compact metric space and $\mathcal{E}$ its Borel $\sigma-$algebra. A Feller transition function on E is a homogeneous transition function $(P_t)_{t \geq 0}$ on the measurable space $(E, \mathcal{E})$ such that for any $t \geq 0$ and any bounded and continuous function $f : E \to \mathbf{R}$ the function $T_t f : E \to \mathbf{R}$ defined by*

$$T_t f(x) = \int_E f(y)P_t(x, dy) \tag{2.2}$$

is continuous.

The following property from [17] builds a gap between a Feller transition function and a semigroup.

Theorem 2.2 *The operators associated to a Feller transition function form a contraction semigroup on the space $C_b(E)$ of bounded continuous functions from E into* $\mathbf{R}$.

Proof. Since a continuous function $f : E \to \mathbf{R}$ is measurable with respect to the Borel $\sigma-$algebras, we have that

$$C_b(E) \subseteq b\mathcal{E}.$$

Furthermore, since

$$T_t f \in C_b(E), \quad \forall f \in C_b(E)$$

we conclude from Theorem 2.1 that $(T_t)_{t\geq 0}$ is a contraction semigroup on the space $C_b(E)$ •

The question that can be raised now is that of how to associate a transition function to a semigroup. This is the subject matter of the next section.

2.1.2 Markov Processes associated to Semigroups

Theorem 2.3 *Let E be a separable compact metric space and let $(T_t)_{t\geq 0}$ be a nonnegative and contraction semigroup on $C(E)$. There exists a unique Feller transition function $(P_t)_{t\geq 0}$ on E such that*

$$T_t f(x) = \int_E P_t(x, dy) f(y) \qquad \forall f \in C(E), \forall x \in E. \tag{2.3}$$

Proof. Let $t \geq 0$ and $x \in E$ and let us consider the function $L : C(E) \to \mathbf{R}$ defined as follows: by

$$L(f) = T_t f(x) \tag{2.4}$$

It is clear that L is linear and nonnegative since T_t is linear and nonnegative. Since E is compact, for any $f \in C(E)$, the support of f is compact. Then we have that the set of continuous functions on E with compact support is equal to $C(E)$, that is

$$C_c(E) = C(E).$$

From the Riesz representation Theorem (Theorem 1.4) there exists a unique measure $P_t(x, dy)$ on $(E, \mathcal{E})$ ($\mathcal{E}$ being the Borel $\sigma-$algebra on E) such that the following properties hold.

1. For any $f \in C(E)$,

$$L(f) = \int_E f(y) P_t(x, dy).$$

2. For any compact subset K of E,

$$P_t(x, K) < \infty.$$

3. For any $A \in \mathcal{E}$,

$$P_t(x, A) = \inf\{P_t(x, V) : A \subseteq V, \quad V \text{ open in } E\}.$$

4. For any $A \in \mathcal{E}$ such that $P_t(x, A) < +\infty$,

$$P_t(x, A) = \sup\{P_t(x, K) : K \subseteq A, K \text{ compact}\}.$$

Then for any $t \geq 0$, P_t is a map from $E \times \mathcal{E}$ into $[0, +\infty]$.
Let us first show that the family $(P_t)_{t \geq 0}$ is a transition function.
1. For any $f \in C(E)$,

$$\begin{aligned} |L(f)| &= |T_t f(x)| \leq \|T_t f\| \\ &\leq \|T_t\|.\|f\| \\ &\leq \|f\|. \end{aligned}$$

Then $|L(f)| \leq \|f\|$ and in particular, since L is nonnegative, we have that:

$$L(1) = |L(1)| \leq \|1\| = 1.$$

Hence,

$$P_t(x, E) = \int_E P_t(x, dy) = L(1) \leq 1.$$

2. Let $A \in \mathcal{E}$, let us show that the function

$$P_t(., A) : (E, \mathcal{E}) \to \mathbf{R}$$

is measurable.
For any measurable function

$$f : (E, \mathcal{E}) \to \mathbf{R},$$

let $P_t(., f) : E \to \mathbf{R}$ be the function defined as follows:

$$P_t(x, f) = \int_E P_t(x, dy) f(y). \tag{2.5}$$

Let us first show that $P_t(., f)$ is measurable.
(i) If f is continuous, then

$$P_t(x, f) = T_t f(x), \quad \forall x \in E.$$

This means that

$$P_t(.,f) = T_t f$$

and therefore

$$P_t(.,f) \in C(E)$$

by definition of T_t. It follows that $P_t(.,f)$ is measurable.
(ii) In the general case, consider the set V of functions $f : E \to \mathbf{R}$ such that $P_t(.,f)$ is measurable and the set C of functions $h : E \to \mathbf{R}$ that can be written as a finite product $f_1.f_2 \ldots f_n$ of continuous functions from E into $\mathbf{R}$. It is clear that C and V verify all the conditions of the Monotone class Theorem (Theorem 1.1). Then V contains all the functions from E to $\mathbf{R}$ measurable with respect to the $\sigma-$algebra $\sigma(C)$ generated by C. But $\sigma(C) = \mathcal{E}$. Indeed, for any continuous function $g : E \to \mathbf{R}$, the $\sigma-$algebra $\sigma(g)$ generated by g is contained into $\mathcal{E}$ and since the product of continuous functions is also continuous, it follows that $\sigma(C) \subseteq \mathcal{E}$.

Conversely, let A be an open subset of E. Consider the sequence $(f_n)_{n\geq 1}$ of functions defined from E to $\mathbf{R}$ as follows [17]:

$$f_n(x) = \min\{n.d(x, E - A), 1\}$$

where d is the distance defined on E. It is clear that all the functions f_n are continuous and the sequence (f_n) is nondecreasing. Furhtermore, the limit f of this sequence is the characteristic function of the subset A in E. Also, we have that:

$$f_n^{-1}(\{1\}) \subseteq f_m^{-1}(\{1\}), \text{ for } m \geq n$$

and

$$A = \cup_{n=1}^{\infty} f_n^{-1}(\{1\}).$$

Since f_n is continuous for any n, it follows that

$$f_n^{-1}(\{1\}) \in \sigma(C)$$

and hence

$$A \in \sigma(C).$$

It follows that $\sigma(C) \supseteq \mathcal{E}$ and therefore $\sigma(C) = \mathcal{E}$.
We conclude that V contains all $\mathcal{E}-$ measurable functions and hence for any $\mathcal{E}-$measurable function $f : E \to \mathbf{R}$, the function $P_t(.,f)$ is also $\mathcal{E}-$measurable.
In particular for any $A \in \mathcal{E}$ the function $P_t(.,A) = P_t(.,1_A)$ is $\mathcal{E}-$ measurable.
3. It remains to verify the Kolmogorov-Chapman equation. Here we will refer to the Riesz representation Theorem (Theorem 1.4).

For any $s, t \geq 0, x \in E, f \in C(E)$,

$$\begin{aligned}
\int_E P_{s+t}(x, dz) f(z) &= T_{t+s} f(x) \\
&= T_t(T_s f)(x) \\
&= \int_E T_s f(y) P_t(x, dy) \\
&= \int_E P_t(x, dy) \int_E f(z) P_s(y, dz) \\
&= \int_E \left[\int_E P_t(x, dy) . P_s(y, dz) \right] f(z).
\end{aligned}$$

Then for any $f \in C(E)$,

$$T_{t+s} f(x) = \int_E \left[\int_E P_t(x, dy) . P_s(y, dz) \right] f(z). \tag{2.6}$$

Consider now the map $\mu : \mathcal{E} \to \mathbf{R}$ defined as follows:

$$\mu(A) = \int_E P_t(x, dy) . P_s(y, A) \tag{2.7}$$

It is obvious that this map is a measure and from equation (2.6) we have that

$$T_{t+s} f(x) = \int_E f(z) \mu(dz). \tag{2.8}$$

Therefore

$$\int_E f(z) \mu(dz) = \int_E P_{s+t}(x, dz) f(z). \tag{2.9}$$

Let us verify that the measure μ satisfies the assumptions of the Riesz representation Theorem (Theorem 1.4).

1. For any compact subset K of E,

$$\begin{aligned}
\mu(K) &= \int_E P_t(x, dy) P_s(y, K) \leq \int_E P_t(x, dy) P_s(y, E) \\
&\leq \int_E P_t(x, dy) . 1 \quad \text{(since } P_s(y, E) \leq 1) \\
&\leq P_t(x, E) \\
&\leq 1
\end{aligned}$$

   and we have that $\mu(K) < \infty$.

2. For any $A \in \mathcal{E}$,

$$P_t(x, A) = \inf\{P_t(x, V) : A \subseteq V, \quad V \text{ open in } E\}$$

   Then

$$\mu(A) = \int_E P_t(x, dy) P_s(y, A)$$

$$
\begin{aligned}
&= \int_E P_t(x,dy).\inf\{P_s(y,V) : A \subseteq V, \quad V \text{ open in } E\} \\
&= \inf\left\{\int_E P_t(x,dy)P_s(y,V) : A \subseteq V, \quad V \text{ open in } E\right\} \\
&= \inf\{\mu(V) : A \subseteq V, \quad V \text{ open in } E\}.
\end{aligned}
$$

3. The same argument can be used to show that for any $A \in \mathcal{E}$ such that $P_t(x,V) < +\infty$,

$$
\begin{aligned}
\mu(A) &= \int_E P_t(x,dy)P_s(y,A) \\
&= \int_E P_t(x,dy)\sup\{P_s(y,K) : K \subseteq A, K \text{ compact}\} \\
&= \sup\left\{\int_E P_t(x,dy)P_s(y,K) : K \subseteq A, K \text{ compact}\right\} \\
&= \sup\{\mu(K) : K \subseteq A, K \text{ compact}\}.
\end{aligned}
$$

By the Riesz representation theorem, we conclude that

$$\mu(A) = P_{t+s}(A). \tag{2.10}$$

That is

$$P_{t+s}(A) = \int_E P_t(x,dy).P_s(y,A), \quad A \in \mathcal{E}. \tag{2.11}$$

This is the Chapman -Kolmogorov equation •

We have then constructed a transition function on E and then a Markov process as desired.

2.2 Infinitesimal generators of Feller semigroups on a Bounded Domain

The purpose of this section is to describe the form of the infinitesimal generator of a Feller semigroup on the closure of a bounded domain of $\mathbf{R}^n$. This infinitesimal generator is the key of relationships between semigroups and partial differential equations.

In this section, D is a bounded domain of $\mathbf{R}^n$ (n ≥ 2). To simplify notations, the closure of D will be denoted E instead of $\bar{D}$ and as usual $\mathcal{E}$ is the corresponding Borel $\sigma-$algebra. We will assume that D is a $C^\infty-$differential manifold defined by its canonical atlas. We begin by considering the case of interior points of the domain and later we will consider points on the boundary of the domain.

2.2.1 Form of infinitesimal generators in the interior of the Domain

The following theorem gives the general form of the infinitesimal generator of a Feller semigroup in the interior of D.

Theorem 2.4 *Let E be the closure of a bounded domain D in $\mathbf{R}^n$. Let $(T_t)_{t\geq 0}$ be a Feller semigroup on E and $\mathcal{U}$ its infinitesimal generator. Suppose that at any point $x_0 \in D$, there exists a local coordinate system $(x_1, x_2, \ldots, x_n)$ and continuous functions $\chi_1, \chi_2, \ldots, \chi_n$ defined from E to $\mathbf{R}$ that extend $x_1, x_2, \ldots, x_n$ respectively such that the functions $1, \chi_1, \chi_2, \ldots, \chi_n$ and $\sum_{i=1}^n \chi_i^2$ belong to the domain $\mathcal{D}(\mathcal{U})$ of $\mathcal{U}$.*
Then for any $f \in \mathcal{D}(\mathcal{U}) \cap C^2(E)$,

$$\begin{aligned}\mathcal{U}f(x_0) = &\sum_{i,j=1}^n a_{ij}(x_0)\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + \sum_{i=1}^n b_i(x_0)\frac{\partial f}{\partial x_i}(x_0) + c(x_0)u(x_0) \\ &+ \int_E e(x_0, dy)\left[f(y) - f(x_0) - \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x_0)(\chi_i(y) - \chi_i(x_0))\right], \end{aligned} \tag{2.12}$$

where

1. *$(a_{ij}(x_0))$ is a symmetric and positive semi-definite matrix,*
2. *$b_i(x_0) = \mathcal{U}(\chi_i - \chi_i(x_0))(x_0)$,*
3. *$c(x_0) = \mathcal{U}(1)(x_0)$,*
4. *$e(x_0, .)$ is a measure on $(E, \mathcal{E})$ such that*

$$\begin{aligned} e(x_0, E\backslash U) &< \infty, \\ \int_U e(x_0, dy)\left[\sum_{i=1}^n (\chi_i(y) - \chi_i(x_0))^2\right] &< \infty, \end{aligned}$$

for any neighborhood U of x_0.

Proof. We give just the main steps of the proof. A complete proof may be found elsewhere [17].

Since $(T_t)_{t\geq 0}$ is a Feller semigroup, then by Theorem 2.3, there exists a unique transition function $(P_t)_{t\geq 0}$ such that for any $f \in C(E)$,

$$T_t f(x) = \int_E P_t(x, dy) f(y). \tag{2.13}$$

Then for any $f \in \mathcal{D}(\mathcal{U}) \cap C^2(E)$ and for any $x_0 \in D$ we have:

$$\mathcal{U}f(x_0) = \lim_{t\downarrow 0} \frac{T_t f(x_0) - f(x_0)}{t}$$

$$= \lim_{t \downarrow 0} \frac{1}{t} \int_E P_t(x, dy) f(y) - f(x_0). \tag{2.14}$$

Putting $f(y)$ in the equivalent form:

$$\begin{aligned} f(y) &= f(x_0) + \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x_0)(\chi_i(y) - \chi_i(x_0)) \\ &\quad + f(y) - f(x_0) - \sum_{i=1}^{n} \frac{\partial u}{\partial x_i}(x_0)(\chi_i(y) - \chi_i(x_0)), \end{aligned}$$

we have that

$$\begin{aligned} \mathcal{U} f(x_0) &= \lim_{t \downarrow 0} \frac{1}{t} \left(\int_E P_t(x_0, dy) - 1 \right) f(x_0) \\ &\quad + \frac{1}{t} \sum_{i=1}^{n} \int_E P_t(x_0, dy)(\chi_i(y) - \chi_i(x_0)) \frac{\partial f}{\partial x_i}(x_0) \\ &\quad + \frac{1}{t} \int_E P_t(x_0, dy)[f(y) - f(x_0) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)(\chi_i(y) - \chi_i(x_0)). \end{aligned} \tag{2.15}$$

Let

$$c(x_0) = \lim_{t \downarrow 0} \frac{T_t 1(x_0) - 1}{t} = \mathcal{U} 1(x_0) \tag{2.16}$$

$$\begin{aligned} b_i(x_0) &= \lim_{t \downarrow 0} \frac{T_t[\chi_i - \chi_i(x_0)](x_0)}{t} \\ &= \mathcal{U}(\chi_i - \chi_i(x_0))(x_0) \quad \text{(since } (\chi_i - \chi_i(x_0))(x_0) = 0) \end{aligned} \tag{2.17}$$

$$d(x_0, y) = \sum_{i=1}^{n} [\chi_i(y) - \chi_i(x_0)]^2 \tag{2.18}$$

$$\tilde{f}(x_0, y) = \frac{f(y) - f(x_0) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0) [\chi_i(y) - \chi_i(x_0)]}{d(x_0, y)}, \tag{2.19}$$

where $y \in E - \{x_0\}$. We get

$$\begin{aligned} \mathcal{U} f(x_0) &= c(x_0) f(x_0) + \sum_{i=1}^{n} b_i(x_0) \frac{\partial f}{\partial x_i}(x_0) \\ &\quad + \lim_{t \downarrow 0} \frac{1}{t} \int_{E - \{x_0\}} P_t(x_0, dy) \tilde{f}(x_0, y) d(x_0, y). \end{aligned} \tag{2.20}$$

Define now a measure $\tilde{P}_t(x_0, .)$ on the measurable space $(E, \mathcal{E})$ as follows

$$\tilde{P}_t(x_0, A) = \frac{1}{t} \int_A P_t(x_0, dy) d(x_0, y), \quad \forall A \in \mathcal{E}. \tag{2.21}$$

Now we have

$$\begin{aligned} \mathcal{U} f(x_0) &= c(x_0) f(x_0) + \sum_{i=1}^{n} b_i(x_0) \frac{\partial f}{\partial x_i}(x_0) \\ &\quad + \lim_{t \downarrow 0} \int_{E - \{x_0\}} \tilde{P}_t(x_0, dy) \tilde{f}(x_0, y). \end{aligned} \tag{2.22}$$

For any sufficiently small $t \geq 0$, one has that $\tilde{P}_t(x_0, E) \leq \lim_{s\downarrow 0} P_s(x_0, E) + 1$. Then,

$$\begin{aligned} \tilde{P}_t(x_0, E) &\leq \lim_{s\downarrow 0} \int_E P_s(x_0, dy) d(x_0, y) + 1 \\ &\leq \mathcal{U}\left(\sum_{i=1}^{n} (\chi_i - \chi_i(x_0))^2\right)(x_0) + 1 \end{aligned} \tag{2.23}$$

by definition of $d(x_0, y)$ and since the function $(\chi_i - \chi_i(x_0))^2$ vanishes at x_0.
Let us now construct a compact space containing $E - \{x_0\}$ in which the function $\tilde{f}(x_0, .)$ can be continuously extended as follows.
For any $i, j = 1, 2, \ldots, n$, define a function $z_{ij}(x_0, .)$ from $E - \{x_0\}$ into $\mathbf{R}$ by

$$z_{ij}(x_0, y) = \frac{(\chi_i(y) - \chi_i(x_0))(\chi_j(y) - \chi_j(x_0))}{d(x_0, y)}. \tag{2.24}$$

By definition of $d(x_0, y)$, one has that

$$\forall y \in E - \{x_0\}, \quad |z_{ij}(x_0, y)| \leq 1 \tag{2.25}$$

and all the principal minors of the matrix $(z_{ij}(x_0, y))$ are nonnegative , that indicates that this matrix is positive semi-definite. This matrix is symmetric since $z_{ij}(x_0, y) = z_{ji}(x_0, y)$.
Let M be the set of positive semi-definite symmetric matrices (z_{ij}) of order n on $\mathbf{R}$ such that $|z_{ij}| \leq 1$.
It is clear that M, considered as a subset of $\mathbf{R}^{n^2}$ is bounded and closed. Consider now the map $\Phi_{x_0} : E - \{x_0\} \to E \times M$ defined as follows

$$\Phi_{x_0}(y) = (y, z_{ij}(x_0, y)). \tag{2.26}$$

It is clear that this map is continuous. It is also one-to-one and hence we can identify $E - \{x_0\}$ with its image $\Phi_{x_0}(E - \{x_0\})$. That is we identify an element y of $E - \{x_0\}$ with the pair $(y, z_{ij}(x_0, y))$.
The function $\tilde{f}(x_0, .)$ can now be considered as a function from $\Phi_{x_0}(E - \{x_0\})$ to $\mathbf{R}$ by taking

$$\forall (y, z_{ij}(x_0, y)) \in \Phi_{x_0}(E - \{x_0\}), \quad \tilde{f}(x_0, (y, z_{ij}(x_0, y))) = f(x_0, y). \tag{2.27}$$

Consider the closure

$$H_{x_0} = \overline{\Phi_{x_0}(E - \{x_0\})} \tag{2.28}$$

in $E \times M$. We have that H_{x_0} is a compact subset of the compact space $E \times M$ which is itself a compact subspace of $\mathbf{R}^{n+n^2}$.
Let $(y_n, z_{ij}(x_0, y_n))$ be a sequence in $\Phi_{x_0}(E - \{x_0\})$ that converges to (x_0, z_{ij}). Taking a development of $f(y_n)$ in a neighborhood of x_0 yields:

$$f(y_n) = f(x_0) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)(\chi_i(y_n) - \chi_i(x_0))$$

$$+\sum_{i,j=1}^{n}\int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0+\theta(y_n-x))(1-\theta)d\theta \times$$
$$(\chi_i(y_n)-\chi_i(x_0))(\chi_j(y_n)-\chi_j(x_0)). \tag{2.29}$$

Then by definition of $\tilde{f}(x_0, y_n)$, and $z_{ij}(x_0, y_n)$ (see equations (2.19) and (2.24)) one has:

$$\tilde{f}(x_0,y_n)=\sum_{i,j=1}^{n}\int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0+\theta(y_n-x))(1-\theta)d\theta z_{ij}(x_0,y_n). \tag{2.30}$$

Then

$$\begin{aligned}\lim_{n\to\infty}\tilde{f}(x_0,y_n) &= \sum_{i,j=1}^{n}\int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)(1-\theta)d\theta z_{ij} \\ &= \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)z_{ij}.\end{aligned} \tag{2.31}$$

Then we can extend the function $\tilde{f}(x_0, .)$ on H_{x_0} as follows:

$$\tilde{f}(x_0,(x_0,z_{ij}))=\frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)z_{ij}. \tag{2.32}$$

Then we obtain a function

$$\tilde{f}(x_0,.) : H_{x_0} \to \mathbf{R}$$

defined as follows: for any $h = (y, z_{ij}) \in H_{x_0}$,

$$\tilde{f}(x_0,h)=\begin{cases}\frac{f(y)-f(x_0)-\sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_0)(\chi_i(y)-\chi_i(x_0))}{d(x_0,y)} & \text{if } h\in\Phi_{x_0}(E-\{x_0\}) \\ \frac{1}{2}\sum_{i,j=1}^{n}\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0)z_{ij} & \text{otherwise.}\end{cases} \tag{2.33}$$

Now we consider the measurable space $(H_{x_0}, \mathcal{H})$ where $\mathcal{H}$ denotes the Borel $\sigma-$algebra on the topological space H_{x_0}. The injection Φ_{x_0} is measurable since it is continuous and we can transfer the measure $\tilde{P}_t(x_0, .)$ on $(H_{x_0}, \mathcal{H})$ by defining a new measure $\hat{P}_t(x_0, .)$ as follows:

$$\hat{P}_t(x_0,A)=\tilde{P}_t(x_0,\Phi_{x_0}^{-1}(A)), \quad \forall A\in\mathcal{H}. \tag{2.34}$$

Then

$$\begin{aligned}\hat{P}_t(x_0,H_{x_0}) &= \tilde{P}_t(x_0,\Phi_{x_0}^{-1}(H_{x_0})) \\ &\le \tilde{P}_t(x_0,E) \\ &\le \lim_{s\downarrow 0}\tilde{P}_s(x_0,E) \\ &= \mathcal{U}\left(\sum_{i=1}^{n}(\chi_i-\chi_i(x_0))^2\right)(x_0)+1 \quad \text{(from equation (2.23))}\end{aligned}$$

where the first inequality comes from the fact that

$$\Phi_{x_0}^{-1}(H_{x_0}) \subseteq \; E - \{x_0\} \subset E.$$

Since the space H_{x_0} is compact, from Theorem 1.3, we have that the family $(P_t(x_0,.))_{t\geq 0}$ contains a subsequence that converges weakly to a measure on H_{x_0}. More precisely, there exists a nonincreasing sequence (t_n) of nonnegative real numbers converging to zero such that the measures $\hat{P}_{t_n}(x_0,.)$ converges to a measure $\hat{P}(x_0,.)$ on H_{x_0}, in the sense that: for any continuous function $f : H_{x_0} \to \mathbf{R}$,

$$\lim_{n\to\infty} \int_{H_{x_0}} f(h)\hat{P}_{t_n}(x_0, dh) = \int_{H_{x_0}} f(h)\hat{P}(x_0, dh). \tag{2.35}$$

Returning to $\mathcal{U}f(x_0)$, the limit term in equation (2.22) can be calculated by noting that:

$$\int_{E-\{x_0\}} \tilde{P}_t(x_0, dy)\tilde{f}(x_0, y) = \int_{H_{x_0}} \hat{P}_t(x_0, dh)\tilde{f}(x_0, h)$$

since for any Borel set A of H_{x_0},

$$\hat{P}_t(x_0, A) = \tilde{P}_t(x_0, \Phi_{x_0}^{-1}(A)).$$

Then

$$\begin{aligned}
\lim_{t\downarrow 0} \int_{E-\{x_0\}} \tilde{P}_t(x_0, dy)\tilde{f}(x_0, y) &= \lim_{t\downarrow 0} \int_{H_{x_0}} \hat{P}_t(x_0, dh)\tilde{f}(x_0, h) \\
&= \lim_{n\to\infty} \int_{H_{x_0}} \hat{P}_{t_n}(x_0, dh)\tilde{f}(x_0, h) \\
&= \int_{H_{x_0}} \hat{P}(x_0, dh)\tilde{f}(x_0, h).
\end{aligned}$$

And therefore

$$\begin{aligned}
\mathcal{U}f(x_0) &= c(x_0)f(x_0) + \sum_{i=1}^{n} b_i(x_0)\frac{\partial f}{\partial x_i}(x_0) \\
&\quad + \int_{H_{x_0}} \hat{P}(x_0, dh)\tilde{f}(x_0, h).
\end{aligned} \tag{2.36}$$

Now we give a clear meaning to the integral term in equation (2.36).
Let $\tilde{P}(x_0,.)$ be the measure defined on $E-\{x_0\}$ as follows: for any Borel set A of $E-\{x_0\}$,

$$\tilde{P}(x_0, A) = \hat{P}(x_0, \Phi_{x_0}(A)). \tag{2.37}$$

Let $Z : E \times M \to M$ be the function defined by:

$$Z_{ij}(h) = (z_{ij}), \quad \forall h = (y, (z_{ij})) \in E \times M. \tag{2.38}$$

Then

$$\int_{H_{x_0}} \hat{P}(x_0, dh)\tilde{f}(x_0, h) = \int_{H_{x_0}-\Phi_{x_0}(E-\{x_0\})} \hat{P}(x_0, dh)\tilde{f}(x_0, h) + \int_{\Phi_{x_0}(E-\{x_0\})} \hat{P}(x_0, dh)\tilde{f}(x_0, h).$$

Since for $h = (y, (z_{ij})) \in H_{x_0} - \Phi_{x_0}(E - \{x_0\})$,

$$\tilde{f}(x_0, h) = \frac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) z_{ij},$$

we have that

$$\int_{H_{x_0}-\Phi_{x_0}(E-\{x_0\})} \hat{P}(x_0, dh)\tilde{f}(x_0, h) = \frac{1}{2}\sum_{i,j=1}^{n} \int_{H_{x_0}-\Phi_{x_0}(E-\{x_0\})} \hat{P}(x_0, dh) \times Z_{ij}(h) \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0).$$

We have also that that

$$\int_{\Phi_{x_0}(E-\{x_0\})} \hat{P}(x_0, dh)\tilde{f}(x_0, h) = \int_{E-\{x_0\}} \tilde{P}(x_0, dy)\tilde{f}(x_0, y)$$

Now letting

$$a_{ij}(x_0) = \frac{1}{2}\int_{H_{x_0}-\Phi_{x_0}(E-\{x_0\})} \hat{P}(x_0, dh) Z_{ij}(h), \tag{2.39}$$

we get

$$\mathcal{U}f(x_0) = c(x_0)f(x_0) + \sum_{i=1}^{n} b_i(x_0)\frac{\partial f}{\partial x_i}(x_0) + \sum_{i=1}^{n} a_{ij}(x_0)\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + \int_{(E-\{x_0\})} \tilde{P}(x_0, dy)\tilde{f}(x_0, y). \tag{2.40}$$

Define now a measure $e(x_0, .)$ on $(E, \mathcal{E})$ as follows:

$$e(x_0, \{x_0\}) = 0 \tag{2.41}$$

$$e(x_0, A) = \int_{A-\{x_0\}} \tilde{P}(x_0, dy)\left(\frac{1}{d(x_0, y)}\right), \quad \forall A \in \mathcal{E}. \tag{2.42}$$

Then we have that

$$\int_{E-\{x_0\}} \tilde{P}(x_0, dy)\tilde{f}(x_0, y) = \int_E e(x_0, dy)[f(y) - f(x_0) - \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(x_0)(\chi_i(y) - \chi_i(x_0))].$$

Now we have the formula

$$\mathcal{U}f(x_0) = \sum_{i,j=1}^{n} a_{ij}(x_0)\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + \sum_{i=1}^{n} b_i(x_0)\frac{\partial f}{\partial x_i}(x_0) + c(x_0)u(x_0)$$
$$+ \int_E e(x_0, dy)\left[f(y) - f(x_0) - \sum_{i=1}^{n}\frac{\partial f}{\partial x_i}(x_0)(\chi_i(y) - \chi_i(x_0))\right]. \qquad (2.43)$$

The matrix $(a_{ij}(x_0))$ is symmetric and positive semidefinite since the function Z takes values in M $\bullet$

Remark 2.1 *It is worth mentioning that if there exists a continuous Markov process that admits the transition function $(P_t)_{t\geq 0}$ defined by the Feller semigroup $(T_t)_{t\geq 0}$, then the integral part of relation* (2.43) *above vanishes* [17]. *In this case the infinitesimal generator of $(T_t)_{t\geq 0}$ is such that: for any $u \in \mathcal{D}(\mathcal{U}) \cap C^2(E)$, for any x_0 in the interior of E,*

$$\mathcal{U}f(x_0) = \sum_{i,j=1}^{n} a_{ij}(x_0)\frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) + \sum_{i=1}^{n} b_i(x_0)\frac{\partial f}{\partial x_i}(x_0) + c(x_0)f(x_0) \qquad (2.44)$$

and therefore $\mathcal{U}$ is a differential operator of second order in the interior of E.

In the proof of Theorem 2.4, the closure H_{x_0} of the image $\Phi_{x_0}(E-\{x_0\})$ in $E\times M$ has been intensively used. Let us illustrate the construction in dimension two. We know that the closure of a subset A of $\mathbf{R}^n$ is formed by the limits of converging sequences (x_n) such that all the points x_n are in A.We recall also that for $i, j = 1, 2$, and for any $y \in E-\{x_0\}$,

$$z_{ij}(x_0, y) = \frac{(\chi_i(y) - \chi_i(x_0))(\chi_j(y) - \chi_j(x_0))}{d(x_0, y)} \in \mathbf{R}.$$

Let $(y, (z_{ij})) \in H_{x_0}$. Then, there exists a sequence $(y_n, (z_{ij}(x_0, y_n)))$ in $\Phi_{x_0}(E - \{x_0\}$ that converges to $(y, (z_{ij}))$. If $y \neq x_0$, then we have that

$$(y_n, (z_{ij}(x_0, y_n)) = \Phi_{x_0}(y_n) \to \Phi_{x_0}(y)$$

since Φ_{x_0} is continuous. Then

$$\Phi_{x_0}(y) = (y, (z_{ij}))$$

by the unicity of the limitand hence

$$(y, (z_{ij})) \in \Phi_{x_0}(E - \{x_0\}).$$

If $y = x_0$, then

$$(y, (z_{ij})) \in H_{x_0} - \Phi_{x_0}(E - \{x_0\}).$$

It follows that $H_{x_0} - \Phi_{x_0}(E - \{x_0\})$ is the set of elements of the form $\lim(y_n, (z_{ij}(x_0, y_n)))$ where (y_n) is a sequence in $E - \{x_0\}$ converging to x_0 such that the sequence $(z_{ij}(x_0, y_n))$ also converges.

This means that $H_{x_0} - \Phi_{x_0}(E - \{x_0\})$ is the set of elements of the form $(x_0, \lim(z_{ij}(x_0, y_n)))$ where (y_n) is a sequence in $E - \{x_0\}$ converging to x_0 such that the sequence $(z_{ij}(x_0, y_n))$ also converges.
To characterize this set we first state and prove the following lemma.

Lemma 2.1 *With the notations used in Theorem 2.4, for any sequence (y_n) in $E - \{x_0\}$, converging to x_0, the sequence of matrices $(z_{ij}(x_0, y_n))$ where*

$$z_{ij}(x_0, y_n) = \frac{(\chi_i(y_n) - \chi_i(x_0))(\chi_j(y_n) - \chi_j(x_0))}{d(x_0, y_n)}$$

converges , if and only if , the quotient

$$\frac{\chi_2(y_n) - \chi_2(x_0)}{\chi_1(y_n) - \chi_1(x_0)}$$

converges to a finite limit or its square tends to infinity.

Proof. Suppose that

$$\lim z_{11}(x_0, y_n) = l, \quad \text{and } \lim z_{12}(x_0, y_n) = a. \tag{2.45}$$

The first equality means that

$$\lim \frac{(\chi_1(y_n) - \chi_1(x_0))^2}{(\chi_1(y_n) - \chi_1(x_0))^2 + (\chi_2(y_n) - \chi_2(x_0))^2} = l.$$

Then by dividing the terms of this fraction by $(\chi_1(y_n) - \chi_1(x_0))^2$ (note that this term does not vanish for any $y \in E - \{x_0\}$)we get

$$\lim \frac{1}{1 + \frac{(\chi_2(y_n) - \chi_2(x_0))^2}{(\chi_1(y_n) - \chi_1(x_0))^2}} = l. \tag{2.46}$$

If $l \neq 0$, then

$$\lim \frac{(\chi_2(y_n) - \chi_2(x_0))^2}{(\chi_1(y_n) - \chi_1(x_0))^2} = \frac{1}{l} - 1.$$

By the same procedure, we get that

$$\lim \frac{(\chi_2(y_n) - \chi_2(x_0))/(\chi_1(y_n) - \chi_1(x_0)}{1 + (\chi_2(y_n) - \chi_2(x_0))^2/(\chi_1(y_n) - \chi_1(x_0))^2} = a$$

and hence

$$\lim \frac{\chi_2(y_n) - \chi_2(x_0)}{\chi_1(y_n) - \chi_1(x_0)} = \frac{a}{l}.$$

If $l = 0$, then equation (2.46) implies that

$$\lim \frac{(\chi_2(y_n) - \chi_2(x_0))^2}{(\chi_1(y_n) - \chi_1(x_0))^2} = \infty.$$

Conversely, if

$$\frac{\chi_2(y_n) - \chi_2(x_0)}{\chi_1(y_n) - \chi_1(x_0)}$$

converges to $\lambda \in \mathbf{R}$ then clearly

$$\begin{aligned}
\lim z_{11}(x_0, y_n) &= \frac{1}{1+\lambda^2}, \\
\lim z_{12}(x_0, y_n) &= \frac{\lambda}{1+\lambda^2}, \\
\lim z_{22}(x_0, y_n) &= \frac{\lambda^2}{1+\lambda^2}.
\end{aligned}$$

Also if

$$\left(\frac{\chi_2(y_n) - \chi_2(x_0)}{\chi_1(y_n) - \chi_1(x_0)}\right)^2 \to \infty,$$

then

$$\lim z_{11}(x_0, y_n) = 0, \quad \lim z_{12}(x_0, y_n) = 0 \quad \text{and} \quad \lim z_{22}(x_0, y_n) = 1.$$

The same situation occurs if

$$\frac{\chi_2(y_n) - \chi_2(x_0)}{\chi_1(y_n) - \chi_1(x_0)} \to \infty \quad \bullet$$

From this lemma, we have that $H_{x_0} - \Phi_{x_0}(E - \{x_0\})$ is the set of elements of the form $(x_0, \lim(z_{ij}(x_0, y_n)))$ where (y_n) is a sequence in $E - \{x_0\}$ such that

$$\lim \frac{\chi_2(y_n) - \chi_2(x_0)}{\chi_1(y_n) - \chi_1(x_0)} = \lambda \in [-\infty, +\infty].$$

In the case where for any $\lambda \in [-\infty, \infty]$ one can construct a sequence (y_n) such that

$$\lim \frac{\chi_2(y_n) - \chi_2(x_0)}{\chi_1(y_n) - \chi_1(x_0)} = \lambda,$$

we have that $H_{x_0} - \Phi_{x_0}(E - \{x_0\})$ is identified to the set

$$\left\{ \left(x_0, \frac{1}{1+\lambda^2}, \frac{\lambda}{1+\lambda^2}, \frac{\lambda}{1+\lambda^2}, \frac{\lambda^2}{1+\lambda^2} \right) : \lambda \in [-\infty, \infty] \right\}.$$

We have now the following theorem (using notations of Theorem 2.4)

Theorem 2.5 *If for any $\lambda \in [-\infty, \infty]$ there exists a sequence (y_n) such that*

$$\lim \frac{\chi_2(y_n) - \chi_2(x_0)}{\chi_1(y_n) - \chi_1(x_0)} = \lambda$$

then the space $H_{x_0} - \Phi_{x_0}(E - \{x_0\})$ is equal to the set

$$\left\{ \left(x_0, \frac{1}{1+\lambda^2}, \frac{\lambda}{1+\lambda^2}, \frac{\lambda}{1+\lambda^2}, \frac{\lambda^2}{1+\lambda^2} \right) : \lambda \in [-\infty, \infty] \right\}. \tag{2.47}$$

This set can be written as $x_0 \times C$ where C is a curve situated on the unity sphere of $\mathbf{R}^4$.

In the next section, we investigate the form of the infinitesimal generator on the boundary of E.

2.2.2 Form of the infinitesimal generator on the boundary of the Domain

As in the previous section, we suppose that E is the closure of a bounded domain D of $\mathbf{R}^n$ of class C^∞. Having seen the form of the infinitesimal generator in the domain D, this section is intended to give the general form of the infinitesimal generator on the boundary ∂D.

Let $(T_t)_{t\geq 0}$ be a Feller semigroup on E and let $\mathcal{U}$ be its infinitesimal generator.

As in [3], [18] let us suppose that for any point $x' \in \partial D$, there exists an open neighborhood U of x' in $\bar{D}$ and a bijection φ from U onto an open subset of the subspace

$$\mathbf{R}^n_+ = \{(a_1, a_2, \dots, a_n) : a_n \geq 0\}$$

defined by $\varphi(x) = (x_1(x), x_2(x), \dots, x_{n-1}(x), x_n(x))$ such that the following properties hold.

1. for any $x \in U$, $x \in U \cap D \iff x_n(x) > 0$ and $x \in U \cap \partial D \iff x_n(x) = 0$,
2. the functions $(x_1, x_2, \dots, x_{n-1}, x_n)$ form a local coordinate system of $U \cap D$, meaning that
$$(U \cap D, \varphi|_{U\cap D})$$
is a chart of D (considered as a $C^\infty-$differential manifold as in Example 1.1),
3. the functions $(x_1, x_2, \dots, x_{n-1})$ form a local coordinate system of $U \cap \partial D$, meaning that
$$(U \cap \partial D, \varphi|_{U\cap\partial D})$$
is a chart of ∂D considered also as a $C^\infty-$differential manifold.

Suppose also that the functions $(x_1, x_2, \dots, x_{n-1}, x_n)$ can be extended respectively to $C^\infty-$functions $\chi_1, \chi_2, \dots, \chi_{n-1}, \chi_n$ defined from $\mathbf{R}^n$ to $\mathbf{R}$ such that for any $x' \in U \cap \partial D$

$$d(x', y) = \chi_n(y) + \sum_{i=1}^{n-1} (\chi_i(y) - \chi_i(x'))^2 > 0, \quad \forall y \in E - \{x'\} \tag{2.48}$$

and

$$\chi_n(y) \geq 0, \quad \forall y \in \mathbf{R}^n. \tag{2.49}$$

Then we have the following theorem [17].

Theorem 2.6 *Any function $f \in \mathcal{D}(\mathcal{U}) \cap C^2(E)$, verifies at each $x' \in \partial D$, a condition of the form:*

$$\sum_{i,j=1}^{n-1} \alpha_{ij}(x') \frac{\partial^2 f}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{n-1} \beta_i(x_0) \frac{\partial f}{\partial x_i}(x') + \gamma(x') f(x')$$
$$+\mu(x') \frac{\partial f}{\partial x_n}(x') - \delta(x') \mathcal{U} f(x')$$
$$+ \int_E \nu(x', dy)[f(y) - f(x') - \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x')(\chi_i(y) - \chi_i(x'))] = 0 \tag{2.50}$$

where: for any $x' \in \partial D$,

1. *the matrix $\alpha_{ij}(x')$ is symmetric and positive semi-definite,*
2. $\gamma(x') \leq 0$,
3. $\mu(x') \geq 0$,
4. $\delta(x') \geq 0$,
5. *$\nu(x', .)$ is a measure on $(E, \mathcal{E})$ such that for any neighborhood W of x' in $\mathbf{R}^n$,*

$$\begin{aligned} \nu(x', E - W) &< \infty, \\ \int_{W \cap E} \nu(x', dy) \left[\chi_n(y) + \sum_{i=1}^{n-1} (\chi_i(y) - \chi_i(x'))^2 \right] &< \infty. \end{aligned}$$

Proof. For the sake of space we give here only the main steps of the proof. A detailled proof is provided in [17]. Another form of this theorem is given in [20].

We consider the unique transition function $(P_t)_{t \geq 0}$ on E such that: for any $f \in C(E)$,

$$T_t f(x) = \int_E P_t(x, dy) f(y).$$

Then for any $f \in \mathcal{D}(\mathcal{U}) \cap C^2(E)$, and for any $x' \in \partial D$, from the identity

$$\begin{aligned} f(y) &= f(x') + \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x')(\chi_i(y) - \chi_i(x'))] \\ &\quad + f(y) - f(x') - \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x')(\chi_i(y) - \chi_i(x'))], \end{aligned}$$

one has, for any $t > 0$,

$$\begin{aligned} \frac{1}{t}(T_t f(x') - f(x')) &= \frac{1}{t}\left(P_t(x', E) - 1\right) f(x') + \\ &\quad + \frac{1}{t} \sum_{i=1}^{n-1} \int_E P_t(x', dy)(\chi_i(y) - \chi_i(x')) \frac{\partial f}{\partial x_i}(x') \end{aligned}$$

$$+\frac{1}{t}\int_E P_t(x',dy)[f(y)-f(x')-\sum_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x')(\chi_i(y)-\chi_i(x'))]. \tag{2.51}$$

Define now the following functions:

$$\gamma_t(x') = \frac{1}{t}\left(P_t(x',E)-1\right), \tag{2.52}$$

$$\beta_j^t(x') = \frac{1}{t}\int_E P_t(x',dy)(\chi_j(y)-\chi_j(x')), \tag{2.53}$$

$$d(x',y) = \chi_n(y)+\sum_{i=1}^{n-1}(\chi_i(y)-\chi_i(x'))^2, \forall y\in E-\{x'\}, \tag{2.54}$$

$$\tilde f(x',y) = \frac{f(y)-f(x')-\sum_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x')(\chi_i(y)-\chi_i(x'))}{d(x',y)} \tag{2.55}$$

where $y\in E-\{x'\}$.
Then we get

$$\frac{1}{t}\left(T_t f(x')-f(x')\right) = \gamma_t(x')f(x')+\sum_{i=1}^{n-1}\beta_j^t(x')\frac{\partial f}{\partial x_i}(x')+ \\ +\frac{1}{t}\int_{E-\{x'\}}P_t(x',dy)\tilde f(x',y)d(x',y). \tag{2.56}$$

Now introduce the function

$$l_t(x') = \frac{1}{t}\int_E P_t(x',dy)d(x',y). \tag{2.57}$$

Since by hypothesis $d(x',y)>0$ for any $y\in E-\{x'\}$, we have that

$$l_t(x)\geq 0, \quad \forall x'\in\partial D.$$

Now we consider two cases: $l_t(x')>0$ and $l_t(x')=0$.
Suppose first that $l_t(x')>0$. Define a measure $\tilde q_t(x',.)$ on $(E,\mathcal{E})$ as follows:

$$\tilde q_t(x',A) = \frac{1}{tl_t(x')}\int_A P_t(x',dy)d(x',y). \tag{2.58}$$

Then the last term in equation (2.56) is:

$$\frac{1}{t}\int_{E-\{x'\}}P_t(x',dy)\tilde f(x',y)d(x',y) = \frac{1}{t}tl_t(x')\int_{E-\{x'\}}\tilde q_t(x',dy)\tilde f(x',y) \\ = l_t(x')\int_{E-\{x'\}}\tilde q_t(x',dy)\tilde f(x',y). \tag{2.59}$$

Note that the measure $\tilde{q}_t(x', .)$ is such that

$$\begin{aligned}\tilde{q}_t(x', E - \{x'\}) &= \frac{1}{tl_t(x')} \int_{E-\{x'\}} P_t(x', dy) d(x', y) \\ &= \frac{1}{tl_t(x')} \int_E P_t(x', dy) d(x', y) \\ &= 1\end{aligned}$$

where the second equality follows the fact that $d(x', x') = 0$ and the last comes from the to the definition of $l_t(x')$ (2.57).
Let us consider the case where $l_t(x') = 0$. From equation (2.57)), we have that

$$\int_E P_t(x', dy) d(x', y) = \int_{E-\{x'\}} P_t(x', dy) d(x', y) = 0 \tag{2.60}$$

since $d(x', x') = 0$. Since the function $d(x', .)$ is positive on $E - \{x'\}$, the last equality in relation (2.60) implies that:

$$P_t(x', E - \{x'\}) = 0.$$

Let x_0 be a fixed interior point of E. Consider the measure $\tilde{q}_t(x', .)$ on $(E, \mathcal{E})$ defined as follows:

$$\forall A \in \mathcal{E}, \qquad \tilde{q}_t(x', A) = \begin{cases} 1 & \text{if } x_0 \in A \\ 0 & \text{otherwise.} \end{cases} \tag{2.61}$$

It follows, as in the first case, that

$$\frac{1}{t} \int_{E-\{x'\}} P_t(x', dy) \tilde{f}(x', y) d(x', y) = l_t(x') \int_{E-\{x'\}} \tilde{q}_t(x', dy) \tilde{f}(x', y)$$

since the two terms are zero. It is also clear that

$$\tilde{q}(x', E - \{x'\}) = 1 \quad \text{ since } x^{'} \neq 0.$$

Now in the two cases there exists a measure $\tilde{q}_t(x', .)$ such that

$$\frac{1}{t} \int_{E-\{x'\}} P_t(x', dy) \tilde{f}(x', y) d(x', y) = l_t(x') \int_{E-\{x'\}} \tilde{q}_t(x', dy) \tilde{f}(x', y) \tag{2.62}$$

and

$$\tilde{q}_t(x', E - \{x'\}) = 1. \tag{2.63}$$

It follows that

$$\begin{aligned}\frac{1}{t}\left(T_t f(x') - f(x')\right) &= \gamma_t(x') f(x') + \sum_{i=1}^{n-1} \beta_j^t(x') \frac{\partial f}{\partial x_i}(x') + \\ &\quad + l_t(x') \int_{E-\{x'\}} \tilde{q}_t(x', dy) \tilde{f}(x', y).\end{aligned} \tag{2.64}$$

Now as in the previous proof, the function $\tilde{f}(x', .)$ should be extended to a compact space containing x'. To this end we consider the functions $w(x', .)$ and $z_{ij}(x', .)$ defined for $y \in E - \{x'\}$ as follows:

$$
\begin{aligned}
w(x', y) &= \frac{\chi_n(y)}{d(x', y)} \\
z_{ij}(x', y) &= \frac{(\chi_i(y) - \chi_i(x'))(\chi_j(y) - \chi_j(x'))}{d(x', y)} \\
\forall i, j = 1, 2, \ldots, n-1.
\end{aligned}
$$

Clearly we have that:

1. $0 \leq w(x', y) \leq 1, \quad \forall y \in E - \{x'\}$.

2. For any $i, j = 1, 2, \ldots, n-1, \quad |z_{ij}(x', y)| \leq 1$ (by using the inequality $|ab| \leq a^2 + b^2$ for any $a, b \in \mathbf{R}$.)

3. $w(x', y) + \sum_{i=1}^{n-1} z_{ii}(x', y) = 1$.

4. The matrix $(z_{ij}(x', y))$ is symmetric and positive semi-definite as in the previous proof.

Let M be the set of symmetric and positive semi-definite matrices (z_{ij}) of order $n-1$ on $\mathbf{R}$ such that $|z_{ij}| \leq 1$ and let H be the subset of $E \times [0, 1] \times M$ defined as follows:

$$
(y, w, (z_{ij})) \in H \iff w + \sum_{i=1}^{n-1} z_{ij} = 1. \tag{2.65}
$$

Let $\Phi_{x'}$ be the injection defined from $E - \{x'\}$ to H as follows:

$$
\Phi_{x'}(y) = (y, w(x', y), (z_{ij}(x', y))). \tag{2.66}
$$

Let

$$
H_{x'} = \overline{\Phi'_x(E - \{x'\})} \tag{2.67}
$$

be the closure of $\Phi'_x(E - \{x'\})$ in H. As before, we identify the set $E - \{x'\}$ and its image $\Phi'_x(E - \{x'\})$, by identifying $y \in E - \{x'\}$ with $(y, w(x', y), (z_{ij}(x', y)))$.
Let $(y_m, w(x', y), z_{ij}(x', y_m))$ be a sequence in $\Phi_{x'}(E - \{x'\})$ that converges to (x', w, z_{ij}) in H. We can take a Taylor's development of $f(y_m)$ in a neighborhood of x' and get:

$$
\begin{aligned}
f(y_m) &= f(x') + \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x')(\chi_i(y_m) - \chi_i(x')) \\
&\quad + \frac{\partial f}{\partial x_n}(x')(\chi_n(y_m)
\end{aligned}
$$

$$+\sum_{i,j=1}^{n-1}\int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(x'+\theta(y_m-x'))(1-\theta)d\theta \times$$
$$(\chi_i(y_m)-\chi_i(x'))(\chi_j(y_m)-\chi_j(x')). \tag{2.68}$$

Then by definition of $\tilde{f}(x',.)$,and $z_{ij}(x_0,.)$ we have:

$$\tilde{f}(x',y_m) = \frac{\frac{\partial f}{\partial x_n}(x')\chi_n(y_m)}{d(x',y_m)} + \sum_{i,j=1}^{n-1}\int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(x'+\theta(y_m-x))(1-\theta)d\theta z_{ij}(x',y_m).$$

Moreover by definition of $w(x',.)$, we have that:

$$\tilde{f}(x',y_m) = \frac{\partial f}{\partial x_n}(x')w(x',y_m)+\sum_{i,j=1}^{n-1}\int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(x'+\theta(y_m-x)) \times (1-\theta)d\theta \times z_{ij}(x',y_m). \tag{2.69}$$

By letting $m\to\infty$, we get

$$f(x',y_m)\to \frac{\partial f}{\partial x_n}(x')w+\frac{1}{2}\sum_{i=1}^{n-1}\frac{\partial^2 f}{\partial x_i \partial x_j}(x')z_{ij}. \tag{2.70}$$

We can therefore extend the function $\tilde{f}(x',y)$ on $H_{x'}$ by taking:

$$\tilde{f}(x',(x',w,z_{ij}))=\frac{\partial f}{\partial x_n}(x')w+\frac{1}{2}\sum_{i,j=1}^{n-1}\frac{\partial^2 f}{\partial x_i \partial x_j}(x')z_{ij}. \tag{2.71}$$

Then we obtain a function

$$\tilde{f}(x',.):H_{x'}\to \mathbf{R}$$

defined as follows: for any $h=(y,w,z_{ij})\in H_{x'}$,

$$\tilde{f}(x',h)=\begin{cases} \frac{f(y)-f(x')-\sum_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x')(\chi_i(y)-\chi_i(x'))}{d(x',y)} & \text{if } h\in\Phi_{x'}(E-\{x_0\}) \\ \frac{\partial f}{\partial x_n}(x')w+\frac{1}{2}\sum_{i,j=1}^{n-1}\frac{\partial^2 f}{\partial x_i \partial x_j}(x')z_{ij} & \text{otherwise.}\end{cases} \tag{2.72}$$

Let us define a measure $\hat{q}_t$ on $H_{x'}$ as follows:

$$\hat{q}_t(x',A)=\tilde{q}_t(x',\Phi_{x'}^{-1}(A)) \tag{2.73}$$

for any Borel set A of $H_{x'}$.
We have that

$$\hat{q}_t(x',H_{x'})=\tilde{q}_t(x',E-\{x'\})=1 \tag{2.74}$$

that means that $\hat{q}_t(x',.)$ is a probability measure. We have also that

$$\begin{aligned}\frac{1}{t}\left(T_t f(x') - f(x')\right) &= \gamma_t(x')f(x') + \sum_{i=1}^{n-1}\beta_j^t(x')\frac{\partial f}{\partial x_i}(x') \\ &\quad + l_t(x')\int_{H_{x'}}\hat{q}_t(x',dh)\tilde{f}(x',h). \end{aligned} \tag{2.75}$$

Now consider the following functions:

$$\theta_m(x') = -\gamma_{1/m}(x') + \sum_{i=1}^{n-1}|\beta_j^{1/m}(x')| + l_{1/m}(x'), \quad m = 1, 2, \dots. \tag{2.76}$$

We have that $\theta_m(x') \geq 0$ since $\gamma_t(x') \leq 0$ and $l_t(x') \geq 0$ for all $x' \in \partial D$. Two cases are to be considered:

Case 1. Suppose that there exists a subsequence (θ_{m_k}) of (θ_m) that converges to zero. Then we have from equation (2.76) that

$$\gamma_{1/m_k}(x') \to 0, \quad \beta_j^{1/m_k}(x') \to 0 \text{ and } \quad l_{1/m_k}(x') \to 0.$$

Taking $t = 1/m_k$ in equation (2.75) and letting $m_k \to \infty$, we get:

$$\mathcal{U}f(x') = 0.$$

In this case, we can take

$$\begin{aligned}\alpha_{ij}(x') &= \beta_j(x') = \gamma(x') = \mu(x') = 0, \\ \delta(x') = 1, \quad \nu(x',.) = 0 & \end{aligned}$$

and get (2.50). Case 2. Suppose that there exists a subsequence (θ_{m_k}) of $(\theta_m(x'))$ that converges to a real number $\theta(x') > 0$. At this step, let us take $t = 1/m_k$ in equation (2.75). By dividing the resulting equation by θ_{m_k} one gets

$$\begin{aligned}\bar{\delta}_k(x')\frac{T_t f(x') - f(x')}{t_k} &= \bar{\gamma}_k(x')f(x') + \sum_{i=1}^{n-1}\bar{\beta}_j^k(x')\frac{\partial f}{\partial x_i}(x') \\ &\quad + \bar{l}_t(x')\int_{H_{x'}}\bar{q}_k(x',dh)\tilde{f}(x',h),\end{aligned}$$

where

$$\begin{aligned} t_k &= \frac{1}{m_k}, \\ \bar{\delta}_k(x') &= \frac{1}{\theta_{m_k}(x')}, \\ \bar{\gamma}_k(x') &= \frac{\gamma_{t_k}(x')}{\theta_{m_k}(x')}, \end{aligned}$$

$$\begin{aligned}
\bar{\beta}_j^k(x') &= \frac{\beta_j^{t_k}(x')}{\theta_{m_k}(x')},\\
\bar{l}_k(x') &= \frac{l_{t_k}(x')}{\theta_{m_k}(x')}\\
\bar{q}_k(x',.) &= \hat{q}_{t_k}(x',.).
\end{aligned}$$

Letting $m_k \to \infty$, we get the relation:

$$\begin{aligned}
\delta(x')\mathcal{U}f(x') &= \gamma(x')f(x') + \sum_{i=1}^{n-1} \beta_j(x')\frac{\partial f}{\partial x_i}(x')\\
&+ l(x')\int_{H_{x'}} \hat{q}(x',dh)\tilde{f}(x',h)
\end{aligned} \tag{2.77}$$

where $\delta(x') = \lim \bar{\delta}_k(x')$, $\quad \gamma(x') = \lim \bar{\gamma}_k(x')$
$\beta_j(x') = \lim \bar{\beta}_j^k(x')$, $\quad l(x') = \lim \bar{l}_k(x')$,
$\hat{q}(x',.) = \lim \hat{q}_k(x',.)$
At this step, we define a measure $\tilde{q}(x',.)$ on $E - \{x'\}$ by taking, for any Borel set A of $E - \{x'\}$:

$$\tilde{q}(x',A) = \hat{q}(x',\Phi_{x'}(A)). \tag{2.78}$$

Consider the functions W and Z from $E \times [0,1] \times M$ to $[0,1]$ and M respectively defined as follows:

$$W(h) = w \quad \text{ and } Z(h) = (z_{ij}), \quad \forall h = (y, w, (z_{ij})) \in E \times [0,1] \times M. \tag{2.79}$$

Now we have that

$$\begin{aligned}
l(x')\int_{H_{x'}} \hat{q}(x',dh)\tilde{f}(x',h) = l(x')&\int_{H_{x'}-\Phi_{x'}(E-\{x'\})} \hat{q}(x',dh)\tilde{f}(x',h)\\
&+l(x')\int_{\Phi_{x'}(E-\{x'\})} \hat{q}(x',dh)\tilde{f}(x',h)\\
= l(x')&\int_{H_{x'}-\Phi_{x'}(E-\{x'\})} \hat{q}(x',dh)W(h)\frac{\partial f}{\partial x_n(x')}\\
+l(x')\frac{1}{2}\sum_{i,j=1}^{n-1}&\int_{H_{x'}-\Phi_{x'}(E-\{x'\})} \hat{q}(x',dh)Z_{ij}(h)\frac{\partial^2 f}{\partial x_i \partial x_j}(x')\\
&+l(x')\int_{(E-\{x'\}} \tilde{q}(x',dh)\tilde{f}(x',h).
\end{aligned} \tag{2.80}$$

Let us take

$$\mu(x') = l(x')\int_{H_{x'}-\Phi_{x'}(E-\{x'\})} \hat{q}(x',dh)W(h), \tag{2.81}$$

$$\alpha_{ij}(x') = \frac{l(x')}{2}\int_{H_{x'}-\Phi_{x'}(E-\{x'\})} \hat{q}(x',dh)Z_{ij}(h). \tag{2.82}$$

Define the measure $\nu(x',.)$ on $(E,\mathcal{E})$ as follows:

$$\left\{\begin{array}{lcl}\nu(x',\{x'\}) & = & 0 \\ \nu(x',A) & = & \int_{A-\{x'\}}\tilde{P}(x',dy)\left(\frac{1}{d(x',y)}\right).\end{array}\right. \tag{2.83}$$

We get

$$\begin{aligned} l(x')\int_{H_{x'}}\hat{q}(x',dh)\tilde{f}(x',h) &= \sum_{i=1}^{n-1}\alpha_{ij}(x')\frac{\partial^2 f}{\partial x_i\partial x_j}(x') + \\ &+ \mu(x')\frac{\partial f}{\partial x_n}(x') + \int_E \nu(x',dy)[f(y)-f(x') \\ &- \sum_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x')(\chi_i(y)-\chi_i(x'))], \end{aligned}$$

and hence

$$\begin{aligned} \sum_{i,j=1}^{n-1}\alpha_{ij}(x')\frac{\partial^2 f}{\partial x_i\partial x_j}(x') + \sum_{i=1}^{n-1}\beta_i(x')\frac{\partial f}{\partial x_i}(x') + \gamma(x')f(x') \\ +\mu(x')\frac{\partial f}{\partial x_n}(x') - \delta(x')\mathcal{U}f(x') \\ + \int_E \nu(x',dy)[f(y)-f(x') - \sum_{i=1}^{n-1}\frac{\partial f}{\partial x_i}(x')(\chi_i(y)-\chi_i(x'))] = 0 \end{aligned}$$

as desired •

Remark 2.2 *As previously let us illustrate the construction of the subspace,*

$$H_{x'} - \Phi_{x'}(E-\{x'\})$$

in dimension 2. We will illustrate this set in the case of two dimensions as previously.

With the same argument as in Remark 2.1, we have that $H_{x'} - \Phi_{x'}(E-\{x'\})$ is the set of elements of the form

$$(x', w(x', y_n), \lim(z_{ij}(x', y_n)))$$

where (y_n) is a sequence in $E-\{x'\}$ converging to x' such that the sequences $w(x',y_n)$ and $(z_{ij}(x',y_n))$ also converge. As in Lemma 2.1, we have that the sequences $w(x',y_n)$ and $(z_{11}(x',y_n))$ converge if and only if the fraction

$$\frac{(\chi_2(y_n))^{1/2}}{\chi_1(y_n)-\chi_1(x')} \to \lambda \in [-\infty,\infty]$$

or its square tends to infinity (as $n\to\infty$). And if this happens, then

$$\begin{aligned} \lim z_{11} &= \frac{1}{1+\lambda^2}, \\ \lim w(x',y_n) &= \frac{\lambda^2}{1+\lambda^2} \end{aligned}$$

with the obvious extension if λ is infinite. In the case where for any $\lambda \in [-\infty, \infty]$ one can construct a sequence (y_n) such that

$$\frac{(\chi_2(y_n))^{1/2}}{\chi_1(y_n) - \chi_1(x')} \to \lambda,$$

the space

$$H_{x'} - \Phi_{x'}(E - \{x'\})$$

is the set

$$\left\{ \left(x', \frac{1}{1+\lambda^2}, \frac{\lambda^2}{1+\lambda^2} \right) : \lambda \in [-\infty, +\infty] \right\}.$$

Then, we have the following theorem (using notations in Theorem 2.6):

Theorem 2.7 *In dimension two, if for any $\lambda \in [-\infty, \infty]$ there exists a sequence (y_n) such that*

$$\frac{(\chi_2(y_n))^{1/2}}{\chi_1(y_n) - \chi_1(x')} \to \lambda \in [-\infty, +\infty] \tag{2.84}$$

then the space $H_{x'} - \Phi_{x'}(E - \{x'\})$ is equal to the set

$$\left\{ \left(x', \frac{1}{1+\lambda^2}, \frac{\lambda^2}{1+\lambda^2} \right) : \lambda \in [-\infty, \infty] \right\}. \tag{2.85}$$

In the particular case where there are no integral terms in equations (2.12) and (2.50), the Feller semigroup (or its generator) is characterized by the differential operators, $\mathcal{A}$ and $\mathcal{L}$ such that

$$\mathcal{A}u(x) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x), \quad x \in \bar{D} \tag{2.86}$$

$$\begin{aligned} \mathcal{L}u(x) &= \sum_{i,j=1}^{n-1} \alpha_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n-1} \beta_i(x_0)\frac{\partial u}{\partial x_i}(x) + \gamma(x)u(x) \\ &\quad + \mu(x)\frac{\partial u}{\partial x_n}(x) - \delta(x)\mathcal{A}u(x) = 0, \quad x' \in \partial D. \end{aligned} \tag{2.87}$$

The converse problem can now be posed and this is the subject matter of the next section.

2.3 Feller Semigroups and Bounded value problems

This section is intended to explain the relationships between Feller semigroups and boundary value problems. It was shown that the infinitesimal generator of Feller semigroup on the closure of a bounded domain in $\mathbf{R}^n$ $(n \geq 2)$ can be described by a differential operator in the interior of the domain and a boundary condition. In this section, the converse problem is investigated. Under which conditions a second order linear differential operator in a domain D and a boundary condition on the boundary ∂D of D, determine a

Feller semigroup on D.

2.3.1 Statement of the Problem

Let D be a bounded domain of $\mathbf{R}^n$ of class C^∞. One can show that this implies that the boundary ∂D of D is a $C^\infty-$differential manifold of dimension n. We will suppose [3], [18] that for any point $x' \in \partial D$, there exists an open neighborhood U of x' in $\bar{D}$ and a bijection φ from U onto an open subset of the subspace

$$\mathbf{R}_+^n = \{(a_1, a_2, \ldots, a_n) : a_n \geq 0\}$$

defined by $\varphi(x) = (x_1(x), x_2(x), \ldots, x_{n-1}(x), x_n(x))$ such that the following properties hold:

1. for any $x \in U$, $x \in U \cap D \iff x_n(x) > 0$ and $x \in U \cap \partial D \iff x_n(x) = 0$,
2. the functions $(x_1, x_2, \ldots, x_{n-1}, x_n)$ form a local coordinate system of $U \cap D$, meaning that
$$(U \cap D, \varphi|_{U \cap D})$$
is a chart of D (considered as a $C^\infty-$differential manifold as in Example 1.1),
3. the functions $(x_1, x_2, \ldots, x_{n-1})$ form a local coordinate system of $U \cap \partial D$, meaning that
$$(U \cap \partial D, \varphi|_{U \cap \partial D})$$
is a chart of ∂D (considered also as a $C^\infty-$differential manifold).

We will suppose, in particular, that the function x_n is the distance to the boundary ∂D, that is
$$x_n(x) = dist(x, \partial D) = \inf\{|x - y| : y \in \partial D\}$$

Consider a second-order elliptic operator of the form:

$$Au(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i}(x) + c(x)u(x) \tag{2.88}$$

which verifies the following conditions:

1. $a_{ij} \in C^\infty(\mathbf{R}^n)$, $\quad a_{ij} = a_{ji}$ and there exists a constant $a_0 > 0$ such that
$$\sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq a_0|\xi|^2, \quad \forall x, \xi \in \mathbf{R}^n, \tag{2.89}$$
2. $b_i \in C^\infty(\mathbf{R}^n)$,
3. $c \in C^\infty(\mathbf{R}^n)$ and $c \leq 0$ on $\bar{D}$.

Furthermore, consider a boundary condition of the form:

$$\begin{aligned} Lu(x') &= \sum_{i,j=1}^{n-1} \alpha_{ij}(x') \frac{\partial^2 u}{\partial x_i \partial x_j}(x') \\ &+ \sum_{i=1}^{n-1} \beta_i(x_0) \frac{\partial u}{\partial x_i}(x') + \gamma(x')u(x') \\ &+ \mu(x') \frac{\partial u}{\partial x_n}(x') - \delta(x')Au(x') = 0, \end{aligned} \tag{2.90}$$

which satisfies the following conditions.

1. $\alpha_{ij} \in C^\infty(\partial D)$, $\alpha_{ij} = \alpha_{ji}$, the α_{ij} are components of a 2-contravariant tensor on ∂D (a tensor on a manifold at a given point is a tensor on the tangent space of this manifold at this point) and for any $x' \in \partial D$,

$$\sum_{i=1}^{n-1} \alpha_{ij}(x')\eta_i\eta_j \geq 0, \quad \forall \eta = \sum_{i=1}^{n-1} \eta_i dx_i \in T^*_{x'}(\partial D). \tag{2.91}$$

2. $\beta_i \in C^\infty(\partial D)$.

3. $\gamma \in C^\infty(\partial D)$ and $\gamma(x') \leq 0, \forall x' \in \partial D$.

4. $\mu \in C^\infty(\partial D)$ and $\mu(x') \geq 0, \forall x' \in \partial D$.

5. $\delta \in C^\infty(\partial D)$ and $\delta(x') \geq 0, \forall x' \in \partial D$.

Now the problem is posed as follows:

Doest it exist a Feller semigroup on $\bar{D}$ whose infinitesimal generator is equal to A and satisfies the boundary condition $Lu(x') = 0$ on ∂D?

The condition L is called a Ventcel' boundary condition.

2.3.2 Existence of Feller semigroups

The purpose of this subsection is to derive sufficients conditions on the operators A and L under which Feller semigroups exist. The results are stated without complete proofs and we refer the reader to [17] for broad and detailed arguments.

Here we summarize, in different steps, the solution of the problem in the case $n \geq 2$ as presented in [17].

Consider the following Dirichlet problem: Let α be a fixed positive constant and let f and φ be given functions. Find a function u defined in $\bar{D}$ such that

$$\begin{cases} (\alpha - A)u &= f \quad \text{in } D \\ u &= \varphi \quad \text{on } \partial D \end{cases} \tag{2.92}$$

We know from Theorem 1.18 that for any fixed

$$\theta \in (0,1), \quad f \in C^{\theta}(\bar{D}), \quad \varphi \in C^{2+\theta}(\partial D)$$

this problem has a unique solution $u \in C^{2+\theta}(\bar{D})$. In particular, for any $f \in C^{\theta}(\bar{D})$, let

$$G^0_\alpha f \in C^{2+\theta}(\bar{D})$$

be the solution of the problem (2.92) for $\varphi = 0$, and for any $\varphi \in C^{2+\theta}(\partial D)$, let

$$H_\alpha \varphi \in C^{2+\theta}(\bar{D})$$

be the solution of problem (2.92) for $f = 0$. Then we have then defined two linear operators:

$$\begin{aligned} G^0_\alpha &: C^{\theta}(\bar{D}) \to C^{2+\theta}(\bar{D}) \\ H_\alpha &: C^{2+\theta}(\partial D) \to C^{2+\theta}(\bar{D}). \end{aligned}$$

The operator G^0_α can be seen as an operator from $C(\bar{D})$ to itself with domain $C^{\theta}(\bar{D})$ and the operator H_α can be considered as an operator from $C(\partial D)$ to $C(\bar{D})$ with domain $C^{2+\theta}(\bar{D})$. Seen in this form, they have the following properties:

1. The operators G^0_α and H_α are nonnegative, continuous and verify:

$$\begin{aligned} \|G^0_\alpha\| &= \|G^0_\alpha 1\| = \sup_{x\in\bar{D}} G^0_\alpha 1(x), \\ \|H_\alpha\| &= \|H_\alpha 1\| = \sup_{x\in\bar{D}} H_\alpha 1(x). \end{aligned}$$

2. The operator G^0_α has a unique nonnegative bounded linear extension, from $C(\bar{D})$ to itself denoted also G^0_α, with domain $C(\bar{D})$ such that

$$\|G^0_\alpha\| = \|G^0_\alpha 1\| \leq \frac{1}{\alpha}.$$

   It has also the following properties: for any $f \in C(\bar{D})$,

   (a) $G^0_\alpha f = 0$ on ∂D,

   (b) for any $\beta > 0$,

$$G^0_\alpha f - G^0_\beta f + (\alpha - \beta) G^0_\alpha f G^0_\beta f = 0, \tag{2.93}$$

   (c) for any $x \in D$,

$$\lim_{\alpha\to\infty} \alpha G^0_\alpha f(x) = f(x)$$

and if $f = 0$ on ∂D, then

$$\lim_{\alpha\to\infty} \|G_\alpha^0 f - f\| = 0$$

(d) for any non-negative integer k, if $f \in C^{k+\theta}(\bar{D})$ then $G_\alpha^0 f \in C^{k+2+\theta}(\bar{D})$.

3. The operator H_α has also a unique nonnegative bounded linear extension, from $C(\partial D)$ to $C(\bar{D})$, denoted again H_α, with domain $C(\partial D)$ and norm

$$\|H_\alpha\| = 1.$$

Furthermore it has the following properties: for any $\varphi \in C(\partial D)$,

(a) $H_\alpha\varphi = \varphi$ on ∂D,

(b) for any $\beta > 0$, $H_\alpha\varphi - H_\beta\varphi + (\alpha - \beta)H_\alpha\varphi H_\beta\varphi = 0$,

(c) for any non-negative integer k, if $\varphi \in C^{k+\theta}(\partial D)$ then $H_\alpha\varphi \in C^{k+2+\theta}(\bar{D})$.

Let us consider the operator $A : C(\bar{D}) \to C(\bar{D})$ with domain $C^2(\bar{D})$ defined as follows:

$$Au = \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i \frac{\partial u}{\partial x_i} + cu.$$

We have the following property:

Theorem 2.8 *The operator A is closable and its minimum closed extension, $\bar{A}$, is an operator from $C(D)$ to itself.*

proof. Suppose that a function $u \in C^2(\bar{D})$ attains its positive maximum at a point $x_0 \in D$. Then the gradient

$$\left(\frac{\partial u}{\partial x_1}(x_0), \frac{\partial u}{\partial x_2}(x_0), \ldots, \frac{\partial u}{\partial x_n}(x_0)\right)$$

vanishes at x_0 and the Hessian

$$H(x_0) = \left(\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)\right)_{i,j}$$

is negative semi-definite [8]. Note that

$$\sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0)$$

is the trace of the product of matrix $H(x_0)$ by the matrix $(a_{ij}(x_0))$. Since the first is negative semi-definite and the latter is positive semi-definite then the product is negative semi-definite and hence its trace is nonpositive. It follows that:

$$Au(x_0) = \sum_{i,j=1}^{n} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) + c(x_0)u(x_0) \leq 0$$

since $c \leq 0$ in D.
Then from Theorem 1.17, A is closable in $C(\bar{D})$ and in particular its minimum closed extension is an operator from $C(\bar{D})$ to itself •

For any $u \in C(\bar{D})$, we can consider u as a distribution, that is, we identify u with the function from $C^\infty(D)$ to $\mathbf{R}$, denoted again u defined as follows:

$$u(g) = \int_D u(x)g(x)dx.$$

In this sense, for any multi-index α, the derivative $D^\alpha u$ of u is the distribution on D defined as follows:

$$D^\alpha u(g) = (-1)^{|\alpha|} \int_D f(x) D^\alpha g(x), \quad g \in C^\infty(D).$$

This will be the sense of derivatives when functions are considered only continuous. More details on the theory of Distributions can be found elsewhere [12], [17].

Some properties of G_α^0 and H_α are given below [17]:

Theorem 2.9 *Let $\mathcal{D}(\bar{A})$ denote the domain of $\bar{A}$. Then the operators $G_\alpha^0 : C(\bar{D}) \to C(\bar{D})$ and $H_\alpha : C(\bar{D}) \to C(\bar{D})$ are such that:*

1. *for any $f \in C(\bar{D})$,*

$$G_\alpha^0 f \in \mathcal{D}(\bar{A}), \quad \textit{and } (\alpha I - \bar{A})G_\alpha^0 f = f \quad \textit{in } D,$$

2. *for any $\varphi \in C(\partial D)$,*

$$H_\alpha \varphi \in \mathcal{D}(\bar{A}), \quad \textit{and } (\alpha I - \bar{A})H_\alpha \varphi = 0 \quad \textit{in } D,$$

3. *for any $u \in \mathcal{D}(\bar{A})$,*

$$u = G_\alpha^0 u_1 + H_\alpha u_2$$

where

$$u_1 = (\alpha I - \bar{A})u, \quad \textit{and} \quad u_2 = u \quad \textit{on} \quad \partial D.$$

Proof. These properties are based mainly on the fact that the spaces $C^\theta(\bar{D})$ and $C^{2+\theta}(\partial D)$ are dense respectively in the spaces $C(\bar{D})$ and $C(\bar{D})$ [17].

For any $f \in C(\bar{D})$ there exists a sequence (f_k) in $C^\theta(\bar{D})$ that converges to f. Then

$$\lim_{k\to\infty} G^0_\alpha f_k = G^0_\alpha f$$

since the operator G^0_α is continuous. Furthermore we have, from the definition of G^0_α, that

$$(\alpha - A)G^0_\alpha f_k = f_k \quad \text{in } D.$$

Since $\bar{A} = A$ on $C^\theta(\bar{D})$, we get

$$\lim_{k\to\infty} (\alpha - \bar{A})G^0_\alpha f_k = f.$$

Since $\bar{A}$ is closed then from Theorem 1.12, we get that

$$G^0_\alpha f \in \mathcal{D}(\bar{A}), \quad \text{and } (\alpha I - \bar{A})G^0_\alpha f = f \quad \text{in } D.$$

The same argument can be used to show the second property.

Consider the linear operator $LG^0_\alpha : C(\bar{D}) \to C(\partial D)$ whose domain is

$$\mathcal{D}(LG^0_\alpha) = \{f \in C(\bar{D}) : G^0_\alpha f \in C^2(\bar{D})\}$$

and defined as follows:

$$LG^0_\alpha f = L(G^0_\alpha f)$$

where L is the boundary condition (2.90).
We have that for any $f \in C^\theta(\bar{D})$,

$$G^0_\alpha f \in C^{2+\theta}(\bar{D}) \subseteq C^2(\bar{D})$$

and hence

$$C^\theta(\bar{D}) \subseteq \mathcal{D}(LG^0_\alpha).$$

Further this operator has the following properties:

Theorem 2.10 *There exists a unique non-negative and bounded linear operator*

$$\overline{LG^0_\alpha} : C(\bar{D}) \to C(\partial D)$$

that extends the operator LG^0_α. Furthermore, for any $f \in C(\bar{D})$, for any $\beta > 0$

$$\overline{LG^0_\alpha} f - \overline{LG^0_\beta} f + (\alpha - \beta)\overline{LG^0_\alpha} G^0_\beta f = 0. \tag{2.94}$$

Proof. Since $C^\theta(\bar{D})$ is dense in $C(\bar{D})$ and $C^\theta(\bar{D}) \subseteq \mathcal{D}(LG^0_\alpha)$, we can take:

$$\overline{LG^0_\alpha} f = \lim_{k\to\infty} LG^0_\alpha f_k$$

where (f_k) is a sequence of $C^\theta(\bar{D})$ that converges to f in $C(\bar{D})$. Equation (2.94) is an immediate consequence of equation (2.93) •

As before consider the linear operator $LH_\alpha : C(\partial D) \to C(\partial D)$ whose domain is

$$\mathcal{D}(LH_\alpha) = C^{2+\theta}(\partial D)$$

and defined from the boundary condition (2.90) as follows:

$$LH_\alpha \psi = L(H_\alpha \psi).$$

The following result holds:

Theorem 2.11 *This operator enjoys the following properties*

1. *For any $\psi \in \mathcal{D}(LH_\alpha)$, if ψ takes its positive maximum at some point $x' \in \partial D$, then $LH_\alpha(x') \leq 0$.*
2. *The operator LH_α is closable and its minimal closed extension $\overline{LH_\alpha}$ in an operator from $C(\partial D)$ to itself.*
3. *Furthermore, for any $\beta > 0$, the operators $\overline{LH_\alpha}$ and $\overline{LH_\beta}$ has the same domain and for any ψ in the domain:*

$$\overline{LH_\alpha}\psi - \overline{LH_\beta}\psi + (\alpha - \beta)\overline{LG^0_\alpha} H_\beta \psi = 0. \tag{2.95}$$

We are now ready to link Feller semigroups on the boundary of the domain D to boundary value problems. We have the following result:

Therorem 2.12 *Let $\alpha > 0$. If the operator $\overline{LH_\alpha} : C(\partial D) \to C(\partial D)$ is the infinitesimal generator of a Feller semigroup on ∂D then for any constant $\lambda > 0$, there exists a dense subset K of $C(\partial D)$ such that for any $\varphi \in K$, the problem*

$$\begin{cases} (\alpha - A)u &= 0 \quad \text{in } D \\ (\lambda - L)u &= \varphi \quad \text{on } \partial D \end{cases} \tag{2.96}$$

has a solution u in $C^{2+\theta}(\bar{D})$.
Conversely, if there exists $\lambda \geq 0$ and a dense subset K of $C(\partial D)$ such that for any $\varphi \in K$, the problem above has a solution u in $C^{2+\theta}(\bar{D})$, then the operator $\overline{LH_\alpha}$ is the infinitesimal generator of a Feller semigroup on ∂D.

Proof. Suppose that the operator $\overline{LH_\alpha}$ is the infinitesimal generator of a Feller semigroup on ∂D. Then since a Feller semigroup is strongly continuous, we have from Theorem 1.14 that for any $\lambda > 0$, the operator $\lambda I - \overline{LH_\alpha}$ is a bijection from its domain to $C(\partial D)$. Then its range $R(\lambda I - \overline{LH_\alpha})$ is equal to $C(\partial D)$. Since $\lambda I - \overline{LH_\alpha}$ is equal to the minimum closed extension of $\lambda I - LH_\alpha$ we have from Corollary 1.1 that

$$R(\lambda I - \overline{LH_\alpha}) \subseteq \overline{R(\lambda I - LH_\alpha)}$$

and hence

$$\overline{R(\lambda I - LH_\alpha)} = C(\partial D).$$

Then $R(\lambda I - LH_\alpha)$ is a dense subset of $C(\partial D)$.

Let $\varphi \in R(\lambda I - LH_\alpha)$. Since the domain of LH_α is $C^{2+\theta}(\partial D)$ then there exists $\psi \in C^{2+\theta}(\partial D)$ such that

$$\varphi = (\lambda I - LH_\alpha)(\psi) = (\lambda - L)H_\alpha\psi,$$

where the last equality follows the fact that

$$LH_\alpha\psi \in C(\partial D).$$

From the definition of H_α, we have that

$$u = H_\alpha\psi \in C^{2+\theta}(\bar{D})$$

is a solution of Problem (2.96).
The converse is a consequence of Theorem 1.17. Let us suppose that there exists $\lambda \geq 0$ and a dense subset K of $C(\partial D)$ such that for any $\varphi \in K$, Problem (2.96) has a solution u in $C^{2+\theta}(\bar{D})$. Let $\varphi \in K$ and let $u \in C^{2+\theta}(\bar{D})$ be a solution of the problem

$$\begin{cases} (\alpha - A)u &= 0 \quad \text{in } D \\ (\lambda - L)u &= \varphi \quad \text{on } \partial D \end{cases} \tag{2.97}$$

Since $u \in C^{2+\theta}(\bar{D})$, then the restriction $u|_{\partial D}$ of u on ∂D belongs to $C^{2+\theta}(\partial D)$ and we have that

$$u = H_\alpha(u|_{\partial D}).$$

Then we have from equation (2.97) that:

$$\lambda u|_{\partial D} - Lu = \varphi$$

and then

$$\lambda u|_{\partial D} - LH_\alpha(u|_{\partial D}) = \varphi.$$

Then

$$(\lambda - LH_\alpha)u|_{\partial D} = \varphi$$

and therefore

$$\varphi \in R(\lambda - LH_\alpha).$$

It follows that

$$K \subseteq R(\lambda - LH_\alpha)$$

and hence the range $R(\lambda - LH_\alpha)$ of $\lambda - LH_\alpha$ is also dense in $C(\partial D)$. Now adding the Property 1 of Theorem 2.11 we get by Theorem 1.17 that $\lambda I - \overline{LH_\alpha}$ is the infinitesimal generator of some Feller semigroup on ∂D •

The following definition will be used in the sequel.

Definition 2.2 *The Ventecel' boundary condition L* (2.90) *is said to be transversal if*

$$\mu(x') + \delta(x') > 0, \forall x' \in \partial D.$$

Now we are ready to state the theorem which gives the solution to the problem [17].

Theorem 2.13 *Suppose that the differential operator A given by equation* (2.88) *satisfies conditions* (2.89) *and the boundary condition L defined by equation* (2.90) *satisfies all the conditions* (2.91) *and is transversal on ∂D. Suppose also that the following conditions are satisfied:*

1. *There exist constants $\alpha \geq 0$ and $\lambda \geq 0$ and a dense subset K of $C(\partial D)$ such that for any $\varphi \in K$, the boundary value problem*

$$\begin{cases} (\alpha - A)u &= 0 \quad \text{in } D \\ (\lambda - L)u &= \varphi \quad \text{on } \partial D \end{cases} \tag{2.98}$$

   has a solution u in $C(\bar{D})$.

2. *There exists a constant $\alpha > 0$ such that any solution to the problem*

$$\begin{cases} (\alpha - A)u &= 0 \quad \text{in } D \\ Lu &= 0 \quad \text{on } \partial D \end{cases} \tag{2.99}$$

   that belongs to $C(\bar{D})$ vanishes in the domain D.

Then there exists a Feller semigroup $(T_t)_{t\geq 0}$ on $\bar{D}$ whose infintesimal generator $\mathcal{U}$ has domain

$$\mathcal{D}(\mathcal{U}) = \{u \in C(\bar{D}) : Au \in C(\bar{D}), Lu = 0\}$$

and such that

$$\mathcal{U}u = Au$$

where Au and Lu are taken in the sense of distribution.

It is not easy to apply directly this result to construct Feller semigroups since the existence of solutions of the given bounded value problem can be very difficult to handle. There is another result deduced from this theorem that will be used in the sequal [17]. First, we have the following definitions.

Definition 2.3 *Let $x' \in \partial D$ and let $v \in T_{x'}(\partial D)$. Then v can be written as*

$$v = \sum_{i,j=1}^{n-1} v_j \frac{\partial}{\partial x_i}(x')$$

where v_j are constants. The vector v is said to be subunit for the operator

$$L_0 = \sum_{i,j=1}^{n-1} \alpha_{ij} \frac{\partial^2}{\partial x_i \partial x_j}$$

if

$$\left(\sum_{i=1}^{n-1} v_i \eta_i \right)^2 \leq \sum_{i,j=1}^{n-1} \alpha_{ij}(x')\eta_i\eta_j, \quad \forall \eta = \sum_{i=1}^{n-1} \eta_i dx_j \in T^*_{x'}(\partial D). \tag{2.100}$$

Definition 2.4 *Let $\rho > 0$. A path $\gamma : [0, \rho] \to \partial D$ is said to be Lipschitz if there exists a constant $C > 0$ such that for any t, s*

$$|\gamma(t) - \gamma(s)| \leq C|t - s|.$$

Definition 2.5 *For any $\rho > 0$, let $B_{L_0}(x', \rho)$ be the set of all points $y \in \partial D$ such that there exists a Lipschitz path $\gamma : [0, \rho] \to \partial D$ joining y to y for which the tangent vector $\gamma'(t)$ to ∂D at the point $\gamma(t)$ is subunit for L_0 for almost every t (with respect to the Lebesgue measure).*

The set $B_{L_0}(x', \rho)$ is called the non-Euclidean ball of radius ρ about x' defined by the operator L_0.
For any $\rho > 0$, let $B_E(x', \rho)$ be the set defined as follows:

$$B_E(x', \rho) = \{y \in \partial D : |x' - y| \leq \rho\}.$$

$B_E(x', \rho)$ is the closed Euclidean ball of ∂D of radius ρ about x'.

We have the following theorem [17].

Theorem 2.14 *Suppose that the differential operator A given by equation* (2.88) *satisfies conditions* (2.89) *and the boundary condition L (equation* (2.90) *satisfies all the conditions*

(2.91) *and is transversal on* ∂D.
If there exist constants $0 < \epsilon \leq 1, C > 0$ *such that for any sufficiently small* $\rho > 0$, *the following property holds*

$$B_E(x', \rho) \subset B_{L_0}(x', C\rho^\epsilon), \quad \forall x' \in M = \{x' \in \partial D : \mu(x') = 0\} \tag{2.101}$$

then there exists a Feller semigroup $(T_t)_{t\geq 0}$ *on* $\bar{D}$ *whose infinitesimal generator* $\mathcal{U}$ *is such that its domain is*

$$\mathcal{D}(\mathcal{U}) = \{u \in C(\bar{D}) : Au \in C(\bar{D}), Lu = 0\} \tag{2.102}$$

and verifies the following property:

$$\mathcal{U}u = Au, \quad \forall u \in \mathcal{D}(\mathcal{U}). \tag{2.103}$$

The generator $\mathcal{U}$ *is equal to the minimum closed extension in* $C(\bar{D})$ *of the restriction of* A *to the space* $\{u \in C^2(\bar{D}) : Lu = 0\}$.

We will apply this theorem to construct some Feller semigroup and hence Markov processes on the closed unity disk of $\mathbf{R}^2$ in Section 3.2.
For further details on the existence of Feller semigroups, we refer the reader to [3], [17].

Chapter 3

Applications

3.1 Semigroups associated to Markov Chains

3.1.1 Discrete-Time Markov Chains

Definition 3.1 *A Markov chain is a Markov process $(X_t)_{t\in T}$ whose state space E is discrete (that is finite or countably infinite). If, in addition, the index set T is discrete , a Markov chain is termed a discrete-time Markov chain and if T is not discrete, the term continuous-time Markov chain is used.*

We will suppose that the discrete state space E is endowed with its discrete topology (any subset of E is an open set). In this case, any subset of E is also a Borel set of E. We will denote as usual the Borel $\sigma - algebra$ on E by $\mathcal{E}$. From Theorem 1.11, we can simply define a discrete-time Markov chain as follows [1],[13].

Definition 3.2 *Let $(X_0, X_1, X_2, \dots,)$ be a sequence of random variables defined from a probability space $(\Omega, \mathcal{F}, P)$ to a discrete topological space E. The stochastic process $\chi = (X_n)_{n\geq 0}$ is said to be a (discrete-time) Markov chain if for any positive integer n and any states $i_0, i_1, \dots, i_n, j \in E$, one has :*

$$P\{X_{n+1} = j|X_n = i_n, X_{n-1} = i_{n-1}, \dots, X_0 = i_0\} = P\{X_{n+1} = j|X_n = i_n\}.$$

The conditional probabilities

$$p_{ij}(n) = P\{X_{n+1} = j|X_n = i\}$$

are called one-step transition probabilities. If this probabilities does not depend on n, the chain is said to be homogeneous.
We will always assume that this is the case and denote p_{ij} instead of $p_{ij}(n)$.
The matrix

$$Q = (p_{ij})_{i,j\in E}$$

is called the one-step transition matrix of the chain.
More generally, the conditional probabilities

$$p_n(i,j) = P\{X_n = j | X_0 = i\}$$

are called the n-step transition probabilities of the Markov chain χ. These probabilities verify the Chapman-Kolmogorov equations [1], [7] [13]:

$$p_{n+m}(i,j) = \sum_{k \in E} p_n(i,k) p_m(k,j) \tag{3.1}$$

and hence the matrix $(p_n(i,j))_{ij}$ is equal to Q^n.

Let us define the functions

$$p_n : E \times \mathcal{E} \to \mathbf{R}, \qquad n = 0, 1, 2, \ldots$$

as follows:

$$p_n(i,A) = P\{X_n \in A | X_0 = i\}. \tag{3.2}$$

We have that

$$p_1(i,j) = p_{ij}, \forall i, j.$$

The following result holds:

Theorem 3.1 *The family* $(p_n)_{n \geq 0}$ *defined in equation* (3.2) *is the transition function of the Markov chain* $\chi = (X_n)_{n \geq 0}$.

Proof. It is clear that for any $n \in \mathbf{N}$ and any $A \in \mathcal{E}$ the function $p_n(., A)$ from E into $\mathbf{R}$ is measurable since in general any function defined on E is measurable. Also, for any $x \in E$, the function $p_n(x, .)$ defined from $\mathcal{E}$ into $\mathbf{R}$ by

$$p_n(x,A) = P\{X_n \in A | X_0 = i\}$$

is a measure since P is itself a measure. Furthermore, we have that:

$$p_n(x,E) = P\{X_n \in E | X_0 = i\} = 1.$$

This means that p_n is a Markov kernel on $(E, \mathcal{E})$.
Adding to this the Chapman-Kolmogorov equation, we get that the family $(p_n)_n \geq 0$ is a transition function on $(E, \mathcal{E})$. It remains to prove the validity of the equation (1.45).
Let $f : E \to \mathbf{R}$ be a bounded function.

$$E[f \circ X_n | X_m] = (p_{n-m} f) \circ X_m \quad \text{a.s.} \tag{3.3}$$

Let $\mathcal{H}$ be the $\sigma-$algebra generated by the random variable X_m. Then we have to show that the function $(p_{n-m}f) \circ X_m$ is $\mathcal{H}-$measurable and that for any $A \in \mathcal{H}$,

$$\int_A f \circ X_n dP = \int_A p_{n-m} f \circ X_m dP. \tag{3.4}$$

The $\mathcal{H}-$measurability of $(p_{n-m}f) \circ X_m$ is obvious since $p_{n-m}f$ is measurable. Let $A \in \mathcal{H}$. we have that:

$$\int_A f \circ X_n dP = \sum_{j \in E} P\{X_n^{-1}(j) \cap A\} f(j) \tag{3.5}$$

Since for any $i \in E$,

$$p_{n-m}f(i) = \int_E p_{n-m}(i, dy) f(y) = \sum_{j \in E} p_{n-m}(i, j) f(j)$$

we have that for any $x \in \Omega$,

$$p_{n-m}f \circ X_m(x) = \sum_{j \in E} p_{n-m}(X_m(x), j) f(j).$$

Then

$$\begin{aligned} \int_A p_{n-m} f \circ X_m dP &= \sum_{j \in E} f(j) \int_A p_{n-m}(X_m(x), j) dP(x) \\ &= \sum_{j \in E} f(j) \sum_{i \in E} p_{n-m}(i, j) P\{X_m^{-1}(i) \cap A\}. \end{aligned}$$

Since $A \in \mathcal{H} = X_m^{-1}(\mathcal{E})$ then there exists $K \subseteq E$ such that $A = X_m^{-1}(K)$. Therefore

$$\begin{cases} X_m^{-1}(i) \cap A = X_m^{-1}(i) & \text{if } i \in K \\ X_m^{-1}(i) \cap A = \emptyset & \text{otherwise.} \end{cases}$$

Then we get

$$\begin{aligned} \sum_{i \in E} p_{n-m}(i, j) P\{X_m^{-1}(i) \cap A\} &= \sum_{i \in K} p_{n-m}(i, j) P\{X_m^{-1}(i)\} \\ &= \sum_{i \in K} P\{X_n = j | X_m = i\}.P\{X_m = i\} \\ &= \sum_{i \in K} P(\{X_n = j\} \cap \{X_m = i\}) \\ &= P\{\{X_n = j\} \cap [\cup_{i \in K}\{X_m = i\}]\} \\ &= P\left\{X_n^{-1}(j) \cap A\right\}. \end{aligned}$$

Then

$$\int_A p_{n-m} f \circ X_m dP = \sum_{j \in E} P\left\{X_n^{-1}(j) \cap A\right\} f(j). \tag{3.6}$$

From equations (3.5) and (3.6), we have that (3.4) holds true •

We can now construct the semigroup associated to a Markov chain. Let $b\mathcal{E}$ the set of bounded and $\mathcal{E}-$ measurable functions $f : E \to \mathbf{R}$. Since $\mathcal{E}$ contains all subsets of E, it is clear that $b\mathcal{E}$ contains all bounded functions defined on E. In the special case where E is also finite, $b\mathcal{E}$ contains all functions defined on E.
Let $(T_n)_{n\geq 0}$ be the semigroup on $b\mathcal{E}$ associated with the Markov chain $\chi = (X_n)_{n\geq 0}$. Then from Theorem 2.1 we have that: for any $n \geq 0$, for any $f \in b\mathcal{E}$, for any $x \in E$,

$$T_n f(x) = \int_E p_n(x, dy) f(y)$$

and then

$$T_n f(x) = \sum_{y\in E} p_n(x, y) f(y). \tag{3.7}$$

Now if we take $E = \{1, 2, \ldots\}$ and identify any function $f \in b\mathcal{E}$, with the sequence $(f(0), f(1), f(2), \ldots,)$ [6], we can see the right hand side of equation (3.7) as a product of matrices and then

$$T_n f = Q^n f, \qquad \forall n \in \mathbf{N}, \tag{3.8}$$

since the probabilities $p_n(x, y)$ are the elements of Q^n where Q is the one-step transition matrix of the chain. It follows that

$$T_n = Q^n. \tag{3.9}$$

Now we state:

Theorem 3.2 *The semigroup of a discrete-time Markov chain with one-step transition matrix Q is the sequence $(Q^n)_{n\geq 0}$ of the nonnegative powers of the matrix Q.*

Let us check whether the converse of this result is also true.
Suppose that E is a finite set and as before suppose that $\mathcal{E}$ is the discrete topology on E. Let us now study the form of contractive nonnegative semigroups $(T_0, T_1, T_2, T_3, \ldots)$ on E and the associated Markov chains. First we have the following definition.

Definition 3.3 *Let A be a square matrix of nonnegative real numbers. A is said to be a substochastic matrix if the entries of each row sum up to a number less or equal to 1. If the sum of the entries of each row is equal to 1, A is said to be a stochastic matrix.*

Let us suppose that $E = \{1, \ldots, m\}$. Since E is finite and its topology is discrete, any function $f : E \to \mathbf{R}$ is bounded and continuous and as before, f is identified with the vector

$$(f(1), f(2), \ldots, f(m)) \in \mathbf{R}^m.$$

The set of continuous and bounded functions defined on E is therefore identified with the vector space $\mathbf{R}^m$. Let

$$(T_n)_{n\geq 0} = (T_0, T_1, T_2, \ldots)$$

be a semigroup on E which is contractive and nonnegative. With the identification above, for any $n \in \mathbf{N}$, T_n is a linear map on $\mathbf{R}^m$. Let Q be the matrix of T_1 with respect to the canonical basis of $\mathbf{R}^m$. (We will simply write $T_1 = Q$ since by fixing a basis any linear map is represented by one matrix and conversely any matrix represents one linear map). Then for any nonnegative vector

$$x = (x_1, x_2, \ldots, x_m)^T \in \mathbf{R}^m,$$

$$T_1(x) = Qx \geq 0$$

since T_1 is nonnegative. Then the matrix Q has only nonnegative entries. Also we have that:

$$\|T_1\| = \|Q\| = \sup\{\|Qx\| : \|x\| = 1\} \leq 1$$

where

$$\|x\| = \sup\{|x_i| : i = 1, 2, \ldots, m\}.$$

Then by taking $x = (1, 1, \ldots, 1)^T$, we get that the sum of all the entries of each row of the matrix Q is less or equal to 1. Then the matrix Q is substochastic.
From

$$T_{n+m} = T_n.T_m$$

we have that:

$$T_2 = T_1.T_1 = Q^2$$

and by induction

$$T_n = Q^n, \quad \forall n \in \mathbf{N}.$$

From Theorem 2.3, there exists a unique transition function $p_n(x, dy)$ on E such that for any $n \geq 0$:

$$T_n f(x) = \int_E p_n(x, dy) f(y) \qquad \forall f \in C(E), \forall x \in E.$$

In the current situation, this can be written as:

$$(Q^n x)_i = \sum_{j \in E} p_n(i, j) x_j \quad \forall x \in \mathbf{R}^m,$$

where $(Q^n x)_i$ is the ith component of the vector $Q^n x$, meaning that, if $Q^n = (a_{ik})$ then

$$(Q^n x)_i = \sum_{k=1}^{m} a_{ik} x_k$$

By taking successively $x = (1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots (0, 0, \ldots, 1)$ we get

$$p_n(i, 1) = a_{i1}, \quad p_n(i, 2) = a_{i2}, \ldots, \quad p_n(i, m) = a_{im}$$

that is

$$p_n(i, j) = a_{ij}.$$

We conclude that the transition function p_n is such that for any states i, j $p_n(i, j)$ is the $(i, j)-$ entry of the matrix Q^n. We can now state:

Theorem 3.3 *For any finite and discrete topological space E, any contractive and non-negative semigroup $(T_1, T_2, T_3, \ldots)$ on $C(E)$ is of the form Q^n where Q is a substochastic matrix of order $m = |E|$. If the matrix Q is stochastic, then any Markov chain associated to this semigroup is such that its one-step transition matrix is equal to Q.*

If we take for example

$$E = \{1, 2, 3\},$$

we have that the space of continuous function defined on E is identified with $\mathbf{R}^3$, that is

$$C(E) = \mathbf{R}^3.$$

For any substochastic matrix Q of order 3, the sequence $(Q^n)_{n=1,2,\ldots}$ is a contractive and nonnegative semigrouop on $\mathbf{R}^3$. Conversely, any contractive and nonnegative semigroup $(T_1, T_2, T_3, \ldots)$ on $C(E)$ is of the form

$$T_n = Q^n,$$

where Q is a substochastic matrix Q of order 3.
Let us consider for example,

$$Q = \begin{pmatrix} 1/2 & 1/3 & 1/6 \\ 1/4 & 1/2 & 1/4 \\ 1/3 & 1/3 & 1/3 \end{pmatrix}.$$

Q is a stochastic matrix and $\{I, Q, Q^2, Q^3, \ldots\}$ is a contraction semigroup and any Markov chain on $\{1, 2, 3\}$ associated to this semigroup is that its one-step transition matrix is equal to Q.

We cannot do more for a discrete-time Markov chain since the notion of infinitesimal semigroup requires at least that the semigroup must be defined on a continuous index set.

3.1.2 Continuous-time Markov chains

Definition 3.4 *Let $(X_t)_{t \geq 0}$ be a stochastic process whose base space is the probability space $(\Omega, \mathcal{F}, P)$ and state space is a discrete topological space E. The process $(X_t)_{t \geq 0}$ is said to be a continuous-time Markov chain if for any real numbers $0 \leq s \leq t$ and any states $j, i, i_u (0 \leq u < s)$:*

$$P\{X_t = j | X_s = i, X_u = i_u (0 \leq u < s)\} = P\{X_t = j | X_s = i\}. \tag{3.10}$$

If the probabilities in (3.10) depend only on the difference $t - s$, the Markov chain is said to be homogeneous. In the sequel, we consider only homogeneous Markov chains.

We also suppose that $E = \{0, 1, 2, \ldots\}$ and we will denote by $\mathbf{R}^E$ the set of all sequences of terms in $\mathbf{R}$ indexed by elements of E, that is

$$\mathbf{R}^E = \{(x_i)_{i \in E} : x_i \in \mathbf{R}, \forall i \in E\}.$$

The conditional probabilities

$$p_{ij}(t) = P\{X_t = j | X_0 = i\}, \quad i, j \in E$$

are called the transition probabilities of the process. We will also assume that:

$$\lim_{t \to 0} p_{ij}(t) = \delta_i^j. \tag{3.11}$$

and that

$$\sum_{j \in E} p_{ij}(t) = 1, \quad \forall i \in E, \quad \forall t \geq 0. \tag{3.12}$$

Let

$$Q(t) = (p_{ij}(t)).$$

The probabilities $p_{ij}(t)$ verify the Chapman-Kolmogorov equation [1], [7],[13]:

$$p_{ij}(t + s) = \sum_{k \in E} p_{ik}(t) p_{kj}(s). \tag{3.13}$$

Let us define the functions

$$p_t : E \times \mathcal{E} \to \mathbf{R}$$

as follows:

$$p_t(i, A) = P\{X_t \in A | X_0 = i\}, \quad t \geq 0.$$

It is now clear that the functions $(p_t)_{t \geq 0}$ form a transition function on the measurable space $(E, \mathcal{E})$ and we have that:

Theorem 3.4 *The family $(p_t)_{t \geq 0}$ is the transition function of the Markov chain $(X_t)_{t \geq 0}$.*

Let $(T_t)_{t \geq 0}$ be the semigroup on $b\mathcal{E}$ (the space of bounded functions defined on E) associated to the transition function $(p_t)_{t \geq 0}$. As before, for any $t \geq 0$, the operator T_t is identified with the transition matrix

$$Q(t) = (p_{ij}(t)). \tag{3.14}$$

This means that, for any bounded sequence

$$x = (x_0, x_1, x_2, \ldots) \in \mathbf{R}^E,$$

$$T_t x = (y_0, y_1, y_2, \ldots), \text{ with } y_i = \sum_{j \in E} p_{ij}(t) x_j, \quad \forall i \in E. \tag{3.15}$$

Now we have the following result [6].

Theorem 3.5 *The semigroup $(T_t)_{t\geq 0}$ associated to a continuous-time Markov chain with transition probabilities $p_{ij}(t)$ verifying equation (3.11) is strongly continuous and contractive.*

Proof. For any $t \geq 0$, we have that:

$$\|T_t\| = \|Q(t)\| = \sup\{\|Q(t)x\| : \|x\| = 1\} = 1 \tag{3.16}$$

since

$$\sum_{j\in E} p_{ij}(t) = 1$$

and the maximum value in equation (3.16) is attained for $x = (1, 1, \ldots, 1, \ldots)$. Furthermore, for any bounded sequence

$$x = (x_0, x_1, x_2, \ldots) \in \mathbf{R}^E$$

we have that:

$$\forall i \in E, \lim_{t\downarrow 0} \left| \sum_{j\in E} p_{ij}(t)x_j - x_i \right| = \left| \sum_{j\in E} \delta_i^j x_j - x_i \right| = 0$$

Then since all the components of $T_t x - x$ converge to 0, it follows that

$$\lim_{t\downarrow 0} \|T_t x - x\| = 0 \quad \bullet$$

Remark 3.1 *For the special case where E is finite, the semigroup $(T_t)_{t\geq 0}$ is clearly a Feller semigroup since in this case E is compact and the space $C(E)$ of continuous functions on E is equal to the space $b\mathcal{E}$ (of bounded measurable functions on E).*

Let us now study the infinitesimal generator of this semigroup.
Let $\mathcal{U}$ be the infinitesimal generator of the semigroup $(T_t)_{t\geq 0}$. It is a linear operator on the space $b\mathcal{E}$ and can be seen as an operator on the space of bounded sequences of real numbers. We know that

$$\mathcal{U}x = \lim_{t\downarrow 0} \frac{T_t x - x}{t} \tag{3.17}$$

and its domain is the set of sequences x for which the limit in equation (3.17) exists.
Then

$$\begin{aligned} \mathcal{U}x &= \lim_{t\downarrow 0} \frac{Q(t)x - x}{t} \\ &= \lim_{t\downarrow 0} \frac{Q(t)x - Q(0)x}{t}. \end{aligned}$$

If

$$\begin{aligned} x &= (x_0, x_1, x_2, \ldots) \text{ and} \\ \mathcal{U}x &= (u_0, u_1, u_2, \ldots), \end{aligned}$$

then

$$u_i = \lim_{t \downarrow 0} \sum_{j \in E} \frac{p_{ij}(t) - p_{ij}(0)}{t} x_j.$$

Now if all the functions $p_{ij}(t)$ are differentiable at zero, we get:

$$u_i = \sum_{j \in E} p'_{ij}(0) x_j.$$

Then we can identify the operator $\mathcal{U}$ with the matrix

$$Q'(0) = (p'_{ij}(0)). \tag{3.18}$$

Remark 3.2 *The time that a continuous-time Markov chain spends in each state before making a transition is a random variable with exponential distribution.*
Suppose now that for each state $i \in E$, the rate of this exponential random variable is v_i. Let p_{ij} be the probability that the next state of the Markov chain will be j when it leaves state i. We have that:

$$\forall i \in E, \quad p_{ii} = 0, \quad \text{and} \quad \sum_{j \in E} p_{ij} = 1.$$

The discrete-time Markov chain on E with one-step transition matrix

$$P = (p_{ij})$$

is called the embedded Markov chain of the continuous-time Markov chain (X_t). The quantities p_{ij}, v_i along with the initial distribution determine completely a continuous Markov chain on a set E [13].

Let

$$q_{ij} = v_i p_{ij}. \tag{3.19}$$

This quantity represents the rate at which the chain makes a transition in state j if it is currently in state i and are called "instantaneous transition rates". The following properties hold [13].

Theorem 3.6 *With the notations above*

$$\lim_{t \downarrow 0} \frac{1 - p_{ii}(t)}{t} = v_i \text{ and} \tag{3.20}$$

$$\lim_{t \downarrow 0} \frac{p_{ij}(t)}{t} = q_{ij} \text{ for } i \neq j. \tag{3.21}$$

Since

$$\lim_{t\downarrow 0} p_{ij}(t) = \delta_i^j,$$

we get:

$$\begin{aligned} p'_{ii}(0) &= -v_i & (3.22) \\ p'_{ij}(0) &= q_{ij} \quad \text{for } i \neq j & (3.23) \end{aligned}$$

and the infinitesimal generator $\mathcal{U}$ of the semigroup (T_t) is fully determined:

$$\mathcal{U} = \begin{pmatrix} -v_0 & v_0 p_{01} & v_0 p_{02} & \cdots \\ v_1 p_{10} & -v_1 & v_1 p_{12} & \cdots \\ v_2 p_{20} & v_2 p_{21} & -v_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can therefore compute the semigroup (T_t) from $\mathcal{U}$ as follows:

$$T_t = e^{\mathcal{U}t}, \quad t \geq 0. \tag{3.24}$$

Conversely, let $v_0, v_1, v_2, \ldots$ be real numbers and $p_{ij}, (i, j \in E)$ be nonnegative real numbers such that

$$\sum_{j \in E - \{i\}} p_{ij} = 1, \forall i \in E.$$

Let

$$\mathcal{U} = \begin{pmatrix} -v_0 & v_0 p_{01} & v_0 p_{02} & \cdots \\ v_1 p_{10} & -v_1 & v_1 p_{12} & \cdots \\ v_2 p_{20} & v_2 p_{21} & -v_2 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.25}$$

Then $\mathcal{U}$ is the infinitesimal generator of the semigroup associated to a continuous-time Markov chain with state space E. This Markov chain is such that the time that it spends in each state before making a transition is a random variable with exponential distribution of rate v_i. Furthermore, the probability that the next state of the Markov chain will be j when it leaves state i is p_{ij}.
Then any matrix

$$A = (h_{ij}) \tag{3.26}$$

where

1. $h_{ii} < 0$,

2. $h_{ij} \geq 0$, for all $i, j \in E$ with $i \neq j$,

3. $\sum_{j \in E} h_{ij} = 0$, for any $i \in E$,

generates a contraction and strongly continuous semigroup on E and hence a continuous

Markov chain.

Remark 3.3 *Let $(X_t)_{t\geq 0}$ be a continuous-time Markov chain on E with transition probabilities $(p_{ij}(t))$ and instantaneous transition rates q_{ij} and let v_i be the rates of the amounts of time spent in states i whenever the chain enters state i. Under suitable conditions, the functions $(p_{ij}(t))$ verify the following differential equations called respectively the Kolmogorov's Backward equations and the Kolmogorov's forward equations* [7], [13].

$$p'_{ij}(t) = \sum_{k\in E-\{i\}} q_{ij}p_{kj}(t) - v_i p_{ij}(t) \quad (3.27)$$

$$p'_{ij}(t) = \sum_{k\in E-\{j\}} q_{kj}p_{ik}(t) - v_j p_{ij}(t) \quad (3.28)$$

$\forall i, j \in E, \forall t > 0.$

Using the infinitesimal generator $\mathcal{U}$ of the semigroup associated to the chain $(X_t)_{t\geq 0}$, these equations can be written as follows:

$$Q'(t) = \mathcal{U}.Q(t), \quad \forall t \geq 0 \quad (3.29)$$

$$Q'(t) = Q(t).\mathcal{U}, \quad \forall t \geq 0, \quad (3.30)$$

where

$$Q(t) = (p_{ij}(t)).$$

Remark 3.4 *Suppose that the chain $(X_t)_{t\geq 0}$ is such that starting in any state, it is possible to enter any other state and starting in any state the expected amount of time to return to the same state is finite. Then for any $j \in E$, the limit*

$$p_j = \lim_{t\to\infty} p_{ij}(t) \quad (3.31)$$

exists and is independent of the initial state i, and p_j represents the long-run proportion of time that the process is in state j. In addition, the p_j's are the solutions of the system:

$$v_j p_j - \sum_{k\in E-j} q_{kj}p_k = 0, \quad \forall j \in E \quad (3.32)$$

$$\sum_{j\in E} p_j = 1 \quad (3.33)$$

obtained from the Kolmogorov's forward equations [13].

The first equation of this system can be written in term of the infinitesimal generator $\mathcal{U}$ as:

$$L.\mathcal{U} = 0 \quad (3.34)$$

where L is the matrix of order n such that all the entries of each row are $p_0, p_1, p_2, \ldots$. We have therefore that:

$$\lim_{t\to\infty} Q(t) = L. \quad (3.35)$$

Let us consider the following application to discrete-time Markov chain.
Consider a discrete-time Markov chain $(Y_m)_{m\geq 0}$ on $E = \{1, 2, \dots, n\}$ with transition matrix

$$P = (a_{ij}).$$

We know that P is a stochastic matrix. Then the matrix

$$\mathcal{U} = P - I \tag{3.36}$$

(I is the identity matrix) has the form of equation (3.25) whith

$$v_i = 1 - a_{ii} \tag{3.37}$$

and

$$p_{ij} = \begin{cases} \frac{a_{ij}}{v_i} & \text{if } v_i \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

Then there exists a continuous-time Markov chain $(X_t)_{t\geq 0}$ such that its associated semi-group $Q(t)$ is generated by $\mathcal{U}$. This Markov chain is such that:

1. If the chain enters state i, the time it spends in this state is exponentially distributed with rate $v_i = 1 - a_{ii}$.

2. If the chain is to leave state i, then it enters state j $(i \neq j)$ with probability $p_{ij} = a_{ij}/v_i$.

Now we have the following result:

Theoreme 3.7 *The expected amount of time that the discrete-time chain (Y_m) spends in state i whenever it enters it, is the same as the expected amount of time that the continuous-time Markov chain (X_t) spends in state i (whenever it enters it).*
If the two chains (Y_m) and (X_t) has limiting probabilities, then they are equal, that is:

$$\lim_{n\to\infty} P^n = \lim_{t\to\infty} Q(t). \tag{3.38}$$

Proof. Let T be the amount of time that the discrete-time Markov chain (Y_m) spends in state i whenever it enters it. We have that for any positive integer n, $T = n$ if and only if $n-1$ successive transitions have left the chain in state i and the $n-$th transition takes the process into another state. Since the probability to remain in state i for each transition is a_{ii} and the probability to leave it is $1 - a_{ii}$, by the Markov property, we have that:

$$P\{T = n\} = a_{ii}^{n-1}(1 - a_{ii}). \tag{3.39}$$

This means that T has the geometric distribution with parameter $p = 1 - a_{ii}$. Then the expectation of T is

$$E[T] = \frac{1}{1 - a_{ii}}. \tag{3.40}$$

Let G be the amount of time that the continuous-time Markov chain (X_t) spends in state i whenever it enters it. We know that G is exponentially distributed with rate $v_i = 1 - a_{ii}$ and then its expectation is

$$E[G] = \frac{1}{v_i} = \frac{1}{1 - a_{ii}} = E[T]. \tag{3.41}$$

Let

$$L = \lim_{n \to \infty} P^n.$$

Then we have that:

$$L.\mathcal{U} = L.(P - I) = L.P - L = L - L = 0. \tag{3.42}$$

Then

$$\lim_{t \to \infty} Q(t) = L \quad \bullet$$

We can also start by a continuous-time Markov chain (X_t) on E with a bounded infinitesimal generator

$$\mathcal{U} = (u_{ij}). \tag{3.43}$$

Consider the stochastic matrix P obtained by taking

$$P = \frac{1}{\lambda}\mathcal{U} + I \tag{3.44}$$

where λ is a positive real number such that:

$$|u_{ij}| \leq \lambda, \quad \forall i, j \in E.$$

Then we have that:

$$\mathcal{U} = \lambda(P - I). \tag{3.45}$$

We have the following result:

Theorem 3.8 *Let (Y_n) be a discrete-chain Markov chain on E whose one-step transition matrix is P. Let also (X_t) be a continuous-time Markov chain on E whose infinitesimal generator is $\mathcal{U}$ such that equation (3.45) holds. If the two chains (Y_m) and (X_t) have limiting probabilities, then they are equal, that is:*

$$\lim_{n \to \infty} P^n = \lim_{t \to \infty} Q(t) = \lim_{t \to \infty} e^{\mathcal{U}t}. \tag{3.46}$$

Proof. Let $L = \lim_{n\to\infty} P^n$. Then

$$\begin{aligned} L.\mathcal{U} &= L.\lambda(P-I) \\ &= \lambda(L.P-L) \\ &= \lambda(L-L) \\ &= 0. \end{aligned}$$

And therefore

$$\lim_{t\to\infty} Q(t) = L \quad \bullet$$

An illustative example is given at the end of the next subsection.

3.1.3 Infinitesimal generators of birth and death processes

Definition 3.5 *Let $(\lambda_n)_{n\geq 0}$ and $(\mu_n)_{n\geq 1}$ be sequences of nonnegative real numbers. For any $t \geq 0$, let X_t be the number of people in a given system at time t. Suppose that whenever there are n people in the system, the amount of time required for a new arrival (called a birth) in the system is an exponential random variable with rate λ_n. Suppose also that the amount of time required for the new departure (called a death) is an exponentially distributed random variable with rate μ_n.*
The process X_t is called a "birth and death process" of birth rates $(\lambda_n)_{n\geq 0}$ and death rates $(\mu_n)_{n\geq 1}$.

It is clear that the process $(X_t)_{t\geq 0}$ is a continuous-time Markov chain with state space $E = \{0, 1, 2, 3, \ldots\}$.
Whenever there is 0 people in the system, the rate of arrival is equal to λ_0. If there are n ($n \geq 1$) people in the system, then the process remains in state n up to the new event (a birth or a death). This means that the amount of time that the process spends in state n is the minimum of two exponential random variables with rates λ_n and μ_n. It is also an exponential random variable with rate $\lambda_n + \mu_n$ since the minimum of two exponentially distributed random variable is also an exponentially distributed random variable [13]. Let v_n be the rate of the amount of time spent in state n. Then

$$\begin{aligned} v_0 &= \lambda_0, \\ v_n &= \lambda_n + \mu_n, \quad \text{for } n \geq 1. \end{aligned}$$

Let p_{ij} be the probability that the next state of the Markov chain will be j when it leaves state i. We have that:

$$p_{01} = 1, \tag{3.47}$$

$$p_{i,i+1} = \frac{\lambda_i}{\lambda_i + \mu_i}, \quad \text{for } i \geq 1, \tag{3.48}$$

$$
\begin{aligned}
p_{i,i-1} &= \frac{\mu_i}{\lambda_i + \mu_i}, \quad \text{for } i \geq 1, && (3.49)\\
p_{ij} &= 0 \quad \text{for } j \in E - \{i-1, i+1\}. && (3.50)
\end{aligned}
$$

The second and the third equations are based on the fact that if X and Y are two exponential random variables defined on $(\Omega, P, \mathcal{F})$ and taking values in the same space E, then [13]:

$$
P\{\min(X,Y) = X\} = \frac{E[X]}{E[X] + E[Y]}
$$

where the letter E stands for the expectation.
It is also clear that:

$$
\lim_{t \to 0} p_{ij}(t) = \delta_i^j.
$$

Then from Theorem 3.6, we have that

$$
\begin{aligned}
p'_{11}(0) &= -v_0 = -\lambda_0,\\
p'_{ii}(0) &= -v_i = -\lambda_i - \mu_i \quad \text{for } i \geq 1,\\
p'_{i,i-1}(0) &= q_{i,i-1} = v_i p_{i,i-1} = \mu_i \quad \text{for } i \geq 1,\\
p'_{i,i+1}(0) &= q_{i,i+1} = v_i p_{i,i+1} = \lambda_i \quad \text{for } i \geq 1,\\
p'_{ij}(0) &= 0 \quad \text{for } j \in E - \{i-1, i+1\}.
\end{aligned}
$$

Then the infinitesimal generator $\mathcal{U}$ of the semigroup associated to the birth and death process is the matrix

$$
\mathcal{U} = \begin{pmatrix}
-\lambda_0 & \lambda_0 & 0 & 0 & 0 & \dots & 0 & \dots\\
\mu_1 & -\lambda_1 - \mu_1 & \lambda_1 & 0 & 0 & \dots & 0 & \dots\\
0 & \mu_2 & -\lambda_2 - \mu_2 & \lambda_2 & 0 & \dots & 0 & \dots\\
0 & 0 & \mu_3 & -\lambda_3 - \mu_3 & \lambda_3 & \dots & 0 & \dots\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
$$

From $\mathcal{U}$ we can now deduce the semigroup $(T_t)_{t \geq 0}$ associated to the birth and death process by the usual formula:

$$
T_t = e^{\mathcal{U}t}
$$

and then determine the $p_{ij}(t)$.

As a particular case, we consider the Poisson process

3.1.4 Particular case: Poisson process

A birth and death process for which $\mu_n = 0$ and $\lambda_n = \lambda$ where λ is a positive constant is called a Poisson process of rate λ. The infinitesimal generator of the semigroup associated to the Poisson process can be identified to the matrix:

$$\mathcal{U} = \begin{pmatrix} -\lambda & \lambda & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & -\lambda & \lambda & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & -\lambda & \lambda & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & -\lambda & \lambda & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We have that for any sequence $x = (x_1, x_1, x_2, \dots)$ of real numbers:

$$\mathcal{U}x = \lambda(x_2 - x_1, x_3 - x_2, \dots, x_n - x_{n-1}, \dots) \tag{3.51}$$

By considering the linear operator A on $\mathbf{R}^E$ defined as follows:

$$\forall x = (x_1, x_2, x_3, \dots) \in \mathbf{R}^E, Ax = (x_2, x_3, \dots) \tag{3.52}$$

we get that

$$\mathcal{U}x = \lambda(Ax - x) \tag{3.53}$$

and then

$$\mathcal{U} = -\lambda(I - A). \tag{3.54}$$

3.1.5 Illustrative Example

Consider the following problem [13].

Example 3.1 *A job shop consists of three machines and two repairmen. The amount of time a machine works before breaking down is exponentially distributed with mean 10. If the amount of time it takes a single repairman to fix a machine is exponentially distributed with mean 8, then*

1. *what is the average number of machines not in use?*
2. *what proportion of time are both repairmen busy?*

We can model this system by a birth and death process.
We will suppose that the lifetimes of these machines are independent, the repair times for the two repairmen are also independent and independent with the lifetimes. Let us suppose that the system is in state i whenever there are i machines down. Let us also suppose that a birth occurs if a machine breaks down and a death occurs whenever a machine is repaired. Then the state space is $E = \{0, 1, 2, 3\}$. We can now determine the infinitesimal generator of the corresponding semigroup.
It is important to note that the minimum of independent random variables exponentially distributed with rates $a_1, a_2, \dots, a_n$ also is a random variable exponentially distributed with rate $a_1 + a_2 + \dots + a_n$ [13]. If there are 0 machine down then all the three machines

are working and the system remains in this state until a failure occurs. Since the lifetime of each machine is a random variable with exponential distribution of mean 10 then the amount of time up to the first failure is also a random variable exponentially distributed with rate 3/10. Then $\lambda_0 = 3/10$.
By the same procedure, we have that: $\lambda_1 = 2/10, \lambda_2 = 1/10, \lambda_3 = 0$ and $\mu_0 = 0, \mu_1 = 1/8, \mu_2 = 2/8, \mu_3 = 2/8$. The infinitesimal generator is therefore:

$$\mathcal{U} = \begin{pmatrix} -3/10 & 3/10 & 0 & 0 \\ 1/8 & -13/40 & 2/10 & 0 \\ 0 & 2/8 & -14/40 & 1/10 \\ 0 & 0 & 2/8 & -2/8 \end{pmatrix}.$$

We can now compute the transition probabilities by the formula

$$(p_{ij}(t)) = e^{t\mathcal{U}}, \quad \forall t \geq 0$$

or to get at least approximative values.
Using equation (3.34) and the fact that the limiting probabilities should sum up to 1, we get that the limiting probabilities of the chain are:

$$p_0 = 125/761, p_1 = 300/761, p_2 = 240/761, p_3 = 96/761.$$

Then the average number of machines not in use is

$$0p_0 + 1p_1 + 2p_2 + 3p_3 + 4p_4 = 1.4113$$

and the proportion of time both repairman are busy is $p_2 + p_3 = 44.15\%$.

In order to illustrate the results of Theorem 3.7 and Theorem 3.8, let us consider the matrix

$$P = I + \mathcal{U} = \begin{pmatrix} 7/10 & 3/10 & 0 & 0 \\ 1/8 & 27/40 & 2/10 & 0 \\ 0 & 2/8 & 26/40 & 1/10 \\ 0 & 0 & 2/8 & 6/8 \end{pmatrix}.$$

It is a stochastic matrix and hence defines a discrete-time Markov chain. Let us suppose that transitions occur each minute, that is one transition per minute. If this chain enters state 0, the amount of time it spends in this state before making a transition is a random variable of geometric distribution with parameter $1 - 7/10 = 3/10$. The mean of this variable is therefore 10/3 minutes. This is exactly the same value for the given continuous chain.

Furthemore, we have that:

$$\lim_{n\to\infty} P^n = \lim_{t\to\infty} e^{t\mathcal{U}} = \begin{pmatrix} 125/761 & 300/761 & 240/761 & 96/761 \\ 125/761 & 300/761 & 240/761 & 96/761 \\ 125/761 & 300/761 & 240/761 & 96/761 \\ 125/761 & 300/761 & 240/761 & 96/761 \end{pmatrix}.$$

The one-step transition matrix of the embedded chain doesn't verify this propery. Indeed, it is given by

$$M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 5/13 & 0 & 8/13 & 0 \\ 0 & 5/7 & 0 & 2/7 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and

$$\lim_{n\to\infty} M^n = \begin{pmatrix} 0.308641975 & 0 & 0.691358025 & 0 \\ 0 & 0.802469136 & 0 & 0.197530864 \\ 0.308641975 & 0 & 0.691358025 & 0 \\ 0 & 0.802469136 & 0 & 0.197530864 \end{pmatrix}.$$

Then the chain that we have constructed represents better the continuous-time Markov chain well than the embedded chain.

3.2 Feller semigroups on a closed disk

In this section, we want to construct some examples of Feller semigroups and hence Markov processes on the closed unity disk of $\mathbf{R}^2$ by using Theorem 2.14. The main difficulty on applying this theorem is the estimation of the non-Euclidean ball $B_{L_0}(x', \rho)$.
This set has interesting intuitive interpretation [17].
We will use notations and hypotheses of Section 2.3.

3.2.1 Atlas on the unity circle

Consider the open unity disk

$$D = \{(x, y) : x^2 + y^2 < 1\} \subset \mathbf{R}^2.$$

We have that

$$\bar{D} = \{(x, y) : x^2 + y^2 \leq 1\}, \quad \partial D = \{(x, y) : x^2 + y^2 = 1\}$$

We recall that

$$\mathbf{R}^2_+ = \{(a_1, a_2) : a_2 \geq 0\}.$$

We begin by defining on ∂D an atlas having the required form described in Section 2.3. Consider the following open subsets of $\bar{D}$:

- $U_1 = \{(x,y) \in \bar{D} : y > 0\}$,
- $U_2 = \{(x,y) \in \bar{D} : y < 0\}$,
- $U_3 = \{(x,y) \in \bar{D} : x > 0\}$,
- $U_4 = \{(x,y) \in \bar{D} : x < 0\}$.

Consider also the maps $\varphi_i : U_i \to \mathbf{R}^2_+$ defined for any $P = (x,y)$as follows:

$$\varphi_i(x,y) = (x, dist(P, \partial D) = \left(x, 1 - \sqrt{x^2+y^2}\right), \quad \text{for } i = 1,2$$

and

$$\varphi_i(x,y) = (y, dist(P, \partial D) = \left(y, 1 - \sqrt{x^2+y^2}\right), \quad \text{for } i = 3,4$$

where *dist* stands for *distance*. It is clear that for any i, φ_i is one-to-one and

$$\begin{aligned}
\varphi_i(U_i) &= \left\{\left(x, 1-\sqrt{x^2+y^2}\right) : \quad x^2+y^2 \leq 1 \text{ and } y > 0\right\} \\
&= \{(x,z) : z = 1 - \sqrt{x^2+y^2}, \quad x^2+y^2 \leq 1 \text{ and } y > 0\} \\
&= \{(x,z) : \quad 0 \leq z < 1 - |x|, \quad |x| < 1\}
\end{aligned}$$

where the last equality follows the fact that by fixing x such that $|x| < 1$, the function z varies from 0 to $1 - |x|$.
It follows that φ_i is a bijection from U_i onto $V = \varphi_i(U_i)$ and clearly V is an open subset of $\mathbf{R}^2_+$.
Its inverse is given by

$$\varphi_i^{-1}(x,y) = \begin{cases} \left(x, \sqrt{(1-y)^2 - x^2}\right) & \text{for i= 1, 2} \\ \left(\sqrt{(1-y)^2 - x^2}, x\right) & \text{for i= 3, 4} \end{cases}$$

Furthermore,

$$\{(U_i \cap D, \varphi_i|_{U_i \cap D})_{i=1,\dots,4}\}$$

is an atlas of D equivalent to its canonical atlas defined in Example 1.1.
Also for any $(x,y) \in U_i \cap \partial D$, we have that

$$\varphi_i(x,y) = \begin{cases} (x,0) & \text{if i = 1, 2} \\ (y,0) & \text{if i = 3, 4} \end{cases}$$

and then $\varphi_i|_{U \cap \partial D}(x,y)$ can be identified to x for $i = 1,2$ and to y for $i = 3,4$. It is an easy matter to show that

$$\mathcal{B} = \{(U_i \cap \partial D, \varphi_i|_{U_i \cap \partial D})_{i=1,2,3,4}\}$$

form an atlas on the circle ∂D which is compatible with its canonical atlas defined in Example 1.1.
In the sequel, we will suppose that ∂D is endowed with atlas $\mathcal{B}$.
This construction can be generalized to the unity open ball D of $\mathbf{R}^n$.

3.2.2 Euclidean and non-Euclidean balls on the unity circle

Consider an operator A of form (2.88):

$$Au(x) = \sum_{i,j=1}^{2} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{2} b_i(x)\frac{\partial u}{\partial x_i}(x) + c(x)u(x) \tag{3.55}$$

and which verifies the conditions (2.89). Let us also consider a boundary condition L of the form

$$\begin{aligned} Lu(x') &= \alpha(x')\frac{\partial^2 u}{\partial x_1^2}(x') + \beta(x_0)\frac{\partial u}{\partial x_1}(x') + \gamma(x')u(x') && (3.56)\\ &+\mu(x')\frac{\partial u}{\partial x_2}(x') - \delta(x')Au(x') && (3.57)\end{aligned}$$

verifying conditions (2.91).
In this section, we will consider the particular case where the function α is the constant function 1. We want to construct a Feller semigroup on $\bar{D}$ defined by data (A, L) using Theorem 2.14.
In this case, the operator L_0 associated to L in Definition 2.3 is defined as follows:

$$L_0 = \frac{\partial^2}{\partial x_1^2}. \tag{3.58}$$

For any $\rho > 0$, and for any

$$x' = (\cos\theta, \sin\theta) \in \partial D, \quad \theta \in [0, 2\pi],$$

we have first to determine the Euclean ball $B_E(x', \rho)$ on the unity circle ∂D.
We know that $B_E(x', \rho)$ is obtained by taking the intersection of the closed disk of $\mathbf{R}^2$ of radius ρ and centre x' with the circle ∂D.
If $\rho < 2$, then this intersection is the path of ∂D that contains x' and whose extremities (x, y) are solutions of the following system:

$$\begin{cases} (x-\cos\theta)^2 + (y-\sin\theta)^2 &= \rho^2 \\ x^2+y^2 &= 1. \end{cases}$$

The solutions of this system are

$$(x, y) = \left(\frac{(2-\rho^2)\cos\theta \pm \rho\sqrt{4-\rho^2}\sin\theta}{2}, \quad \frac{(2-\rho^2)\sin\theta \mp \rho\sqrt{4-\rho^2}\cos\theta}{2}\right).$$

This means that

$$(x,y) \quad = \quad (\cos(\theta \pm \tau), \sin(\theta \pm \tau))$$

where

$$\tau = \cos^{-1}\left(1-\rho^2/2\right).$$

Then for $\rho < 2$, $B_E(x',\rho)$ is the path of ∂D containing x' and with end points

$$(\cos(\theta \pm \tau), \sin(\theta \pm \tau)).$$

We have therefore that:

$$B_E(x',\rho) = \left\{(\cos(\theta \pm t), \sin(\theta \pm t)) \in \mathbf{R}^2 : 0 \leq t \leq \cos^{-1}\left(1-\rho^2/2\right)\right\}. \tag{3.59}$$

Secondly, we discuss the non-Euclidean ball $B_{L_0}(x',\rho)$ on the unity circle ∂D. The determination of this set is not an easy task.

We begin by the case $x' = (0,1)$ in order to show the origin of our procedure. It is assumed that ρ is a small positive number ($\rho < 1$). Points in $B_{L_0}(x',\rho)$ should be found on ∂D by looking in both sides from x'. This means that points on this set are of the form

$$(\cos(\pi/2 \pm \theta), \sin(\pi/2 \pm \theta)), \qquad \text{whith } \theta \geq 0.$$

We consider therefore a path γ on ∂D of the form

$$\gamma(t) = (\cos f(t), \sin f(t)), \quad t \in [0,\rho] \tag{3.60}$$

where f is a differentiable function such that the curve γ is Lipschitz and $f(0) = \pi/2$. It will be also assumed that γ is at least of class C^3 in an open interval containing 0. We want to derive the form of the function f for which we can reach the maximum number of points of ∂D from the point x'. The reader should note first that the function f is restricted to the subunit property as follows.

In the local chart $(U_1 \cap \partial D, \varphi_1|_{U_1 \cap \partial D})$, we have that

$$\begin{aligned} \gamma'(t) &= \frac{d}{dt}(\varphi_1 \circ \gamma)(t)\frac{\partial}{\partial x_1} \\ &= \frac{d}{dt}(\cos f)(t)\frac{\partial}{\partial x_1} \\ &= -f'(t)\sin(f(t))\frac{\partial}{\partial x_1}. \end{aligned} \tag{3.61}$$

From the definition of L_0, a tangent vector

$$v = v_1\frac{\partial}{\partial x_1}$$

is subunit to the operator L_0 if and only if for any

$$\eta = \eta_1 dx_1 \in T^*_{x'}(\partial D),$$

one has

$$(v_1\eta_1)^2 \leq \eta_1^2.$$

Then v is subunit for L_0 if and only if $|v_1| \leq 1$. Hence $v = \gamma'(t)$ is subunit for L_0 if and only if

$$|f'(t)\sin(f(t))| \leq 1. \tag{3.62}$$

A Taylor expansion of the function f in a small neighborhood of 0 yields:

$$f(t) = \frac{\pi}{2} + f'(0)t + f''(0)\frac{t^2}{2} + f'''(0)\frac{t^3}{6} + o(t^3). \tag{3.63}$$

In order to get a large path, we need to take $f'(0)$ as large as possible.
If we consider the particular function

$$f(t) = \frac{\pi}{2} + t$$

we have that for any $0 \leq t \leq \rho$, inequality (3.62) holds. Thus we should take $f'(0) \geq 1$.
If we suppose that $f'(0) > 1$, then since

$$\begin{aligned} f'(t) &= f'(0) + f''(0)t + f'''(0)\frac{t^2}{2} + o(t^2) \quad \text{and} \\ \sin f(t) &= \cos\left(f'(0)t + f''(0)\frac{t^2}{2} + f'''(0)\frac{t^3}{6} + o(t^3)\right) \\ &= 1 - \frac{t^2}{2}(f'(0))^2 + o(t^2), \end{aligned}$$

we have that for sufficiently small $t > 0$

$$f'(t)\sin f(t) = f'(0) + f''(0)t + f'''(0)\frac{t^2}{2} + o(t^2). \tag{3.64}$$

Therefore

$$f'(t)\sin(f(t)) > 1, \quad \text{in a small open interval } (0,\alpha), \quad \alpha > 0.$$

Then $\gamma'(t)$ is not subunit for L_0 for any $t \in (0,\alpha)$. Since the interval $(0,\alpha)$ has measure $\alpha \neq 0$, from Definition 2.5 we reject the case $f'(0) > 1$. We will therefore assume that $f'(0) = 1$.
Furthermore, since we want to have a large curve, the number $f''(0)$ should be nonnegative because if $f''(0) < 0$ then the function f yields a small curve than $t + \frac{\pi}{2}$. If we take

$f''(0) > 0$, then from equation (3.64), we have as before that

$$f'(t)\sin(f(t)) > 1, \quad \text{in a small open interval } (0,\alpha).$$

This implies that we should take $f''(0) = 0$.
Then we get

$$\begin{aligned} f'(t) &= 1 + f'''(0)\frac{t^2}{2} + o(t^2), \\ f'(t)\sin(f(t)) &= 1 + (f'''(0) - 1)\frac{t^2}{2} + o(t^2). \end{aligned}$$

For the same reason as above we will have $f'''(0) \leq 1$ and then an optimal choice is $f'''(0) = 1$.
Therefore, we have that the function f is of the form

$$f(t) = \frac{\pi}{2} + t + \frac{t^3}{6} + o(t^3) \tag{3.65}$$

and it can be exactly determined by developing to high orders.

In particular the exact function

$$f(t) = \frac{\pi}{2} + t + \frac{t^3}{6} \tag{3.66}$$

is such that for sufficiently small $\rho > 0$, the vector $\gamma'(t)$ is subunit for L_0 for every $t \in [0,\rho]$.
We conclude that for small $\rho > 0$, the set $B_{L_0}(x',\rho)$ contains the path

$$\left(\cos\left(\frac{\pi}{2} + t + \frac{t^3}{6}\right), \quad \sin\left(\frac{\pi}{2} + t + \frac{t^3}{6}\right)\right), \quad t \in [0,\rho].$$

This set contains also the path

$$\left(\cos\left(\frac{\pi}{2} - t - \frac{t^3}{6}\right), \quad \sin\left(\frac{\pi}{2} - t - \frac{t^3}{6}\right)\right), \quad t \in [0,\rho]$$

since

$$\begin{aligned} \sin\left(\frac{\pi}{2} + t + \frac{t^3}{6}\right) &= \cos\left(t + \frac{t^3}{6}\right) \\ &= 1 - \frac{t^2}{2} - \frac{3t^4}{24} + o(t^4) \end{aligned}$$

and hence

$$f'(t)\sin\left(\frac{\pi}{2} + t + \frac{t^3}{6}\right) = 1 - \frac{9}{24}t^4 + o(t^4) \leq 1 \quad \text{in a small interval } [0,\rho].$$

It follows that

$$B_{L_0}(x',\rho) \supseteq \left\{ \left(\cos\left(\frac{\pi}{2} \pm (t + \frac{t^3}{6}) \right), \quad \sin\left(\frac{\pi}{2} \pm (t + \frac{t^3}{6}) \right) \right), \quad t \in [0,\rho] \right\}. \tag{3.67}$$

We can now compare the balls $B_E(x',\rho)$ and $B_{L_0}(x',\rho)$. To this end, let us denote $d_E(\rho)$ and $d_L(\rho)$ the lengths of the paths $B_E(x',\rho)$ and $B_{L_0}(x',\rho)$ respectively. From (3.59) and (3.67), we have that for sufficiently small $\rho > 0$,

$$d_E(\rho) \;=\; 2\cos^{-1}(1-\rho^2/2) \text{ and} \tag{3.68}$$

$$d_L(\rho) \;\geq\; 2\left(\rho + \frac{\rho^3}{6}\right). \tag{3.69}$$

Expanding $d_E(\rho)$, we get, for sufficiently small $\rho > 0$,

$$d_E(\rho) = 2\cos^{-1}(1-\rho^2/2) = 2(\rho + o(\rho^3)) \leq 2\left(\rho + \frac{\rho^3}{6}\right). \tag{3.70}$$

It follows that for sufficiently small $\rho > 0$ we have:

$$B_E(x',\rho) \subset B_{L_0}(x',\rho). \tag{3.71}$$

The discussion above can be generalized to any point $x' \in \partial D$.
If we take

$$x' = (\cos\theta, \sin\theta), \quad \text{with } 0 < \theta < \pi \text{ or } \pi < \theta < 2\pi,$$

then the function

$$f(t) = \theta + t + \frac{t^3}{6}, \quad t \in [0,\rho]$$

determines a path of ∂D contained in $B_{L_0}(x',\rho)$. It is so since

$$f'(t)\sin(f(t)) = \sin\theta + t\cos\theta + \frac{8}{18}t^3\cos\theta + o(t^3)$$

and for sufficiently small ρ and $\theta \neq \pi/2$, we have that

$$|f'(t)\sin(f(t))| \leq 1.$$

If we take $\theta = 0$, that is, $x' = (1,0)$ then we can consider the local chart $(U_3 \cap \partial D, \varphi_3|_{U_3\cap\partial D})$ that contains x'. The path

$$\gamma(t) = (\cos f(t), \sin f(t)), \quad t \in [0,\rho]$$

where

$$f(t) = t + \frac{t^3}{6}$$

is such that

$$\begin{aligned} \gamma'(t) &= \frac{d}{dt}(\varphi_3 \circ \gamma)(t)\frac{\partial}{\partial x_1} \\ &= \frac{d}{dt}(\sin f)(t)\frac{\partial}{\partial x_1} \\ &= f'(t)\cos(f(t))\frac{\partial}{\partial x_1}. \end{aligned}$$

We have that

$$\begin{aligned} f'(t)\cos f(t) &= \left(1 - \frac{t^2}{2}\right)\left(1 - \frac{t^2}{2} - \frac{7t^4}{24} + o(t^4)\right) \\ &= 1 - \frac{13t^4}{24} + o(t^4). \end{aligned}$$

Hence

$$|f^{'}(t)\cos f(t)| \leq 1, \quad 0 \leq t \leq \rho$$

for sufficiently small $\rho > 0$.

For the case $\theta = \pi$, that is, $x' = (-1, 0)$ we can consider the chart $(U_4 \cap \partial D, \varphi_4|_{U_4 \cap \partial D})$ and the function

$$f(t) = \pi + t + \frac{t^3}{6}.$$

We have that

$$\begin{aligned} \gamma(t) &= f'(t)\cos f(t)\frac{\partial}{\partial x_1} \\ &= -\left(1 + \frac{t^2}{2}\right)\cos\left(t + \frac{t^3}{6}\right)\frac{\partial}{\partial x_1} \end{aligned}$$

and we conclude as before.

We can summarize the above construction by the following theorem.

Theorem 3.9 *For any sufficiently small number* $\rho > 0$*, and any*

$$x' = (\cos\theta, \sin\theta), \quad 0 \leq \theta \leq 2\pi,$$

if

$$L_0 = \frac{\partial^2}{\partial x_1^2}$$

then $B_{L_0}(x', \rho)$ *contains the path*

$$\left\{ \left(\cos \left(\theta \pm (t + \frac{t^3}{6}) \right), \quad \sin \left(\theta \pm (t + \frac{t^3}{6}) \right) \right), \quad t \in [0, \rho] \right\}$$

and

$$B_{L_0}(x', \rho) \supset B_E(x', \rho).$$

3.2.3 Examples of Feller semigroups

We can now apply Theorem 3.9 and Theorem 2.14 to construct Feller semigroups on the disk $\bar{D}$. We have the following result.

Theorem 3.10 *For any operator*

$$Au(x) = \sum_{i,j=1}^{2} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \partial x_j}(x) + \sum_{i=1}^{2} b_i \frac{\partial u}{\partial x_i}(x) + cu(x) \tag{3.72}$$

such that conditions (2.89) holds and for any transversal boundary condition

$$\begin{aligned} Lu(x') &= \frac{\partial^2 u}{\partial x_1^2}(x') + \beta(x_0) \frac{\partial u}{\partial x_1}(x') + \gamma(x')u(x') \\ &+ \mu(x') \frac{\partial u}{\partial x_2}(x') - \delta(x')Au(x') \end{aligned} \tag{3.73}$$

on ∂D *verifying conditions (2.91), there exists a Feller semigroup* $(T_t)_{t \geq 0}$ *on* $\bar{D}$ *whose infinitesimal generator is defined as follows:*

1. *the domain of* $\mathcal{U}$ *is*

$$\mathcal{D}(\mathcal{U}) = \{u \in C(\bar{D}) : Au \in C(\bar{D}), Lu = 0\} \tag{3.74}$$

2.

$$\mathcal{U}u = Au, \quad \forall u \in \mathcal{D}(\mathcal{U}). \tag{3.75}$$

Let us give now some numerical examples.

Example 3.2 *Consider the operator* A *and the boundary conditions* L *defined as follows:*

$$\begin{aligned} Au(x, y) &= \Delta u(x, y) = \frac{\partial^2 u}{\partial x_1^2}(x, y) + \frac{\partial^2 u}{\partial x_2^2}(x, y), \quad (x, y) \in \bar{D}. \\ Lu(x, y)) &= \frac{\partial^2 u}{\partial x_1^2}(x, y) + \left(x^2 + (y-1)^2\right) \frac{\partial u}{\partial x_2}(x, y) - Au(x, y), \quad (x, y) \in \partial D. \end{aligned}$$

There exists a Feller semigroup $(T_t)_{t \geq 0}$ in the disk $\bar{D}$ whose infinitesimal generator $\mathcal{U}$ has domain

$$\mathcal{D}(\mathcal{U}) = \{u \in C(\bar{D}) : \Delta u \in C(\bar{D}), Lu = 0\}$$

and such that

$$\mathcal{U}u = \Delta u.$$

The Feller semigroup generated by $\mathcal{U}$ is an example of what is called $\Delta-$diffusion process on $\bar{D}$ [18].

Example 3.3 *Consider the operator A and the boundary condition L defined by*

$$\begin{aligned} Au(x,y) &= e^x \frac{\partial^2 u}{\partial x_1^2}(x,y) + e^y \frac{\partial^2 u}{\partial x_2^2}(x,y) - u(x,y), \quad (x,y) \in \bar{D}. \\ Lu(x,y) &= \frac{\partial^2 u}{\partial x_1^2}(x,y) - e^{x+y} Au(x,y), \quad (x,y) \in \partial D. \end{aligned}$$

As before, there exists a Feller semigroup $(T_t)_{t\geq 0}$ in the disk $\bar{D}$ whose infinitesimal generator $\mathcal{U}$ has domain

$$\mathcal{D}(\mathcal{U}) = \{u \in C(\bar{D}) : Au \in C(\bar{D}), Lu = 0\}$$

and such that

$$\mathcal{U}u = Au.$$

From the operator $\mathcal{U}$ one can determine the operators T_t using Theorem 1.15. Further by Theorem 2.3 one can determine the transition function associated to this semigroup (at least numerically). Any realization of this transition function is a Markov process on $\bar{D}$ determined by the operator A with the boundary condition L.

Concluding remarks

In this dissertation we have highlighted relationships between three completely different subjects of Mathematics, namely:Markov processes, Semigroups and Partial differential equations. We have then illustrated these relationships in the particular case of Markov chains and applied them to construct examples of Feller semigroups and hence Markov processes on the unity disk of $\mathbf{R}^2$.
The main lessons that can be drawn from our investigations are as follows:

1. For any Markov process $(X_t)_{t\geq 0}$ on a topological space E with transition function $(P_t)_{t\geq 0}$, the linear operators T_t defined on the Banach space $b\mathcal{E}$ of bounded and measurable functions on E as follows:

$$T_t(f)(x) = \int_E f(y)P_t(x, dy), \quad f \in b\mathcal{E}, x \in E,$$

form a semigroup on $b\mathcal{E}$.

2. Conversely, if E is a separable compact metric space and if $(T_t)_{t\geq 0}$ is a nonnegative and contraction semigroup on the Banach space $C(E)$ of continuous functions on E, then there exists a unique Feller transition function $(P_t)_{t\geq 0}$ on E such that

$$T_t f(x) = \int_E P_t(x, dy)f(y), \qquad \forall f \in C(E), \forall x \in E.$$

This transition function induces a Markov process on E if an initial distribution is given.

3. Under some regularity conditions, the form of the infinitesimal generator of a Feller semigroup in the closure of a bounded domain of $\mathbf{R}^n$ (with $n \geq 2$) is of the form:

$$\begin{aligned}\mathcal{U}u(x_0) = &\sum_{i,j=1}^{n} a_{ij}(x_0)\frac{\partial^2 u}{\partial x_i \partial x_j}(x_0) + \sum_{i=1}^{n} b_i(x_0)\frac{\partial u}{\partial x_i}(x_0) + c(x_0)u(x_0)\\ &+ \int_E e(x_0, dy)\left[u(y) - u(x_0) - \sum_{i=1}^{n}\frac{\partial u}{\partial x_i}(x_0)(\chi_i(y) - \chi_i(x_0))\right]\end{aligned}$$

in the interior of the domain and verifies a boundary condition of the form

$$\sum_{i,j=1}^{n-1} \alpha_{ij}(x') \frac{\partial^2 f}{\partial x_i \partial x_j}(x') + \sum_{i=1}^{n-1} \beta_i(x_0) \frac{\partial f}{\partial x_i}(x') + \gamma(x') f(x')$$
$$+\mu(x') \frac{\partial f}{\partial x_n}(x') - \delta(x') \mathcal{U} f(x')$$
$$+ \int_E \nu(x', dy)[f(y) - f(x') - \sum_{i=1}^{n-1} \frac{\partial f}{\partial x_i}(x')(\chi_i(y) - \chi_i(x'))] = 0$$

on the boundary of the domain. In the special case where there are no integral terms in these equations, these integro-differential operators reduce to differential operators. Computing of the coefficient functions $a_{ij}(x_0)$ and (α_{ij}) in these operators uses integration over the spaces $H_{x_0} - \Phi_{x_0}(E - \{x_0\})$ and $H_{x'} - \Phi_{x'}(E - \{x'\})$ and we have explicitly given the elements of these spaces in particular cases.

4. Conversely given a second order elliptic differential operator A and a boundary condition L on the closure of a bounded domain of $\mathbf{R}^n$ the existence of Feller semigroup defined by A and L can be studied by bounded value problems.

5. Semigroups associated to discrete Markov processes are the sets of its transition matrices and the corresponding infinitesimal generators can also be identified with matrices. In finite case, it is possible to associate to a continuous-time Markov chain, using its infinitesimal generator, a discrete-time Markov chain which has similar properties with the first chain. This has been illustrated in the case of birth and death processes.

6. If D is the open unity disk of $\mathbf{R}^2$, then for any sufficiently small number $\rho > 0$, and any x' of the circle ∂D if

$$L_0 = \frac{\partial^2}{\partial x_1^2},$$

then the non-Euclidean ball $B_{L_0}(x', \rho)$ of radius ρ about x' of ∂D contains the Euclidean ball $B_E(x', \rho)$ of centre x' and radius x'. Feller semigroups and Markov processes can be constructed on the disk $\bar{D}$ from this result.

The non-Euclidean balls $B_{L_0}(x', \rho)$ play a fundamental role in the construction of Markov processes. Since they are not fully described, further investigations should be made in this direction. It should also be important to investigate how the relationships between Markov processes, Semigroups and Partial differential equations discussed in this disseretation can be applied to model real life systems.

Bibliography

[1] BEICHELT, F. E. and FATTI, L. P.(2002). *Stochastic processes and their applications.* New York: Taylor & Francis.

[2] BERGER, M. and GOSTIAUX, B. (1987). *Géométrie différentielle: variétés, courbes et surfaces.* Paris: Presses Universitaires de France.

[3] BONY, J.-M.,COURREGE, P., PRIOURET,P. (1968). Semi-groupes de Feller sur une variété à bord compacte et problèmes aux limites intégro-différentiels du second ordre donnant lieu au principe du maximum. *Ann. Inst. Fourier*(Grenoble) 18, 369-521.

[4] DELLACHERIE, C. and MEYER, P-A. (1975). *Probalitité et potentiel, Chapitres I à IV.* Paris: Herman.

[5] DELLACHERIE, C. and MEYER, P-A. (1983). *Probalitité et potentiel, Chapitres IX à XI, Théorie discrète du potentiel.* Paris: Herman.

[6] DELLACHERIE, C. and MEYER, P-A. (1987). *Probalitité et potentiel, Chapitres XII à XIV, Théorie du potentiel associée à une résolvente, Théorie des processus de Markov.* Paris: Herman.

[7] GUIKHMAN, I. and SKOROKHOD, A. (1980). *Introduction à la théorie des processus aléatoires.* Moscou: Mir.

[8] LUENBERGER, D. G. (1984). *Linear and Nonlinear programming.* Massachusets: Addison-Wesley.

[9] MARLE, C-M. (1974). *Mesures et Probabilités.* Paris: Herman.

[10] MONFORT, A. (1980). *Cours de probabilités.* Paris: Economica.

[11] QIAN, Z. and YING, J. G. (2005). *Markov semigroups and Markov processes.* University of Oxford: Mimeo.

[12] RENARDY, M. and ROGERS, C.R. (2004). *Introduction to Partial differential equations.* 2^{nd} edition. New-York: 1993 Springer-Verlag, Inc.

[13] ROSS, S.M. (2002). *Introduction to probability models* 7^{th} edition. Boston: Academic Press.

[14] SHALIZI, C. (2007). *Advanced Probability II.* Retrieved August 10, 2007, from http://www.stat.cmu.edu/ cshalizi/754/notes/all.pdf

[15] SMOLLER, J. (1983). *Shock Waves and Reaction-Diffusion Equations.* New York: Springer-Verlag.

[16] STERNBERG, S. (1964). *Lectures on Differential Geometry.* New Jersey: Prentice-Hall, Inc.

[17] TAIRA, K. (1988). *Diffusion processes and Partial differential equations.* Ibaraki: Academic Press, Inc.

[18] TAIRA, K. (1979). Sur l'existence de processus de diffusion, *Ann. Inst. Fourier* (Grenoble) 29, 99-126.

[19] THOMSEN, K. *The Riesz representation theorem.* Retrieved July 30, 2007, from www.imf.au.dk/kurser/advanalyse/F06/Riesz.pdf

[20] VENTTSEL', A. D. (1959).
On boundary conditions for multidimensional diffusion processes. *Theory of Probability and its applications*, 4 (2), 164-177.