SOME STOCHASTIC PROBLEMS IN RELIABILITY AND INVENTORY

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Summary

An attempt is made in this thesis to study some stochastic models of both reliability and inventory systems with reference to the following aspects:

(i) the confidence limits with the introduction of common-cause failures.
(ii) the Erlangian repair time distributions.
(iii) the product interactions and demand interactions.
(iv) the products are perishable.

This thesis contains six chapters.

Chaper 1 is introductory in nature and gives a review of the literature and the techniques used in the analysis of reliability systems.

Chapter 2 is a study of component common-cause failure systems. Such failures may greatly reduce the reliability indices. Two models of such systems (series and parallel) have been studied in this chapter. The expressions such as, reliability, availability and expected number of repairs have been obtained. The confidence limits for the steady state availability of these two systems have also been obtained. A numerical example illustrates the results.

A $100\,(1-\alpha)\,\%$ confidence limit for the steady state availability of a two unit hot and warm standby system has been studied, when the failure of an online unit is constant and the repair time of a failed unit is Erlangian.

The general introduction of various inventory systems and the techniques used in the analysis of such systems have been explained in chapter 4.

Chapter 5 provides two models of two component continuous review inventory systems. Here we assume that demand occurs according to a poisson process and that a demand can be satisfied only if both the components are available in inventory. Back-orders are not permitted. The two components are bought from outside suppliers and are replenished according to $(s,\ S)$ policy. In model 1 we assume that the lead-time of the components follow an exponential distribution. By identifying the inventory level as a Markov process, a system of difference-differential equations at any time and the steady-state for the state of inventory level are obtained. In model 2 we assume that the lead-time distribution of one product is arbitrary and the other is exponential. Identifying the underlying process as a semi-regenerative process we find the stationary distribution of the inventory level. For both these models, we find out the performance measures such as the mean stationary rate of the number of lost demands, the demands satisfied and the reorders made. Numerical examples for the two models are also considered.

Chaper 6 is devoted to the study of a two perishable product inventory model in which the products are substitutable. The perishable rates of product 1 and product 2 are two different constants. Demand for product 1 and product 2 follow two independent Poisson processes. For replenishment of product 1 $(s,\ S)$ ordering policy is followed and the associated lead-time is arbitrary. Replenishment of product 2 is instantaneous. A demand for product 1 which occurs during its stock-out period can be substituted by product 2 with some probability. Expressions are derived for the stationary distribution of the inventory level by identifying the underlying stochastic process as a semi-regenerative process. An expression for the expected profit rate is obtained. A numerical illustration is provided and an optimal reordering level maximising the profit rate is also studied.

To sum up, this thesis is an effort to improve the state the of art of (i) complex reliability systems and their estimation study (ii) multiproduct inventory systems. The salient features of the thesis are:

(i) Analysis of a two-component reliability system with common-cause failures.
(ii) Estimation study of a complex system in which the repair time for both hot standby and warm standby systems are assumed to be Erlangian.
(iii) A multi-product continuous review inventory system with product interaction, with a $(s,\ S)$ policy.
(iv) Introduction of the concept of substitutability for products.
(v) Derivation of expressions for various statistical measures.
(vi) Effective use of the regeneration point technique in deriving various measures for both reliability and inventory systems.
(vii) Illustration of the various results by extensive numerical work.
(vii) Consideration of relevant optimization problems.

CHAPTER 1

Introduction to reliability

1.1 Introduction

"Reliability, as a human attribute, has been sought after and praised since time immemorial. Few would disparage the reliable man or decry his worth in industry, commerce or society generally. Trustworthy, dependable and consistent are almost synonymous adjectives which may be applied to those on whom reliance can be placed and these give an indication of why the characteristic of reliability is so valued. Inherently, man always finds comfort where there is trust and craves for those things which are consistent and predictable.

Although reliability in human behaviour is valued, it is difficult to define the characteristic precisely or to measure its exact worth. On the other hand, there is obviously a degree of reliability and no definite line can be drawn between the man who is reliable and the man who is not. It is nearly always possible to adjudicate on a comparative basis and decide whether one particular individual is more reliable than another. This judgement becomes easier when it is related to some definite human function. For instance, the degree of punctuality of individuals in arriving at work or attending meetings could be used as a measure of their reliability in performing this particular function.

It is interesting to note, in fact, that the characteristic of reliability is usually used to describe some function or task. In the widest sense, it may be said to be a measure of performance. The man, who, as an emissary, always says the right things, may be described as reliable because he is good at this diplomatic task. The man who always completes his work in the scheduled time may be said to be reliable because he succeeds in this type of activity. The man who is always in the right place at the right time may be called reliable because he is functioning in some particular desired manner. Generally, then, the quality of man's performance and also the time at which or in which he performs may be a measure of his reliability in association with any particular task.

It is easy to see, with this general concept of reliability, how man has applied the term "reliability" not only to human activity but also to the performance of functional objects of his own make or invention. Just as a man feels let down by his fellow men, so he may feel frustrated or disappointed if the function objects with which he deals do not perform in the manner desired. With objects, perhaps more than with people, the lack of reliability may lead to more than just a feeling of disenchantment. Unreliability of functional objects can waste man's time, cost him money or even endanger his life. As the consequences of this type of unreliable behaviour becomes more severe, so man's interest in reliability and his desire of reliable products becomes more acute." (Green & Bourne (1978).)

Technological developments lead to an increase in the number of complicated systems as well as an increase in the compliexity of the systems themselves. With remarkable advancements made in electronics and communications, systems became more and more sophisticated. Because of their varied natures, these problems have attracted the attention of scientists from various disciplines especially the systems engineers, software engineers and applied probabilists. An overall scientific discipline, called reliability theory that deals with the methods and techniques to ensure the maximum effectiveness of systems (from known qualities of their component parts) has developed. Reliability theory introduces quantitative indices of the quality of production (Gnedenko et al (1969)) and these are carried through from the design and subsequent manufacturing process to the use and storage of technological devices. Engineers, Scientists and Government leaders are all concerned with increasing the reliabiilty of manufactured goods and operating systems. As "Unreliability has consequences in cost, time wasted, the psychological effect of inconvenience, and in certain instances personal and national security" (Lloyd and Lopow (1962)). In 1963 the first journal on reliability, IEEE-Transactions on Reliability saw the light.

Due to the very nature of the subjects, the methods of Probability theory and Mathematical statistics (information theory, queueing theory, linear and nonlinear programming, mathematical logic, the methods of statistical simulation on electronic computers, demography, etc.), play an important role in the problem solving of reliability theory consider contemporary medicine, reliable software systems, geoastononmy, irregularities in newonal activity, interactions of physiological systems, spontaneous single newon discharge, phase dependence of population growth, fluctuations in business investments, and many more. In human behaviour mathematical models based on probability theory and stochastic processes are helpful in rendering realistic modeling for social mobility of individuals, industrial mobility of labour, educational advancements, diffusion of information and social networks. In the biological sciences stochastic models are first used by Watson and Galton (1874) in a study of extinction of families. Research on population genetics, branching process, birth and death processes, recovery, relapse, cell survival after irradiation, the flow of particles through organs, etc. then followed. In business management analytical models evolved for the purchasing behaviour of the individual consumer, credit risk and term structure, income determination under uncertainty and many more relating subjects. Traffic flow theory is a well known field for stochastic models and studies have been developed for traffic of pedestrians, freeways, parking lots, intersections, etc.

Problems ecnountered in the design of highly reliable technical systems have led to the development of high-accuracy methods of reliability analysis. Two major problems can be identified, namely:

- creating classes of probability-statistical models that can be used in the description of the reliability behaviour of the system, and
- developing mathematical methods for the examination of the reliability characteristic of a class of systems.

Considering only redundant systems the classical examples are the models of Markov processes with a finite set of states (in particular birth and death processes) (Gnedenko et al (1969), Barlow (1984), Gertsbakh (1989) and Kovalenko et al (1997)), the renewal process method (Cox (1962)) the semi-Markov process method and its generalizations (Cinlar (1975)), Generalized semi-Markov process (GSMP) method (Rubenstein (1981)), spacial models for coherent systems (Aven (1996)) and systems in random and variable environment (Ozekici (1996) and Finkelstein (1999a, b, c)).

Depending on the nature of the research, the applicable form of reliability theory can be introduced to each. A stochastic analysis is made based on some good probability model, but ultimately the goal is to give a numerical estimate of the reliability characteristics. It is, however, not simply a case of changing terminology in standard probability theory (say, "random variable" changes to "lifetime"), but reliability distringuishes itself by providing answers and solutions to a series of new problems not solved in the "standard" probability theory framework. Gertsbakh (1989) points out that reliability

- of a system is based on the information regarding the reliability of the system's components
- gives a mathematical description of the aging process with the introduction of several formal notations of aging (failure rate, etc.)
- introduces well-developed techniques of renewal theory
- introduces redundancy to achieve optimal use of standby components (an excellent introduction to redundant systems is given in Gnedenko et al (1969))
- includes the theory of optimal preventative maintenance (Beichelt & Fischer (1980))
- is a study of statistical inference (often from censored data)

Generally, the mathematical problems of lifetime studies of technical objects (reliability theory) and of biological entities (survival analysis) are similar, differing only in the notation. The term "lifetime" therefore does not apply to lifetimes in the strictest literal sense, but can be used in the figurative sense. The idea is that the statisical analysis done in this thesis should be true in any of the applicable disciplines, although the notation is mostly as for engineering (systems, components, units, etc.). With minor modifications the discipline can be changed to biological, or financial, etc.

1.2 Failure

"A failure is a result of a joint action of many unpredictable, random processes going on inside the operating system as well as in the environment in which the system is operating" (Gertsbakh 1989). Functioning is therefore seriously impeded or completely stopped at a certain moment in time and all failures have a stochastic nature. In some cases the time of failure is easily observed, but if units deteriorate continuously determination of the moment of failure is not an easy task. In this study we assume that failure of a unit can be obtained exactly. Failure of a system is called a disappointment or a death and failure results in the system being in the down state. This can also be referred to as a breakdown (Finkelstein (1999a)).

Zacks (1992) points out that there are two types of data to consider, namely:

- data from continuous monitoring of a unit failure is observed
- data from observations made at discrete time points, therefore failure counts

Villemeur (1992) gives an extensive list of possible failures and their causes, naming two categories, namely random, individual, independent failures and inter-dependent failures. There are catastrophic failures, determined by a sharp change in the parameters and drift failures (the result of wear or fatigue), arising as a result of a gradual change in the values of the parameters.

1.3 Repairable systems

Failed units of a system may be replaced by new ones, but this may prove to be expensive. To repair the failed units is usually a more cost-effective option and failed units are sent to a repair facility. A ***repairable (or renewable) system*** can be described as one where, when the system (or a unit) is down as the result of a failure, a repair facility is available where the system can be made operable again. If a system can be renewed, the reliability is increased, resulting in an increase in its time of service. If no repair facility is free, failed units queue up for repair. The life time of a unit while on-line, while in standby as well as the repair times are all independent random variables. It is assumed that the distributions of these random variables are known and that they have probability density functions.

Repairable systems have been the subject of intensive investigation for a long time. Different random variables can form the basis for research, such as the

- availability (or non-availability) and reliability
- time necessary for repair
- number of repairs that can be handled
- switch over time to and from the repair facility
- possibility of a vacation time in the repair facility, and many more.

Barlow (1962) considered some "repair man" (or repair facility) problems and they have much in common with queueing problems. Rau (1964) analyzed the problem of finding the optimum value of an m-out-of-n : G system for maximum reliability. Ascher (1968) has pointed out some inconsistencies in modelling of repairable systems by renewal theory. Several authors, notably Buzacott (1970), Shooman (1968), Barlow & Proschan (1965), Sandler (1963) and Doyan & Berssenbrugge (1968) have used continuous time discrete state Markov process models for describing the behavior of a repairable system. These models, although conceptually simple, are not practically feasible in the case of a large number of states. Gaver (1963), Gnedenko et al (1969), Srinivasan (1966) and Osaki (1970a) have used semi-Markov processes for calculation of the reliability of a system with exponential failures. Osaki (1969) has used signal flow graphs to discuss a two-unit system. With the use of semi-Markov processes Kumagi (1971) studied the effect of different failure distributions on the availability through numerical calculations. Branson & Shah (1971) also used semi-Markov process analysis to study repairable systems with arbitrary distributions. Srinivasan & Subramanian (1980), Venkatakrishnan (1975), Ravichandran (1979), Natarajan (1980) and Sarma (1982) have used regeneration point techniques to analyze repairable systems with arbitrary distributions. More references in this and related topics can be found in various review papers by Subba Rao & Natarajan (1970), Osaki & Nakagawa (1976), Pierskalla & Voelker (1976), Lie et al (1977), Kumar & Aggarwal (1980), Birolini (1985) and Yearout et al (1986) and Finkelstein (1993a, 1993b). Jain & Jain (1994) have considered the regulation of 'up' and 'down' times of a repairable system to improve the efficiency of the system.

1.4 Redundancy and different types of redundant systems

In a redundant system more units are built into it than is actually necessary for proper system performance. Redundancy can be applied in more than one way and a definite distinction can be made between parallel and standby (sequential) redundancy. In parallel redundancy the redundant units form part of the system from the start, whereas in a standby system, the redundant units do not form part of the system (until they are needed).

1.4.1 Parallel systems

A parallel redundant system with n units is one in which all units operate simultaneously, although system operation requires at least one unit to be in operation.

Hence, a system failure only occurs when all the components have failed.

Let k be a non-negative integer, such that $k \leq n$, counting the number of units in an n-unit system. It is customary to refer to such a system as a k-out-of-n system.

k-out-of-n: F-system

If a k-out-of-n system fails when k units fail, it is called an F-system. The functioning of a minimum number of units ensures that the system is up (Sfakianakis and Papastavridis (1993)). A survey of such systems has been studied by Chao et al (1995).

k-out-of-n: G-system

A G-system is operational if and only if at least k units out of the n units of the system are operational. Resent work related to this topic can be seen in Zhang and Lam (1998) and Liu (1998). Suppose a radar network has n radar control stations covering a certain area: the system can be operable if and only if at least k of these stations are operable. In other words, to ensure functioning of the system it is essential that a minimal number of units, k, are functioning.

Lately attention moved to load-sharing k-out-of-n: G systems, where

- the serving units share the load
- the failure rate of a component is affected by the magnitude of the load it shares.

n-out-of-n: G-system

A series system that consists of n units and when the failure of any one unit causes the system to fail. Although this type of system is not a redundant system, as all the units are in series and have to be operational, it can still be considered as a special case of a k-out-of-n system..

There are many papers on the reliability of these types of systems. Scheuer (1988) studied reliability for shared-load k-out-of-n: G systems, where there is an increasing failure rate in survivors, assuming i.d. components with constant failure rates. Shao & Laberson (1991) considered the same scenario, but with imperfect switching. Then Huamin (1998) published a paper on the influence of work-load sharing in non-identical, non-repairable components, each having an arbitrary failure time distribution. He assumed that the failure time distribution of the components can be represented by the accelerated failure time model, which is also a proportional hazards model when base-line reliability is Weibull.

1.4.2 Standby redundancy

Standby redundancy consists in attaching to an operating unit one or more redundant (standby) units, which can, on failure of the operating unit, be switched on-line (if operable). Gnedenko et al (1969) classifies standby units as cold, warm or hot.

1. **A cold standby** is completely inactive and because it is not hooked up, it cannot (in theory) fail until it is put in place of the primary unit it replaces. Also assume that, having been in a non-operating state it's reliability will not change when it is put into an operating state.
2. **A warm standby** has a diminished load because it is only partially energized. The standby unit is not subject to the same loading conditions as the on-line unit and failure is generally due to some extraneous random influence. So, although such a warm standby can fail, the probability of it failing is smaller than the probability of the unit on-line failing. This is the most general type of standby because of hot standby's failure rate and cold standby's possible time lapse before it is operable.
3. **A hot standby** is fully active in the system (although redundant) and the probability of loss of operational ability of a hot standby is the same as that of an operating unit. An operating unit in the standby state. The reliability of a hot standby is independent of the instant at which it takes the place of the operable unit.

1.5 Measures of system performance

In the previous sections a brief discussion was given of the various types of redundant systems as discussed in the literature. In this section the discussion is about important *measures* of system performance as applicable in different contexts. (Barlow & Proschan (1965) Gnedenko et al (1969)).

1.5.1 Reliability

Reliability engineering has developed, and advanced substantially during the past 45 years, mainly due to the use of high risk and complex systems. *Reliability* is a quantitative measure to ensure operational efficiency. 'The reliability of a product is the measure of its ability to perform its function, when required, for a specific time, in a particular environment. It is measured as a probability,'. This implies that reliability contains four parts, namely

- the expected function of a system
- the environment of a system (climate, packaging, transportation, storage, installation, pollution etc.)
- time, which is often negatively correlated with reliability
- probability, which is time-dependent, thus causing the need for a statistical analysis.

One can distinguish between mission reliability, when a device is constructed for the performance of one mission only and operational reliability, when a system is turned on and off intermittently for the purpose of performing a certain function. In the latter case we refer to an intermittently used system.

Ordinarily the period of time intended is $(0, t]$.

Let $\{\phi(t), t \geqslant 0\}$ be the performance process of the system.

For fixed t this $\phi(t)$ is a binary random variable, defined as follows:

$$\phi(t) = \begin{cases} 0 & \text{if the system is functioning at time } t. \\ 1 & \text{if the system is in the failed state at time } t. \end{cases}$$

1.5.1a The reliability function

The reliability function, $R(t)$ gives the probability that the system does not fail up to t, i.e.

$$\begin{aligned} R(t) &= P\,[\text{system is functioning in } (0,t]] \\ &= P\,[\phi(u) = 0 \ \ \forall\, u \text{ such that } 0 < u \leq t] \end{aligned}$$

1.5.1b Interval reliability

If the number of system failures in the interval $(t, t+x]$ is considered, the performance measure

$$R(t,x) = P[\phi(u) = 0 \text{ for } \forall \ \ t < u \leq t + x]$$

is referred to as the interval reliability.

If $t = 0$ the interval reliability becomes the reliability $R(x)$.

1.5.1c Limiting interval reliability

Limiting interval reliability is defined as the limit of $R(t,x)$ as $t \to \infty$, and is denoted as $R_{\infty}(x)$

1.5.1d Mean time to system failure

The expectation of the random variable representing the duration of time, measured from the point the system starts operating, till the instant it fails for the first time is called mean time to system failure (MTSF). This is obtained from the relation

$$\text{MTSF} = \int_0^{\infty} R(u)du$$

1.5.2 Availability

This measure of system performance '...denotes the probability that the system is available for use (in operable condition) at any arbitrary instant t'. Availability is therefore the probability that, at the given time t, the system will be operational. It combines aspects of reliability, maintainability and maintenance support and implies that the system is either in active operation or is able to operate if required.

Availability pertains only to systems which undergo repair and are restored after failure, or to intermittently used systems. As such it is eminently reasonable to introduce an availability function $A(t)$. In theory $A(t)$ should be 100%, but even equipment coming directly out of storage may be defective. A high availability can be obtained either by increasing the average operational time until the next failure, or by improving the maintainability of the system. Gnedenko and Uskakov (1995) defines different coefficients of availability for one-unit systems.

1.5.2a Instantaneous or pointwise availability

This is a point function which describes the probability that a system will be able to operate at a given instant of time (Klassen &Van Peppen (1989) and Beasley (1991)). In symbols:

$$A(t) = P\left[\phi(t) = 0\right].$$

1.5.2b Asymptotic or steady-state or limiting availability

The limiting availability, A_∞ is the expected fraction of time that the system operates satisfactorily in the long run (Barlow & Proschan (1965)) : it is the probability that the system will be in an operational state at time t, when t is considered to be infinitely large

$$A_\infty = \lim_{t \to \infty} A(t)$$

1.5.3 Mean number of events in $(0, t]$

Let $N(a, t)$ denote the number of a particular type of a event (e.g. a disappointment, system recovery, system down, etc.) in $(0, t]$. The mean number of events in $(0, t]$ is then given by

$$E\left[N(a,t)\right] = \int_0^t h_1(u)du$$

where $h_1(u)$ is the first order product density of the events (product densities are defined in a subsequent section of this chapter).

The mean stationary rate of occurrence of these events is given by

$$E\left[N(a)\right] = \lim_{t\to\infty} \frac{E[N(a,t)]}{t}$$

1.5.4 Confidence limits for the steady state availability

A 100(1-α)% confidence interval for A_∞ is defined by

$$P\left[a < A_\infty < b\right] = 1 - \alpha$$

where the numbers a and b $(a < b)$ are determined using the appropriate statistical tables. It may be noted that A_∞ is a function of the parameters of operating time distribution, repair time, need and no-need period distributions etc.

1.6 Stochastic processes used in the analysis of redundant systems

Previous sections briefly looked at different types of redundant systems and the various measures of system performance. In this section the techniques used in the analysis of redundant repairable systems will be summarized.

1.6.1 Renewal theory

In renewal theory there exists times, usually random, from which onward the future of the process is a probabilistic replica of the original process and interest is in the lifetime (a stochastic variable) of a unit. At time $t = 0$ a repairable unit is put into operation and is functioning. At each failure the unit is replaced by a new one of the same type, or subjected to maintenance that completely restores it to an 'as good as new' condition. This process is repeated and replacement time taken as negligible. The result is a sequence of lifetimes, and the study is restricted to these renewal points. The probability object in these sums of non-negative i.i.d. random variables lies in the number of renewals N_t up to some time t.

Renewal processes are extensively used by many researchers to study specific reliability problems. The homogeneous Poisson process is the simplest renewal process and has received considerable attention. As in all other processes, the time parameter can be considered as either discrete or continuous. Feller (1950) gave a proper lead for the discrete case and this was followed by the very lucid account of Cox (1962) for the continuous case (he provided an introduction to renewal theory in the case of a repair facility not being available and failed units queueing up for repair). Barlow (1962) applied queueing theory in his research on repairable systems. Srinivasan (1971) studied some operating characteristics of a one unit system, Gnedenko et al (1969) obtained the mean time to system failure of a two-unit standby system, Buzacott (1971) studied some priority redundant systems etc.

Although renewals can take on different forms, the system starts a new cycle after each renewal (which is independent of the previous ones). If repair time is not negligible, each cycle consists of a lifetime and a repair time and both are random variables with individual distributions (repair time can also be considered as a fixed time). The process is called

- an ordinary renewal process if the time origin is the initial installation of the system and the repair time is considered negligibly small in comparison with the lifetime of the unit - renewal is taken as instantaneous, or

- a general renewal process if the time origin is some point subsequent to the initial installation of the system (Cox (1962)). Høyland & Rausand (1994) calls this a modified renewal process, while Feller (1957) refers to such a process considering the residual life time of a system at an arbitrary chosen time origin as a delayed renewal process (see 1.7.3).

1.6.1a Ordinary renewal process : instantaneous renewal

Consider a basic model of continuous operation where a unit begins operating at instant $t = 0$ and stays operational for a random time T_1 and then fails. At this instant it is replaced by a new and statistically identical unit, which operates for a length of time T_2, then fails and is again replaced etc. These random component life lengths $T_1, T_2, ...T_r...$ of the identical units are independent, non-negative and identically distributed random variables that constitute a random flow or ordinary renewal process.

Let

$$P\left[T_i \leq t\right] = F(t)\ ;\ t > 0,\ \ i = 1, 2, ...$$

be the underlying distribution of the renewal process.

The time until the rth renewal is given by

$$t_r = T_1 + T_2 + ... + T_r = \sum_{i=1}^{r} T_i.$$

Let the random variable $N(t) = \max\{r; R_r \leqslant t\}$ indicate the number of times a renewal takes place in the interval $(0; t]$, then the number of renewals in an arbitrary time interval $(t_1, t_2]$ is equal to

$N(t_2) - N(t_1)$.

A renewal function $H(t)$, which is the expected value of $N(t)$ in the time interval $(0; t]$, can now be defined as

$$\begin{aligned} H(t) &= E[N(t)] \\ &= \sum_{r=1}^{\infty} F^{(r)}(t) \end{aligned}$$

where $F^{(r)}(\cdot)$ is the r-fold convolution of F.

Furthermore

$$H(t) = F(t) + \int_0^t H(t-x)dF(x)$$

The renewal density function is

$$h(t) = \sum_{n=1}^{\infty} f^{(r)}(t)$$

and the renewal density function $h\,(t)$ satisfies the equation

$$h(t) = f(t) + \int_0^t h(t-x)f(x)dx$$

it implies that the renewal density $h(t)$ basically differs from the hazard rate $h^o(t)$, as

$$h^o(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1 - F(t)}$$

1.6.1b Random renewal time

Suppose the time for a renewal is not instantaneous but considered as a random variable that is included in the consecutive time-periods, or cycles, of the systems' performance. Each cycle then consists of a time to failure and a time to repair and both are stochastic variables. Instants of failure and cycles of renewal can be identified.

Let $F(t)$ be the life distribution and $G(x)$ be the repair length function with respective probability density functions $f(t)$ and $g(x)$, then the density function of the cycles C of the life time and repair time, say $k(t)$ is obtained by the convolution formula

$$k(t) = \int_0^t f(x)g(t-x)dx$$

If $N_F(t)$ counts the number of failures and $N_R(t)$ the number of repairs in $(0; t]$, define

$$W(t) = E\left[N_F(t)\right]$$

and

$$V(t) = E\left[N_R(t)\right]$$

and let $Q(t) = W(t) - V(t)$; $\forall\, t$, assuming that $w(t) = W'(t)$ and $v(t) = V'(t)$.

The failure and repair intensities can then respectively be defined as

$$\lambda(t) = \frac{w(t)}{A(t)}$$

where $A(t)$ is the availability function

$$\mu(t) = \frac{v(t)}{Q(t)}$$

1.6.1c Alternating renewal processes

Alternating renewal processes were first studied in detail by Takács (1957) and are discussed in many textbooks (Ross (1970)). A generalization of the ordinary renewal process discussed previously where the state of the unit is given by the binary variable

$$X(t) = \begin{cases} 0 & \text{if the unit is functioning at time } t \\ 1 & \text{otherwise} \end{cases}$$

The two alternating states may be 'system up' and 'system down'. If these alternating independent renewal processes are distributed according to $F(x)$ and $G(x)$, there are two renewal processes embedded in them for the different transitions from 'system up' to 'system down'.

One-item repairable structures are generally described by alternating renewal processes with the assumption that after each repair the item is like new.

1.6.1d The age and remaining lifetime of a unit

In the notation of 1.7.1a, let t_r indicate the random component life lengths, i.e. $t_r = \sum_{i=1}^{r} T_i$

Let $R_r, \quad r \in N,$ represent the length of the rth repair time, then the sequence $T_1,\ R_1,\ T_2,\ R_2, ...$ forms an alternating renewal process. Define

$$t_n = T_1 + \sum_{r=1}^{n-1}(R_r + T_{r+1}) \ ; \quad n \in \mathbb{N}$$

and

$$t_n^o = \sum_{r=1}^{n}(R_r + T_r)$$

and set $t_0 = t_0^o = 0$.

This sequence t_n generates a delayed renewal process.

If $B_1(t)$ denotes the forward recurrence time at time t, then

$$B_1(t) = t_{N_t+1} - t \qquad \text{or} \qquad B_1(t) = t_{N_t^o+1} - t$$

Hence,

- $B_1(t)$ equals the time to the next failure time if the system is up at time t, or
- $B_1(t)$ equals the time to complete the repair if the system is down at time t.

Hence,

- $B_2(t)$ equals the age of the unit if the system is up at time t, or
- $B_2(t)$ equals the duration of the repair if the system is down at time t.

Returning to the renewal function $H(t)$, define the elementary renewal theorem (Feller (1941)), stating that, for an ordinary renewal process with underlying exponential distribution (parameter λ and $H(t) = \lambda t$)

$$\lim_{t \to \infty} \frac{H(t)}{t} = \frac{1}{\mu}$$

with $\mu = E(T_i) = \frac{1}{\lambda}$, the mean lifetime.

If the renewals correspond to component failures, the mean number of failures in $(0, t]$ is approximately (for t large)

$$H(t) = E[N(t)] \approx \frac{1}{\mu} = \frac{1}{\text{MTSF}}$$

1.6.2 Semi-Markov and Markov renewal processes

Consider a general description of a process where a system

- moves from one state to another with random sojourn times in between
- the successive states visited form a Markov chain
- the sojourn times have a distribution which depend on the present state as well as the next state to be entered.

This describes a Markov chain if all the sojourn times are equal to one, a Markov process if the distribution of the sojourn times are all exponential and independent of the next state and a renewal process if there is only one state (then allowing an arbitrary distribution of the sojourn times).

Denote the state space by the set of non-negative integers $\{0, 1, 2...\}$ and the transition probabilities by p_{ij} , $i, j = 0, 1, 2...$ If $F_{ij}(t)$, $t > 0$ is the conditional distribution function of the sojourn time in state i , given that the next transition will be into state j, let

$$Q_{ij}(t) = p_{ij}F_{ij}(t) , \quad i, j = 0, 1, 2...$$

denote the probability that the process makes a transition into state j in an amount of time less than or equal to t, given that it just entered state i at $t = 0$. The functions $Q_{ij}(t)$ satisfy the following conditions

$$\begin{aligned} Q_{ij}(0) &= 0, \qquad Q_{ij}(\infty) = p_{ij} \\ Q_{ij}(t) &\geq 0, \qquad i, j = 0, 1, 2... \\ \sum_{j=0}^{\infty} Q_{ij}(t) &= 1 \end{aligned}$$

Let J_0 and J_n respectively denote the initial state and the state after the nth transition occurred. The embedded Markov chain $\{J_n,\ n = 0, 1, 2...\}$ then describes a Markov chain with transition probabilities p_{ij}.

Let $N_i(t)$ denote the number of transitions into state i in $(0, t]$ and

$$N(t) = \sum_{i=0}^{\infty} N_i(t)$$

The stochastic process $\{X(t), t \geqslant 0\}$ with $X(t) = i$ denoting that the process is in state i at time t is called a semi-Markov process (SMP) and it is clear that $X(t) = J_{N(t)}$. A SMP is a pure jump process and all the states are regeneration states. The consecutive states form a time-homogeneous Markov chain, but it is a process without memory at the transition point from one state to the next.

The vector stochastic process $\{N_1(t), N_2(t)...\}$ for $t \geqslant 0$ is called a Markov renewal process (MRP). This implies that the SMP records the state of the process at each time point, while the MRP is a counting process keeping track of the number of visits to each state.

Assume that the time-intervals in which the r.v. $X(t)$ continues to remain in the n-point state are independently distributed such that

$$\lim_{\Delta \to 0} P[X(t+x) = j,\ X(t+u) = i : \forall u \leq x \mid X(t) = i, X(t-\Delta) \neq i]$$

$$= f_{ij}(x) \qquad ; \qquad i, j = 1, 2, ..., n$$

If the transition of $X(t)$ is characterized by a change of state, then the quantities $f_{ii}(\cdot)$ are zero functions. Such a process which is a Markov chain with a randomly transformed time scale is called a MRP.

To remove the consequence that $f_{ii}(\cdot) = 0$, another definition of a MRP can be given, namely defining it as a regenerative stochastic process $\{X(t)\}$ in which the epochs at which $X(t)$ visits any member of a certain countable set of states are regeneration points; the visits being regenerative events.

In a combination of a Markov chain and a renewal process to form a SMP, the purpose is to create a tool that is more powerful than what either could provide individually. SMP were independently introduced by Lévy (1954) and Smith (1955). Detailed use of SMP and MRP can be found in Pyke (1961a, 1961b), Cinlar (1975) and Ross (1970). Barlow and Proschan (1965) used these processes to determine the MTSF of a two-unit system. Cinlar (1975), Osaki (1970a), Arora (1976), Nakagawa & Osaki (1974,1976) and Nakagawa (1974) have used the theory of SMP to discuss certain reliability problems.

1.6.3 Regenerative processes

In a regenerative stochastic process $X(t)$ there exists a sequence $t_0, t_1, \ldots$ of stopping times such that $t = \{t_n; n \in N\}$ is a renewal process. If a point of regeneration happens at $t = t_1$, then the knowledge of the history of the process prior to t_1 loses its predictive value; the future of the process is totally independent of the past. Thus $X(t)$ regenerates itself repeatedly at these stopping times and the times between consecutive renewals are called regeneration times. The application of renewal theory to regenerative processes makes renewal theory such an important tool in elementary probability theory.

A delayed renewal process is defined as follows: if $\hat{t} = \{t_n - t_0; n \in N\}$ is a renewal process such that $t_0 \geqslant 0$ is independent of $\hat{t}$, (implying that the time t_0 of the first renewal is not necessarily the time origin) it is called a delayed renewal process. A delayed regenerative process is a process with a sequence $t = \{t_n; n \in N\}$ of stopping times which form a delayed renewal process. As an example: for any initial state i, the times of successive entrances to a fixed state j in a Markov process form a delayed renewal process.

In some cases non-exponentially distributed repair times and/or failure free operating times may lead to semi-Markov processes, but in general it leads to processes with only a few regeneration states (or even to non-regenerative processes). Recent research in this field is concerned with Brownian motion with the interest on the random set of all regeneration times and on the excursions of the process between generations.

1.6.4 Stochastic point processes

Among discrete stochastic processes, point processes are widely used in reliability theory to describe the appearance of events in time. A renewal process is a well known type of point process, used as a mathematical model to describe the flow of failures in time. It is a point process with restricted memory and each event is a regeneration point. In practical reliability problems, the interest is often in the behavior of a renewal process in a stationary regime, i.e., when $t \to \infty$, as repairable systems enter an 'almost stationary' regime very quickly. A generalization of a renewal process is the so-called alternating renewal process, which consists of two types of i.i.d. random variables alternating with each other in turn.

This theory of recurrent events has a huge variety of applications ranging from classical physics, biology, management sciences, cybernetics and many other areas. The result is that point processes have been defined differently by individuals in the different fields of application. The properties of stationary point processes were first studied by Wold (1948) and Bartlett (1954), to whom we owe the current terminology. Moyal (1962) gave a formal and well-knit theory of the subject that even provides an extension to cover non-Euclidean spaces. Srinivasan (1974), Srinivasan & Subramanian (1980) and Finkelstein (1998, 1999c) extensively used point processes in reliability theory and applications.

Our interest in point processes lies in those applications which, in general, lead to the development of multivariate point processes. For this purpose we can define a point process as a stochastic process 'whose realizations are related to a series of point events occurring in a continuous one-dimensional parameter space (such as time, etc.)' The sequence of times $\{t_n\}$ are the "renewal" epochs which generate the point process and the two random variables of interest are

- the number of points that fall in the interval $(t; t+x]$
- the time that has lapsed since the nth point after (or before) t.

The characterization property of stationarity applies to certain point processes, namely that the density function of the number of observed events in a time interval does not depend on its position on the time axis, but only on the length of the interval. There are different types of stationarity that can be defined, namely simply stationary, weakly stationary and completely stationary (Srinivasan and Subramanian (1980)).

Furthermore, define $p(n; t, x) = P[N(t, x) = n]$ and if $\sum_{n \geq 2} P(n; t, t+\Delta) = o(\Delta)$ for small Δ, the point process is said to be orderly or regular (there are no multiple events, or clusters of events with probability one).

1.6.4a Multivariate point processes

Applications for multivariate stationary point processes can be found in many fields and the properties of these processes have been studied in depth by Cox & Lewis (1970).

If the constraint of independence of the intervals in a stationary renewal process is relaxed, a stationary point process is obtained; if the same constraint is removed in the case of a Markov renewal process a multivariate stationary point process is obtained.

Product densities

Ramakrishnan (1954) developed, analyzed and perfected the product density technique as a sophisticated tool for the study of point processes. A point process is described by the triplet $(\Phi, \mathbf{B}, P)$, where P is a probability distribution on some σ-field $\mathbf{B}$ of subsets of the space Φ of all states. Describe the state of a set of objects by a point x of a fixed set of points X. Assume for this discussion that X is the real number line. Define A_k as intervals and $N(.)$ as a counting measure which is uniquely associated with a sequence of points $\{t_i\}$ such that:

$$\begin{aligned} N(A) &= \text{the number of points of the sequence } \{t_i : t_i \in A\} \\ N(t,x) &= \text{the number of points (events) in the interval } (t; t+x] \\ N(t,x) &= \text{the number of points (events) in } (t+x; t+x+\Delta] \end{aligned}$$

The central quantity of interest in the product density technique is this $N'(t,x)$, denoting the number of entities with parametric values between x and $x+\Delta$ at time t.

From the factorial moment distribution the product density of order n, which represents the probability of an event in each of the intervals $(x_1, x_1+\Delta_1)$, $(x_2, x_2+\Delta_2), ..., (x_n, x_n+\Delta_n)$, can be defined. It is expressed as the product of the density of expectation measures at different points, namely

$$h_n(x_1, x_2, ..., x_n) = \lim_{\Delta_1, \Delta_2, ..., \Delta_n \to 0} \frac{E\left[\prod_{i=1}^{n} N(x_i, \Delta_i)\right]}{\Delta_1 \Delta_2 ... \Delta_n} \; ; \quad x_1 \neq x_2 \neq ... \neq x_n$$

or, equivalently

$$h_n(x_1, x_2, ..., x_n) = \lim_{\Delta_1, \Delta_2, ..., \Delta_n \to 0} \frac{P\left[N(x_i, \Delta_i) \geqslant 1, \;\; i = 1, 2, ..., n\right]}{\Delta_1 \Delta_2 ... \Delta_n} \; ; \quad x_1 \neq x_2 \neq ... \neq x_n$$

Since $h_n(...)$ is a product of the density of expectation measures at different points, the density is aptly called the product density.

Considering the ordinary renewal process as defined in 1.7.1a, the renewal function $H(t)$ is the expected number of random points in the interval $(0; t]$. Modify the process by allocation of all integral values to $\{t_i\}$ and consider a corresponding sequence of points on the real line. In the point process then generated by the random variables $\{t_i\}$, the counting process $N(t,x)$ represents the number of points in the interval $(t,t+x]$ and the product density is

$$h_m(t,t_1,t_2,...,t_m) = E\left[N'(t,t_1)N'(t,t_2)...N'(t,t_m)\right]$$

The product density of degree m is

$$h_m(t,t_1,t_2,...t_m) = h_1(t,t_1)h(t_2-t_1)h(t_3-t_2)...h(t_m-t_{m-1})$$

$$(t_1 < t_2 < ... < t_m)$$

1.7 General notation

λ_i :	Demand rate of product i, $i = 1,\ 2$
μ_i :	Service rate of product i, $i = 1,\ 2$
S_i :	Maximum inventory level of product i, $i = 1, 2$
s_i :	Reorder level of product i, $i = 1,\ 2$
$S_i - s_i$:	Quantity of product i reordered, $i = 1, 2$
d_i :	Event that a demand for product i is satisfied with product i, $i = 1, 2$
g :	Event that a demand for product 1 is satisfied by product 2
l_i :	Event that a demand for product i is lost, $i = 1, 2$
$N(\eta,\ t)$:	Number of η events in the interval $(0, t]$

δ_{ij} :	Kronecker delta, i.e. $\delta_{ij} = \begin{bmatrix} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{bmatrix}$
$H(i)$:	$= \begin{bmatrix} 0 \text{ if } i < 0 \\ 1 \text{ otherwise} \end{bmatrix}$
ⓒ :	Convolution symbol
$\xi(s)$:	Laplace transform of an arbitrary function $\xi(t)$
$f^{(n)}(t)$:	n fold convolution of an arbitrary function $f(t)$ with itself
$\overline{F}(t)$:	$1 - \int_0^t f(u)\,du$ for an arbitrary function $f(u)$
$X(\cdot)$:	A stochastic process describing the state of a system
$p.d.f.$:	Probability density function
$r.v.$:	Random variable
$f(\cdot)$:	The p.d.f. of the lifetime of a unit while on-line
$g(\cdot)$:	The p.d.f. of the repair time of a unit
$f^*(s)$:	Laplace transform of the function $f(t)$
$F(t)$:	Cumulative distribution function: $\int_0^t f(u)\,du$
$\overline{F}(t)$:	$1 - F(t)$
E_i :	Regenerative event of type i
A :	Availability
$A_i(t)$:	P (system is up at t/E_i at $t = 0$)
A_∞ :	Steady state availability
R :	Reliability
$R_i(t)$:	P (system is up in $(0,\ t]/E_i$ at $t = 0$)

MLE	maximum likelihood estimator
MTSF	mean time to system failure (also MTTF)
MTSR	mean time to system repair
MTFD	mean time to first disappointment
SMP	semi-Markov process
MRP	Markov renewal process
FCFS	first-come-first-served

CHAPTER 2

Inferential statistics for the availability of a two component system in the presence of common-cause failures

A modified version of this chapter has been communicated to Asia-Pacific Journal of Operations Research (2002).

2.1 Introduction

The scope of reliability engineering is extremely wide. It helps to obtain reliable transportation and telecommunication systems, provide a steady supply of power, and ensure successful operations of robotics, and so on. The growth of knowledge in several areas of reliability engineering and its applications has become increasingly important (Chung, 1990).

The common-cause failures have gained considerable attention in the field of reliability (Dhillon (1979), Chung (1988a, 1988b, 1990), Shooman (1971), Dhillon (1981) etc.)

Reliability of a system is fairly simple when units fail independently of each other. In the presence of common-cause failures, the reliability calculation requires a set of simultaneous linear differential equations. Some of the reasons for systems with common-cause failures are:

(i) equipment design deficiencies

(ii) unforeseen external abnormal environments - dust, humidity, temperature

(iii) operations and maintenance errors

(iv) external catastrophe

(v) functional deficiencies

(vi) common power source

When components of a system fail, they do not necessarily fail independently of each other. The failures may be synchronised, and these cases have a common cause. This type of failure has been identified by reliability analysts. Common-cause failures greatly reduce the reliability indices. Billinton and Allen (1983) discussed the role of common-cause shock (CCS) failures.

Vesely (1977) discussed the binomial failure rate model (BFR). Atwood (1986), Atwood and Stevenson (1982) and Meachum and Atwood (1983) have applied the BFR model for CCS failures to the data associated with nuclear power plants and given in nuclear regulatory commission reports. They discussed the quantification and estimation of CCS failure rates. Chari (1988), Chari et al (1991) have extended the concept of CCS failures to arrive at expressions of the reliability indices; i.e. the reliability function, mean time between failures and availability. The expressions of these measures were derived using a Markovian approach. In this chapter, we derive the formulae for failure frequency, as this is another important measure in the reliability analysis of some systems which are assumed to be affected by the presence of CCS failures, and also for the availability. The above study is done for both parallel and series systems. Confidence limits for the steady state availability are also obtained for both models.

2.2 System description & notation

1. The system has two S-independent and identical components.
2. The system is affected by both individual and CCS failures.
3. The components in the system may fail singly (individual failure) at the rate of λ_a, and the chance of occurrence of such failures is C_1.
4. The components also fail simultaneously when CCS hits the system, at the rate λ_c. The chance of such failures is C_2; $C_1 + C_2 = 1$.
5. The times between individual failures and between shock failures follow an exponential distribution.
6. The individual failures and shock failures occur independently of each other.
7. The failed components are serviced singly and service times follow an exponential distribution

$R_s^*(t)$: Reliability of the series configuration in the presence of chronic CCS failures.

$R_p^*(t)$: Reliability of parallel configuration in the presence of CCS failures.

$A_S^*(t)$: Availability of the series configuration in the presence of CCS failures.

$A_p^*(t)$: Availability of the parallel configuration in the presence of CCS failures.

f_{C_i} : Frequency of encountering state i in the case of CCS model; $i = 0,\ 1,\ 2.$

$f_{C_{ij}}$: Frequency of transit from state i to j in the case of the CCS model $i = 0,\ 1;\ \ j = 1,\ 2.$

UP : Component is up.

DN : Component is down.

2.3 Model

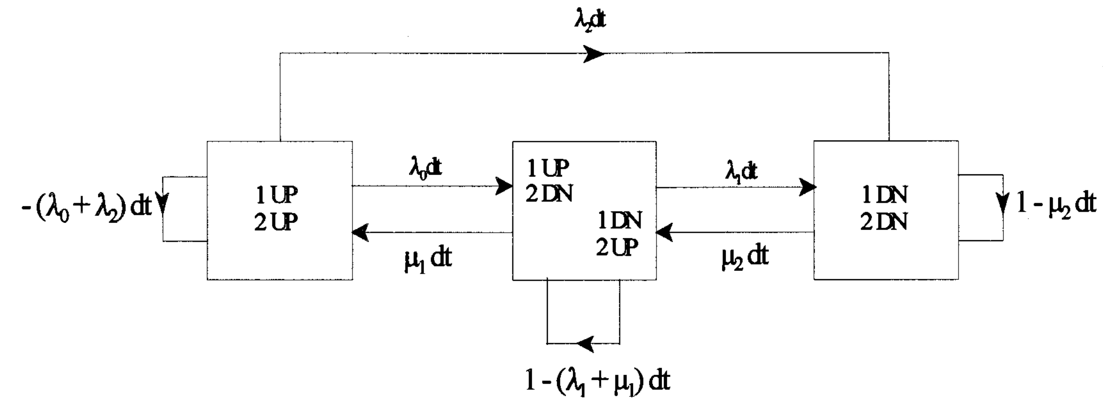

Figure 2.1

Using Figure 2.1, we can formulate a Markov model to derive the availability function $A(t)$ and the frequency of encountering different states of the system under the influence of CCS failures in addition to individual failures. The quantities $\lambda_0,\ \lambda_1,\ \lambda_2,\ \mu_1$ and μ_2 in Figure 2.1 are defined as follows:

$$\lambda_0 = 2\lambda_a C_1;\ \ \lambda_1 = \lambda_a C_1;\ \ \lambda_2 = \lambda_C C_2;\ \ \mu_1 = \mu;\ \ \mu_2 = 2\mu \tag{2.1}$$

The following set of different differential equations can be obtained using Figure 2.1

$$\begin{aligned} P_0'(t) &= (\lambda_0 + \lambda_2) P_2(t) + \mu_1 P_1(t) \\ P_1'(t) &= (\lambda_1 + \mu_1) P_1(t) + \mu_2 P_2(t) + \lambda_0 P_0(t) \\ P_2'(t) &= \mu_2 P_2(t) + \lambda_1 P_1(t) + \lambda_2 P_0(t) \end{aligned} \tag{2.2}$$

Using the initial conditions, $P_0(0) = 1;\ P_1(0) = P_2(0) = 0$ to sovle (2.2), we get

$$P_0(t) = \frac{\mu_1 \mu_2}{r_1 r_2} + \frac{[I_1 e^{r_1 t} - I_2 e^{r_2 t}]}{r_1 r_2} \tag{2.3}$$

$$P_1(t) = \frac{(\lambda_0 \mu_2 + \lambda_2 \mu_2)}{r_1 r_2} + \frac{[K_1 e^{r_1 t} - K_2 e^{r_2 t}]}{r_1 r_2} \tag{2.4}$$

$$P_2(t) = 1 - P_0(t) - P_1(t) \tag{2.5}$$

$$\text{where } r_1,\ r_2 = \frac{-b \pm \sqrt{b^2 - 4C}}{2} \tag{2.6}$$

$$\begin{aligned} b &= \mu_1 + \mu_2 + \lambda_0 + \lambda_1 + \lambda_2 \\ &= \mu_1 \mu_2 + \mu_2 \lambda_0 + \mu_2 \lambda_2 + \mu_1 \lambda_2 + \lambda_0 \lambda_1 + \lambda_1 \lambda_2 \end{aligned}$$

$$\begin{aligned} \text{and } I_1 &= \frac{r_1^2 + r_1(\mu_1 + \mu_2 + \lambda_1) + \mu_1 \mu_2}{r_1} \\ I_2 &= \frac{r_2^2 + r_2(\mu_1 + \mu_2 + \lambda_1) + \mu_1 \mu_2}{r_2} \end{aligned} \tag{2.7}$$

$$\begin{aligned} K_1 &= \frac{r_1 \lambda_0 + \lambda_0 \mu_2 + \lambda_2 \mu_2}{r_1} \\ K_2 &= \frac{r_2 \lambda_0 + \lambda_0 \mu_2 + \lambda_2 \mu_2}{r_2} \end{aligned} \tag{2.8}$$

2.4 Availability analysis

The two component system shown in Figure 2.1 has either a series or a parallel configuration.

2.4.1 Time dependent availability

We derive the time-dependent availability for both configurations in the case of common-cause failures as well as individual failures.

2.4.1.1 Series configuration

For a series system, state 1 itself is an absorbing state and hence no transition is allowed from state 1 to state 2. Hence $\lambda_1 = 0$. Therefore, the time dependent availability for a series system in the case of CCS failures as well as individual failures is

$$A_s^*(t) = P_0(t) = P_0(t) = \frac{\mu_1\mu_2}{r_1 r_2} + \frac{I_1 e^{r_1 t} - I_2 e^{r_2 t}}{r_1 - r_2} \tag{2.9}$$

and $r_1 r_2 = \mu_1\mu_2 + \mu_2\lambda_0 + \mu_2\lambda_2 + \mu_1\lambda_2$.

$I_1,\ I_2$ are given by (2.7). The quantities $\lambda_0,\ \lambda_2,\ \mu_1,\ \mu_2$ are to be substituted as seen in (2.9) after simplifying that expression. Availability of the series system in the case of CCS and individual failures is

$$A_s^*(t) = \frac{2\mu_2}{2\mu_2 + 4\lambda_a C_1\mu + 3\lambda_C C_2\mu} + \frac{G_1 e^{r_1 t} - G_2 e^{r_2 t}}{(r_1 - r_2)} \tag{2.10}$$

$$\text{where } G_1 = \frac{r_1^2 + 3r_1\mu + 2\mu^2}{r_1}$$

$$G_2 = \frac{r_2^2 + 3\mu r_2 + 2\mu^2}{r_2}$$

$$r_1,\ r_2 = \frac{1}{2}\left[-(3\mu + 2\lambda_a C_1 + \lambda_C C_2) \pm \sqrt{(\mu - 2\lambda_a C_1 \lambda_C C_2)^2 - 4\lambda_C C_2 \mu}\right]$$

In the case of a series system, it is seen that the expression for availability given in (2.10) agrees with the expression of availability by Balagurusamy (1988), when individual failures only affect the system (this implies that CCS failures are not affecting the system, i.e. $C_2 = 0$ or $\lambda_C = 0$).

2.4.1.2 Parallel configuration

The time-dependent availability of the parallel system for the CCS failures in addition to individual failures can be obtained by

$$A_p^*(t) = P_0(t) + P_1(t)$$

where $P_0(t)$ and $P_1(t)$ are given in expressions (2.3) and (2.4). After substituting the quantities $\lambda_0,\ \lambda_1,\ \lambda_2,\ \mu_1$ and μ_2 the formula for availability in the case of CCS failures as well as individual failures is

$$A_p^*(t) = \frac{2\mu\left(\mu + 2\lambda_a C_1 + \lambda_C C_2\right)}{2\mu^2 + 4\lambda_a C_1\mu + 3\lambda_C C_2\mu + 2\lambda_a^2 C_1^2 + \lambda_a\lambda_C C_1 C_2} + \frac{H_1 e^{r_1 t} - H_2 e^{r_2 t}}{r_1 - r_2} \quad (2.11)$$

where

$$H_1 = \frac{r_1^2 + 3r_1\left(\mu + \lambda_a C_1\right) + 2\mu\left(2\lambda_a C_1 + \lambda_C C_2 + \mu\right)}{r_1}$$

$$H_2 = \frac{r_2^2 + 3r_2\left(\mu + \lambda_a C_1\right) + 2\mu\left(2\lambda_a C_1 + \lambda_C C_2 + \mu\right)}{r_2}$$

$$r_1,\ r_2 = \frac{1}{2}\left[-\left(3\mu + 3\lambda_a C_1 + \lambda_C C_2 \pm \sqrt{\left(\mu + \lambda_a C_1 - \lambda_C C_2\right)^2 - 8\lambda_C C_2\mu}\right)\right]$$

Even in the case of a parallel system, it is interesting to see that the formula for the availability given in (2.12) agrees with that found when individual failures alone are acting on the system (i.e. $C_2 = 0/\lambda_C = 0$). There results an inagreement with Balagurusamy (1988).

2.5 Steady-state availability

We now derive the steady-state availability for both configurations in the case of CCS failures and individual failures.

2.5.1 Series configuration

As time becomes large, the steady-state availability of the system can be obtained by using the final value theorem of the Laplace transform i.e.

$$\begin{aligned} A_s^*(\infty) = \lim_{t\to\infty} A_s^*(t) &= \lim_{t\to\infty} P_0(t) \\ &= \lim_{s\to 0+} s\, P_0^*(s) \end{aligned}$$

Therefore

$$A_s^*(\infty) = \frac{\mu_1\mu_2}{\mu_1\mu_2 + \mu_2\lambda_0 + \mu_2\lambda_2 + \mu_1\lambda_2 + \lambda_1\lambda_0 + \lambda_1\lambda_2}$$

putting $\lambda_1 = 0$ and then substituting the quantities λ_0, λ_2, μ_1 and μ_2 as in (2.1) in the formula, the availability of the series system in the case of CCS failures in addition to individual failures is given by

$$A_s^*(\infty) = \frac{2\mu^2}{2\mu_2 + 4\lambda_a C_1\mu + 3\lambda_C C_2\mu} \tag{2.12}$$

It is interesting to note that the formula for availability given in (2.14) agrees with the formula of availability in the case of a series system when individual failures only are occurring in the system. This is in agreement with Balagurusamy (1988).

2.5.2 Parallel configuration

After a long-run usage of the system, the steady-state availability of the parallel system is arrived at as a limiting case of the availability

$$\begin{aligned} A_p^*(\infty) = \lim_{t\to\infty} A_p^*(t) &= \lim_{t\to\infty}\left[P_0(t) + P_1(t)\right] \\ &= \lim_{s\to 0+}\left[P_0^*(s) + P_1^*(s)\right] \end{aligned}$$

Using the final value theorem of the Laplace transform,

$$A_p^*(\infty) = \frac{2\mu\left(2\lambda_a C_1 + \lambda_C C_2 + \mu\right)}{2\mu\left(\mu + 2\lambda_a C_1 + \lambda_C C_2 + \mu\lambda_C C_2 + \lambda_a C_1\left(2\lambda_a C_1 + \lambda_C C_2\right)\right)} \tag{2.13}$$

As for the parallel configuration, the formula for availability given (2.13) agrees with that found when individual failures alone affect the system if we let $C_2 = 0/\lambda = 0$. This is in agreement with Balagurusamy (1988).

2.6 Frequency of encountering different states - CSS model

The frequency of encountering different states of the system in the case of the CCS model is evaluated in terms of the steady-state probability of the different states. This is given by

$$\begin{aligned} P_0 &= \frac{\mu_1\mu_2}{r_1 r_2}; \qquad\qquad P_1 = \frac{\mu_2\left(\lambda_0 + \lambda_2\right)}{r_1 r_2} \\ P_2 &= \frac{\mu_1\lambda_2 + \lambda_1\left(\lambda_0 + \lambda_2\right)}{r_1 r_2} \end{aligned} \tag{2.14}$$

where $\lambda_0,\ \lambda 1,\ \lambda_2,\ \mu_1,\ \mu_2$ and $r_1,\ r_2$ are to be substituted as in (2.1) and (2.6) respectively. The frequencies of encountering the different states in the case of the CCS model are

$$
\begin{aligned}
f_{C_0} &= \frac{2\mu^2 (2\lambda_a C_1 + \lambda_C C_2)}{D} \\
f_{C_1} &= \frac{[2\mu (2\lambda_a C_1 + \lambda_C C_2) (\lambda_a C_1 + \mu)]}{D} \\
f_{C_2} &= \frac{2\mu \left[\lambda_C C_2 (\mu + \lambda_a C_1) + 2\lambda_a^2 C_1^2\right]}{D} \\
f_{C_{12}} &= \frac{2\mu \left[4\lambda_a^2 C_1^2 + \lambda_C C_2 (2\lambda_a C_1 + \mu)\right]}{D} \qquad (2.15)
\end{aligned}
$$

where

$$D = 2\mu (\mu + 2\lambda_a C_2 + \lambda_C C_2) + \mu\lambda_C C_2 + \lambda_a C_1 (2\lambda_a C_1 + \lambda_C C_2) \qquad (2.16)$$

The frequency of down-state.ŝl3Ut ate two-component series and parallel configurations in the case of the CCS model will be evaluated in the following sections. The down-state and up-state of these configurations can be represented in terms of the frequency of encountering different states as defined in (2.15).

2.6.1 Series configuration - CCS model

The frequency of down-state and up-state are

$$
\begin{aligned}
f_{C_s}(down) &= f_{C_1} + f_{C_2} - f_{C_{12}} \\
&= \frac{2\mu^2 [2\lambda_a C_1 + \lambda_C C_2]}{D} \qquad (2.17) \\
f_{C_s}(up) &= f_{C_0} \\
&= \frac{2\mu^2 [2\lambda_a C_1 + \lambda_C C_2]}{D} \qquad (2.18)
\end{aligned}
$$

where D is defined as in (2.16). The two expressions above are the same, as teh system is represented by only two states, namely good and bad (see Billinton & Allen, 1983).

It is interesting to note that this expression agrees with that derived already for a series system when only individual failures affect the system. This is in agreement with Billinton & Allen (1983).

2.6.2 The frequencies of down- and up-states of a two-component parallel configuration are represented by the frequencies of encountering different states, developed. The frequencies of down-state and up-state of a parallel system in the case of the CCS model are

$$f_{C_p}(down) = f_{c_2} = \frac{2\mu\left[\lambda_C C_2\left(\mu + \lambda_a C_1\right) + 2\lambda_a^2 C_1^2\right]}{D} \tag{2.19}$$

$$\begin{aligned} f_{C_p}(up) &= f_{C_1} + f_{C_2} - f_{C_{12}} \\ &= \frac{2\mu 2\mu\left[\lambda_C C_2\left(\mu + \lambda_a C_1\right) + 2\lambda_a^2 C_1^2\right]}{D} \end{aligned} \tag{2.20}$$

where D is defined in (2.16).

Even in this case, we observe that these expressions, under the assumption that there are no common-cause failures (i.e. $C_2 = 0/\lambda_2 = 0$), will agree with that already developed in the literature for the case of individual failure (see Billinton & Allen, 1983).

Frequency of failures can be calculated using (2.17) and (2.19) for the CCS model for series and parallel configurations respectively.

2.7 Confidence limits for the availability

We now study the inferential statistics for the series and parallel configurations (when CCS is considered).

2.7.1 Series configuration

We know that

$$A_s^*(\infty) = \frac{\mu_1\mu_2}{\mu_1\mu_2 + \mu_2\lambda_0 + \mu_2\lambda_2 + \mu_1\lambda_2 + \lambda_1\lambda_0 + \lambda_1\lambda_2}$$

using the relation (2.1) in the above equation, we get

$$A_s^*(\infty) = \frac{2\mu^2}{2\mu^2 + 4\lambda_a C_1\mu + 3\lambda_C C_2\mu + 2\lambda_a^2 C_1^2 + \lambda_a\lambda_C C_1 C_2}$$

Let $(U_{11},\ U_{12},\ ...,\ U_{1n_1})$; $(U_{21},\ U_{22},\ ...,\ U_{2n_2})$ be random samples of individual system failures and common cause system failures with sample sizes n_1 and n_2 respectively. Let $(U_{31},\ U_{32},\ ...,\ U_{3n_3})$ be a random sample of repairs of the component with size n_3. All these samples are drawn from exponential populations. In particular we assume $n_i = n$; $i = 1,\ 2,\ 3$. Let $\overline{U}_1,\ \overline{U}_2,\ \overline{U}_3$ be the sample means. It can be shown that $\overline{U}_1,\ \overline{U}_2,\ \overline{U}_3$ are the maximum likelihood estimates of $\frac{1}{\lambda_a}$, $\frac{1}{\lambda_C}$ and $\frac{1}{\mu}$ respectively. Let $\theta_1 = \frac{1}{\lambda_a}$, $\theta_2 = \frac{1}{\lambda_C}$ and $\theta_3 = \frac{1}{\mu}$. The steady-state availability $A_s^*(\infty)$ reduces to

$$A_s^*(\infty) = \frac{A_s}{A_s + B_s}$$

where

$$A_s = 2\theta_1^2\theta_2$$

$$B_s = 4C_1\theta_1\theta_2\theta_3 + 3C_2\theta_1^2\theta_3 + 2C_1^2\theta_3^2\theta_2 + C_1C_2\theta_1\theta_3^2$$

and hence the MLE of $A_s^*(\infty)$ is given by

$$\hat{A}_s^*(\infty) = \frac{A}{A+B}$$

$$\text{where } A = 2\overline{U}_1^2\overline{U}_2$$
$$B = 4C_1\overline{U}_1\overline{U}_2\overline{U}_3 + 3C_2\overline{U}_1^2\overline{U}_3 + 2C_1^2\overline{U}_3^2\overline{U}_2 + C_1C_2\overline{U}_1\overline{U}_3^2$$

It may be noted that $\hat{A}_s^*(\infty)$ is a real valued function in $\overline{U}_1,\ \overline{U}_2,\ \overline{U}_3$, which are also differentiable. Now, consider the following application of multivariate central limit theorem (Rao, 1974).

Suppose $\Upsilon_1^*,\ \Upsilon_2^*,\ \Upsilon_3^*,\ ...$ are independent and identically distributed k-dimensional random variables such that

$$\Upsilon_n^* = (\Upsilon_{1n},\ \Upsilon_{2n},\ ...,\ \Upsilon_{kn});\ \ n = 1,\ 2,\ ...$$

having first and second order moments $E(\Upsilon_n) = m$, and $D(\Upsilon_n) = \Sigma$. Define the sequence of random variables

$$\overline{\Upsilon}_n = \left(\overline{\Upsilon}_{1n},\ \overline{\Upsilon}_{2n},\ ...,\ \overline{\Upsilon}_{kn}\right);\ \ n = 1,\ 2,\ 3,\ ...$$

where $\overline{\Upsilon}_n = \frac{1}{n}\sum_{j=1}^{n}\Upsilon_{ij};\quad i = 1,\ 2,\ ...,\ k$
$\quad j = 1,\ 2,\ ...,\ n.$

Then $\sqrt{n}\left(\Upsilon_n - m_1\right) \xrightarrow{d} N_k\left(0,\ \Sigma_1\right)$ as $n \to \infty$.

By applying the above application of multivariate control limit theorem, it readily follows that

$$\sqrt{n}\left[\left(\overline{U}_1,\ \overline{U}_2,\ \overline{U}_3\right) - \left(\theta_1,\ \theta_2,\ \theta_3\right)\right] \xrightarrow{d} N_3\left(0,\ \Sigma_1\right) \text{ as } n \to \infty$$

where the dispersion matrix $\Sigma = \left(\sigma_{ij}\right)_{3\times 3}$ is given by $\Sigma = diag\left(\theta_1^2,\ \theta_2^2,\ \theta_3^2\right)$. Again from Rao (1974), we have

$$\sqrt{n}\left[\hat{A}_s^*\left(\infty\right) - A_s^*\left(\infty\right)\right] \to N\left(0,\ \sigma^2\left(\theta\right)\right) \text{ as } n \to \infty$$

here $\theta^2\left(\theta_1,\ \theta_2,\ \theta_3\right)$ and

$$\begin{aligned} \sigma^2\left(\theta\right) &= \sum_{i=1}^{3}\left(\frac{\partial A_s^*\left(\infty\right)}{\partial\theta_i}\right)^2 \\ \sigma_{ii} &= \sum_{i=1}^{3}\left(\frac{\partial A_s^*\left(\infty\right)}{\partial\theta_i}\right)^2 \theta_i^2 \end{aligned}$$

Substituting for $\dfrac{\sigma A_s^*\left(\infty\right)}{\partial\theta_i}$; $i = 1,\ 2,\ 3,\ i\ n$ the above equation, we obtain $\sigma^2\left(\theta\right)$. Hence $\hat{A}_s\left(\infty\right)$ is a consistently asymptotic normal estimator of $A_s^*\left(\infty\right)$. Let $\sigma^2\left(\hat{\theta}\right)$ be the estimator of $\sigma^2\left(\theta\right)$ obtained by replacing θ by a consistant estimator $\hat{\theta}$ namely $\hat{\theta} = \left(\overline{U}_1,\ \overline{U}_2,\ \overline{U}_3\right)$. Let $\hat{\sigma}^2 = \sigma^2\left(\hat{\theta}\right)$. Since $\sigma^2\left(\theta\right)$ is a continuous function of θ, $\hat{\sigma}^2$ is a consistent estimator of $\sigma^2\left(\theta\right)$, i.e. $\hat{\sigma}^2 \to \sigma^2\left(\theta\right)$ as n$\to \infty$. By Slutzky theorem, we have $\sqrt{n}\dfrac{\left[\hat{A}_s\left(\infty\right) - A_s\left(\infty\right)\right]}{\sigma} \to N\left(0,\ 1\right)$

$$\text{i.e. } P\left[-K_{\frac{\alpha}{2}} < \frac{\sqrt{n}\left[\hat{A}_s^*(\infty) - A_s(\infty)\right]}{\sigma} < K_{\frac{\alpha}{2}}\right]$$

where $K_{\frac{\alpha}{2}}$ is obtained from normal tables. Hence $100(1-\alpha)\,\%$ asymptotic confidence limits for $A_s^*(\infty)$ are given by

$$A_s^*(\infty) \pm K_{\frac{\alpha}{2}} \cdot \frac{\sigma}{\sqrt{n}}.$$

2.7.2 Parallel system

$$A_p^*(\infty) = \frac{2\mu\,(2\lambda_a C_1 + \lambda_C C_2 + \mu)}{2\mu\,(2\lambda_a C_1 + \lambda_C C_2 + \mu) + \lambda_a C_1\,(2\lambda_a C_1 + \lambda_C C_2)}$$

Let $(V_{11},\ V_{12, \dots,}\ V_{1n_1})$ and $(V_{21},\ V_{22},\ \dots,\ V_{2n_2})$ be random samples of individual system failures and common cause system failures with sample sizes n_1 and n_2 respectively. Let $(V_{31},\ V_{32},\ \dots,\ V_{3n_3})$ be a random sample of repair of a component with size n_3. All these samples are drawn from exponential populations. In particular we assume $n_i = n;\quad i = 1,\ 2,\ 3$. Let $(\overline{V}_1,\ \overline{V}_2,\ \overline{V}_3)$ be the sample means of the above samples. It can be shown that $(\overline{V}_1,\ \overline{V}_2,\ \overline{V}_3)$ are the maximum likelihood estimates of $\frac{1}{\lambda_a}$, $\frac{1}{\lambda_C}$ and $\frac{1}{\mu}$ respectively. Let $\nu_1 = \frac{1}{\lambda_a};\ \nu_2 = \frac{1}{\lambda_C};\ \nu_3 = \frac{1}{\mu}$. The steady-state availability $A_p^*(\infty)$ reduces to

$$A_p^*(\infty) = \frac{A_p}{A_p + B_p}$$

$$\begin{aligned} \text{where } A_p &= 2\nu_1\,(2C_1\nu_2\nu_3 + C_2\nu_1\nu_3 + \nu_1\nu_2) \\ B_p &= C_2\nu_1^2\nu_3 + C_1\nu_3^2\,(2C_1\nu_2 + C_2\nu_1) \end{aligned}$$

and hence MLE of $\hat{A}_p^*(\infty)$ is given by

$$\hat{A}_p^*(\infty) = \frac{A_2}{A_2 + B_2}$$

$$\begin{aligned} A_2 &= 2\overline{v}_1 \left(2C_1\overline{v}_2\overline{v}_3 + C_2\overline{v}_1\overline{v}_3 + \overline{v}_1\overline{v}_2\right) \\ B_2 &= C_2\overline{v}_1^2\overline{v}_3 + C_1\overline{v}_3^2 \left(2C_1\overline{v}_2 + C_2\overline{v}_1\right) \end{aligned}$$

It may be noted that $A_p^*(\infty)$ is a real valued function in $\overline{v}_1,\ \overline{v}_2$ and $\overline{v}_3$, which are also differentiable. Now, consider the following application of multivariate control limit theorem (see Rao, 1974).

Using the above theorem stated for series system, and apply the same to the parallel system, we get

$$\sqrt{n}\left[(\overline{v}_1,\ \overline{v}_2,\ \overline{v}_3) - (\nu_1,\ \nu_2,\ \nu_3)\right] \xrightarrow{d} N_3\left(0,\ \Sigma_2\right) \text{ as } n \to \infty$$

where the dispersion matrix $\Sigma_2 = \left(\rho_{ij}\right)_{3\times3}$ is given by $\Sigma = diag\left(\nu_1^2,\ \nu_2^2,\ \nu_3^2\right)$

$$\therefore \sqrt{n}\left[\hat{A}_p^*(\infty) - A_p^*(\infty)\right] \to N\left(0,\ \rho^2(\nu)\right) \text{ as } n \to \infty$$

hence

$$\begin{aligned} \nu &= (v_1,\ v_2,\ v_3) \text{ and} \\ \rho^2(\nu) &= \sum_{i=1}^{3} \left(\frac{\partial A_p^*(\infty)}{\partial \nu_i}\right)^2 \\ \rho_{ii} &= \sum_{i=1}^{3} \left(\frac{\partial A_p^*(\infty)}{\partial \nu_i}\right)^2 \nu_i^2 \end{aligned}$$

substituting for $\frac{\partial A_p^*(\infty)}{\partial \nu_i}$; $i = 1,\ 2,\ 3$ in the above equation, we obtain $\rho^2(\nu) \cdot A_p^*(\infty)$. Let $\rho^2(\hat{\nu})$ be the estimator of $\rho^2(\nu)$ obtained by replacing v by a consistent estimator $\hat{v}$ namely $\hat{\nu} = (\overline{v}_1,\ \overline{v}_2,\ \overline{v}_3)$. Let $\hat{\rho}^2 = \rho^2(\hat{\nu})$. Since $\rho^2(\nu)$ is a continuous function of ν, $\hat{\rho}^2$ is a consistent estimator of $\rho^2(\nu)$ i.e. $\hat{\rho}^2 \rightarrow \rho^2(v)$ as $n \rightarrow \infty$. By Slutzky theorem, we have

$$\frac{\sqrt{n}\left[\hat{A}_p^*(\infty) - A_p^*(\infty)\right]}{\rho} \rightarrow N(0,\ 1)\text{, i.e.}$$

$$P\left[-K_{\frac{\alpha}{2}} < \frac{\sqrt{n}\left[\hat{A}_p^*(\infty) - A_p^*(\infty)\right]}{\rho} < K_{\frac{\alpha}{2}}\right] = 1 - \alpha$$

where $K_{\frac{\alpha}{2}}$ is obtained from normal tables. Hence $100(1-\alpha)\,\%$ asymptotic confidence limits for $A_p^*(\infty)$ are given by

$$\hat{A}_p^*(\infty) \pm K_{\frac{\alpha}{2}} \frac{\rho}{\sqrt{n}}.$$

2.8 Numerical illustration

For fixed values of $\lambda_a,\ \lambda_C,\ C_1,\ C_2,$ values of t, the availability is shown in Figure 1. Figure 1 shows three different curves for different values of μ.

Table 1 presents the confidence limits for different values of μ and for different sample sizes.

Table 1

Confidence intervals with $\lambda_a = 0.1,\ \lambda_C = 0.2,\ C_1 = 0,\ c_2 = 1$		
$\mu = 5$	$\mu = 6$	$\mu = 7$
(0.923426; 0.963367)	(0.93241; 0.972352)	(0.938933; 0.978875)
(0.929275; 0.957518)	(0.93826; 0.966502)	(0.944783; 0.973026)
(0.931866; 0.954926)	(0.940851; 0.963911)	(0.947374; 0.970434)
(0.933411; 0.953382)	(0.942396; 0.962366)	(0.948919; 0.968889)
(0.934465; 0.952327)	(0.94345; 0.961312)	(0.949973; 0.967835)

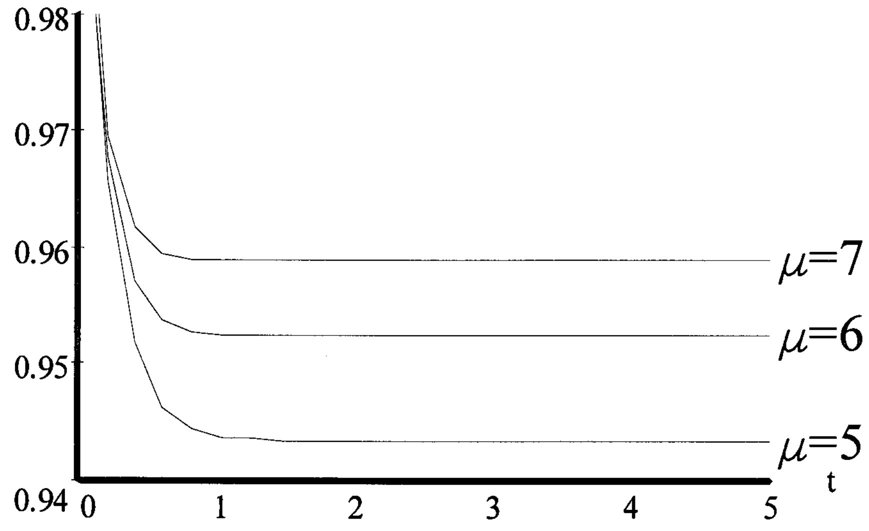

Figure 1

CHAPTER 3

A study of a two unit standby system with Erlangian repair

3.1 Introduction

It is well known that the steady state availability is a statisfactory measure for systems, which are operated continuously (e.g. detection radar system). A point estimate of steady state availability is usually the only statistic calculated, although decisions about the true steady state availability of the system should take uncertainly into account. Since

$$A_\infty = \frac{MTBF}{MTBF + MTTR}$$

the uncertainties in the values of the MTBF and MTTR reflect an uncertainty in the values of the point steady state availability.

By treating these uncertain parameters as random variables, we can obtain the distribution of point steady state availability by combining the distribution of operation and repair times. Hence we can construct estimators and confidence limit intervals for the steady state availability, which are consistent with equivalent statements on the operating time and repair time parameters. Thomson (1966) has derived techniques for placing a lower confidence limit on the system steady state availability and for deciding if the true system steady state availability differ significantly from a specified value, when MTBF and MTTR are estimated from test data.

Gray and Lewis (1967) have established the exact confidence interval for steady state availability of systems assuming that the time between failures is described by an exponential random variable and that the time to repair is described by a lognormal random variable.

Butterworth and Nikolaisen (1973) have obtained the bounds on the availability function for the general repair time distribution. Masters and Lewis (1987) have derived exact confidence limits for the system steady state availability with Gamma lifetime and lognormal repair time. Masters et al (1992) have proposed a method for establishing an exact confidence interval for steady state availability of systems when the time between failure and time to repair are independent Weibull and lognormal random variables respectively.

Abu-Salih et al (1990) have derived $100\,(1-\alpha)\,\%$ confidence limts for the steady state availability of a two unit parallel system with the assumption that the failure time distribution is exponential and the repair time has a two stage Erlangian distribution. They have also assumed that an operable unit will not fail when the other unit is in the second stage of repair. In general, the failure time and repair time are independent random variables. Chandrasekhar and Natarajan (1994a, b) have considered an n-unit parallel system with the assumption that the failure time distribution is exponential and the repair time has a two state Erlangian distribution. Further, they assumed that an operable unit can also fail while the other unit is in the second stage of repair. In particular they have derived a $100\,(1-\alpha)\,\%$ confidence interval for the steady state availability of a two unit parallel system.

A $100\,(1-\alpha)\,\%$ confidence limit for the steady state availability of a two unit cold-standby system, when the failure rate of an online unit is constant and repair time of a failed unit has an Erlangian distribution was obtained by Chandrasekhar and Natarajan (1992). Also, Chandrasekhar el at (1993) have derived a $100\,(1-\alpha)\,\%$ confidence interval for the steady state availability of a system when

(i) both the operating time and the repair distributions are lognormal and

(ii) the operating time distribution is Inverse Gaussian (IG) and the repair time distribution is lognormal.

Table 3.1 indicates the state-of-art of the earlier work in this direction on systems with several oprating time and repair time distributions.

Table 3.1

Author	System	Operating time distribution	Repair time distribution
Thompson (1966)	One unit system	Exponential	Exponential
Gray & Lewis (1967)	One unit system	Exponential	Lognormal
Butterworth & Nikolaisen (1973)	One unit system	Exponential	General
Masters & Lewis (1987)	One unit system	Gamma	Lognormal
Mohammed Abu-salih et al (1990)	Parallel system	Exponential	Erlangian
Masters et al (1992)	One unit system	Weibull	Lognormal
Chandrasekhar & Natarajan (1994a)	Standby system	Exponential	Erlangian
Chandrasekhar & Natarajan (1994b)	Parallel system	Exponential	Erlangian

In the following sections (3.2) and (3.3), $100\,(1-\alpha)\,\%$ confidence limits for the steady state and availability of two unit hot standby and warm standby systems are derived separately. The model and assumptions are discussed in the following:

3.2 Model I (Hot standby system)

3.2.1 The model and assumptions

The system under consideration is a two-unit hot standby system with a single repair facility. We have precisely the following assumptions.

(i) The units are identical and statistically independent. Each unit has a constant failure rate, say λ.

(ii) Failure rate of a unit while in standby is the same as that of the online unit.

(iii) There is only one repair facility and the repair time distribution is a two stage Erlangian distribution with p.d.f. given by

$$h(u) = \mu^2 u e^{-\mu u}; \qquad u > 0,\ \mu > 0. \tag{3.2.1}$$

(iv) Each units is new after repair.

(v) Switch is perfect and the switchover is instantaneous.

3.2.2 Analysis of the system

To analyse the behaviour of the system we note that at any time t, the system will be found in any one of the following states:

S_0 : one unit is operating online and the other kept as standby

S_1 : one unit is operating online and the other is in the first stage of repair

S_2 : one unit is operating online and the other is in the second stage of repair

S_3 : one unit is in the first stage of repair and the other is waiting for repair

S_4 : one unit is in the second stage of repair and the other is waiting for repair

Let $p_i(t)$ denote the probability that the system is in state s_i; $i = 0,\ 1,\ 2,\ 3,\ 4$ at time t. The transition probability matrix for the two unit hot standby system in the interval $(t,\ t+dt)$ may be given below

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1-2\lambda & 2\lambda & 0 & 0 & 0 \\ 0 & [1-(\lambda+\mu)] & \mu & 0 & \lambda \\ \mu & 0 & 1-(\lambda+\mu) & 0 & \lambda \\ 0 & 0 & 0 & 1-\mu & \mu \\ 0 & \mu & 0 & 0 & 1-\mu \end{bmatrix} \end{array}$$

From the transition probability matrix we readily obtain the following system of differential equations.

$$P_0'(t) = -2\lambda p_0(t) + \mu p_0(t) \quad (3.2.2)$$

$$P_1'(t) = -(\lambda+\mu) p_1(t) + 2\lambda p_0(t) + \mu p_4(t) \quad (3.2.3)$$

$$P_2'(t) = -(\lambda+\mu) p_2(t) + \mu p_1(t) \quad (3.2.4)$$

$$P_3'(t) = -\mu p_3(t) + \lambda p_1(t) \quad (3.2.5)$$

$$P_4'(t) = -\mu p_4(t) + \lambda p_2(t) + \mu p_2(t) \quad (3.2.6)$$

Solving the above equations $(3.2.2) - (3.2.6)$, and taking the limit as $t \to \infty$, i.e.

$$p_i = \lim_{t\to\infty} p_i(t); \qquad \sum_{i=0}^{4} p_i = 1$$

$$p_0 = \frac{1}{1+4\theta(1+\theta)^2} \quad (3.2.7)$$

$$p_1 = \frac{2\theta(1+\theta)}{1+4\theta(1+\theta)^2} \quad (3.2.8)$$

$$p_2 = \frac{2\theta}{1+4\theta(1+\theta)^2} \quad (3.2.9)$$

$$p_3 = \frac{2\theta^2(1+\theta)}{1+4\pi(1+\pi)^2} \quad (3.2.10)$$

where $\theta = \dfrac{\lambda}{\mu}$.

Since S_3 and S_4 correspond to the system down states, the steady state availability of the system is given by

$$\begin{aligned} A_\infty &= 1 - p_3 - p_4 \\ &= 1 - \frac{2\theta^2(3+2\theta)}{4\theta^3+8\theta^2-4\theta+1} \end{aligned} \quad (3.2.11)$$

In the following section, we obtain the $100\,(1-\alpha)\,\%$ confidence limits for the steady state availability of the system.

3.2.3 Confidence limits for the steady state availability of the system

Let $X_1,\ X_2,\ ...,\ X_n,$ be a random sample of times to failure with p.d.f.

$$f\,(x) = \lambda e^{-\lambda x};\quad x > 0,\ \lambda > 0 \tag{3.2.12}$$

Let $Y_1,\ Y_2,\ ...,\ Y_{n_2}$ be random sample of times to failure with p.d.f. given by

$$g\,(y) = \lambda e^{-\lambda y};\quad \lambda > 0,\ y > 0; \tag{3.2.13}$$

Let $Z_1,\ Z_2,\ ...,\ Z_{n_3}$ be another random sample of times to repair with p.d.f. as in (3.2.1).

It is clear that
$$E\left(\overline{X}\right) = \frac{1}{\lambda} = \left(\overline{Y}\right)$$
$$E\left(\overline{Y}\right) = \frac{1}{\mu}$$

where $\overline{X},\ \overline{Y},\ \overline{Z}$ are the sample means of time to failure while the unit is online, in standby and time to repair respectively. It can be shown that $\dfrac{n_1\overline{X}+n_2\overline{Y}}{n_1+n2}$ and $\dfrac{\overline{Z}}{2}$ are the maximum likelihood estimators (MLE) of $\dfrac{1}{\lambda}$ and $\dfrac{1}{\mu}$ respectively. The MLE of $\theta = \dfrac{\lambda}{\mu}$ is given by

$$\hat{\theta} = \frac{n_1+n_2}{n_1\overline{X}+n_2\overline{Y}}\cdot\frac{\overline{Z}}{2} \tag{3.2.14}$$

and the MLE of A_∞ is given by

$$\hat{A}_\infty = 1 - \frac{2\hat{\theta}^2\left(3+2\hat{\theta}\right)}{4\hat{\theta}^3+8\hat{\theta}^2+4\hat{\theta}+1} \tag{3.2.15}$$

It can be shown that $2\lambda\left(n_1\overline{X}+n_2\overline{Y}\right)$ and $2n_3\mu\overline{Z}$ are two independent ψ^2 variates with $2\left(n_1+n_2\right)$ and $4n_3$ degrees of freedom respectively. Now,

$$\begin{aligned} F^* &= \frac{2\lambda\left(n_1\overline{X}+n_2\overline{Y}\right)}{2\left(n_1+n_2\right)}/\frac{2n_3\mu\overline{Z}}{4n_3} \\ &= \frac{\theta}{\hat{\theta}} \end{aligned} \tag{3.2.16}$$

$\frac{\theta}{\hat{\theta}}$ has an F-distribution with $2\left(n_1+n_2\right)$ degrees of freedom. Let

$F_\alpha\left(2\left(n_1+n_2\right),\ 4n_3\right)$ represent the α-percentile of $F\left(2\left(n_1 n_2\right),\ 4n_3\right)$.

Now a $100\left(1-\alpha\right)\%$ upper confidence limit for system steady state availability is obtained as follows:

$$\begin{aligned} &1-\alpha \\ &= P\left[F^* \geq F_\alpha\left(2\left(n_1+n_2\right),\ 4n_3\right)\right] \\ &= P\left[\theta \geq \hat{\theta}F_\alpha\left(2\left(n_1+n_2\right),\ 4n_3\right)\right] \\ &= P\left[A_\infty \leq 1- \frac{\left\{2\hat{\theta}^2 F_\alpha^2\left(2\left(n_1+n_2\right),\ 4n_3\right)\right\}\left\{2\hat{\theta}F_\alpha\left(2\left(n_1+n_2\right),\ 4n_3\right)\right\}}{\left\{4\hat{\theta}F_\alpha^3\left(2\left(n_1 n_2\right),\ 4n_3\right)+8\hat{\theta}^2 F_\alpha^2\left(2\left(n_1+n_2\right),\ 4n_3\right)+4\hat{\theta}f_\alpha\left(2\left(n_1+n_2\right),\ 4n_3\right)+1\right\}}\right] \end{aligned}$$

Therefore $100\left(1-\alpha\right)\%$ upper confidence limit (UCL) for A_∞ is given by

$$UCL = 1- \frac{2\hat{\theta}^2\left\{2\hat{\theta}+3F_{1-\alpha}\left(4n_3,\ 2\left(n_1+n_2\right)\right)\right\}}{\left[4\hat{\theta}^3+8\hat{\theta}F_{1-\alpha}(4n_3, 2\left(n_1+n_2\right))+4\hat{\theta}F_{1-\alpha}^2(4n_3, 2\left(n_1+n_2\right))+F_{1-\alpha}^3(4n_3, 2\left(n_1+n_2\right))\right]} \tag{3.2.17}$$

Similarly, a $100\,(1-\alpha)\,\%$ lower confidence limit (LCL) for A_∞ is given by

$$LCL = 1 - \frac{2\hat{\theta}^2\left\{2\hat{\theta} + 3F_\alpha\left(4n_3,\ 2\left(n_1+n_2\right)\right)\right\}}{\left[4\hat{\theta}^3 + 8\hat{\theta}F_\alpha(4n_3,2\left(n_1+n_2\right)) + 4\hat{\theta}F_\alpha^2\left(4n_3,2\left(n_1+n_2\right)\right) + F_\alpha^3\left(4n_3,2\left(n_1+n_2\right)\right)\right]} \qquad (3.2.18)$$

3.3 Model II (Warm standby system)

3.3.1 Model & assumptions

The system under consideration is a two unit warm standby system with a single repair facility. The assumptions of the model are the same as in model I except that the failure rate of a unit while in standby is a constant, say v where $v < \lambda$.

3.3.2 Analysis of the system

To analyse the behaviour of the system, we note that at any time t, the system will be found in any one of the following states:

S_0 : one unit is operating online and the other kept in warm standby

S_1 : one unit is operating online and the other is in the first stage of repair

S_2 : one unit is operating online and the other is in the second stage of repair

S_3 : one unit is in the first stage of repair and the other is waiting for repair

S_4 : one unit is in the second stage of repair and the other is waiting for repair

Let $p_i\,(t)$ denote the probability that the system is in state s_i; $i = 0,\ 1,\ 2,\ 3,\ 4$ at time t. The transition probabiilty matrix for the two-unit standby system in the interval $(t,\ t+dt)$ is given by

$$P = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \left[\begin{array}{ccccc} 1-(\lambda+\nu) & (\lambda+\nu) & 0 & 0 & 0 \\ 0 & 1-(\lambda+\mu) & \mu & \lambda & 0 \\ v & 0 & 1-(\lambda+\mu) & 0 & \lambda \\ 0 & 0 & 0 & 1-\mu & \mu \\ 0 & \mu & 0 & 0 & 1-\mu \end{array}\right] \end{array}$$

From the transition probability matrix we can readily obtain the following system of differential equations.

$$P_0'(t) = -(\lambda+\nu)\,p_0(t) + \mu p_2(t) \tag{3.3.1}$$

$$P_1'(t) = -(\lambda+\mu)\,p_1(t) + (\lambda+\nu)\,p_0(t) + \mu p_4(t) \tag{3.3.2}$$

$$P_2'(t) = -(\lambda+\mu)\,p_2(t) + \mu p_1(t) \tag{3.3.3}$$

$$P_3'(t) = -\mu p_3(t) + \lambda p_1(t) \tag{3.3.4}$$

$$P_4'(t) = -\mu p_4(t) + \mu p_3(t) + \lambda p_4(t) \tag{3.3.5}$$

Solving the above equations $(3.3.1) - (3.3.5)$ for $p_i(t)$, and taking the limit as $t \to \infty$, we get

$$p_i = \lim_{t\to\infty} p_i(t)\,; \qquad \sum_{i=0}^{4} p_i = 1$$

Then

$$p_0 = \frac{\mu^3}{\left[\mu^3 + 2(\lambda+\mu)^2(\lambda+\nu)\right]} \tag{3.3.6}$$

$$p_1 = \frac{\mu(\lambda+\mu)(\lambda+\nu)}{\left[\mu^3 + 2(\lambda+\mu)^2(\lambda+\nu)\right]} \tag{3.3.7}$$

$$p_2 = \frac{\mu^2(\lambda+\nu)}{\left[\mu^3 + 2(\lambda+\mu)^2(\lambda+\nu)\right]} \tag{3.3.8}$$

$$p_3 = \frac{\lambda(\lambda+\mu)(\lambda+\nu)}{\left[\mu^3 + 2(\lambda+\mu)^2(\lambda+\nu)\right]} \tag{3.3.9}$$

$$p_4 = \frac{\lambda(\lambda+2\mu)(\lambda+\nu)}{\left[\mu^3 + 2(\lambda+\mu)^2(\lambda+\nu)\right]} \tag{3.3.10}$$

Since S_3 and S_4 correspond to the system down states, the steady state availability A_∞ is given by

$$\begin{aligned} A_\infty &= 1 - p_3 - p_4 \\ &= \frac{\mu\left[\mu^2 + (\lambda+2\mu)(\lambda+\nu)\right]}{\mu^3 + 2(\lambda+\mu)^2(\lambda+\nu)} \end{aligned} \tag{3.3.11}$$

3.3.3 Confidence limits for steady state availability of the system

Let $X_1,\ X_2,\ \ldots,\ X_n$, be a random sample of times to failure with p.d.f.

$$f(x) = \lambda e^{-\lambda x}; \quad \lambda > 0,\ x > 0$$

Let $Y_1,\ Y_2,\ ...,\ Y_{n_2}$ be another random sample of times to failure with p.d.f.

$$g\left(y\right)=\nu^{-\nu y};\quad y>0,\ v>0; \tag{3.2.12}$$

Let $Z_1,\ Z_2,\ ...,\ Z_{n_3}$ be another random sample of times to repair with p.d.f. as in (3.2.1).

We know that
$$\begin{aligned} E\left(\overline{X}\right) &= \frac{1}{\lambda} \\ E\left(\overline{Y}\right) &= \frac{1}{\nu} \\ E\left(\overline{Z}\right) &= \frac{1}{\mu} \end{aligned}$$

where $\overline{X},\ \overline{Y},\ \overline{Z}$ are the sample means of time to failure where the unit is online, in standby and time to repair respectively. It can be shown that $\overline{X},\ \overline{Y}$ and $\overline{Z}$ are the MLEs of $\frac{1}{\lambda},\ \frac{1}{\nu},\ \frac{1}{\mu}$ respectively. Let $\theta_1=\frac{1}{\lambda};\quad \theta_2=\frac{1}{\nu};\quad \theta_3=\frac{1}{\mu}$.

Clearly the steady state availability A_∞ reduces to

$$A_\infty=\frac{\theta_1^2\theta_2+\theta_3\left(\theta_1+\theta_2\right)\left(2\theta_1+\theta_2\right)}{\theta_1^2\theta_2+2\theta_3^2\left(\theta_1+\theta_2\right)\left(\theta_1+\theta_3\right)^2} \tag{3.3.14}$$

and hence the MLE of A_∞ is given by

$$\hat{A}_\infty=\frac{2\left[4\overline{X}^2\overline{Y}+\overline{Z}\left(\overline{X}+\overline{Y}\right)\left(4\overline{X}+\overline{Z}\right)\right]}{8\overline{X}^2\overline{Y}+\overline{Z}^2\left(\overline{X}+\overline{Y}\right)\left(2\overline{X}+\overline{Y}\right)^2} \tag{3.3.15}$$

It may be noted that $\hat{A}_\infty$ is a real valued continuous function in $\overline{X},\ \overline{Y}$ and $\overline{Z}$, which is also differentiable.

By an application of multivariate central limit theorem (see Rao (1974)), it follows that

$$\sqrt{n}\left[\left(\overline{X},\ \overline{Y},\ \frac{\overline{Z}}{2}\right)-(\theta_1,\ \theta_2,\ \theta_3)\right]\xrightarrow{d} N_3\,(0,\ \Sigma) \text{ as } n\to\infty,$$

where the dispertion matrix $\Sigma=(\sigma_{ij})_{3\times 3}$ is given by

$$\Sigma = diag\left(\theta_1^2,\ \theta_2^2,\ \frac{\theta_3^2}{2}\right)$$

From Rao (1974), we have

$$\sqrt{n}\left[\hat{A}_\infty - A_\infty\right]\xrightarrow{d} N_1\,(0,\ \sigma^2\,(\theta)) \text{ as } n\to\infty$$

where $\theta=(\theta_1,\ \theta_2,\ \theta_3)$

$$\begin{aligned}\sigma^2\,(\theta) &= \sum_{i=1}^{3}\left(\frac{\partial A_\infty}{\partial\theta_I}\right)^2\sigma_{ii}\\ &= \theta_1^2\left(\frac{\partial A_\infty}{\theta_1}\right)^2+\theta_2^2\left(\frac{\partial A_\infty}{\partial\theta_2}\right)^2+\frac{\theta_3^2}{2}\left(\frac{\partial A_\infty}{\partial\theta_3}\right)^2 \qquad (3.3.16)\end{aligned}$$

It can be shown that

$$\frac{\partial A_\infty}{\partial\theta_1} = \frac{2\theta_1\theta_2+\theta_3\,(4\theta_1+2\theta_2+\theta_3)}{A}-\frac{B\left[2\theta_1\theta_2+2\theta_3^2\,(\theta_1+\theta_3)\,(\theta_3+2\theta_2+3\theta_1)\right]}{A^2}$$

$$\frac{\partial A_\infty}{\partial\theta_2} = \frac{\left[\theta_1^2+\theta_3^2+2\theta_1\theta_2\right]}{A}-\frac{B\left[\theta_1^2+2\theta_3^2\,(\theta_1+\theta_3)^2\right]}{A^2}$$

$$\frac{\partial A_\infty}{\partial\theta_3} = \frac{2\,(\theta_1+\theta_2)\,(\theta_1+\theta_3)}{A}-\frac{B\,[4\theta_3\,(\theta_1+\theta_2)\,(\theta_1+\theta_3)\,(\theta_1+2\theta_3)]}{A^2}$$

where

$$\begin{aligned}A &= \theta_1^2\theta_2+2\,(\theta_1\theta_3)^2\,(\theta_1+\theta_2)^2\,(\theta_1+\theta_2)\,\theta_3^2\\ B &= \theta_1^2\theta_2+(\theta_1+\theta_2)+(\theta_1+\theta_2)\,(\theta_3+2\theta_1)\,\theta_3\end{aligned}$$

substituting the above in (3.3.16) we get $a^2(\theta)$. Hence,

$$\sqrt{n}\left[\hat{A}_{\infty} - A_{\infty}\right] \xrightarrow{d} N_1\left(0,\ \sigma^2(\theta)\right) \text{ as } n \to \infty$$

Let $\hat{\sigma}^2(\theta)$ be the estimator of $\sigma^2(\theta)$ obtained by replacing θ by the consistent estimator of $\hat{\theta}$ namely $\hat{\theta} = \left(\overline{X},\ \overline{Y},\ \frac{\overline{Z}}{2}\right)$. Hence

$$\sqrt{n}\frac{\left[\hat{A}_{\infty} - A_{\infty}\right]}{\sigma^2(\theta)} \xrightarrow{d} N(0,\ 1) \text{ as } n \to \infty$$

Since $\sigma^2(\theta)$ is a continuous function of θ, $\hat{\sigma}^2(\theta)$ is a consistent estimator of $\sigma^2(\theta)$.

$$\text{i.e. } \hat{\sigma}^2 \xrightarrow{p} \sigma^2(\theta) \text{ as } n \to \infty$$

By Slutzky's theorem, a $100(1-\alpha)\%$ confidence interval for A_{∞} is given by

$$P\left[-k_{\alpha} < \frac{\hat{A}_{\infty}}{\hat{\sigma}(\theta)} < k_{\alpha}\right] = 1 - \alpha$$

where k_{α} is obtained from normal tables, i.e. $100(1-\alpha)\%$ confidence interval is given by

$$\hat{A}_{\infty} \pm K_{\alpha}\hat{\sigma}(\theta)$$

CHAPTER 4

Introduction to inventory models

4.1 Inventory models

A storage point into and out of which commodities move or flow is termed an inventory. The inflow is characterised by replenishment from production sources and the outflow is induced by demand processes. The netflow generates a cascade of problems pertaining to the control and maintenance of inventory systems. There are innumerable factors pertaining to the functioning of an inventory system and inclusion of even a very few of them in the formulation of a model makes the model complex. Accordingly, it is quite impossible to obtain a tractable mathematical model which will truly reflect the behaviour of an inventory system. However, several nearly realistic models have been proposed and studied extensively in the past giving importance to the inherent stochastic nature of these systems. Most of thes models assume that the organisations keeping the inventories have control in determining when and in what quantity, the inventories have to be replenished but have no control over the demand process. A systematic account of the analysis of stochastic inventory systems can be found in Arrow, Karlin and Scarf (1958), Beckman (1961) and Hadley and Whitin (1963). As the study of these systems progressed over time, several reviews have appeared from time to time to highlight the state-of-art (see, for example Aggarwal (1974), Nahmias (1978), Silver (1981) and Raafat (1991).

4.2 Types of inventory models

The various models of stochastic analysis of inventory systems are broadly classified into two types, namely, periodic review systems and continuous review systems. In periodic review systems, the state of the system is examined only at equally spaced time points and decisions such as placing of orders and the quantity to be added to the inventory are made only at these review points. On the other hand, in continuous review systems, all events associated with the time evolution of the inventory are recorded and the stock level is reviewed continuously at the occurrence of each demand for the product in the inventory. Continuous review systems have occupied a wider scope for applications since failure of review of the inventory level even at a single time point may prove a disastrous result for an organsation such as military, industry, etc. Further, classification of inventory systems is made as single product inventory systems and multi-product inventory systems according to the inventory stocks single product or varieties of products.

4.3 Single product inventory systems

Several models for single product inventories have been proposed, optimal ordering policies developed and studied extensively in the past by several researches both for periodic and continuous review cases (for example, see Beckmann (1961), Dirickx and Kolvocts (1977), Wijingaard and Winkel (1979), Kalpakam and Arivarignan (1985, 1988, 994), Horrowitz and Daganso (1986), Beckman and Srinivasan (1987), Ramanarayanan and Jacob (1987), Ravichandran (1988), Weiss (1988), Srinivasan (1989), Krishnamoorthy and Laxmy (1990), Krishnamoorthy and Manoharan (1990) and Kalpakam and Sapna (1996)).

4.4 Multi-product inventory systems

We have many real life situations in which multi-product inventories are required. For example, a pharmacist keeps a number of medicines of different brands, a ready-made clothes shop keeps dresses of different designs in different colours and in different sizes, a shoe store stocks shoes of various models and sizes. Hence, the study of a multi-product inventory model has drawn special attention recently. Page and Paul (1976), Chakravarthy (1981), Holt Albert (1986), Sung and Chang (1986), Oneiva and Larraneta (1987), Aksoy and Erengue (1988), Amiya and Martin (1988), Goyal (1988), Hall Nicholas (1988) and Correnu (1990) have analysed multi-product inventory systems. An inventory problem for two slow-moving substitutable products have been studied by Gürler & Kara (1996), Pasternak & Drezner (1991) and Anabazhazan (2002).

4.4.1 Ordering policies

In a multi-product inventory system, the inventory control policies, the nature of demands may be different from that of a single product system.

First we consider inventory control policies. The inventory of each product may be controllable independently or there may exist an interaction among the items in some manner and a joint control of the inventory may be required. For example, a demand for a tyre of a two-wheeler will not affect the demand for a tyre of a truck even though both may be available with the same dealer. Inventory of such items can be controlled individually. Hence, we may have the following two types of reordering policies for the control of the inventory of the products:

(i) "INDIVIDUAL ORDERING POLICY" under which each item is ordered according to its own single-item policy, or

(ii) "JOINT ORDERING POLICY" under which whenever a product is ordered, every other product is also ordered along with it, irrespective of its inventory level. That is, whenever a replenishment occurs, every product is brought up to a specified inventory level.

4.4.2 Demand interaction

Next, we consider the nature of demands. A demand may be for one product or several products. For example, consider the case of a new car-seller whose inventory consists of, apart from new cars, many car accessories like car-air-conditioners, car-fans, fancy lamps, maintenance kits, etc., and the buyer has the option to take one or more of these accessories.

It is also possible that a demand for a particular product during its stock-out period may be substituted with another similar product in the inventory. Examples of such products having at least partial substitutability include:

(i) Consumer products such as different brands of toothpastes and different varieties of food items;

(ii) Hardware products such as different brands of paints and containers of different sizes of the same brands;

(iii) Dresses of the same design and same brand but in different colours;

(iv) Fluorescent light bulbs of different makes and ceiling fans of different brands.

When this type of interaction occurs, large stock of a particular product can be avoided, as it is substitutable. Further, the available total storage space for the inventories of these products can be shared optimally so as to reduce the loss of demands. Kamat (1971) has studeid substitutability of demands by considering a two substitutable product inventory model with a prescribed order period and obtained a cost function. McGillivray and Silver (1978) have investigated the effect of substitutable demands on stocking control rules and developed a heuristic approach for establishing the value of control parameters (the order up to levels) for the case of two products. Parlar and Goyal (1984) considered a model of two substitutable products as an extension of the classical single period newsboy problem. They have shown that the optimal order quantities can be found for each product by maximising the expected profit function which is strictly concave for a wide range of parameter values. Parlar (1988) has used game-theoretic concepts (two-person continuous game) to analyse an inventory problem with two substitutable products having random demands.

4.5 Techniques used in the study of inventory models

4.5.1 Renewal theory

One of the important types of stochastic processes is the renewal process. Outstanding contributions have been made by several researchers in the theory of renewal processes (see, for example, Feller (1941), Cox and Smith (1954), Smith (1958) and Neuts (1978). A systematic account of renewal theory and its applications to diversified fields can be found in Cox (1962), Parzen (1962), Prabhu (1965), Feller (1968) and Medhi (1994).

4.5.1.1 Definition

Let $\{X_n;\ \ n = 1,\ 2,\ ...\}$ be a collection of non-negative random variables which are independent and identically distributed. Then $\{X_n\}$ is called a renewal process.

We assume that each of the random variables X_i has a finite mean μ. A renewal process is completely determined by $f(\cdot)$, the p.d.f. of X_i. Let

$$\begin{aligned} S_0 &= 0 \\ S_n &= X_1X_2 + ... + X_n,\ \ n = 1,\ 2,\ ... \\ N(t) &= \max\{n : S_n \leq t\},\ t > 0 \end{aligned}$$

Then $N(t)$ is called the number of renewals upto time t. The expected value of $N(t)$ namely $E(N(t))$ is called the renewal function and is denoted by $H(t)$. The derivative $H(t)$, whenever it exists, is called the renewal density and is denoted by $h(t)$.

4.5.1.2 Renewal equation

The quantity $h(t)\,dt$ has the probabilistic interpretation that it denotes the probability that a renweal occurs in the interval $(t,\ t + dt)$. Since this renewal may be either the first or the subsequent renewal, the function $h(t)$ satisfies the equation:

$$h(t) = f(t) + \int_0^t h(u) f(t-u)\, du.$$

This equation is called the renewal equation.

4.5.1.3 Key renewal theorem

Let $Q(t)$ be non-negative and non-increasing for $t > 0$ such that $\int_0^t Q(t)\, dt < \infty$. Then

$$\lim_{t\to\infty} \int_0^t Q(t-x)\, dH(x) = \frac{1}{\mu} \int_0^\infty Q(u)\, du,$$

where $\mu = E(X_i)$.

4.5.2 Markov renewal processes

These stochastic processes are generalizations of renewal processes and have become indispensable in inventory applications. A systematic and deep study of them can be found in Pyke (1961a, b), Cinlar (1975a, b) and Medhi (1994).

Let E be a finite set, N the set of non-negative integers and $\mathbb{R}_+ = [0, +\infty)$. Suppose we have, in a probability space $(\Omega,\ X,\ P)$, random variables $X_n : \Omega \to E,\ T_n : \Omega \to \mathbb{R}_+$ defined for each $n \in N$, so that $0 = T_0 \le T_1 \le T_2 \le ...$

Definition 1. The stochastic process $(X,\ T) = \{X_n,\ T_n,\ n \in N\}$ is said to be a Markov renewal process with state space E provided that $P\{X_{n+1} = j,\ T_{n+1} - T_n \le t/X_0,\ ...,\ X_n;\ T_0,\ ...,\ T_n\} = P\{X_{n+1}\} = j,\ T_{n+1} - T_n \le t/X_n$ for all $n \in N,\ j \in E$ and $t \in \mathbb{R}_+$.

We assume that $(X,\ T)$ is time homogeneous: that is, for and $i,\ j \in E$, and $t \in \mathbb{R}_+,\ P\{X_{n+1} = j,\ T_{n+1} - T_n \le t/X_n = i\} = Q(i,\ j,\ t)$ is independent of

n. The family of probabilities $Q = \{Q(i,\ j,\ t) : i,\ j \in E,\ t \in \mathbb{R}_{+}\}$ is called a semi-Markov kernel over E. We assume that $Q(i,\ j,\ 0)$ for all $i,\ j$ in E.

For each pair $(i,\ j)$, the function $t \rightarrow Q(i,\ j,\ t)$ has all the properties of a distribution function except that:

$$P(i,\ j) = \lim_{t \to \infty} Q(i,\ j,\ t)$$

is not necessarily 1. It is easy to see that

$$P(i,\ j) \geq 0, \qquad \sum_{j \in E} P(i,\ j) = 1;$$

that is, the $P(i,.\ j)$ are the transition probabilities for some Markov chain with state space E. It follows from the definition 1 and the above that

$$P\{X_{n+1} = j|\, X_0,\ \ldots,\ X_n;\ \ T_0,\ \ldots,\ T_n\} = P(X_n,\ j) \text{ for all } n \in N,\ j \in E.$$

This implies that $X = \{X_n;\ n \in N\}$ is a Markov chain with state space E and transition matrix P.

We write $P_i\{A\}$ for the conditional probability $P\{A|\, X_0 = i\}$ and similarly $E_i(X)$ for the conditional expectation of X given $\{X_0 = i\}$. We also assume that $P_i\{T_0 = T_1 = T_2 = \ldots 0\}$. We define

$Q^n(i,\ j,\ t) = P_i\{X_n = j,\ T_n \leq t\}$, $I,\ j \in E,\ t \in \mathbb{R}_{+}$, for all $n \in N$. Then

$$Q^0(i,\ j,\ t) = \sigma_{ij} = \begin{matrix} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{matrix}$$

for all $t \geq 0$; and, for all $n \geq 0$, we have the recursive relation.

$$Q^{n+1}(i,\ k,\ t) = \sum_{j\in E} \int_0^t Q(i,\ j,\ du)\, Q^n(j,\ k,\ t-u),$$

where the investigation is over $[0,\ t]$.

The expression $R(i,\ j,\ t)$ which gives the expected number of renewals of the position j in the interval $[0,\ t]$ is given by

$$R(i,\ j,\ t) = \sum_{n=0}^{\infty} Q^n(i,\ j,\ t).$$

This is finite for any $i,\ j \in N$ and $t < \infty$. The $R(i,\ j,\ \cdot)$ are called Markov renewal functions and the collection $R = \{R(i,\ j,\ t) : i,\ j \in E,\ t \in \mathbb{R}_+\}$ of these functions is called the Markov renewal kernel corresponding to Q. We note that for fixed $i,\ j \in E$ the function $t \to R(i,\ j,\ t)$ is a renewal function.

We can easily see from the various expressions above that $R_\alpha = (I = Q_\alpha)^{-1}$, where I is the unit matrix and

$$R_\alpha(i,\ j) = \int_0^{\infty} e^{-\alpha t} R(i,\ j)\, dt, \quad \alpha > 0$$

The class B of functions which we will be working with, is the set of all functions $f : E \times \mathbb{R}_+ \to \mathbb{R}$ such that for every $i \in E$, the function $t \to f(i,\ t)$ is Borel measurable and bounded over finite intervals, and for every fixed $j \in E$, the functions $(i,\ j) \to Q^n(i,\ j,\ t)$ and $(i,\ j) \to R(i,\ j,\ t)$ both belong to B.

For any function $f \in B$, the function $Q©f$ defined by:

$$Q©f(i,\ t) = \sum_{j\in E} \int_0^t Q(i,\ j,\ ds)\, f(j,\ t-s)$$

is well defined and $Q©f$ belongs to B again. Hence, the operation can be repeated, and the n-th iterate is given by

$$Q^n \copyright f(i,\ t) = \sum_{j\in E} \int_0^t Q^n(i,\ j,\ ds)\, f(j,\ t-s)\,.$$

We can replace Q by R and note that $R\copyright f$ is again a well defined function; that is, for $f \in B$,

$$R\copyright f = \sum_{j\in E} \int_0^t R(i,\ j,\ ds)\, f(j,\ t-s)\,.$$

A function $f \in B$ is said to satisfy a Markov renewal equation if for all $i \in E$ and $t \in \mathbb{R}_+$,

$$f(i,\ t) = g(i,\ t) + \sum_{j\in E} \int_0^t Q(i,\ j,\ ds)\, f(j,\ t-s)$$

for some function $f \in B$.

Limiting ourselves to functions $f,\ g \in B$ which are non-negative, and denoting this set by B_+, the Markov renewal equation now becomes $f = g + Q\copyright f,\ f,\ g \in B_+$.

This Markov renewal equation has a solution $R\copyright g$. Every solution f is of the form $f = R\copyright g + h$ where h satisfies

$$h = Q\copyright h,\ h \in B_+.$$

4.5.3 Semi-Markov processes

Let $(x,\ T)$ be a Markov renewal process with state space E and semi-Markov kernel Q. Define $L = T_n$. Then L is the lifetime of $(X,\ T)$. If E is finite or if X is irreducible recurrent, then $L = +\infty$ almost surely. By weeding out those $w \in \Omega$ for which $T_n(w) < \infty$, we assume that $T_n(w) = +\infty$ for all w. Then for any

$w \in \Omega$ and $t \in \mathbb{R}_+$, there is some integer $n \in N$ such that $T_n(w) \leq t \leq T_{n+1}(w)$. We can therefore define a continuous time parameter $Y = (Y_t)\, t \in \mathbb{R}$ with state space E by putting $Y_t = X_n$ on $\{T_n \leq t < T_{n+1}\}$. The process $Y = (Y_t)\, t \in \mathbb{R}_+ Sv$ defined is called a semi-Markov process with state space E and semi-Markov transition kernel $Q = \{Q(i,\ j,\ t)\}$.

4.5.4 Semi-regenerative processes

Let a stochastic process $Z = (Z_t)\, t \in \mathbb{R}_+$ be a stochastic process with a topological state space F, and suppose that the function $t \rightarrow Z(w)$ is right continuous and has left-hand limits for almost all $w \in \Omega$.

A random variable $T : \Omega \rightarrow [0,\ \infty]$ is called a stopping time for Z provided that, for any $t \in \mathbb{R}_+$, the occurrence or non-occurrence of the event $\{T \leq t\}$ can be determined once the history $H_t = \sigma(Z_u;\ u \leq t)$ of Z before t is known. If t is the stopping time of Z, then we denote by H_T the history of Z before T.

The process $Z = \{Z_t;\ t \geq 0\}$ is called regenerative if there exists a sequence $S_0,\ S_1,\ S_2,\ ...,$ of stopping times such that $(a)\, S = \{S_n;\ n \in N\}$ is a renewal process (b) for any $n,\ m \in N,\ t_1,\ t_2,\ ...,\ t_n \in \mathbb{R}_+$ and any bounded function f defined on E^n. $E[f(Z_{s_m} + t_1,\ Z_{s_m} + t_2,\ ...,\ Z_{s_m} + t_n / Z_u;\ U \leq S_m]] = E[f(Z_{t_1},\ Z_{t_2},\ ...,\ Z_{t_n})]$.

Definition: Let $Z = (Z_t)\, t \in \mathbb{R}_+$ be a stochastic process with a topological state space F, and suppose that the function $t \rightarrow Z(w)$ is right continuous and has left hand limits for almost all w. The process Z is said to be semi-regenerative if there exists a Markov renewal process $(X,\ T)$ with infinite lifetime satisfying the following:

(i) for each $n \in N,\ T_n$ is a stopping time for Z;

(ii) for each $n \in N,\ _n$ is determined by $\{Z_u;\ u \leq T_n\}$;

(iii) for each $n \in N,\ m \geq 1,\ 0 \leq t_1 < t_2 < ... < t_m$, and function f defined on F^m,

$$E_I\left[f\left(Z_{Tn+t_1},\ Z_{Tn+t_2},\ ...,\ Z_{Tn+t_m} \middle| Z_u;\ U \leq T_m\right)\right] = E_j\left[f\left(Z_{t_1},\ Z_{t_2},\ ...,\ Z_{t_m}\right)\right] \text{ on } \{X_n = j\}.$$

In this definition E_i and E_j refer to the expectations given the initial state for the Markov chain X.

Detailed treatments of SMP and MRP can be found in Pyke (1961a, 1961b), Levy (1954), Cinlar (1975b) and Ross (1970). The survey of Cinlar (1975b) demonstrates the usefulness of the theory of MRP and SMP in applications.

4.5.5 Stochastic point process

Stochastic point processes for a class of processes more general than those considered in the previous sections. Since point processes have been studied by many with varying backgrounds, there have been several definitions of the point processes each appearing quite natural from the veiw point of the particular problem under study (see, for example, Bhabha (1950), Khintchine (1960), Harris (1963) and Bartlett (1966). A stochastic point process is the mathematical abstraction which arises from considering such phenomena as a randomly located population or a sequence of events in time. Typically, there is envisaged a state space X and a set of points X_n from X representing the locations of the different members of the population or the times at which the event occur. Because a realization (or sample path) of any of these phenomena is just a set of points in time or space, a family of such realizations has come to be called a point process (Daley and Vere-Jones (1971)).

A comprehensive definition of point processes is due to Moyal (1962) who deals with such processes is an general space which is not necessarily Euclidean. Consider a set of objects each of whose state is described by a point x of a fixed set X of points.

Such a collection of objects which we may call a population may be stochastic if there exists a well defined probability distribution P on some σ field B of subsets of the space ϕ of all states. We shall assume that members of the population are indistinguishable from one another. The state of the population is defined as an unordered set $x^n = \{x_1,\ x_2, ...,\ x_n\}$ representing the situation where the population has n members with one each in the states $x_1,\ x_2,\ ...,\ x_n$. Thus, the population state space ϕ is the collection of all x^n with $n = 0,\ 1,\ 2,\ ...$, where x^0 denotes the empty population. A point process is defined to be the triplet $(\phi,\ B,\ P)$. For a detailed treatment of stochastic point processes with special reference to their applications, the reader is referred to Srinivasan (1974). A point process is called a regular point process if the probability of occurrence of more than one event in $(0,\ \Delta)$, where Δ is small.

4.5.5.1 Product densities

One of the ways of characterising a general stochastic point process is through product densities (Ramakrishnan (1950, 1958), Srinivasan (1974)). These densities are analogous to the renewal density in the case of non-renewal processes.

Let $N(t,\ x)$ denote the random variable representing the number of events in the interval $(t,\ t+x)$, $d_x N(t,\ x)$ the events in the interval $(t+x,\ t+x+dx)$ and $p(n,\ t,\ x) = \Pr\{N(t,\ x) = n\}$. The product density of order n is defined as:

$$h_n\left(x_1,\ x_2,\ ...,\ x_n = \lim_{\Delta_1,\Delta_2...\Delta_n \to 0} \frac{\Pr\{N(x_i,\ \Delta_j) \geq 1,\ i = 1,\ y^n\}}{\Delta_1, \Delta_2...\Delta_n}\right);$$

$$x_1 \neq x_2 \neq: ... \neq x_n,$$

or equivalently for a regular process

$$h_n(x_1,\ x_2,\ ...,\ x_n) = \lim_{\Delta_1,\Delta_2...\Delta_n \to 0} \frac{E\left[\prod_{i=1}^{n} N(x_i,\ \Delta_i)\right]}{\Delta_1, \Delta_2...\Delta_n};\ x_1 \neq x_2 \neq ... \neq x_n.$$

These densities represent the probability of an event in each of the intervals $(x_1,\ x_2,+\Delta x_1)$, $(x_2,\ x_2+\Delta x_2)$, ..., $(x_n,\ x_n+\Delta x_n)$. Even though the functions $h_n(x_1,\ ...,\ x_n)$ are called densities it is important to note that their integration will not give possibilities but will yield the factorial moments. The ordinary moments can be obtained by relaxing the condition that all the x_i are different.

4.6 Measures of system performance

In this section some of the important measures of inventory systems are explained.

Let $I(t)$ be the inventory level at time t and S be the maximum capacity of the inventory. Then the net inventory level distribution $P(I,\ t|\,k)$ at any time t is given by:

$$P(i,\ t|\,k) = \Pr\{I(t) = i|\,(0) = k\};\ \ I,\ k = 0,\ 1,\ ...,\ S$$

The limiting distribution $P(i)$ (if it exists) is defined as $P(i) = \lim_{t\to\infty} P(i,\ t|\,k)$

For a two product system let the state of the system be represented by the ordered pairs $x(t)$, $Y(t)$, where $X(t)$ is the inventory level of product 1 and $Y(t)$ is the inventory level of product 2. Then, the inventory level distribution $P(i,\ j,\ t|\,k,\ 1)$ at time t is given by:

$$P(i,\ j,\ t|\,k,\ 1) = \Pr\{x(t),\ Y(t) = (i,\ j)|\,x(0),\ Y(0) = (k,\ 1)\},\ i,\ k = 0,\ 1,\ 2,\ ...,\ S_1;\ j,\ 1 = 0,\ 1,\ 2,\ ...s_2,$$

Where S_1 and S_2 are the maximum inventory levels of product 1 and product 2 respectively.

The limiting distribution $P(i,\ j)$ (if it exists) is defined as:

$$P(i,\ j) = \lim_{t\to\infty} P(i,\ j,\ t|\, k,\ 1)$$

The expected stock on hand or mean inventory level $E(L)$, at any time for a single product system in the steady state is given by:

$$E(L) = \sum_{i=0}^{S} iP(i)$$

In an inventory model, apart from the distribution of the inventory level, the mean number of reorders placed, replenishments made, demands satisfied, demands lost in an arbitrary interval of time are also some of the important measures.

In the context of a multi-product system allowing substitutablility, the number of demands for a particular product satisfied by a different product deserves consideration. The stationary rate of these events are used in the cost analysis of the system. To find these measures we follow the procedure given below.

Let $N(\eta,\ t)$ denote the number of a specific type of event η (like reorders, replenishment, demand for a product satisfied by the same product, demand for a product satisfied by the other product, demands lost etc.) in $(0,\ t)$. Then, the expected number of η events in $(0,\ t)$ is given by:

$$E[N(\eta,\ t)] = \int_0^t h(u)\, du,$$

Where $h(u)$ is the first order product density corresponding to the event under consideration. In the long run, the stationary rate of η events is given by:

$$E(\eta) = \lim_{t\to\infty} \frac{E(N)(\eta,\, t)}{t} = \lim_{t\to\infty} h(t)$$

4.7 Cost analysis

4.7.1 Inventory related costs

We consider the following costs in the analysis of the inventory models:

(i) Holding costs: This not only includes the expenses incurred for storage facilities but also the amount invested could have earned a return elsewhere. This cost at any time depends upon the stock on hand.

(ii) Reordering costs: When the stock in hand comes down to a level where a reorder is necessary, a reorder is placed. This involves additional expenditure such as the expenses on the paper work, inspection and the material handling costs.

(iii) Cost for demands lost: When a demand is not met and not also back-ordered, the profit that would have come is lost, together with the goodwill.

(iv) Procurement cost: This is the cost at which the items are bought either from a manufacturer or from the market. Most inventory control procedures recognise price fluctuations, and so they are treated in this thesis.

4.7.2 Cost optimisation

There are a number of objectives that may be sought by inventory managers. These usually involve the minimization (maximization) of costs (profit) function which could be either discounted or undiscounted. The planning period or horizon may be finite or infinite. In stochastic models the mean value of costs are measured and the criterion consists in the minimization of the total expected cost per unit time or of the expected discount cost over a finite or infinite horizon. The cost function will, in general, consist of the additive contribution of the procurement cost, the holding cost and the storage cost.

Under the $(s,\ S)$ policy, the objective function will, in general, be expressible as a function of two variables s and S. The resultant optimization problem consists in determining the optimal values of s and S to achieve the selected criterion. For a multi-product system the maximum inventory levels of the various products and the reorder levels can be considered as variables for optimization.

In this connection, it should be pointed out that there are two distinct approaches in formulating and solving the stochastic inventory problems both in theory and in practice. In the first approach the system is viewed as a multi-stage decision process and the technique of dynamic programming is employed in finding the optimal policy that minimises the total expected cost over the duration of the process. The following second approach known as stationary approach is often used when the duration of the process is infinite; an ordering policy of a given type is chosen and the stationary behaviour of the inventory levels is analysed without any reference to the cost structure of the problem. Such entities as the expected frequencies of orders and the expected quantity on hand etc. are computed. A cost structure is then imposed on the system and the stationary total expected cost rate for operating the inventory system is minimized. In this thesis, the stationary approach is adopted for optimal analysis.

If $C(t)$ represents the total cost in $(0,\ t)$, then the expected cost rate, $E(C)$, is given by:

$$E(C) = \lim_{t \to \infty} E(C(t))/t.$$

CHAPTER 5

Stochastic model of a two-product inventory system with product interaction

1. A modified version of this chapter has been published in "Management Dynamics", Vol. 10, No. 3, pp. 29-39, 2001.
2. One more paper from this chapter has been communicated to Journal of Industrial Engineering.

5.1 Introduction

A common phenomenon that occurs in the maintenance of multi-product inventory systems, is that of product interaction. The demand for a particular product may not be satisfied even though the product is available, for the simple reason that another complementary product is unavailable. For instance, every automobile vehicle is supplied with maintenance kits and some essential spare parts when it is sold. The non-availability of the maintenance kit or some of the spares may prevent the sale of the vehicle. Again, similar type of problems occur in multi-product inventory systems where the end-product is an assembly of several components.

Separate inventory systems for each component is maintained and controlled. These type inventories are abundant. For example, a dispensing chemist stores several chemicals and, upon receiving the prescription, prepares the compound mixture and then dispenses it to the customer. The non-availability of one of the required chemicals may delay the supply process. As another example, a cycle manufacturer needs a wheel rim, hub and spokes to assemble a wheel and again the non-availability of even one component may inhibit the production of cycles.

In these multi-product inventory systems, equal inventories cannot be maintained for the sub-products due to various costs arising from storage, procurement and stock-out, lead-times, holding costs, opportunity costs and replenishment costs which may be different depending on the nature of the sub-products. This necessitates the adoption of an optimal ordering policy for such an inventory system.

Schmidt and Nahmias (1985) studied a two-product inventory problem where the end-product is assembled from two components. However, the paper confines itself to the examination of periodic review systems. Yano (1987) considered a two-part assembly system where the procurement lead-times of both the parts are stochastic. In this chapter, the author ascertains planned lead-times for procurement and for assembly production so that the total inventory holding costs for parts and the assembled product, and the tardiness costs for the assembled product, are minimized. Kumar (1989) has also studied a multi-part assembly system with stochastic lead-times for the parts and determines the re-order points of each part in order for the total cost to be minimised. The probability distribution for inventory level and mean reorder for a single commodity continuous review inventory system has been obtained by Perumal et al (2001), Elango & Arivrignan (2001).

Fujiwara and Sedarage (1997) have considered an EOQ-type model for a production system, where a number of parts are acquired to produce a single product and where the lead-times are random. The author's objective is to determine when to order each part and what lot size to produce so that the average total cost per unit time is minimised. However, attention needs to be focussed on the study of multi-product continuous review inventory systems with product interaction.

In this chapter, an attempt is made to fill the gap by presenting a model of a multi-product continuous-review inventory system with product interaction. For simplicity, we assume that two sub-products are assembled together to produce an end-product instantaneously for which demands from outside occur. The demands for the end-products occur according to a Poisson process which is a counting process and can be satisfied only if both the products are available in the inventory. Back-orders are not permitted. The two sub-products are brought from outside and are replenished according to a $(s,\ S)$ policy.

The chapter is organized as follows:

In Section 5.2, the analysis of the model in which the lead-times for the two products are exponentially distributed because of the Markovian nature of the distribution is presented. Numerical results are illustrated in Section 5.3.

5.2 System description

5.2.1 Assumptions & notation

A two-product continuous-review assembly inventory system with the following assumptions and notations are considered:

1. The maximum inventory level of a product i is $S_i,\ i = 1,\ 2$
2. The re-ordering policy for the product i is $(s_i,\ S_i)$, where $S_i > 2s_i,\ i = 1,\ 2$
3. The lead-times of the products are independent and are exponentially distributed with parameters μ_1 and μ_2.
4. The demands for the end-products occur according to a Poisson process with parameter $\lambda,\ \lambda > 0$.
5. Backlogging is not permitted.

R_i : Event that a re-order is placed for product i; $i = 1,\ 2$.

ℓ : Event that a demand is lost.

d : Event that a demand is satisfied.

$N_d(t)$: Random variable representing the number of demands satisfied in the interval, $(0,\ t]$.

$L_i(t)$: The inventory level of the product i at any time $t,\ t \geq 0$; $i = 1,\ 2$.

$Z(t)$: $(L_1(t),\ L_2(t))$, the vector process representing the state of the system at time t.

E_0 : Event that denotes the initial condition that a reorder is placed for Product 1 and the inventory level of product 2 is j_0; $j_0 = 1,\ 2,\ ...,\ s_2,\ ...,\ S_2$.

$$P_{i,j}(t) = P[Z(t) = (i,\ j)|\, E_0]; \quad \begin{matrix} i = 0,\ 1,\ ...,\ s_1,\ ...,\ S_1 \\ j = 0,\ 1,\ ...,\ s_2,\ ...,\ S_2 \end{matrix}$$

© Convolution symbol

$\overline{G}(t) = 1 - G(t)$

$$P_{i,j}(t) = P[Z(t) = (i,\ j)|\, E_0]; \quad \begin{matrix} i = 0,\ 1,\ ...,\ s_1,\ ...,\ S_1 \\ j = 0,\ 1,\ ...,\ s_2,\ ...,\ S_2 \end{matrix}$$

5.2.2 The inventory level distribution

The lead-times of the products are independent and exponentially distributed, and the demands occur according to a Poisson process. Consequentially, the stochastic process $\{Z(t),\ t \geq 0\}$, representing the inventory level at any time is a Markov process. Since a demand can only be satisfied when $L_1(t) > 0$ and $L_2(t) > 0$, and a replenishment for product 1 can only occur when $0 \leq L_1(t) \leq s_1$, and for product 2 when $0 \leq L_2(t) \leq s_2$, by using standard probabilistic arguments, the following system of differential-difference equations satisfied by $P_{i,j}(t)$, is obtained:

$$\begin{aligned} P'_{i,j}(t) = &\ -\{\lambda H(i-1) H(j-1) + \mu_1 H(s_1 - i) + \mu_2 H(s_2 - j)\} P_{i,j}(t) \\ &+\lambda H(S_1 - i - 1) H(S_2 - j - 1) P_{i+1,j+1}(t) \\ &+\mu_1 H(i - S_1 + s_1) H(S_2 - j) P_{i-S_1+s_1,j}(t) \\ &+\mu_2 H(S_1 - i) H(j - S_2 + s_2) P_{i,j-S_2+s_2} \end{aligned} \tag{5.1}$$

where $i = 0,\ 1,\ 2,\ ...,\ s_1,\ ...,\ S_1$ and $j = 0,\ 1,\ ...,\ s_2,\ ...,\ S_2$ and where, $H(i)$ is a Heaviside function, i.e.

$$H(i) = \begin{cases} 1 & \text{if } i > 0 \\ 0 & \text{otherwise} \end{cases}$$

Letting $P(t)$ denote the $1 \text{ X } (S_1 + 1)(S_2 + 1)$ order row vector

$(P_{0,0}(t),\ P_{1,0}(t),\ ...,P_{S_1,S_2}(t))$ the system of equations (5.1) can be written in the following matrix form

$$P'(t) = P(t)\,Q \tag{5.2}$$

where Q represents the $(S_1+1)(S_2+1)\ X\ (S_1+1)(S_2+1)$ order coefficient matrix of (5.1). Since Q is independent of time t equation (5.2) can be solved and

$$P(t) = P(0)\,e^{Qt}$$

is obtained where $P(0)$ is the vector of initial state probabilities.

5.2.3 Steady-state results

The stationary distribution of the Markov process $\{Z(t),\ t \geq 0\}$ is defined by

$$P_{i,j} = \lim_{t\to\infty} P_{i,j}(t)$$

Denoting $P = (P_{0,0};\ P_{0,1},\ ...,\ P_{S_1,S_2})$, we observe from the equation (5.2) it is observed that P is the solution of the system of homogeneous linear equations $P.$ $Q = 0$, subject to the total probability constraint $P.\ \rho = 1$, where $\rho = (1,\ 1,\ ...,\ 1)^T$.

5.2.4 Mean number of lost demands

To derive an expression for the mean number of lost demands, the first-order product density of ℓ-events defined by

$$h_\ell(t) = \lim_{\Delta\to 0} \frac{P[\ell-\text{ event in } (t,\, t+\Delta)|\, E_0]}{\Delta}$$

is considered.

It is observed that a demand is lost even if one product is not available in the inventory, and hence we obtain

$$h_\ell(t) = \sum_{i=0}^{s_1} P_{i,0}(t)\,\lambda + \sum_{j=0}^{S_2} P_{0,j}(t)\,\lambda.$$

Now, the mean number of lost demands in the interval $(0, t]$ is given by

$$\int_0^t h_\ell(u)\,du,$$

and hence the stationary mean rate of lost demands is given by

$$\begin{aligned} E(\ell) &= \lim_{t\to\infty} \frac{1}{t}\int_0^t h_\ell(u)\,du = \lim_{t\to\infty} h_\ell(t) \\ &= \lambda \sum_{i=0}^{S_1} P_{i,0} + \lambda \sum_{j=0}^{S_2} P_{0,j} \end{aligned}$$

5.2.5 Mean number of re-orders

The first order product densities of

$$h_{R_i}(t) = \lim_{\Delta\to 0} \frac{P\left[R_i - \text{ event in } (t,\ t+\Delta)|\, E_0\right]}{\Delta},\ i = 1,\ 2.$$

Since a re-order is placed whenever the inventory level of the product enters s_i the following is obtained:

$$\begin{aligned} h_{R_1}(t) &= \sum_{j=1}^{S_2} P_{s_1+1,j}(t) \\ h_{R_2}(t) &= \sum_{i=1}^{S_1} P_{i,S_2+1,j}(t) \end{aligned}$$

Therefore, the mean stationary rates of R_i-events are given by

$$\begin{aligned} E(R_1) &= \sum_{j=1}^{S_2} P_{s_1+1,j}\lambda \\ E(R_2) &= \sum_{i=1}^{S_1} P_{i,s_2+1}\lambda \end{aligned}$$

5.2.6 Mean number of demands

The first-order product density of the d-events is defined as

$$h_d(t) = \lim_{\Delta \to 0} \frac{P[d - \text{ event in } (t,\ t+\Delta) |\, E_0]}{\Delta}$$

Since a demand can be satisfied when both the sub-products are available, we have:

$$h_d(t) = \sum_{i=1}^{S_1} \sum_{j=1}^{S_2} P_{i,j}(t)\, \lambda$$

Consequently, the mean stationary rate of demands satisfied is given by

$$E(d) = \sum_{i=1}^{S_1} \sum_{j=1}^{S_2} P_{i,j} \lambda$$

5.2.7 Cost analysis

Maintenance of an inventory necessarily involves several costs and these costs are intrinsic in nature, subject to which the optimal re-order levels of the inventory has to be decided. The analysis of cost plays a central role in the study of inventory systems. Here the following are considered:

C_{R_i} : the cost corresponding to the re-order of the product $i = 1,\ 2$;
C_{ℓ}: the cost associated with a lost demand;
C_{Q_i} : the purchasing cost of product i;
C_i : the selling price of the end product;
C_T : the selling price of the end product.

Since the total cost $C(S_1,\ S_2,\ s_1,\ s_2)$ is made up of the costs relating to re-ordering, lost demands, holding of sub-products in the inventory, and replenishment, the following results:

$$C(S_1,\,S_2,\,s_1,\,s_2) \;=\; \sum_{i=1}^{2} C_{R_i}E(R_i) + C_\ell E(\ell) + C_1\sum_{i=0}^{S_1}\sum_{j=0}^{S_2} iP_{i,j} + C_2\sum_{i=0}^{S_1}\sum_{j=0}^{S_2} jP_{i,j}$$

$$+\sum_{i=1}^{2} E(R_i)\,C_{Q_i}(S_i - s_i)$$

Then the profit function $PROFIT(S_1,\,S_2,\,s_1,\,s_2)$ is given by

$$PROFIT(S_1,\,S_2,\,s_1,\,s_2) = E(d)\,C_T - C(S_1,\,S_2,\,s_1,\,s_2)$$

5.3 Numerical illustration

The nature of the product interaction in the above is more clearly understood when considering a numerical illustration. For this purpose, let us consider

$$S_1 = 7,\; s_1 = 3,\; S_2 = 5,\; s_2 = 2;\;\; \mu_1 = 0,2;\;\; \mu_2 = 2.0$$

Varying λ from 10 to 30 in increments of 2, the mean number of re-orders for each of the products are obtained and presented in Table 5.1. It is observed that, as λ increases, the mean stationary rate of demand is satisfied and the re-orders of each product also increase.

Next, for the same values of $S_1,\; S_2,\; s_1,\; s_2,\; \lambda = 10.0$ is used; $\mu_2 = 2.0$, and varying μ_1 from 10.0 to 30.0; the mean stationary rates of lost demands for the end-product is obtained and presented in Table 5.2. The mean stationary rate of the lost demands decrease and that of the demands satisfied, increases.

As a third illustration, the following values are used:

Varying $s_1,\; s_2$ in the rectangle $0 \leq s_1 \leq 3$ and $0 \leq s_2 \leq 2$, the total cost and profit are obtained. In Table 5.3, the values of both the cost and the profit functions are presented. It is observed that the profit attains its maximum value and the cost its minimum value when $s_1 = 2\;\; s_2 = 0$.

TABLE 5.1

Increasing demand rate

λ	$E(\ell)$	$E(R_1)$	$E(R_2)$	$E(d)$
10.0000	9.4153	0.2270	3.0873	0.7767
12.0000	11.4668	0.2348	3.6877	0.7797
14.0000	13.5205	0.2423	4.2870	0.7819
16.0000	15.5755	0.2496	4.8856	0.7835
18.0000	17.6314	0.2568	5.4836	0.7847
20.0000	19.6880	0.2639	6.0813	0.7857
22.0000	21.7451	0.2708	6.6788	0.7865
24.0000	23.8026	0.2778	7.2760	0.7871
26.0000	25.8604	0.2846	7.8730	0.7877
28.0000	27.9184	0.2915	8.4699	0.7882
30.0000	29.9766	0.2983	9.0667	0.7886

TABLE 5.2

Increasing replenishment rate of product 1

μ	$E(\ell)$	$E(R_1)$	$E(R_2)$	$E(d)$
10.0000	5.3879	2.4810	1.5485	4.6262
12.0000	5.3785	2.4885	1.5469	4.6303
14.0000	5.3735	2.4926	1.5461	4.6323
16.0000	5.3706	2.4950	1.5457	4.6334
18.0000	5.3688	2.4965	1.5455	4.6340
20.0000	5.3677	2.4975	1.5453	4.6344
22.0000	5.3669	2.4981	1.5453	4.6347
24.0000	5.3664	2.4986	1.5452	4.6348
26.0000	5.3660	2.4989	1.5452	4.6349
28.0000	5.3658	2.4991	1.5451	4.6350
30.0000	5.3656	2.4993	1.5451	4.6350

TABLE 5.3

Optimal re-order profit

s_1	s_2	Cost	Profit
0	0	580.3572	5419.6430
0	1	649.6544	5463.1420
0	2	692.1176	5451.7270
1	0	608.3080	5428.2780
1	1	1015.2550	4967.1390
1	2	733.3414	5297.2330
2	0	321.9494	6125.6520
2	1	653.7461	5534.5830
2	2	707.6093	5494.5480
3	0	627.5200	5294.1770
3	1	1165.9970	4733.3940
3	2	699.7847	5275.1420

5.4 Model 2

This model is an extension of model 1 and is given here in order to highlight the fact that even relaxing one assumption of model 1 renders complexity in the structureal analysis of the inventory system leading to the usage of more general stochastic processes such as Markov Renewal Process.

5.4.1 Assumptions & notation

We consider a two-product continuous reveiw inventory system, where the two products are assembled to produce an end product for which external demands occur according to a Poisson process with parameter λ. The model assumptions and notation are the same as in model 1 except the assumption 3 of model 1 which is modified as follows:

Assumption 3′: The lead-time of product 1 has an arbitrary distribution with pdf $g(\cdot)$ and that of product 2 has exponential distribution with parameter μ.

5.4.2 Auxiliary function

To derive an expression for the inventory level distribution of the two products, we first consider the behaviour of the one-product inventory system of product 2 as a one-product system. For this, we define the auxiliary function:

$$\phi(i,\ j,\ k,\ t) = P\left[L_2(t) = d,\ N_d(t) = k \mid L(0) = i\right]$$

Since the domain of the usage of this function is restricted, we note that the number k of the demands satisfied in the period can be at most s_1. Using probabilistic arguments, we obtain $\phi(i,\ j,\ k,\ t)$ are given below:

For $k = 0$, we have

$\phi(0,\ 0,\ 0,\ t) = e^{-\mu t}$

$\phi(0,\ S_2 - s_2,\ 0,\ t) = \mu e^{-\mu t} \copyright e^{-\lambda t}$

$\phi(i,\ i,\ 0,\ t) = e^{-(\lambda+\mu)t};\ \ 1 \le i \le s_2$

$\phi(i,\ i + S_2 - s_2,\ 0,\ t) = \mu e^{-\mu t} \copyright e^{-\lambda t};\ \ 1 \le i \le s_2$

$\phi(i,\ i,\ 0,\ t) = e^{-\lambda t};\ \ s_2 + 1 \le i \le S_2$

For $1 \le k \le i \le s_2$, we have

$$\phi(i,\ i-k,\ k,\ t) = \frac{e^{-(\lambda+\mu)t}(\lambda t)^k}{k!},\quad k = 0,\ 1,\ 2,\ \ldots$$

$$\phi(i,\ i-k+S_2-s_2,\ k,\ t) = \left[1 - e^{-\mu t}\right] e^{-\lambda t}\frac{(\lambda t)^k}{k!}$$

For $1 \le k \le i - s_2 - 1$, we have

$$\phi(i,\ i-k,\ k,\ t) = \frac{e^{-\lambda t}(\lambda t)^k}{k!};\quad s_2 < i \le S_2$$

For $i - s_2 \leq k \leq i$, we have

$$\phi(i,\ j,\ k,\ t) = \frac{e^{-\lambda t}(\lambda t)^{i-s_2-1}}{(i-s_2-1)!} \copyright \phi(s_2,\ j,\ k-i+s_2,\ t)\,; \quad s_2+1 \leq i \leq S_2; \quad 0 \leq j \leq S_2$$

For $i + 1 \leq k \leq S_1$, we have

$$\phi(i,\ j,\ k,\ t) = \mu e^{-\mu t} \sum_{\ell=0}^{i-1} \frac{e^{-\lambda t}(\lambda t)^{\ell}}{\ell!} \copyright \phi(i-1+S_2-s_2,\ j,\ k-\ell,\ t)$$

$$+\mu e^{-\mu t}\left[1 - \sum_{\ell=0}^{i-1} \frac{e^{-\lambda t}(\lambda t)^{\ell}}{\ell!}\right] \copyright \phi(S_2-s_2,\ j,\ k-\ell,\ t)\,; \quad 1 \leq i \leq s_2; \quad 0 \leq j \leq S_2$$

$$\phi(i,\ j,\ k,\ t) = \frac{e^{-\lambda t}(\lambda t)^{i-s_2-1}}{(i-s_2-1)!} \copyright \phi(s_2,\ j,\ k-i+s_2,\ t)\,, \quad s_2+1 \leq i \leq S_2; \quad 0 \leq j \leq S_2$$

5.4.3 Inventory level distribution

Let $0 = T_0,\ T_1,\ T_2,\ \ldots$ be the time points at which reorders for product 1 are placed, and let $X_n = L_2(T_n+)$. Then, we observe that the process

$$(X,\ T) = \{(X_n,\ T_n)\,;\quad n = 0,\ 1,\ 2,\ \ldots\}$$

is a MRP with a state space

$$E = \{0,\ 1,\ 2,\ \ldots,\ s_2,\ \ldots,\ S_2 - 1\}\,.$$

The semi-Markov kernel (see Cinlar, 1975) of this process is the family of probabilities defined by

$$\phi(i,\ j,\ t) = P\left[X_{n+1} = j,\ T_{n+1} - T_n < t \mid X_n = i\right] \quad \text{where } i,\ j \in E$$

To derive an expression for $\phi(i,\ j,\ t)$, we consider the following mutually exclusive and exhaustive possibilities:

(i) Replenishment for product 1 occurs when the inventory level of product 1 is not zero.

(ii) Replenishment for product 1 occurs when the inventory level of product 1 is zero.

Accordingly, we have

$$\phi(i,\ j,\ t) = \int_0^t \left[\sum_{j'=0}^{S_2} \sum_{\ell=0}^{s_1-1} \{g(u)\,\phi(i,\ j',\ \ell,\ u)\} ©\phi(j',\ j+1,\ S_1-\ell-s_1-1,\ u)\,\lambda \right.$$

$$\left. + \sum_{j'=1}^{S_2} \left\{ g(u) \int_0^u \phi(i,\ j',\ s_1-1\ v)\,\lambda dv \right\} ©\phi(j'-1,\ j+1,\ S_1-2s_1-1,\ u)\,\lambda \right] du$$

We now define a Markov renewal function of $(X,\ T)$ as

$$R(i,\ j,\ t) = \sum_{n=0}^{\infty} \phi(i,\ j,\ t)\,; \quad i,\ j \in E$$

The Markov renewal kernel of the $(X,\ T)$ is the matrix $R(t) = [R(i,\ j,\ t)]$. Setting $Q(t)$ as the $(S_2+1)\,X\,(S_2+1)$ order matrix $[Q(i,\ j,\ t)]$, we obtain from the theory of Markov renewal processes.

$$R^*(s) = [I - Q^*(s)]^{-1}$$

Now to determine the distribution of two product inventory level, we consider the vector process

$$Z(t) = (L_1(t)\,,\ L_2(t))\,;\quad t \geq 0.$$

This process is clearly a semi-regenerative process on the state space

$$F = \{(i,\ j)|\ \ i = 0,\ 1,\ 2,\ ...,\ s_1,\ ...,\ S_1; \\ j = 0,\ 1,\ 2,\ ...,\ s_2,\ ...,\ S_2\}.$$

It is evident that the MRP $(X,\ T)$ is embedded in $Z\,(t)$ (see Cinlar, 1975). We define for any $(i,\ j) \in F$

$$P\,(i,\ j,\ t|\,s_1,\ j_0) = P\,[Z\,(t) = (i,\ j)|\,E_0]\,.$$

The function $P\,(i,\ j,\ t|\,s_1,\ j_0)$ gives the two-product invntory level distribution at any time t, and to derive the expression for it, we consider the function on $k\,(i,\ j,\ t|\,s_1,\ j_0)$ defined by

$$K\,(i,\ j,\ t|\,s_1,\ j_0) = P\,[Z\,(t) = (i,\ j)\,;\ \ T_1 > t|\,E_0]$$

We observe that $K\,(i,\ j,\ t|\,s_1,\ j_0)$ gives the probability that given a reorder is made at time T_0 and that the inventory level then is $(s_1,\ j_0)$, the subsequent reorder is placed only after time t and the inventory level at time t is $(i,\ j)$. To derive $K\,(i,\ j,\ t|\,s_1,\ j_0)$, we have the following cases:

Case 1: $i \leq s_1$

In this case, we observe that no replenishment for product 1 can occur in $(0,\ t)$ and exactly $(s_1 - i)$ demands are satisfied in $(0,\ t)$. Hence, we have

$$K\,(i,\ j,\ t|\,s_1,\ j_0) = \overline{G}\,(t)\,\phi\,(j_0,\ j,\ s_1 - i,\ t)$$

Case 2: $i > S_1 - s_1$

Here we note that a replenishment for product 1 should occur before t and only $0 \leq \ell \leq S_1 - i$ demands can be satisfied as the inventory level of product 1 after replenishment should be above i. Hence we have

$$K(i,\ j,\ t|\ s_1,\ j_0) = \sum_{j'=0}^{S_2} \sum_{\ell=0}^{S_1-i} g(t)\,\phi(j_0,\ j',\ \ell,\ t) \copyright \phi(j',\ j,\ S_1-\ell-i,\ t)$$

Case 3: $s_1 < i < S_1 - s_1$

In this case, a replenishment for product 1 occurs before t and it occurs either before or after the inventory level of product 1 becomes zero. Hence

$$K(i,\ j,\ t|\ s_1,\ j_0) = \sum_{j'=0}^{S_2} \sum_{\ell=0}^{s_1-1} g(t)\,\phi(j_0,\ j',\ \ell,\ t) \copyright \phi(j',\ j,\ S_1-\ell-i,\ t)$$

$$+\sum_{j'=1}^{S_2}\left\{g(t)\int_0^t \phi(j_0,\ j',\ s_1-1,\ u)\,du\right\} \copyright \phi(j'-1,\ j,\ S_1-s_1-i,\ t)$$

Now to obtain an expression for $P(i,\ j,\ t|\ s_1,\ j_0)$, we condition on T_1 and use the regenerative property of $Z(t)$. Accordingly, we get

$$P(i,\ j,\ t|\ s_1,\ j_0)\,t = K(i,\ j,\ t|\ s_1,\ j_0) + \sum_{j'=0}^{S_2}\int_0^t \phi(j_0,\ j',\ du)\,P(i,\ j,\ t-u|\ s_1,\ j')$$

From the theory of MRP, we obtain

$$P(i,\ j,\ t|\ s_1,\ j_0) = \int_0^t R(i,\ j,\ du|\ j_0,\ j')\,K(i,\ j,\ t-u|\ s_1,\ j)$$

the distribution of inventory level of the system, at any time t.

5.4.4 Limiting distribution of the inventory level

Let $\Pi = (\Pi_1,\ \Pi_2,\ ...,\ \Pi_{S_2})$ be the stationary distribution of a Markov chain Z_n. Then Π is the solution of the equation $\Pi\ \overline{Q} = \Pi$ where $\overline{Q}$ is the $(S_2+1)\,X\,(S_2+1)$ order matrix with its elements given by

$$\overline{Q}(i,\ j) = \lim_{t\to\infty} Q(i,\ j,\ t)$$

Let $m(i) = E[T_{n+1} - T_n | Z_n = i]$ be the mean sojourn time in the state i of the MRP (Z, T). Then, we have

$$m(i) = \int_0^\infty \left[1 - \sum_{i,j \in E} Q(i, j, t) \right] dt$$

Hence applying a theorem of semi-regenerative processes, we get

$$\lim_{t \to \infty} P(i, j, t | s_1, j_0) = P(i, j) = \sum_{k \in E} \frac{\Pi(k) \int_0^\infty K(i, j, t | s_1, k) \, dt}{\Pi m}$$

where $m = [m(1), m(2), \ldots, m(S_2)]$.

5.4.5 Measures of system performance

In this section, we proceed to obtain certain measures of performance of the system.

5.4.5.1 Mean number of lost demands

The first order product density of the lost demands is given by

$$h_{j_0}^{\ell}(t) = \sum_{i=0}^{S_1} P(i, 0, t | s_1, j_0) \lambda + \sum_{j=1}^{S_2} P(0, j, t | s_1, j_0) \lambda$$

So the mean number of demands lost in the interval $(0, t]$ is $\int_0^t h_{j_0}^{\ell}(u) \, du$. Hence the stationary mean rate of demands lost is given by

$$E(\ell) = \sum_{i=0}^{S_1} P(i, 0) \lambda + \sum_{j=1}^{S_2} P(0, j) \lambda$$

5.4.5.2 Mean number of reorders

The first order product densities of R_i, $i = 1, 2$ events are given by

$$h_{j_0}^{R_1}(t) = \sum_{j=1}^{S_2} P(s_1 + 1, j, t | s_1, j_0) \lambda.$$

$$h_{j_0}^{R_2}(t) = \sum_{i=1}^{S_1} P(i,\ s_2+1,\ t|\, s_1,\ j_0)\,\lambda.$$

and the respective stationary mean rates of R_1 and R_2 are

$$E(R_1) = \sum_{j=1}^{S_2} P(s_1+1,\ j)\,\lambda.$$

$$E(R_2) = \sum_{i=1}^{S_1} P(i,\ s_2+1)\,\lambda.$$

5.4.5.3 Mean number of demands satisfied

The first order product density of the process of d-events is

$$h_{j_0}^{d}(t) = \sum_{\ell=1}^{S_1} \sum_{j=1}^{S_2} P(i,\ j,\ t|\, s_0,\ j_0)\,\lambda.$$

and the mean stationary rate of the demands satisfied is

$$E(d) = \lambda \sum_{i=1}^{S_1} \sum_{j=1}^{S_2} P(i,\ j)\,.$$

5.4.6 Cost analysis

We note that $P(i,\ j)$ is the fraction of time the process $Z(t)$ spends in $(i,\ j)$. The expected holding cost per unit time is $(c_1 i + c_2 j)\,P(i,\ j)$. The total expected cost per unit time is

$$C(S_1,\ S_2,\ s_1,\ s_2) = \sum_{i=0}^{S_1} \sum_{j=0}^{S_2} (c_1 i + c_2 j)\,P(i,\ j) + E(\ell)\,C_2 + \sum_{i=1}^{S_2} E(R_i)\,c_{R_i} c_{q_i}$$

where c_{R_i} is the reordering cost product i and c_{q_i} is the buying price of product i. Therefore $PROFIT = E(d)\,c_{t_i} - C(S_1,\ S_2,\ s_1,\ s_2)\,.$

5.4.7 Numerical illustration

For the purpose of illustration, we assume

$$g(t) = \frac{ab}{b-a}\left[e^{-at} - e^{-bt}\right], \ a > 0, \ b > 0$$

$a = 0.1; \quad b = 0.01; \quad S_1 = 6; \quad s_1 = 2; \quad S_2 = 4; \quad s_2 = 1; \quad \lambda = 0.5$ and $\mu = 0.5$

We compute the measures of performance of the system by randing λ from 0.5 to 5.5 and obtain table 5.4. In this table we observe that as the mean demand rate increases, the mean stationary rate of reorders for both the products and that of demands lost and satisfied also increase. In table 5.5 we present the stationary rates of cost and the profit for various values of reorder levels of the two products. We note that the optimum value corresponds to $s_1 = 2, \ s_2 = 1$.

Table 5.4

Increasing demand rate

λ	LOST	REORD 1	REORD 2	SATIS
0.5000	0.1332	0.0807	0.1198	0.3594
1.0000	0.3336	0.1266	0.2158	0.6474
1.5000	0.5612	0.1553	0.3010	0.9030
2.0000	0.8091	0.1751	0.3811	1.1434
2.5000	1.0674	0.1895	0.4585	1.3756
3.0000	1.3321	0.2006	0.5343	1.6028
3.5000	1.6013	0.2093	0.6090	1.8270
4.0000	1.8736	0.2164	0.6830	2.0490
4.5000	2.1483	0.2223	0.7565	2.2695
5.0000	2.4247	0.2272	0.8296	2.4889
5.5000	2.7025	0.2315	0.9025	2.7074

Table 5.5

Optimum reorder level

s_1	s_2	COST	PROFIT
2	1	326.6472	503.5375
2	2	365.9118	481.5439
2	3	413.3092	444.3516
3	2	372.3246	457.5576
3	3	421.8203	417.3643
4	1	352.0535	439.0807
4	2	400.3319	412.2556
4	3	434.8828	381.9120

CHAPTER 6

Substitutable perishable inventory products

A modified version of this chapter has been communicated to the conference on "Stochastic Modelling", University of Melbourne, July 2002.
A modified version of this chapter has been communicated to Stochastic Analysis and Applications.

6.1 Introduction

Most of the multi product inventory models do not take into account the substitutability of one product with another. Substitution among perishable products is a common feature in some products especially when the products are perishable in nature and so suitable models for them are absolutely essential.

The perishing of many products like fish, vegetables etc. are continuous and depends upon so many factors including humidity, heat etc. Most of these products are also substitutable to a large extent. Several attempts have been made to systematically study some aspects of perishable inventories. A review of the work on perishable inventory is provided by Nahmias(1982). Baker (1983), Parlar (1985), Nahmias & Schmidt (1986), Pegels (1986), Perry and Posner (1990) and many others have contributed to the development of the study.

In this chapter we study a continuous review of a two perishable product inventory system. The products have constant perishable rates. Allowing substitution for the first by the second and assuming the lead time for the replenishment of product 1, to be a random variable with arbitrary distribution we derive expressions for the stationary distribution of the inventory level by identifying the underlying stochastic process as a semi-regenerative process. We have also derived an expression for the expected profit rate. Maximization of this profit rate is also considered. A numerical illustration is provided to observe the effect of substitution. The model under consideration is described in the following section.

6.2 Model: assumptions & notation

We consider a two perisable product continuous review inventory model with the following assumptions.

1. The maximum inventory level of product i $(i = 1,\ 2)$ is S_i.
2. The reordering policy for product 1 is $S - s$ policy with associated lead time following an arbitrary distrubtion with pdf $f(\cdot)$, i.e. $S_1 - s_1$ quantities are ordered when the inventory level reaches the state s_1.
3. Replenishment of product 2 is instantaneous for $S_2 + 1$ units and is made at the epoch of occurrence of the first demand for this product when its inventory level is zero.
4. The demand process of the products are two independent poisson processes with parameters λ_1 and λ_2.
5. A demand for product 1 which occurs druing its stock out period may be satisfied with product 2 with probability p_{12} and is not backlogged.
6. The products 1 and 2 perish at constant rates μ_1 and μ_2 respectively.

$X(t)$: Two valued stochastic process taking the values 1 and 0 according as a reorder for product 1 is pending or not.

$L_i(t)$: Inventory level of product i at time t; $i = 1,\ 2$.

μ_i : Perishable rate of product i; $i = 1,\ 2$.

$f(\cdot)$: Lead time distribution of product 1.

b : Event that a replenishment for product 1 occurs.

d : Event that a replenishment for product 2 occurs.

a : Event that a reorder is placed for product 1.

c : Event that a reorder is placed for product 2.

p_i : Event that product i perishes, $i = 1,\ 2$.

C_i : Holding cost per unit time of product $i,\ i = 1,\ 2$.

C_{ℓ_1} : Cost per lost demand for product 1.

C_a : Cost incurred per reorder for product 1.

C_2 : Cost incurred per reorder for product 2.

C_{p_i} : Salvage cost per unit of product 1.

C_{q_i} : Buying price of product i; $i = 1,\ 2$.

C_{t_i} : Selling price of product i; $i = 1,\ 2$.

$E(\eta)$: Stationary mean rate of n events. $n = \ell,\ a,\ c,\ p_1,\ p_2$

p_{12} : Probability for a demand for product 1 to be satisfied by product 2 during the stock-out period of product 1.

6.3 Auxiliary functions

We can identify the stochastic process underlying the behaviour of the system with a Markov Renewal Process (MRP) and study the behaviour of the system in intervals of time in which $X(t) = 0$ and $X(t) = 1$ separately. For this purpose we define and obtain expressions for the following auxiliary functions.

6.3.1 Function $w_{ij}(t)$:

To study the behaviour of the process $L_1(t)$ in an interval between any two successive transitions of the process $X(t)$, we define the function $w_{ij}(t)$ as follows:

$$w_{ij}(t) = P\left[L_1(t) = j \middle| L_1(0) = i\right];$$

$$i,\ j = 0,\ 1,\ 2,\ ...,\ s_1 \text{ or } i,\ j = s_1 + 1,\ s_2 + 2,\ ....,\ S_1$$

In an interval under consideration the process $L_1(t)$ behaves like a death process with state dependent death rate. Hence using probabilistic arguments we derive the following expressions for $w_{ij}(t)$.

$$\text{For } j > i,\ w_{ij}(t) = 0 \tag{6.1}$$

$$\text{For } 0 \leq j \leq i;\ \ w_{ij}(t) = (\lambda_1 + i\mu_1)\, e^{-(\lambda_1 + i\mu_1)t} \copyright w_{i-1,j}(t) \tag{6.2}$$

$$\text{For } i \neq 0;\ \ w_{ij}(t) = e^{-(\lambda_1 + i\mu_1)t} \text{ and } w_{00}(t) = 1 \tag{6.3}$$

Solving the equations $(5.1) - (5.3)$ after taking Laplace Transforms and inverting we get

$$w_{ij}(t) = \sum_{k=0}^{i-j} A_{kij}(t)\, e^{-\alpha_{kij}(t)} \tag{6.4}$$

$$\text{where } j > i,\ A_{kij} = 0;\ \ \alpha_{kij} = 0 \tag{6.5}$$

$$\text{for } j = i \neq 0,\ A_{kij} = 1\ \ \alpha_{kij} = \lambda_1 + (i-k)\,\mu_1 \tag{6.6}$$

$$\text{for } j = i = 0,\ A_{kij} = 1;\ \ \alpha_{kij} = 0 \tag{6.7}$$

$$\text{for } 0 < j < i,\ A_{kij} = \frac{(-1)^{i-j-ki}}{k1(i-j-k)!} \prod_{\ell=j+1}^{i} \left[\frac{\lambda + \ell\mu_1}{\mu}\right];\ \ \alpha_{kij} = \lambda_1 + (i-k)\,\mu_1 \tag{6.8}$$

$$\text{for } j = 0 \neq i,\ k \neq 0,\ A_{kij} = \frac{(-1)^k \mu_1}{(k-1)!(i-k)!(\lambda_1 + k\mu_1)} \prod_{\ell=1}^{i} \left[\frac{\lambda_1 + \ell\mu_1}{\mu_1}\right];$$

$$\alpha_{kij} = \lambda_1 + (i-k)\,\mu_1 \tag{6.9}$$

$$\text{for } j = 0 \neq i,\ k = 0;\ \ A_{0i0} = 1;\ \ \alpha_{0i0} = 1 \tag{6.10}$$

6.3.2 Functions ${}_dU_{ij},\ U_{ij},\ {}_dV_{ij},\ V_{ij}$:

Next we study the behaviour of the process $L_2(t)$ in an interval between two successive transitions of the $X(t)$ process. The process $L_2(t)$ also behaves like a death process in the interval between any two reorders. However, the rate of transition of this process at any time t depends upon the state of the process $L_1(t)$ also. Hence we introduce the functions ${}_dU_{ij}(t)$ and $U_{ij}(t)$ to describe the behaviour of the process $L_2(t)$ in an interval in which $L_1(t) \neq 0$ throughout and the functions ${}_dV_{ij}(t)$ and $V_{ij}(t)$ to describe it in an interval in which $L_1(t) = 0$. If for all $u \in (0,\ t],\ L_1(u) \neq 0$, define

$$
\begin{aligned}
{}_dU_{ij}(t) &= P\left[L_2(t) = j,\ N(d,\ t) = 0 \mid L_2(0) = i\right] \\
U_{ij}(t) &= P\left[L_2(t) = j \mid L_2(0) = i\right];\quad i,\ j = 0,\ 1,\ 2,\ \ldots,\ S_2
\end{aligned}
$$

and if for all $u \in (0,\ t],\ L_1(u) = 0$, define

$$
\begin{aligned}
{}_dV_{ij}(t) &= P\left[L_2(t) = j,\ N(d,\ t) = 0 \mid L_2(0) = i\right] \\
V_{ij}(t) &= P\left[L_2(t) = j \mid L_2(0) = i\right];\quad i,\ j = 0,\ 1,\ 2,\ \ldots,\ S_2
\end{aligned}
$$

Using arguments similar to the derivtion of equations $(6.1) - (6.3)$ we get the following expressions for ${}_dU^*_{ij}(s)$

$$
{}_dU^*_{ij}(s) = \begin{cases} 0 & \\ \dfrac{1}{s + \lambda_2 + i\mu_2}; & j = i \\ \dfrac{\prod\limits_{\ell=j+1}^{i} (\lambda_2 + \ell\mu_2)}{\prod\limits_{\ell=j+1}^{i} (s + \lambda_2 + \ell\mu_2)}; & 0 \le j < i \end{cases} \tag{6.11}
$$

Next we derive an expression for $U_{ij}(t)$. To get this we note that the 'd' events consitute a renewal process and that the occurrence of 'd' events correspond to the occurrence of a demand for product 2 when its inventory level is zero.

Let $f_d(t)$ be the interval between any two successive 'd' events. Then

$$f_d(t) = {}_dU_{S_2 0}(t)\,\lambda_2 \tag{6.12}$$

Let $h(t)$ be the renewal density (Cox (1962)) corresponding to this renewal process. Then

$$h(t) = \sum_{n=1}^{\infty} f_d^{(n)}(t) \tag{6.13}$$

Using the fact that a 'd' event occurs or not in the interval $(0,\ t]$ we get,

$$U_{ij}(t) = H(i-j)\ {}_dU_{ij}(t) + \lambda_2\,{}_dU_{i0}(t)\,©\,{}_dU_{S_2 j}(t)$$

$$+\lambda_2\,{}_dU_{i0}(t)\,©\,h(t)\,©\,{}_dU_{S_2 j}(t) \tag{6.14}$$

Taking Laplace transform on both sides of (6.14) and using (6.12), we obtain after simplification

$$U_{ij}^{*}(s) = \frac{\prod\limits_{\ell=j+1}^{i}(\lambda_2+\ell\mu_2)\prod\limits_{m=0}^{j-1}(s+\lambda_2+m\mu_2)\prod\limits_{n=x+1}^{S_2}(s+\lambda_2+n\mu_2)}{\prod\limits_{\ell=0}^{S_2}(s+\lambda_2+\ell\mu_2)-\prod\limits_{n=0}^{S_2}(\lambda_2+n\mu_2)}$$

Now we derive expressions for the functions ${}_dV_{ij}(t)$ and $V_{ij}(t)$. To this end we note that a transition in the $L_2(t)$ process takes place at the epoch of occurrence of any one of the following events:

(i) a demand for product 2 occurs

(ii) a unit of product 2 perishes

(iii) a demand for product 1 occurs and is satisfied with product 2.

Hence the expressions for ${}_dV_{ij}(t)$ and $V_{ij}(t)$ can be obtained from the corresponding expressions for ${}_dU_{ij}(t)$ and $U_{ij}(t)$ by replacing λ_2 by $(p_{12}\lambda_1+\lambda_2)$

6.4 The inventory level

Let $T_0,\ T_1,\ T_2,\ \ldots$ be the epochs at which the process $X(t)$ changes its state. Define $X_n = X(T_n+);\ L_{1_n} = L_1(T_n+),\ L_{2_n} = L_2(T_n+)$ for $n = 0,\ 1,\ 2,\ 3,\ \ldots$ We can easily see that the process $Z_n = (X_n,\ L_{1_n},\ L_{2_n})$ is a MRP (Cinlar, 1975) with state space $E_1 = E_2 \cup E_3$ where

$E_2 = \{(1,\ s_1,\ i),\ \ i = 0,\ 1,\ 2,\ \ldots,\ S_2\}$ and

$E_3 = \{(0,\ j,\ i),\ \ j = S_1 - s_1,\ S_1 - s_1 + 1,\ \ldots,\ S_1;\ \ i = 0,\ 1,\ 2,\ \ldots,\ S_2\}$

The semi Markov Kernel of this process is defined as

$$Q(j_1,\ j_2,\ j_3,\ t|\, i_1,\ i_2,\ i_3) =$$
$$P[X_{n+1} = j_1,\ L_{1_{n+1}} = j_2,\ L_{2_{n+1}}\ j_3,\ T_{n+1} - T_n \le t|\, X_n = i_1,\ L_{1_n} = i_2,\ L_{2_n} = i_3];$$
$$(i_1,\ i_2,\ i_3),\ (j_1,\ j_2,\ j_3) \in E_1.$$

We now derive an expression for this semi Markov Kernel.

Since T_n are the epochs of transitions of the process $X(t)$, we get

$$\text{for } i_1 = j_1,\ \ Q(j_1,\ j_2,\ j_3,\ t|\, i_1,\ i_2,\ i_3) = 0 \tag{6.16}$$

Since the probability that the process $L_1(t)$ enters the state s_1 in $(u,\ u + du)$ is $w_{i_2 s_1+1}(u)\{\lambda_1 + (s_1 + 1)\mu_1\}$, for $(0,\ i_2,\ i_3) \in E_3$ and $(1,\ s_1,\ j_3) \in E_2$ we have

$$Q(1,\ s_1,\ j_3,\ t|\, 0,\ i_2,\ i_3) = \int_0^t w_{i_2 s_1+1}(u)\, U_{i_3 j_3}(u)\{\lambda_1 + (s_1 + 1)\mu_1\}\, du \tag{6.17}$$

Next we derive an expression for $Q(0,\ j_2,\ j_3,\ t|\, 1,\ s_1,\ i_3)$ we have to consider the following cases:

(i) For product 1 the replenishment occurs before it is out of stock or

(ii) during its stock out period.

Accordingly we have, for $j_2 > S_1 - s_1$

$$Q(0,\ j_2,\ j_3,\ t|\ 1,\ s_2,\ i_3) = \int_0^t w_{s_1 j_2 - S_1 + s_1}(u)\, U_{i_3 j_3}(u)\, dF(u) \tag{6.18}$$

and

$$Q(0,\ S_1 - s_1,\ j_3,\ t|\ 1,\ s_1,\ i_3)$$

$$= \sum_{k=0}^{S_2} \int_0^t f(v)\, dv \int_0^t w_{s_1 1}(u)\, U_{i_3 k}(u)\, V_{k j_3}(v-u)\, (\lambda_1 + \mu_1)\, du \tag{6.19}$$

From (5.16) it follows that semi Markov Kernel is of the form

$$Q(t) = \begin{array}{c} \\ E_2 \\ E_3 \end{array} \begin{array}{c} \begin{array}{cc} E_2 & E_3 \end{array} \\ \left[\begin{array}{cc} 0 & A(t) \\ B(t) & 0 \end{array} \right] \end{array} \tag{6.20}$$

The matrix $A(t)$ is of order $(S_2 + 1) \times s_1 (S_2 + 1)$ and its elements are given by (6.18) and (6.19). The matrix $B(t)$ is of order $s_1 (S_2 + 1) \times (S_2 + 1)$ and its elements are given by (6.17).

Let $A = \lim\limits_{t \to \infty} A(t)$ and $B = \lim\limits_{t \to \infty} B(t)$. We see that the one step transition probablity matrix of the Markov chain $\{(X_n,\ L_{1_n},\ L_{2_n}),\ n \geq 0\}$ is given by

$$Q = \left[\begin{array}{cc} 0 & A \\ B & 0 \end{array} \right] \tag{6.21}$$

The structure of Q at once reveals that the chain is periodic with period 2. Also it can be seen that every element in A is greater than zero, and hence the Markov chain $(X_n,\ L_{1_n},\ L_{2_n})$ is irreducible. As a consequence we see that the stationary distribution of the Markov chain exists.

Let $\underline{\Pi} = (\underline{\Pi}_1,\ \underline{\Pi}_2)$ be the stationary distribution of the Markov chain $(X_n,\ L_{1_n},\ L_{2_n})$ where

$$\underline{\Pi}_1 = (\Pi(1,\ s_1,\ 0),\ \Pi(1,\ s_1,\ 1),\ ...,\ \Pi(0,\ s_1,\ S_2)) \tag{6.22}$$

and $\underline{\Pi}_2 = (\Pi(0,\ S_1 - s_1,\ 0),\ \Pi(0,\ S_1 - s_1,\ 1),\ ...,\ \Pi(0,\ S_1,\ S_2))$ (6.23)

Taking $\underline{\Pi}_1$ to be a solution of $\underline{\Pi}AB = \underline{\Pi}_1$ and solving $\underline{\Pi}_2 = \underline{\Pi}_1 A$ and using the normalizing condition $\underline{\Pi}\,\underline{e} = 1$ we obtain $\underline{\Pi}$.

Let $R(j_1,\ j_2,\ j_3,\ t|\,i_1,\ i_2,\ i_3)$ be the Markov renewal function corresponding to $Q(j_1,\ j_2,\ j_3,\ t|\,i_1,\ i_2,\ i_3)$. Then $R(j_1,\ j_2,\ j_3,\ t|\,i_1,\ i_2,\ i_3)$ is called the Markov renewal kernel of the process. From the theory of Markov Renewal process we have

$$R^*(s) = [I - Q^*(s)]^{-1}$$

$$= \begin{bmatrix} [I - A^*(s)\,B^*(s)]^{-1} & A^*(s)\,[I - A^*(s)\,B^*(s)]^{-1} \\ B^*(s)\,[I - A^*(s)\,B^*(s)]^{-1} & [I - A^*(s)\,B^*(s)]^{-1} \end{bmatrix} \tag{6.24}$$

Now we study the vector process $Z(t) = (X(t),\ L_1(t),\ L_2(t))$ to get the distribution of the inventory level. This process is a semi-regenerative process (Cinlar, 1975) on the state space $E = E_4\ U\ E_5$ where

$$E_4 = ((1,\ i,\ j);\ \ i = 0,\ 1,\ 2,\ ...,\ s_1,\ j = 0,\ 1,\ 2,\ ...,\ S_2) \tag{6.25}$$

$$E_5 = ((0,\ i,\ j);\ \ i = s_1 + 1,\ s_1 + 2,\ ...,\ S_1;\ \ j = 0,\ 1,\ 2,\ ...,\ S_2) \tag{6.26}$$

and $((X_n,\ L_{1_n},\ L_{2_n}),\ T_n)$ is the MRP embedded in it.

Define

$$P[j_1,\ j_2,\ j_3,\ t|\,i_1,\ i_2,\ i_3] =$$
$$P[X(t) = j_1,\ L_1(t) = j_2;\ \ L_2(t) = j_3|\,X(0) = i_1,\ L_1(0) = i_2,\ L_2(0) = i_3];$$
$$(i_1,\ i_2,\ i_3) \in E_1;\ \ (j_1,\ j_2,\ j_3) \in E$$

Since $T_1 > t$, we have

$$K(j_1,\ j_2,\ j_3,\ t|\,i_1,\ i_2,\ i_3) = 0 \quad \text{for } i_1 \neq j, \tag{6.27}$$

when $i_1 = j_1 = 1$ we have to deal with the cases $j_2 \neq 0$ and $j_2 = 0$ separately. Accordingly, using simple probabilistic arguments we have for $(i_1, i_2, i_3) \in E_2$ and $0 < j_2 \leq s_1$.

$$K(1, j_2, j_3, t| 1, s_1, i_3) = w_{s_1 j_2}(t) U_{i_3 j_3}(t) \overline{F}(t) \quad (6.28)$$

For $(i_1, i_2, i_3) \in E_2$ and $j_2 = 0$.

$$K(1, 0, j_3, t| 1, s_1, i_3)$$

$$= \sum_{k=0}^{S_2} \overline{F}(t)(\lambda_1 + \mu_1) \int_0^t w_{s_1 1}(u) U_{i_3 k}(u) v_{k j_3}(t-u) du \quad (6.29)$$

For $(0, i_2, i_3) \in E_3$ and $(j_1, j_2, j_3) \in E_5$

we have $K(0, j_2, j_3, t| 0, i_2, i_3) = w_{i_2 j_2}(t) U_{i_3 j_3}(t)$ (6.30)

We next obtain the expression for the function $P(j_1, j_2, j_3, t| i_1, i_2, i_3)$.
For this purpose we derive the following Markov renewal equation for $P(j_1, j_2, j_3, t| i_1, i_2, i_3)$ by conditioning on T_1 and using the regenerative property.

$$P[j_1, j_2, j_3, t| i_1, i_2, i_3] = K(j_1, j_2, j_3, t| i_1, i_2, i_3) +$$

$$\sum_{(n_1, n_2, n_3) \in E_1} \int_0^t Q(n_1, n_2, n_3, du| i_1, i_2, i_3) P(j_1, j_2, j_3, t-u| n_1, n_2, n_3) \quad (6.31)$$

The solution of which is given by

$$P[j_1, j_2, j_3, t| i_1, i_2, i_3] = \sum_{(n_1, n_2, n_3) \in E_1} \int_0^t R(n_1, n_2, n_3, du| i_1, i_2, i_3)$$

$$K(j_1, j_2, j_3, t-u| n_1, n_2, n_3) \quad (6.32)$$

Let $P[j_1,\ j_2,\ j_3,\ t|\ i_1,\ i_2,\ i_3] = P[L_1(t) = j_2;\ \ L_2(t) = j_3|\ X(0) = i_1,$

$L_1(0) = i_2,\ L_2(0) = i_3];\ \ (i_1,\ i_2,\ i_3) \in E_1;\ \ 0 \le j_2 \le S_1;\ \ 0 \le j_3 \le S_2$

Then using the standard probabilistic arguments we get the following expressions for the distribution of the inventory level.

$$P[j_1,\ j_2,\ j_3,\ t|\ i_1,\ i_2,\ i_3] = \sum_{j_1=0}^{1} P[j_1,\ j_2,\ j_3,\ t|\ i_1,\ i_2,\ i_3] \tag{6.33}$$

6.4.1 Limiting distribution of the inventory level

Next we consider the limiting distribution of the inventory level. Let

$$m(i_1,\ i_2,\ i_3) = E[T_{n+1} - T_n|\ (X_n,\ L_{1_n},\ L_{2_n}) = (i_1,\ i_2,\ i_3)];\ i_1,\ i_2,\ i_3 \in E_1 \tag{6.34}$$

Then $m(i_1,\ i_2,\ i_3)$ is the mean sojourn time in the state $(i_1,\ i_2,\ i_3)$. Hence we have

$$m(i_1,\ i_2,\ i_3) = \int_0^\infty \left[1 - \sum_{(j_1,\ j_2,\ j_3) \in E_1} Q(j_1,\ j_2,\ j_3,\ t|\ i_1,\ i_2,\ i_3)\right] dt \tag{6.35}$$

From this we observe that the process $\{(X_n,\ L_{1_n},\ L_{2_n}),\ T_n\}$ is an irreducible aperiodic Markov renewal chain. Hence applying a theorem on semi-regenerative process we get

$$\lim_{t\to\infty} P[j_1,\ j_2,\ j_3,\ t|\ i_1,\ i_2,\ i_3] = P(j_1,\ j_2,\ j_3)$$

$$= \frac{\sum_{(i_1,\ i_2,\ i_3) \in E_1} \Pi(i_1,\ i_2,\ i_3) \int_0^\infty K(j_1,\ j_2,\ j_3,\ t|\ i_1,\ i_2,\ i_3)\, dt}{\underline{\Pi}\, \underline{m}}$$

where

$$\underline{m} = (m(0,\ 0,\ 0),\ ...,\ m(1,\ S_1,\ S_2)) \tag{6.36}$$

Let

$$P(j_2,\ j_3) = \lim_{t\to\infty} P(j_2,\ j_3,\ t|\ i_1,\ i_2,\ i_3); \quad \begin{array}{l} j_2 = 0,\ 1,\ 2,\ \ldots,\ S_1 \\ j_3 = 0,\ 1,\ 2,\ \ldots,\ S_2 \\ (i_1,\ i_2,\ i_3) \in E_1 \end{array} \qquad (6.37)$$

Hence using (6.36) in (6.33) and from (6.37) we get

$$P(j_2,\ j_3) = \sum_{j_1=0}^{1} P(j_1,\ j_2,\ j_3)$$

Thus the stationary distribution of the inventory level is obtained.

6.5 Mean number of demands lost

From the assumptions of the model it is clear that no demand for product 2 is lost. Let ℓ_1 be the event that a demand for product 1 is lost. Define

$$h^{\ell_1}_{i_1,\ i_2,\ i_3}(t) = \lim_{\Delta\to 0} P\,[\text{`}\ell_1\text{', event in } (t,\ t+\Delta)|\ Z_0 = (i_1,\ i_2,\ i_3)]; \quad (i_1,\ i_2,\ i_3) \in E_1$$

Then we easily see that

$$h^{\ell_1}_{i_1,\ i_2,\ i_3}(t) = \sum_{j_3=0}^{S_2} P(1,\ 0,\ j_3,\ t|\ i_1,\ i_2,\ i_3)\,(1-p_{12})\,\lambda_1 \qquad (6.38)$$

Since $h^{\ell_1}_{i_1,\ i_2,\ i_3}(t)$ is the first order product density (Srinivasan (1974)) of the ℓ_1 events. The mean number of demands lost in the interval $(0,\ t]$ is given by

$$\int_0^t h^{\ell_1}_{i_1,\, i_2,\, i_3}(u)\, du.$$

Hence the stationary mean rate of demands lost is given by

$$E(\ell_1) = \lim_{t \to \infty} h^{\ell_1}_{i_1,\, i_2,\, i_3}(t)\,.$$

Using (6.38) we get

$$E(\ell_1) = \sum_{j_3=0}^{S_2} P(1,\ 0,\ j_3)\,(1 - p_{12})\,\lambda_1 \tag{6.39}$$

6.6 Mean number of reorders

Since an 'a' event and a 'c' event respectively denote that a reorder is placed for product 1 and product 2 we define the following functions to get an expression for the number of reorders placed.

Let

$h^{a}_{i_1,\, i_2,\, i_3}(t) = \lim_{\Delta \to 0} P\,[\text{'}a\text{' event in }(t,\ t + \Delta)|\ Z_0 = (i_1,\ i_2,\ i_3)]\,;\quad (i_1,\ i_2,\ i_3) \in E_1$

and

$h^{c}_{i_1,\, i_2,\, i_3}(t) = \lim_{\Delta \to 0} P\,[\text{'}c\text{' event in }(t,\ t + \Delta)|\ Z_0 = (i_1,\ i_2,\ i_3)]\,;\quad (i_1,\ i_2,\ i_3) \in E_1$

We note that 'a' event occurs when the process $X(t)$ enters the state 1 from 0. Hence we have

$$h^{a}_{i_1,\, i_2,\, i_3}(t) = \sum_{j_3=0}^{S_2} P\,[0,\ s_1 + 1,\ j_3,\ t|\, i_1,\ i_2,\ i_3]\,[\lambda_1 + (s_1 + 1)\,\mu_1] \tag{6.40}$$

From the asumptions of the model we note that a 'c' event will occur when

(i) inventory level of product 1 is greater than zero, that of product 2 is zero and a demand for product 2 occurs.

or

(ii) the inventory level of both product 1 and product 2 is zero and either a demand for product 2 occurs or a demand for product 1 which can be satisfied with product 2 occurs.

Hence we have

$$h^{c}_{i_1,\, i_2,\, i_3}(t) = \sum_{j_2=0}^{s_1} P(1,\; j_2,\; 0,\; t|\, i_1,\; i_2,\; i_3)\,(\lambda_2 + d_{j_20}p_{12}\lambda_1) +$$

$$\sum_{j_2=S_1-s_1}^{S_1} P(0,\; j_2,\; 0,\; t|\, i_1,\; i_2,\; i_3)\,\lambda_2 \qquad (6.41)$$

Mean number of reorders for product 1 and product 2 are respectively given by $\int\limits_0^t h^{a}_{i_1,\, i_2,\, i_3}(u)\,du$ and $\int\limits_0^t h^{c}_{i_1,\, i_2,\, i_3}(u)\,du$. In the limiting case, we get

$$E(a) = \sum_{j_3=0}^{S_2} P(0,\; s_1+1,\; j_3)\,[\lambda_1 + (s_1+1)\,\mu_1] \qquad (6.42)$$

and

$$E(c) = P(1,\; 0,\; 0)\,(\lambda_2 + q\lambda_1) + \sum_{j_2=1}^{s_1} P(1,\; j_2,\; 0)\,\lambda_2 + \sum_{j_2=s_1+1}^{S_1} P(0,\; j_2,\; 0)\,\lambda_2 \qquad (6.43)$$

6.7 Mean number of perished items

We note that when the process $Z(u)$ is in state $(j_1,\; j_2,\; j_3)$ at time u an item of product 1 will perish in $(u,\; u+du)$ with probability $j_2\mu_1 du$ and an item of product 2 will perish with probability $j_3\mu_2 du$. Hence the mean number of product 1 perished

$$= \sum_{j_2=0}^{S_1} \int\limits_0^t P(j_1,\; j_2,\; j_3,\; u|\, i_1,\; i_2,\; i_3)\, j_2\mu_1 du$$

mean stationary perishable rate of product 1

$$E(p_1) = \sum_{j_2=1}^{S_1} P(j_1,\; j_2,\; j_3)\, j_2\mu_1 \qquad (6.44)$$

similarly the mean stationary perishable rate of product 2 is

$$E(p_2) = \sum_{j_3=1}^{S_2} P(j_1,\; j_2,\; j_3)\, j_3\mu_2 \qquad (6.45)$$

6.8 Mean number of demands satisfied

Demands for product 1 can be satisfied by product 1 itself if its inventory level is greater than zero when a demand occurs. So, the mean number of demands for product 1 satisfied by product 1 can easily be seen to be

$$E(e_1) = \sum_{j_1=0}^{1} \sum_{j_2=1}^{S_1} \sum_{j_3=0}^{S_2} P(j_1, j_2, j_3)\,\lambda_1 \tag{6.46}$$

Since the replenishment for product 2 is instantaneous, a demand can be satisfied even during the stock-out period of product 2. Hence we get the mean number of demands for product 2 satisfied by product 2 as

$$E(e_2) = \sum_{j_1=0}^{1} \sum_{j_2=0}^{S_1} \sum_{j_3=0}^{S_2} P(j_1, j_2, j_3)\,\lambda_2 = \lambda_2 \tag{6.47}$$

Also the mean number of demands for product 1 satisfied by product 2 is

$$E(g_1) = \sum_{j_1=0}^{1} \sum_{j_3=0}^{S_2} P(j_1, 0, j_3)\,\lambda_1 p_{12} \tag{6.48}$$

6.9 Cost analysis

It is evident that the total expected cost per unit time is the sum of the holding cost, the reordering cost, the salvage cost, for perished items and the cost due to demands lost. Since $P(i, j_1, j_2)$ can be considered as the fraction of time the process $Z(t)$ is in state (i, j_1, j_2), the expected holding cost corresponding to this can be taken as $(C_1 j_1 + C_2 j_2)\,P(i, j_1, j_2)$. Hence the total expected cost per unit time is

$$C(S_1, S_2, s_1) = \sum_{j_1=0}^{s_1} \sum_{j_2=0}^{S_2} (C_1 j_1 + C_2 j_2)\,P(i, j_1, j_2)$$

$$+ \sum_{j_1=s_1+1}^{S_1} \sum_{j_2=0}^{S_2} (C_1 j_1 + C_2 j_2)\,P(0, j_1, j_2) + E(a)\,C_a + E(C)\,C_c + E(p_1)\,C_{p_1}$$

$$+E(p_2)\,C_{p_2} + E(\ell_1)\,C_{\ell_1} \tag{6.49}$$

In order to find the profit we have to take into consideration the demands satisfied as well as the buying and selling prices.

The profit function $PF\left(S_1,\ S_2,\ s_1\right)$ can be seen as

$$PF\left(S_1,\ S_2,\ s_1\right) = E\left(e_1\right) C_{t_1} + \left(E\left(e_2\right) + E\left(g_1\right)\right) C_{t_2} - C\left(S_1,\ S_2,\ s_1\right)$$

$$-E\left(a\right) C_{q_1}\left(S_1 - s_1\right) - E\left(C\right) C_{q_2} S_2 \qquad (6.50)$$

Therefore the optimal ordering level for product 1 maximising the total expected profit can be obtained.

6.10 Numerical illustration

To illustate the various findings in the system we give a numerical example.

Let $f\left(u\right) = \frac{xy}{y-x}\left[e^{-xu} - e^{-yu}\right]$. Let the procurement price of the products 1 and 2 be c_{q1} and c_{q2} and their selling price be C_{t_1} and C_{t_2} respectively.

The values of the parameters for the example are given below:

$\lambda_1 = 40$	$\lambda_2 = 2.6$
$S_1 = 9$	$y = 3$
$x = 5$	$C_c = 80$
$C_{\ell_1} = 60$	$\mu_2 = 5$
$C_a = 50$	$C_{p2} = 100$
$\mu_1 = 30$	$C_{q2} = 80$
$C_{p1} = 120$	$C_{t_2} = 180$
$C_{q_1} = 100$	
$C_{t_1} = 220$	

we consider the values of the mean rates of

(i) the demands for product 1 lost

(ii) the number of reorders placed

(iii) the number of items that perish in unit time and

(iv) profit

corresponding to

(a) various reorder levels of product 1 (see table 6.3)

(b) various substitution probabilities (see tables 6.1 and 6.2)

(a) As p_{12} increases, the cost of

(i) product 1 demands lost decreases (tables 6.1 and 6.2)

(ii) reorders for product 1 remains the same whereas that of product 2 increases (see tables 6.1 and 6.2)

(iii) item perished for product 1 remains the same whereas that for product 2 increases

(iv) the profit increases (see table 6.1 and 6.2).

(b) when the reorder level of product 1 increases, the cost of

(i) the damends for product 1 lost decreases first and then increases

(ii) reorders for product 1 increases whereas that for product 2 increases first and then decreases or increases throughout (see table 6.3)

(iii) items of the product 1 perished first decreases and then increases and that of product 2 fluctuates depending on the value of the parameters

(iv) the profit fluctuates depending on the value of the parameters (see table 6.3)

For the values considered $S_1 = 9,\ S_2 = 1,\ p = 1,\ s_1 = 1$ corresponds to the maximum profit.

Table 6.1

	p_{12}	Cost of demands lost	Cost of reorder product 1	Cost of reorder product 2	Perishing cost product 1	Perishing cost product 2	PROFIT
	0.0	27.658	1.296	0.986	4.627	1.143	206.87
$S_1 = 9$	0.1	24.892	1.296	2.524	4.627	1.444	435.81
$S_2 = 1$	0.2	22.127	1.296	3.962	4.627	1.554	712.92
$s_1 = 0$	0.3	19.361	1.296	5.373	4.627	1.611	1003.27
	0.4	16.595	1.296	6.773	4.627	1.646	1299.09
	0.5	13.829	1.296	9.558	4.627	1.686	1597.68
	0.6	11.063	1.296	9.558	4.627	1.686	1897.87
	0.7	8.297	1.296	10.947	4.627	1.699	2199.07
	0.8	5.531	1.296	12.334	4.627	1.709	2500.94
	0.9	2.765	1.296	13.721	4.627	1.717	2803.29
	1.0	0.000	1.296	15.107	4.627	1.723	3105.97
	0.0	29.473	1.507	1.047	3.722	1.328	-194.88
$S_1 = 9$	0.1	26.526	1.507	2.662	3.722	1.605	58.71
$S_2 = 1$	0.2	23.579	1.507	4.190	3.722	1.707	357.06
$s_1 = 2$	0.3	20.632	1.507	5.697	3.722	1.760	667.91
	0.4	17.684	1.507	7.182	3.722	1.793	983.97
	0.5	14.737	1.507	8.667	3.722	1.815	1302.71
	0.6	11.789	1.507	10.149	3.722	1.830	1622.98
	0.7	8.842	1.507	11.628	3.722	1.842	1944.23
	0.8	5.894	1.507	13.107	3.722	1.859	2266.13
	0.9	2.947	1.507	14.584	3.722	1.859	2588.50
	1.0	0.000	1.507	16.061	3.722	1.865	2911.20

Table 6.2

	p_{12}	Cost of demands lost	Cost of reorder product 1	Cost of reorder product 2	Perishing cost product 1	Perishing cost product 2	PROFIT
	0.0	29.019	1.483	0.652	3.961	4.057	-323.55
$S_1 = 9$	0.1	26.117	1.483	1.591	3.961	4.847	37.02
$S_2 = 3$	0.2	23.216	1.483	2.411	3.961	5.205	478.75
$s_1 = 2$	0.3	20.314	1.483	3.190	3.961	5.413	948.65
	0.4	17.412	1.483	3.950	3.961	5.548	1431.76
	0.5	14.510	1.483	4.700	3.961	5.644	1922.20
	0.6	11.608	1.483	5.444	3.961	5.715	2417.12
	0.7	8.706	1.483	6.183	3.961	5.770	2914.98
	0.8	5.804	1.483	6.920	3.961	5.814	3414.88
	0.9	2.902	1.483	7.654	3.961	5.850	3916.26
	1.0	0.000	1.483	8.387	3.961	5.880	4418.73
	0.0	29.317	1.625	0.648	3.731	4.460	-415.74
$S_1 = 9$	0.1	26.386	1.625	1.557	3.731	5.167	-35.93
$S_2 = 3$	0.2	23.454	1.625	2.378	3.731	5.493	416.31
$s_1 = 4$	0.3	20.523	1.625	3.162	3.731	5.683	894.81
	0.4	17.591	1.625	3.928	3.731	5.807	1384.22
	0.5	14.659	1.625	4.684	3.731	5.896	1881.02
	0.6	11.727	1.625	5.434	3.731	5.961	2381.99
	0.7	8.795	1.625	6.181	3.731	6.012	2885.99
	0.8	5.863	1.625	6.924	3.731	6.053	3391.31
	0.9	2.931	1.625	7.666	3.731	6.086	3891.30
	1.0	0.000	1.625	8.406	3.731	6.114	4406.32

Table 6.3

	s_1	Cost of demands lost	Cost of reorder product 1	Cost of reorder product 2	Perishing cost product 1	Perishing cost product 2	PROFIT
	0	27.658	1.296	0.985	4.627	1.543	206.87
$S_1 = 9$	1	27.416	1.343	1.000	4.712	1.080	261.84
$S_2 = 1$	2	29.473	1.507	1.041	3.723	1.328	-194.88
$p = 0$	3	29.591	1.576	1.056	3.620	1.370	-219.21
	4	29.347	1.627	1.070	3.712	1.364	-166.95
	0	13.829	1.296	8.167	4.627	1.670	1597.68
$S_1 = 9$	1	13.708	1.343	8.138	4.711	1.640	1631.07
$S_2 = 1$	2	14.737	1.507	8.667	3.722	1.815	1302.71
$p = 0.5$	3	14.797	1.527	8.704	3.618	1.834	1289.38
	4	14.677	1.627	8.656	3.707	1.822	1328.96
	0	0.000	1.296	15.107	4.627	1.723	3105.97
$S_1 = 9$	1	0.000	1.343	15.020	4.711	1.697	3125.29
$S_2 = 1$	2	0.000	1.507	16.061	3.722	1.865	2911.20
$p = 1$	3	0.000	1.577	16.129	3.617	1.883	2904.82
	4	0.000	1.627	16.020	3.706	1.870	2931.24

Bibliography

Abu-Salih, M., Anakesh, N.N. & Ahmed, M.S. (1990) Confidence limits for system steady state availability. *Pakistan Journal of Statistics*, Vol. 6, No. 2, pp. 189-196.

Aggarwal, S.C. (1974) A review of current inventory theory and its applications. *Int. J. Product. Res.*, 12, pp. 443-482.

Aksoy, Y. & Erengue, S.S. (1988) Multi-item inventory models with coordinated replenishment: A survey. *IJOPM*, 8, pp. 63-73.

Amiya, K. & Martin, G.E. (1988) Optimal multi-product inventory grouping for coordinated periodic replenishment under stochastic demand. *Computer and Opns. Res.*, 15, pp. 263-270.

Anbazhagan, N. (2002) Analysis of two commodity stochastic inventory systems. PhD thesis, Madurai-Kamaraj University, India.

Arora, J.A. (1976) Reliability of a two unit priority standby redundant systems with preventative maintenance. *IEEE Trans. Rel.*, Vol. R-25, 205-207.

Arrow, K.J., Karlin, S. & Scarf, H. (1958) Studies in mathematical theory of inventory and production. Stanford University Press.

Ascher, H.E. (1968) Evaluation of repairable system reliability using "bad as old" concept. *IEEE Trans. Rel.*, Vol. R-17, 103-110.

Atwood, C.L. (1986) The binomial failure rate common cause model. *Technometrics*, Vol. 28, pp. 139-147. (1986).

Atwood, C.L. & Stevenson, J. A. (1982) Common cause rates for diesel generators. *U.S. nuclear regulatory commission report.* NUREG/CR-2099, EGG-EA-5359 (Rev. 1).

Aven, T. (1996) Availability analysis of monotone systems. Reliability and maintenance of complex systems; *NATO ASI series, Series F: computer and system sciences*. Springer, Berlin, 154, 206-223.

Baker (1983) Optimal ordering and inventory control of perishable goods. *Naval Research Logistics Quarterly, (U.S.)* Vol. 30 (Dec), pp. 609-625.

Balagurusamy, E. (1984) *Reliability Engineering*. Tata, McGraw-Hill, New Delhi.

Barlow, R.E. (1962) *Repair man problems; studies in Applied probability and Management Science.* Stanford University Press, Stanford.

Barlow, R.E. (1984) *Repair man problems, studies in Applied Probability and Management Science.* Stanford University Press, Stanford.

Barlow R.E. & Proschan, F. (1965) *Statistical theory of reliability and life testing: probability models*, Holt, Rinehart & Winston, Inc.

Bartlett, M.S. (1954) Processus stochastiques ponctuels. *Ann. Inst. H Poinccarè*, 14, 36-60.

Bartlett, M.S. (1966) *An introduction to stochastic processes (2nd edition).* Cambridge University Press.

Beasley, E.I.M. (1991) *Reliability for engineers.* Hampshire, Macmillian.

Beichelt, F. & Fischer, K. (1980) General failure model applied to preventative maintenance policies. *IEEE Trans. Rel.*, Vol. 29, 39-41.

Beckman, M.J. (1961) An inventory model with arbitrary interval and quantity distributions of demands. *Mgmt. Sci.*, 8, pp. 35-57.

Beckman, M.J. & Srinivasan, S.K. (1987) An (s, S) inventory system with Poisson demand and exponential lead time. *O.R. Spektrum*, Vol. 9, pp. 213-217.

Bhabha, H.J. (1950) On the stochastic theory of continuous parametric systems and its application to electron-photon cascades. *Proc. Roy. Soci. London.* A., 202, pp. 301-332.

Billinton, R. & Allen, R.N. (1983) *Evaluation of engineering systems, concepts and techniques*. Phenum, New York.

Birolini, A. (1985) *Lecture notes in Economics and Mathematical systems.* No. 252, Springer-Verlag, Berlin.

Branson, M.H. & Shah, B. (1971) Reliability analysis of a system comprised of units with arbitrary distribution, *IEEE Trans. Rel.*, Vol 20, 217-223.

Butterworth, J.A.B. & Nikolaisen, T. (1973) Bounds on the availability function. *Naval Res. Log. Quarterly*, R-20, pp. 60-63.

Buzacott, J.A. (1970) Markov approach to finding failure times of repairable systems. *IEEE Trans. Rel.*, Vol. R-19, 128-134.

Buzacott, J. A. (1971) Availability of priority standby redundant systems. *IEEE Trans. Rel.*, Vol. R-20, 60-63.

Chakravarthy, A.K. (1981) Multi-item inventory aggregation into groups. *Journal of Operation Research Society*, 32, pp. 19-26.

Chandrasekhar, P. & Natarajan, R. (1994a) Confidence limits for steady state availability of a two unit standby system. *Microelectronics and Reliability*, Vol. 34, No. 7, pp. 1249-1251.

Chandrasekhar, P. & Natarajan, R. (1994b) Confidence limits for steady state availability of a parallel system. *Microelectronics and Reliability*, Vol. 34, No. 11, pp. 1847-1851.

Chandrasekhar, P. & Raghavan, K.S. (1992) Expected total sojourn times in a complex one-unit system; IAPQR Transactions. *J. Ind. Assoc. for productivity, quality and reliability*, Vol. 18, No. 2, pp. 21-28.

Chao, M.T., Fu, J.C. & Koutras, M.V. (1995) Survey of reliability studies of consecutive k out of $n : F$ related systems. *IEEE Trans. Rel.*, Vol. 44, No. 1, March 1995, pp. 120-127.

Chari, A.A. (1988) *A Markovian approach to system reliability and availability measures with common cause failures.* Ph.D. thesis, S.V. University, Tirurpati.

Chari, A.A., Sastry, M.P. & Verma, S.M. (1991) Reliability analysis in the presence of chance common cause shock failures. *Microelectronics & Reliability*, Vol. 31, pp. 15-19.

Chung, W.K. (1988a) A k out of $n : G$ redundant system with dependent failure rates and common-cause failures. *Microelectronics & Reliability*, Vol. 28, p. 201.

Chung, W.K. (1988b) An availability analysis of a k out of $n : G$ redundant system with dependent failure rates and common-cause failures. *Microelectronics & Reliability*, Vol. 28, p. 391.

Chung, W.K. (1990) Common-cause failures and critical human errors in repairable and non-repairable systems. *Microelectronics & Reliability*, Vol. 30, No. 2, pp. 243-247.

Çinlar, E. (1975) *Introduction to Stochastic Processes*, Englewood Cliffs: Prentice-Hall.

Çinlar, E. (1975a) Markov renewal theory; a survey. *Mgmt. Sci.*, 21, pp. 727-752.

Çinlar, E. (1975b) *Introduction to stochastic processes*. Prentice Hall.

Correnu, J.J. (1990) Economic lot scheduling for multiple products as parallel processors. *Mgmt. Sci.*, 36, pp. 348-358.

Cox, D.R. (1962) *Renewal Theory*. London: Methuen.

Cox, D.R. & Lewis, P.A.W. (1970) Multivariate point processes. *Proc. 6th Berkley Symp. Math. Statist, and Prob.*

Cox, D.R. & Smith, W.L. (1954) On the superposition of renewal processes. *Biometrika*, 41, pp. 91-99.

Daley, D.J. & Vere-Jones (1971) *A summary of the theory of point processes in stochastic processes*. (ed) A.W. Lewis. John Wiley & Sns. pp. 299-383.

Dhillon, B.S. (1979) On common-cause failures - a bibliography. *Microelectronics & Reliability*, Vol. 18, p. 533.

Dhillon, B.S. & Singh, C. (1981). *Engineering reliability: new techniques and applications*. John Wiley & Sons, New York, N.Y.

Dirickx, Y.M.I. & Kolvocts, D. (1977) A continuous review inventory model with compounds poisson demand process and stochastic lead time. *Nav. Logist. Quart.* 24, pp. 577-585.

Doyan, L.R. & Berssenbrugge, M.W. (1968) Solving complex reliability models by computer. *Proc. IEEE Ann. Symp. Rel.*, pp. 431-438.

Erlango, C. & Arivarignani, G. (2001) A lost sales inventory system with a set of reorder levels (in Russian). *Electronne Modelirovaine*, Vol. 23, No. 4, pp. 74-81.

Feller, W. (1950, 1957) *An introduction to probability theory and its applications.* New York, Wiley.

Feller, W. (1941) On the integral equation of renewal theory. *Ann. Math Stat ist.*, 12, pp. 243-267.

Feller, W. (1949) Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.*, 67, pp. 98-119.

Feller, W. (1968) *An introduction to Probability theory and its applications.* Vol. 1, 3rd Edn., John Wiley-Interscience, New York.

Finkelstein, M.S. (1993a) A scale model of general repair. *Microelectronicss and reliability*, Vol. 33, No. 1, pp. 41-44.

Finkelstein, M.S. (1993b) On some models of general repair. *Microelectronicss and reliability*, Vol. 33, No. 5, pp. 663-666.

Finkelstein, M.S. (1998) A point process stochastic model with application to safety at sea. *Reliability Engineering and System Safety*, 60N3, pp. 227-234.

Finkelstein, M.S. (1999a) Multiple availability on stochastic demand. *IEEE Trans. Rel.*, Vol. 48, No. 1, pp. 19-24.

Finkelstein, M.S. (1999b) Wearing-out of components in a variable environment. *Reliability Engineering & System Safety*, Vol. 66, No. 3, pp. 235-42.

Finkelstein, M.S. (1999c) A point process model for software reliability. *Reliability Engineering and System Safety*, Vol. 63N1, pp. 67-71.

Fujiwara, O. & Sedarage, D. (1997) An optimal (Q, r) policy for a mulitpart assembly system under stochastic part procurement lead times. *Euro. Jour. Oper. Res.*, Vol. 100, pp. 550-556.

Gaver, D.P. (1963) Time to failure and availability of parallel systems with repair. *IEEE Transactions on Reliability*, R-1:30-38.

Gertsbakh, I.B. (1989) *Statistical Reliability Theory.* New York & Basel: Marcel Dekker.

Gnedenko, Y.K. & Belyayev, Yu. K. & Solovyev, A.D. (1969) *Mathematical Methods of Reliability Theory.* New York & London: Academic Press.

Gnedenko, B.V. & Ushakov, I. (1995) *Probabilistic Reliability Engineering.* New York: Wiley & Sons.

Goyal, S.K. (1988) Economic ordering policy for jointly replenished items. *Int. J. Product. Res.*, 26, pp. 1237-1240.

Gray, H.L. & Lewis, T.O. (1967) Confidence interval for the availability rate. *Technometrics*, 9, 465.

Green, A.E. & Bourne, A.J. (1978) *Reliability technology.* John Wiley & Sons, New York.

Gürler, V. & Kara, A. (1996) An inventory problem with slow-moving substitutable products; *Proceedings: Stochastic models in engineering, technology and management*. pp. 167-176.

Hadley, G. & Whitin, T.M. (1963) *Analysis of Inventory systems.* Prentice Hall, Englewood Cliffs, N.J.

Hall Nicholas, G. (1988) Separable vs Joint Replenishment policies with maximum storage requirement costs. *Eur. J. Opl. Res.*, 36, pp. 180-185.

Harris, T.E. (1963) *The theory of branching processes.* Springer-Verlag, Berlin.

Holt Albert, C. (1986) Multi-item inventory control for fluctuating reorder intervals. *Interfaces*, 3, pp. 60-67.

Horrowitz, Abraham, D & Daganso, Carlos, S. (1986) A graphical method for optimizing a continuous review inventory system. *Production and inventory Management*, 27, pp. 30-45.

HØyland, A. & Rausand, M. (1994) *System Reliability Theory.* New York: John Wiley & Sons.

Huamin, L. (1998) Reliability of a Load-Sharing k-out-of-n : G System: Non-i.i.d Components with Arbitrary Distributions. *IEEE Transactions on Reliability*, Vol. 47, No. 3, pp. 279-284.

Jain, S. & Jain, R.K. (1994) Reliability analysis of a Markovian deteriorating system. *Microelectronics and Reliability*, Vol. 34, No. 12, pp. 1939-1941.

Kalpakam, S. & Arivarignan, G. (1985) A continuous review inventory system with arbitrary inter arrival times between demands and an exhibiting item subject to random failure. *Opsearch*, 22, pp. 153-168.

Kalpakam, S. & Arivarignan, G. (1994) Semi-Markov models in inventory systems. *Journal of Mathematical and Physical Sciences*, Vol. 5, pp. 51-517.

Kalpakam, S. & Arivarignan, G. (1988) A continuous review perishable inventory model. *Statistics*, 19, pp. 389-398.

Kalpakam, S. & Sapna, K.P. (1994) Continuous review $(s,\ S)$ inventory system with random lead times and postive lead times. *OR Letters*, 16, pp. 115-119.

Kalpakam, S. & Sapna, K.P. (1996) An $(s,\ S)$ Perishable system with arbitrary distributed lead times. *Opsearch*, 33, pp. 1-19.

Kamat, S.J. (1971) *A two-product inventory control model with substitution.* Presented at the 39th ORSA meeting, May 5-7 (1971).

Khitchine, A.J. (1960) *Mathematical methods in the theory of queueing.* Griffin, London.

Klassen, B. & Van Peppen, J.C.L. (1989) *System reliability: concepts and applications.* New York: Edward Arnold.

Kovalenco, I.N., Kuznetsoy, N. Yu. & Pegg, Ph. A. (1997) *Mathematical theory of reliability theory of time dependent systems with practical applications.* New York: John Wiley & Sons.

Krishnamoorthy, A. & Laxmy, B. (1990) An inventory system with Markov dependent reordering levels. *Opsearch*, 27, pp. 39-45.

Krishnamoorthy, A. & Manoharan, M. (1990) An $(s,\ S)$ inventory policy with independent non-identically distributed arrival demand times and random lead time. *Opsearch*, 27, pp. 93-101.

Kumagi, M. (1971) Reliability analysis for systems with repair. *J. Oper. Re. Soc.*, Japan. Vol. 14, pp. 53-71.

Kumar, A. (1989) Component inventory costs in an assembly problem with uncertain supplier lead-times. *IIE Transactions*, Vol. 21, pp. 112-121.

Kumar, A. & Agarwal, M.L. (1980) A review of standby systems. *IEEE Trans. Rel.*, R-29, 290-294.

Kumaraswamy, S. & Sankarasubramanian, E. (1981) A continuous review of $S - s$ inventory systems in which depletion is due to demand and failure of units. *Journal of Operational Research Society*, 32, pp. 997-1001.

Lévy, R.D. (1954) Processus semi-Markoviens. *Proc. Intern. Congr. Math.* (Amsterdam). 3, pp. 416-426.

Lie, C.H., Hwang, C.I. & Tillman, F.A. (1977) Availability of maintained systems: A state-of-the-art survey. *AIIE Trans.*, Vol. 9, No. 3, pp. 247-259.

Liu, H. (1998) Reliability of a load-sharing k-out-of-n : G system: non-iid components with arbitrary distributions. *IEEE Trans. Rel.*, Vol. 47, No. 3, pp. 279-284.

Lloyd, D.K. & Lipow, M. (1962) *Reliability: Management Methods and Mathmatics.* Prentice-Hall, Englewood Cliffs, NJ.

Masters, B.N. & Lewis, T.O. (1987) A note on the confidence interval for the availability ratio. *Microelectronics and Reliability*, Vol. 27, p. 247.

Masters, B.N., Lewis, T.O. & Kolariok, W.J. (1992) A confidence interval for the availability ratio for systems with Weibull operating time and lognormal repair time. *Microelectronics and Reliability*, Vol. 32, pp. 89-99.

McGillivray & Silver, E.A. (1978) Some concepts for inventory control under substitutable demands. *Information system and Operational Research*, 23, pp. 47-63.

Medhi, J. (1994) *Stochastic Processes*. Wiley Eastern Ltd.

Meachum, T.R. & Atwood, C.L. (1983) *Common cause fault rates for instrumentation and control assemblies*. NUREG/CR-3289, EGG-2258.

Moyal, J.E. (1962) The general theory of stochastic population processes. *Act. Math.*, 108, pp. 1-31.

Nahmias, S. (1978) Inventory models. *In the Encyclopedia of Computer Science and Tech.*, Vol. 9, Belzer, A., Holzman, A. and Kent, A., (eds.), Mercel Dekker, New York.

Nahmias, Stevens, (1982) Perishable inventory theory: A review. *Opns. Res. (U.S.)* 30, 4 (July-August) pp. 680-708.

Nahmias, Stevens & Schmidt, C.A. (1986) An application of the theory of weak convergence to the dynamic perishable inventory problem with discrete demand. *MOR (U.S.)* 11, (Feb) pp. 62-69.

Nakagawa, T. (1974) The expected number of visits to state *k* before a total system failure of complex system with repair maintenance. *Oper. Res.*, 22, pp. 108-116.

Nakagawa, T. & Osaki, S. (1974) Stochastic behaviour of two-unit standby redundant system. *INFOR* 12, pp. 66-70.

Nakagawa, T. & Osaki, S. (1976) Markov renewal processes with some non-regeneration points and their application to renewal theory. *Micro. Rel.*-15, pp. 633-636.

Natarajan, R. (1980) *Stochastic models of standby redundant systems*, Ph D Thesis, Dept. of Mathematics, I.I.T., Madras, Indian.

Neuts, M.F. (1978) Renewal processes of phase type. *Naval Res. Log. Quart.*, 25, pp. 445-454.

Osaki, S. (1969) *System reliability and signal flow graphs*, presented at the meeting of the Operations Research Society of Japan, May 21-22. Tokyo, Japan.

Osaki, S. (1970a) System reliability analysis by Markov renewal processes. *J. Oper. Res. Soc. Japan*, Vol. 12, pp. 127-188.

Osaki, S. (1970b) Reliability analysis of a two-unit standby redundant system with priority. *CORSJ*, Vol. 8, pp. 60-62.

Osaki, S. & Nakagawa, T. (1976) Bibliography for reliability of stochastic systems. *IEEE Trans. Rel.*, Vol. R-25, pp. 284-287.

Ozekici, S. (1996) Complex systems in random environments. Reliability and maintenance of complex systems. *NATO ASI series, Series F: computer and system sciences.* Springer, Berlin, 154, 137-157.

Oneiva, Luis & Larraneta, Juan. (1987) Methods para la determinacion del tamano del lote en articulos subjectos a ordenes conjutas (Heuristic rules in the joint ordering replenishment problem). *QUESTIIO,* 11, pp. 61-83.

Page, E. & Paul, R.J. (1976) Multi-product inventory situation with one restriction. *Operat. Res. Quart.*, 27, pp. 815-834.

Parlar, M. (1985) Optimal ordering policies for a perishable and substitutable product - a Markov decision model. *INFor (Canada)*, 23, pp. 182-195.

Parlar, M. (1988) Game theoretic analysis of the substitutable product in inventory problems with random demands. *Naval Res. Log. Quart.*, 35, pp. 397-409.

Parlar, M. (1984) Optimal ordering policies for two substitutable products with stochastic demands. *Opsearch*, 21, pp. 1-15.

Parlar, M. & Goyal, S.K. (1984) Optimal ordering policies for two substitutable products with stochastic demands. *Opsearch*, Vol. 21, pp.1-15.

Parzen, E. (1962) *Stochastic Processes.* Holden-Day, McGraw Hill, New York.

Pasternak, B.A. & Drezner, Z.D. (1991) Optimal policies for substitutable commodities with stochastic demand. *Naval Research Logistics Quarterly, (U.S.)* Vol. 38, pp. 221-240.

Perry & Posner, M.J.M. (1990) Control of input and demand rates in inventory systems of perishable commodities. *NRL (U.S.)* 37, 1 (Feb) pp. 85-98.

Perumal, V., Anbazhagan, N. & Arivarignan, G. *A Markovian inventory system with set of reorders and state dependent lead times.* Proceedings, conference on Mathematical Computational Models, Dec. 2001, Coimbatore, pp. 73-76.

Pierskalla, P. & Voelker, A. (1976) A survey of maintenance models: the control of surveillance of deteriorating systems. *Naval Res. Logistics Qly.*, Vol. 23, No. 3, pp. 353-388.

Prabhu, N.U. (1965) *Stochastic Processes: Basic theory and its applications.* The Macmillan Co., New York.

Pyke, R. (1961a) Markov renewal processes: definitions and preliminary properties. *Ann. Math. Statist.* 32, pp. 1231-1242.

Pyke, R. (1961b) Markov renewal processes with finitely many states. *Ann. Math. Statist.* 32, pp. 1243-1259.

Raafat, F. (1991) Survey of Literature on continuously deteriorating inventory models. *J. Opl. Res. Soc.*, 42, pp. 27-37.

Ramakrishnan, A. (1950) Stochastic processes relating to particles distribution in an infinity of states. *Proc. Camb. Phil. Soc.*, 46, pp. 595-602.

Ramakrishnan, A. (1958) *Probability and stochastic processes*, in Handbuch des physik, 13, Springer-Verlag, Berlin.

Ramanarayanan, R. & Jacob, M.J. (1987) General analysis of (S, s) inventory system with random lead times and bulk demand. *CCERO*, 29, pp. 239-246.

Rao, C.R. (1974) *Linear statistical inference and its applications.* New York: John Wiley & Sons.

Rau, J.G. (1964) *Optimization and probability in systems engineering.* Princeton, Von Nostrand.

Ravichandran, N. (1979) *Reliability analysis of redundant repairable systems*. Thesis for Ph D in Mathematics, I.I.T., Madras, India.

Ravichandran, N. (1988) Probablistic analysis of a continuos review perishable inventory system with Markovian demand, Erlangian life and non-instantaneous leadtime. *OR Spektrum*, 10, pp. 23-27.

Ross, S.M. (1970) *Applied probability models with optimization applications*. San Francisco: Holden-Day.

Rubenstein, R. (1981) *Simulation and Monte Carlo method.* Wiley, New York.

Sandler, G.H. (1963) *System reliability engineering*. Englewood Cliffs, N.J. Prentice Hall.

Sarma, K.V.S. (1987) A deterministic order level inventory model for deteriorating items with two storage facilities. *Eur. J. Opl. Res.*, 29, pp. 70-73.

Sarma, Y.S.S. (1982) *Stochastic models of redundant repairable systems*. Ph D Thesis, I.I.T., Madras, India.

Scheuer, E.M. (1988) Reliability of an m-out-of-n system when component failure induces higher failure rates in survivors. *IEEE Transactions on Reliability*, Vol. 37, March, 73-74.

Schmidt, C.P. & Nahmias, S. (1985) Optimal policy for a two-stage assembly system under random demand. *Oper. Res.*, Vol. 33, pp. 1130-1145.

Sfakianakis, M.E. & Papastavridis, S.G. (1993) Reliability of a general consecutive-k-out-of-n : F system. *IEEE Trans. Rel.,* Vol. 42, No. 3, pp. 491-495.

Shao, J. & Lamberson, L.R. (1991) Modeling a shared-load k-out-of-n : G system. *IEEE Transactions on Reliability*, Vol. 40, June, pp. 205-209.

Shooman, M.L. (1968) *Probabilistic reliability: an engineering approach.* McGraw-Hill Co., New York, U.S.A.

Silver, E.A. (1981) Operations research in inventory management. A review critique. *Opns. Res.*, 29, pp. 628-645.

Smith, W.L. (1955) Regenerative stochastic processes. *Proc. Roy. Soc. London Ser*: A232, 6-31.

Smith, W.L. (1958) Renewal theory and its ramification. *J. R. Statist. Soc.*, B, 20, 243-302.

Srinivasan, S.K. (1971) Stochastic point processes and statistical physics. *J. Math. Phy. Sci.*, 5, pp. 291-316.

Srinivasan, S.K. (1974) *Stochastic point processes and their applications.* Griffin, London.

Srinivasan, S.K. (1989) Analysis of $(S,\ s)$ inventory systems with general lead time and demand distribution. *Optimization*, 19, pp. 557-576.

Srinivasan, S.K. & Subramanian, R. (1980) *Probabilistic Analysis of Redundant Sytems.* Lecture notes in Economics and Mathematical systems, no. 175. Springer-Verlag, Berlin.

Srinivasan, V.S. (1966) The effect of standby redundancy in systems failure with repair maintenance. *Oper. Res.*, Vol. 14, 1024-1036.

Subba Rao, S. & Natarajan, R. (1970) Reliability with standbys, *Opserach,* Vol. 7, pp. 23-36.

Sung, C.S. & Chang, S.H. (1986) A capacity planning model for multi-product-facility. *Engineering Optimization*, 10, pp. 263-270.

Takács, L. (1957) On certain problems concerning the theory of counters. *Act. Math. Hung*, 8, pp. 127-138.

Thomson, M. (1966) Lower confidence limits and a test of hypothesis for a system availability. *IEEE Trans. Rel.*, R-15, pp. 32-36.

Venkatakrishnan, K. (1975) *Probabilistic analysis of repair redundant systems.* Thesis for Ph D in Mathematics, I.I.T., Madras, India.

Veseley, W.E. (1977) *Estimating common cause failure probabilities in reliability and risk analysis: Marshall-Olkin specializations in nuclear systems reliability engineering and risk assessment (edited by J. B. Fussell and G. R. Burdick).* pp. 314-341, Society for industrial and applied mathematics. Philadelphia.

Villemeur, A. (1992) *Reliability Availability, Maintainability and Safety Assessment.* Chichester: John Wiley & Sons.

Watson, H.W. & Galton, F. (1874) On the probability of extinction of families. *J. Anthropol. Inst.* Great Britain and Ireland, 4, pp. 138-44.

Weiss, H.J. (1988) Sensitivity of continuous review (S, s) inventory systems to ordering delays. *Eur. J. Opl. Res.*, 36, pp. 174-179.

Wijingaard, J. & Winkel, E.G.F. Van.,(1979) Average cost in a continuous review (S, s) inventory system with exponentially distributed lead time. *Opns. Res.*, 27, pp. 396-401.

Wold, H. (1948) Sur les processus stationnairies ponctuels. *Colloques Internationaux, C.N.R.S.* 13, pp. 75-86.

Yadavalli, V.S.S., Botha, M. & Bekker, A. *A symtotic confidence limits for the steady state availability of a two unit parallel system with "preparation time" for the repair factility.* Accepted for publication in Asia-Pacific Journal of Operations Research (to appear in 2002).

Yano, C.A. (1987) Stochastic lead-times in two-level assembly systems. *IIE Transactions*, Vol. 19, pp. 371-378.

Yearout, R.D., Reddy, P. & Grosh, D.L. (1986) Standby redundancy in reliability - a review. *IEEE Trans. Rel.*, R-35, pp. 285-292.

Zacks, S. (1992) *Introduction to Reliability Analysis.* Berlin: Springer-Verlag.

Zhang, Y.L. & Lam, Y. (1998) Reliability of consecutive-k-out-of-n : G repairable system. *International journal of Systems Science* Vol. 29, No. 12, pp. 1375-1379.