MAXIMAL NONTRACEABLE GRAPHS

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Summary

A graph G is *maximal nontraceable* (MNT) (*maximal nonhamiltonian* (MNH)) if G is not traceable (hamiltonian), i.e. does not contain a hamiltonian path (cycle), but $G+xy$ is traceable (hamiltonian) for all nonadjacent vertices x and y in G. A graph G is *hypohamiltonian* if G is not hamiltonian, but every vertex deleted subgraph $G-v$ of G is hamiltonian. A graph which is maximal nonhamiltonian and hypohamiltonian is called *maximal hypohamiltonian* (MHH).

Until recently, not much has appeared in the literature about MNT graphs, although there is an extensive literature on MNH graphs. In 1998 Zelinka constructed two classes of MNT graphs and made the conjecture, which he later retracted, that every MNT graph belongs to one of these classes. We show that there are many different types of MNT graphs that cannot be constructed by Zelinka's methods.

Although we have not been able to characterize MNT graphs in general, our attempt at characterizing MNT graphs with a specified number of blocks and cut-vertices enabled us to construct infinite families of non-Zelinka MNT graphs which have either two or three blocks.

We consider MNT graphs with toughness less than one, obtaining results leading to interesting constructions of MNT graphs, some based on MHH graphs. One result led us to discover a non-Zelinka MNT graph of smallest order, namely of order 8. We also present examples of MNT graphs with toughness at least one, including an infinite family of 2-connected, claw-free graphs.

We find a lower bound for the size of *2-connected* MNT graphs of order n. We construct an infinite family of 2-connected cubic MNT graphs of order n, using MHH graphs as building blocks. We thus find the minimum size of *2-connected* MNT graphs for infinitely many values of n. We also present a construction, based on MHH graphs, of an infinite family of MNT graphs that are almost cubic. We establish the minimum size of MNT graphs of order n, for all except 26 values of n, and we present a table of MNT graphs of possible smallest size for the excluded 26 values of n.

Key terms:

graph theory; hamiltonian path; traceable; nontraceable; maximal nontraceable; hamiltonian cycle; hamiltonian; nonhamiltonian; maximal nonhamiltonian; hypohamiltonian; maximal hypohamiltonian; hamiltonian-connected; maximal nonhamiltonian-connected

Chapter 1

Introduction

In the first section of this chapter we present the notation and a number of basic definitions that will be used throughout this thesis. In the second section we give the background to the problem of characterizing and constructing maximal nontraceable graphs of various types, and in the third section we provide an overview of the contents of this thesis.

1.1 Definitions

A *simple graph* G with n vertices and m edges consists of a vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and an edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, where each edge is an unordered pair of distinct vertices. Since $E(G)$ is a set, in a simple graph no edge is repeated. We consider only simple graphs, and for brevity we use the term *graph* to mean simple graph. An edge $e = \{u, v\}$ is said to join the vertices u and v, and we denote $\{u, v\}$ by uv or vu. The *complement* of a graph G, denoted by $\overline{G}$, is the graph with vertex set $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $u \in V(G)$, $v \in V(G)$ and $uv \notin E(G)$.

We denote the cardinality of any set S by $|S|$. The cardinalities $|V(G)|$ and $|E(G)|$ are called the *order* and *size* of the graph G, respectively, and we denote them by $v(G)$ and $e(G)$.

An edge $e \in E(G)$ is *incident* with a vertex $v \in V(G)$ if $v \in e$, and the *degree* of a vertex v is the number of edges incident with v, which is denoted by $\deg_G(v)$ or simply by $\deg(v)$ if no confusion can result. The *maximum degree*, denoted by $\Delta(G)$, and the *minimum degree*, denoted by $\delta(G)$, of a graph G are, respectively, the maximum and minimum degrees of the vertices of G. If $uv \in E(G)$ then we say that the vertices u and v are adjacent, or that u is a *neighbour* of v. The *open neighbourhood* of a vertex v in $V(G)$ is the set $N_G(v) = \{x \in V(G) : vx \in E(G)\}$. Again, if no confusion can arise, we use $N(v)$ to denote the open neighbourhood of v. If H is a subgraph of G and $x \in G - V(H)$, then $N_H(x)$ denotes $N_G(x) \cap V(H)$. A vertex v in $V(G)$ is a *universal vertex* of G if it is adjacent to all vertices in $V(G)$. If e_1 and e_2 are distinct edges of G which are incident with a common vertex, then e_1 and e_2 are *adjacent edges*.

A graph G is *regular of degree r* if $\deg(v) = r$ for each $v \in V(G)$. Such a graph is called *r-regular*. A 3-regular graph is also called a *cubic graph*.

Two graphs G and H are *isomorphic* if there exists a bijection $\psi : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $\psi(u)\psi(v) \in E(H)$. A graph which is isomorphic to a graph G is called a *copy* of G. A graph H is a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A graph that is a copy of a subgraph of G will also be called a subgraph of G. If H is a subgraph of G we write $H \subseteq G$.

A subgraph of G having the same order as G is called a *spanning subgraph* of G. If $U \subseteq V(G)$ is nonempty, then the subgraph of G *induced* by U has vertex set U and edge set $\{uv : u \in U, v \in U, uv \in E(G)\}$. We denote the subgraph of G induced by $U \subseteq V(G)$ by $G\langle U\rangle$. If no confusion can result we simply write $\langle U\rangle$. A subgraph H of G is called *vertex-induced* or simply *induced* if $H = \langle U\rangle$ for some $U \subseteq V(G)$. Similarly, if $X \subseteq E(G)$ is nonempty, then the subgraph $\langle X\rangle$ *induced* by X has vertex set $\{u : u \in e, e \in X\}$ and edge set X. A subgraph H of G is *edge-induced* if $H = \langle X\rangle$ for some $X \subseteq E(G)$. If $H \subseteq V(G)$ then we write $G - H$ for the graph $G\langle V(G) \setminus H\rangle$, and if $H = \{w\}$ we write $G - w$ instead of

$G - \{w\}$. If $u \in V(G)$ and $v \in V(G)$, but $uv \notin E(G)$, then $G + uv$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup \{uv\}$. Similarly, if $uv \in E(G)$, then $G - uv$ is the graph with vertex set $V(G)$ and edge set $E(G) \setminus \{uv\}$.

The *join* G of two graphs G_1 and G_2, denoted by $G = G_1 + G_2$, is such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2) \cup \{uv : u \in V(G_1) \text{ and } v \in V(G_2)\}$.

The *line graph* $L(G)$ of a graph G is such that there is a one-to-one correspondence between the vertices of $L(G)$ and the edges of G so that two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G are adjacent.

A graph G is *complete* if $uv \in E(G)$ for all distinct vertices u, v in $V(G)$. The complete graph of order n is denoted by K_n. A K_3 is also known as a *triangle*. A complete subgraph of a graph G is called a *clique*.

A *path* is a sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ of vertices and edges such that $e_i = v_{i-1}v_i$ for $i = 1, 2, ..., k$ and no vertex is repeated. We usually denote a path P by listing the vertices in the path as follows; $P : v_0v_1v_2 \ldots v_k$. The vertices v_0 and v_k are called *endvertices* of P, and the vertices of P which are not endvertices are called *internal* vertices of P. We say that P is a path from v_0 to v_k. The size of a path P is also called the *length* of P.

If u, v are endvertices of a path P we say that u is *joined* to v by P, or u and v are joined by P. We also refer to P as a $u - v$ path. A graph G is *connected* if any two vertices u, v in $V(G)$ are joined by a path in G. A graph that is not connected is called *disconnected*.

A *component* of a graph G is a maximal connected induced subgraph of G. A *cut-vertex* of G is a vertex $v \in V(G)$ such that $G - v$ has more components than G and a *vertex-cut* of G is a set S of vertices of G such that $G - S$ has more components than G. A nontrivial connected graph with no cut-vertices is called a *nonseparable* graph. A graph G is *2-connected* if G is nonseparable and $v(G) \geq 3$. A *block* of a graph G is a maximal nonseparable induced subgraph of G. Every two blocks of G have at most one vertex in common, namely a cut-vertex.

A *cycle* of order n, denoted by C_n, consists of n vertices $v_1, v_2, \ldots, v_n$ such that v_i is adjacent to v_{i+1} for $i = 1, 2, \ldots, n-1$ and v_n is adjacent to v_1. The *girth* $g(G)$ and the *circumference* $c(G)$ are, respectively, the order of a shortest and a longest cycle in G.

A cycle in a graph G is called a *hamiltonian cycle* if it contains all the vertices of G. In such a case the graph G is called a *hamiltonian graph*. A graph G is *maximal nonhamiltonian* (MNH) if G is not hamiltonian, but $G + e$ is hamiltonian for all $e \in E(\overline{G})$.

A path in a graph G is called a *hamiltonian path* if it contains all the vertices of G. In such a case the graph G is called a *traceable graph*. If $v \in V(G)$ is an endvertex of a hamiltonian path in G, we say that G is traceable from v. A graph G is *homogeneously traceable* if it is traceable from each $v \in V(G)$. A graph G is *maximal nontraceable* (MNT) if G is not traceable, but $G + e$ is traceable for all $e \in E(\overline{G})$.

It follows from the above definitions that if a graph G is MNH then any two nonadjacent vertices are the endvertices of a hamiltonian path in G. Thus an MNH graph is traceable.

The *detour order of a vertex* $v \in V(G)$ is the order of a longest path P in G having v as an endvertex, and is denoted by $\tau_G(v)$, or by $\tau(v)$ if no confusion can arise. The *detour order of a graph* G is the order of a longest path in G and is denoted by $\tau(G)$, i.e. $\tau(G) = \max\{\tau(v) | v \in V(G)\}$.

Finally, a graph G is *hypohamiltonian* if G is not hamiltonian, but $G - v$ is hamiltonian for all $v \in V(G)$. We say that a graph G is *maximal hypohamiltonian* (MHH) if G is MNH and hypohamiltonian.

Other definitions will be given where they are needed. For any concept not defined here we use the definition given in Chartrand and Lesniak [7].

1.2 Background

Until recently, not much has appeared in the literature about MNT graphs, although there is an extensive literature on MNH graphs (see, for example, [4], [9], [10], [11], [18] and [24]).

In 1998 Zelinka [26], constructed two classes of MNT graphs and made the conjecture, which he later retracted, that every MNT graph belongs to one of these classes. These graphs have toughness less than one and contain fairly large cliques and are therefore quite dense. At the Fourth Cracow Conference on Graph Theory (Czorsztyn) in 2002 two different constructions were presented for infinite families of MNT graphs which do not belong to either of Zelinka's classes. See [12] and [6].

We, thus, became interested in characterizing MNT graphs. Although we have not succeeded in characterizing MNT graphs in general, we have managed to characterize certain types of MNT graphs and the results obtained have led to some interesting constructions of MNT graphs. One result has led to the construction of a non-Zelinka MNT graph of smallest order, namely of order 8.

We have also constructed an infinite family of 2-connected *cubic* MNT graphs. Although several constructions of cubic MNH graphs appear in the literature, we have found no references to cubic MNT graphs. The only cubic Zelinka MNT graph is the disconnected graph $K_4 \cup K_4$.

Another problem which we investigated was determining the least number of edges, $g(n)$, in an MNT graph of order n. The analogous problem for MNH graphs has been completely solved by combined results of Bondy [4], Clark, Entringer and Shapiro [10], [11], and Lin, Jiang, Zhang and Yang [18]. We have not been able to determine $g(n)$ for all values of n, but have found $g(n)$ for all except 26 values of n.

1.3 Overview

In Chapter 2 we present some results concerning maximal hypohamiltonian graphs as well as maximal nonhamiltonian-connected graphs, since these graphs play a crucial role in certain constructions of MNT graphs.

In Chapter 3 we discuss some properties of MNT graphs which we require in later chapters and introduce the concept of a saturation operation.

In Chapter 4 we describe Zelinka's constructions of MNT graphs. We then characterize MNT graphs with a specified number of blocks and cut-vertices. We give examples of Zelinka MNT graphs, and construct infinite families of non-Zelinka MNT graphs which have either three or two blocks. The final section is on claw-free MNT graphs.

Chapter 5 deals with MNT graphs of small size. We find a lower bound for the size of 2-connected MNT graphs of order n, and, by constructing an infinite family of 2-connected cubic MNT graphs, we find the minimum size for such graphs for infinitely many values of n. We also present a construction of an infinite family of MNT graphs that are almost cubic. We then establish the minimum size of MNT graphs of order n for all except 26 values of n and present a table of MNT graphs of possible smallest size for the omitted values of n.

In the final chapter, Chapter 6, we consider MNT graphs with toughness less than one, and we obtain results which lead to interesting constructions of MNT graphs. We then give examples of MNT graphs that are 1-tough and an example of one that is 2-tough.

Chapter 2

Preliminaries

In this chapter we consider a few properties of MHH graphs because some of these graphs form the building blocks of certain of our constructions of MNT graphs which we present in Chapters 5 and 6. Since some of these MHH graphs are also maximal nonhamiltonian-connected, and hence have additional properties, we also discuss some properties of maximal nonhamiltonian-connected graphs.

2.1 Maximal hypohamiltonian graphs

We first note that the minimum degree $\delta(G)$ of a hypohamiltonian graph G is at least 3. (If u is a vertex of G such that $\deg_G(u) \leq 2$ and v is a neighbour of u, then $\deg_{G-v}(u) \leq 1$ and hence $G - v$ cannot contain a hamiltonian cycle.) Thus cubic hypohamiltonian, and hence cubic MHH graphs, are of special interest to us, since they have the minimum possible size for their order. We give a few examples of well-known cubic MHH graphs, after considering the following results which we require for the construction of cubic MNT graphs in Section 5.2.

Lemma 2.1.1 *Suppose H is a hypohamiltonian graph and $z \in V(H)$. Put $F = H - z$. Then*

(i) for every $v \in V(F)$, there is a hamiltonian path in F with endvertex v.

(ii) there is no hamiltonian path in F with both endvertices in $N_H(z)$.

(iii) if $\deg(z) = 3$, then for any $y \in N_H(z)$ there exists a hamiltonian path in $F - y$ whose endvertices are the other two vertices of $N_H(z)$.

Proof.

(i) This follows directly from the fact that F has a hamiltonian cycle.

(ii) Suppose $v...w$ is a hamiltonian path in F with $v, w \in N_H(z)$. Then $zv...wz$ is a hamiltonian cycle in H, which is a contradiction.

(iii) Since $H-y$ is hamiltonian there is a hamiltonian cycle in $H-y$ containing the path vzw, where $v, w \in N_H(z) - y$. Thus there is a hamiltonian path in $F - y$ with endvertices v and w. ■

Lemma 2.1.2 *Suppose H is an MNH graph having a vertex z of degree 3. Put $F = H - z$. If u_1 and u_2 are nonadjacent vertices in F, then $F + u_1u_2$ has a hamiltonian path with both endvertices in $N_H(z)$.*

Proof. There exists a hamiltonian cycle in $H + u_1u_2$ which contains the path vzw, where $v, w \in N_H(z)$. Thus there exists a hamiltonian path in $F + u_1u_2$ with endvertices v and w. ■

The following graphs are all cubic MHH (see [4] and [10]):

1. The *Petersen* graph with 10 vertices and girth 5 which is depicted in Figure 2.1.

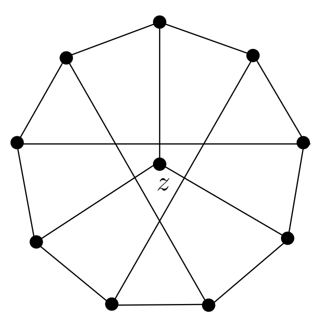


Figure 2.1: Petersen graph

2. The *Coxeter* graph with 28 vertices and girth 7, depicted in Figure 2.2.

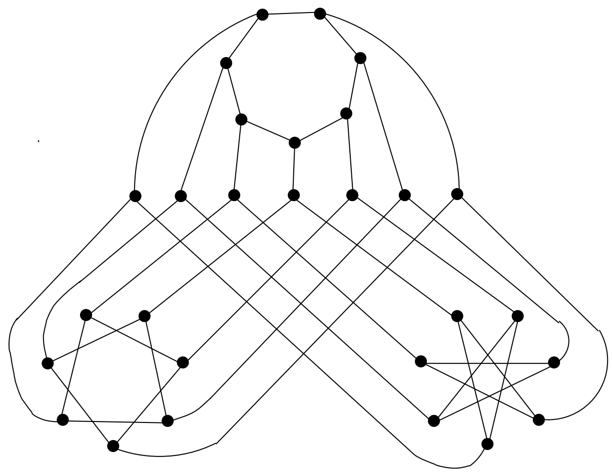


Figure 2.2: Coxeter graph

For our purposes we find Biggs' [2] representation of the Coxeter graph, which is given in Figure 2.3, more useful.

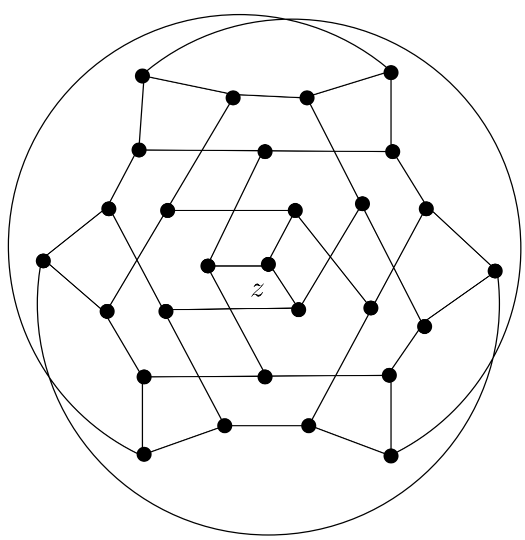


Figure 2.3: Coxeter graph - another representation

3. The *Isaacs' snarks* J_k of order $4k$ for odd $k \geq 5$ defined as follows:

   The graph J_k has vertex set $\{v_0, v_1, ..., v_{4k-1}\}$ and edge set $E_0 \cup E_1 \cup E_2 \cup E_3$, where

$$\begin{aligned} E_0 &= \bigcup_{j=0}^{k-1}\{v_{4j}v_{4j+1}, v_{4j}v_{4j+2}, v_{4j}v_{4j+3}\} \\ E_1 &= \bigcup_{j=0}^{k-1}\{v_{4j+1}v_{4j+7} : 0 \leq j \leq k-1\} \\ E_2 &= \bigcup_{j=0}^{k-1}\{v_{4j+2}v_{4j+6} : 0 \leq j \leq k-1\} \\ E_3 &= \bigcup_{j=0}^{k-1}\{v_{4j+3}v_{4j+5} : 0 \leq j \leq k-1\} \end{aligned}$$

   and where subscripts should be read as modulo $4k$.

   The graphs J_5 and J_7 are depicted in, Figure 2.4, where each vertex is identified with its subscript.

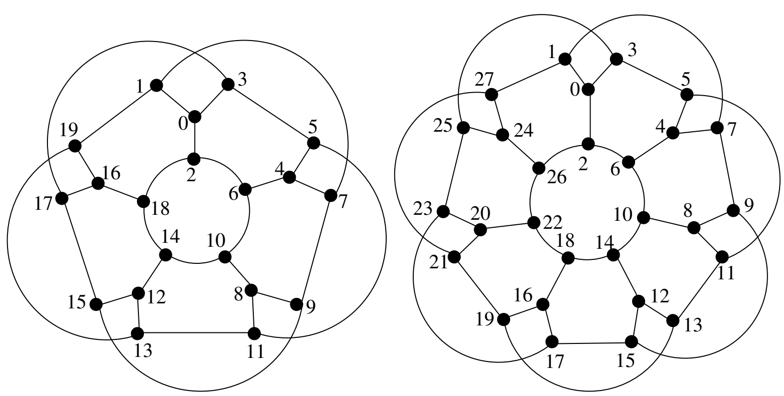


Figure 2.4: Isaacs' snarks J_5 and J_7

The Isaacs' snarks J_k have girth 5 for $k = 5$ and girth 6 for $k \geq 7$.

We determined, by considering symmetry and using the Graph Manipulation Package developed by Siqinfu and Sheng Bau,[1] that the snark G_3 of order 22, reported by Chisala [8] and depicted in Figure 2.5, is also MHH. We call this snark, Chisala's G_3-snark.

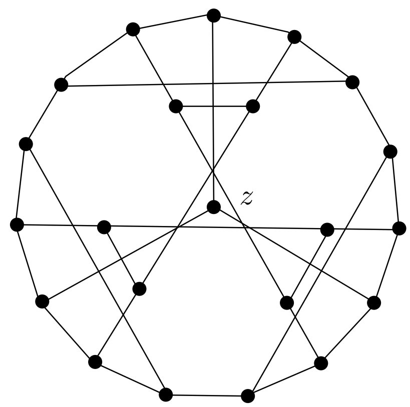


Figure 2.5: Chisala's G_3-snark

The Graph Manipulation Package allows one to sketch a graph on a computer screen by placing vertices and adding edges. On request the programme will either draw in a hamiltonian cycle or state that the graph is non-hamiltonian.

The following theorem given in [10] gives us another property of Isaacs' snarks which we require at a later stage.

Theorem 2.1.3 *(Clark and Entringer [10])*

If z and v are nonadjacent vertices of J_k, $k \geq 5$, then for every $u \in N_{J_k}(z)$ the edge uz lies in some hamiltonian $z - v$ path of J_k.

It follows directly from this theorem that if $z \in V(J_k)$ and $v \notin N_{J_k}(z)$, then for every $u \in N_{J_k}(z)$ the edge uz lies in a hamiltonian cycle of $J_k + zv$. This holds for all $z \in J_k$.

[1] We wish to thank Sheng Bau for allowing us the use of the programme, Graph Manipulation Package Version 1.0 (1996), Siqinfu and Sheng Bau, Inner Mongolia Institute of Finance and Economics, Huhhot, CN-010051, People's Republic of China.

We found, by using the Graph Manipulation Package, that this condition is also satisfied by the Petersen graph, the Coxeter graph and Chisala's G_3-snark for the specified vertex z in Figures 2.1, 2.3 and 2.5.

2.2 Maximal nonhamiltonian-connected graphs

A graph G is *hamiltonian-connected* (Hc) if for every two distinct vertices $u, v \in V(G)$ there is a hamiltonian $u - v$ path. If G is not hamiltonian-connected, but $G + e$ is hamiltonian-connected for every $e \in E(\overline{G})$, then G is said to be *maximal nonhamiltonian-connected* (MnHc).

We wish to thank Z. Skupień for his correspondence after reading our papers, [14] and [15], and bringing to our attention that he had already proved that the Petersen graph [22], the Coxeter graph [23], and the Isaacs's snarks J_k for odd $k > 5$ [17] are MnHc. Thus, according to Lemma 2.2.1 these graphs automatically satisfy the conditions of Theorem 2.1.3. He also showed in [23] that J_5 is not MnHc.

The following two lemmas will be of use to us in Section 5.2.

Lemma 2.2.1 *(Skupień [23])*

A nonhamiltonian graph G of order at least 3 is MnHc if and only if, for every two non-adjacent vertices $u, v \in V(G)$ and any edge $e \in E(G)$, there exists a hamiltonian $u - v$ path which contains e.

Lemma 2.2.2 *(Skupień [23])*

Suppose G is nonhamiltonian, $v(G) > 3$, and G has the property that for any two nonadjacent vertices $u, v \in V(G)$ and any edge $e \in E(G)$, there exists a hamiltonian $u - v$ path which contains e. Then G is hypohamiltonian.

Chapter 3

Properties of maximal nontraceable graphs

In this short chapter we state certain properties of MNT graphs which we require in later chapters.

3.1 Vertex-cuts and blocks of MNT graphs

We first present a lemma, which we require in Sections 5.4 and 6.2 about the relationship between the cardinality of a vertex-cut S of a connected MNT graph G and the number of components of $G - S$.

Lemma 3.1.1 *Suppose S is a vertex-cut of a connected graph G and $A_1, ..., A_k$ are components of $G - S$.*

(i) If $k \geq |S| + 2$, then G is nontraceable.

(ii) If G is MNT, then $k \leq |S| + 2$.

(iii) Suppose $k = |S|+2$. Then G is MNT if and only if $\langle S \cup A_i \rangle$ is complete for $i = 1, 2, ..., k$.

Proof.

(i) A path in G can contain vertices from at most $|S| + 1$ components of $G - S$.

(ii) Let u and v be vertices in different components of $G - S$. Then $G + uv$ contains a hamiltonian path and hence $G - S$ has at most $|S| + 2$ components.

(iii) Suppose G is MNT and that there exists an i such that $\langle S \cup A_i \rangle$ has two nonadjacent vertices x and y. Then S is a vertex-cut of the graph $G + xy$ and $(G + xy) - S$ has $|S| + 2$ components and hence, by (i), $G + xy$ is nontraceable.

Now suppose the converse holds. It follows from (i) that G is not traceable. Since $\langle S \cup A_i \rangle$ is complete for $i = 1, 2, ..., k$ we need only show that $G + uv$ is traceable for $u \in \langle A_i \rangle$, $v \in \langle A_j \rangle$, where $i \neq j$ and $i, j = 1, ..., k$. Without loss of generality let $i = k - 1$ and $j = k$. Let $S = \{x_1, ..., x_{k-2}\}$. We construct a hamiltonian path in $G + uv$ as follows:

We start with a hamiltonian path in $\langle A_1 \rangle$, followed by x_1, followed by a hamiltonian path in $\langle A_2 \rangle$, followed by x_2. We continue the path in a similar way until we end at x_{k-2}. We then continue with a hamiltonian path in $\langle A_{k-1} \rangle$ ending at u, followed by the edge uv and then a hamiltonian path in $\langle A_k \rangle$ starting at v. ■

We now present a similar result, but in terms of blocks and cut-vertices.

Lemma 3.1.2 *Suppose B is a block of a connected graph G.*

(i) If B contains more than two cut-vertices of G, then G is nontraceable.

(ii) If G is MNT, then B contains at most three cut-vertices of G.

(iii) Suppose B contains exactly three cut-vertices of G. Then G is MNT if and only if G consists of exactly four blocks, each of which is complete.

Proof.

(i) A path in G can pass through at most two cut-vertices of B.

(ii) Suppose the blocks B and B_i have x_i as a common cut-vertex. Then a path in $G + uv$, where $u \in V(B)$, $v \in V(B_i)$ and $u \neq x_i$, $v \neq x_i$, can pass through at most three cut-vertices of G in B.

(iii) Suppose B contains exactly three cut-vertices x_1, x_2, x_3 of G and x_i is common to the blocks B_i and B for $i = 1, 2, 3$.

Suppose G is MNT. If G has more than four blocks then a fifth block, B_4, shares a cut-vertex x with one of the B_i's, say B_1. (Note that x may be x_1.) Then a path in $G + uv$, where $u \in V(B_1)$, $v \in V(B_4)$ and $u \neq x$, $v \neq x$, can pass through at most two cut-vertices of G in B. Hence G consists of exactly four blocks. Suppose B is not complete. Then a longest path in $G + uv$, $u, v \in V(B)$ and $uv \notin E(G)$, misses the vertices in one of the $(B_i - x_i)$'s. The same reasoning applies if one of the B_i's is not complete.

We now assume that G consists of exactly four blocks, each of which is complete. It follows from (i) that G is not traceable. We show that $G + uv$ is traceable for $u, v \in V(G)$ and $uv \notin E(G)$. Without loss of generality we need consider only the following two cases.

Case 1. $u \in V(B_1) - \{x_1\}$ and $v \in V(B) - \{x_1\}$.

If $v \notin \{x_2, x_3\}$, then there is a hamiltonian path in $G + uv$ consisting of a hamiltonian path in B_2 ending at x_2, followed by v, then a hamiltonian path in B_1, starting at u and ending at x_1, followed by a hamiltonian path in $B - \{x_1, x_2, v\}$, ending at x_3 and then a hamiltonian path in $B_3 - x_3$. If $v = x_2$, then we obtain a hamiltonian path by identifying v and x_2 in the hamiltonian path described above. If $v = x_3$, then we can find a hamiltonian path in $G + uv$ in a similar way as above, but beginning with a hamiltonian path in B_3 ending at v.

Case 2. $u \in V(B_1) - \{x_1\}, v \in V(B_2) - \{x_2\}$.

There is a hamiltonian path in $G + uv$ consisting of a hamiltonian path in B_1 ending at u, followed by a hamiltonian path in B_2, starting at v and ending at x_2, followed by a hamiltonian path in $B - \{x_1, x_2\}$, ending at x_3 and then a hamiltonian path in $B_3 - x_3$.

■

3.2 A saturation operation

The following useful lemma is due to my promoter. Using it makes the proofs of a number of the lemmas in Chapter 5 much shorter and more elegant.

Lemma 3.2.1 *Let Q be a path in an MNT graph G. If $\langle V(Q)\rangle$ is not complete, then some internal vertex of Q has a neighbour in $G - V(Q)$.*

Proof. Let u and v be two nonadjacent vertices of $\langle V(Q)\rangle$. Then $G + uv$ has a hamiltonian path P. Let x and y be the two endvertices of Q and suppose no internal vertex of Q has a neighbour in $G - V(Q)$. Then P contains a subpath R in $\langle V(Q)\rangle + uv$ such that $V(R) = V(Q)$ or a disjoint union of subpaths R_1 and R_2 in $\langle V(Q)\rangle + uv$ such that $V(R_1 \cup R_2) = V(Q)$.

In the first case, the path obtained from P by replacing R with Q is a hamiltonian path in G. Now consider the second case. If P has both endvertices in Q, then on replacing $R_1 \cup R_2$ with Q in P, a hamiltonian cycle is obtained in G, which is a contradiction. Thus P has only one endvertex in Q, i.e. the neighbours in P of one of the endvertices of Q, say x, are not in Q. Then either R_1 or R_2, say R_1, consists of x alone. The path obtained from P by replacing R_2 with the subpath Q' of Q obtained by deleting x from Q is a hamiltonian path in G. ■

This lemma led us to define a *saturation operation* on a graph G in the following way: We say that a path P in a graph G is an *eligible path* if no internal vertex of P has a neighbour in $G - V(P)$ and $\langle V(P)\rangle$ is not complete. Suppose $x, y \in V(P)$ and $xy \notin E(G)$. Then by joining x and y, we say that we are "adding a missing edge to $\langle V(P)\rangle$". The operation of adding all missing edges to an eligible path of G is called the *local saturation* of G on P and is denoted by $G(P)$. The *saturation* $s(G)$ of G is the graph obtained from G by performing the local saturation operation on all eligible paths in G. Since adding missing edges to an eligible path does not affect the eligibility of any maximal eligible path, the saturation of a given graph is unique. The saturation of a graph has the following properties.

Lemma 3.2.2

(i) $\tau(s(G)) = \tau(G)$ *for every graph* G.

(ii) If a graph G *is maximal nontraceable, then* $s(G) = G$.

(iii) A graph G *is traceable if and only if* $s(G)$ *is a complete graph.*

Proof.

(i) As shown in the proof of Lemma 3.2.1, adding a missing edge to an eligible path does not increase the detour order of G.

(ii) If G is MNT, then it follows from Lemma 3.2.1 that G has no eligible paths and hence $s(G) = G$.

(iii) Suppose G is traceable. Then G has a hamiltonian path P. Since $G - V(P) = \emptyset$, P is an eligible path of G. Hence $s(G) = K_{|V(P)|}$. The converse statement follows from (i).

■

From the results of the above lemma one can see the potential use of the saturation operation. Since the operation "preserves" the detour order of a graph the saturation operation may lead to constructions of MNT graphs from nontraceable graphs. See, for example, the construction of an MNT claw-free graph just after Corollary 4.5.1. Also, if it is unclear whether or not a graph G is traceable, it may become clearer after having performed the local saturation operation on a number of eligible paths.

Chapter 4

Some Classes of Maximal Nontraceable Graphs

In this chapter we describe Zelinka's constructions of MNT graphs, discuss MNT graphs with a specified number of blocks and cut-vertices, and finally consider claw-free MNT graphs. We give a number of constructions of infinite families of MNT graphs that cannot be obtained from Zelinka's constructions, and we present a non-Zelinka MNT graph of order 8. All MNT graphs of order less than 8 are Zelinka graphs. Several of the constructions presented in this chapter appear again in the next chapter, where we attempt to determine the minimum size of an MNT graph of order n for every positive integer n.

4.1 Zelinka's constructions

As stated in the Introduction, Zelinka [26] constructed two classes of MNT graphs and made the conjecture, which he later retracted, that every MNT graph belongs to one of these classes. We describe the constructions below.

Construction I: Zelinka Type I graphs

Suppose p is a non-negative integer and $a_1, ..., a_k$, where $k = p + 2$ are positive integers. Let $U_0, U_1, ..., U_k$ be pairwise disjoint sets of vertices such that $|U_0| = p$ and $|U_i| = a_i$ for $i = 1, ..., k$. Let the graph G have $V(G) = \bigcup_{i=0}^{k} U_i$ and $E(G)$ be such that the induced subgraphs $G\langle U_0 \cup U_i \rangle$ for $i = 1, ..., k$ are complete graphs. We call such a graph G a *Zelinka Type I* graph.

This construction is represented diagrammatically in Figure 4.1.

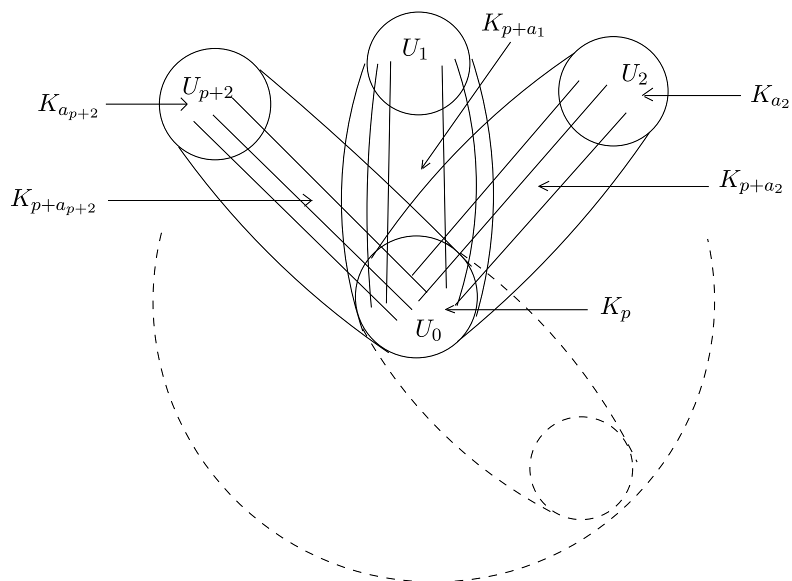


Figure 4.1: Zelinka Type I graph

Remarks 4.1.1

1. *All disconnected MNT graphs are of the form $K_m \cup K_n$ and are Zelinka Type I graphs with $p = 0$.*

2. *If $p = 1, a_1 = a_2 = a_3 = 1$, then we obtain the smallest connected MNT graph, the claw, $K_{1,3}$, which is depicted in Figure 4.3.*

Construction II: Zelinka Type II graphs

Suppose $p, q, r, a_1, ..., a_p, b_1, ..., b_q, c_1, ..., c_r$ are positive integers and s a non-negative integer. Let $U_0, U_1, ..., U_p, V_0, V_1, ..., V_q, W_0, W_1, ..., W_r, X$ be pairwise disjoint sets of vertices such that $|U_0| = p, |U_i| = a_i$ for $i = 1, ..., p, |V_0| = q, |V_i| = b_i$ for $i = 1, ..., q, |W_0| = r, |W_i| = c_i$ for $i = 1, ..., r$ and $|X| = s$.
Let the graph G have $V(G) = (\bigcup_{i=0}^{p} U_i) \cup (\bigcup_{i=0}^{q} V_i) \cup (\bigcup_{i=0}^{r} W_i) \cup X$ and $E(G)$ be such that the induced subgraphs $G\langle U_0 \cup U_i \rangle$ for $i = 1, ..., p, G\langle V_0 \cup V_i \rangle$ for $i = 1, ..., q, G\langle W_0 \cup W_i \rangle$ for $i = 1, ..., r$, and $G\langle U_0 \cup V_0 \cup W_0 \cup X \rangle$ are all complete graphs. We call such a graph G a *Zelinka Type II* graph.

This construction is represented diagrammatically in Figure 4.2.

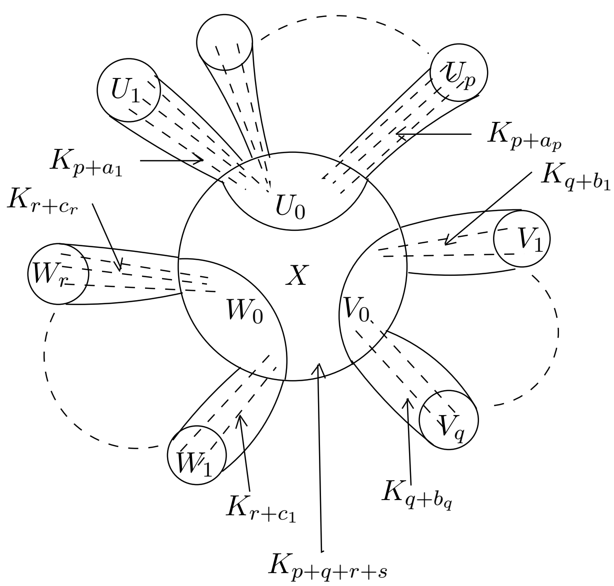


Figure 4.2: Zelinka Type II graph

Remark 4.1.2 *The smallest connected Zelinka Type II graph is the net N, depicted in Figure 4.3, which is obtained when $p = q = r = 1, a_1 = b_1 = c_1 = 1, s = 0$.*

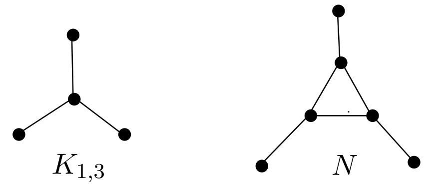


Figure 4.3: The claw and the net

Remark 4.1.3 *By consulting [20], An Atlas of Graphs, we see that all MNT graphs with fewer than 8 vertices are Zelinka graphs.*

4.2 Maximal nontraceable graphs having exactly four blocks

In [26], Zelinka also proved the following characterization of MNT graphs whose blocks are all complete.

Theorem 4.2.1 *(Zelinka [26])*

If G is a graph in which all the blocks are complete, then G is MNT if and only if

(i) G has exactly three pairwise neighbouring blocks; or

(ii) G has exactly four blocks, three of which are pairwise non-neighbouring and the fourth is neighbouring to all of them. (See Figure 4.4.)

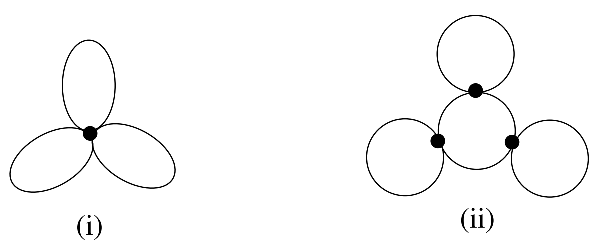


Figure 4.4: Diagram for Theorem 4.2.1

Remarks 4.2.2

1. *If G is an MNT graph that has a structure as depicted in Figure 4.4 (i) or (ii), then every block of G is complete.*

2. *If G is an MNT graph with at least four blocks, then no proper induced subgraph of G has the structure depicted in Figure 4.4 (i) and if G has at least five blocks, then no proper induced subgraph of G has the structure depicted in Figure 4.4 (ii).*

We say that a graph G has a *linear block structure* if G is connected and every block of G contains at most two cut-vertices of G and a cut-vertex of G lies in at most two blocks.

We have the following result concerning MNT graphs that have linear block structures.

Lemma 4.2.3 *Suppose G is an MNT graph with a linear block structure. Then G has at most three blocks.*

Proof. Suppose G is an MNT graph with a linear block structure that has four blocks, B_1, B_2, B_3, B_4. Let y_i be the cut-vertex that is common to B_i and B_{i+1} for $i = 1, 2, 3$. Such a graph is depicted in Figure 4.5.

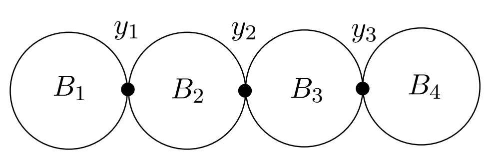


Figure 4.5: Diagram for Lemma 4.2.3

Consider $G + y_1y_3$. Any path in $G + y_1y_3$ containing the edge y_1y_3 will miss vertices in at least one of the blocks. This is a contradiction. ■

It follows from Remarks 4.2.2 and Lemma 4.2.3 that no MNT graph has more than four blocks and those with exactly four blocks can be characterized as follows.

Theorem 4.2.4 *Suppose G is a connected graph with exactly four blocks. Then G is MNT if and only if one of the blocks has three cut-vertices and each of the blocks is complete.*

4.3 Maximal nontraceable graphs having exactly three blocks

According to Lemma 3.1.1 a graph having exactly three blocks and one cut-vertex is MNT if and only if all the blocks are complete. In this section we consider MNT graphs with three blocks and two cut-vertices.

We first define the following concept and notation:

A subgraph F of a graph G is called a 2-*path cover* of G if $V(F) = V(G)$, F has at most two components and each component of F is a path (including paths consisting of one vertex).

If F is a 2-path cover of a graph G, and consists of two paths, then we use F_G^1 and F_G^2 to denote the two paths. Also for $i = 1, 2$

$F_G^i(v, w)$ denotes a path in G from v to w;

$F_G^i(-, w)$ denotes a path in G ending at w and may consist of the vertex w alone;

$F_G^i(v, -)$ denotes a path in G beginning at v and may consist of the vertex v alone.

We use the following to denote hamiltonian paths in a graph G:

P_G denotes a hamiltonian path in G;

$P_G(v, w)$ denotes a hamiltonian path in G from v to w;

$P_G(-, w)$ denotes a hamiltonian path in G ending at w;

$P_G(v, -)$ denotes a hamiltonian path in G beginning at v.

Recall that we denote a path by listing *only* the *vertices* in that path. Thus, for example, the notation $P_G(u, v)P_H(w, z)$ represents a path obtained by following a hamiltonian path in G from u to v, followed by the edge vw, followed by a hamiltonian path in H from w to z.

Theorem 4.3.1 *Let G be a connected graph having exactly three blocks B_1, B and B_2 and exactly two cut vertices y_1 and y_2, where y_i is common to B and B_i for $i = 1, 2$. Then G is* *maximal nontraceable if and only if the following hold.*

1. *Both B_1 and B_2 are complete.*

2. *$y_1y_2 \in E(G)$.*

3. *B satisfies the following conditions:*

   (a) *B has no hamiltonian path with endvertices y_1 and y_2, but $B + uv$, where $u, v \in V(B)$ and $uv \notin E(G)$ does have such a hamiltonian path.*

   (b) *$B - y_i$ is traceable from y_j, where $i \neq j$ and $i, j = 1, 2$.*

   (c) *For $u \in V(B) - \{y_1, y_2\}$ there exists either*

      (i) *a 2-path cover of B with y_1 and y_2 being the endvertices of one path and u an endvertex of the other path; or*

      (ii) *two 2-path covers of B in which each cover has a path with endvertices u and y_i and the other with endvertex y_j, where $i \neq j$, $i, j = 1, 2$.*

Proof. The graph G is depicted in Figure 4.6.

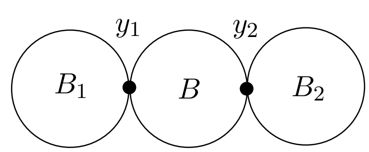


Figure 4.6: MNT graph with three blocks and two cut-vertices

Suppose G is MNT. Not all three blocks are complete, otherwise G is traceable. Since $G + uy_2$, $u \in V(B_1)$ and $u \neq y_1$, is traceable it follows that B_i is traceable from y_i for $i = 1, 2$. Now B is not complete, otherwise G would be traceable. Suppose that B_1 is not complete. Then $G + uv$, where $u, v \in V(B_1)$ and $uv \notin E(G)$, is traceable and so B is traceable from y_1 to y_2 and B_2 is traceable from y_2. But, since B_1 is traceable from y_1 it then follows that G is traceable. Hence B_1 is complete. Similarly B_2 is complete. Thus condition 1 holds.

Suppose $y_1y_2 \notin E(G)$. Then $G + y_1y_2$ has a hamiltonian path with one endvertex in B_1 and the other in B_2 and contains the edge y_1y_2, which is a contradiction since no other vertices in B can be contained in the path. Thus condition 2 holds.

The block B has no hamiltonian path with endvertices y_1 and y_2, otherwise G would be traceable. Now, since $G + uv$, where $u, v \in V(B)$ and $uv \notin E(G)$ is traceable, $B + uv$ has a hamiltonian path with endvertices y_1 and y_2. Thus we have shown that condition 3(a) holds.

Consider $G + uy_i$, where $i = 1, 2$ and $u \in V(B_j)$, $i \neq j$, $j = 1, 2$. Since $G + uy_i$ is traceable, it follows that $B - y_i$ is traceable from y_j. Thus condition 3(b) holds.

For any $u \in V(B) - \{y_1, y_2\}$ and $v \in V(B_i)$, $i = 1, 2$, $G + uv$ has a hamiltonian path P containing uv. Then P may enter B once or twice. Thus $F = \langle E(P) \cap E(B) \rangle$ is a 2-path cover, in which y_1 and y_2 are endvertices of one path and u is the endvertex of the other path (a hamiltonian path in B with endvertices y_j and u is included in this type of 2-path cover), or $F = \langle E(P) \cap E(B) \rangle$ is a 2-path cover, in which y_j and u are endvertices of one path and y_i, $i \neq j$ is the endvertex of the other path. Thus we have shown that condition 3(c) holds.

Conversely, suppose G satisfies conditions 1-3.
Since B has no hamiltonian path with endvertices y_1 and y_2, it follows that G is not traceable. We now prove that $G + uv$ is traceable for all $u, v \in V(G)$ where $uv \notin E(G)$.

Suppose $u, v \in V(B)$. It then follows from 3(a) that $B + uv$ has a hamiltonian path with endvertices y_1 and y_2 and hence $G + uv$ has a hamiltonian path.

Suppose $u \in V(B_i)$ and $v = y_j$, $i \neq j$, $i, j = 1, 2$. We note that $u \neq y_i$ since y_1 and y_2 are adjacent. By 3(b) we have a hamiltonian path in $G + uv$ of the form

$$P_{G+uv} = P_{B_j}(-, y_j)P_{B_i - y_i}(u, -)P_{B-y_j}(y_i, -).$$

Suppose $u \in V(B_1)$, $v \in V(B_2)$ and $u \neq y_1$, $v \neq y_2$. From 3(b) we have a hamiltonian path

$$P_{G+uv} = P_{B_2}(y_2, v)P_{B_1 - y_1}(u, -)P_{B-y_2}(y_1, -).$$

Suppose $u \in V(B) - \{y_1, y_2\}$ and $v \in V(B_i)$, $i = 1, 2$.

If 3(c)(i) holds, then we have a hamiltonian path

$$P_{G+uv} = F_B^1(-, u)P_{B_i - y_i}(v, -)F_B^2(y_i, y_j)P_{B_j - y_j}.$$

If 3(c)(ii) holds, then we have a hamiltonian path

$$P_{G+uv} = F_B^1(-, y_i)P_{B_i - y_i}(-, v)F_B^2(u, y_j)P_{B_j - y_j}.$$

Thus G is MNT. ∎

We shall call the block B in Figure 4.6 the middle block of G. Note that it follows from condition 3(a) of Theorem 4.3.1 that the middle block B of an MNT graph G with exactly three blocks and two cut-vertices is either hamiltonian (but no hamiltonian cycle in B contains the edge joining the two cut-vertices) or MNH. In the next two subsections we give examples illustrating these two possibilities.

4.3.1 Examples of MNT graphs with exactly three blocks and two cut-vertices in which the middle block is maximal nonhamiltonian

1. Zelinka's construction

The smallest MNT graph of the type described in Theorem 4.3.1 is a Zelinka Type II graph of order 8 (see Section 4.1) in which $p = q = 1, r = 2, s = 0, a_1 = b_1 = c_1 = c_2 = 1$. This graph is depicted in Figure 4.7.

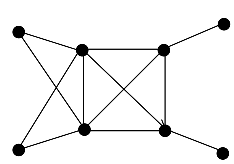


Figure 4.7: Smallest MNT graph with three blocks and two cut-vertices

In this case, we have $B_1 \cong B_2 \cong K_2$ and B is MNH. It is easy to check that this graph satisfies the conditions of Theorem 4.3.1.

The graph depicted in Figure 4.7 can be generalized to a family of MNT graphs satisfying the conditions of Theorem 4.3.1. A graph in this family is of the following form:

A Zelinka Type II graph, in which $p = q = 1, r \geq 2, a_1 \geq 1, b_1 \geq 1, c_i \geq 1$ for $i = 1, 2, ..., r$.

2. Construction by Dudek, Katona and Wojda

In [13] Dudek, Katona and Wojda construct, for every $n \geq 54$ as well as for every $n \in I = \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$, an MNT graph of size $\lceil \frac{3n-2}{2} \rceil$ in the following way:

Consider a cubic MNH graph B with the properties that

D(1): there is an edge y_1y_2 of B, such that $N(y_1) \cap N(y_2) = \emptyset$, and

D(2): $B + e$ has a hamiltonian cycle containing y_1y_2 for every $e \in E(\overline{B})$.

Take two graphs H_1 and H_2, with $H_1 \cong K_1$ and $H_2 \cong K_1$ or $H_2 \cong K_2$ and join each vertex of H_i to y_i; $i = 1, 2$. We shall call graphs constructed in this manner *DKW-graphs*. A DKW-graph is an MNT graph of order $v(B) + 2$ and size $e(B) + 2$ or of order $v(B) + 3$ and size $e(B) + 4$.

We show that the graphs constructed by Dudek, Katona and Wojda satisfy the conditions of Theorem 4.3.1.

Theorem 4.3.2 *A DKW-graph satisfies the conditions of Theorem 4.3.1.*

Proof. Let G be a DKW-graph. Then G consists of three blocks, B_1, B (the middle block) and B_2, where $B_1 \cong K_2$ and $B_2 \cong K_2$ or $B_2 \cong K_3$, and two cut-vertices y_1 and y_2 such that $y_1y_2 \in E(G)$. Conditions 1 and 2 are thus satisfied.

The block B does not have a hamiltonian path with endvertices y_1 and y_2, otherwise B would be hamiltonian. However, since $B + uv$, where $u, v \in V(B)$ and $uv \notin E(B)$ has a hamiltonian cycle containing y_1y_2, it has a hamiltonian path with endvertices y_1 and y_2. Thus B satisfies condition 3(a).

Suppose $z \in N_B(y_j)$, $j = 1, 2$. Then $z \notin N_B(y_i)$, $i \neq j$, $i = 1, 2$. Thus $B + zy_i$ has a hamiltonian cycle containing zy_iy_j and hence B has a hamiltonian path of the form $y_iy_j...z$. Thus $B - y_i$ has a hamiltonian path of the form $y_j...z$ and is consequently traceable from y_j. Thus condition 3(b) is satisfied.

Let $u \in V(B) - \{y_1, y_2\}$. Since $N_B(y_1) \cap N_B(y_2) = \emptyset$ it follows that $uy_i \notin E(B)$ for at least one of the values of i, say $i = 1$. Then $u...wy_2y_1$ is a hamiltonian path in B and thus $u...w$ and y_2y_1 are the two components of a 2-path cover as described in condition 3(c)(i). ■

In [10] and [11] Clark, Entringer and Shapiro prove that the Isaacs' snarks of order $n = 4(2l + 1)$, where $l \geq 2$ and adaptations of the Isaacs' snarks of order n, for even $n \geq 54$ and $n \neq 4(2l + 1)$ as well as for $n \in \{38, 40, 46, 48\}$, have properties D(1) and D(2). Thus it follows that for every even $n \geq 52$ as well as for $n \in \{20, 28, 36, 38, 40, 44, 46, 48\}$ there exists a cubic MNH graph of order n with properties D(1) and D(2). Thus this construction provides MNT graphs of order n and size $\lceil \frac{3n-2}{2} \rceil$ for every $n \geq 54$ as well as for every $n \in I$.

It is easy, by using sketches, to show that the Petersen graph also has properties D(1) and D(2). (This fact also follows from the discussion in Section 2.2.) Hence, according to the above construction, there are also MNT graphs of order n and size $\lceil \frac{3n-2}{2} \rceil$ for $n = 12, 13$. The MNT graph obtained by using the Petersen graph as B and $H_i \cong K_1$ for $i = 1, 2$ is depicted in Figure 4.8.

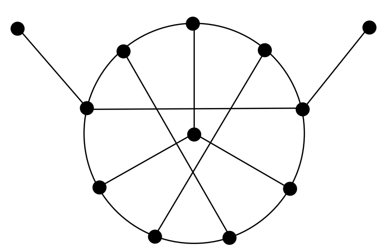


Figure 4.8: An MNT graph with the Petersen graph as the middle block

Thus we have the following corollary.

Corollary 4.3.3 *There exists an MNT graph of order n and size $\lceil \frac{3n-2}{2} \rceil$ for every $n \geq 54$ as well as for every $n \in \{12, 13, 22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$.*

DKW-graphs can be generalized by letting $H_1 \cong K_n$ and $H_2 \cong K_m$, with $n \geq 1$ and $m \geq 1$.

4.3.2 Examples of MNT graphs with exactly three blocks and two cut-vertices in which the middle block is hamiltonian

We shall prove that the least order of an MNT graph with exactly three blocks and two cut-vertices in which the middle block is hamiltonian is 10. We also show that the graph depicted in Figure 4.9, which has order 10 and size 15, is an example of such a graph. We call this graph the *sputnik*.

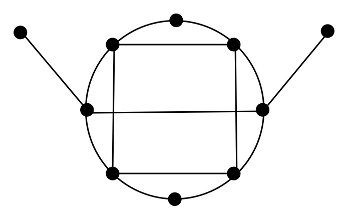


Figure 4.9: The sputnik

We use the following notation in our poof. We suppose that a cycle C of a graph G has an orientation. If $u \in V(C)$ and $v \in V(C)$ we denote the path on C from u to v by $C[u, v]$ and the other path on C from u to v by $\overline{C}[u, v]$. The paths obtained by deleting the endvertices u, v are denoted by $C(u, v)$ and $\overline{C}(u, v)$, respectively.

Theorem 4.3.4 *Suppose G is an MNT graph that consists of three blocks B_1, B, B_2 and two cut-vertices y_1, y_2 in which the middle block B is hamiltonian. Then G has order at least 10.*

Proof. Since we are only concerned with graphs G of least order, it follows from condition 1 of Theorem 4.3.1 that we may assume that $B_1 \cong B_2 \cong K_2$. Let $V(B_i) = \{x_i, y_i\}$, $i = 1, 2$.

Let C be a hamiltonian cycle in B. Since G is not traceable we have $|V(C(y_1, y_2))| > 0$ and $|V(\overline{C}(y_1, y_2))| > 0$. Furthermore, there must be at least one vertex on $C(y_1, y_2)$ which is adjacent to a vertex on $\overline{C}(y_1, y_2)$. (Consider, for example, $G + x_1y_2$. Then a hamiltonian path P, if one exists, in $G + x_1y_2$ would begin as follows: $x_2y_2x_1y_1$. It is obvious that if no vertex on $C(y_1, y_2)$ is adjacent to a vertex on $\overline{C}(y_1, y_2)$, then P cannot contain all the vertices of C.) It follows that $|V(C)| \geq 6$, otherwise G is traceable.

Suppose $|V(C)| = 6$, i.e. G has order 8. Then G has one of the two graphs depicted in Figure 4.10 as a spanning subgraph.

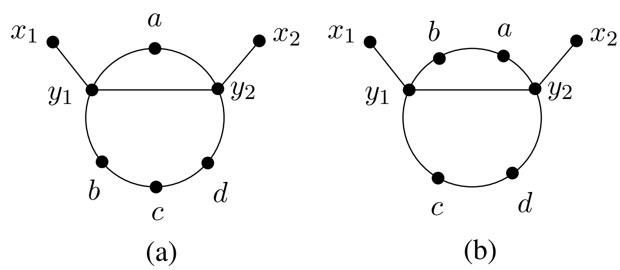


Figure 4.10: Spanning subgraphs for possible MNT graph of order 8

Subcase depicted in Figure 4.10(a)

Firstly $ab, ad \notin E(G)$, otherwise G is traceable. Suppose $ac \in E(G)$. Then $bd \notin E(G)$. No matter what else y_1 is adjacent to, it is easy to see that $G + x_1y_2$ does not contain a hamiltonian path. If a hamiltonian path existed then it would begin as follows: $x_2y_2x_1y_1$. If this path continued to a it would miss either b or d. If it continued to b (d) it would miss either a or d (b). If it continued to c, then it would miss at least one of $\{a, b, d\}$. Hence G is not MNT.

Subcase depicted in Figure 4.10(b)

Firstly $ac, bd \notin E(G)$, otherwise G is traceable. Hence we may assume, without loss of generality, that $ad \in E(G)$. Then $bc, y_2b, y_2c \notin E(G)$. It is easy to see that $G + x_2y_1$ does not contain a hamiltonian path and hence G is not MNT.

Thus $|V(C)| > 6$, i.e. G has order greater than 8.

Finally suppose $|V(C)| = 7$, i.e. G has order 9. Then G has one of the two graphs depicted in Figure 4.11 as a spanning subgraph.

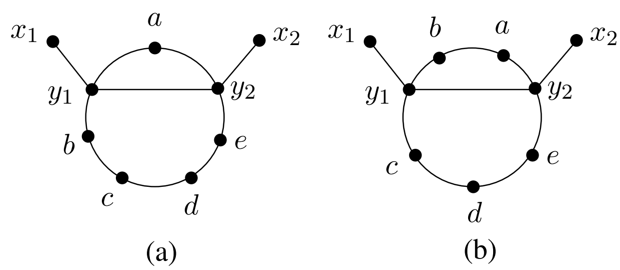


Figure 4.11: Spanning subgraphs for possible MNT graph of order 9

Subcase depicted in Figure 4.11(a)

Firstly $ab, ae \notin E(G)$, otherwise G is traceable. Hence we may assume, without loss of generality, that $ac \in E(G)$. Then $ad, bd, be \notin E(G)$. It can be seen that $G + x_1y_2$ is not traceable and hence G is not MNT.

Subcase depicted in Figure 4.11(b)

Firstly $ac, be \notin E(G)$, otherwise G is traceable.

(I) Suppose $ae \in E(G)$. Then $y_2b, y_2c, bc, bd \notin E(G)$.

- (i) Suppose $ad \in E(G)$. Then $ce \notin E(G)$. Now $G + x_2y_1$ is not traceable and hence G is not MNT.
- (ii) Suppose $ad \notin E(G)$. Then again, it is obvious that there is no hamiltonian path in $G + x_2y_1$ and hence G is not MNT.

(II) Suppose $ae \notin E(G)$.

- (i) Suppose $ad \in E(G)$. Then $ce, bc \notin E(G)$. However, it can be seen that $G + x_1y_2$ is not traceable and hence G is not MNT.

(ii) Suppose $ad \notin E(G)$. Then it is obvious that there is no hamiltonian path in $G + x_1y_2$.

This proves that $|V(C)| \geq 8$, i.e. G has order at least 10. ■

Theorem 4.3.5 *The sputnik is MNT.*

Proof. Let G be the sputnik. We prove that G satisfies the conditions of Theorem 4.3.1. We label the vertices in the sputnik as shown in Figure 4.12.

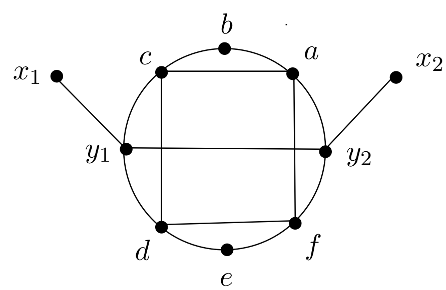


Figure 4.12: Graph for Theorem 4.3.5

The graph G consists of three blocks, B_1, B and B_2, where $B_1 \cong K_2$, $B_2 \cong K_2$, $B = G - \{x_1, x_2\}$ and the cut-vertices y_1 and y_2 of G are such that $y_1y_2 \in E(G)$. Conditions 1 and 2 are thus satisfied.

A longest path in B with endvertices y_1 and y_2 will miss either b or e and thus there is no hamiltonian path in B with endvertices y_1 and y_2. We now show that $B + uv$, where $u, v \in V(B)$ and $uv \notin E(G)$ has a hamiltonian path with endvertices y_1 and y_2. Due to symmetry we need only consider the following cases.

Case 1. $u = y_1$ and $v \in \{a, b\}$.

If $v = a$, then a hamiltonian path is $y_1abcdefy_2$.

If $v = b$, then a hamiltonian path is $y_1bcdefay_2$.

Case 2. $u = a$ and $v \in \{d, e\}$.

If $v = d$, then a hamiltonian path is $y_1cbadefy_2$.

If $v = e$, then a hamiltonian path is $y_1dcbaefy_2$.

Case 3. $u = b$ and $v \in \{d, e\}$.

If $v = d$, then a hamiltonian path is $y_1cbdefay_2$.

If $v = e$, then a hamiltonian path is $y_1cdebafy_2$.

Thus condition 3(a) is satisfied.

Clearly, $B - y_1$ is traceable from y_2 and $B - y_2$ is traceable from y_1. Thus condition 3(b) is satisfied. Also, since it is obvious that $B - \{y_1, y_2\}$ has a hamiltonian path with endvertex u for all $u \in B - \{y_1, y_2\}$, it follows that condition 3(c)(i) is satisfied. ■

Corollary 4.3.6 *The sputnik is an MNT graph of least order with three blocks and two cut-vertices in which the middle block is hamiltonian.*

By replacing each triangle and each B_i, $i = 1, 2$ of the sputnik by a complete graph of arbitrary order, in the manner shown in Figure 4.13, we obtain an MNT graph, consisting of three blocks and two cut-vertices in which the middle block is hamiltonian, of order n for each $n \geq 10$. We call such a graph a *generalized sputnik*. It is rather tedious, but one can show that a generalized sputnik satisfies the conditions of Theorem 4.3.1, and is thus MNT.

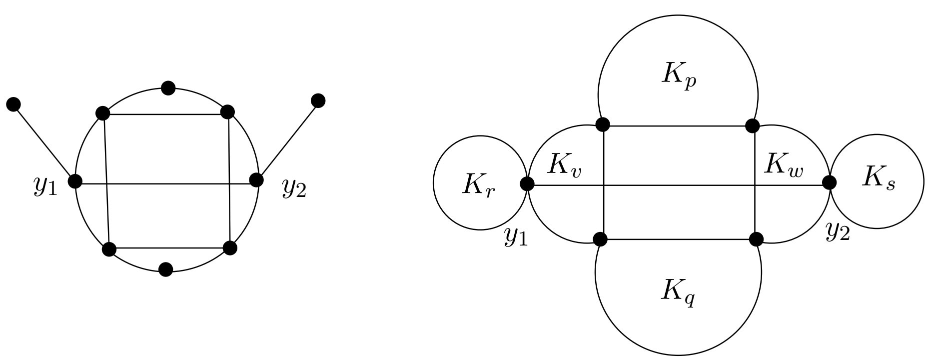


Figure 4.13: The sputnik and a generalized sputnik

4.4 Maximal nontraceable graphs having exactly two blocks

We pointed out in Section 4.1 that each disconnected MNT graph G consists of two components, both of which are complete, and that G is a Zelinka Type I graph with $p = 0$.

We now consider connected MNT graphs having exactly two blocks.

Theorem 4.4.1 *Let G be a connected graph having exactly two blocks A and B with cut-vertex x. Then G is MNT if and only if the following conditions hold.*

1. *One of the blocks, say A, is complete.*
2. *B satisfies the following:*
   - *(a) B is not traceable from x, but if $u, v \in V(B)$ and $uv \notin E(G)$, then $B + uv$ is traceable from x.*
   - *(b) For each $u \in V(B)$, $u \neq x$ there exists a 2-path cover F of B in which x is an endvertex of one path and u is an endvertex of the other path.*

Proof. The graph G is depicted in Figure 4.14.

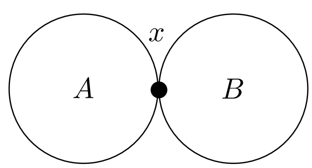


Figure 4.14: MNT graph with two blocks and one cut-vertex

Suppose G is MNT. Not both blocks are complete, otherwise G is traceable. Suppose neither of the blocks is complete. Then $G + u_1u_2$, $u_1, u_2 \in V(A)$, $u_1u_2 \notin E(G)$ is traceable and hence B is traceable from x. Similarly $G + v_1v_2$, $v_1, v_2 \in V(B)$, $v_1v_2 \notin E(G)$ is traceable and hence A is traceable from x. Thus G is traceable. Hence exactly one of A and B is complete. Thus condition 1 holds. Suppose A is complete.

The block B is not traceable from x, otherwise G is traceable. Consider $G + uv$, $u, v \in V(B)$, $uv \notin E(G)$. If $B + uv$ is not traceable from x, then $G + uv$ is not traceable, which is a contradiction. Thus we have shown that condition 2(a) holds.

For any $u \in V(B)$, $u \neq x$ and $v \in V(A)$, $v \neq x$, $G + uv$ has a hamiltonian path P containing uv. Since B is not traceable from x, it follows that P visits B twice. Then $F = \langle E(P) \cap E(B) \rangle$ is a 2-path cover of B in which u is an endvertex of one path and x is an endvertex of the other path. Hence condition 2(b) holds.

Conversely, suppose conditions 1-2 hold. Since B is not traceable from x it follows that G is not traceable.

We now prove that $G + uv$ is traceable for all $u, v \in V(G)$ where $uv \notin E(G)$.
Suppose $u, v \in V(B)$. From 2(a) we have a hamiltonian path in $G + uv$ of the form

$$P_{G+uv} = P_{A-x} P_{B+uv}(x, -).$$

Now suppose $u \in V(B)$, $u \neq x$ and $v \in V(A)$, $v \neq x$. From 2(b) we have a hamiltonian path

$$P_{G+uv} = F_B^1(-, u) P_{A-x}(v, -) F_B^2(x, -).$$

Hence G is MNT. ■

Remark 4.4.2 *Note that it follows from condition 2(a) of Theorem 4.4.1 that the noncomplete block of a connected MNT graph with exactly two blocks is either traceable (but not from the cut-vertex) or is MNT. We have only found examples of these type of graphs in which the noncomplete block is traceable. In the next subsection we present some of these graphs. It is an open question if any connected MNT graphs with only two blocks exist in which one is MNT.*

4.4.1 Examples of connected MNT graphs with two blocks

Zelinka graphs

The smallest connected Zelinka MNT graph with exactly two blocks is a Zelinka Type II graph (see Section 4.1) in which $p = 1, q = r = 2, s = 0, a_1 = b_1 = b_2 = c_1 = c_2 = 1$. This graph is of order 10 and size 19 and is depicted in Figure 4.15.

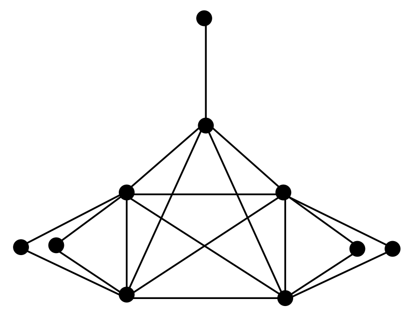


Figure 4.15: Smallest Zelinka MNT graph with two blocks

In this case, we have $A \cong K_2$ and B is traceable, but not from the cut-vertex. It is easy to check that the graph satisfies the conditions of Theorem 4.4.1.

The above graph can be generalized to a family of MNT graphs satisfying the conditions of Theorem 4.4.1. A graph in this family is of the following form:
A Zelinka Type II graph, in which $p = 1, q \geq 2, r \geq 2, a_1 \geq 1, b_i \geq 1$ for $i = 1, ..., q$, $c_j \geq 1$ for $j = 1, ..., r$.

Non-Zelinka graphs

Non-Zelinka MNT graph of least order

The smallest non-Zelinka MNT graph with two blocks that we have constructed is the graph depicted in Figure 4.16, which has order 8 and size 13. We call this graph the *propeller*. It follows from Remark 4.1.3 that this graph is a non-Zelinka MNT graph with the least number of vertices.

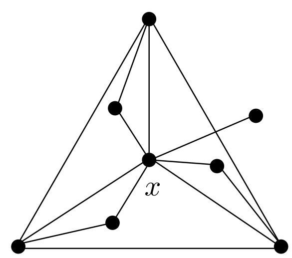


Figure 4.16: The propeller, a non-Zelinka MNT graph of smallest order

In this case, we have $A \cong K_2$ and B is traceable, but not from the cut-vertex x. We note that x is a universal vertex. It is easy to check that the graph satisfies the conditions of Theorem 4.4.1. We can describe B in the following way: Let $x \in V(K_4)$. Then B is obtained by subdividing the three edges of K_4 incident with x, and adding the relevant edges to make x a universal vertex.

Remark 4.4.3 *It is interesting to note that if we consider the block B of the propeller and replace the universal vertex by a K_3, the graph so formed (depicted in Figure 4.17) is the smallest MNH graph of order 9. See [18].*

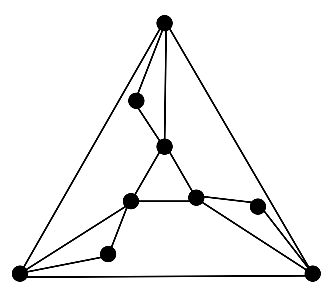


Figure 4.17: The smallest MNH graph of order 9

General constructions

Construction I

The propeller can be generalized to an MNT graph of order $n \geq 8$, depicted in Figure 4.18, in the following way: Let $A \cong K_s$, where $s \geq 2$. For B, replace the three triangles incident with x in the propeller with graphs G_1, G_2 and G_3, where $G_1 \cong K_p$, $G_2 \cong K_q$ and $G_3 \cong K_r$, where $p \geq 3$, $q \geq 3$, $r \geq 3$ and $p+q+r+s-3 = n$. Such a graph is called a *generalized propeller*. Thus we obtain non-Zelinka MNT of all orders greater than or equal to 8.

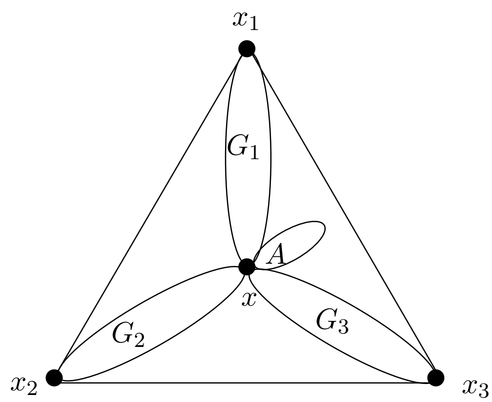


Figure 4.18: A generalized propeller

We note that the propeller is obtained from a generalized propeller when $s = 2$ and $p = q = r = 3$.

Theorem 4.4.4 *A generalized propeller G is maximal nontraceable.*

Proof. We prove that G satisfies the conditions of Theorem 4.4.1. The graph G consists of one cut-vertex x and two blocks, $A \cong K_s$ and $B = G - (V(A) - x)$. Thus condition 1 is satisfied.

A longest path in B with endvertex x will miss some vertices in one of the graphs G_i for $i = 1, 2, 3$ and thus B is not traceable from x.

We now show that $B + uv$ is traceable from x for all nonadjacent vertices u and v in B. Again we note that x is a universal vertex. Taking the symmetry of B into account we need only consider the following:

Case 1. $u \in V(G_1 - x)$, $v \in V(G_2) - \{x, x_2\}$.
A hamiltonian path is

$$P_{G_3}(x, x_3)P_{G_2-x}(x_2, v)P_{G_1-x}(u, -).$$

Case 2. $u \in V(G_1) - \{x, x_1\}$, $v = x_2$.
In this case a hamiltonian path is

$$P_{G_3}(x, x_3)P_{G_1-x}(x_1, u)P_{G_2-x}(v, -).$$

Thus condition 2(a) is satisfied.

Now suppose $u \in V(B - x)$. Without loss of generality, let $u \in V(G_1 - x)$. Then there is a 2-path cover, where $F_B^1 = P_{G_1-x}(u, -)$ and $F_B^2 = P_{G_2}(x, x_2)P_{G_3-x}(x_3, -)$, and hence condition 2(b) is satisfied. ∎

The construction given above can be further generalized by starting with any K_n, with $n \geq 5$, instead of K_4 and replacing any three edges incident with $x \in V(K_n)$ with complete graphs.

The fact that the graphs constructed above are MNT also follows directly from Theorem 6.2.2 and Remark 6.2.3 which we state in Chapter 6.

Construction 2

We now construct another family of non-Zelinka MNT graphs, different from the generalized propellers, which also have two blocks. This construction resulted from a graph in this family of order 14 that was first constructed by a colleague, Susan van Aardt. The graph with the smallest order in this family, which we call the *tarantula*, has order 12 and size 22 and is depicted in Figure 4.19. We note that the generalized propellers of order 12 have at least 24 edges.

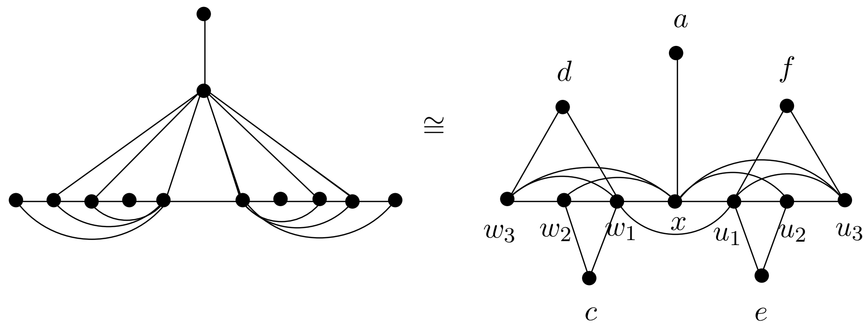


Figure 4.19: The tarantula

We note that in the tarantula both $\langle x, w_1, w_2, w_3\rangle$ and $\langle x, u_1, u_2, u_3\rangle$ are complete graphs. We generalize the tarantula as depicted in Figure 4.20.

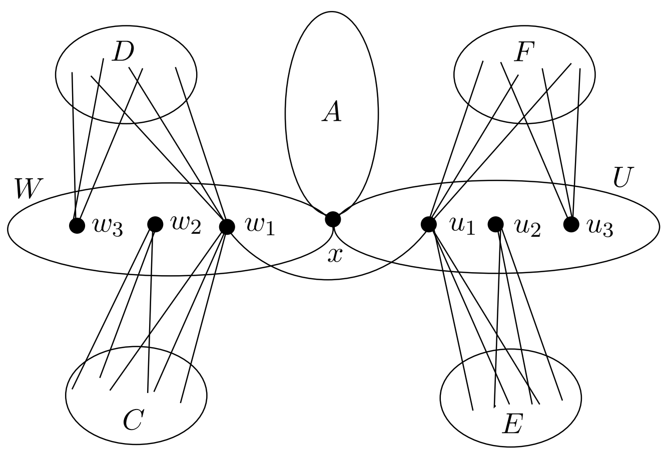


Figure 4.20: Generalized tarantula

A generalized tarantula G contains three complete graphs, A (of order at least 2), W and U (both of order at least 4) which share a single common vertex x and four mutually disjoint complete graphs, C, D, E and F which have no vertices in common with $V(A) \cup V(W) \cup V(U)$. The graph W has three distinguished vertices w_1, w_2, w_3 and U has three distinguished vertices u_1, u_2, u_3. The graph G has the following edges in addition to the edges in the complete graphs:

The vertices w_1 and w_2 are adjacent to all vertices in C, w_1 and w_3 are adjacent to all vertices in D, u_1 and u_2 are adjacent to all vertices in E, u_1 and u_3 are adjacent to all vertices in F, and w_1 is adjacent to u_1.

Theorem 4.4.5 *If G is a generalized tarantula, then G is MNT.*

Proof. We show that G satisfies the conditions of Theorem 4.4.1. The graph G consists of one cut-vertex x and two blocks, A and B. The block A is complete and thus condition 1 is satisfied.

The block B is not traceable from x, since every hamiltonian path with endvertex x contains the edge w_1u_1. But such a path misses all vertices from at least one of C, D, E or F. We now show that $B + uv$ is traceable from x for all nonadjacent vertices u and v in B. Taking the symmetry of B into account we need consider only the following six cases:

Case 1. $u = x, v \in C$

Case 2. $u \in W - x, v \in C$

Case 3. $u \in W - x, v \in E$

Case 4. $u \in W - x, v \in U - x$

Case 5. $u \in C, v \in D$

Case 6. $u \in C, v \in E$.

Since it is rather tedious to prove all the cases and the reasoning used in each of them is similar, we shall consider only two of the cases.

Case 2. $u \in W - x, v \in C$.

Then $u \notin \{w_1, w_2\}$. If $u \neq w_3$, then a hamiltonian path in $B + uv$ is

$$P_{W-\{w_1,w_3,u\}}(x, w_2)P_C(-, v)uw_3P_D(-,-)w_1u_1P_F(-,-)P_{U-\{x,u_1\}}(u_3, u_2)P_E(-,-).$$

If $u = w_3$, then we obtain a hamiltonian path by identifying u and w_3 in the above hamiltonian path.

Case 4. $u \in W - x, v \in U - x$.

If $u = w_1$, then $v \neq u_1$ and if $v = u_1$, then $u \neq w_1$. Suppose, without loss of generality, that $v \neq u_1$. If $u \notin \{w_1, w_2, w_3\}$ and $v \notin \{u_1, u_2, u_3\}$, then a hamiltonian path in $B + uv$ is

$$P_{W-\{w_1,w_3,u\}}(x, w_2)P_C(-,-)w_1P_D(-,-)w_3uP_{U-\{x,u_1\}}(v, u_2)P_E(-,-)u_1P_F(-,-).$$

If $u = w_1$ and $v = u_3$, then a hamiltonian path in $B + uv$ is

$$P_{U-\{u_1,u_3\}}(x, u_2)P_E(-,-)u_1P_F(-,-)vuP_C(-,-)P_{W-\{x,w_1\}}(w_2, w_3)P_D(-,-).$$

All other subcases are proved similarly.

Thus condition 2(a) is satisfied. Obviously condition 2(b) is also satisfied and hence G is MNT. ■

4.5 Claw-free maximal nontraceable graphs

A graph G is *claw-free* if G does not contain an induced subgraph isomorphic to the claw, $K_{1,3}$. Every disconnected MNT graph is claw-free as it is of the form $K_n \cup K_m$. Also, the line graph $L(G)$ of a graph G is always claw-free (see [7]).

4.5.1 Properties of claw-free MNT graphs

A vertex of a claw-free graph G is called an *eligible vertex* if $\langle N_G(x)\rangle$ is a connected, non-complete graph. The operation of joining every pair of nonadjacent vertices in $\langle N_G(x)\rangle$ by an edge is called the *local completion* of G at x.

In [21] Ryjáček defined the *closure*, $cl(G)$, of a claw-free graph G to be the graph obtained by recursively performing the local completion to eligible vertices of G until no eligible vertex remains, and proved that the graph so obtained is well defined, i.e. it is independent of the order of the eligible vertices used during the construction. A claw-free graph G is said to be *closed* if $cl(G) = G$.

The following results concerning $cl(G)$ are proved in [21] and [5].

Theorem 4.5.1 *(Brandt, Favaron and Ryjáček [21], [5])*

Let G be a claw-free graph. Then the following hold:

(i) $cl(G)$ *is claw-free.*

(ii) $c(cl(G)) = c(G)$.

(iii) $\tau(cl(G)) = \tau(G)$.

(iv) $cl(G)$ *is the line graph of some triangle free graph.*

Corollary 4.5.2 *If G is a closed claw-free graph, then we have the following:*

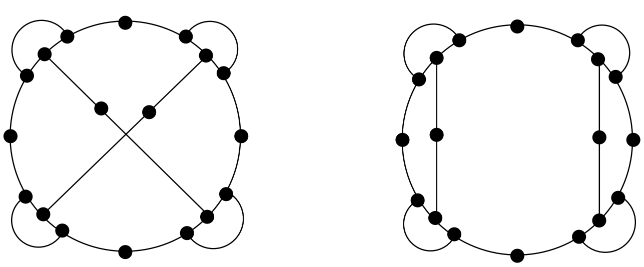
(i) For every $v \in V(G)$ the graph $\langle N_G(v) \rangle$ is either a complete graph or the disjoint union of two complete graphs.

(ii) If X and Y are two maximal cliques in G, then $|V(X) \cap V(Y)| \leq 1$.

The following corollary is a direct consequence of Theorem 4.5.1(iii).

Corollary 4.5.3 *Every MNT claw-free graph G is closed.*

Thus, according to Lemma 3.2.2(ii), every MNT claw-free graph is closed and is its own saturation. However, not every closed claw-free graph is its own saturation. In [6] two graphs, each of order 18 and size 24, are constructed and shown to be the smallest claw-free, 2-connected nontraceable graphs. These graphs A and B are depicted in Figure 4.21.



A B

Figure 4.21: Graphs A and B

Graphs A and B do not contain eligible vertices and hence are closed. However, neither of the graphs is saturated since each vertex of degree 2 is contained in an eligible path of order 3.

We apply the saturation operation to the graphs A and B, which in each case, reduces to adding the edges between the neighbours of each vertex of degree 2. We obtain $s(A)$ and $s(B)$, as shown in Figure 4.22. It can be shown that $s(A)$ is MNT, but that $s(B)$ is not. We have to add another two edges to $s(B)$ to obtain $s^*(B)$, which is MNT.

It can be seen that $s(A)$ is claw-free, but $s^*(B)$ is not. Hence, not every claw-free, nontraceable graph is a spanning subgraph of a claw-free MNT graph.

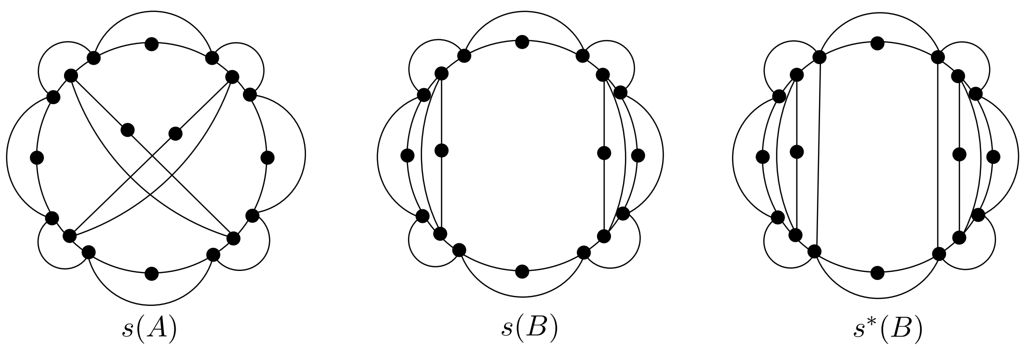


Figure 4.22: Graphs $s(A)$, $s(B)$ and $s^*(B)$

Remark 4.5.4 *It is interesting to note that $s^*(B)$ is obtained by joining two sputniks (see Figure 4.9), by identifying each of the vertices of degree one in the one graph with the corresponding vertex of degree one in the other graph and then adding the edges between the neighbours of the two vertices of degree 2.*

4.5.2 Claw-free MNT graphs with connectivity 1

We firstly note that the smallest connected Zelinka Type I MNT graph is the claw. It is obvious, from Figure 4.1, that no connected Zelinka Type I MNT graph is claw-free. For example,

$\langle\{v_0, v_1, v_2, v_3\}\rangle$, where $v_i \in U_i$, and $i = 0, 1, 2, 3$ is a claw.

The net, the smallest Zelinka Type II graph, is claw-free. It can also be seen, from Figure 4.2, that Zelinka Type II graphs in which $p = q = r = 1$ are claw-free. These graphs all have connectivity 1. A Zelinka Type II graph in which at least one of p, q or r is at least 2 is not claw-free. Consider, for example, $p \geq 2$. Then $\langle\{v_0, v_1, v_2, v_3\}\rangle$, where $v_i \in U_i$, for $i = 0, 1, 2$ and $v_3 \in V_0$ is a claw. Thus there are no claw-free Zelinka MNT graphs which are 2-connected.

We also use a toughness argument in Chapter 6 to show that there are no Zelinka MNT graphs that are claw-free and 2-connected.

4.5.3 2-connected claw-free MNT graphs

From Section 4.5.1, $s(A)$ is the smallest 2-connected, claw-free, MNT graph and can be drawn as shown in Figure 4.23.

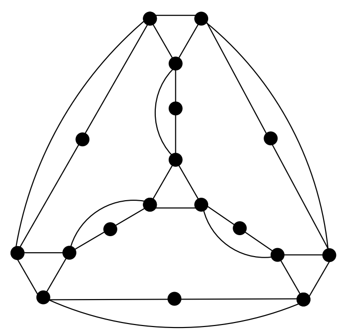


Figure 4.23: Smallest 2-connected, claw-free MNT graph

We can construct an infinite family of 2-connected, claw-free, MNT graphs of order greater than 18 as follows: We construct a graph of order $18 + m$ for every $m \geq 1$ by joining the vertices of a new K_m to every vertex of one of the triangles in the graph in Figure 4.23. (We can also produce such graphs by joining complete graphs of appropriate order to some or all of the triangles.)

Thus, at present, the only connected claw-free MNT graphs that we know of are either of the type described above or Zelinka Type II graphs as discussed in Subsection 4.5.2. The clique structures of all known claw-free MNT graphs are shown in Figure 4.24. In the figure circles and ellipses represent cliques and a dot in the center of a circle (ellipse) indicates that the clique has at least one more vertex than that indicated by the other dots in the circle (ellipse).

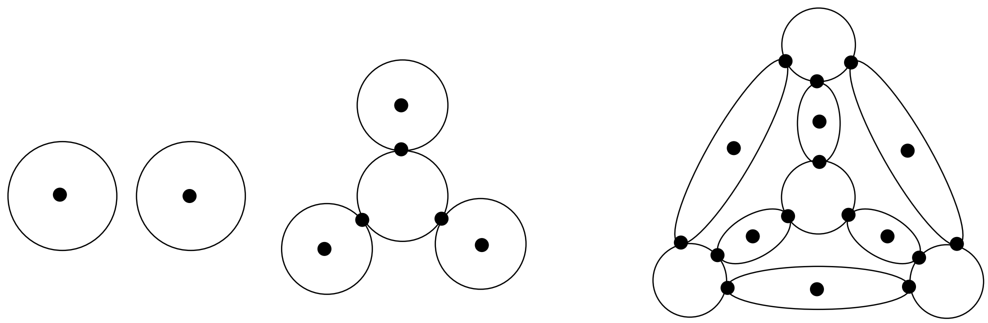


Figure 4.24: Clique structures of all the known claw-free MNT graphs

Chapter 5

Maximal Nontraceable Graphs of Small Size

5.1 Degrees of vertices in maximal nontraceable graphs

In this section we investigate the degrees of vertices in MNT graphs in order to obtain a lower bound for the size of 2-connected MNT graphs, as well as MNT graphs in general. We note that since 2-connected graphs do not contain cut-vertices, such graphs do not contain vertices of degree 1 or adjacent vertices of degree 2 with a common neighbour.

Lemma 5.1.1 *If G is a connected MNT graph and $v \in V(G)$ with $\deg(v) = 2$, then the neighbours of v are adjacent. Also, one of the neighbours has degree at least 4 and the other neighbour has degree 2 or at least 4.*

Proof. Let $N_G(v) = \{x_1, x_2\}$ and let Q be the path x_1vx_2. Since $N_G(v) \subseteq V(Q)$, it follows from Lemma 3.2.1 that $\langle V(Q)\rangle$ is a complete graph; hence x_1 and x_2 are adjacent.

Since G is connected and nontraceable, at least one of x_1 and x_2 has degree greater than 2. Suppose $\deg(x_1) > 2$ and let $z \in N(x_1) - \{v, x_2\}$. If Q is the path zx_1vx_2 then, since

$\deg(v) = 2$, the graph $\langle V(Q)\rangle$ is not complete and hence it follows from Lemma 3.2.1 that $\deg(x_1) \geq 4$. Similarly, if $\deg(x_2) > 2$, then $\deg(x_2) \geq 4$. ■

Corollary 5.1.2 *If G is a 2-connected MNT graph and $v \in V(G)$ with $\deg(v) = 2$, then each neighbour of v has degree at least 4.*

Proof. A 2-connected graph does not contain adjacent vertices of degree 2 with a common neighbour. ■

Lemma 5.1.3 *Suppose G is a connected MNT graph with distinct nonadjacent vertices v_1 and v_2 such that $\deg(v_1) = \deg(v_2) = 2$. If v_1 and v_2 have exactly one common neighbour x, then $\deg(x) \geq 5$.*

Proof. Let $N(v_i) = \{x, y_i\}$, $i = 1, 2$. It follows from Lemma 5.1.1 that x is adjacent to y_i, $i = 1, 2$. Let Q be the path $y_1v_1xv_2y_2$. Since $\langle V(Q)\rangle$ is not complete, it follows from Lemma 3.2.1 that x has a neighbour in $G - V(Q)$. Hence $\deg(x) \geq 5$. ■

Lemma 5.1.4 *Suppose G is a connected MNT graph and $v_1, v_2 \in V(G)$ such that $\deg(v_1) = \deg(v_2) = 2$. If v_1 and v_2 have the same two neighbours x_1 and x_2, then $N_G(x_1) - \{x_2\} = N_G(x_2) - \{x_1\}$ and $\deg(x_1) = \deg(x_2) \geq 5$.*

Proof. From Lemma 5.1.1 it follows that x_1 and x_2 are adjacent. Let Q be the path $x_2v_1x_1v_2$. $\langle V(Q)\rangle$ is not complete since v_1 and v_2 are nonadjacent. Thus it follows from Lemma 3.2.1 that x_1 has a neighbour in $G - V(Q)$. Now suppose $p \in N_{G-V(Q)}(x_1)$ and $p \notin N_G(x_2)$. Then a hamiltonian path P in $G + px_2$ contains a subpath of either of the forms given in the first column of Table 5.1. Note that $i, j \in \{1, 2\}$; $i \neq j$ and that L represents a subpath of P in $G - \{x_1, x_2, v_1, v_2, p\}$. If each of the subpaths is replaced by the corresponding subpath in the second column of the table we obtain a hamiltonian path P' in G, which leads to a contradiction.

Subpath of P	Replace with
$v_i x_1 v_j x_2 p$	$v_i x_2 v_j x_1 p$
$v_i x_1 L p x_2 v_j$	$v_i x_2 v_j x_1 L p$

Table 5.1

Hence $p \in N_G(x_2)$. Thus $N_G(x_1) - \{x_2\} \subseteq N_G(x_2) - \{x_1\}$. Similarly $N_G(x_2) - \{x_1\} \subseteq N_G(x_1) - \{x_2\}$. Thus $N_G(x_1) - \{x_2\} = N_G(x_2) - \{x_1\}$, and hence $\deg(x_1) = \deg(x_2)$. Now let Q be the path $px_1v_1x_2v_2$. Since $\langle V(Q) \rangle$ is not complete, it follows from Lemma 3.2.1 that x_1 or x_2 has a neighbour in $G - V(Q)$. Hence $\deg(x_1) = \deg(x_2) \geq 5$. ■

Lemma 5.1.5 *Suppose G is a connected MNT graph of order $n \geq 6$ and that v_1, v_2 and v_3 are vertices of degree 2 in G having the same neighbours, x_1 and x_2. Then $G - \{v_1, v_2, v_3\}$ is complete and hence $e(G) = \frac{1}{2}(n^2 - 7n + 24)$.*

Proof. The set $\{x_1, x_2\}$ is a vertex-cut of G. Thus, according to Lemma 3.1.1, we have $G - \{v_1, v_2, v_3\} = K_{n-3}$. Hence $e(G) = \frac{1}{2}(n-3)(n-4) + 6$. ■

By combining the previous four lemmas we obtain the following theorem.

Theorem 5.1.6 *Suppose G is a connected MNT graph without vertices of degree 1 or adjacent vertices of degree 2. If G has order $n \geq 7$ and m vertices of degree 2, then $e(G) \geq \frac{1}{2}(3n+m)$.*

Proof. If G has three vertices of degree 2 having the same two neighbours then, by Lemma 5.1.5, since $m = 3$ we have

$$e(G) = \frac{1}{2}(n^2 - 7n + 24) \geq \frac{1}{2}(3n + m) \text{ when } n \geq 7.$$

We now assume that G does not have three vertices of degree 2 that have the same two neighbours. Let $v_1, ..., v_m$ be the vertices of degree 2 in G and let $H = G - \{v_1, ..., v_m\}$. Then, by Lemmas 5.1.1, 5.1.3 and 5.1.4, the minimum degree, $\delta(H)$ of H is at least 3. Hence

$$e(G) = e(H) + 2m \geq \frac{3}{2}(n - m) + 2m = \frac{1}{2}(3n + m).$$ ■

Let $g_2(n)$ denote the minimum number of edges of a 2-connected MNT graph of order n. It follows from Theorem 5.1.6 that $g_2(n) \geq \frac{1}{2}(3n + m)$ for $n \geq 7$. For $m \geq 1$ this bound is realized for $n = 7$ by a Zelinka Type I graph and for $n = 18$ by the smallest 2-connected claw-free MNT graph. These graphs are depicted in Figure 5.1. For $m = 0$ the bound is realized by the cubic graphs presented in the next section.

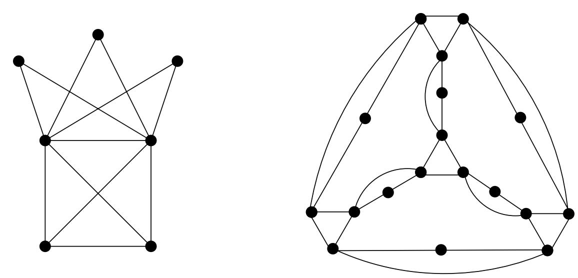


Figure 5.1: MNT graphs with $\frac{1}{2}(3n + m)$ edges

5.2 Cubic maximal nontraceable graphs

It follows from the previous section that $g_2(n) \geq \frac{3n}{2}$ for $n \geq 7$. We now construct an infinite family of 2-connected cubic MNT graphs of order n, showing that $g_2(n) = \frac{3n}{2}$ for infinitely many values of n.

Construction of the graph $K_4[H_1, H_2, H_3]$

For $i = 1, 2, 3$, let H_i be a cubic graph, with a vertex z_i with neighbours a_i, b_i and c_i.

In the same sense as Grünbaum [16] we use $H_i \backslash z_i$ to denote H_i "opened up" at z_i (see Figure 5.2).

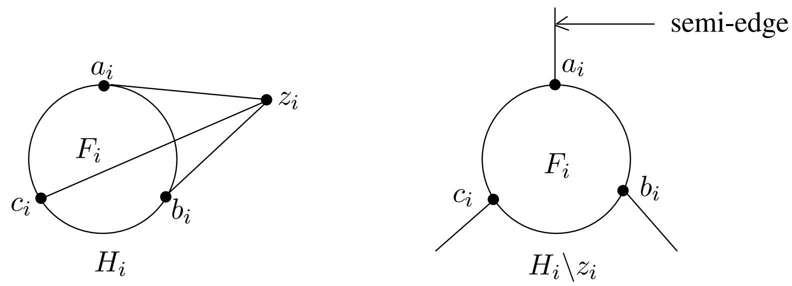


Figure 5.2: H_i "opened up" at z_i

Let $K_4[H_1, H_2, H_3]$ be an inflated K_4 obtained from $H_i\backslash z_i$; $i = 1, 2, 3$ and a vertex x by joining x to the semi-edge incident with a_i for $i = 1, 2, 3$ and joining the remaining semi-edges as depicted in Figure 5.3. Let F_i denote $H_i - z_i$; $i = 1, 2, 3$. We call a_i, b_i and c_i the *exit vertices* of F_i.

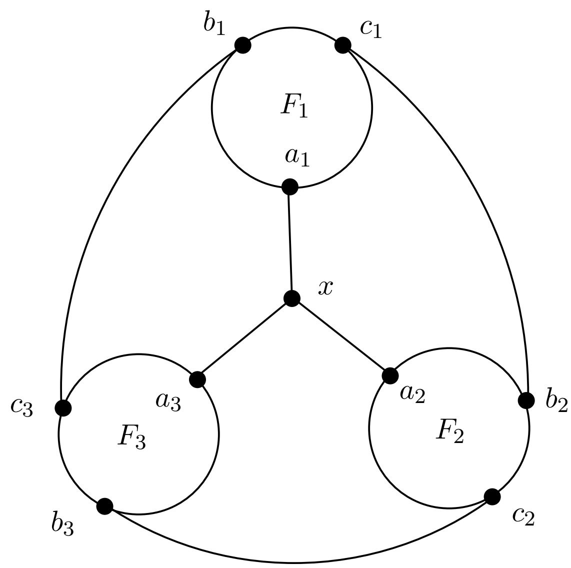


Figure 5.3: Cubic MNT graph

In the proof of the theorem that follows we use the notation for hamiltonian paths which we introduced at the beginning of Section 4.3.

Theorem 5.2.1 *Suppose each cubic graph H_i, $i = 1, 2, 3$, described in the construction of $K_4[H_1, H_2, H_3]$ satisfies the following conditions:*

(i) Every $v \in V(F_i)$ is an endvertex of a hamiltonian path in F_i.

(ii) There is no hamiltonian path in F_i with both endvertices in $N_{H_i}(z_i)$.

(iii) For any $y \in N_{H_i}(z_i)$ there exists a hamiltonian path in $F_i - y$ with endvertices the other two vertices of $N_{H_i}(z_i)$.

(iv) If u_1 and u_2 are nonadjacent vertices in F_i, then $F_i + u_1u_2$ has a hamiltonian path with both endvertices in $N_{H_i}(z_i)$.

(v) For every vertex $u_i \notin N_{H_i}(z_i)$, the graph $H_i + z_iu_i$ has a hamiltonian cycle containing the edge a_iz_i as well as a hamiltonian cycle containing either the edge b_iz_i or the edge c_iz_i.

Then the graph $G = K_4[H_1, H_2, H_3]$ is a cubic MNT graph.

Proof. It is obvious from the construction that G is cubic.

We now show that G is nontraceable. Suppose P is a hamiltonian path of G. Then at least one of the subgraphs F_i, say F_2, does not contain an endvertex of P. Thus P passes through F_2, using two of the exit vertices of F_2. However, by (ii) such a path cannot contain all the vertices of F_2.

We now show that $G + uv$ is traceable for all nonadjacent vertices u and v in G.

Case 1. $u, v \in V(F_i)$; $i \in \{1, 2, 3\}$.

Without loss of generality consider $i = 2$. By (iv) there is a hamiltonian path in $F_2 + uv$ with both endvertices from the set $\{a_2, b_2, c_2\}$.

Subcase 1.1. Suppose the endvertices are a_2 and c_2. (A similar proof holds for a_2 and b_2.) By using (i) we obtain a hamiltonian path

$$P_{G+uv} = P_{F_1}(-, a_1)xP_{F_2+uv}(a_2, c_2)P_{F_3}(b_3, -).$$

Subcase 1.2. Suppose the endvertices are b_2 and c_2. By using (iii) we obtain a hamiltonian path

$$P_{G+uv} = a_1 x P_{F_3-c_3}(a_3, b_3) P_{F_2+uv}(c_2, b_2) P_{F_1-a_1}(c_1, b_1) c_3.$$

Case 2. $u \in \{a_i, b_i, c_i\}$ and $v \in \{a_j, b_j, c_j\}$; $i, j \in \{1, 2, 3\}$; $i \neq j$.
Without loss of generality we choose $i = 2$ and $j = 3$. By using (i) and (iii) we find a hamiltonian path P_{G+uv} in $G + uv$. All subcases can be reduced to the following:
Subcase 2.1. $u = a_2, v = a_3$.

$$P_{G+uv} = a_2 a_3 x P_{F_1-c_1}(a_1, b_1) P_{F_3-a_3}(c_3, b_3) P_{F_2-a_2}(c_2, b_2) c_1.$$

Subcase 2.2. $u = a_2, v = b_3$.

$$P_{G+uv} = c_2 b_3 P_{F_2-c_2}(a_2, b_2) P_{F_1-b_1}(c_1, a_1) x P_{F_3-b_3}(a_3, c_3) b_1.$$

Subcase 2.3. $u = a_2, v = c_3$.

$$P_{G+uv} = b_1 c_3 P_{F_2-c_2}(a_2, b_2) P_{F_1-b_1}(c_1, a_1) x P_{F_3-c_3}(a_3, b_3) c_2.$$

Subcase 2.4. $u = b_2, v = b_3$.

$$P_{G+uv} = c_2 b_3 P_{F_2-c_2}(b_2, a_2) x P_{F_3-b_3}(a_3, c_3) P_{F_1}(b_1, -).$$

Subcase 2.5. $u = b_2, v = c_3$.

$$P_{G+uv} = c_1 b_2 c_3 P_{F_1-c_1}(b_1, a_1) x P_{F_2-b_2}(a_2, c_2) P_{F_3-c_3}(b_3, a_3).$$

Case 3. $u \in V(F_i) - \{a_i, b_i, c_i\}$ and $v \in V(F_j)$; $i, j \in \{1, 2, 3\}$; $i \neq j$.
Without loss of generality we choose $i = 2$ and $j = 3$. Let F_2^* be the graph obtained from G by contracting $G - V(F_2)$ to a single vertex z_2^*. Then F_2^* is isomorphic to H_2 and hence, it follows from (v) that $F_2^* + uz_2^*$ has a hamiltonian cycle containing the path $uz_2^*a_2$. Thus F_2 has a hamiltonian path with endvertices u and a_2. Using this fact and (i) we construct a hamiltonian path

$$P_{G+uv} = P_{F_3}(-, v) P_{F_2}(u, a_2) x P_{F_1}(a_1, -).$$

Case 4. $u = x$ and $v \in V(F_i)$; $i \in \{1, 2, 3\}$.

Without loss of generality we choose $i = 2$.

Subcase 4.1. $v \in \{b_2, c_2\}$.

Consider $v = b_2$. (The case $v = c_2$ follows similarly.) By using (i) and (iii) we obtain a hamiltonian path

$$P_{G+uv} = P_{F_3}(-, b_3)P_{F_2-b_2}(c_2, a_2)xb_2P_{F_1}(c_1, -).$$

Subcase 4.2. $v \in V(F_2) - \{a_2, b_2, c_2\}$.

According to (v) and an argument similar to that in Case 3, there is a hamiltonian path in F_2 with endvertices v and d, where $d \in \{b_2, c_2\}$. Suppose $d = b_2$. (A similar proof holds for $d = c_2$.) Using this fact and (i) we construct a hamiltonian path

$$P_{G+uv} = P_{F_3}(-, a_3)xP_{F_2}(v, b_2)P_{F_1}(c_1, -). \qquad \blacksquare$$

Theorem 5.2.2 *We have* $g_2(n) = \frac{3n}{2}$ *for* $n = 28, 38, 40$ *and all even* $n \geq 46$.

Proof. By Lemmas 2.1.1 and 2.1.2, every MHH graph with a vertex of degree 3 satisfies conditions (i)-(iv) of Theorem 5.2.1.

From Section 2.1 we see that the Petersen graph ($n = 10$), Chisala's G_3-snark ($n = 22$), the Coxeter graph ($n = 28$) and the Isaacs'snarks J_k ($n = 4k$ for odd $k \geq 5$) are all cubic MHH graphs that satisfy condition (v).

Thus, by using various combinations of these MHH graphs, we can produce cubic MNT graphs of order

$$n = \begin{cases} 8p & p \geq 5 \\ 8p+2 & p \geq 6 \\ 8p+4 & p = 3,\ p \geq 6 \\ 8p+6 & p \geq 4. \end{cases}$$

Thus $g_2(n) = \frac{3n}{2}$ for $n = 28, 38, 40$ and all even $n \geq 46$. $\blacksquare$

The graph $K_4[H_1, H_2, H_3]$ with H_i isomorphic to the Petersen graph for all $i = 1, 2, 3$ is depicted in Figure 5.4.

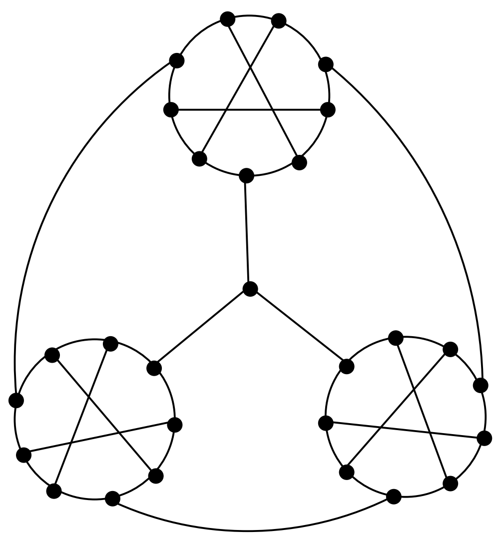


Figure 5.4: Cubic MNT graph, using the Petersen graph

Remark 5.2.3 *We thank Z. Skupień for pointing out that cubic nonhamiltonian MnHc graphs are cubic MHH graphs that satisfy condition (v) of Theorem 5.2.1 - see Section 2.2.*

Remark 5.2.4 *Our construction yields MNT graphs of girths 5, 6 and 7. We do not know whether MNT graphs with girth greater than 7 exist.*

5.3 Almost cubic maximal nontraceable graphs

In this section we construct an infinite family of MNT graphs of order n and size $\frac{3n}{2} + 1$ for even n, by using the building blocks that we used to construct the cubic graphs in the previous section.

Construction of the graph $G[H_1, H_2, H_3, H_4]$

For $i = 1, 2, 3, 4$, let H_i be a cubic graph, with a vertex z_i with neighbours a_i, b_i and c_i, and let F_i denote $H_i - z_i$. Let $G[H_1, H_2, H_3, H_4]$ be the graph obtained by using $H_i \backslash z_i$; $i = 1, 2, 3, 4$ (see Figure 5.2), and identifying the vertices a_1 and a_4 as vertex $a_{1,4}$ and the vertices a_2 and a_3 as vertex $a_{2,3}$, and joining the remaining semi-edges as depicted in Figure 5.5.

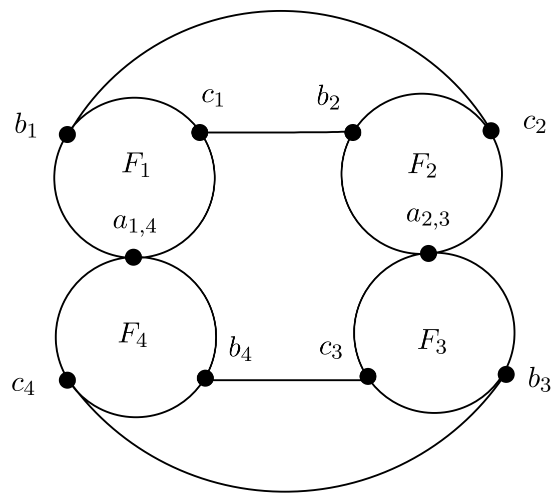


Figure 5.5: Almost cubic MNT graph

We require the same conditions that we used to prove that $K_4[H_1, H_2, H_3]$ is maximal nontraceable to prove that $G = G[H_1, H_2, H_3, H_4]$ is maximal nontraceable. For completeness sake we repeat these conditions in the next theorem.

Theorem 5.3.1 *Suppose each cubic graph H_i, $i = 1, 2, 3, 4$, described in the construction of $G[H_1, H_2, H_3, H_4]$ satisfies the following conditions:*

(i) Every $v \in V(F_i)$ is the endvertex of a hamiltonian path in F_i.

(ii) There is no hamiltonian path in F_i with both endvertices in $N_{H_i}(z_i)$.

(iii) For any $y \in N_{H_i}(z_i)$ there exists a hamiltonian path in $F_i - y$ with endvertices the other two vertices of $N_{H_i}(z_i)$.

(iv) If u_1 and u_2 are nonadjacent vertices in F_i, then $F_i + u_1u_2$ has a hamiltonian path with both endvertices in $N_{H_i}(z_i)$.

(v) For every vertex $u_i \notin N_{H_i}(z_i)$, the graph $H_i + z_iu_i$ has a hamiltonian cycle containing the edge a_iz_i as well as a hamiltonian cycle containing either the edge b_iz_i or the edge c_iz_i.

Then the graph $G = G[H_1, H_2, H_3, H_4]$ is maximal nontraceable.

Proof. By inspection, a longest path in G misses one vertex and hence G is not traceable. We now show that $G + uv$, $u, v \in V(G)$ and $uv \notin E(G)$ is traceable. We use both a_i and a_j to denote $a_{i,j}$, where $(i, j) = (1, 4)$ or $(i, j) = (2, 3)$. Due to the symmetry of G we need only consider the following cases:

Case 1. $u, v \in V(F_3)$

Case 2. $u = c_3$, $v \in \{a_1, b_1, c_1, b_2, c_2, c_4\}$

$u = c_3$, $v \in V(F_i) - \{a_i, b_i, c_i\}$, $i = 1, 2, 4$

Case 3. $u = a_{2,3}$, $v \in \{a_{1,4}, b_1\}$

$u = a_{2,3}$, $v \in V(F_1) - \{a_1, b_1, c_1\}$

Case 4. $u = b_3$, $v \in \{b_1, c_2\}$

$u = b_3$, $v \in V(F_i) - \{a_i, b_i, c_i\}$, $i = 1, 2, 4$

Case 5. $u \in V(F_3) - \{a_3, b_3, c_3\}$, $v \in V(F_i) - \{a_i, b_i, c_i\}$, $i = 1, 2, 4$

In all cases except where $u \in V(F_3) - \{a_3, b_3, c_3\}$ and $v \in V(F_2) - \{a_2, b_2, c_2\}$, we use the same reasoning as in Theorem 5.2.1 to find a hamiltonian path in $G + uv$. We consider two such examples:

1. Let $u = c_3$ and $v = c_1$. Then by using (i) and (iii) we construct the following hamiltonian

path in $G+uv$:

$$P_{G+uv} = P_{F_4}(-,c_4)P_{F_3-a_{2,3}}(b_3,u)P_{F_1-a_{1,4}}(v,b_1)P_{F_2}(c_2,-).$$

2. Let $u = c_3$ and $v \in V(F_4) - \{a_4, b_4, c_4\}$. Using (i), (iii) and (v) we construct the following hamiltonian path in $G+uv$:

$$P_{G+uv} = P_{F_3}(-,u)P_{F_4}(v,a_{1,4})P_{F_1-b_1}(a_{1,4},c_1)P_{F_2-a_{2,3}}(b_2,c_2)b_1.$$

We return to the case where $u \in V(F_3) - \{a_3, b_3, c_3\}$ and $v \in V(F_2) - \{a_2, b_2, c_2\}$. From condition (iii) it follows that $F_3 - a_{2,3} + b_3c_3$ has a hamiltonian cycle C. Suppose x is a neighbour of u on C. Then $C - ux$ is a hamiltonian path in $F_3 - a_{2,3} + b_3c_3$. Hence $F_3 - a_{2,3}$ has a 2-path cover, the one path $F^1_{F_3-a_{2,3}}$ with endvertices u and either b_3 or c_3, say b_3, and the other path $F^2_{F_3-a_{2,3}}$ with c_3 as endvertex. Thus there is a hamiltonian path

$$P_{G+uv} = P_{F_1}(-,b_1 \text{ or } c_1)P_{F_2}(c_2 \text{ or } b_2,v)F^1_{F_3-a_{2,3}}(u,b_3)P_{F_4-a_{1,4}}(c_4,b_4)F^2_{F_3-a_{2,3}}(c_3,-).$$

■

Theorem 5.3.2 *There are maximal nontraceable graphs of order n and size $\frac{3n}{2}+1$ for $n = 38, 50, 62$ and all even $n \geq 68$.*

Proof. As stated in Theorem 5.2.2 the Petersen graph ($n = 10$), Chisala's G_3-snark ($n = 22$), the Coxeter graph ($n = 28$) and the Isaacs'snarks J_k ($n = 4k$ for odd $k \geq 5$) are all cubic graphs that satisfy the conditions of Theorem 5.3.1.

Thus, by using various combinations of these MHH graphs, we can produce graphs of order n and size $\frac{3n}{2}+1$ for $n = 38, 50, 62$ and all even $n \geq 68$. ■

The graph $G[H_1, H_2, H_3, H_4]$ with H_i isomorphic to the Petersen graph for all $i = 1, 2, 3, 4$ is depicted in Figure 5.6. It has 34 vertices and was first constructed by Thomassen [25], as an example of a hypotraceable graph. (A graph G is *hypotraceable* if G is not traceable, but every vertex deleted subgraph $G - v$ is traceable.)

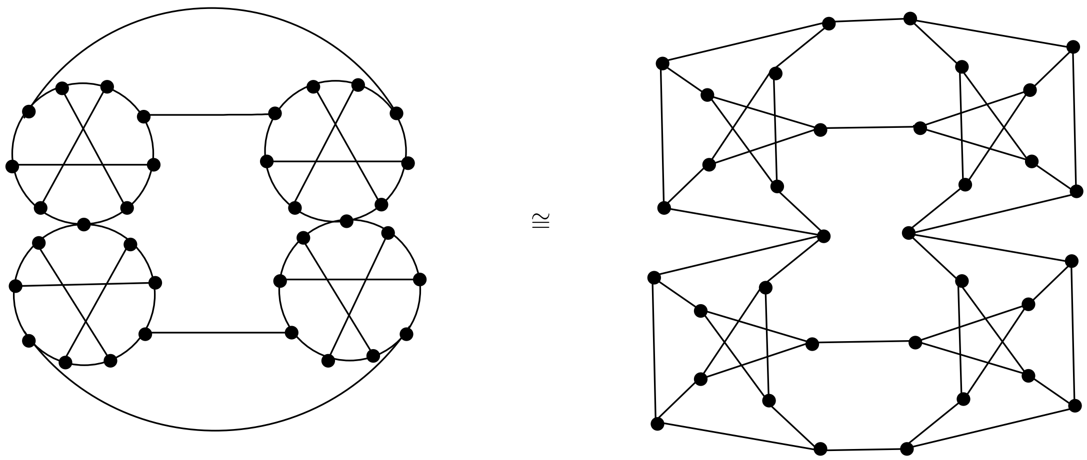


Figure 5.6: Thomassen graph

Remark 5.3.3 *Remark 5.2.3 also applies to the graphs used in the construction of* $G[H_1, H_2, H_3, H_4]$.

5.4 Minimum size of maximal nontraceable graphs of order n

Let $g(n)$ denote the minimum number of edges in an MNT graph of order n. Dudek, Katona and Wojda [13] proved, by considering a special case of m–path saturated graphs, that

$$g(n) \geq \lceil \tfrac{3n-2}{2} \rceil - 2 \text{ for } n \geq 20$$

and showed, by construction (see Construction 2 of Subsection 4.3.1), that

$$g(n) \leq \lceil \tfrac{3n-2}{2} \rceil \text{ for } n \geq 54$$

as well as for $n \in I = \{22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$.

In this section we prove, using a method different from that in [13], that

$$g(n) \geq \lceil \tfrac{3n-2}{2} \rceil \text{ for } n \geq 10.$$

Since in Construction 2 of Subsection 4.3.1 we showed that graphs of order $n = 12, 13$ with $\lceil \frac{3n-2}{2} \rceil$ edges exist, we will thus prove that

$$g(n) = \lceil \tfrac{3n-2}{2} \rceil \text{ for } n \geq 54 \text{ as well as for } n \in I \cup \{12, 13\}.$$

We also determine the exact value of $g(n)$ for $n \leq 10$.

We require the following lemma in the proof of the theorem that follows. This lemma combines results proved in [4] and [18].

Lemma 5.4.1 *(Bondy and Lin, Jiang, Zhang and Yang [4], [18])*

If G is an MNH graph of order n, then $e(G) \geq \frac{3n}{2}$ for $n \geq 6$.

By consulting [20], An Atlas of Graphs, one can see, by inspection, that $g(2) = 0$, $g(3) = 1$, $g(4) = 2$, $g(5) = 4$, $g(6) = 6$ and $g(7) = 8$ (see Figure 5.9).

We now give a lower bound for $g(n)$ for $n \geq 8$.

Theorem 5.4.2 *If G is an MNT graph of order n, then*

$$e(G) \geq \begin{cases} 10 & \text{if } n = 8 \\ 12 & \text{if } n = 9 \\ \frac{3n-2}{2} & \text{if } n \geq 10. \end{cases}$$

Proof. If G is not connected, then $G = K_k \cup K_{n-k}$ for some positive integer $k < n$ and then, clearly, $e(G) > \frac{3n-2}{2}$ for $n \geq 8$. Thus we assume that G is connected.

In order to determine a lower bound for $g(n)$ we need at times to consider the degrees of vertices of G. In view of Theorem 5.1.6, we let

$$M = \{v \in V(G) \mid \deg(v) = 2 \text{ and no neighbour of } v \text{ has degree } 2\}.$$

The remaining vertices of degree 2 can be dealt with simultaneously with the vertices of degree 1. We let

$$S = \{v \in V(G) - M \mid \deg(v) = 2 \text{ or } \deg(v) = 1\}.$$

If $S = \emptyset$, then it follows from Theorem 5.1.6 that $e(G) \geq \frac{1}{2}(3n + m)$. Thus we assume that $S \neq \emptyset$.

We observe by Lemma 3.2.1 that, if H is a component of the graph $\langle S \rangle$, then either $H \cong K_1$ or $H \cong K_2$ and $N_G(H) - V(H)$ consists of a single vertex, which is a cut-vertex of G.

An example of such a graph G is depicted in Figure 5.7.

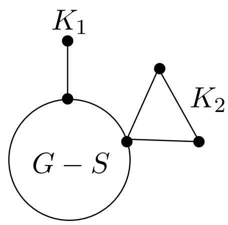


Figure 5.7: Graph for Theorem 5.4.2

Let $s = |S|$. Clearly the graph $\langle S \rangle$ has at most three components. We thus have three cases:

Case 1. $\langle S \rangle$ has exactly three components, say H_1, H_2, H_3:
In this case G has exactly four blocks since an MNT graph has at most four blocks. It thus follows from Theorem 4.2.4 that the neighbourhoods of H_1, H_2, H_3 are pairwise disjoint and that $G - S$ is a complete graph of order at least 3. Furthermore, for every possible value of s, the number of edges in G incident with the vertices in S is $2s - 3$. Thus

$$e(G) = \binom{n-s}{2} + 2s - 3 \text{ for } s = 3, 4, 5 \text{ or } 6;\ s \leq n - 3.$$

An easy calculation shows that, for each possible value of s,

$$e(G) \geq \begin{cases} 10 & \text{if } n = 8 \\ 12 & \text{if } n = 9 \\ \frac{3n-2}{2} & \text{if } n \geq 10. \end{cases}$$

All graphs of this type are Zelinka Type II graphs. Those of smallest size of orders 8 and 9 are depicted in Figure 5.9.

Case 2. $\langle S \rangle$ has exactly two components, say H_1, H_2:
In this case the number of edges in G incident with the vertices in S is $2s-2$.

Subcase 2.1. $N_G(H_1) = N_G(H_2)$:
It follows from Lemma 3.1.1 that $G - S$ is a complete graph. Hence

$$e(G) = \binom{n-s}{2} + 2s - 2 \text{ for } s = 2, 3 \text{ or } 4.$$

Thus

$$e(G) \geq \begin{cases} 12 & \text{if } n = 8 \\ 16 & \text{if } n = 9 \\ \frac{3n-2}{2} & \text{if } n \geq 10. \end{cases}$$

All graphs of this type are Zelinka Type I graphs.

Subcase 2.2. $N_G(H_1) \neq N_G(H_2)$:
Let $N_G(H_i) = y_i$, $i = 1, 2$ and $y_1 \neq y_2$.
If $y_1y_2 \notin E(G)$ then $G + y_1y_2$ has a hamiltonian path P. But then P has one endvertex in H_1 and the other in H_2 and contains the edge y_1y_2; hence $V(G-S) = \{y_1, y_2\}$. But then G is disconnected. This contradiction shows that $y_1y_2 \in E(G)$.

Now $G - S$ is not complete, otherwise G would be traceable. Since $G + vw$, where v and w are nonadjacent vertices in $V(G-S)$, contains a hamiltonian path with one endvertex in H_1 and the other in H_2 and $y_1y_2 \in E(G)$, it follows that $(G-S) + vw$ has a hamiltonian cycle. Hence $G - S$ is either hamiltonian or MNH. We consider these two cases separately.

Subcase 2.2.1. $G - S$ is hamiltonian:
Since no hamiltonian cycle in $G - S$ contains y_1y_2 we have $\deg_{G-S}(y_i) \geq 3$ for $i = 1, 2$.

It also follows from Lemma 3.1.1(iii) that no vertex $v \in M$ can be adjacent to both y_1 and y_2, since the graph $\langle V(H_i) \cup T \rangle$, where $T = \{y_1, y_2\}$ is not complete, for $i = 1, 2$. If $v \in M$ is adjacent to to one of the vertices y_i for $i = 1, 2$, say y_1, then, since the neighbours of v are adjacent, it follows that $\deg_{G-M-S}(y_1) \geq 3$.

It follows from our definition of M and S that $N_G(M) \cap S = \emptyset$. Since $G - M$ is not a complete graph, it follows from Lemma 5.1.5 that M does not have three vertices that have the same neighbourhood in G. Hence, by Lemmas 5.1.1, 5.1.3 and 5.1.4, the minimum degree of the graph $G - M - S$ is at least 3.

Now, for $n \geq 8$

$$\begin{aligned} e(G) &= e(G - M - S) + 2m + 2s - 2 \\ &\geq \frac{1}{2}\left(3\left(n - m - s\right)\right) + 2m + 2s - 2 \\ &= \frac{1}{2}\left(3n + m + s - 4\right) \\ &\geq \frac{3n - 2}{2}, \text{ since } s \geq 2. \end{aligned}$$

Subcase 2.2.2. $G - S$ is maximal nonhamiltonian:

Since $G - S$ is MNH it follows from Theorem 5.4.1, that $e(G - S) \geq \frac{3}{2}(n - s)$ for $n - s \geq 6$.

Thus, for $n - s \geq 6$ and $n \geq 8$

$$\begin{aligned} e(G) &= e(G - S) + 2s - 2 \\ &\geq \frac{1}{2}(3(n - s)) + 2s - 2 \\ &= \frac{1}{2}(3n + s - 4) \\ &\geq \frac{3n - 2}{2}, \text{ since } s \geq 2. \end{aligned}$$

From [18] we have

$$e(G - S) \geq \begin{cases} 6 & \text{for } n - s = 5 \\ 4 & \text{for } n - s = 4. \end{cases}$$

Thus

$$e(G) \geq \begin{cases} 12 & \text{for } n = 9 \text{ and } n - s = 5 \\ 10 & \text{for } n = 8 \text{ and } n - s = 5 \text{ or } n - s = 4. \end{cases}$$

The smallest MNH graphs F_4 and F_5 of order 4 and 5 respectively, are depicted in Figure 5.8; cf. [18]. The graphs G_8 and G_9 (see Figure 5.9) are obtained, respectively, by using F_4 with $s = 4$ or F_5 with $s = 3$, and F_5 with $s = 4$.

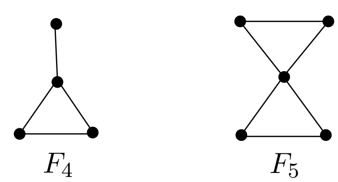


Figure 5.8: Smallest MNH graphs of order 4 and 5

Case 3. $\langle S \rangle$ has exactly one component, say H:

Since

$$\sum_{v \in S} \deg_G(v) = 3s - 2, \text{ for } s = 1, 2$$

it follows that

$$\begin{aligned} e(G) &= e(G - M) + 2m \\ &= \frac{1}{2}\left(\sum_{v \in V(G-M)-S} \deg_{G-M}(v) + \sum_{v \in S} \deg_{G-M}(v)\right) + 2m \\ &\geq \frac{1}{2}\left(3\left(n - m - s\right) + 3s - 2\right) + 2m \\ &= \frac{1}{2}\left(3n + m - 2\right) \\ &\geq \frac{3n - 2}{2}. \end{aligned}$$

■

Corollary 5.4.3 *$g(n) = \lceil \frac{3n-2}{2} \rceil$ for $n \geq 54$ as well as for every $n \in \{12, 13, 22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51\}$.*

It follows from Theorem 5.4.2 and the fact that the graphs G_8 and G_9 shown in Figure 5.9 are MNT, that $g(8) = 10$ and $g(9) = 12$. Maximal nontraceable graphs G_n of order n with $g(n)$ edges, for $n \leq 9$, are depicted in Figure 5.9.

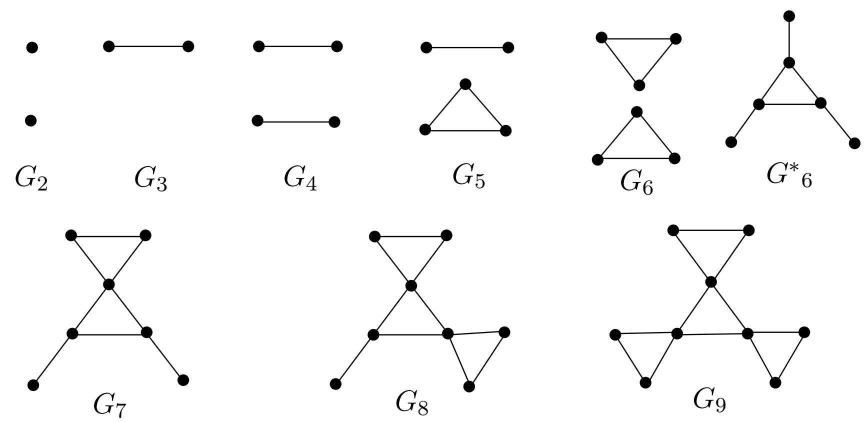


Figure 5.9: Smallest MNT graphs of orders 2 to 9

Theorem 5.4.2 implies that $g(10) \geq 14$. We now show that there are no MNT graphs of order 10 and size 14 and that, in fact, $g(10) = 15$.

We first give a few definitions and auxiliary results before proving the result.

We recall that the circumference $c(G)$ of a graph G is the order of a longest cycle, a circumference cycle, of G. We also recall that if $u \in V(C)$ and $v \in V(C)$, where C is a cycle with orientation, then we denote the path on C from u to v by $C[u, v]$ and the other path on C from u to v by $\overline{C}[u, v]$. The paths obtained by deleting the endvertices u, v are denoted by $C(u, v)$ and $\overline{C}(u, v)$, respectively.

A vertex of a subgraph H of a graph G that has a neighbour in $G - V(H)$ is called an *attachment vertex* of H.

Lemma 5.4.4 *Suppose C is a circumference cycle of a graph G and $G - V(C)$ has a path $P = yLz$ with endvertices y and z, where we also allow $V(L) = \emptyset$ as well as $y = z$. If y is adjacent to $u \in V(C)$ and z is adjacent to $v \in V(C)$, then*

$$|V(C(u,v)| \geq |V(P)| \text{ and } |V(\overline{C}(u,v)| \geq |V(P)|.$$

Proof. Suppose $|V(C(u,v))| < |V(P)|$. Then $C' = y\overline{C}[u,v]zLy$ is a cycle such that $|V(C')| > |V(C)|$, which is a contradiction, since C is a circumference cycle.

A similar argument holds if $|V(\overline{C}(u,v))| < |V(P)|$. ∎

Remark 5.4.5 *If P consists of the vertex y alone, then y is not adjacent to neighbours on C.*

The following lemma is a direct result of Lemma 5.4.4.

Lemma 5.4.6 *Suppose $c(G) = k$. Let C be a circumference cycle of G.*
If $u, v \in V(G - V(C))$ are endvertices of a path in $G - V(C)$ of order at least $\lfloor \frac{k}{2} \rfloor$, then not both u and v are attachment vertices of $G - V(C)$.

We also require the following lemma for the proof of the next theorem.

Lemma 5.4.7 *Suppose G is an MNT graph as described in Case 3 of Theorem 5.4.2, with $H \cong K_1$ and $\deg_G(v) = 3$ for all $v \in G - S$. Then $G - S$ is 2-connected.*

Proof. Suppose $G - S$ is not 2-connected. Then G has a structure as depicted in Figure 5.10(a) or (b), where A, B and C are blocks of G.

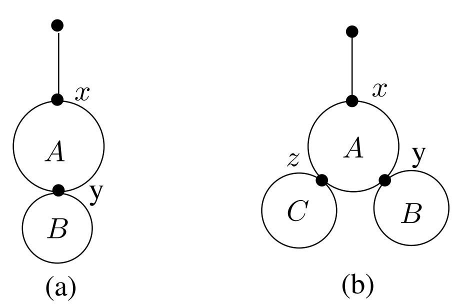


Figure 5.10: Graph for Lemma 5.4.7

If G has the structure as depicted in Figure 5.10(a), then it follows from Theorem 4.3.1 that B is complete. If $B = K_2$ or $B = K_3$, then B has vertices of degree less than 3 in G, and if $B = K_4$, then $\deg_G(y) > 3$. If G has the structure as depicted in Figure 5.10(b), then clearly A, B and C are all complete and, as before, we get a contradiction. Hence $G - S$ is 2-connected. ■

Theorem 5.4.8 *The minimum size of an MNT graph of order 10 is 15.*

Proof. According to Theorem 5.4.2 we have $g(10) \geq 14$. We show, by considering the various cases given in the proof of that theorem, that there is no MNT graph of order 10 with 14 edges.

Case 1.

The smallest graph of order 10 for this case is the Zelinka Type II graph depicted in Figure 5.11 which has 15 edges.

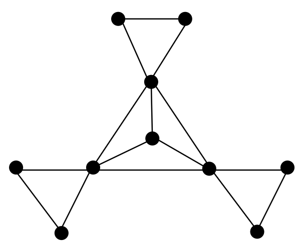


Figure 5.11: Graph for Theorem 5.4.8, Case 1

Case 2.1.

The smallest graph for this case has 21 edges and is depicted in Figure 5.12.

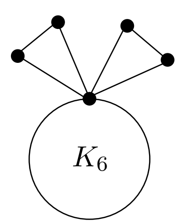


Figure 5.12: Graph for Theorem 5.4.8, Case 2.1

Case 2.2.1.

Since $e(G) \geq \frac{1}{2}(3n + m + s - 4)$ (see Case 2.2.1 of Theorem 5.4.2), it follows that the only values of m and s that will produce an MNT graph G of order 10 and 14 edges are $m = 0$ and $s = 2$, i.e. $G - S$ has no vertices of degree 2.

The graph G has one of the three graphs H_i, $i = 1, 2, 3$ depicted in Figure 5.13 as a spanning subgraph.

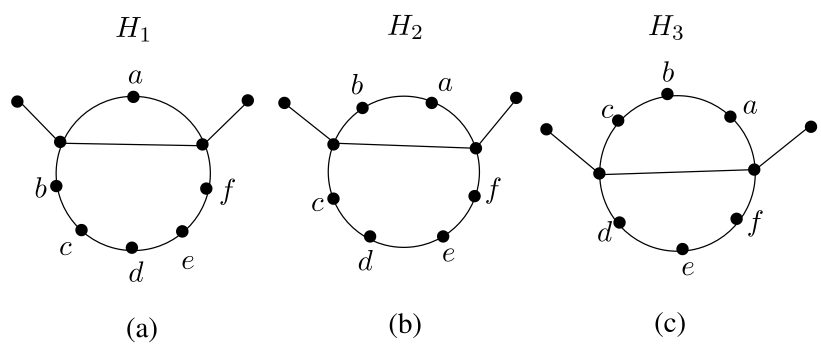


Figure 5.13: Graph for Theorem 5.4.8, Case 2.2.1

In order to have 14 edges and no vertices of degree 2 in $G - S$, each vertex v of degree 2 in H_i, $i = 1, 2, 3$ must be adjacent in G to a vertex of degree 2 in H_i which is not a neighbour, i.e. $G - S$ is a cubic graph of order 8.

Subcase depicted in Figure 5.13(a)

Firstly $ab, af \notin E(G)$, otherwise G is traceable. Due to symmetry all subcases are covered by the following two cases.

(i) $ac \in E(G)$:

The only possibility is $be, df \in E(G)$, but then G is traceable.

(ii) $ad \in E(G)$: Then $be, cf \notin E(G)$. The only possibility is $bf, ce \in E(G)$, but then G is traceable.

Subcase depicted in Figure 5.13(b)

Firstly $ac, bf \notin E(G)$, otherwise G is traceable. Due to symmetry all subcases are covered by the following three cases.

(i) $ad \in E(G)$: Then $bc, be \notin E(G)$. Hence no graph G is possible.

(ii) $ae \in E(G)$: Then $bd \notin E(G)$. If $bc, df \in E(G)$, then G is traceable.

(iii) $af \in E(G)$: Then $bc, be \notin E(G)$. If $bd, ce \in E(G)$, then G is traceable.

Subcase depicted in Figure 5.13(c)

Firstly $ad, cf \notin E(G)$, otherwise G is traceable. Due to symmetry all subcases are covered by the following three cases.

(i) $ac \in E(G)$: Then $bd, bf \notin E(G)$. If $be, df \in E(G)$, then G is traceable.

(ii) $ae \in E(G)$: The only possibility is $bf, cd \in E(G)$, but then G is traceable.

(iii) $af \in E(G)$: Then $bd \notin E(G)$. If $be, cd \in E(G)$, then G is traceable.

Thus there is no MNT graph G with 14 edges having one of the graphs in Figure 5.13 as a spanning subgraph.

Case 2.2.2.

By using the smallest MNH graph of order 6, cf. [18] we obtain either the graph depicted in Figure 5.11 or in Figure 5.14, both of which have 15 edges.

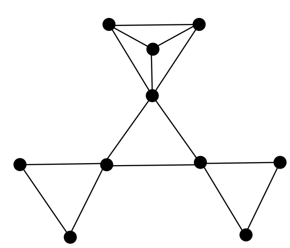


Figure 5.14: Graph for Theorem 5.4.8, Case 2.2.2

The smallest MNH graphs of orders 7 and 8 have 12 and 15 edges, respectively (see [18]). Thus we need only consider the one of order 7 and this graph produces an MNT graph of order 10 with 16 edges.

Case 3.

Since $e(G) \geq \frac{1}{2}(3n+m-2)$ (see Case 3 of Theorem 5.4.2), the only possibility for a graph of order 10 to have 14 edges is when $m = 0$, i.e. when $\deg_G(v) \neq 2$ for all $v \in V(G) - S$.
For $s = 1$, $e(G) = 14 \iff \deg_G(v) = 3$ for all $v \in V(G) - S$.
For $s = 2$, using Lemma 5.1.1, we have

$$\begin{aligned} e(G) &= \frac{1}{2}\left(\sum_{v \in S} \deg_G(v) + \sum_{v \in V(G)-S} \deg_G(v)\right) \\ &\geq \frac{1}{2}\left(4 + (4 + 6(3) + 4)\right) = 15. \end{aligned}$$

Thus we only consider $s = 1$, i.e. $H \cong K_1$.

Let x denote the vertex adjacent to H in G, and $F = G - S$. Then $\deg_F(x) = 2$ and the other 8 vertices of F have degree 3. Note that F is not traceable from x, otherwise G is traceable. We show that such an MNT graph G does not exist by considering circumference cycles of F.

Let C be a circumference cycle of F and $D = F - V(C)$. We define, respectively, the degree deficiency, $dd(C)$, of C in F and the degree deficiency, $dd(D)$, of D in F as

$$dd(C) = \sum_{v \in V(C)} \deg_F(v) - 2|V(C)|$$

and

$$dd(D) = \sum_{v \in V(D)} \deg_F(v) - \sum_{v \in V(D)} \deg_D(v).$$

Thus

$$dd(C) = \begin{cases} |V(C)| & \text{if } x \notin V(C) \\ |V(C)| - 1 & \text{if } x \in V(C) \end{cases}$$

and $dd(D)$ equals the number of edges in $E(F)$ having one endvertex in D, and the other endvertex in $V(C)$. Therefore, since F is connected, $dd(D) > 0$ and, since vertices on C may have neighbours on C, we have

$$dd(C) \geq dd(D) > 0.$$

Since F is not traceable from x, $c(F) \leq 8$. Suppose C is a circumference cycle of F of order k, where $k \leq 8$. We denote the vertices on C in an anti-clockwise direction by $x_0, x_1, ..., x_{k-1}$, where $x_0 = x$ when $x \in V(C)$. We denote the vertices of D by $v_0, v_1, ..., v_{8-k}$, where $v_0 = x$ when $x \in V(D)$.

Case 3.1. $c(F) \leq 5$:

Clearly at least one component of D, say A, is not K_1. Since F is 2-connected (Lemma 5.4.7), there exists two distinct vertices in A which are attachment vertices of D. According to Lemma 5.4.6 this is impossible.

Case 3.2. $c(F) = 6$:

(a) $x \in V(C)$.

Now $dd(C) = 5$ and the only candidates for D with $0 < dd(D) \leq 5$ are P_3 and K_3. In each case, there is a path of order 3 with endvertices which are attachment vertices of D. According to Lemma 5.4.6 this is impossible.

(b) $x \in D$.

Now $dd(C) = 6$ and the only candidates for D with $0 < dd(D) \leq 6$ are $K_1 \cup K_2$, P_3 and K_3. Using reasoning similar to that in (a) it can be seen that D cannot be isomorphic to P_3 or K_3. Now consider $D \cong K_1 \cup K_2$. Let $V(K_1) = \{w_1\}$ and $V(K_2) = \{w_2, w_3\}$, where $x = w_1$ or $x = w_2$. Then, according to Lemma 5.4.4, without loss of generality, $w_2x_0, w_3x_3 \in E(F)$, but then $w_3x_i \notin E(F)$ for $i = 1, 2, 4, 5$ and hence $\deg_F(w_3) < 3$, a contradiction.

Case 3.3. $c(F) = 7$:

(a) $x \in V(C)$.

(i) $v_0v_1 \notin E(F)$.

Then $|N_C(v_0)| = |N_C(v_1)| = 3$. Suppose, without loss of generality, that $v_0x_1 \in E(F)$. Then, by Lemma 5.4.4, we have $v_0x_i \in E(F)$ for $i = 3, 5$ and $v_1x_i \in E(F)$ for $i = 2, 4, 6$. But, then F is traceable from x.

(ii) $v_0v_1 \in E(F)$.

Then $|N_C(v_0)| = |N_C(v_1)| = 2$. Also, $v_0x_1, v_0x_6, v_1x_1, v_1x_6 \notin E(F)$, otherwise F is traceable from x. By Lemma 5.4.4, $v_0x_2, v_1x_5 \in E(F)$; but then $v_0x_i \notin E(F)$ for $i = 3, 4$. Hence $\deg_F(v_0) < 3$.

(b) $x \in V(D)$.

(i) $xv_1 \notin E(F)$.

Then $|N_C(x)| = 2$ and $|N_C(v_1)| = 3$. Suppose $v_1x_i \in E(F)$ for $i = 0, 2, 4$. (Other cases are similar.) Then $xx_i \notin E(F)$ for $i = 1, 3, 5, 6$, otherwise F is traceable from x. Thus $\deg_F(x) = 0$.

(ii) $xv_1 \in E(F)$.

Then $|N_C(x)| = 1$ and $|N_C(v_1)| = 2$. Suppose, without loss of generality, that $xx_0 \in E(F)$. Then by Lemma 5.4.4, $v_1x_i \notin E(F)$ for $i = 1, 2, 5, 6$. Also by Lemma 5.4.4 not both v_1x_3 and v_1x_4 can be in $E(F)$. Hence $\deg_F(v_1) < 3$.

Case 3.4. $c(F) = 8$:

(a) $x \in V(C)$.

In this case $V(D) = \{v_0\}$ and $|N_C(v_0)| = 3$. Also, $x_iv_0 \notin E(F)$ for $i = 1, 7$, otherwise F is traceable from x. Then, by Lemma 5.4.4, $v_0x_i \in E(F)$ for $i = 2, 4, 6$. But then $x_1x_i \notin E(F)$ for $i = 3, 5, 7$, otherwise F is traceable from x. Hence $\deg_F(x_1) = 2$.

(b) $x \in D$.

It is obvious that x cannot be an attachment vertex of D.

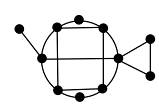
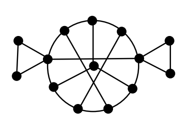
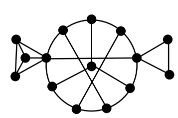
We have thus shown that no MNT graphs of order 10 and size 14 exist and since there are MNT graphs of order 10 and size 15 it follows that $g(10) = 15$. ■

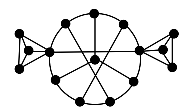
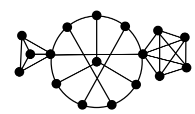
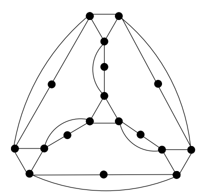
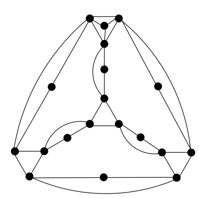
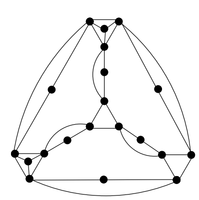
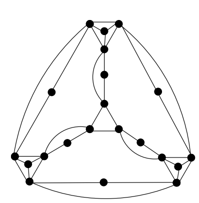
Remark 5.4.9 *We note that there are two Zelinka graphs (see Figures 5.11 and 5.14) and a non-Zelinka graph, the sputnik (see Figure 4.9), each of which has order 10 and size 15.*

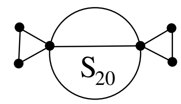
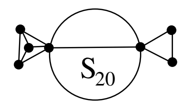
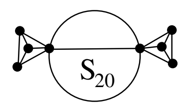
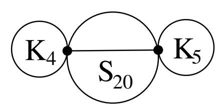
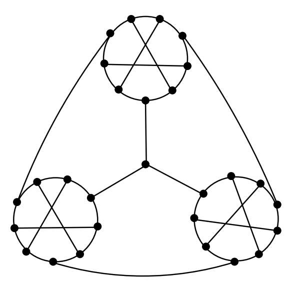
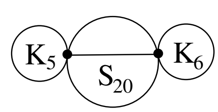
5.5 Candidates for the missing cases

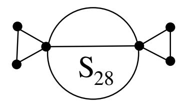
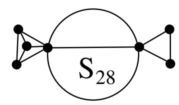
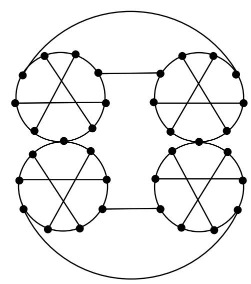
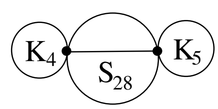
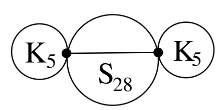
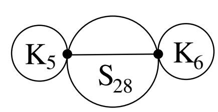
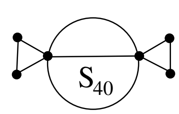
It remains an open problem to find $g(n)$ for $n = 11$ and those values of n between 13 and 54 which are not in $\{13, 22, 23, 30, 31, 38, 39, 40, 41, 42, 43, 46, 47, 48, 49, 50, 51, 54\}$.

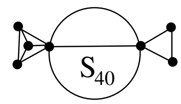
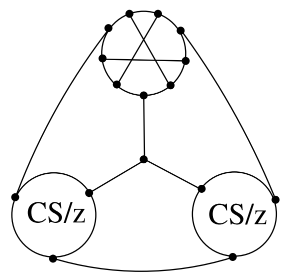
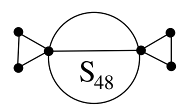
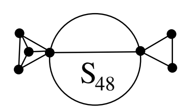
In the table below we give examples of MNT graphs of the orders mentioned above which are candidates for the ones of smallest size. In the table S_n denotes a snark of order n (as mentioned in [13]) and $CS\backslash z$ denotes Chisala's G_3-snark "opened up" at the vertex z (see Section 5.2).

n	$\lceil \frac{3n-2}{2} \rceil$	$e(G)$	Graph G
11	16	17	
14	20	21	
15	22	24	

n	$\lceil \frac{3n-2}{2} \rceil$	$e(G)$	Graph G
16	23	27	
17	25	31	
18	26	30	
19	28	33	
20	29	36	
21	31	39	

n	$\lceil \frac{3n-2}{2} \rceil$	$e(G)$	Graph G
24	35	36	
25	37	39	
26	38	42	
27	40	46	
28	41	42	
29	43	55	

n	$\lceil \frac{3n-2}{2} \rceil$	$e(G)$	Graph G
32	47	48	
33	49	51	
34	50	52	
35	52	58	
36	53	62	
37	55	67	
44	65	66	

n	$\lceil \frac{3n-2}{2} \rceil$	$e(G)$	Graph G
45	67	69	
52	77	78	 
53	79	81	

Chapter 6

Maximal Nontraceable Graphs and Toughness

6.1 Introduction

A noncomplete graph G is t-tough if $t \leq |S|/\kappa(G-S)$ for every vertex-cut $S \subset V(G)$, where $\kappa(G-S)$ denotes the number of components in $G-S$ and t is a nonnegative real number. It thus follows that G is s-tough if G is t-tough and $0 < s < t$.

The maximum real number t for which G is t-tough is called the *toughness* of G and denoted by $t(G)$. Thus

$$t(G) = \min\{|S|/\kappa(G-S)\}$$

where the minimum is taken over all vertex-cuts S of G.

In [7] toughness is described as “a measure of how tightly the subgraphs of G are held together”. For example, a 1-tough graph has the property that breaking the graph into k components (if this is possible) requires removing at least k vertices, whereas breaking a 2-tough graph into the same number of components requires removing at least $2k$ vertices.

6.2 Maximal nontraceable graphs with toughness less than one

A number of the results proved in this section are results obtained at a Graph Theory Workshop held at Salt Rock, South Africa, during March 2004 in which I and the following persons participated: Marietjie Frick, Kieka Mynhardt, Carol Whitehead, Susan van Aardt, Gabriel Semanišin and Frank Bullock. In particular, certain results were combined by Kieka Mynhardt to produce Theorem 6.2.2.

If G is an MNT graph with $t(G) < 1$, then it follows from Lemma 3.1.1(ii) that G has a vertex-cut S such that $\kappa(G - S) = |S| + 2$ or $\kappa(G - S) = |S| + 1$. We now characterize the first of these two cases.

Theorem 6.2.1 *G is an MNT graph having a subset S such that $\kappa(G - S) = |S| + 2$ if and only if G is a Zelinka Type I graph.*

Proof. Let G be a Zelinka Type I graph as depicted in Figure 4.1. Then $\kappa(G-U_0) = |U_0|+2$. The converse follows from Lemma 3.1.1(iii). ■

If G is a Zelinka Type II graph G as depicted in Figure 4.2, then $\kappa(G-U_0) = |U_0|+1$ and every component of $G - U_0$ except for one is complete. We suspected at first that the Zelinka Type II graphs are the only ones with this property. However, the following theorem enabled us to find non-Zelinka graphs with this property.

Theorem 6.2.2 *Let G be a graph with a minimal vertex-cut S such that $|S| = k$ and $G - S$ has $k + 1$ components $G_1, G_2, ..., G_k, H$ all of which are complete except for H. Then G is MNT if and only if the following conditions hold:*

(i) $\langle S \cup V(G_i)\rangle$ is complete for each $i = 1, 2, ..., k$, and each vertex in $N = V(H) \cap N(S)$ is adjacent to every vertex in S.

(ii) H is not traceable from any vertex in N.

(iii) *H is traceable from each vertex in $V(H) - N$.*

(iv) *If $u, v \in V(H)$ and $uv \in E(\overline{G})$, then there exists a vertex $w \in N$ such that $H + uv$ is traceable from w.*

(v) *H has no universal vertices, and whenever v is a universal vertex of $\langle N \rangle$, there is a hamiltonian path of H in which v is adjacent to a vertex $u \in N$.*

Proof. Suppose G is MNT. If $x, y \in S$ such that $xy \notin E(G)$, then a longest path in $G + xy$ containing xy can contain vertices from at most k components of $G - S$. Thus $G + xy$ has no hamiltonian path. Suppose for some $j = 1, 2, ..., k$ there exist $x \in S$ and $v \in V(G_j)$ such that $vx \notin E(G)$. Since S is a minimal vertex-cut, x is adjacent to some $u \in V(G_j)$, otherwise $S - x$ is a vertex-cut of G. Let P be a hamiltonian path in $G + vx$. Then P visits each of G_i, $i = 1, ..., k$ and H exactly once. Now $P_j = \langle E(P) \cap E(G_j) \rangle$ is a hamiltonian path in G_j, with endvertex v and the other endvertex, say, w. Since G_j is complete there is a $u - w$ hamiltonian path of G_j. If in P we replace the edge xv by xu and the path P_j by a $u - w$ hamiltonian path in G_j we obtain a hamiltonian path in G. Thus the first part of (i) holds.

It is clear that $G - V(H)$ is homogeneously traceable. Suppose H is traceable from some vertex $v \in N$ and that $x \in N(v) \cap S$. Then $P = P_{G-V(H)}(-, x)P_H(v, -)$ is a hamiltonian path in G. Thus (ii) holds. Now, if there exist $v \in N$ and $x \in S$ such that $vx \notin E(G)$, then any hamiltonian path in $G + vx$ visits each G_i and H only once, and thus H is traceable from v. Since this contradicts (ii) it follows that the second part of (i) holds.

If H is not traceable from some vertex $v \in V(H) - N$, then for any $x \in S$, $G + vx$ does not have a hamiltonian path. Hence (iii) holds. Similarly, (iv) holds.

For any $v \in N$ and any $y \in G_i$ for $i = 1, .., k$, $G + vy$ has a hamiltonian path P containing vy. It follows from (ii) that P visits H more than once and since $\kappa(G - S) = k + 1$, P visits H exactly twice. Then $F = \langle E(P) \cap E(H) \rangle$ is a 2-path cover of H in which v is an endvertex of one path F^1 and the other path F^2 has $u \in N$ as endvertex. If v is a universal vertex of $\langle N \rangle$, then $F + uv$ is a hamiltonian path of H which satisfies the condition in (v). In this case it follows from (ii) that u is not the only vertex of F^2. Let the other endvertex of F^2 be w. It follows from (ii) that $w \in H - N$. If $vw \in E(G)$, then $F + vw$ is a hamiltonian path in H with endvertex $u \in N$ which contradicts (ii). Thus $vw \notin E(G)$ and hence v is not a universal vertex of H.

Suppose $v \in V(H) - N$ is a universal vertex of H. By (ii) and (iii) there exists a $v - u$ hamiltonian path P of H, where $u \in V(H) - N$. Then $P + uv$ is a hamiltonian cycle in H. But then H is homogeneously traceable, which contradicts (ii).

Conversely, suppose G satisfies (i) - (v). If G is traceable, then it follows from the fact that $|S| = k$ and $\kappa(G-S) = k+1$ that any hamiltonian path visits each G_i and H exactly once and that the endvertices of the path are in two of the components of $G - S$. Thus each component of $G - S$ (and in particular H) is traceable from a vertex in $N(S)$. This contradicts (ii). Hence G is not traceable. However, it follows from (i) that $G - H$ is homogeneously traceable.

To show that G is MNT we need to show that $G + uv$ is traceable for all $u, v \in V(G)$, where $uv \notin E(G)$.

Case 1. $u, v \in V(H)$:

It follows from (i) and (iv) that $G + uv$ is traceable.

Case 2. $u \in V(H), v \in S$:

By (i) $u \in V(H) - N$. By (i) and (iii) it follows that $G + uv$ is traceable.

Case 3. $u \in V(H) - N, v \in V(G_i), i = 1, ..., k$:

According to (i) and (iii) $G + uv$ is traceable.

Case 4. $u \in N, v \in V(G_i), i = 1, ..., k$:

Suppose that u is not a universal vertex of $\langle N \rangle$. Then there exists $w \in N$ such that $uw \notin E(G)$ and by (iv) $H + uw$ has a hamiltonian path P containing the edge uw. Thus $F = P - uw$ is a 2-path cover of H such that u and w are endvertices of two different components of F. Suppose u is a universal vertex of $\langle N \rangle$. Then by (v) there is a hamiltonian path P' in H such that u is adjacent, on P', to a vertex $w \in N$. In this case $F = P - uw$ is a 2-path cover of

H such that u and w are endvertices of two different components of F. Let F^1 and F^2 be the components of F containing u and w, respectively. Choose an arbitrary $x \in S$. Then a hamiltonian path in $G + uv$ is

$$P_{G+uv} = F^1(-, u)P_{G-V(H)}(v, x)F^2(w, -).$$

Case 5. Consider $k \geq 2$. Let $u \in V(G_i)$ and $v \in V(G_j)$, $i \neq j$, $i, j = 1, ..., k$:
Let $z \in N$ be arbitrary and define F^1 and F^2 as the components of a 2-path cover F of H as described in Case 4, but where z and $w \in N$ are, respectively, endvertices of F^1 and F^2. Choose arbitrary $x, y \in S$. Then a hamiltonian path in $G + uv$ is

$$P_{G+uv} = F^1(-, z)xP_{G_i}(-, u)P_{G-V(H)-V(G_i)-x}(v, y)F^2(w, -). \qquad \blacksquare$$

Remark 6.2.3 *Let G be an MNT graph that has the structure as described in Theorem 6.2.2. Condition (iii) of Theorem 6.2.2 implies that if $N \neq V(H)$, then H is traceable. If $N = V(H)$, then (ii) and (iv) imply that H is MNT and (i) implies that each vertex in S is a universal vertex in G. We now give examples illustrating these two possibilities.*

Examples

I. Examples of MNT graphs that have the structure as described in Theorem 6.2.2 with H being traceable

1. **Zelinka Type II graphs**

   Zelinka Type II graphs are of this type - see Figure 4.2 and let $S = U_0$.

2. **Construction of Dudek, Katona and Wojda using the Petersen graph**

   The graph depicted in Figure 4.8 can also be depicted as in Figure 6.1.

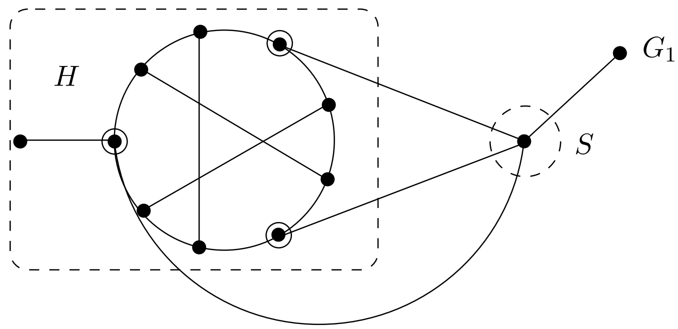


Figure 6.1: Construction of Dudek, Katona and Wojda using the Petersen graph

Here S is a cut-vertex, $G_1 \cong K_1$ and H is a vertex deleted Petersen graph with a K_1 attached to a vertex of degree 2. The vertices which are circled (with solid lines) are vertices in $N_H(S)$. By using Lemmas 2.1.1 and 2.1.2 it is easy to check that H satisfies the conditions of Theorem 6.2.2.

3. **The sputnik**

   We depict the sputnik in the following sketch.

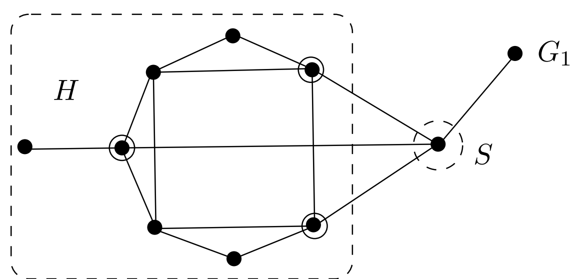


Figure 6.2: The sputnik

4. **The tarantula**

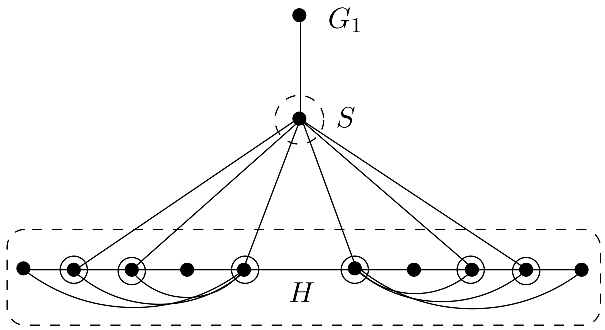


Figure 6.3: The tarantula

II. Examples of MNT graphs that have the structure as described in Theorem 6.2.2 with H being MNT

1. Let H be the net, N (the smallest Zelinka MNT graph without a universal vertex), $S = \{x\}$ and $G_1 \cong K_1$. Join each vertex in H and G_1 to x. The resulting graph is depicted in Figure 6.4 and is isomorphic to the propeller, a non-Zelinka MNT graph of smallest order (see Figure 4.16).

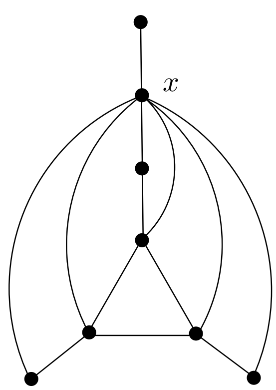


Figure 6.4: A graph isomorphic to the propeller.

2. The generalized propellers, based on a K_4 as described in Construction 1 of non-Zelinka graphs in Subsection 4.4.1, can be constructed by using the method described in 1, with any Zelinka Type II graph H in which $p = q = r = 1$, $s = 0$ (see Section 4.1). If $s > 0$, then we obtain generalized propellers based on a K_n, where $n \geq 5$.

If an MNT graph has a cut-vertex x such that $G - x$ has exactly two components, then it follows from Theorem 4.4.1 that exactly one component of $G - x$ is complete, and hence G has the structure given in Theorem 6.2.2.

There also exist MNT graphs having a subset S with $|S| = k$ and $\kappa(G - S) = k + 1$ such that no component of $G - S$ is complete. Such graphs seem difficult to characterize in general, and thus we consider here only the case $k = 2$. We first show that if $k = 2$ and $G - S$ has three components of which two are not complete, then the third component is also not complete.

Theorem 6.2.4 *Suppose G is a graph with a minimal vertex-cut S such that $|S| = 2$ and $G - S$ has three components G_1, G_2, G_3. If G is MNT then either exactly two of the components are complete or none of the components are complete.*

Proof. Let $S = \{x_1, x_2\}$. If all three components are complete then G is traceable.

Assume that exactly one of the components, say G_3 is complete. Then it follows from the proof of the first part of (i) in Theorem 6.2.2 that $\langle S \cup V(G_3)\rangle$ is complete. Now not both $\langle V(G_1) \cup \{x_i\}\rangle$ and $\langle V(G_2) \cup \{x_j\}\rangle$, $i \neq j$, $i, j = 1, 2$ can be traceable, respectively, from x_i and x_j, otherwise G is traceable.

Suppose $u, v \in V(G_1)$ and $uv \notin E(G)$. Then since $G + uv$ is traceable,

(i) $\langle V(G_1) \cup \{x_1\}\rangle + uv$ is traceable from x_1 and $\langle V(G_2) \cup \{x_2\}\rangle$ is traceable from x_2; or

(ii) $\langle V(G_1) \cup \{x_2\}\rangle + uv$ is traceable from x_2 and $\langle V(G_2) \cup \{x_1\}\rangle$ is traceable from x_1.

Without loss of generality we assume the first case is true. Now suppose $z, w \in V(G_2)$ and $zw \notin E(G)$. Then $\langle V(G_2) \cup \{x_1\}\rangle + zw$ is traceable from x_1 and $\langle V(G_1) \cup \{x_2\}\rangle$ is traceable from x_2. (If $\langle V(G_1) \cup \{x_1\} \rangle$ was traceable from x_1, then G would be traceable.) So $\langle V(G_i) \cup \{x_2\} \rangle$, $i = 1, 2$ is traceable from x_2. Also G_i is not traceable from any vertex in $N_i(x_1) = N_G(x_1) \cap V(G_i)$ for $i = 1, 2$. Consider $G + uv$, where $u \in N_1(x_1)$ and $v \in N_2(x_1)$. Since G_1 and G_2 are not traceable, respectively, from u and v a hamiltonian path in $G + uv$ has to visit each of G_1 and G_2 at least twice and G_3 at least once and this is impossible. Thus our assumption is false. ■

Before we consider the structure of an MNT graph G in which the minimal vertex-cut S is such that $|S| = 2$ and $G - S$ has three noncomplete components, we prove the following lemma.

Lemma 6.2.5 *Suppose H is a maximal nonhamiltonian graph. Let $x \in V(H)$ and let $G = H - x$ and $N = N_H(x)$. If $V(G) \neq N$, then the following hold:*

(i) For each $v \in V(G) - N$, there exists a hamiltonian path in G with v as an endvertex and the other endvertex in N.

(ii) If v is not a universal vertex of $\langle N \rangle$ then H has a hamiltonian path with endvertices v and w, where $w \in N$.

(iii) No hamiltonian path in G has both endvertices in N.

Proof. Suppose $v \in V(G) - N$. Then $vx \notin E(H)$ and hence there is a hamiltonian cycle C in $H + vx$. Let $u \neq v$ be adjacent to x on C. Then $u \in N$. Hence $C - x$ is a hamiltonian path in G with endvertices v and u. Thus (i) holds.

Suppose v is not a universal vertex of $\langle N \rangle$. Then there exists a vertex $w \in N$ such that $vw \notin E(\langle N \rangle)$. Hence $H + vw$ has a hamiltonian cycle and hence H has a hamiltonian path with endvertices v and w. Thus (ii) holds.

Suppose G has a hamiltonian path P with both endvertices in N. Then xPx is a hamiltonian cycle in H, which is a contradiction. Hence (iii) holds. ■

Theorem 6.2.6 *Let G be a graph with minimal vertex-cut $S = \{x, y\}$ such that $G - S$ consists of three noncomplete components G_1, G_2, G_3 and $N_{G_i}(x) = N_{G_i}(y) = N_i \neq V(G_i)$, for $i = 1, 2, 3$.*

Then G is maximal nontraceable if and only if the following hold:

(i) $xy \in E(G)$.

(ii) $H_i = \langle V(G_i) \cup \{x\}\rangle$ *is maximal nonhamiltonian for* $i = 1, 2, 3$.

(iii) At most one of the graphs $\langle N_i\rangle$ *has a universal vertex.*

(iv) All three graphs G_i *are traceable and at least two of them are homogeneously traceable.*

(v) If u *is a universal vertex of* $\langle N_i\rangle$ *for some* $i \in \{1, 2, 3\}$*, then* G_i *is traceable from* u.

Proof. Suppose G is MNT. If $xy \notin E(G)$, then a longest path in $G + xy$ can contain vertices from at most two components of $G - S$. Hence (i) holds.

We show that no H_i is hamiltonian. Without loss of generality, suppose that H_2 is hamiltonian. Let $e \in V(\overline{G_2})$. Since $G + e$ has a hamiltonian path, there exists a hamiltonian path in H_1 with endvertex x and a hamiltonian path in $\langle V(G_3) \cup \{y\}\rangle$ with endvertex y. Since H_2 is hamiltonian and $N_{G_2}(x) = N_{G_2}(y)$ there exists a hamiltonian path in $\langle H_2 \cup \{y\}\rangle$ with endvertices x and y. Hence there is a hamiltonian path in G, which is a contradiction.

We now prove that H_i is MNH for $i = 1, 2, 3$. Consider $G + uv$, where $u, v \in V(H_i)$ and $uv \notin E(G)$ for some $i \in \{1, 2, 3\}$. Without loss of generality, suppose $i = 2$. Since each H_i is nonhamiltonian, a hamiltonian path in $G + uv$ consists of a hamiltonian path in G_1, followed by a hamiltonian path $xw...z$ in $H_2 + uv$, where $w, z \in N_2$, followed by y followed by a hamiltonian path in G_3. Hence $xw...zx$ is a hamiltonian cycle in $H_2 + uv$ and thus, since H_2 is nonhamiltonian, H_2 is MNH.

We now prove that at most one of the graphs $\langle N_i\rangle$ has a universal vertex. Consider $G + v_iv_j$, $v_i \in N_i$,$v_j \in N_j$, $i \neq j$, $i, j = 1, 2, 3$. Let $i = 1$ and $j = 2$. Other cases are proved similarly.

From Lemma 6.2.5(iii) it follows that a hamiltonian path in $G + v_1v_2$ has the structure shown in Figure 6.5, in which G_1 and G_2 can be interchanged.

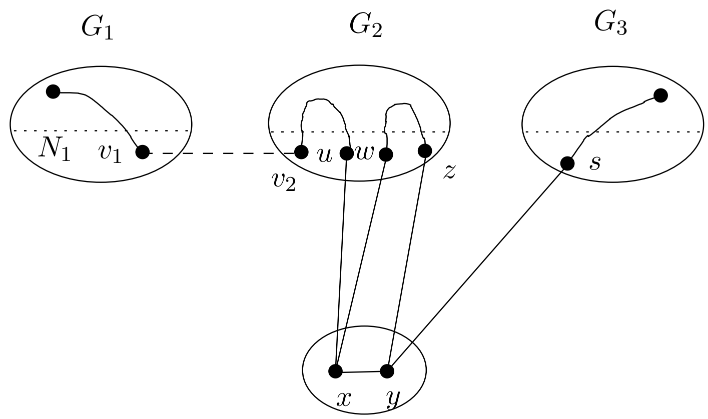


Figure 6.5: Sketch for Theorem 6.2.6

Since a hamiltonian path visits G_2 twice and contains all the vertices of $V(G_2)$, we have a 2-path cover F of G_2, the one path, F^1, with endvertices v_2 and u and the other path, F^2, with endvertices w and z, where all the endvertices are in N_2. A hamiltonian path P in $G + v_1v_2$ consists of a hamiltonian path in G_1 ending at v_1, followed by $F^1(v_2, u)$, then x, then $F^2(w, z)$, then y, and finally by a hamiltonian path in G_3.

Suppose v_i is a universal vertex of $\langle N_i \rangle$, for $i = 1, 2$. Then v_2 is adjacent to w. If in P we replace v_1v_2 with v_1x, $F^1(v_2, u)$ with $F^1(u, v_2)$, then we obtain a hamiltonian path in G, which is a contradiction. Thus $\langle N_2 \rangle$ has no universal vertices. Similarly, for any choice of i and j, $i \neq j$, $i, j = 1, 2, 3$, it follows that at least one of $\langle N_i \rangle$ and $\langle N_j \rangle$ has no universal vertices. Hence at most one of the graphs $\langle N_k \rangle$, $k = 1, 2, 3$ has a universal vertex.

According to Lemma 6.2.5(i) each G_i is traceable. Suppose one of the components G_i, say G_2, is not homogeneously traceable. Let v_2 be a vertex in G_2 that is not the endvertex of any hamiltonian path in G_2. Then, by Lemma 6.2.5(i), $v_2 \in N_2$. Now let $v_1 \in N_1$. Then $G + v_1v_2$ has a hamiltonian path with the structure shown in Figure 6.5 and so G_1 has a hamiltonian path with v_1 as endvertex. Since by Lemma 6.2.5(i), G_1 is also traceable from each vertex in $V(G_1) - N_1$ it follows that G_1 is homogeneously traceable. Similarly, G_3 is homogeneously traceable.

Suppose one of the induced subgraphs $\langle N_i \rangle$, say $\langle N_2 \rangle$, has a universal vertex v_2. Suppose also that G_2 is not traceable from v_2. Then, as before, it follows that $G + v_1v_2$, where $v_1 \in N_1$, has a hamiltonian path with the structure shown in Figure 6.5. By using reasoning similar to that in the proof of (iii) it follows that G has a hamiltonian path; hence a contradiction. Thus G_2 is traceable from v_2.

Conversely, suppose G satisfies conditions (i)-(v). It follows from (ii) and Lemma 6.2.5(iii) that G_i has no hamiltonian path with both endvertices in N_i and hence G is not traceable.

It also follows from (ii) and Lemma 6.2.5(ii) that, if $v \in N_i$ is not a universal vertex of $\langle N_i \rangle$, then $\langle V(G_i) \cup \{x, y\} \rangle$ has a hamiltonian path Q with endvertices v and y.

We now show that $G + uv$ is traceable for $u, v \in V(G)$ and $uv \notin E(G)$.

Case 1. $u, v \in V(G_i)$, say $i = 2$.
From (ii) it follows that $\langle V(G_2) \cup \{x\} \rangle + uv$ has a hamiltonian cycle and hence there is a hamiltonian path P in $\langle V(G_2) \cup \{x, y\} \rangle + uv$ with endvertices x and y. Thus, according to Lemma 6.2.5(i), there is a hamiltonian path in $G + uv$, consisting of a hamiltonian path in G_1 ending at some vertex in N_1, followed by P and then followed by a hamiltonian path in G_3 starting at some vertex in N_3.

Case 2. $u \in V(G_i) - N_i$ and $v \in \{x, y\}$, say $i = 2$ and $v = x$.
It follows immediately from Lemma 6.2.5(i) that there is a hamiltonian path in $G + uv$.

Case 3. $u \in V(G_i) - N_i$, $v \in V(G_j) - N_j$, $i \neq j$, $i, j = 1, 2$, say $i = 1$ and $j = 2$.
It follows immediately from Lemma 6.2.5(i) that there is a hamiltonian path in $G + uv$.

Case 4. $u \in N_i$, $v \in N_j$, $i \neq j$, $i, j = 1, 2$, say $i = 1$ and $j = 2$.
At least one of G_1 and G_2, say G_1, is homogeneously traceable. Hence G_1 has a hamiltonian path P with endvertex u. If v is not a universal vertex of $\langle N_2 \rangle$, then by applying

Lemma 6.2.5(ii), it follows that $\langle V(G_2) \cup \{x, y\} \rangle$ has a hamiltonian path Q with endvertices v and y. Also, by Lemma 6.2.5(i), G_3 has a hamiltonian path R with an endvertex in N_3. Thus the path consisting of P, followed by Q, followed by R is a hamiltonian path in $G+uv$. If v is a universal vertex of $\langle N_2 \rangle$, then G_2 is traceable from v and hence G_2 has a hamiltonian path P' with endvertex v. Furthermore, u is not a universal vertex of $\langle N_1 \rangle$ and hence $\langle V(G_1) \cup \{x, y\} \rangle$ has a hamiltonian path Q' with endvertices u and y. Thus the path consisting of P', followed by Q', followed by R is a hamiltonian path in $G + uv$.

Case 5. $u \in N_i$, $v \in V(G_j) - N_j$, $i \neq j$, $i, j = 1, 2$, say $i = 1$ and $j = 2$.
If u is a universal vertex of $\langle N_1 \rangle$, then G_1 is traceable from u. By using this fact and Lemma 6.2.5(i) and (ii) we can construct a hamiltonian path in $G + uv$.

If u is not a universal vertex of $\langle N_1 \rangle$, then $\langle V(G_1) \cup \{x, y\} \rangle$ has a hamiltonian path Q with endvertices u and y. By using this fact and Lemma 6.2.5(i) we can construct a hamiltonian path in $G + uv$. ■

An example of a graph, due to my promoter, that satisfies the conditions of Theorem 6.2.6 is depicted in Figure 6.6. In this graph, $G_1 \cong G_2 \cong G_3$ is homogeneously traceable and $\langle V(G_i) \cup \{x\} \rangle$ is isomorphic to the Petersen graph which is MNH.

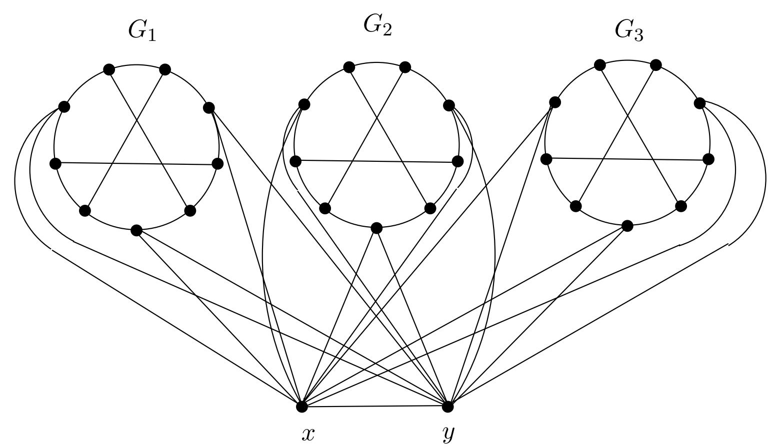


Figure 6.6: Graph that satisfies the conditions of Theorem 6.2.6

Remark 6.2.7 *Suppose H is an MHH graph and that $x \in V(H)$. By Lemma 2.1.1(i) $G = H - x$ is homogeneously traceable. Thus we can replace any or all of the induced subgraphs $\langle V(G_i) \cup \{x\} \rangle$ in the graph depicted in Figure 6.6, by H, where H is isomorphic to the Coxeter graph, Chisala's G_3-snark, or the Isaacs' snarks J_k for odd $k \geq 5$.*

Theorem 6.2.8 *Let G be a graph with a minimal vertex-cut $S = \{x, y\}$ such that $G - S$ consists of three noncomplete components G_1, G_2, G_3 and $N_{G_i}(x) \cap N_{G_i}(y) = \emptyset$, for $i = 1, 2, 3$. Let $H_i = \langle V(G_i) \cup \{x, y\} \rangle$.*

Then G is maximal nontraceable if and only if the following hold.

(i) $xy \in E(G)$.

(ii) *For each $i = 1, 2, 3$ there is no hamiltonian cycle in H_i containing the edge xy, but $H_i + e$ has a hamiltonian cycle containing xy for each $e \in E(\overline{H_i})$.*

Proof. Suppose G is MNT. As in the proof of Theorem 6.2.6 it follows that $xy \in E(G)$.

Suppose, without loss of generality, that H_2 has a hamiltonian cycle containing xy. Let $e \in E(\overline{G_2})$. Since $G + e$ is traceable there is a hamiltonian path in $H_1 - y$ with endvertex x and a hamiltonian path in $H_3 - x$ with endvertex y. Since H_2 has a hamiltonian path with endvertices x and y, it follows that G is traceable, a contradiction. Thus no H_i has a hamiltonian cycle containing xy. Now consider $G + e$, where $e \in E(\overline{H_i})$. Then a hamiltonian path in $G + e$ contains a hamiltonian path in $H_i + e$ with endvertices x and y, and thus a hamiltonian cycle containing xy.

Suppose that conditions (i) and (ii) hold. If G was traceable, then some H_i would contain a hamiltonian path with endvertices x and y and hence H_i would contain a hamiltonian cycle containing the edge xy, a contradiction. Thus G is nontraceable.

Since S is a minimal vertex-cut of G it follows that x and y each have at least one neighbour in each G_i, $i = 1, 2, 3$. Then, since $N_{G_i}(x) \cap N_{G_i}(y) = \emptyset$, there is at least one vertex in G_i not adjacent to x and at least one vertex in G_i not adjacent to y. Suppose $v \in V(G_i)$

is not adjacent to x. Then, by (ii), $H_i + vx$ has a hamiltonian cycle containing vxy, so G_i has a hamiltonian path with v and w as endvertices, where w is adjacent to y. Similarly, if $u \in V(G_i)$ is not adjacent to y, then G_i has a hamiltonian path with u and z as endvertices, where z is adjacent to x. Thus each G_i has two hamiltonian paths, one with endvertex adjacent to x and the other with endvertex adjacent to y.

We now prove that $G + uv$ is traceable for $u, v \in V(G)$ and $uv \notin E(G)$.

Case 1. $u, v \in V(H_i)$, say $i = 2$.
Then $G+uv$ has a hamiltonian path obtained from a hamiltonian path in G_1 with one endvertex adjacent to x, followed by a hamiltonian path in $H_2 + uv$ with endvertices x and y, followed by a hamiltonian path in G_3 with endvertex adjacent to y.

Case 2. $u \in V(G_i), v \in V(G_j), i \neq j$ and $i, j = 1, 2$, say $i = 1$ and $j = 2$.
Suppose, without loss of generality, that v is not adjacent to x. Since u is nonadjacent to either x or y, it follows from condition (ii) that G_1 has a hamiltonian path ending at u. Also, G_2 has a hamiltonian path P with one endvertex v and the other endvertex adjacent to y. Then $G+uv$ has a hamiltonian path obtained from a hamiltonian path in G_1 with endvertex u, followed by P, then y, x, followed by a hamiltonian path in G_3 with endvertex adjacent to x. ■

It follows from condition (ii) of the above theorem that each graph H_i, $i = 1, 2, 3$ is either hamiltonian (but no hamiltonian cycle contains the edge xy) or MNH. We present two examples of graphs G that satisfy the conditions of Theorem 6.2.8, the first in which all the graphs H_i are MNH (see Figure 6.7) and the second in which all the graphs H_i are hamiltonian (see Figure 6.8). Note that in Figure 6.7 each $H_i = \langle V(G_i) \cup \{x, y\} \rangle$ is isomorphic to the Petersen graph and in Figure 6.8 each $H_i = \langle V(G_i) \cup \{x, y\} \rangle$ is isomorphic to the middle block of the sputnik.

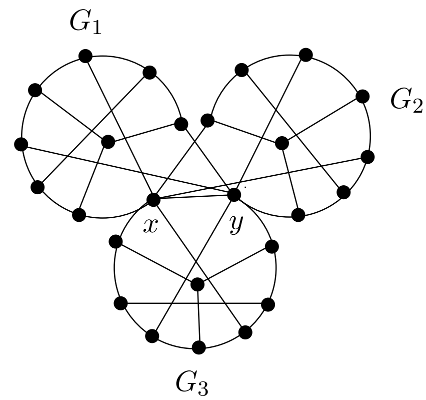


Figure 6.7: First example of graph satisfying the conditions of Theorem 6.2.8

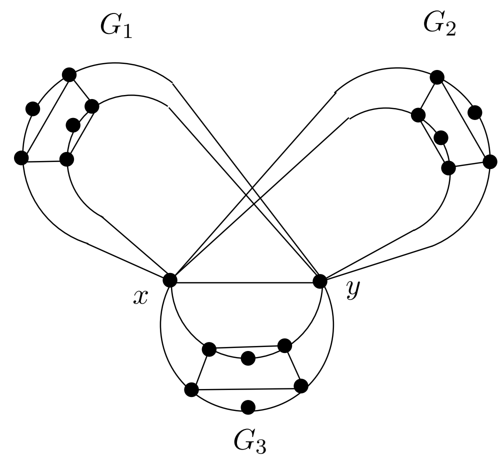


Figure 6.8: Second example of graph satisfying the conditions of Theorem 6.2.8

Remarks 6.2.9

1. *Suppose H is a nonhamiltonian MnHc graph. Let $x, y \in V(H)$ and $xy \in E(H)$. Then, according to the definition of MnHc graphs (see Section 2.2), $H + e$, $e \in E(\overline{H})$ has a hamiltonian $x - y$ path and hence a hamiltonian cycle containing xy. Thus we can replace any or all of the subgraphs H_i in the graphs depicted in Figures 6.7 and 6.8, by H, where H is isomorphic to the Coxeter graph or the Isaacs' snarks J_k for odd $k \geq 7$.*

2. *Likewise, we can replace any or all of the subgraphs H_i in the graphs depicted in Figures 6.7 and 6.8 by the middle block of any generalized sputnik.*

At this stage we do not know whether there exists an MNT graph G having a minimal vertex-cut $S = \{x, y\}$ such that $G - S$ consists of three noncomplete components G_1, G_2, G_3 with $N_{G_i}(x)$ and $N_{G_i}(y)$ being neither disjoint nor equal for some i.

6.3 Maximal nontraceable graphs with toughness at least one

The non-Zelinka claw-free graphs presented in Section 4.5 are 1-tough since all 2-connected, claw-free graphs are 1-tough (cf. [19]). Thus, since the Zelinka MNT graphs are not 1-tough, it follows that no 2-connected Zelinka MNT graph is claw-free.

We also have the following result.

Lemma 6.3.1 *If G is a cubic 2-connected MNT graph, then G is 1-tough.*

Proof. Suppose to the contrary that G is not 1-tough, i.e. G has a vertex-cut S and $\kappa(G-S) > |S|$. Since G is 2-connected, $|S| \geq 2$. If $x_1, x_2 \in S$ and $x_1x_2 \notin E(G)$, then a path in $G+x_1x_2$ containing x_1x_2 can visit at most $|S|$ components, and thus there is no hamiltonian path in $G + x_1x_2$. Hence $\langle S \rangle$ is complete, and $|S| = 2$ or $|S| = 3$.

Suppose $|S| = 2$ and $S = \{x_1, x_2\}$. Then $\deg_{\langle S \rangle} x_i = 1$ for $i = 1, 2$. Now x_1 and x_2 must each have neighbours in each component, otherwise G is not 2-connected. Hence $\deg_G x_i > 3$ for $i = 1, 2$ which is a contradiction.

Suppose $|S| = 3$ and $S = \{x_1, x_2, x_3\}$. Since $\deg_{\langle S \rangle} x_i = 2$ and $\deg_G x_i = 3$ for $i = 1, 2, 3$, it follows that the neighbours of S are in at most three components of $G - S$, which is a contradiction. ■

We now consider an MNT graph that has toughness 2. In [1] Bauer, Broersma and Veldman construct $(\frac{9}{4} - \epsilon)$-tough $(0 < \epsilon \leq \frac{1}{4})$ nontraceable graphs, thereby refuting the conjecture, usually attributed to Chvátal, that every 2-tough graph is hamiltonian. We look at the construction given in [1].

For a given graph H and $x, y \in V(H)$ the graph $G(H, x, y, l, m)$, l, m natural numbers is defined as follows.
Take m disjoint copies $H_1, ..., H_m$ of H, with $x_i, y_i \in V(H_i)$ corresponding to $x, y \in V(H)$ for $i = 1, ..., m$. Let F_m be the graph obtained from $H_1 \cup ... \cup H_m$ by adding edges between all possible pairs of vertices in $\{x_1, ..., x_m, y_1, ..., y_m\}$. Let $T = K_l$ and let $G(H, x, y, l, m)$ be the join $T + F_m$ of T and F_m. The following theorem is proved in [1].

Theorem 6.3.2 *(Bauer, Broersma and Veldman [1])*

Let H be a graph and $x, y \in V(H)$ be vertices which are not joined by a hamiltonian path in H. If $m \geq 2l + 3$, then $G(H, x, y, l, m)$ is nontraceable.

The graph L which is depicted in Figure 6.9 has vertices u and v which are not joined by a hamiltonian path.

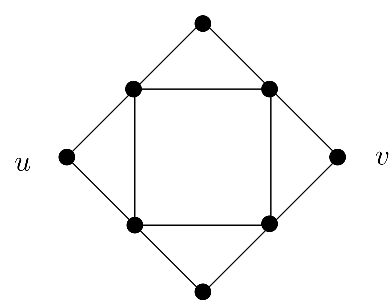


Figure 6.9: The graph L

Remark 6.3.3 *We note that $L + uv$ is isomorphic to the middle block of the sputnik (see Figure 4.9).*

Bauer, Broersma and Veldman also proved the following.

Theorem 6.3.4 *(Bauer, Broersma and Veldman [1])*

For $l \geq 2$ and $m \geq 1$,

$$t(G(L, u, v, l, m)) = \frac{l + 4m}{2m + 1}$$

and hence the graph $G(L, u, v, l, 2l + 3)$ is nontraceable and has toughness $(9l + 12)/(4l + 7) = \frac{9}{4} - \epsilon$.

Remark 6.3.5 *The graph with the smallest order in the family of nontraceable graphs $G(L, u, v, l, 2l+3)$, $l \geq 2$, is $G(L, u, v, 2, 7)$, has 58 vertices and toughness 2, and is depicted in Figure 6.10. This graph is also maximal nontraceable.*

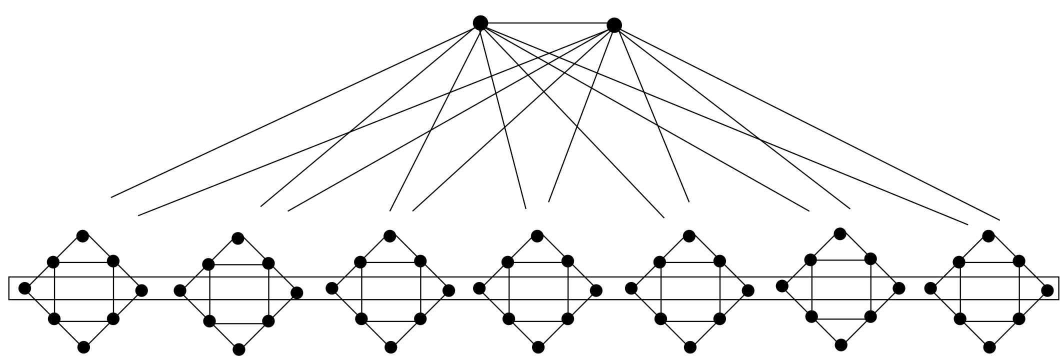


Figure 6.10: The graph $G(L, u, v, 2, 7)$

In Figure 6.10 the edges joining the pairs of vertices in $\{u_1, ..., u_7, v_1, ..., v_7\}$ are indicated by the rectangle enclosing u_i, v_i, $i = 7$.

The question now arises: How tough can a maximal nontraceable graph be?

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