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Citation

A characterization of single-peaked preferences via random social choice functions

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This paper proves the following result: every path-connected domain of preferences that admits a strategy-proof, unanimous, tops-only random social choice function satisfying a compromise property is single-peaked. Conversely, every single-peaked domain admits a random social choice function satisfying these properties. Single-peakedness is defined with respect to arbitrary trees. The paper provides a justification of the salience of single-peaked preferences and evidence in favor of the Gul conjecture (Barberà 2010).

KEYWORDS. Random social choice functions, strategy-proofness, compromise, single-peaked preferences.

JEL CLASSIFICATION. D71.

1. INTRODUCTION

Single-peaked preferences are the cornerstone of several models in political economy and social choice theory. They were proposed initially by Black (1948) and Inada (1964), and can be informally described as follows. The set of alternatives is endowed with a structure that enables one to say for some triples of alternatives, say a, b, and c, that b is "closer" to a than c. On a preference order that is single-peaked, if an alternative b is closer to the maximal element in the preference than another alternative c, it must be the case that b is ranked above c in the preference. A domain of preferences is single-peaked if there is a *common* structure on alternatives such that all preference orders in

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the domain are single-peaked with respect to that structure. Single-peaked preferences arise naturally in various settings. However, their main attraction is that they allow successful preference aggregation both in the Arrovian and the strategic sense (Moulin 1980, Barberà 2010), for instance, by the median voter aggregator/social choice function. Our goal in this paper is to provide a converse result with the following flavor: any "rich" preference domain that admits a suitably "well behaved" solution to the strategic voting problem *must* be a single-peaked domain.[1]

Our model consists of a finite number of voters and alternatives.[2] We consider *random* social choice functions (RSCFs) defined on a suitably rich but arbitrary domain of preference orders. A RSCF associates a probability distribution over alternatives with every profile of voters' preference orders in the domain. Following Gibbard (1977), a RSCF is *strategy-proof* if truth-telling by a voter results in a probability distribution that first-order stochastically dominates the probability distribution that arises from any misrepresentation by the voter. Moreover, this holds for every possible profile of other voters so that truth-telling is a (weakly) dominant strategy in the revelation game. In addition to strategy-proofness, we impose three other requirements on RSCFs under consideration. Two of these, *unanimity* and *tops-onlyness*, are standard in the literature on voting. The third axiom is the *compromise* property. Consider a preference profile where the set of voters is split into two equal groups[3] and the following conditions are satisfied: (i) all voters within a group have identical preferences; (ii) the peaks of the preferences of the two groups are different; (iii) there is an alternative that is second-ranked according to the preferences of both groups. This commonly second-ranked alternative can be regarded as a *compromise* alternative and the axiom requires that this alternative receives strictly positive probability in the profile. According to our main result, any rich domain that admits a strategy-proof, tops-only RSCF satisfying unanimity and the compromise property must be single-peaked. Conversely, any single-peaked domain admits a strategy-proof, tops-only RSCF satisfying ex post efficiency (a stronger version of unanimity) and the compromise property.

The single-peaked domain characterized by our result is more general than the usual one (for example, in Moulin 1980). These preferences were introduced in Demange (1982) and Danilov (1994), and are defined on arbitrary *trees*.

It is natural to allow for randomization whenever there are conflicts of interest among agents. Randomization also facilitates truth-telling because the evaluation of lotteries using the expected utility hypothesis imposes preference restrictions. Recently Chatterji et al. (2014) have shown that randomization can significantly enlarge the class of strategy-proof and unanimous rules in dictatorial domains. Results characterizing the class of strategy-proof and unanimous RSCFs for single-peaked domains (on the line) have been obtained in Ehlers et al. (2002), Peters et al. (2014), and Pycia and Ünver (2015).

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[1]Claims of this nature have been referred to as the Gul conjecture in Barberà (2010) and attributed to Faruk Gul. The precise formulation of the conjecture can take several forms. Our result can be regarded as further evidence in favor of the conjecture.

[2]The number of alternatives is assumed to be at least three.

[3]If there is an odd number of voters, the two groups are "almost" equal.

Another consequence of considering RSCFs is that the anonymity requirement (anonymity implies that the names of voters do not matter and reshuffling preferences across voters does not affect the social outcome) imposed on *deterministic* social choice functions (DSCFs) to rule out dictatorship must be replaced, because it is always possible to design a strategy-proof RSCF that satisfies anonymity. To see this, consider an arbitrary domain and the RSCF that picks the top-ranked alternative of voter i with probability $1/N$ at each profile (N is the number of voters).[4] This RSCF is a particular instance of a random dictatorship (Gibbard 1977) and is strategy-proof, anonymous, ex post efficient, and tops-only. However, it suffers from a well recognized defect: it does not permit society to put positive probability on an alternative unless it is top-ranked for some voter, even though the alternative may be highly ranked (say second-ranked) for all voters. The present paper introduces the compromise axiom, which is a natural way to ensure that social decisions give strictly positive probability to a compromise alternative whenever it exists. In conjunction with the other assumptions on the RSCF, we find that it implies that the domain must be single-peaked.

A paper related to ours is Chatterji et al. (2013). That paper investigated preference domains that admit well behaved and strategy-proof DSCFs. In particular, it showed that every rich domain that admits a strategy-proof, unanimous, anonymous, and tops-only DSCF with an even number of voters is *semi-single-peaked*.[5] These preferences are also defined on trees but are significantly less restrictive than single-peaked preferences. Our paper demonstrates that two important objectives can be met by considering RSCFs rather than DSCFs. The first is that a characterization of single-peaked rather than semi-single-peaked preferences can be obtained naturally. The second is that the awkward assumption regarding the even number of voters in Chatterji et al. (2013) can be removed.

The paper is organized as follows. Section 2 and various subsections present the model, definitions, and axioms. Section 3 contains the characterization result for single-peaked domains and demonstrates the indispensability of each of our axioms and the richness condition. Section 4 concludes. Appendices A and B contain some additional discussion and an omitted proof.

2. Model and notation

Let $A = \{a_1, a_2, \ldots, a_m\}$ be a finite set of alternatives with $m \geq 3$. Let $\Delta(A)$ denote the lottery space induced by A. An element of $\Delta(A)$ is a *lottery* or probability distribution over the elements of A. For every $a_j \in A$, let $e_j \in \Delta(A)$ denote the degenerate lottery where alternative a_j gets probability 1.

Let $I = \{1, \ldots, N\}$ be a finite set of voters with $|I| = N \geq 2$. Each voter i has a (strict preference) order P_i over A that is antisymmetric, complete and transitive, i.e., a linear

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[4]This description is valid for profiles where the peaks of voters are distinct. A complete description can be found in the definition of a random dictatorship at the end of Section 2.1.

[5]The notion of richness is exactly the same as that in our paper.

order. For any $a_j, a_k \in A$, $a_j P_i a_k$ is interpreted as "a_j is *strictly* preferred to a_k according to P_i." Let $\mathbb{P}$ denote the set containing *all* linear orders over A. The set of all admissible orders is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as the *preference domain*. A *preference profile* $P \equiv (P_1, P_2, \ldots, P_N) \in \mathbb{D}^N$ is an N-tuple of orders.

For any $P_i \in \mathbb{D}$, $r_k(P_i)$ denotes the kth ranked alternative in P_i, $k = 1, \ldots, m$. For any $P \in \mathbb{D}^N$, $r_1(P) = \bigcup_{i \in I}\{r_1(P_i)\}$ denotes the set of voters' peaks or first-ranked alternatives.

2.1 *Random social choice functions and their properties*

A *random social choice function* (RSCF) is a map $\varphi : \mathbb{D}^N \to \Delta(A)$. At every profile $P \in \mathbb{D}^N$, $\varphi(P)$ is the "socially desirable" lottery. For any $a_j \in A$, $\varphi_j(P)$ is the probability with which a_j will be chosen in the lottery $\varphi(P)$. Thus, $\varphi_j(P) \geq 0$ and $\sum_{j=1}^m \varphi_j(P) = 1$.

A *deterministic social choice function* (DSCF) is a RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ where the outcome at every preference profile is a degenerate probability distribution, i.e., $\varphi(P) = e_j$ for some $a_j \in A$ at profile P.

A RSCF satisfies *unanimity* if it assigns probability 1 to any alternative that is ranked first by all voters, i.e., RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies unanimity if $[r_1(P_i) = a_j$ for all $i \in I] \Rightarrow [\varphi(P) = e_j]$ for all $a_j \in A$ and $P \in \mathbb{D}^N$.

An axiom stronger than unanimity is *ex post efficiency*. It requires all Pareto-dominated outcomes to never be chosen. Formally, the RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is ex post efficient if for all $a_j, a_k \in A$ and $P \in \mathbb{D}^N$, $[a_j P_i a_k$ for all $i \in I] \Rightarrow [\varphi_k(P) = 0]$.

Voters' preferences are assumed to be private information. It is important therefore for voters to have appropriate incentives to reveal their private information *truthfully*. A RSCF is *strategy-proof* if truth-telling is a (weakly) dominant strategy for every voter, i.e., the truth-telling lottery first-order stochastically dominates the lotteries arising from misrepresentation. Formally, the RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is *strategy-proof* if $\sum_{k=1}^t \varphi_{r_k(P_i)}(P_i, P_{-i}) \geq \sum_{k=1}^t \varphi_{r_k(P_i)}(P_i', P_{-i})$, $t = 1, \ldots, m$, holds for all $i \in I$; $P_i, P_i' \in \mathbb{D}$, and $P_{-i} \in \mathbb{D}^{N-1}$. This notion of strategy-proofness was first formulated in Gibbard (1977). It is equivalent to requiring a voter's expected utility from truth-telling to be no less than her expected utility from misrepresentation of *any* cardinal representation of her true preferences. We omit these details, which may be found in Gibbard (1977).

A prominent class of RSCFs is the class of *tops-only* RSCFs. The value of these RSCFs at any profile depends only on voters' peaks at that profile. The RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies the *tops-only* property if $[r_1(P_i) = r_1(P_i')$ for all $i \in I] \Rightarrow [\varphi(P) = \varphi(P')]$ for all $P, P' \in \mathbb{D}^N$. Tops-only RSCFs have obvious informational and computational advantages; for this reason, they (more accurately, DSCFs) have received a great deal of attention in the literature (see Weymark 2008 and Chatterji and Sen 2011).

The notions of unanimity, ex post efficiency, strategy-proofness, and tops-onlyness are standard axioms in the literature on mechanism design in voting environments. Below, we first introduce a mechanism that satisfies all axioms mentioned above.

Consider the RSCF known as *random dictatorship*. Each voter is assigned a nonnegative weight, with the sum of weights across voters being 1. At any profile, the probability with which an arbitrary alternative a_j is chosen is the sum of the probability weights of voters for whom a_j is the first-ranked alternative. Random dictatorships satisfy all the

properties discussed above: they are ex post efficient, tops-only, and strategy-proof. If the weights are $1/N$, they also satisfy the property of *anonymity*, i.e., they do not depend on the "names" of voters.[6] Yet they suffer from an important and well known infirmity: they do not admit *compromise*. Imagine a two-voter world with several alternatives (say, a thousand). Consider a profile where voter 1's first-ranked and thousandth-ranked alternatives are a_j and a_k, respectively. Alternatively, voter 2's first-ranked and thousandth-ranked alternative are a_k and a_j, respectively. Suppose, in addition, that there is an alternative say a_r that is highly ranked by *both* voters, for instance, ranked second by both. A reasonable RSCF should put at least some probability weight on a_r, but no random dictatorship would.

2.2 *The compromise property*

We introduce a new axiom so as to deal with the difficulties associated with random dictatorships outlined above. The axiom requires some compromise alternatives in certain profiles to be selected by the RSCF with strictly positive probability.

Let $P_i, P_j \in \mathbb{D}$ be such that $r_1(P_i) \neq r_1(P_j)$. Let $C(P_i, P_j) = \{a_r \in A \mid a_r = r_2(P_i) = r_2(P_j)\}$. Note that $C(P_i, P_j)$ is either empty or contains a singleton.

Let $\hat{I} \subset I$ be a nonempty strict subset of voters.   For any $P_i, P_j \in \mathbb{D}$, let $(\underbrace{P_i, \ldots, P_i}_{\hat{I}}, \underbrace{P_j, \ldots, P_j}_{I \setminus \hat{I}})$ denote the profile where all voters in $\hat{I}$ have the order P_i while those not in $\hat{I}$ have P_j.

DEFINITION 1. A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies the compromise property if there exists $\hat{I} \subseteq I$ with $|\hat{I}| = N/2$ if N is even and $|\hat{I}| = (N+1)/2$ if N is odd, such that for all $P_i, P_j \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P_j)$ and $C(P_i, P_j) \equiv \{a_r\}$, we have $\varphi_r(\underbrace{P_i, \ldots, P_i}_{\hat{I}}, \underbrace{P_j, \ldots, P_j}_{I \setminus \hat{I}}) > 0$.

The axiom requires the existence of a subset of voters $\hat{I}$ that is approximately half the size of the set of voters. Pick an arbitrary profile where all voters in $\hat{I}$ have identical preferences as do voters in the complement set $I \setminus \hat{I}$. Suppose the common preferences in $\hat{I}$ and $I \setminus \hat{I}$ have distinct peaks but have a common second-ranked alternative a_r. According to the axiom, the RSCF must give a_r strictly positive probability at the profile.

We believe that the axiom is both weak and natural. It is weak because it applies to a very narrow class of profiles. It is natural because in the profile where it applies, the alternative to which strictly positive probability is assigned according to the axiom, is an obvious compromise between the two groups of voters.

We make two remarks about the set $\hat{I}$ in Definition 1. The first is that Definition 1 merely requires the existence of one such set of voters. A stronger but equally plausible axiom would require the property to hold for all subsets $\hat{I}$ such that $|\hat{I}| = N/2$ if N is even and $|\hat{I}| = (N + 1)/2$ if N is odd. We make the weaker assumption because the stronger one is not required for our result. Note however that once $\hat{I}$ is fixed,

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[6] A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is *anonymous* if for every permutation $\sigma : I \to I$ and $P \in \mathbb{D}^N$, $\varphi(P_1, \ldots, P_N) = \varphi(P_{\sigma(1)}, \ldots, P_{\sigma(N)})$.

the strictly positive probability requirement on the compromise applies to *all* profiles
$\underbrace{(P_i, \ldots, P_i}_{\hat{I}}, \underbrace{P_j, \ldots, P_j)}_{I \setminus \hat{I}}$.

The second remark is to point out that the choice of the cardinality of $\hat{I}$ in Definition 1 is arbitrary. As footnote 11 points out, *any* choice of the cardinality of $\hat{I}$ works for our proof, provided $0 < |\hat{I}| < N$. We could have assumed, for instance, that $|\hat{I}| = 2$ or $|\hat{I}| = N - 1$. We could even have left $|\hat{I}|$ unspecified. We have however chosen $|\hat{I}|$ to be approximately half of N because we feel that it is the compelling case for the axiom to hold.

2.3 *Domains*

Our goal in this paper is to characterize preference domains that admit RSCFs satisfying the properties described in the previous subsection. However, we need to restrict attention to domains that satisfy a regularity condition that we call *path-connectedness*.

The path-connectedness condition was introduced in Chatterji et al. (2013).[7] Fix a domain $\mathbb{D}$. A pair of distinct alternatives $a_j, a_k \in A$ satisfies the *free pair at the top* (FPT) property if there exist $P_i, P_i' \in \mathbb{D}$ such that (i) $r_1(P_i) = r_2(P_i') = a_j$, (ii) $r_2(P_i) = r_1(P_i') = a_k$, and (iii) $r_t(P_i) = r_t(P_i')$, $t = 3, \ldots, m$. In other words, two alternatives satisfy the FPT property if there exists a pair of admissible orders where the alternatives are at the top of both orders and are locally switched, i.e., all alternatives other than the specified pair are ranked in the same way in both orders. Let FPT($\mathbb{D}$) denote the set of alternative pairs that satisfy the FPT property. The domain $\mathbb{D}$ is *path-connected* if for every pair of alternatives $a_j, a_k \in A$, there exists a sequence $\{x_t\}_{t=1}^{T} \subseteq A$, $T \geq 2$, such that $x_1 = a_j$, $x_T = a_k$ and $(x_t, x_{t+1}) \in$ FPT($\mathbb{D}$), $t = 1, \ldots, T - 1$.

The path-connectedness assumption imposes structure on the domain. It allows the construction of paths between admissible orders by switching preferences at the top of the orders. Very similar conditions have been identified in Carroll (2012) and Sato (2013) as being critical for the purpose of identifying domains where *local incentive-compatibility* ensures strategy-proofness.[8]

Chatterji et al. (2013) provide an extensive discussion of well known domains that satisfy the path-connectedness assumption. The complete domain and the single-peaked domain are path-connected. Maximal single-crossing domains (Saporiti 2009) are path-connected provided that every alternative is first-ranked in some order in the domain. A generalized single-peaked domain (Nehring and Puppe 2007) may or may not be path-connected. Alternatively, the separable domain (Barberà et al. 1991, Le Breton and Sen 1999) and the multidimensional single-peaked domain (Barberà et al. 1993) are not path-connected. For details the reader is referred to Examples 1, 2, and 3 in Chatterji et al. (2013).

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[7]Slightly different names were used in Chatterji et al. (2013) for the free pair at the top property and path-connectedness. We believe that the new names are more apposite.

[8]Assume that every alternative is first-ranked in some preference. Then domains of ordinal preferences studied in both Carroll (2012) and Sato (2013) are path-connected.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
a_1	a_1	a_2	a_2	a_2	a_3	a_3	a_4	a_4
a_2	a_2	a_1	a_4	a_3	a_2	a_2	a_2	a_2
a_4	a_3	a_4	a_3	a_4	a_4	a_1	a_3	a_1
a_3	a_4	a_3	a_1	a_1	a_1	a_4	a_1	a_3

TABLE 1. Domain $\bar{\mathbb{D}}$.

A domain of central importance in collective choice theory is the *single-peaked* domain. It was originally introduced in Black (1948) and Inada (1964). Here we consider a generalization due to Demange (1982) and Danilov (1994).

An undirected graph $G = \langle V, E \rangle$ is a set of vertices V and a set of edges E. The set E consists of pairs vertices, i.e., $E \subseteq \{(u, v) \mid u, v \in V \text{ and } u \neq v\}$. If $(u, v) \in E$, we say that (u, v) is an edge in G.[9] A path in G is a sequence $\{v_k\}_{k=1}^s \subseteq V$ where $s \geq 2$ and $(v_k, v_{k+1}) \in E$, $k = 1, \ldots, s - 1$. The graph G is *connected* if there exists a path between every pair of vertices, i.e., for all $u, v \in V$ with $u \neq v$, there exists a path $\{v_k\}_{k=1}^s$ such that $u = v_1$ and $v = v_s$. The connected graph G is a tree if the path between every pair of vertices is unique. Let G be a tree and let $u, v \in V$ be a pair of vertices. We denote the unique path between them by $\langle u, v \rangle$.[10]

In what follows, we shall consider graphs G of the kind $G = \langle A, E \rangle$, i.e., whose vertex set is the set of alternatives.

DEFINITION 2. Let $G = \langle A, E \rangle$ be a tree. The order P_i is single-peaked on G if for all $a_j, a_k \in A$,

$$\left[a_j \in \langle r_1(P_i), a_k \rangle \setminus \{a_k\} \right] \Rightarrow [a_j \, P_i \, a_k].$$

Pick a preference P_i and an arbitrary alternative a_k. Since the graph is a tree, there is a unique path between $r_1(P_i)$ and a_k. The order P_i is single-peaked if every alternative a_j on this path that is distinct from a_k is strictly preferred to a_k according to P_i.

A domain $\mathbb{D}$ is single-peaked if there exists a tree G such that $P_i \in \mathbb{D}$ implies P_i is single-peaked on G.

A case of special interest is the one where the graph $G = \langle A, E \rangle$ is a *line*. Formally, G is a line if there exists a permutation $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that $E = \{(a_{\sigma(k)}, a_{\sigma(k+1)})\}_{k=1}^{m-1}$. The standard definition of a single-peaked domain is one where the underlying graph is a line. We illustrate these notions with some examples below.

EXAMPLE 1. Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\bar{\mathbb{D}}$ is described in Table 1.

The domain $\bar{\mathbb{D}}$ is single-peaked on the tree G^T shown in Figure 1.

Note that there are orders that are single-peaked on G^T but not included in $\bar{\mathbb{D}}$, for instance, $a_2 \, P_{10} \, a_1 \, P_{10} \, a_3 \, P_{10} \, a_4$. The largest single-peaked domain on G^T contains 12 orders. $\diamond$

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[9] In an undirected graph, (u, v) and (v, u) represent a same edge.

[10] In particular, if $u = v$, $\langle u, v \rangle = \{u\}$ is a singleton set.
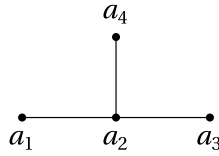
FIGURE 1. Tree G^T.

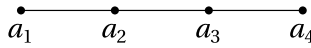| P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a_1	a_2	a_2	a_2	a_3	a_3	a_3	a_4
a_2	a_1	a_3	a_3	a_2	a_2	a_4	a_3
a_3	a_3	a_4	a_1	a_1	a_4	a_2	a_2
a_4	a_4	a_1	a_4	a_4	a_1	a_1	a_1

TABLE 2. Domain $\hat{\mathbb{D}}$.

FIGURE 2. Line G^L.

EXAMPLE 2. Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\hat{\mathbb{D}}$ is described in Table 2.

The domain $\hat{\mathbb{D}}$ is single-peaked on the line G^L shown in Figure 2.

In contrast to domain $\bar{\mathbb{D}}$ in Example 1, domain $\hat{\mathbb{D}}$ includes *all* orders that are single-peaked on G^L. Observe also that $\bar{\mathbb{D}}$ is not single-peaked on a line; neither is $\hat{\mathbb{D}}$ single-peaked on G^T. To verify the former claim, observe that any domain that is single-peaked on a line must have at least two alternatives that have unique orders where these alternatives are peaks (these are the alternatives at either end of the line); there are no alternatives with this property in $\bar{\mathbb{D}}$. Alternatively, the maximal number of alternatives that can be second-ranked to a given alternative on any domain that is single-peaked on a line is two, whereas on domain $\bar{\mathbb{D}}$, the alternative a_2 has three distinct second-ranked alternatives a_1, a_3, and a_4. ◇

3. MAIN RESULT: SINGLE-PEAKEDNESS

Our main result characterizes single-peaked domains.

THEOREM. *Every path-connected domain that admits a unanimous, tops-only, and strategy-proof RSCF satisfying the compromise property is single-peaked. Conversely, every single-peaked domain admits an ex post efficient, tops-only, and strategy-proof RSCF satisfying the compromise property.*

PROOF. We first prove necessity. Assume that $\mathbb{D}$ is path-connected. In addition, there exists a RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ that is tops-only, strategy-proof, and unanimous, and satisfies the compromise property. We will show that there exists a tree G such that $\mathbb{D}$ is single-peaked on G.

The first four lemmas establish critical properties of the RSCF φ.

LEMMA 1. *Let $a_j, a_k \in A$ with $(a_j, a_k) \in \text{FPT}(\mathbb{D})$. Let $P_i, P_i' \in \mathbb{D}$ be such that (i) $r_1(P_i) = r_2(P_i') = a_j$, (ii) $r_2(P_i) = r_1(P_i') = a_k$, and (iii) $r_t(P_i) = r_t(P_i')$, $t = 3, \ldots, m$. Then, for all $P_{-i} \in \mathbb{D}^{N-1}$, $\varphi_j(P_i, P_{-i}) + \varphi_k(P_i, P_{-i}) = \varphi_j(P_i', P_{-i}) + \varphi_k(P_i', P_{-i})$ and $\varphi_t(P_i, P_{-i}) = \varphi_t(P_i', P_{-i})$ for all $a_t \in A \setminus \{a_j, a_k\}$.*

Suppose voter i switches her order from P_i to P_i', a move that involves the reshuffling of the top two alternatives, say a_j and a_k, while leaving all other alternatives unaffected. According to Lemma 1, the switch leaves the probabilities of alternatives other than a_j and a_k, and the *sum* of probabilities of a_j and a_k, unchanged. Lemma 1 is a special case of Lemma 2 in Gibbard (1977). It is a consequence of strategy-proofness and we omit its elementary proof.

LEMMA 2. *If domain $\mathbb{D}$ admits a unanimous, tops-only, and strategy-proof RSCF satisfying the compromise property, then it admits a two-voter unanimous, tops-only, and strategy-proof RSCF satisfying the compromise property.*

PROOF. Let $\varphi : \mathbb{D}^N \to \Delta(A)$ denote an unanimous, tops-only, and strategy-proof RSCF satisfying the compromise property. We know that there exists $\hat{I} \subseteq I$ with $|\hat{I}| = N/2$ if N is even and $|\hat{I}| = (N+1)/2$ if N is odd such that, for all $P_i, P_j \in \mathbb{D}$ with (i) $r_1(P_i) \neq r_1(P_j)$ and (ii) $a_r \equiv r_2(P_i) = r_2(P_j)$, we have $\varphi_r(\underbrace{P_i, \ldots, P_i}_{\hat{I}}, \underbrace{P_j, \ldots, P_j}_{I \setminus \hat{I}}) > 0$.

Construct a two-voter RSCF $\phi : \mathbb{D}^2 \to \Delta(A)$ as $\phi(P_1, P_2) = \varphi(\underbrace{P_1, \ldots, P_1}_{\hat{I}}, \underbrace{P_2, \ldots, P_2}_{I \setminus \hat{I}})$

for all $P_1, P_2 \in \mathbb{D}$. In other words, ϕ is constructed by "merging" all voters in $\hat{I}$ and all voters in $I \setminus \hat{I}$ in φ.[11] Clearly, ϕ is a RSCF satisfying unanimity and the tops-only property. It is also strategy-proof (see the proof of Lemma 3 in Sen 2011). We show that ϕ satisfies the compromise property.

Let $\bar{I} = \{1\}$ in the two-voter model. Let $P_1, P_2 \in \mathbb{D}$ with (i) $r_1(P_1) \neq r_1(P_2)$ and (ii) $a_r \equiv r_2(P_1) = r_2(P_2)$. Then $\phi_r(P_1, P_2) = \varphi_r(\underbrace{P_1, \ldots, P_1}_{\hat{I}}, \underbrace{P_2, \ldots, P_2}_{I \setminus \hat{I}}) > 0$, where the equality follows from the construction of ϕ and the inequality follows from the fact that φ satisfies the compromise property. Therefore ϕ satisfies the compromise property. This completes the proof of the lemma.   ◁

In view of Lemma 2, we can assume without loss of generality that the set of voters is $\{1, 2\}$ and φ is a RSCF $\varphi : \mathbb{D}^2 \to \Delta(A)$ that is unanimous, tops-only, and strategy-proof, and satisfies the compromise property. We make a further simplification in notation. Since φ is tops-only, we can represent a profile $P \in \mathbb{D}^2$ by a pair of alternatives a_j and a_k, where $r_1(P_1) = a_j$ and $r_1(P_2) = a_k$. We shall also occasionally let (a_j, P_2) denote a preference profile (P_1, P_2), where $r_1(P_1) = a_j$.

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[11]Any choice of the cardinality of $\hat{I}$ works for our proof, provided $0 < |\hat{I}| < N$. We could have assumed, for instance, that $|\hat{I}| = 2$ or $|\hat{I}| = N - 1$.

LEMMA 3. *Let $a_j, a_k \in A$ with $(a_j, a_k) \in \text{FPT}(\mathbb{D})$. There exists $\beta \in [0, 1]$ such that $\varphi(a_j, a_k) = \beta e_j + (1 - \beta)e_k$.*

PROOF. Let $P_1, P_1' \in \mathbb{D}$ be such that (i) $r_1(P_1) = r_2(P_1') = a_j$, (ii) $r_2(P_1) = r_1(P_1') = a_k$, and (iii) $r_t(P_1) = r_t(P_1')$, $t = 3, \ldots, m$ (such two preferences exist since $(a_j, a_k) \in \text{FPT}(\mathbb{D})$). We then have

$$\varphi_j(a_j, a_k) + \varphi_k(a_j, a_k) = \varphi_j(P_1, a_k) + \varphi_k(P_1, a_k) \quad \text{(by the tops-only property)}$$

$$= \varphi_j(P_1', a_k) + \varphi_k(P_1', a_k) \quad \text{(by Lemma 1)}$$

$$= \varphi_k(a_k, a_k) = 1 \quad \text{(by unanimity)}.$$

Let $\varphi_j(a_j, a_k) = \beta$. Thus, $\varphi(a_j, a_k) = \beta e_j + (1 - \beta)e_k$ as required.   ◁

The next lemma considers situations that are more general than those considered in the previous one. We illustrate it with an example. Suppose $(a_1, a_2), (a_2, a_3) \in \text{FPT}(\mathbb{D})$. We know from Lemma 3 that there exist $\beta_1, \beta_2 \in [0, 1]$ such that $\varphi(a_1, a_2) = \beta_1 e_1 + (1 - \beta_1)e_2$ and $\varphi(a_2, a_3) = \beta_2 e_2 + (1 - \beta_2)e_3$. The next lemma shows that $\beta_2 > \beta_1$ and $\varphi(a_1, a_3) = \beta_1 e_1 + (\beta_2 - \beta_1)e_2 + (1 - \beta_2)e_3$.

LEMMA 4. *Let $\{a_k\}_{k=1}^s \subseteq A$, $s \geq 3$, be such that $(a_k, a_{k+1}) \in \text{FPT}(\mathbb{D})$, $k = 1, \ldots, s - 1$. Let $\beta_k = \varphi_k(a_k, a_{k+1})$, $k = 1, \ldots, s - 1$. Then the following two conditions hold.*

 (i) *We have $0 \leq \beta_k < \beta_{k+1} \leq 1$, $k = 1, \ldots, s - 2$.*

 (ii) *For all $1 \leq i < j \leq s$, $\varphi(a_i, a_j) = \beta_i e_i + \sum_{k=i+1}^{j-1}(\beta_k - \beta_{k-1})e_k + (1 - \beta_{j-1})e_j$.*

PROOF. We know from Lemma 3 that $\varphi(a_k, a_{k+1}) = \beta_k e_k + (1 - \beta_k)e_{k+1}$, $k = 1, \ldots, s - 1$. Pick k with $1 \leq k \leq s - 2$. Since $(a_{k+1}, a_{k+2}) \in \text{FPT}(\mathbb{D})$ and $a_k \notin \{a_{k+1}, a_{k+2}\}$, Lemma 1 implies $\varphi_{k+1}(a_k, a_{k+2}) + \varphi_{k+2}(a_k, a_{k+2}) = \varphi_{k+1}(a_k, a_{k+1}) + \varphi_{k+2}(a_k, a_{k+1}) = \varphi_{k+1}(a_k, a_{k+1}) = 1 - \beta_k$ and $\varphi_k(a_k, a_{k+2}) = \varphi_k(a_k, a_{k+1}) = \beta_k$. Also, since $(a_k, a_{k+1}) \in \text{FPT}(\mathbb{D})$, Lemma 1 implies $\varphi_k(a_k, a_{k+2}) + \varphi_{k+1}(a_k, a_{k+2}) = \varphi_k(a_{k+1}, a_{k+2}) + \varphi_{k+1}(a_{k+1}, a_{k+2}) = \varphi_{k+1}(a_{k+1}, a_{k+2}) = \beta_{k+1}$. Therefore, $\varphi_{k+1}(a_k, a_{k+2}) = \beta_{k+1} - \varphi_k(a_k, a_{k+2}) = \beta_{k+1} - \beta_k$ and $\varphi_{k+2}(a_k, a_{k+2}) = 1 - \beta_k - \varphi_{k+1}(a_k, a_{k+2}) = 1 - \beta_{k+1}$. Therefore, $\varphi_k(a_k, a_{k+2}) + \varphi_{k+1}(a_k, a_{k+2}) + \varphi_{k+2}(a_k, a_{k+2}) = 1$ and $\varphi(a_k, a_{k+2}) = \beta_k e_k + (\beta_{k+1} - \beta_k)e_{k+1} + (1 - \beta_{k+1})e_{k+2}$. Therefore $\beta_{k+1} \geq \beta_k$. We conclude the argument by showing that the inequality must be strict.

Since $(a_k, a_{k+1}), (a_{k+1}, a_{k+2}) \in \text{FPT}(\mathbb{D})$, we have $P_1^*, P_2^* \in \mathbb{D}$ such that $r_1(P_1^*) = a_k$, $r_1(P_2^*) = a_{k+2}$, and $r_2(P_1^*) = r_2(P_2^*) = a_{k+1}$. Thus, $C(P_1^*, P_2^*) = \{a_{k+1}\}$. Then the tops-only property and the compromise property imply that $\beta_{k+1} - \beta_k = \varphi_{k+1}(a_k, a_{k+2}) = \varphi_{k+1}(P_1^*, P_2^*) > 0$ as required. This completes the verification of part (i) of the lemma.

Pick a_i, a_j in the sequence $\{a_k\}_{k=1}^s$ such that $i < j$. We will prove part (ii) by induction on the value of $l = j - i$. Observe that part (ii) has already been proved for the cases $l = 1$ (Lemma 3) and $l = 2$ (in the proof of part (i)). Assume therefore that $3 \leq l \leq s - 1$. We

impose the following induction hypothesis: for all $1 \leq \underline{i} < \underline{j} \leq s$,

$$[\underline{j} - \underline{i} < l] \Rightarrow \left[\varphi(x_{\underline{i}}, x_{\underline{j}}) = \beta_{\underline{i}} e_{x_{\underline{i}}} + \sum_{k=\underline{i}+1}^{j-1} (\beta_k - \beta_{k-1}) e_{x_k} + (1 - \beta_{\underline{j}-1}) e_{x_{\underline{j}}} \right].$$

We complete the proof by showing that part (ii) holds for all i, j with $1 \leq i < j \leq s$ and $j - i = l$.

Since $j - i = l \geq 3$, we know that $i < i + 1 < j - 1 < j$. Also $(j - 1) - i = l - 1 < l$ and $j - (i + 1) = l - 1 < l$. The induction hypothesis can then be applied to the profiles (a_i, a_{j-1}) and (a_{i+1}, a_j). Hence

$$\varphi(a_i, a_{j-1}) = \beta_i e_i + \sum_{k=i+1}^{j-2} (\beta_k - \beta_{k-1}) e_k + (1 - \beta_{j-2}) e_{j-1} \quad \text{and}$$

$$\varphi(a_{i+1}, a_j) = \beta_{i+1} e_{i+1} + \sum_{k=i+2}^{j-1} (\beta_k - \beta_{k-1}) e_k + (1 - \beta_{j-1}) e_j.$$

Since $(a_j, a_{j-1}) \in \mathrm{FPT}(\mathbb{D})$ and $a_i, \ldots, a_{j-2}$ are distinct from a_{j-1} and a_j, Lemma 1 implies $\varphi_i(a_i, a_j) = \varphi_i(a_i, a_{j-1}) = \beta_i$ and $\varphi_k(a_i, a_j) = \varphi_k(a_i, a_{j-1}) = \beta_k - \beta_{k-1}$, $k = i + 1, \ldots, j - 2$. Similarly, since $(a_i, a_{i+1}) \in \mathrm{FPT}(\mathbb{D})$, a_{j-1} and a_j are distinct from a_i and a_{i+1}, Lemma 1 implies $\varphi_{j-1}(a_i, a_j) = \varphi_{j-1}(a_{i+1}, a_j) = \beta_{j-1} - \beta_{j-2}$ and $\varphi_j(a_i, a_j) = \varphi_j(a_{i+1}, a_j) = 1 - \beta_{j-1}$. Thus, $\sum_{k=i}^{j} \varphi_k(a_i, a_j) = 1$ and $\varphi(a_i, a_j) = \beta_i e_i + \sum_{k=i+1}^{j-1} (\beta_k - \beta_{k-1}) e_k + (1 - \beta_{j-1}) e_j$ as required. This completes the verification of the induction hypothesis and hence part (ii) of the lemma. ◁

To demonstrate that $\mathbb{D}$ is single-peaked, we need to construct a tree $G = \langle A, E \rangle$ and show that $P_i \in \mathbb{D}$ implies P_i is single-peaked on G.

Let $G(\mathbb{D}) = \langle A, \mathrm{FPT}(\mathbb{D}) \rangle$ be a graph, i.e., let $a_j, a_k \in A$ constitute an edge in $G(\mathbb{D})$ only if they satisfy the FPT property. Since $\mathbb{D}$ is path-connected, graph $G(\mathbb{D})$ is connected. The following lemma shows that $G(\mathbb{D})$ is a tree.

LEMMA 5. *The graph $G(\mathbb{D})$ is a tree.*

PROOF. Suppose not, i.e., suppose there exists a sequence $\{a_k\}_{k=1}^{s} \subseteq A$, $s \geq 3$, such that $(a_k, a_{k+1}) \in \mathrm{FPT}(\mathbb{D})$, $k = 1, \ldots, s$, where $a_{s+1} = a_1$. Let $\beta_k = \varphi_k(a_k, a_{k+1})$, $k = 1, \ldots, s - 1$. Since $(a_k, a_{k+1}) \in \mathrm{FPT}(\mathbb{D})$, $k = 1, \ldots, s - 1$, Lemma 4 implies $\varphi(a_1, a_s) = \beta_1 e_1 + \sum_{k=2}^{s-1} (\beta_k - \beta_{k-1}) e_k + (1 - \beta_{s-1}) e_s$, where $0 \leq \beta_k < \beta_{k+1} \leq 1$, $k = 1, \ldots, s - 2$. However, since $(a_1, a_s) \in \mathrm{FPT}(\mathbb{D})$, Lemma 3 implies $\varphi_k(a_1, a_s) = 0$ for all $a_k \neq a_1, a_s$. We have a contradiction. ◁

LEMMA 6. *The inclusion $[P_i \in \mathbb{D}] \Rightarrow [P_i$ is single-peaked on $G(\mathbb{D})]$.*

PROOF. Suppose $a_t, a_j, a_k \in A$ are such that $r_1(P_i) = a_t$ and $a_j \in \langle a_t, a_k \rangle \setminus \{a_k\}$. Let $\langle a_t, a_k \rangle = \{x_r\}_{r=1}^{T}$, where $x_1 = a_t$, $x_T = a_k$, and $a_j = x_l$ for some $1 \leq l < T$. If $a_j = a_t$,

$a_j\,P_i\,a_k$ follows trivially. Assume therefore that $a_j \neq a_t$. Thus, $T \geq 3$. Suppose $a_k\,P_i\,a_j$. Consider the profile $P = (a_t, a_k)$ and $\varphi(P)$. According to Lemma 4, all alternatives in the sequence $\{x_r\}_{r=2}^{T-1}$ get strictly positive probability. Hence $\varphi_j(a_t, a_k) > 0$. Since φ satisfies unanimity, $\varphi(a_k, a_k) = 1$. Then voter i can obtain a strictly higher probability on the set of alternatives at least as preferred to a_j under P_i (this set includes a_k by hypothesis) by putting a_k on top of her order. This contradicts the strategy-proofness of φ. Therefore, $a_j\,P_i\,a_k$ as required.   ◁

This completes the verification of the necessity part of the theorem.

To demonstrate sufficiency, let $\mathbb{D}$ be a single-peaked domain on a tree $G = \langle A, E \rangle$. We construct a RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ that is strategy-proof, tops-only, and ex post efficient, and satisfies the compromise property. We proceed as follows: in the first step, we use the idea in Chatterji et al. (2013) to construct a specific DSCF (see the proof of the sufficiency part of the theorem in Chatterji et al. 2013); in the second step, we consider randomization over such DSCFs.

For any set $B \subseteq A$, let $G(B)$ be the *minimal subgraph* of G that contains B as vertices. More formally, $G(B)$ is the unique graph that satisfies the following properties.

1. The set of vertices in $G(B)$ contains B.

2. Let $a_j, a_k \in B$. Graph $G(B)$ has an edge (a_j, a_k) only if (a_j, a_k) is an edge in G.

3. The graph $G(B)$ is connected.

4. We have $a_k \in G(B)$ if and only if $a_k \in \langle a_r, a_j \rangle$, where $a_r, a_j \in B$.

Fix a profile $P \in \mathbb{D}^N$ and an alternative $a_k \in A$. Consider the graph $G(r_1(P))$. Suppose $a_k \notin G(r_1(P))$. Since G is a tree and contains no cycles, there exists a *unique* alternative in $G(r_1(P))$ that belongs to *every* path from a_k to any vertex in $G(r_1(P))$. Let this alternative be denoted by $a_{\beta(a_k, P)}$.[12] Then define the alternative $a_{\pi(a_k, P)}$ as

$$a_{\pi(a_k, P)} = \begin{cases} a_k & \text{if } a_k \in G(r_1(P)) \\ a_{\beta(a_k, P)} & \text{if } a_k \notin G(r_1(P)). \end{cases}$$

Consider Example 1. Suppose $I = \{1, 2, 3\}$. Let a_k be the alternative a_4 and let P be a profile such that $r_1(P) = \{a_1, a_2, a_3\}$. Then $G(r_1(P))$ is the graph consisting of the vertices $\{a_1, a_2, a_3\}$ and the edges (a_1, a_2) and (a_2, a_3). Then $a_{\pi(a_k, P)} = a_{\beta(a_4, P)} = a_2$. Further examples can be found in Chatterji et al. (2013).

For every $a_k \in A$, the RSCF $\phi^{a_k} : \mathbb{D}^N \to \Delta(A)$ is defined as follows: for all $P \in \mathbb{D}^N$, $\phi^{a_k}(P) = e_{\pi(a_k, P)}$. Evidently, ϕ^{a_k} is a DSCF. Its outcome at profile P is the "projection" of a_k on the minimal subgraph of G generated by the set of the first-ranked alternatives in P.

In the next step, we construct the RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ as follows: for all $P \in \mathbb{D}^N$, $\varphi(P) = \sum_{a_k \in A} \lambda^{a_k} \phi^{a_k}(P)$, where $\lambda^{a_k} > 0$ for all $a_k \in A$ and $\sum_{a_k \in A} \lambda^{a_k} = 1$. The RSCF is

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[12]It would have been more appropriate to write $a_{\beta(a_k, G(r_1(P)))}$ but we choose to suppress the dependence of this alternative on G for notational convenience.

obtained by choosing over the DSCFs ϕ^{a_k}, $k = 1, \ldots, m$, according to a fixed probability distribution where the probability of choosing each such DSCF is strictly positive. We call RSCF φ a *weighted projection rule*. We claim that φ satisfies all the required properties.[13]

LEMMA 7. *The RSCF φ is tops-only and strategy-proof.*

PROOF. According to Proposition 1 in Chatterji et al. (2013), a single-peaked domain is semi-single-peaked where every alternative can be taken to be a threshold in the definition of semi-single-peakedness. The sufficiency part of the theorem in Chatterji et al. (2013) shows that for any threshold $a_k \in A$, ϕ^{a_k} is strategy-proof and tops-only, and satisfies unanimity over a semi-single-peaked domain. Consequently, each ϕ^{a_k} is tops-only and strategy-proof. Therefore, φ, which is a convex combination of distinct tops-only and strategy-proof RSCFs, is also a tops-only and strategy-proof RSCF.[14]   ◁

LEMMA 8. *The RSCF φ is ex post efficient.*

PROOF. Suppose the lemma is false, i.e., there exist $P \in \mathbb{D}^N$ and $a_j, a_k \in A$ such that $a_j P_i a_k$ for all $i \in I$ and $\varphi_k(P) > 0$. Evidently, $a_k \notin r_1(P)$. Since φ satisfies unanimity, $\varphi_k(P) > 0$ implies that $|r_1(P)| > 1$. Observe that $a_{\pi(a_t,P)} \in G(r_1(P))$ for all $a_t \in A$. Hence, by construction of φ, if a_r is not included in the vertex set of $G(r_1(P))$, then $\varphi_r(P) = 0$. Therefore, a_k belongs to the vertex set of $G(r_1(P))$.

Let $\text{Ext}(G(r_1(P)))$ denote the set of vertices in $G(r_1(P))$ with degree 1, i.e., $a_t \in \text{Ext}(G(r_1(P)))$ if there exists a unique $a_s \in A$ such that (a_t, a_s) is an edge in $G(r_1(P))$. Observe that $\text{Ext}(G(r_1(P))) \subseteq r_1(P)$. (Suppose $a_t \in \text{Ext}(G(r_1(P)))$ but $a_t \notin r_1(P)$. Then a_t can be deleted as a vertex in $G(r_1(P))$ contradicting the assumption that $G(r_1(P))$ is minimal.) In other words, the vertices at the ends of every maximal path in $G(r_1(P))$ must be some elements of $r_1(P)$.

It follows from the arguments in the two previous paragraphs that $a_k \in G(r_1(P)) \setminus \text{Ext}(G(r_1(P)))$. Consequently, there exist $i, i' \in I$ such that $r_1(P_i) \neq r_1(P_{i'})$, $a_k \in \langle r_1(P_i), r_1(P_{i'}) \rangle$, and $a_k \neq r_1(P_i), r_1(P_{i'})$. Let a_t be the projection of a_j on the interval $\langle r_1(P_i), r_1(P_{i'}) \rangle$. By assumption, $a_t \in \langle r_1(P_i), r_1(P_{i'}) \rangle$. Hence, either $a_k \in \langle r_1(P_i), a_t \rangle$ or $a_k \in \langle r_1(P_{i'}), a_t \rangle$ must hold. Therefore either $a_k \in \langle r_1(P_i), a_j \rangle$ or $a_k \in \langle r_1(P_{i'}), a_j \rangle$ must hold, i.e., either $a_k P_i a_j$ or $a_k P_{i'} a_j$ must hold by single-peakedness of $\mathbb{D}$. We have a contradiction to our initial hypothesis that $a_j P_i a_k$ for all $i \in I$. Therefore, φ is ex post efficient.   ◁

LEMMA 9. *The RSCF φ satisfies the compromise property.*

PROOF. Let $P_i, P_j \in \mathbb{D}$ be such that $a_k = r_1(P_i) \neq r_1(P_j) = a_l$ and $C(P_i, P_j) = \{a_r\}$. Let $\hat{I} \subset I$ be such that $|\hat{I}| = N/2$ if N is even and $|\hat{I}| = (N+1)/2$ if N is odd. Let $\bar{P} \in \mathbb{D}^N$ be the profile $(\underbrace{P_i, \ldots, P_i}_{\hat{I}}, \underbrace{P_j, \ldots, P_j}_{I \setminus \hat{I}})$. We will show that $\varphi_r(\bar{P}) > 0$.

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[13]Some further properties of weighted projection rules are discussed in Appendix A.
[14]These arguments are routine and therefore omitted.

P_1	P_2	P_3	P_4	P_5	P_6
a_1	a_2	a_2	a_3	a_3	a_4
a_2	a_1	a_3	a_2	a_4	a_3
a_4	a_4	a_4	a_4	a_2	a_2
a_3	a_3	a_1	a_1	a_1	a_1

TABLE 3. Domain $\mathbb{D}^3$.

Since $\mathbb{D}$ is single-peaked on the tree $G = \langle A, E \rangle$, it follows that $(a_k, a_r), (a_l, a_r) \in E$. Hence $a_r \in G(r_1(\bar{P}))$ and $\phi^{a_r}(\bar{P}) = e_r$. Therefore, $\varphi_r(\bar{P}) \geq \lambda^{a_r} > 0$.   ◁

This completes the proof of the sufficiency part of the theorem.   □

3.1 *Discussion: Indispensability of the axioms and the richness condition*

In this section, we show that our axioms and richness assumption are indispensable for the theorem. In Examples 3, 4, 5, and 6, we drop, respectively, the compromise, tops-onlyness, unanimity, and strategy-proofness axioms in turn, and demonstrate the existence of a non-single-peaked domain that admits RSCFs satisfying the remaining axioms. In Example 7, we show that the separable domain violates path-connectedness but admits a unanimous, tops-only, strategy-proof RSCF satisfying the compromise property.

EXAMPLE 3 (Dropping the compromise property). Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\mathbb{D}^3$ is described in Table 3.

Since $(a_1, a_2), (a_2, a_3), (a_3, a_4) \in \mathrm{FPT}(\mathbb{D}^3)$, domain $\mathbb{D}^3$ is path-connected. In view of the path-connectivity structure, the only candidate for a graph with respect to which $\mathbb{D}^3$ could be single-peaked is the line G^L in Figure 2. However, preferences P_1 and P_2 violate single-peakedness in this case. Hence $\mathbb{D}^3$ is not single-peaked.

The domain $\mathbb{D}^3$ is, however, semi-single-peaked (Chatterji et al. 2013) with respect to (G^L, a_2). Consequently, the projection rule $\phi^{a_2} : [\mathbb{D}^3]^2 \to \Delta(A)$ is unanimous, tops-only, and strategy-proof. (This can also be verified directly.)

Note that $C(P_i, P_j) = \varnothing$ for all profile pairs with distinct peaks except for the pairs (P_1, P_4) and (P_3, P_6). Accordingly, $C(P_1, P_4) = \{a_2\}$, $C(P_3, P_6) = \{a_3\}$, and $\varphi_2(P_1, P_4) = \varphi_2(P_4, P_1) = 1 > 0$, *but* $\varphi_3(P_3, P_6) = \varphi_3(P_6, P_3) = 0$. Therefore, RSCF ϕ^{a_2} violates the compromise property.   ◇

EXAMPLE 4 (Dropping the tops-only property). Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\mathbb{D}^4$ is described in Table 4.

Once again, domain $\mathbb{D}^4$ is path-connected since $(a_1, a_2), (a_2, a_3), (a_3, a_4) \in \mathrm{FPT}(\mathbb{D}^4)$. Using the same arguments as in Example 3, it follows that $\mathbb{D}$ is not single-peaked.

P_1	P_2	P_3	P_4	P_5	P_6
a_1	a_2	a_2	a_3	a_3	a_4
a_2	a_1	a_3	a_2	a_4	a_3
a_4	a_4	a_4	a_4	a_1	a_1
a_3	a_3	a_1	a_1	a_2	a_2

TABLE 4. Domain $\mathbb{D}^4$.

Let $\phi^{a_k} : [\mathbb{D}^4]^2 \to \Delta(A)$, $k = 1, 2, 3, 4$, denote the four projection rules on the line G^L in Figure 2. We specify a unanimous RSCF $\varphi : [\mathbb{D}^4]^2 \to \Delta(A)$ as

$$\varphi(P_i, P_j) = \begin{cases} \frac{1}{2}e_{r_1(P_i)} + \frac{1}{2}e_{r_1(P_j)} \\ \quad \text{if } (P_i, P_j) \in \{P_1, P_2\} \times \{P_5, P_6\} \text{ or } \{P_5, P_6\} \times \{P_1, P_2\} \\ \frac{1}{3}\phi^{a_1}(P_i, P_j) + \frac{1}{6}\phi^{a_2}(P_i, P_j) + \frac{1}{6}\phi^{a_3}(P_i, P_j) + \frac{1}{3}\phi^{a_4}(P_i, P_j) \\ \quad \text{otherwise.} \end{cases}$$

The RSCF φ is an equal weight random dictatorship when a preference profile belongs to the subdomain $\{\{P_1, P_2\} \times \{P_5, P_6\}\} \cup \{\{P_5, P_6\} \times \{P_1, P_2\}\}$; otherwise it is a specific weighted projection rule on the line G^L. The RSCF φ is also strategy-proof; this can be verified by showing that in every possible manipulation, probabilities are transferred from preferred alternatives to less preferred alternatives in the true preference while probabilities assigned to other alternatives are unchanged. The details of the verification are found in Appendix B.

Note that $r_1(P_4) = r_1(P_5) = a_3$ and $\varphi_2(P_1, P_4) = \frac{1}{6} \neq 0 = \varphi_2(P_1, P_5)$. Therefore φ violates the tops-only property. Note that $C(P_i, P_j) = \varnothing$ for all profile pairs with distinct peaks except for pairs (P_1, P_4) and (P_3, P_6). Accordingly, $C(P_1, P_4) = \{a_2\}$, $C(P_3, P_6) = \{a_3\}$, $\varphi_2(P_4, P_1) = \varphi_2(P_1, P_4) = \frac{1}{6} > 0$, and $\varphi_3(P_3, P_6) = \varphi_3(P_6, P_3) = \frac{1}{6} > 0$. Hence, the compromise property is satisfied. ◇

EXAMPLE 5 (Dropping unanimity). Consider the complete domain $\mathbb{P}$. Fix a collection $[\lambda^{a_k}]_{k=1}^m \in \mathbb{R}_{++}^m$ with $\sum_{k=1}^m \lambda^{a_k} = 1$, and construct the RSCF $\varphi : \mathbb{P}^N \to \Delta(A)$ as

$$\varphi(P) = \sum_{k=1}^m \lambda^{a_k} e_k \quad \text{for all } P \in \mathbb{P}^N.$$

The RSCF φ is tops-only and strategy-proof, and satisfies the compromise property. Since it is a convex combination of all *constant* DSCFs, it violates unanimity. ◇

EXAMPLE 6 (Dropping strategy-proofness). Consider the complete domain $\mathbb{P}$. Fix a collection $[\lambda^{a_k}]_{k=1}^m \in \mathbb{R}_{++}^m$ with $\sum_{k=1}^m \lambda^{a_k} = 1$, and construct the RSCF $\varphi : \mathbb{P}^N \to \Delta(A)$ as

$$\varphi(P) = \begin{cases} e_k & \text{if } r_1(P) = \{a_k\} \text{ for some } a_k \in A \\ \sum_{k=1}^m \lambda^{a_k} e_k & \text{otherwise.} \end{cases}$$

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4
a_2	a_4	a_1	a_3	a_2	a_4	a_1	a_3
a_4	a_2	a_3	a_1	a_4	a_2	a_3	a_1
a_3	a_3	a_4	a_4	a_1	a_1	a_2	a_2

TABLE 5. Domain $\mathbb{D}^7$.

The RSCF φ picks alternative a_k for sure if a_k is the peak for all voters in a profile. In all other profiles, it is a convex combination of all constant DSCFs. It is unanimous and tops-only, and satisfies the compromise property but not strategy-proofness.   ◇

EXAMPLE 7 (Dropping path-connectedness). Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\mathbb{D}^7$ is specified in Table 5.

A convenient way to represent these preferences is to regard each alternative a_k, as comprising two components (a_k^1, a_k^2). Specifically, $A^1 = \{0, 1\}$, $A^2 = \{0, 1\}$, $a_1 = (0, 0)$, $a_2 = (1, 0)$, $a_3 = (1, 1)$, and $a_4 = (0, 1)$. Then domain $\mathbb{D}^7$ is a separable domain (Barberà et al. 1991, Le Breton and Sen 1999). Apparently, FPT($\mathbb{D}^7$) $= \varnothing$ and hence domain $\mathbb{D}^7$ is not path-connected.

For all $P_i, P_j \in \mathbb{D}^7$, let $r_1(P_i) = a_i \equiv (a_i^1, a_i^2)$ and $r_1(P_j) = a_j \equiv (a_j^1, a_j^2)$. Accordingly, $\mathbb{D}^7$ admits the following four DSCFs: for all $P_i, P_j \in \mathbb{D}^7$,

$$\phi^{a_1}(P_i, P_j) = e_{(\min(a_i^1, a_j^1), \min(a_i^2, a_j^2))} \qquad \phi^{a_2}(P_i, P_j) = e_{(\max(a_i^1, a_j^1), \min(a_i^2, a_j^2))}$$

$$\phi^{a_3}(P_i, P_j) = e_{(\max(a_i^1, a_j^1) \max(a_i^2, a_j^2))}, \qquad \phi^{a_4}(P_i, P_j) = e_{(\min(a_i^1, a_j^1), \max(a_i^2, a_j^2))}.$$

The DSCFs ϕ^{a_1}, ϕ^{a_2}, ϕ^{a_3}, ϕ^{a_4} are unanimous, anonymous, tops-only, and strategy-proof.

Pick $\lambda^{a_k} > 0$, $k = 1, 2, 3, 4$, with $\sum_{k=1}^4 \lambda^{a_k} = 1$, and define RSCF $\varphi : [\mathbb{D}^7]^2 \to \Delta(A)$ as

$$\varphi(P) = \sum_{k=1}^4 \lambda^{a_k} \phi^{a_k}(P) \quad \text{for all } P \in [\mathbb{D}^7]^2.$$

Since it is a convex combination of DSCFs satisfying unanimity, anonymity, tops-onlyness and strategy-proofness, φ also satisfies these properties. Finally, observe that $C(P_1, P_5) = \{a_2\}$, $C(P_2, P_6) = \{a_4\}$, $C(P_3, P_7) = \{a_1\}$, $C(P_4, P_8) = \{a_3\}$, and $C(P_i, P_j) = \varnothing$ for all other pairs (P_i, P_j) with $r_1(P_i) \neq r_1(P_j)$. Since $\varphi_2(P_1, P_5) = \varphi_2(P_5, P_1) = \lambda^{a_2} > 0$, $\varphi_4(P_2, P_6) = \varphi_4(P_6, P_2) = \lambda^{a_4} > 0$, $\varphi_1(P_3, P_7) = \varphi_1(P_7, P_3) = \lambda^{a_1} > 0$, and $\varphi_3(P_4, P_8) = \varphi_3(P_8, P_4) = \lambda^{a_3} > 0$, RSCF φ satisfies the compromise property.   ◇

4. CONCLUSION

We have characterized domains of single-peaked preferences as the only domains that admit well behaved random social choice functions.

| | | $P_i \in \bar{\mathbb{D}}$ | | |
		$r_1(P_i) = a_1$	$r_1(P_i) = a_2$	$r_1(P_i) = a_3$	$r_1(P_i) = a_4$
$P_j \in \bar{\mathbb{D}}$	$r_1(P_j) = a_1$	e_1	$\frac{1}{3}e_1 + \frac{2}{3}e_2$	$\frac{1}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3$	$\frac{1}{3}e_1 + \frac{1}{6}e_2 + \frac{1}{2}e_4$
	$r_1(P_j) = a_2$	$\frac{1}{3}e_1 + \frac{2}{3}e_2$	e_2	$\frac{2}{3}e_2 + \frac{1}{3}e_3$	$\frac{1}{2}e_2 + \frac{1}{2}e_4$
	$r_1(P_j) = a_3$	$\frac{1}{3}e_1 + \frac{1}{3}e_2 + \frac{1}{3}e_3$	$\frac{2}{3}e_2 + \frac{1}{3}e_3$	e_3	$\frac{1}{6}e_2 + \frac{1}{3}e_3 + \frac{1}{2}e_4$
	$r_1(P_j) = a_4$	$\frac{1}{3}e_1 + \frac{1}{6}e_2 + \frac{1}{2}e_4$	$\frac{1}{2}e_2 + \frac{1}{2}e_4$	$\frac{1}{6}e_2 + \frac{1}{3}e_3 + \frac{1}{2}e_4$	e_4

TABLE 6. RSCF $\varphi : \bar{\mathbb{D}}^2 \to \Delta(A)$.

APPENDIX A: THE WEIGHTED PROJECTION RULE

In the verification of the sufficiency part of the theorem, we constructed a weighted projection rule. In this section, we briefly describe some important features of such rules.

A projection rule is a DSCF that is strategy-proof, efficient, tops-only, and anonymous. A weighted projection rule is a convex combination of all projection rules and inherits all the properties of projection rules mentioned above and satisfies the compromise property. If the weights are chosen to be $1/|A|$, a weighted projection rule also satisfies neutrality.[15]

Weighted projection rules *are not* the only RSCFs that satisfy the required properties in the theorem on single-peaked domains on a tree. One way to see this is to note that a projection rule on a line is a particular case of a phantom voter rule (see Moulin 1980, Border and Jordan 1983, and Schummer and Vohra 2002) where all phantom voters have the same peak.[16] Consider the single-peaked domain on a line (see Example 2) and let the RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$, $N \geq 3$, be a convex combination of all phantom voter rules on the line where every phantom voter rule has strictly positive weight. It is easy to show that φ is ex post efficient, anonymous, tops-only, and strategy-proof, and satisfies the compromise property. However, RSCF φ is not a weighted projection rule since it includes some phantom voter rules with distinct peaks of phantom voters. In the case of two voters, efficiency reduces the number of phantom voters to one. However even in this case, there exist strategy-proof, ex post efficient, tops-only RSCFs satisfying the compromise property that are *not* weighted projection rules. This is shown in Example 8 below.

EXAMPLE 8. Consider domain $\bar{\mathbb{D}}$ in Example 1. Note that for all $P_i, P_j \in \bar{\mathbb{D}}$ with $r_1(P_i) \neq r_1(P_j)$, either $C(P_i, P_j) = \{a_2\}$ or $C(P_i, P_j) = \varnothing$. Domain $\bar{\mathbb{D}}$ admits the RSCF $\varphi : \bar{\mathbb{D}}^2 \to \Delta(A)$ specified in Table 6. It is easy to verify that φ is ex post efficient, anonymous, and tops-only, and satisfies the compromise property.

There are three maximal paths in G^T: $\{a_1, a_2, a_3\}$, $\{a_1, a_2, a_4\}$, and $\{a_3, a_2, a_4\}$. Accordingly, we have three subdomains: $\bar{\mathbb{D}}_1 = \{P_i \in \bar{\mathbb{D}} \mid r_1(P_i) \in \{a_1, a_2, a_3\}\}$, $\bar{\mathbb{D}}_2 =$

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[15]A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is *neutral* if for every permutation $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ and $P, P' \in \mathbb{D}^N$ with $[a_k P_i a_l] \Leftrightarrow [a_{\sigma(k)} P_i' a_{\sigma(l)}]$ for all $i \in I$ and $k, l \in \{1, \ldots, m\}$, we have $\varphi_r(P) = \varphi_{\sigma(r)}(P')$.

[16]Thomson (1993) and Vohra (1999) show that the family of projection rules is uniquely characterized by Pareto optimality and the axiom of replacement dominance over the single-peaked domain.

$\{P_i \in \bar{\mathbb{D}} \mid r_1(P_i) \in \{a_1, a_2, a_4\}\}$, and $\bar{\mathbb{D}}_3 = \{P_i \in \bar{\mathbb{D}} \mid r_1(P_i) \in \{a_3, a_2, a_4\}\}$. Observe that for every $P \in \bar{\mathbb{D}}^2$, there exists $k \in \{1, 2, 3\}$ (not necessarily unique) such that $P \in \bar{\mathbb{D}}_k^2$.[17]

The RSCF φ is defined by considering a separate weighted projection rule for each of the subdomains $\bar{\mathbb{D}}_1, \bar{\mathbb{D}}_2$, and $\bar{\mathbb{D}}_3$. Specifically, for all $P^1 \in \bar{\mathbb{D}}_1^2$, $P^2 \in \bar{\mathbb{D}}_2^2$, and $P^3 \in \bar{\mathbb{D}}_3^2$,

$$\varphi(P^1) = \tfrac{1}{3}\phi^{a_1}(P^1) + \tfrac{1}{3}\phi^{a_2}(P^1) + \tfrac{1}{3}\phi^{a_3}(P^1)$$

$$\varphi(P^2) = \tfrac{1}{3}\phi^{a_1}(P^2) + \tfrac{1}{6}\phi^{a_2}(P^2) + \tfrac{1}{2}\phi^{a_4}(P^2)$$

$$\varphi(P^3) = \tfrac{1}{3}\phi^{a_3}(P^3) + \tfrac{1}{6}\phi^{a_2}(P^3) + \tfrac{1}{2}\phi^{a_4}(P^3).$$

Note that if $P \in \bar{\mathbb{D}}_k^2$ and $P \in \bar{\mathbb{D}}_{k'}^2$, where $k \neq k'$, $\varphi(P)$ is *identically* induced by the two corresponding distinct weighted projection rules. For instance, $(P_1, P_2) \in \bar{\mathbb{D}}_1^2$ and $(P_1, P_2) \in \bar{\mathbb{D}}_2^2$. According to $\bar{\mathbb{D}}_1$, $\varphi(P_1, P_2) = \tfrac{1}{3}e_1 + \tfrac{2}{3}e_2$, while according to $\bar{\mathbb{D}}_2$, we also have $\varphi(P_1, P_2) = \tfrac{1}{3}e_1 + \tfrac{2}{3}e_2$.

Similar to the verification of strategy-proofness in Example 4, we fix voter i and check all possible manipulations: $(P_i, P_j) \leftrightarrow (P_i', P_j)$. It follows from standard arguments that manipulation never occurs within any of the subdomains $\bar{\mathbb{D}}_1, \bar{\mathbb{D}}_2$, and $\bar{\mathbb{D}}_3$, i.e., if the true preference and the misrepresentation lie within the same subdomain. We will consider every misrepresentation that leads to an outcome according to a different weighted projection rule relative to truth-telling. It covers three situations and we specify the changes of probabilities in each situation that indicate that probabilities are always transferred from the preferred alternatives to less preferred alternatives according to the true preference.

1. We have $(P_i, a_3) \leftrightarrow (P_i', a_3)$, where $r_1(P_i) = a_1$ and $r_1(P_i') = a_4$:

   - Probabilities $\tfrac{1}{3}$ and $\tfrac{1}{6}$ are transferred from a_1 and a_2 to a_4, respectively, where $a_1\, P_i\, a_4$ and $a_2\, P_i\, a_4$ by single-peakedness.

   - Probabilities $\tfrac{1}{3}$ and $\tfrac{1}{6}$ are transferred from a_4 to a_1 and a_2, respectively, where $a_4\, P_i'\, a_1$ and $a_4\, P_i'\, a_2$.

2. We have $(P_i, a_4) \leftrightarrow (P_i', a_4)$, where $r_1(P_i) = a_1$ and $r_1(P_i') = a_3$:

   - Probability $\tfrac{1}{3}$ is transferred from a_1 to a_3, where $a_1\, P_i\, a_3$.

   - Probability $\tfrac{1}{3}$ is transferred from a_3 to a_1, where $a_3\, P_i'\, a_1$.

3. We have $(P_i, a_1) \leftrightarrow (P_i', a_1)$, where $r_1(P_i) = a_3$ and $r_1(P_i') = a_4$:

   - Probabilities $\tfrac{1}{3}$ and $\tfrac{1}{6}$ are transferred from a_3 and a_2 to a_4, respectively, where $a_3\, P_i\, a_4$ and $a_2\, P_i\, a_4$ by single-peakedness.

   - Probabilities $\tfrac{1}{3}$ and $\tfrac{1}{6}$ are transferred from a_4 to a_3 and a_2, respectively, where $a_4\, P_i'\, a_3$ and $a_4\, P_i'\, a_2$.

In conclusion, RSCF φ is strategy-proof.

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[17]For instance, $(P_1, P_2) \in \bar{\mathbb{D}}_1^2$ and $(P_1, P_2) \in \bar{\mathbb{D}}_2^2$.

Last, we verify that φ is *not* a weighted projection rule. Suppose it is not true. Then there exists $\lambda^{a_k} \geq 0$, $k = 1, 2, 3, 4$, with $\sum_{k=1}^{4} \lambda^{a_k} = 1$ such that $\varphi(P) = \sum_{k=1}^{4} \lambda^{a_k} \phi^{a_k}(P)$ for all $P \in \bar{\mathbb{D}}^2$. We must then have (i) $\lambda^{a_1} = \varphi_1(a_1, a_2) = \frac{1}{3}$, (ii) $\lambda^{a_3} = \varphi_3(a_3, a_2) = \frac{1}{3}$, and (iii) $\lambda^{a_4} = \varphi_4(a_4, a_2) = \frac{1}{2}$. Consequently, $\lambda^{a_1} + \lambda^{a_3} + \lambda^{a_4} > 1$ which is a contradiction. Hence, φ is not a weighted projection rule.   ◇

Appendix B: Strategy-proofness in Example 4

To verify that RSCF φ in Example 4 is strategy-proof, it suffices to show that in every possible manipulation, probabilities are transferred from preferred alternatives to less preferred alternatives in the true preference while probabilities assigned to other alternatives are unchanged. Note that since RSCF φ is anonymous, we can fix a voter, say voter i, and consider all possible manipulations, $(P_i, P_j) \leftrightarrow (P'_i, P_j)$.

According to the construction of φ, it is evident that manipulation can never occur if both truth-telling and misrepresentation result in a random dictatorship outcome.

Next, voter i would consider a misrepresentation that makes RSCF φ change from random dictatorship to the weighted projection rule or vice versa. Given $P_i \in \{P_1, P_2\}$, $P_j \in \{P_5, P_6\}$, and $P'_i \in \{P_3, P_4, P_5, P_6\}$, we specify the changes of probabilities in all possible manipulations that indicate that probabilities are always transferred from the preferred alternatives to less preferred alternatives in the true preference.

1. We have $(P_1, P_5) \leftrightarrow (P_3, P_5)$:

   - Probability $\frac{1}{2}$ is transferred from a_1 to a_2, where $a_1 \, P_1 \, a_2$.

   - Probability $\frac{1}{2}$ is transferred from a_2 to a_1, where $a_2 \, P_3 \, a_1$.

2. We have $(P_1, P_5) \leftrightarrow (P_4, P_5)$ or (P_5, P_5):

   - Probability $\frac{1}{2}$ is transferred from a_1 to a_3, where $a_1 \, P_1 \, a_3$.

   - Probability $\frac{1}{2}$ is transferred from a_3 to a_1, where $a_3 \, P_4 \, a_1$ and $a_3 \, P_5 \, a_1$.

3. We have $(P_1, P_5) \leftrightarrow (P_6, P_5)$:

   - Probabilities $\frac{1}{6}$ and $\frac{1}{3}$ are transferred from a_1 to a_3 and a_4, respectively, where $a_1 \, P_1 \, a_3$ and $a_1 \, P_1 \, a_4$.

   - Probabilities $\frac{1}{6}$ and $\frac{1}{3}$ are transferred from a_3 and a_4 to a_1, respectively where $a_3 \, P_6 \, a_1$ and $a_4 \, P_6 \, a_1$.

4. We have $(P_2, P_5) \leftrightarrow (P_3, P_5)$: Probability does not change.

5. We have $(P_2, P_5) \leftrightarrow (P_4, P_5)$ or (P_5, P_5):

   - Probability $\frac{1}{2}$ is transferred from a_2 to a_3, where $a_2 \, P_2 \, a_3$.

   - Probability $\frac{1}{2}$ is transferred from a_3 to a_2, where $a_3 \, P_4 \, a_2$ and $a_3 \, P_5 \, a_2$.

6. We have $(P_2, P_5) \leftrightarrow (P_6, P_5)$:

- Probabilities $\frac{1}{6}$ and $\frac{1}{3}$ are transferred from a_2 to a_3 and a_4, respectively, where $a_2 \, P_2 \, a_3$ and $a_2 \, P_2 \, a_4$.

- Probabilities $\frac{1}{6}$ and $\frac{1}{3}$ are transferred from a_3 and a_4 to a_2, respectively, where $a_3 \, P_6 \, a_2$ and $a_4 \, P_6 \, a_2$.

7. We have $(P_1, P_6) \leftrightarrow (P_3, P_6)$:

- Probabilities $\frac{1}{2}$ and $\frac{1}{6}$ are transferred from a_1 to a_2 and from a_4 to a_3, respectively, where $a_1 \, P_1 \, a_2$ and $a_4 \, P_1 \, a_3$.

- Probabilities $\frac{1}{2}$ and $\frac{1}{6}$ are transferred from a_2 to a_1 and from a_3 to a_4, respectively, where $a_2 \, P_3 \, a_1$ and $a_3 \, P_3 \, a_4$.

8. We have $(P_1, P_6) \leftrightarrow (P_4, P_6)$ or (P_5, P_6):

- Probabilities $\frac{1}{2}$ and $\frac{1}{6}$ are transferred from a_1 and a_4 to a_3, respectively, where $a_1 \, P_1 \, a_3$ and $a_4 \, P_1 \, a_3$.

- Probabilities $\frac{1}{2}$ and $\frac{1}{6}$ are transferred from a_3 to a_1 and a_4, respectively, where $a_3 \, P_4 \, a_1$, $a_3 \, P_4 \, a_4$ and $a_3 \, P_5 \, a_1$, $a_3 \, P_5 \, a_4$.

9. We have $(P_1, P_6) \leftrightarrow (P_6, P_6)$:

- Probability $\frac{1}{2}$ is transferred from a_1 to a_4, where $a_1 \, P_1 \, a_4$.

- Probability $\frac{1}{2}$ is transferred from a_4 to a_1, where $a_4 \, P_6 \, a_1$.

10. We have $(P_2, P_6) \leftrightarrow (P_3, P_6)$:

- Probability $\frac{1}{6}$ is transferred from a_4 to a_3, where $a_4 \, P_2 \, a_3$.

- Probability $\frac{1}{6}$ is transferred from a_3 to a_4, where $a_3 \, P_3 \, a_4$.

11. We have $(P_2, P_6) \leftrightarrow (P_4, P_6)$ or (P_5, P_6):

- Probabilities $\frac{1}{2}$ and $\frac{1}{6}$ are transferred from a_2 and a_4 to a_3, respectively, where $a_2 \, P_2 \, a_3$ and $a_4 \, P_2 \, a_3$.

- Probabilities $\frac{1}{2}$ and $\frac{1}{6}$ are transferred from a_3 to a_2 and a_4, respectively, where $a_3 \, P_4 \, a_2$, $a_3 \, P_4 \, a_4$ and $a_3 \, P_5 \, a_2$, $a_3 \, P_5 \, a_4$.

12. We have $(P_2, P_6) \leftrightarrow (P_6, P_6)$:

- Probability $\frac{1}{2}$ is transferred from a_2 to a_4, where $a_2 \, P_2 \, a_4$.

- Probability $\frac{1}{2}$ is transferred from a_4 to a_2, where $a_4 \, P_6 \, a_2$.

By a symmetric argument, we know that voter i would neither manipulate at $(P_i, P_j) \in \{P_5, P_6\} \times \{P_1, P_2\}$ via $P_i' \in \{P_1, P_2, P_3, P_4\}$ nor manipulate at $(P_i, P_j) \in \{P_1, P_2, P_3, P_4\} \times \{P_1, P_2\}$ via $P_i' \in \{P_5, P_6\}$.

Last, we show that no manipulation occurs within the weighted projection rule. Accordingly, we consider all possible manipulations $(P_i, P_j) \leftrightarrow (P_i', P_j)$ in three jointly exhaustive cases:

(i) $P_i, P_j, P_i' \in \{P_1, P_2, P_3, P_4\}$

(ii) $P_i, P_j, P_i' \in \{P_3, P_4, P_5, P_6\}$

(iii) $P_i \in \{P_1, P_2\}$, $P_j \in \{P_3, P_4\}$ and $P_i' \in \{P_5, P_6\}$.

Note that in case (i), $\varphi_4(P_i, P_j) = 0$ and $\varphi_4(P_i', P_j) = 0$. Since preferences P_1, P_2, P_3, P_4 are single-peaked on the subline $\{a_1, a_2, a_3\}$, possible manipulations via any of the preferences P_1, P_2, P_3, P_4 are not beneficial. Case (ii) is symmetric to case (i).[18] In case (iii), note that $\varphi(P_2, P_j) = \varphi(P_3, P_j)$ and $\varphi(P_5, P_j) = \varphi(P_4, P_j)$. Then, in case (iii), a manipulation of voter i via P_2 or P_5 is identical to a manipulation via P_3 or P_4, respectively, and hence is nonprofitable according to cases (i) and (ii), respectively. Now, we specify the changes of probabilities in the rest of the possible manipulations in case (iii) that also indicate that probabilities are always transferred from the preferred alternatives to less preferred alternatives in the true preference.

1. We have $(P_1, P_j) \leftrightarrow (P_6, P_j)$:

   - Probabilities $\frac{1}{3}$ and $\frac{1}{6}$ are transferred from a_1 to a_4 and from a_2 to a_3, respectively, where $a_1 \, P_1 \, a_4$ and $a_2 \, P_1 \, a_3$.

   - Probabilities $\frac{1}{3}$ and $\frac{1}{6}$ are transferred from a_4 to a_1 and from a_3 to a_2, respectively, where $a_4 \, P_6 \, a_1$ and $a_3 \, P_6 \, a_2$.

2. We have $(P_2, P_j) \rightarrow (P_6, P_j)$: Probabilities $\frac{1}{6}$ and $\frac{1}{3}$ are transferred from a_2 to a_3 and a_4, respectively, where $a_2 \, P_2 \, a_3$ and $a_2 \, P_2 \, a_4$.

3. We have $(P_5, P_j) \rightarrow (P_1, P_j)$: Probabilities $\frac{1}{3}$ and $\frac{1}{6}$ are transferred from a_3 to a_1 and a_2, respectively, where $a_3 \, P_5 \, a_1$ and $a_3 \, P_5 \, a_2$.

In conclusion, RSCF φ is strategy-proof.

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[18]In case (ii), $\varphi_1(P_i, P_j) = 0$, $\varphi_1(P_i', P_j) = 0$, and preferences P_3, P_4, P_5, P_6 are single-peaked on the subline $\{a_2, a_3, a_4\}$.

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