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Huaxia ZENG
Singapore Management University, huaxia.zeng.2011@me.smu.edu.sg

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HUAXIA ZENG

SINGAPORE MANAGEMENT UNIVERSITY

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by
Huaxia Zeng

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Dissertation Committee:

Shurojit Chatterji (Supervisor/Chair)
Professor of Economics
Singapore Management University

Takashi Kunimoto
Associate Professor of Economics
Singapore Management University

Jingyi Xue
Assistant Professor of Economics
Singapore Management University

Arunava Sen
Professor of Economics
Indian Statistical Institute, Delhi

Singapore Management University
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Abstract

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This dissertation studies a standard voting formulation with randomization. Formally, there is a finite set of voters, a finite set of alternatives and a lottery space over the alternative set. Each voter has a strict preference over alternatives. The domain of preferences contains all admissible preferences. Every voter reports a preference in the domain; a preference profile is generated; and the social lottery then is determined by a Random Social Choice Function (or RSCF).

This dissertation focuses on RSCFs which provide every voter incentives to truthfully reveal her preference, and hence follows the formulation of strategy-proofness in [26] which requires that the lottery under truthtelling (first-order) stochastically dominates the lottery under any misrepresentation according to every voter's true preference independently of others' behaviors. Moreover, this dissertation restricts attention to the class of unanimous RSCFs, that is, if the alternative is the best for all voters in a preference profile, it receives probability one. A typical class of unanimous and strategy-proof RSCFs is random dictatorships.

A domain is a random dictatorship domain if every unanimous and strategy-proof RSCF is a random dictatorship. Gibbard [26] showed that the complete domain is a random dictatorship domain. Chapter 2 studies dictatorial domains, i.e., every unanimous and strategy-proof Deterministic Social Choice Function (or DSCF) is a dictatorship, and shows that a dictatorial domain is not necessarily a random dictatorship domain. This result applies to the constrained voting model. Moreover, this chapter shows that substantial strengthenings of Linked Domains (a

class of dictatorial domains introduced in [1]) are needed to restore random dictatorship and such strengthenings are“almost necessary”.

Single-peaked domains are the most attractive among restricted voting domains which can admit a large class of “well-behaved” strategy-proof RSCFs. Chapter 3 studies an inverse question: does the single-peakedness restriction naturally emerge as a consequence of the existence of a well-behaved strategy-proof randomized voting rule? This chapter proves the following result: Every path-connected domain that admits a unanimous, tops-only, strategy-proof RSCF satisfying a compromise property is single-peaked on a tree. Conversely, every single-peaked domain admits such a RSCF satisfying these properties. This result provides a justification of the salience of single-peaked preferences and evidence in favor of the Gul conjecture (see [3]).

One important class of RSCFs is the class of tops-only RSCFs whose social lottery under each preference profile depends only on the peaks of preferences. The tops-only property is widely explored in DSCFs, and more importantly, is usually implied by unanimity and strategy-proofness in DSCFs (e.g., [52], [15]). In Chapter 4, a general condition is identified on domains of preferences (the Interior Property and the Exterior Property), which ensures that every unanimous and strategy-proof RSCF has the tops-only property. Moreover, this chapter provides applications of this sufficient condition and use it to derive new results.

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Chapter 1 Introduction

Randomization is a natural resolution to many instances of conflict of interest in economic settings. It can easily bring fairness to the ex-ante consideration of collective decision making problems. As an example, consider the following economic scenario: two agents have opposite opinions on two indivisible objects, and one object needs to be chosen for production. Tossing a fair coin to decide the choice appears to be the fair solution. It is also true that randomization improves incentives for truthtelling in models with private information. The reason it does so is that outcomes under randomization are lotteries and it is typically assumed that player's preferences over lotteries satisfy von-Neumann-Morgenstern expected utility hypothesis. The mechanism designer can exploit this preference restrictions in order to expand the set of mechanisms that provides agents incentives for truthtelling, e.g., combining several dictatorships with equal weights. More importantly, the injection of randomization has been recently shown to significantly enlarge the scope for designing "well-behaved" mechanisms with nice incentive properties.

This dissertation studies randomization in the *voting* environment where each voter submits an *ordinal strict preference* order over a finite set of alternatives; a preference profile is generated; a "desirable" social lottery over alternatives is accordingly chosen; and monetary compensation for voters is not feasible. Each voter's preference order is her private information and drawn from a set of admissible preference orders which is referred to as the *domain* of preferences. A *Random Social Choice Function* (or RSCF), which is a map from the Cartesian product of domains to the set of lotteries, determines the social lottery under every profile of reported preferences. In particular, if a degenerate lottery (i.e., one alternative is assigned probability one) is chosen under each preference profile, the RSCF is degenerated to a *Deterministic Social Choice Function* (or DSCF).

Since a voter's preference is her private information, she is not obligated to report her preference sincerely, especially if she can benefit from misrepresentation. Therefore, in the literature of mechanism design, incentives for truthfullying revealing private information are prominently at the forefront. Fixing an ordinal preference and a utility function representing this preference, every voter is assumed to evaluate lotteries according to the von-Neumann-Morgenstern expected utility hypothesis. Accordingly, Gibbard [26] established the notion of a strategy-proof RSCF which requires no voter can obtain a strictly higher expected utility by misreporting her preference for any utility representing her sincere preference and any beliefs regarding the reports of other voters. Equivalently, this notion of strategy-proofness can be reformulated in terms of *(first-order) stochastic dominance* which says that for each voter, the social lottery under truthtelling stochastically dominates a lottery induced by any unilateral misrepresentation according to her true preference independently of others' reports. The whole dissertation follows Gibbard's formulation of strategy-proofness. Moreover, this dissertation restricts attention to the class of RSCFs satisfying the mild requirement of *unanimity*, i.e., a unanimous RSCF requires an alternative to be selected with certainty under a preference profile if it is top-ranked by all voters.

According to the classic impossibility theorem in [26], the only unanimous and strategy-proof RSCFs are *random dictatorships*, provided that voters's preferences belong to the complete domain, and the number of alternatives is at least three. A random dictatorship is a fixed probability distribution over dictatorial DSCFs, and therefore is clearly more appealing than dictatorships since it introduces ex-ante fairness and every voter may have the same probability to the chosen dictator. Later, the random dictatorship characterization result is robustly established in different settings (e.g., cardinal preferences in [28]), or by different approaches (e.g., [21]), or under restricted ordinal preferences (e.g., [48]). A domain of preferences is referred to as a *random dictatorship domain* if every unanimous and strategy-proof RSCF is a random dictatorship. Even though a random dictatorship is strategy-proof on arbitrary domains of preferences, it remains unsatisfactory and suffer a significant infirmity - it assigns positive probability on an alternative only if it is top-ranked in some voter's preference, and hence does not admit compromise. For instance, voters may disagree strongly on each other's most preferred alternatives but have a commonly second best alternative. However, this commonly second best alternative is ignored by a random dictatorship. Essentially, this dissertation explores the following fundamental question in different directions: *When can we escape random dictatorships?*

In the set of strategy-proof RSCFs satisfying unanimity, all unanimous and strategy-proof DSCFs are extreme points. One then would ask: *Are there any other extreme points?* Accordingly, if there exists *no* extreme point other than unanimous and strategy-proof DSCFs, one says the domain of preferences satisfies *the extreme point property*. Evidently, on a domain satisfying the extreme point property, every strategy-proof RSCFs satisfying unanimity must be specified by a convex combination of unanimous and strategy-proof DSCFs. The extreme point property has been established on several voting domains, e.g., the complete domain ([26]), the binary domain ([37]), the single-peaked domain ([24], [36], [38]) and the product domain with lexicographically separable preferences ([16]). These results appear to suggest that permitting randomization does not substantially expand possibilities for design strategy-proof RSCFs. In particular, on the class of dictatorial domains (i.e., every unanimous and strategy-proof DSCF is a dictatorship) which is pervasive and includes much sparser domains in addition to the complete domain (linked domains in [1] and circular domains in [43]), one may simply conjecture that the same extreme point property prevails. Equivalently, the following question is proposed:

Is every dictatorial domain a random dictatorship domain?

The primary goal in Chapter 2 of this dissertation is to show that this conjecture is *false*. This chapter does so in the following way. Firstly, identify a sufficient condition on domains, called Condition SC (Successful Compromise) that permits the existence of unanimous and strategy-proof RSCFs which are not random dictatorships. These RSCFs put strictly positive probability on compromise alternatives (second ranked alternatives) in certain cases. Condition SC then is shown to be compatible with the linked domain condition in Aswal et al. [1] that ensures that a domain is dictatorial. This chapter also shows that the domain satisfying Condition SC arises naturally in a model of independent economic interest, that of constrained voting (e.g., [6], [7], [8]).

Furthermore, Chapter 2 considers strengthenings of the linkedness condition that ensures a dictatorial domain be a random dictatorship domain. Two conditions are proposed in this regard. The first is *Condition H* (Hub), which requires the corresponding connectivity graph over alternatives to have an alternative to be connected to all other alternatives in addition to the linkedness property. For the second condition, called *Condition TS* (Two Steps), The connectedness requirement is firstly strengthened underlying the definition of a linked domain to obtain the notion of a strongly linked domain; an additional condition then is imposed whic is however weaker than the counterpart of the hub condition. The additional condition requires the existence of a path of length at most two between any two alternatives in the corresponding strong connectivity graph. It is obvious that Conditions H and TS are very significant strengthenings of the linkedness condition. However, this chapter shows that these conditions in conjunction with a linkedness condition are "almost necessary" for random dictatorship. For example, this chapter constructs linked domains, where there is no hub but "almost all" alternatives are "almost hubs", which satisfy Condition SC and are therefore not random dictatorship domains.

The proofs of the sufficiency result rely on a ramification result that states that a random dictatorship domain when there are two voters is in fact, a random dictatorship domain when there is an arbitrary number of voters. This approach was initiated in [30] in the context of domains that permit non-dictatorial Arrovian aggregation. Corresponding results for dictatorial domains appear in [32], [47] and

[1]. The result for random dictatorship is however, significantly more difficult than its dictatorial domains counterpart. In fact, this chapter is able to prove it only under an additional hypothesis which is fortunately weak and is satisfied by the sufficiency conditions.

Chapter 2 is organized as follows. Section 2.1 introduces linked domains, Section 2.2 comprises three subsections that introduce Condition SC, shows that it can be satisfied by some dictatorial domains and finally applies it to a model of constrained voting. Section 2.3 provides random dictatorship results and results on the necessity of Conditions H and SC. The Appendix contains the Ramification Theorem and the other proofs.

Single-peaked preferences are the cornerstone of several models in political economy and social choice theory. They were proposed initially by [11] and [29]. Single-peaked preferences arise naturally in various setting. However, their main attraction is that they allow successful preference aggregation both in the Arrovian and the strategic sense (see [34], [3]). For instance, under each profile of single-peaked preferences, one can adopt majority voting to induce a desirable social preference (provided odd number of voters), or generate a fair and strategy-proof social outcome which indicates the distinction from dictatorships. Literature has widely explored strategy-proof voting rules over single-peaked preferences (e.g., [34], [19], [18], [24]), however, initialized in [6], an inverse question is proposed:

Does the single-peakedness restriction naturally emerge as a consequence of the existence of a "well-behaved" strategy-proof voting rule?

The question of this nature is referred to as the Gul Conjecture (see [3]). The precise formulation of the conjecture can take several forms, e.g., [6], [10], [35].

Chapter 3 studies the inverse question above in a randomized setting. The goal in this chapter is to provide a result with the following flavor: any rich preference domain that admits a suitably well-behaved randomized solution to the strategic voting problem must be a single-peaked domain. The single-peaked domain characterized in this chapter is more general than the usual one (for example, the single-peakedness in [34]). These preferences were introduced in [20] and [19], and are defined on arbitrary trees.

One consequence of considering RSCFs is that the *anonymity* requirement (the names of voters do not affect the social lottery) imposed on DSCFs to escape dictatorships appears ineffective and must be replaced since it is always possible to design a unanimous and strategy-proof RSCF satisfying anonymity on an arbitrary domain, e.g., a random dictatorship where all voters are endowed with the same weight. Therefore, a new appropriate notion of well-behavedness needs to be formulated. This chapter imposes two additional axioms on RSCFs under consideration. The first one, *the tops-only property*, is standard in the literature on voting which implies that the social lottery under each preference profile depends only on the peaks of preferences. The second axiom, *the compromise property*, is established to deal with the infirmity of random dictatorships mentioned before. Consider a preference profile where the set of voters are split into two (almost) equal groups and the following conditions are satisfied: (i) all voters within a group have identical preferences, (ii) the peaks of two groups' preferences are different, and (iii) there is an alternative that is second-ranked according to the preferences of both groups. This commonly second-ranked alternative can be naturally regarded as a *compromise* alternative and the compromise property requires that the compromise alternative receives strictly positive probability in the profile. This chapter then proves the following result: Every "suitably" rich domain that admits a tops-only, strategy-proof RSCF satisfying unanimity and the compromise property is single-peaked on a tree. Conversely, every single-peaked domain admits a tops-only, strategy-proof RSCF satisfying ex-post efficiency (a stronger version of unanimity) and the compromise property.

A paper related to this chapter is Chatterji et al. [17]. That paper investigated preference domains that admits well-behaved and strategy-proof DSCFs. In particular, it showed that every rich domain that admits a strategy-proof, unanimous, anonymous and tops-only DSCF with an even number of voters, is semi-single-peaked.[1] These preferences are also defined on trees but are significantly less restrictive than single-peaked preferences. This chapter demonstrates that two important objectives can be met by considering RSCFs rather than DSCFs. The first is that a characterization of single-peaked rather than semi-single-peaked preferences can be obtained naturally. The second is that the awkward assumption regarding the even number of voters in [17] can be removed.

Chapter 3 is organized as follows. Section 3.1 introduces the compromise property. Section 3.2 introduces path-connected domains, while Section 3.3 contains the characterization result for single-peaked domains and demonstrates the indispensability of each axiom and the richness condition. The Appendix contains some additional discussion and an omitted proof.

If strategy-proofness is the only concern, one can construct a *constant* RSCF which ignores all information of voters' preferences and fixes a lottery as the social outcome for every preference profile. However, such a RSCF is clearly not desirable. On the other hand, while allowing the social lottery to vary with preference profiles is desirable, maintaining strategy-proofness becomes correspondingly harder as the social lottery begins to depend more Chapter 4 investigates strategy-proof RSCFs which only use the peaks of voters' preferences to calculate the social lottery. This class of RSCFs is referred to as RSCFs satisfying *the tops-only property*, which implies that if the peaks of each voter across two preference profiles are identical, the social lottery remains the same.

The class of tops-only RSCFs has considerable informational and computational advantages from the design point of view; it also reduces the degree of possible manipulations significantly since any misrepresentation using a preference with the same peak as the true preference does not affect the social lottery and hence is not beneficial. More importantly, in much of the literature, the tops-only property is a key necessary step in designing and characterizing strategy-proof RSCFs.[2]

[1] The notion of richness is exactly the same as that in this chapter.

[2] For instance, see dictatorship in [25], [45] and [4], random dictatorship in [26], voting by committees in [5], generalized median voter rule in [6], fixed-probabilistic-ballots rule in [24], voting by

However, once insisting on the tops-only property, one encounters the following designing problem:

Is the scope for designing strategy-proof RSCFs significantly constrained?

Indeed, there may exist other intuitive RSCFs that use some non-top information and have nice incentive properties, e.g., the point voting schemes in [2]. Chapter 4 precludes this possibility by providing a condition on preference domains on which strategy-proofness and unanimity *imply* the tops-only property.[3] Importantly, note that under the sufficient condition, the tops-only property emerges endogenously; the methodology proposed in this chapter allows one to assert the tops-only property without requiring us to explicitly characterize the class of all unanimous and strategy-proof RSCFs.

In the literature, more precisely, in the case of DSCFs (e.g., [52] and [15]), it is well-known that appropriate *richness conditions* are required on domains so that the tops-only property can be endogenously established. This chapter identifies a new sufficient condition on domains which ensures the tops-only property for all unanimous and strategy-proof RSCFs. This condition requires that a particular *Interior Property* and an *Exterior Property*, respectively, hold. The Interior Property is a restriction applied to every subdomain of preferences that shares a common peak while the Exterior Property is a restriction applied across subdomains with distinct peaks.

The Interior Property is formulated in terms of *adjacent connectedness* proposed by Sato [44]. Two distinct preference orders are adjacently connected if there exists exactly one pair of alternatives with contiguous opposite relative rankings while the ranking of every other alternative is identical across these two preferences. The Interior Property requires that for any two distinct preferences with an identical peak, one can construct a sequence of adjacently connected preferences in the correspond-

issue in [35] and generalized random dictatorships in [16].

[3]A consequence of the ordinal version of strategy-proofness is that the primitive of this chapter is an admissible domain of ordinal strict preferences. The principle assumptions will accordingly be imposed on the admissible domain of preferences.

ing sub-domain connecting them.

The Exterior Property considers the relation between two preferences with *distinct* peaks. It is formulated in terms of *isolation*, i.e., a pair of alternatives is isolated in two preferences if one can identify an integer k such that in both preferences, the sets of the top-k ranked alternatives are identical, include one of these two alternatives and exclude the other. Accordingly, a domain satisfies the Exterior Property if fixing a pair of preferences with distinct peaks and a pair of alternatives x and y such that x is ranked above y in both preferences, one can always construct a sequence of preferences connecting these two fixed preferences such that x and y are isolated in every two consecutive preferences in the sequence.

The first result in this chapter states that every unanimous and strategy-proof RSCF defined on a domain satisfying the Interior Property and the Exterior Property must satisfy the tops-only property. As applications, this chapter demonstrates that prominent domains in the literature satisfy the Interior Property and the Exterior Property, e.g., the complete domain, the single-peaked domain, the single-dipped domain, maximal single-crossing domains and the multi-dimensional single-peaked domain. Furthermore, to extend the study to separable preferences, a new notion called *general connectedness* is introduced. Correspondingly, a modification of the Interior Property and the Exterior Property is established which respectively uses general connectedness to replace adjacent connectedness in the Interior Property and imposes general connectedness on the sequence of preferences in the Exterior Property. This chapter then shows that the tops-only property is implied by unanimity and strategy-proofness over a domain of separable preferences satisfying *the Modified Interior Property* and *the Modified Exterior Property*, and moreover, the separable domain is covered by these two modified properties.

In many political and economic settings, the restrictions of multi-dimensional single-peakedness and separability, respectively, arise naturally. For instance, in a political election, each candidate can be described as a combination of positions on various political issues, e.g., expenditure on education, health, etc. Normally, the preference of a voter over all candidates is formulated according to the criteria of "*closeness*", i.e., a candidate with positions "closer" to the voter's ideal political attitude is preferred to another candidate with "further" positions. Hence, multi-dimensional single-peakedness is embedded in voter's preferences. Consider another example where a club decides to recruit k new members from the pool of m applicants where $m \geq k$. A possible recruitment profile can be represented as a m-tuple of zeros and ones where if the hth coordinate of the m-tuple is zero, then applicant h is excluded; otherwise, applicant h is included. Separable preferences arise whenever each member of the recruitment committee has an unambiguous attitude over the inclusion/exclusion of every applicant.

Deterministic strategy-proof voting rules are widely explored over both the multi-dimensional single-peaked domain and the separable domain in the literature, e.g., voting by committee in [5] and [8], generalized median voter rules in [6] and [7], decomposable rules in [13] and voting by issue in [35]. Since the sufficient conditions in this chapter can be applied to induce the tops-only property in RSCFs over these two domains, the results in this chapter can be used to further characterize strategy-proof RSCFs. This chapter first shows that every ex-post efficient (an axiom stronger than unanimity) and strategy-proof RSCF over the multi-dimensional single-peaked domain is a random dictatorship. Similarly, this chapter asserts that every unanimous and strategy-proof RSCF over the separable domain is a generalized random dictatorship. Both characterization results are instances of the extreme point property studied in Chapter 2.

Furthermore, the result in this chapter allows one to study the inverse question specified in Chapter 3 in a particular class of *rich* domains. This chapter strengthens ex-post efficiency to ex-post efficiency* by assigning strictly positive probabilities to all *Pareto-undominated* alternatives under every preference profile, and show that single-peakedness (on a tree) is uniquely characterized by ex-post efficiency* and strategy-proofness.

Chapter 4 is organized as follows. Section 4.1 presents the main result. Section

4.2 provides five applications while Section 4.3 summarizes the relation to the literature, and provides some discussion on the necessity of the conditions. Proofs of lemmas and propositions are gathered in the Appendix.

Last, Chapter 5 concludes all main results established in Chapters 2, 3 and 4.

1.1 Preliminaries and the Model

Let $A = \{a, b, c, \ldots\}$ be a finite set of alternatives with $|A| = m \geq 3$. Sometimes, the alternative set A is assumed to be labeled as $A = \{a_1, \ldots, a_m\}$. Let $\Delta(A)$ denote the lottery space induced by A. An element of $\Delta(A)$ is a *lottery* or probability distribution over the elements of A. In particular, $e_a \in \Delta(A)$ is a degenerate lottery where alternative a is chosen with probability one. Let $I = \{1, \ldots, N\}$ be a finite set of voters with $|I| = N \geq 2$. Each voter i has a (strict preference) order P_i over A which is antisymmetric, complete and transitive, i.e., a linear order.[4] For any $a, b \in A$, aP_ib is interpreted as "a is *strictly* preferred to b according to P_i". Let $r_k(P_i)$ denote the kth ranked alternative in P_i, $k = 1, \ldots, m$. A pair of alternatives $a, b \in A$ is *contiguous* in P_i if $\{a, b\} = \{r_k(P_i), r_{k+1}(P_i)\}$ for some $1 \leq k \leq m-1$. Accordingly, let $aP_i!b$ denote that a and b are contiguous in P_i, and aP_ib. Given $1 \leq k \leq m$ and $P_i \in \mathbb{D}$, $B^k(P_i) = \cup_{t=1}^{k}\{r_t(P_i)\}$ is the set of top-k ranked alternatives. Similarly, given $a \in A$ and $P_i \in \mathbb{D}$, let $B(P_i, a) = \{x \in A | xP_ia\}$ and $W(P_i, a) = \{x \in A | aP_ix\}$ denote respectively the (strictly) upper contour set and the (strictly) lower contour set of a in P_i. Let $\mathbb{P}$ denote the set containing *all* linear orders over A. The set of all admissible orders is a set $\mathbb{D} \subseteq \mathbb{P}$, referred to as the *preference domain*. In particular, $\mathbb{P}$ is called *the complete domain*. A preference profile $P \equiv (P_1, P_2, \ldots, P_N) \equiv (P_i, P_{-i}) \in \mathbb{D}^N$ is an N-tuple of orders. Given nonempty subset $\hat{I} \subseteq I$, let $r_1(P_{\hat{I}}) = \cup_{i \in \hat{I}}\{r_1(P_i)\}$ denote the set of top-ranked alternatives in $P_{\hat{I}}$.[5] For notational convenience, let $\mathbb{D}^a = \{P_i \in \mathbb{D} | r_1(P_i) = a\}$ denote the set of

[4]The whole dissertation only studies the strict preferences.

[5]Throughout the whole dissertation, $\subseteq$ and $\subset$ denote the weak and strict inclusion relations between two sets respectively.

preferences with peak a; $\mathbb{D}^{a,b} = \{P_i \in \mathbb{D} | r_1(P_i) = a$ and $r_2(P_i) = b\}$ denote the set of preferences with peak b and the second best b; and $\mathbb{D}^S = \{P_i \in \mathbb{D} | r_1(P_i) \in S\}$ denote the set of preferences with peak in the subset $S \subseteq A$. In particular, a domain $\mathbb{D}$ is referred to be *minimally rich* if every alternative is top-ranked in some preference in the domain, i.e., $\mathbb{D}^a \neq \emptyset$ for all $a \in A$.

A *Random Social Choice Function* (or RSCF) is a map $\varphi : \mathbb{D}^N \to \Delta(A)$. At every profile $P \in \mathbb{D}^N$, $\varphi(P)$ is referred to as the "socially desirable" lottery. For any $a \in A$, $\varphi_a(P)$ is the probability with which the alternative a will be chosen in $\varphi(P)$. Thus, $\varphi_a(P) \geq 0$ for all $a \in A$ and $\sum_{a \in A} \varphi_a(P) = 1$. A *Deterministic Social Choice Function* (or DSCF) is a particular RSCF where a degenerate lottery is chosen under each preference profile, i.e., $\varphi(P) = e_a$ for some $a \in A$ at profile P. For notational convenience, a DSCF sometimes is simply written as a map $f : \mathbb{D}^N \to A$.

A RSCF satisfies *unanimity* if it assigns probability one to any alternative that is ranked first by all voters.

Definition 1.1.1. *A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is unanimous if $[r_1(P_i) = a$ for all $i \in I] \Rightarrow [\varphi_a(P) = 1]$ for all $a \in A$ and $P \in \mathbb{D}^N$.*

An axiom stronger than unanimity is *ex-post efficiency*. It requires all Pareto-dominated outcomes to never be chosen.

Definition 1.1.2. *A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is ex-post efficient if for all $a, b \in A$ and $P \in \mathbb{D}^N$, $[aP_ib$ for all $i \in I] \Rightarrow [\varphi_b(P) = 0]$.*

If the social lottery does not depend on the "names" of voters, a RSCF is referred to satisfy the property of *anonymity*.

Definition 1.1.3. *A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is anonymous if for every permutation $\sigma : I \to I$ and $P = (P_1, \ldots, P_N) \in \mathbb{D}^N$, $\varphi(P_1, \ldots, P_N) = \varphi(P_{\sigma(1)}, \ldots, P_{\sigma(N)})$.*

A prominent class of RSCFs is the class of *tops-only* RSCFs. The social lottery of a tops-only RSCF at every profile depends only on voters' peaks at that profile.

Definition 1.1.4. *A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies the tops-only property if $[r_1(P_i) = r_1(P'_i)$ for all $i \in I] \Rightarrow [\varphi(P) = \varphi(P')]$ for all $P, P' \in \mathbb{D}^N$.*

Gibbard [26] proposed an ordinal formulation of *strategy-proofness* which requires the social lottery under truthtelling (first-order) stochastically dominates any social lottery under misrepresentation according to every voter's true preference independently of others' behavior.

Definition 1.1.5. *A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is strategy-proof if for all $i \in I$; $P_i, P'_i \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$,*

$$\sum_{t=1}^{k} \varphi_{r_t(P_i)}(P_i, P_{-i}) \geq \sum_{t=1}^{k} \varphi_{r_t(P_i)}(P'_i, P_{-i}), k = 1, \ldots, m.$$

This notion of strategy-proofness is equivalent to requiring a voter's expected utility from truthtelling to be no less than her expected utility from misrepresentation for any cardinal representation of her true preferences independently of other voters' behavior. We omit these details which may be found in [26].

The notions of unanimity, ex-post efficiency, anonymity, tops-onlyness and strategy-proofness are standard axioms in the literature on mechanism design in voting environments. An important class of RSCFs satisfying unanimity (ex-post efficiency), the tops-only property and strategy-proofness is the class of *random dictatorships*. Each voter is assigned a non-negative weight with the sum of weights across voters being one. At any profile, the probability with which an arbitrary alternative a is chosen is the sum of the probability weights of voters for whom a is the top-ranked alternative.

Definition 1.1.6. *A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a random dictatorship if there exists $[\varepsilon_i]_{i \in I} \in \mathbb{R}^N_+$ with $\sum_{i \in I} \varepsilon_i = 1$ such that for all $a \in A$ and $P \in \mathbb{D}^N$,*

$$\varphi_a(P) = \sum_{i \in I : r_1(P_i) = a} \varepsilon_i.$$

In other words, a random dictatorship is a convex combination of a group of dictatorial DSCFs.[6] In particular, if the weight is $\frac{1}{N}$ for every voter, the random dictatorship also satisfies the property of anonymity.

Definition 1.1.7. *A domain $\mathbb{D}$ is a random dictatorship domain, if every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$, $N \geq 2$, is a random dictatorship.*

In particular, given $\mathbb{D}$ and fixing $N \geq 2$, if every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a random dictatorship, $\mathbb{D}$ is referred to as a random dictatorship domain of N voters. A fundamental result in random mechanism design theory proved in [26] is that the complete domain $\mathbb{P}$ is a random dictatorship domain (see also [21] and [48]).

Analogously, a domain is a *dictatorial domain*, if every unanimous and strategy-proof DSCF $f : \mathbb{D}^N \to A$, $N \geq 2$, is a dictatorship. In the deterministic environment, anonymity is the polar opposite of dictatorship. According to the Gibbard-Satterthwaite Theorem ([25] and [45]), the complete domain $\mathbb{P}$ is also a dictatorial domain (see also [41] and [47]).

[6] A DSCF is dictatorial if there exists a fixed voter (the dictator) whose best alternative is the social outcome under every preference profile. Formally, a DSCF $f : \mathbb{D} \to A$ is a *dictatorship* if there exists $i \in I$ such that $f(P) = r_1(P_i)$ for all $P \in \mathbb{D}^N$.

Chapter 2 Random Dictatorship Domains

2.1 Linked Domains

A central concern of this chapter is the relationship between dictatorial and random dictatorship domains. It is evident that a random dictatorship domain is dictatorial. The question of interest is clearly the converse question. As mentioned in Section 1.1, the complete domain is both a dictatorial domain and a random dictatorship domain. Does this relationship hold true generally? In order to investigate this question, this chapter recalls the main result of Aswal et al. [1] on dictatorial domains.

One important type of dictatorial domains is linked domains introduced by [1]. To establish the definition of linked domains, this chapter first introduces the definitions of connectedness and linkedness. Given a domain $\mathbb{D}$, a pair of distinct alternatives a, b is *connected*, denoted $a \sim b$, if $\mathbb{D}^{a,b} \neq \emptyset$ and $\mathbb{D}^{b,a} \neq \emptyset$. Furthermore, Given $B \subset A$ and $a \in A \backslash B$, a is linked to B if there exist two distinct alternatives $b, c \in B$ such that $a \sim b$ and $a \sim c$.

Definition 2.1.1. *The domain $\mathbb{D}$ is linked, if according to the labeled alternative set $A = \{a_1, \ldots, a_m\}$, there exists a bijective function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that*

(i) $a_{\sigma(1)} \sim a_{\sigma(2)}$*;*

(ii) $a_{\sigma(j)}$ *is linked to* $\{a_{\sigma(1)}, \ldots a_{\sigma(j-1)}\}$, $j = 3, \ldots, m$.

The notion of a linked domain is formulated entirely in terms of alternatives that can be ranked first and second according to preferences in the domain. Evidently, a linked domain is minimally rich and the minimal cardinality of a linked domain is $4m-6$.

The reader is referred to [1] for details and numerous examples on linked domains. The following theorem summarizes the main result in [1].

Theorem 2.1.1. *Linked domains are dictatorial domains.*

A natural question is whether linked domains are also random dictatorship domains. This is addressed in the following sections.

2.2 Dictatorial Domains are not Random Dictatorship Domains

This section provides examples of dictatorial domains that are not random dictatorship domains. In fact, a stronger result is shown: there exist domains which are (deterministic) dictatorial but admit anonymous, unanimous and strategy-proof RSCFs that are not random dictatorships.

This section proceeds as follows. A sufficient condition on domains is first identified that ensures the existence of anonymous, unanimous and strategy-proof RSCFs that are not random dictatorships. This section then shows that there are linked domains that satisfy the sufficient condition.

2.2.1 A Sufficient Condition

The idea behind the condition is extremely simple. It ensures strategy-proofness of a special RSCF that puts positive probability on the second ranked alternative of a voter under special circumstances and remains to be a $\frac{1}{N}$ random dictatorship elsewhere.

Fix $\mathbb{D} \subseteq \mathbb{P}$ and $a \in A$. Let $\mathcal{S}(a)$ denote the set of alternatives each of which is ranked second in an admissible preference order where a is the peak, i.e., $[x \in \mathcal{S}(a)] \Leftrightarrow [x = r_2(P_k)$ for some $P_k \in \mathbb{D}^a]$.

Definition 2.2.1. *A domain $\mathbb{D}$ satisfies Condition SC (Successful Compromise), if there exists a nonempty set $B \subset A$ and $y \in A \backslash B$ such that*

(i) *if $|B| > 1$, there exists an preference order whose first and second ranked alternatives belong to B, i.e., $\mathbb{D}^{a,b} \neq \emptyset$ for some $a, b \in B$.*

(ii) *for all $P_k \in \mathbb{D}^B$,*

- *if $|B| = 1$, $[a \in \mathcal{S}(y)] \Rightarrow [aP_k y]$.*
- *if $|B| > 1$, $[r_2(P_k) \in B$ and $a \in \mathcal{S}(y)] \Rightarrow [aP_k y]$ and*

$$[r_2(P_k) \notin B \text{ and } a \in \mathcal{S}(y)] \Rightarrow [yP_k a].$$

(iii) *for all $P_k \in \mathbb{D}^{A\backslash[B\cup\{y\}]}$, $[aP_k y$ for some $a \in \mathcal{S}(y)] \Rightarrow [yP_k z$ for all $z \in B]$.*

In order to interpret Condition SC, it may be helpful to think of the set B as the *test set*, the alternative y as the *test alternative* and every alternative ranked second in any preference order where y is ranked first as a *compromise*. The RSCF will specify $\frac{1}{N}$ random dictatorship *except* at a profile where $N-1$ voters have alternatives in the test set as their first and second ranked alternatives (in case there is only one alternative in the test set, it has to be ranked first by $N-1$ voters) *and* one voter has the test alternative ranked first, then a suitably "small" probability is transferred to the appropriate compromise alternative. The SC conditions impose simple restrictions between the compromise alternatives, the test alternative and the test set in various preference orders.

Condition SC has been presented for two cases: one where the cardinality of the test set is one and another where it is greater than one. These two cases will be used separately in Section 2.3. Condition SC is illustrated in Example 2.2.1.

Example 2.2.1. Let $A = \{a, b, c\}$. Domains $\mathbb{D}^1$ and $\mathbb{D}^2$ satisfy Condition SC with respect to $B = \{a\}$ and b (see Table 2.1), while domains $\mathbb{D}^3$ and $\mathbb{D}^4$ satisfy Condition SC with respect to $B = \{a, c\}$ and b (see Table 2.2).

P_1	P_2	P_3	P_4		P'_1	P'_2	P'_3	P'_4
a	a	b	c		a	b	b	c
b	c	a	b		c	a	c	b
c	b	c	a		b	c	a	a

Table 2.1: Domains $\mathbb{D}^1$ and $\mathbb{D}^2$

$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$	$\bar{P}_4$		$\hat{P}_1$	$\hat{P}_2$	$\hat{P}_3$	$\hat{P}_4$
a	a	b	c		a	b	c	c
b	c	c	a		c	a	a	b
c	b	a	b		b	c	b	a

Table 2.2: Domains $\mathbb{D}^3$ and $\mathbb{D}^4$

□

The following proposition shows that a domain satisfying Condition SC is not a random dictatorship domain.

Proposition 2.2.1. *A domain satisfying Condition SC admits an anonymous, unanimous and strategy-proof RSCF that is not a random dictatorship.*

Proof. Let domain $\mathbb{D}$ satisfy Condition SC with respect to some $B \subset A$ and $y \in A \backslash B$. We consider the cases $|B| = 1$ and $|B| > 1$ separately. For each case, we construct an anonymous, unanimous and strategy-proof RSCF that is not a random dictatorship.

Case A: $|B| = 1$.

Let $B = \{x\}$ and consider the RSCF φ below: for all $P \in \mathbb{D}^N$,

$$\varphi(P) = \begin{cases} (\frac{1}{N} - \alpha)e_y + \alpha\, e_{r_2(P_i)} + \frac{N-1}{N}\, e_x, \\ \qquad \text{if } P_i \in \mathbb{D}^y \text{ for some } i \in I \text{ and } P_j \in \mathbb{D}^x \text{ for all } j \in I\backslash\{i\} \\ \sum\limits_{i \in I} \frac{1}{N}\, e_{r_1(P_i)}, \\ \qquad \text{otherwise.} \end{cases}$$

where $0 < \alpha \leq \frac{1}{N}$.

By construction φ is a random dictatorship with weight $\frac{1}{N}$ on the best alternative of every voter at all profiles *except* at a profile where exactly one voter has peak y and all other voters have peak x. At such a profile, probability α is transferred from y to the second best alternative of the voter with the peak y.

Evidently, φ is anonymous, unanimous and not a random dictatorship. We show that φ is strategy-proof by showing that any possible manipulation always makes probabilities systematically transferred from the preferred alternatives to less preferred alternatives according to the true preference which equivalently indicates stochastic dominance. In view of the construction of φ, only the following two cases need attention:

1. The profile is $(P_i, P_j, P_{-\{i,j\}})$ where $P_i \in \mathbb{D}^y$ and $P_k \in \mathbb{D}^x$ for all $k \in I\backslash\{i\}$. Voter $j \in I\backslash\{i\}$ considers a manipulation via $P_j' \notin \mathbb{D}^x$.

   Since $P_j \in \mathbb{D}^x$, we know $r_2(P_i)P_jy$ by part (ii) of Condition SC. Consequently, we have

$$\varphi(P_i, P_j, P_{-\{i,j\}}) \xrightarrow[\frac{1}{N}]{xP_jr_1(P_j')}, \xrightarrow[\alpha]{r_2(P_i)P_jy} \varphi(P_i, P_j', P_{-\{i,j\}}).[1]$$

2. The profile is $(P_i, P_j, P_{-\{i,j\}})$ where $P_i \in \mathbb{D}^y$, $P_j \notin \mathbb{D}^x$ for some $j \in I\backslash\{i\}$ and $P_k \in \mathbb{D}^x$ for all $k \in I\backslash\{i,j\}$. Voter $j \in I\backslash\{i\}$ considers a manipulation via $P_j' \in \mathbb{D}^x$.

   If $yP_jr_2(P_i)$, we have

$$\varphi(P_i, P_j, P_{-\{i,j\}}) \xrightarrow[\frac{1}{N}]{r_1(P_j)P_jx}, \xrightarrow[\alpha]{yP_jr_2(P_i)} \varphi(P_i, P_j', P_{-\{i,j\}}).$$

   If $r_2(P_i)P_jy$, then $P_j \in \mathbb{D}^{A\backslash\{x,y\}}$, and furthermore, part (iii) of Condition SC

[1]The notation $\varphi(P_i, P_j, P_{-\{i,j\}}) \xrightarrow[\frac{1}{N}]{xP_jr_1(P_j')}, \xrightarrow[\alpha]{r_2(P_i)P_jy} \varphi(P_i, P_j', P_{-\{i,j\}})$ represents that (i) $xP_jr_1(P_j')$ and $r_2(P_i)P_jy$, and (ii) from $\varphi(P_i, P_j, P_{-\{i,j\}})$ to $\varphi(P_i, P_j', P_{-\{i,j\}})$, probabilities $\frac{1}{N}$ and α are transferred from x to $r_1(P_j')$, and from $r_2(P_i)$ to y respectively.

implies yP_jx. Consequently, we have

$$\varphi(P_i, P_j, P_{-\{i,j\}}) \xrightarrow[\frac{1}{N}-\alpha]{r_1(P_j)P_jx}, \xrightarrow[\alpha]{r_1(P_j)P_jr_2(P_i),\text{ or }r_1(P_j)=r_2(P_i)}, \xrightarrow[\alpha]{yP_jx} \varphi(P_i, P'_j, P_{-\{i,j\}}).$$

We conclude that RSCF φ is strategy-proof.

Case B: $|B| > 1$.

Consider the following RSCF φ': for all $P \in \mathbb{D}^N$,

$$\varphi'(P) = \begin{cases} (\frac{1}{N} - \alpha)e_y + \alpha\, e_{r_2(P_i)} + \sum\limits_{j \in I\backslash\{i\}} \frac{1}{N}\, e_{r_1(P_j)}, \\ \qquad \text{if } P_i \in \mathbb{D}^y \text{ for some } i \in I, \text{ and } P_j \in \mathbb{D}^B \text{ with } r_2(P_j) \in B \text{ for all } j \in I\backslash\{i\}; \\ \sum\limits_{i \in I} \frac{1}{N}\, e_{r_1(P_i)}, \\ \qquad \text{otherwise.} \end{cases}$$

where $0 < \alpha \leq \frac{1}{N}$.

Thus, φ' is a random dictatorship with weight $\frac{1}{N}$ on the best alternative of every voter at all profiles *except* at a profile where exactly one voter has peak y and all other voters' first and second ranked alternatives belong to B. At such a profile, probability α is transferred from y to the second best alternative of the voter with the peak y.

As before, φ' is easily shown to be anonymous, unanimous and not a random dictatorship. We only need to show that φ' is strategy-proof. Once again, only the two cases below require attention and the arguments work in virtually the same way as they do in Case A.

1. The profile is $(P_i, P_j, P_{-\{i,j\}})$ where $P_i \in \mathbb{D}^y$ and $P_k \in \mathbb{D}^B$ with $r_2(P_k) \in B$ for all $k \in I\backslash\{i\}$. Voter $j \in I\backslash\{i\}$ considers a manipulation via $P'_j \in \mathbb{D}$ such that $\varphi'(P_i, P'_j, P_{-\{i,j\}}) = \frac{1}{N}e_{r_1(P'_j)} + \sum_{k \in I\backslash\{j\}} \frac{1}{N}e_{r_1(P_k)}$.

   Since $P_j \in \mathbb{D}^B$ and $r_2(P_j) \in B$, part (ii) of Condition SC implies $r_2(P_i)P_jy$.

Consequently, we have

$$\varphi'(P_i, P_j, P_{-\{i,j\}}) \xrightarrow[\frac{1}{N}]{r_1(P_j)P_j r_1(P_j'),\text{ or } r_1(P_j)=r_1(P_j')}, \xrightarrow[\alpha]{r_2(P_i)P_j y} \varphi'(P_i, P_j', P_{-\{i,j\}}).$$

2. The profile is $(P_i, P_j, P_{-\{i,j\}})$ where $P_i \in \mathbb{D}^y$, $P_k \in \mathbb{D}^B$ with $r_2(P_k) \in B$ for all $k \in I\backslash\{i,j\}$ and $P_j \in \mathbb{D}$ such that $\varphi'(P_i, P_j, P_{-\{i,j\}}) = \frac{1}{N}e_y + \frac{1}{N}e_{r_1(P_j)} + \sum_{k\in I\backslash\{i,j\}} \frac{1}{N}e_{r_1(P_k)}$. Voter j considers a manipulation via $P_j' \in \mathbb{D}^B$ with $r_2(P_j') \in B$.

   If $yP_j r_2(P_i)$, we have

$$\varphi'(P_i, P_j, P_{-\{i,j\}}) \xrightarrow[\frac{1}{N}]{r_1(P_j)P_j r_1(P_j'),\text{ or } r_1(P_j)=r_1(P_j')}, \xrightarrow[\alpha]{yP_j r_2(P_j)} \varphi'(P_i, P_j', P_{-\{i,j\}}).$$

   Lastly, assume therefore that $r_2(P_i)P_j y$ holds. Clearly $P_j \notin \mathbb{D}^y$. Suppose $r_1(P_j) \in B$. Our assumption that $\varphi'(P_i, P_j, P_{-\{i,j\}}) = \frac{1}{N}e_y + \frac{1}{N}e_{r_1(P_j)} + \sum_{k\in I\backslash\{i,j\}} \frac{1}{N}e_{r_1(P_k)}$ implies $r_2(P_j) \notin B$. Hence, part (ii) of Condition SC implies $yP_j r_2(P_i)$ which contradicts our initial assumption. Therefore, $r_1(P_j) \notin B \cup \{y\}$. Consequently, part (iii) of Condition SC implies $yP_j r_1(P_j')$. Then, we have

$$\varphi'(P_i, P_j, P_{-\{i,j\}}) \xrightarrow[\frac{1}{N}-\alpha]{r_1(P_j)P_j r_1(P_j')}, \xrightarrow[\alpha]{r_1(P_j)P_j r_2(P_i),\text{ or } r_1(P_j)=r_2(P_i)}, \xrightarrow[\alpha]{yP_j r_1(P_j')} \varphi'(P_i, P_j', P_{-\{i,j\}}).$$

We conclude that RSCF φ' is strategy-proof. □

In the next two subsections, two applications of Condition SC are presented.

2.2.2 Dictatorial Domains satisfying Condition SC

This subsection shows by means of examples that there are linked domains that satisfy Condition SC. An immediate consequence of Theorem 2.1.1 and Proposition 2.2.1 is that there exist dictatorial domains that admit anonymous, unanimous and strategy-proof RSCFs that are not random dictatorships.

Example 2.2.2. Let $A = \{x, y, a, b, c, d\}$. Domain $\hat{\mathbb{D}}$ of preferences over the six alternatives is described in Table 2.3.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}
x	x	a	a	a	b	b	b	b	c	c	c	c	d	d	d	y	y
a	b	x	b	c	x	a	c	d	a	b	d	y	b	c	y	c	d
·	·	·	·	·	·	·	·	·	·	·	y	·	·	y	·	·	·
·	·	·	·	y	·	·	y	y	y	y	·	·	y	·	·	·	·
·	·	y	y	·	y	y	·	·	·	·	·	·	·	·	·	·	·
·	·	c	c	x	c	c	x	x	x	x	·	·	x	·	·	·	·
y	y	d	d	·	d	d	·	·	·	·	·	·	·	·	·	·	·

Table 2.3: Domain $\hat{\mathbb{D}}$ [2]

To check whether a domain is linked, it is convenient to associate to the domain a graph that reflects its connectedness structure. This is done as follows. Fixing a domain $\mathbb{D}$, let $G(\mathbb{D})$ denote the connectivity graph of $\mathbb{D}$ where (i) the set of nodes is A, and (ii) for all $a, b \in A$, (a, b) is an edge if $a \sim b$. Correspondingly, for domain $\hat{\mathbb{D}}$, the connectivity graph $G(\hat{\mathbb{D}})$ is shown in Figure 2.1. It is clear that $\hat{\mathbb{D}}$ is a linked domain: we relabel alternatives in A as $a_1 = x$, $a_2 = a$, $a_3 = b$, $a_4 = c$, $a_5 = d$ and $a_6 = y$, and the one to one function $\sigma : \{1, \dots, 6\} \to \{1, \dots, 6\}$ in Definition 2.1.1 is the identity function.

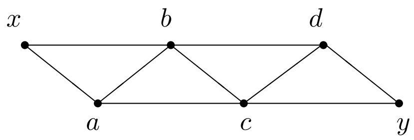

Figure 2.1: Connectivity Graph $G(\hat{\mathbb{D}})$

We claim that $\hat{\mathbb{D}}$ satisfies Condition SC with respect to $B = \{x\}$ and y. Observe that $\mathcal{S}(x) = \{a, b\}$ and $\mathcal{S}(y) = \{c, d\}$. Pick $P_k \in \hat{\mathbb{D}}^x = \{P_1, P_2\}$. Then, $cP_k y$ and $dP_k y$ so that part (ii) of Condition SC is satisfied. To verify part (iii) of Condition SC, note that if $P_k \in \hat{\mathbb{D}}^{A \setminus \{x,y\}}$ and $zP_k y$ for some $z \in \mathcal{S}(y)$, then $P_k \in \{P_5, P_8, P_9, P_{10}, P_{11}, P_{12}, P_{13}, P_{14}, P_{15}, P_{16}\}$. For these preference orders, we have $yP_k x$. $\square$

[2] In Table 2.3, dots in a particular preference order signify that alternatives unspecified are arbitrarily ordered.

The next result shows that the example can be suitably generalized.

Proposition 2.2.2. *Given $m \geq 6$, there exist dictatorial domains that admit anonymous, unanimous and strategy-proof RSCFs that are not random dictatorships.*

Proof. Pick $m \geq 6$. We construct a linked domain satisfying Condition SC in two steps, where the cardinality of the test set is one.

Step 1: We construct a linked domain $\mathbb{D}$ satisfying the following connectedness structure: the one to one function σ in Definition 2.1.1 is the identity function with $a_1 \sim a_2$; $a_k \sim a_{k-1}$ and $a_k \sim a_{k-2}$, $3 \leq k \leq m$. Since $m \geq 6$, we know $\{a_2, a_3\} \cap \{a_{m-2}, a_{m-1}\} = \emptyset$.

Step 2: Let $a_1 = x$, $a_m = y$ and $B = \{x\}$ in Condition SC. We can mimic the preference orders in $\hat{\mathbb{D}}$ (Example 2.2.2) thereby satisfying Condition SC, while remaining compatible with the connectedness structure specified in Step 1. $\square$

Remark 2.2.1. In order to construct a linked domain satisfying Condition SC, the restriction $m \geq 6$ is necessary.[3]

2.2.3 Random Constrained Voting

The voting model with separable preferences was introduced in Barberà et al. [5] (see also [13]). There is a set of voters who wish to elect a subset (possibly empty) of candidates. The set of deterministic, unanimous and strategy-proof SCFs was characterized in [5] and shown to be *decomposable*, i.e., there exist strategy-proof, unanimous, deterministic SCFs for every candidate that determines whether she is elected. Subsequently, several papers (e.g., [6], [7], [8], [49]) have considered variants of the model where certain subsets of candidates are not feasible. For instance, it may be required that at least one candidate is elected and so on. The model with

[3] In an arbitrary linked domain over less than six alternatives, every pair of alternatives is either connected or connected to another common alternative. However, in order to embed Condition SC into a linked domain, there must exist a pair of alternatives "far away" from each other (in the sense of the connectivity graph), i.e., neither connected to each other, not connected to another common alternative.

constrains on the set of feasible alternatives is referred to as the constrained voting model.

This chapter studies the constrained voting model where voters' preferences are assumed *separable*. Separability induces an unambiguous preference over the inclusion/exclusion of every candidate, precluding thereby externalities across candidates. Note that the constrained voting model with the accompanying separability assumption is well-established in the literature (e.g., [1], [5], [8], [13], [49]), and is a natural and tractable model for the purpose of this chapter. Aswal et al. [1] show that certain kinds of constraints on the feasible set lead to linked domains and therefore to dictatorship. This subsection shows the existence of constraints on the feasible set which lead to linked domains but satisfy Condition SC, i.e., permit anonymous, unanimous and strategy-proof RSCFs that are not random dictatorships.

The set of voters is I as before. There is a set of four candidates $\{1, 2, 3, 4\}$. The set of alternatives A is the set of all subsets of candidates and can be represented by $A = \{0, 1\} \times \{0, 1\} \times \{0, 1\} \times \{0, 1\}$ (for instance, $(1, 0, 1, 0)$ represents the set consisting of only Candidates 1 and 3). The alternatives are labeled as follows: $a_0 = (0, 0, 0, 0)$; $a_1 = (0, 0, 1, 0)$, $a_2 = (0, 1, 0, 0)$, $a_3 = (1, 0, 0, 0)$, $a_4 = (0, 0, 0, 1)$; $a_5 = (1, 1, 0, 0)$, $a_6 = (1, 0, 1, 0)$, $a_7 = (1, 0, 0, 1)$, $a_8 = (0, 1, 1, 0)$, $a_9 = (0, 1, 0, 1)$, $a_{10} = (0, 0, 1, 1)$; $a_{11} = (1, 1, 1, 0)$, $a_{12} = (1, 0, 1, 1)$, $a_{13} = (1, 1, 0, 1)$, $a_{14} = (0, 1, 1, 1)$ and $a_{15} = (1, 1, 1, 1)$.

Let $\mathbb{D}_S$ denote the domain of *all* separable preferences over A.[4] Each voter is endowed with the domain of preferences $\mathbb{D}$ specified in Table 2.4. It is true that $\mathbb{D} \subset \mathbb{D}_S$.[5] Note that the dots in the preference orders Table 2.4 *do not* imply that the unspecified alternatives can be ranked arbitrarily. Instead, it indicates that the unspecified alternatives can be ranked in a way that is consistent with separable preference requirement. The rankings of these alternatives are irrelevant for our

[4] Assume that the alternative set follows a Cartesian product structure, i.e., $A = \times_{s \in M} A^s$, where $|M| \geq 2$ and $|A^s| \geq 2$, $s \in M$. An alternative $a \in A$ is expressed as (a^s, a^{-s}). Accordingly, for every $s \in M$, $A^{-s} = \times_{\tau \neq s} A^s$. A preference P_i is separable, if for every $s \in M$ and $a^s, b^s \in A^s$, $[(a^s, x^{-s}) P_i (b^s, x^{-s})$ for some $x^{-s} \in A^{-s}] \Rightarrow [(a^s, y^{-s}) P_i (b^s, y^{-s})$ for all $y^{-s} \in A^{-s}]$.

[5] A complete argument can be found in Appendix 1.

results.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}
a_0	a_0	a_0	a_0	a_5	a_0	a_0	a_6	a_6	a_5	a_6	a_{15}	a_{15}	a_6	a_{15}	a_{15}	a_{15}	a_{15}
a_1	a_1	a_2	a_2	a_2	a_3	a_3	a_3	a_3	a_{11}	a_{11}	a_{11}	a_{11}	a_{12}	a_{12}	a_{12}	a_{13}	a_{13}
a_2	a_3	a_1	a_3	a_{11}	a_1	a_2	a_{11}	a_{12}	a_2	a_3	a_{12}	a_{13}	a_3	a_{11}	a_{13}	a_{11}	a_{12}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	a_{13}	$\vdots$	$\vdots$	a_{13}	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	a_{13}	a_{13}	a_{13}	a_{13}	a_{13}	a_{13}	a_{13}	a_{13}	a_{13}	$\vdots$	$\vdots$	a_{13}	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	a_{11}	a_{11}	$\vdots$	a_{11}	a_{11}	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{13}	a_{13}	a_{12}	a_{12}	a_1	a_{12}	a_{12}	a_1	a_1	a_1	a_1	$\vdots$	$\vdots$	a_1	$\vdots$	$\vdots$	$\vdots$	$\vdots$
a_{15}	a_{15}	a_{15}	a_{15}	$\vdots$	a_{15}	a_{15}	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 2.4: Domain $\mathbb{D}$

The set of feasible alternatives is $X = \{a_1, a_2, a_3, a_{11}, a_{12}, a_{13}\} \subset A$. In other words, the following three restrictions are imposed on the feasible set:

(i) Either only one candidate is elected, or a set of three candidates is elected.

(ii) If only one candidate is elected, it is *never* Candidate 4.

(iii) If a set of three candidates is elected, Candidate 1 is *always* in the elected set.

A (deterministic) constrained voting SCF is a map $f : \mathbb{D}^N \to X$. Assume that f is an onto function. A random constrained voting SCF is a map $\varphi : \mathbb{D}^N \to \Delta^X(A)$, where $\Delta^X(A) = \{\lambda \in \Delta(A) : \lambda_x = 0 \text{ for all } x \in A \backslash X\}$. Correspondingly, the definitions of dictatorship, unanimity and random dictatorship are slightly modified for the present context. The constrained voting SCF $f : \mathbb{D}^N \to X$ is a dictatorship if there exists $i \in I$ such that for all $P \in \mathbb{D}^N$, $f(P) = \max(P_i, X)$.[6] The random constrained voting SCF $\varphi : \mathbb{D}^N \to \Delta^X(A)$ is unanimous if for all $x \in X$ and $P \in \mathbb{D}^N$, $[\max(P_i, X) = x \text{ for all } i \in I] \Rightarrow [\varphi_x(P) = 1]$. The random constrained voting SCF $\varphi : \mathbb{D}^N \to \Delta^X(A)$ is a random dictatorship if there exists $[\varepsilon_i]_{i \in I} \in \mathbb{R}^N_+$ with $\sum_{i \in I} \varepsilon_i = 1$ such that for all $x \in X$ and $P \in \mathbb{D}^N$, $\varphi_x(P) = \sum\limits_{i \in I : \max(P_i, X) = x} \varepsilon_i$.

The next proposition shows that the results in Theorem 2.1.1 and Proposition 2.2.1 apply to the model under consideration.

[6]Given $P_i \in \mathbb{D}$, $\max(P_i, X)$ is the highest ranked alternative in X according to P_i.

Proposition 2.2.3. *The constrained voting SCF $f : \mathbb{D}^N \to X$ is strategy-proof only if it is a dictatorship. However, there exists a random constrained voting SCF $\varphi : \mathbb{D}^N \to \Delta^X(A)$ which is anonymous, unanimous and strategy-proof, but not a random dictatorship.*

Proof. Standard results on strategy-proofness imply that the values of f and φ at any $P \in \mathbb{D}^N$ depend only on profiles induced by P on the feasible set X.[7] We denote this induced domain by $\bar{\mathbb{D}}$, i.e., $\bar{P}_k \in \bar{\mathbb{D}}$ if $\bar{P}_k = (P_k, X)$ for some $P_k \in \mathbb{D}$. The domain $\bar{\mathbb{D}}$ is shown in Table 2.5.

$\bar{P}_1$	$\bar{P}_2$	$\bar{P}_3$	$\bar{P}_4$	$\bar{P}_5$	$\bar{P}_6$	$\bar{P}_7$	$\bar{P}_8$	$\bar{P}_9$	$\bar{P}_{10}$	$\bar{P}_{11}$	$\bar{P}_{12}$	$\bar{P}_{13}$	$\bar{P}_{14}$	$\bar{P}_{15}$	$\bar{P}_{16}$	$\bar{P}_{17}$	$\bar{P}_{18}$
a_1	a_1	a_2	a_2	a_2	a_3	a_3	a_3	a_3	a_{11}	a_{11}	a_{11}	a_{11}	a_{12}	a_{12}	a_{12}	a_{13}	a_{13}
a_2	a_3	a_1	a_3	a_{11}	a_1	a_2	a_{11}	a_{12}	a_2	a_3	a_{12}	a_{13}	a_3	a_{11}	a_{13}	a_{11}	a_{12}
·	·	·	·	·	·	·	·	·	·	·	a_{13}	·	·	a_{13}	·	·	·
·	·	·	·	a_{13}	·	·	a_{13}	a_{13}	a_{13}	a_{13}	·	·	a_{13}	·	·	·	·
·	·	a_{13}	a_{13}	·	a_{13}	a_{13}	·	·	·	·	·	·	·	·	·	·	·
·	·	a_{11}	a_{11}	a_1	a_{11}	a_{11}	a_1	a_1	a_1	a_1	·	·	a_1	·	·	·	·
a_{13}	a_{13}	a_{12}	a_{12}	·	a_{12}	a_{12}	·	·	·	·	·	·	·	·	·	·	·

Table 2.5: Domain $\bar{\mathbb{D}}$

The domain $\bar{\mathbb{D}}$ is, in fact identical to domain $\hat{\mathbb{D}}$ in Example 2.2.2 with the relabeling $a_1 = x$, $a_{13} = y$, $a_2 = a$, $a_3 = b$, $a_{11} = c$ and $a_{12} = d$. The result follows immediately from this observation. $\square$

2.3 Random Dictatorship Results

This section provides two conditions that ensure that a domain is a random dictatorship domain. The first imposes restrictions on the connectedness structure of linked domains. The second strengthens the requirement for the connectedness of two alternatives but imposes a weaker requirement on the connectedness structure. Finally, this section shows that the strengthened conditions are "close to" being necessary using results developed in Section 2.2.1.

[7]The arguments here are routine and tedious; details can be found in Appendix 1.

In the analysis of random dictatorship, the first step is to reduce the dimension of the problem from an arbitrary number of voters to two voters. This chapter establishes a Ramification Theorem (Theorem 5.1 in Appendix 5), which shows that a random dictatorship domain with two voters, is also a random dictatorship domain for an arbitrary number of voters, provided an additional richness condition (Definition 3.1, Appendix 5) is satisfied. A result of this kind was first established in [30] which showed that a domain where all Arrovian social welfare functions are dictatorial when there are two voters also admits only dictatorial Arrovian social welfare functions for an arbitrary number of voters. A similar property has been shown for deterministic strategy-proof SCFs (initiated in [32], see also [47], [1]).

The property of ramification for RSCFs is interesting in its own right. In addition, it is very helpful analytically; in order to determine whether a domain is a random dictatorship domain, it suffices to verify that every two-voter strategy-proof RSCF satisfying unanimity is a random dictatorship. Unfortunately, ramification appears to be a significantly more difficult issue to resolve than its counterpart for the deterministic case. A formal treatment is presented in Appendix 5.

Fortunately, all domains discussed in this chapter are covered by the richness conditions in the Ramification Theorem. Consequently, the whole analysis is restricted to two-voter RSCFs.

2.3.1 Linked Domains with Condition H

The following condition is imposed on the connectedness structure of domains.

Definition 2.3.1. *A domain $\mathbb{D}$ satisfies Condition H (Hub) if there exists $a \in A$ such that $b \sim a$ for all $b \in A \backslash \{a\}$.*

An alternative connected to all other alternatives is referred to as a *hub*. Domains satisfying Condition SC must violate Condition H.[8] The following examples of five

[8] Condition H implies that every alternative is connected to every other alternative in at most two steps, while in order to embed Condition SC in linked domains, there must exist two alternatives which are neither connected to each other nor connected to some other common alternative. A similar argument holds when the domain is not linked; details can be found in Appendix 2.1.

connectivity graphs are presented to illustrate the relation between Condition H and linked domains.

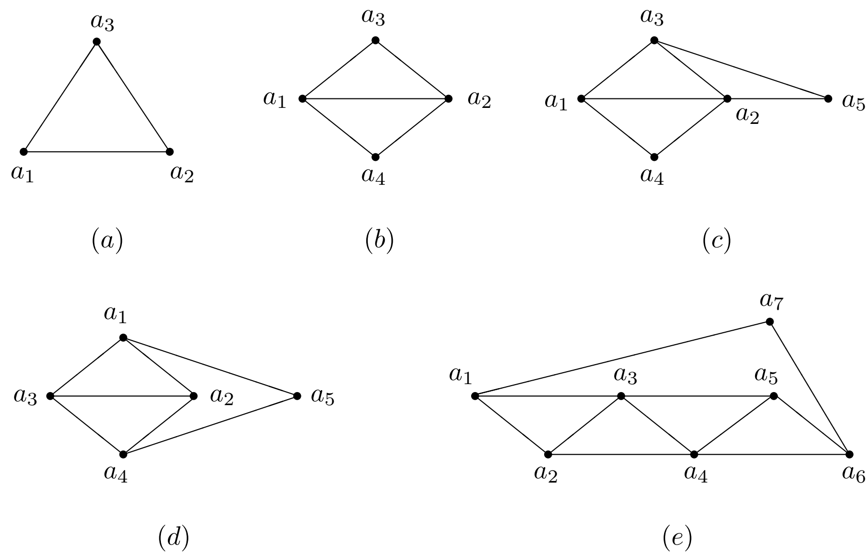

Figure 2.2: Connectivity Graphs and Condition H

Figure 2.2 shows various types of linked domains. The domains corresponding to (a), (b) and (c) satisfy Condition H, while domains related to (d) and (e) violate it. In diagram (a), any alternative is a hub; in diagram (b), it must be either a_1 or a_2, while in diagram (c), the only candidate for the hub is a_2. Observe that in diagrams (d) and (e), for every pair of two alternatives, they are either connected or connected to another common alternative.

The main result in this section is that the assumption of a linked domain in conjunction with Condition H ensures that the domain is a random dictatorship domain.

Theorem 2.3.1. *A linked domain satisfying Condition H is a random dictatorship domain.*

The proof is in Appendix 3.

Remark 2.3.1. The Free Pair at the Top domain (FPT domain) introduced in [1] in which every two alternatives are connected, is a linked domain satisfying Condition H (every alternative is a hub) and is consequently a random dictatorship domain.

This addresses an open question in [48]: is the FPT domain a random dictatorship domain for an arbitrary number of voters?

Remark 2.3.2. It is possible to construct a linked domain of minimal cardinality satisfying Condition H . This can be done as follows: $a_1 \sim a_2$; and $a_k \sim a_1$, $a_k \sim a_2$, $k = 3, \ldots, m$. Therefore "small" random dictatorship domains can be found - those that grow linearly in the number of alternatives.

2.3.2 Strongly Linked Domains with Condition TS

This subsection provides another condition that guarantees random dictatorship. The approach here is to strengthen the notion of connectedness of alternatives along the lines initiated in [17].

Definition 2.3.2. *A pair of distinct alternatives a, b is strongly connected, denoted $a \approx b$, if there exist $P_i \in \mathbb{D}^{a,b}$ and $P_i' \in \mathbb{D}^{b,a}$ such that $r_k(P_i) = r_k(P_i')$, $k = 3, \ldots, m$.*

In other words, a and b are stongly connected if it is possible to find a preference order in the domain where a and b are first and second ranked and it is possible to flip a and b while keeping the positions of all other alternatives fixed.

Definition 2.3.3. *A strongly linked domain is defined in exactly the same way as a linked domain except that the notion of connectedness is replaced by strong connectedness.*

A strongly linked domain has stronger restrictions embedded in it than a linked domain. However, it is not necessarily a random dictatorship domain - a strongly linked domain may satisfy Condition SC (see Example 2.3.1).

Example 2.3.1. Let $A = \{x, y, z, a, b, c, d\}$. The Domain $\check{\mathbb{D}}$ of preferences over the seven alternatives is described in Table 2.6.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}	P_{15}	P_{16}	P_{17}	P_{18}	P_{19}	P_{20}	P_{21}	P_{22}
x	x	x	y	y	z	z	a	a	a	a	b	b	b	b	c	c	c	c	d	d	d
z	a	b	d	c	x	a	x	z	b	c	x	a	d	c	y	a	b	d	y	b	c
$\cdot$	y	y	$\cdot$	$\cdot$	$\cdot$	y	y	y	y	y	y	y	y	y	$\cdot$	y	y	y	$\cdot$	y	y
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
y	$\cdot$	$\cdot$	$\cdot$	$\cdot$	y	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$

Table 2.6: Domain $\check{\mathbb{D}}$

In Table 2.6, dots in a particular preference order signify that alternatives unspecified are arbitrarily ordered. Therefore, strong connectedness can be induced. The strong connectivity graph $G(\check{\mathbb{D}})$ is shown in Figure 2.3. It is clear that $\check{\mathbb{D}}$ is a strongly linked domain.

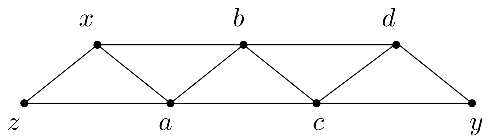

Figure 2.3: Connectivity Graph $G(\check{\mathbb{D}})$

Furthermore, by careful specification of preferences in Table 2.6, domain $\check{\mathbb{D}}$ can satisfy Condition SC with respect to $B = \{x, z\}$ and y:

(i) Set $\mathcal{S}(x) = \{z, a, b\}$, $\mathcal{S}(z) = \{x, a\}$ and $\mathcal{S}(y) = \{c, d\}$.

(ii) For all $P_k \in \check{\mathbb{D}}^B$, $[r_2(P_k) \in B] \Rightarrow [P_k \in \{P_1, P_6\}]$. Set cP_ky and dP_ky;

For all $P_k \in \check{\mathbb{D}}^B$, $[r_2(P_k) \notin B] \Rightarrow [P_k \in \{P_2, P_3, P_7\}]$. Set yP_kc and yP_kd.

(iii) For all $P_k \in \check{\mathbb{D}}^{A\setminus\{x,y,z\}}$,

$[\bar{x}P_ky$ for some $\bar{x} \in \mathcal{S}(y)] \Rightarrow [P_k \in \{P_{11}, P_{14}, P_{15}, P_{16}, P_{17}, P_{18}, P_{19}, P_{20}, P_{21}, P_{22}\}]$.

Set yP_kx and yP_kz.

$\square$

Clearly additional conditions are required to make a strongly linked domain, a random dictatorship domain. This subsection provides such a condition.

Definition 2.3.4. *A domain $\mathbb{D}$ satisfies Condition TS (Two Steps) if for all $a, b \in A$, either $a \approx b$, or $a \approx c$ and $b \approx c$ for some $c \in A\backslash\{a, b\}$.*

In other words, every alternative is strongly connected to any other alternative in *at most* two steps. The counterpart of this condition for connectedness is clearly weaker than Condition H. If diagrams (d) and (e) in Figure 2.2 are interpreted in terms of strong connectedness, then they represent strongly linked domains satisfying Condition TS. In addition, domains satisfying Condition SC must violate Condition TS.[9]

Theorem 2.3.2. *A strongly linked domain satisfying Condition TS is a random dictatorship domain.*

The proof is contained in Appendix 4.

Remark 2.3.3. Note here that in the constrained voting model, introducing more restrictions to the feasible alternative set might lead to a violation of Condition SC and furthermore, lead to random dictatorships. Consider the following additional restriction on the feasible alternative set in the constrained voting model studied in Section 2.2.3: if only one candidate is elected, it is never Candidate 3. Thus, the set of feasible alternatives is $\tilde{X} = \{a_2, a_3, a_{11}, a_{12}, a_{13}\}$. In Figure 2.1, when $x = a_1$ is removed, the remaining connections are unaffected and $c = a_{11}$ turns to be the hub.[10] Therefore, Theorem 2.3.1 implies random dictatorships in the constrained voting model.

One may verify that the induced domain $\tilde{\mathbb{D}}$ over $\tilde{X}$ will be strongly linked for some specification of preferences in Table 2.5. Then the strong connectedness structure of $\tilde{\mathbb{D}}$ is as specified in Figure 2.1 with alternative $x = a_1$ removed, and since every alternative in $\tilde{X}$ is strongly connected to any other alternative in at most two steps, domain $\tilde{\mathbb{D}}$ satisfies Condition TS. Therefore, one can alternatively deduce random dictatorship by Theorem 2.3.2.

[9] The argument here is analogous to that in footnote 8, details can be found in Appendix 2.2.

[10] After removing alternative a_1 in every preference in Table 2.5, the set of preferences $\{\bar{P}_4, \bar{P}_5, \bar{P}_7, \ldots, \bar{P}_{18}\}$ in the induced domain $\tilde{\mathbb{D}}$ over $\tilde{X}$ indicates that $\tilde{\mathbb{D}}$ is linked and satisfies Condition H where the hub is $a_{11} = (1, 1, 1, 0)$.

2.3.3 "Necessity" of Conditions H and TS

Conditions H and TS are obviously strong conditions. Are they necessary for random dictatorship? The question appears to be extremely difficult to resolve completely. However, Examples 2.3.2 and 2.3.3 suggest that they are *close to* being necessary in an appropriate sense.

Example 2.3.2. Consider A with $|A| \geq 6$. Let $x, y \in A$, and $T, T' \subseteq A \backslash \{x, y\}$ be such that $T \cup T' = A \backslash \{x, y\}$, $T \cap T' = \emptyset$ and $|T|, |T'| \geq 2$. Construct $\mathbb{D}^*$ satisfying the following restrictions:

(i) for all $a, b \in A \backslash \{x, y\}$, $a \sim b$;

(ii) for all $a \in T$, $a \sim x$;

(iii) for all $a' \in T'$, $a' \sim y$;

(iv) No connectedness other than those specified in parts (i), (ii) and (iii);

(v) for all $P_k \in \mathbb{D}^{*\,x}$, $r_m(P_k) = y$;

(vi) for all $P_k \in \mathbb{D}^{*\,A \backslash \{x,y\}}$, either $r_2(P_k) = y$ or $r_3(P_k) = y$.

The following schematic diagram illustrates the connectedness structure of $\mathbb{D}^*$.

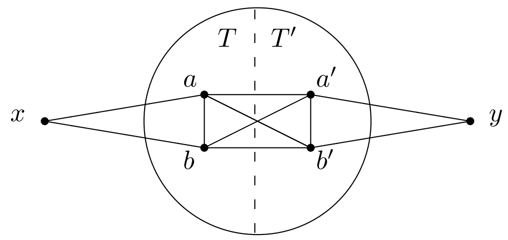

Figure 2.4: Connectedness structure of domain $\mathbb{D}^*$

It is easy to verify that $\mathbb{D}^*$ satisfies Condition SC with respect to $B = \{x\}$ and y (parts (ii) - (vi)) and is linked (parts (i) - (iii)). In addition, domain $\mathbb{D}^*$ violates Condition H but is very close to satisfying Condition H. In particular, "almost" every alternative (those other than x and y) is "almost" a hub as we note below.

(i) Every $z \in T$ is a hub for the sub-domain $\mathbb{D}^{*\,A\backslash\{y\}}$: $[z \in T] \Rightarrow [z \sim a$ for all $a \in A\backslash\{z, y\}]$;

(ii) Every $z' \in T'$ is a hub for the sub-domain $\mathbb{D}^{*\,A\backslash\{x\}}$: $[z' \in T'] \Rightarrow [z' \sim a$ for all $a \in A\backslash\{z', x\}]$.

Finally, since domain $\mathbb{D}^*$ satisfies Condition SC, Proposition 2.2.1 implies that it is not a random dictatorship domain. □

Example 2.3.3. Consider A with $|A| \geq 7$. Let $x, y, z \in A$ and $T, T' \subseteq A\backslash\{x, y, z\}$ with $T \cup T' = A\backslash\{x, y, z\}$, $T \cap T' = \emptyset$ and $|T|, |T'| \geq 2$. Construct $\mathbb{D}^{**}$ satisfying the following restrictions:

(i) for all $a, b \in A\backslash\{x, y, z\}$, $a \approx b$;

(ii) $x \approx z$;

(iii) for all $a \in T$, $a \approx x$ and $a \approx z$;

(iv) for all $a' \in T'$, $a' \approx y$;

(v) No connectedness other than the connectedness induced by the strong connectedness specified in parts (i), (ii) and (iii);

(vi) for all $P_k \in \mathbb{D}^{**\,\{x,z\}}$ with $r_2(P_k) \in \{x, z\}$, $r_m(P_k) = y$;

(vii) for all $P_k \in \mathbb{D}^{**\,\{x,z\}}$ with $r_2(P_k) \notin \{x, z\}$, $r_3(P_k) = y$;

(viii) for all $P_k \in \mathbb{D}^{**\,A\backslash\{x,y,z\}}$, either $r_2(P_k) = y$ or $r_3(P_k) = y$.

We provide the following schematic diagram to illustrate the strong connectedness structure of $\mathbb{D}^{**}$.

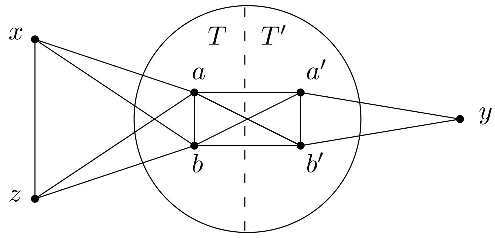

Figure 2.5: Strong connectedness structure of domain $\mathbb{D}^{**}$

It is easy to verify that $\mathbb{D}^{**}$ satisfies Condition SC with respect to $B = \{x, z\}$ and y (parts (ii) - (viii)) and is strongly linked (parts (i), (iii) and (iv)). In addition, domain $\mathbb{D}^{**}$ violates Condition TS but is very close to satisfying Condition TS:

(i) sub-domain $\mathbb{D}^{**\ A\backslash\{y\}}$ satisfies Condition TS: $[a, b \in A\backslash\{y\}] \Rightarrow$ [either $a \approx b$, or $a \approx c$ and $b \approx c$ for some $c \in A\backslash\{y\}$];

(ii) sub-domain $\mathbb{D}^{**\ A\backslash\{x,z\}}$ satisfies Condition TS: $[a, b \in A\backslash\{x, z\}] \Rightarrow$ [either $a \approx b$, or $a \approx c$ and $b \approx c$ for some $c \in A\backslash\{x, z\}$].

Finally, since domain $\mathbb{D}^{**}$ satisfies Condition SC, Proposition 2.2.1 implies that it is not a random dictatorship domain. □

Chapter 3 A Characterization of Single-Peaked Preferences via Random Social Choice Functions

3.1 The Compromise Property

Recall that random dictatorships satisfy anonymity (if weights are equal for all voters), unanimity (ex-post efficiency), the tops-only property and strategy-proofness. However, they suffer from an important and well-known infirmity: they do not admit *compromise*. Imagine a two-voter world with several alternatives (say, a thousand). Consider a profile where voter 1's first-ranked and thousandth-ranked alternatives are a and b, respectively. Alternatively, voter 2's first-ranked and thousandth-ranked alternative are b and a, respectively. Suppose, in addition, that there is an alternative say c that is highly ranked by *both* voters, for instance, ranked second by both. A reasonable RSCF should put at least some probability weight on c, but no random dictatorship would.

This chapter introduces a new axiom so as to deal with the difficulties associated with random dictatorships outlined above. The axiom requires some compromise alternatives in certain profiles to be selected by the RSCF with strictly positive probability.

Let $P_i, P_j \in \mathbb{D}$ be such that $r_1(P_i) \neq r_1(P_j)$. Let $C(P_i, P_j) = \{a_r \in A | a_r = r_2(P_i) = r_2(P_j)\}$. Note that $C(P_i, P_j)$ is either empty or contains a singleton.

Let $\hat{I} \subset I$ be a nonempty strict subset of voters. For any $P_i, P_j \in \mathbb{D}$, let

$\big(\underbrace{P_i,\ldots,P_i}_{\hat{I}},\underbrace{P_j,\ldots,P_j}_{I\setminus\hat{I}}\big)$ denote the profile where all voters in $\hat{I}$ have the order P_i while those not in $\hat{I}$ have P_j.

Definition 3.1.1. *A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ satisfies the compromise property if there exists $\hat{I} \subseteq I$ with $|\hat{I}| = N/2$ if N is even and $|\hat{I}| = (N+1)/2$ if N is odd, such that for all $P_i, P_j \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P_j)$ and $C(P_i, P_j) \equiv \{a\}$, we have $\varphi_a\big(\underbrace{P_i,\ldots,P_i}_{\hat{I}},\underbrace{P_j,\ldots,P_j}_{I\setminus\hat{I}}\big) > 0$.*

The axiom requires the existence of a subset of voters $\hat{I}$ that is approximately half the size of the set of voters. Pick an arbitrary profile where all voters in $\hat{I}$ have identical preferences as do voters in the complement set $I \setminus \hat{I}$. Suppose the common preferences in $\hat{I}$ and $I \setminus \hat{I}$ have distinct peaks but have a common second-ranked alternative a_r. According to the axiom, the RSCF must give a_r strictly positive probability at the profile.

This chapter believes that the axiom is both weak and natural. It is weak because it applies to a very narrow class of profiles. It is natural because in the profile where it applies, the alternative to which strictly positive probability is assigned according to the axiom, is an obvious compromise between the two groups of voters.

Two remarks are made here about the set $\hat{I}$ in Definition 3.1.1. The first is that Definition 3.1.1 merely requires the existence of one such set of voters. A stronger but equally plausible axiom would require the property to hold for all subsets $\hat{I}$ such that $|\hat{I}| = N/2$ if N is even, and $|\hat{I}| = (N+1)/2$ if N is odd. A weaker assumption is made in Definition 3.1.1 because the stronger one is not required for the result. Note however that once $\hat{I}$ is fixed, the strictly positive probability requirement on the compromise applies to *all* profiles $\big(\underbrace{P_i,\ldots,P_i}_{\hat{I}},\underbrace{P_j,\ldots,P_j}_{I\setminus\hat{I}}\big)$.

The second remark is to point out that the choice of the cardinality of $\hat{I}$ in Definition 3.1.1, is arbitrary. As footnote 5 points out, *any* choice of the cardinality of $\hat{I}$ works for our proof, provided $0 < |\hat{I}| < N$. One could have assumed, for instance, $|\hat{I}| = 2$ or $|\hat{I}| = N - 1$. One could even have left $|\hat{I}|$ unspecified. However, $|\hat{I}|$

is chosen to be approximately half of N because it is the compelling case for the axiom to hold.

3.2 Path-connected Domains

The goal in this chapter is to characterize preference domains that admit RSCFs satisfying unanimity (ex-post efficiency), the tops-only property, strategy-proofness and the compromise property. However, this chapter needs to restrict attention to domains that satisfy a regularity condition called *path-connectedness*.

The path-connectedness condition was introduced in Chatterji et al. [17].[1] Fix a domain $\mathbb{D}$. A pair of distinct alternatives $a, b \in A$ satisfies the *Free Pair at the Top* (or FPT) property, if there exist $P_i, P_i' \in \mathbb{D}$ such that (i) $r_1(P_i) = r_2(P_i') = a$, (ii) $r_2(P_i) = r_1(P_i') = b$, and (iii) $r_t(P_i) = r_t(P_i')$, $t = 3, \ldots, m$. The definition of FPT property is identical to the definition of strong connectedness (see Definition 2.3.2 in Chapter 2). In other words, two alternatives satisfy the FPT property if there exists a pair of admissible orders where the alternatives are at the top of both orders and are locally switched, i.e., all alternatives other than the specified pair are ranked in the same way in both orders. Let $FPT(\mathbb{D})$ denote the set of alternative pairs that satisfy the FPT property. The domain $\mathbb{D}$ is *path-connected* if for every pair of alternatives $a, b \in A$, there exists a sequence $\{x_t\}_{t=1}^{T} \subseteq A$, $T \geq 2$, such that $x_1 = a$, $x_T = b$ and $(x_t, x_{t+1}) \in FPT(\mathbb{D})$, $t = 1, \ldots, T-1$.

The path-connectedness assumption imposes structure on the domain. It allows the construction of paths between admissible orders by switching preferences at the top of the orders. Very similar conditions have been identified in [14] and [44] as being critical for the purpose of identifying domains where *local incentive-compatibility* ensures strategy-proofness.[2]

Chatterji et al. [17] provide extensive discussion of well-known domains that

[1]Slightly different names were used in [17] for the Free Pair at the Top property and path-connectedness. We believe that the new names are more apposite.

[2]Assume that domains are minimally rich. Then domains of ordinal preferences studied in both [14] and [44] are path-connected.

satisfy the path-connectedness assumption. The complete domain and the single-peaked domain are path-connected. Maximal single-crossing domains ([42]) are path-connected provided that every alternative is first-ranked in some order in the domain. A generalized single-peaked domain ([35]) may or may not be path-connected. Alternatively, the separable domain ([5] and [13]) and the multi-dimensional single-peaked domain ([6]) are not path-connected. For details the reader is referred to Examples 1, 2 and 3 in [17].

A domain of central importance in collective choice theory is the *single-peaked* domain. It was originally introduced in [11] and [29]. Here this chapter considers a generalization due to [20] and [19].

An undirected graph $G = \langle V, E \rangle$ is a set of vertices V and a set of edges E. The set E consists of pairs vertices, i.e., $E \subseteq \{(u, v) | u, v \in V \text{ and } u \neq v\}$. If $(u, v) \in E$, we say that (u, v) is an edge in G.[3] A path in G is a sequence $\{v_k\}_{k=1}^{s} \subseteq V$ where $s \geq 2$ and $(v_k, v_{k+1}) \in E$, $k = 1, \ldots, s-1$. The graph G is *connected* if there exists a path between every pair of vertices, i.e., for all $u, v \in V$ with $u \neq v$, there exists a path $\{v_k\}_{k=1}^{s}$ such that $u = v_1$ and $v = v_s$. The connected graph G is a tree if the path between every pair of vertices is unique. Let G be a tree and $u, v \in V$ be a pair of vertices. Accordingly, $\langle u, v \rangle$ denotes the unique path between them.[4]

In what follows, graphs G of the kind $G = \langle A, E \rangle$ is considered, i.e., whose vertex set is the set of alternatives.

Definition 3.2.1. *Let $G = \langle A, E \rangle$ be a tree. A preference P_i is single-peaked on G if for all $a, b \in A$,*

$$\left[a \in \langle r_1(P_i), b \rangle \backslash \{b\}\right] \Rightarrow [aP_ib].$$

Pick a preference P_i and an arbitrary alternative b. Since the graph is a tree, there is a unique path between $r_1(P_i)$ and b. The order P_i is single-peaked if every

[3] In an undirected graph, (u, v) and (v, u) represent a same edge.
[4] In particular, if $u = v$, $\langle u, v \rangle = \{u\}$ is a singleton set.

alternative a on this path that is distinct from b is strictly preferred to b according to P_i.

A domain $\mathbb{D}$ is single-peaked if there exists a tree G such that $P_i \in \mathbb{D}$ implies P_i is single-peaked on G.

A case of special interest is the one where the graph $G = \langle A, E\rangle$ is a *line*. Formally, G is a line if according to the labeled alternative set $A = \{a_1, \ldots, a_m\}$, there exists a bijective function $\sigma : \{1, \ldots, m\} \to \{1, \ldots, m\}$ such that $E = \left\{\left(a_{\sigma(k)}, a_{\sigma(k+1)}\right)\right\}_{k=1}^{m-1}$. The standard definition of a single-peaked domain is one where the underlying graph is a line. This section illustrates these notions with Examples 3.2.1 and 3.2.2.

Example 3.2.1. Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\bar{\mathbb{D}}$ is described in Table 3.1.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9
a_1	a_1	a_2	a_2	a_2	a_3	a_3	a_4	a_4
a_2	a_2	a_1	a_4	a_3	a_2	a_2	a_2	a_2
a_4	a_3	a_4	a_3	a_4	a_4	a_1	a_3	a_1
a_3	a_4	a_3	a_1	a_1	a_1	a_4	a_1	a_3

Table 3.1: Domain $\bar{\mathbb{D}}$

The domain $\bar{\mathbb{D}}$ is single-peaked on the tree G^T shown in Figure 3.1.

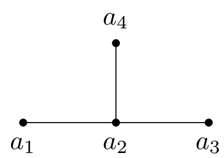

Figure 3.1: Tree G^T

Note that there are orders that are single-peaked on G^T but not included in $\bar{\mathbb{D}}$, for instance, $a_2 P_{10} a_1 P_{10} a_3 P_{10} a_4$. The largest single-peaked domain on G^T contains 12 orders. $\square$

Example 3.2.2. Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\hat{\mathbb{D}}$ is described in Table 3.2.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a_1	a_2	a_2	a_2	a_3	a_3	a_3	a_4
a_2	a_1	a_3	a_3	a_2	a_2	a_4	a_3
a_3	a_3	a_4	a_1	a_1	a_4	a_2	a_2
a_4	a_4	a_1	a_4	a_4	a_1	a_1	a_1

Table 3.2: Domain $\hat{\mathbb{D}}$

The domain $\hat{\mathbb{D}}$ is single-peaked on the line G^L shown in Figure 3.2.

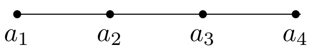

Figure 3.2: Line G^L

In contrast to domain $\bar{\mathbb{D}}$ in Example 3.2.1, domain $\hat{\mathbb{D}}$ includes *all* orders that are single-peaked on G^L. Observe also that $\bar{\mathbb{D}}$ is not single-peaked on a line; neither is $\hat{\mathbb{D}}$ single-peaked on G^T. To verify the former claim, observe that any domain that is single-peaked on a line must have at least two alternatives which have unique orders where these alternatives are peaks (these are the alternatives at either end of the line); there are no alternatives with this property in $\bar{\mathbb{D}}$. Alternatively, the maximal number of alternatives that can be second-ranked to a given alternative on any domain that is single-peaked on a line, is two, whereas on domain $\bar{\mathbb{D}}$, the alternative a_2 has three distinct second-ranked alternatives a_1, a_3 and a_4. □

3.3 Main Result: Single-Peakedness

The main result in this chapter characterizes single-peaked domains.

Theorem 3.3.1. *Every path-connected domain that admits a unanimous, tops-only and strategy-proof RSCF satisfying the compromise property is single-peaked. Conversely, every single-peaked domain admits an ex-post efficient, tops-only and strategy-proof RSCF satisfying the compromise property.*

Proof. We first prove necessity. Assume that $\mathbb{D}$ is path-connected. In addition, there exists a RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ that is tops-only, strategy-proof and unanimous, and satisfies the compromise property. We will show that there exists a tree G such that $\mathbb{D}$ is single-peaked on G.

The first four lemmas establish critical properties of the RSCF φ.

Lemma 3.3.1. *Let $a, b \in A$ with $(a, b) \in FPT(\mathbb{D})$. Let $P_i, P_i' \in \mathbb{D}$ be such that* (i) $r_1(P_i) = r_2(P_i') = a$ (ii) $r_2(P_i) = r_1(P_i') = b$, *and* (iii) $r_t(P_i) = r_t(P_i')$, $t = 3, \ldots, m$. *Then, for all $P_{-i} \in \mathbb{D}^{N-1}$, $\varphi_a(P_i, P_{-i}) + \varphi_b(P_i, P_{-i}) = \varphi_a(P_i', P_{-i}) + \varphi_b(P_i', P_{-i})$ and $\varphi_c(P_i, P_{-i}) = \varphi_c(P_i', P_{-i})$ for all $c \in A\backslash\{a, b\}$.*

Suppose voter i switches her order from P_i to P_i', a move that involves the reshuffling of the top two alternatives, say a and b, while leaving all other alternatives unaffected. According to Lemma 3.3.1, the switch leaves the probabilities of alternatives other than a and b, and the *sum* of probabilities of a and b, unchanged. Lemma 3.3.1 is a special case of Lemma 2 in [26]. It is a consequence of strategy-proofness and we omit its elementary proof.

Lemma 3.3.2. *If domain $\mathbb{D}$ admits a unanimous, tops-only and strategy-proof RSCF satisfying the compromise property, then it admits a two-voter unanimous, tops-only and strategy-proof RSCF satisfying the compromise property.*

Proof. Let $\varphi : \mathbb{D}^N \to \Delta(A)$ denote a unanimous, tops-only and strategy-proof RSCF satisfying the compromise property. We know that there exists $\hat{I} \subseteq I$ with $|\hat{I}| = N/2$ if N is even and $|\hat{I}| = (N + 1)/2$ if N is odd such that for all $P_i, P_j \in \mathbb{D}$ with (i) $r_1(P_i) \neq r_1(P_j)$ and (ii) $a \equiv r_2(P_i) = r_2(P_j)$, we have $\varphi_a\big(\underbrace{P_i, \ldots, P_i}_{\hat{I}}, \underbrace{P_j, \ldots, P_j}_{I\backslash\hat{I}}\big) > 0$.

Construct a two-voter RSCF $\phi : \mathbb{D}^2 \to \Delta(A)$ as $\phi(P_1, P_2) = \varphi\big(\underbrace{P_1, \ldots, P_1}_{\hat{I}}, \underbrace{P_2, \ldots, P_2}_{I\backslash\hat{I}}\big)$ for all $P_1, P_2 \in \mathbb{D}$. In other words, ϕ is constructed by "merging" all voters in $\hat{I}$ and all voters in $I\backslash\hat{I}$ in φ.[5] Clearly, ϕ is a RSCF satisfying unanimity and the tops-only

[5]Any choice of the cardinality of $\hat{I}$ works for our proof, provided $0 < |\hat{I}| < N$. We could have assumed, for instance that $|\hat{I}| = 2$ or $|\hat{I}| = N - 1$.

property. It is also strategy-proof (see the proof of Lemma 3 in [48]). We show that ϕ satisfies the compromise property.

Let $\bar{I} = \{1\}$ in the two-voter model. Let $P_1, P_2 \in \mathbb{D}$ with (i) $r_1(P_1) \neq r_1(P_2)$ and (ii) $a \equiv r_2(P_1) = r_2(P_2)$. Then $\phi_a(P_1, P_2) = \varphi_a\big(\underbrace{P_1, \ldots, P_1}_{\hat{I}}, \underbrace{P_2, \ldots, P_2}_{I \setminus \hat{I}}\big) > 0$ where the equality follows from the construction of ϕ and the inequality follows from the fact that φ satisfies the compromise property. Therefore ϕ satisfies the compromise property. This completes the proof of the lemma. □

In view of Lemma 3.3.2, we can assume without loss of generality that the set of voters is $\{1, 2\}$ and φ is an RSCF $\varphi : \mathbb{D}^2 \to \Delta(A)$ that is unanimous, tops-only and strategy-proof, and satisfies the compromise property. We make a further simplification in notation. Since φ is tops-only, we can represent a profile $P \in \mathbb{D}^2$ by a pair of alternatives a and b where $r_1(P_1) = a$ and $r_1(P_2) = b$. We shall also occasionally let (a, P_2) denote a preference profile (P_1, P_2) where $r_1(P_1) = a$.

Lemma 3.3.3. *Let $a, b \in A$ with $(a, b) \in FPT(\mathbb{D})$. There exists $\beta \in [0, 1]$ such that $\varphi(a, b) = \beta e_a + (1 - \beta)e_b$.*

Proof. Let $P_1, P_1' \in \mathbb{D}$ be such that (i) $r_1(P_1) = r_2(P_1') = a$, (ii) $r_2(P_1) = r_1(P_1') = b$ and (iii) $r_t(P_1) = r_t(P_1')$, $t = 3, \ldots, m$ (such two preferences exist since $(a, b) \in FPT(\mathbb{D})$). We then have

$$\begin{aligned}
\varphi_a(a, b) + \varphi_b(a, b) &= \varphi_a(P_1, b) + \varphi_b(P_1, b) \text{ (by the tops-only property)} \\
&= \varphi_a(P_1', b) + \varphi_b(P_1', b) \text{ (by Lemma 3.3.1)} \\
&= \varphi_b(b, b) = 1 \text{ (by unanimity).}
\end{aligned}$$

Let $\varphi_a(a, b) = \beta$. Thus, $\varphi(a, b) = \beta e_a + (1 - \beta)e_b$ as required. □

The next lemma considers situations that are more general than those considered in the previous one. We illustrate it with an example. Suppose $(a_1, a_2), (a_2, a_3) \in FPT(\mathbb{D})$. We know from Lemma 3.3.3 that there exist $\beta_1, \beta_2 \in [0, 1]$ such that

$\varphi(a_1, a_2) = \beta_1 e_{a_1} + (1 - \beta_1)e_{a_2}$ and $\varphi(a_2, a_3) = \beta_2 e_{a_2} + (1 - \beta_2)e_{a_3}$. The next lemma shows that $\beta_2 > \beta_1$ and $\varphi(a_1, a_3) = \beta_1 e_{a_1} + (\beta_2 - \beta_1)e_{a_2} + (1 - \beta_2)e_{a_3}$.

Lemma 3.3.4. *Let $\{a_k\}_{k=1}^s \subseteq A$, $s \geq 3$, be such that $(a_k, a_{k+1}) \in FPT(\mathbb{D})$, $k = 1, \ldots, s-1$. Let $\beta_k = \varphi_{a_k}(a_k, a_{k+1})$, $k = 1, \ldots, s-1$. Then, the following two conditions hold.*

(i) *We have $0 \leq \beta_k < \beta_{k+1} \leq 1$, $k = 1, \ldots, s-2$.*

(ii) *For all $1 \leq i < j \leq s$, $\varphi(a_i, a_j) = \beta_i e_{a_i} + \sum_{k=i+1}^{j-1} (\beta_k - \beta_{k-1})e_{a_k} + (1 - \beta_{j-1})e_{a_j}$.*

Proof. We know from Lemma 3.3.3 that $\varphi(a_k, a_{k+1}) = \beta_k e_{a_k} + (1 - \beta_k)e_{a_{k+1}}$, $k = 1, \ldots, s-1$. Pick k with $1 \leq k \leq s-2$. Since $(a_{k+1}, a_{k+2}) \in FPT(\mathbb{D})$ and $a_k \notin \{a_{k+1}, a_{k+2}\}$, Lemma 3.3.1 implies $\varphi_{a_{k+1}}(a_k, a_{k+2}) + \varphi_{a_{k+2}}(a_k, a_{k+2}) = \varphi_{a_{k+1}}(a_k, a_{k+1}) + \varphi_{a_{k+2}}(a_k, a_{k+1}) = \varphi_{a_{k+1}}(a_k, a_{k+1}) = 1 - \beta_k$ and $\varphi_{a_k}(a_k, a_{k+2}) = \varphi_{a_k}(a_k, a_{k+1}) = \beta_k$. Also, since $(a_k, a_{k+1}) \in FPT(\mathbb{D})$, Lemma 3.3.1 implies $\varphi_{a_k}(a_k, a_{k+2}) + \varphi_{a_{k+1}}(a_k, a_{k+2}) = \varphi_{a_k}(a_{k+1}, a_{k+2}) + \varphi_{a_{k+1}}(a_{k+1}, a_{k+2}) = \varphi_{a_{k+1}}(a_{k+1}, a_{k+2}) = \beta_{k+1}$. Therefore, $\varphi_{a_{k+1}}(a_k, a_{k+2}) = \beta_{k+1} - \varphi_{a_k}(a_k, a_{k+2}) = \beta_{k+1} - \beta_k$ and $\varphi_{a_{k+2}}(a_k, a_{k+2}) = 1 - \beta_k - \varphi_{a_{k+1}}(a_k, a_{k+2}) = 1 - \beta_{k+1}$. Therefore, $\varphi_{a_k}(a_k, a_{k+2}) + \varphi_{a_{k+1}}(a_k, a_{k+2}) + \varphi_{a_{k+2}}(a_k, a_{k+2}) = 1$ and $\varphi(a_k, a_{k+2}) = \beta_k e_{a_k} + (\beta_{k+1} - \beta_k)e_{a_{k+1}} + (1 - \beta_{k+1})e_{a_{k+2}}$. Therefore $\beta_{k+1} \geq \beta_k$. We conclude the argument by showing that the inequality must be strict.

Since $(a_k, a_{k+1}), (a_{k+1}, a_{k+2}) \in FPT(\mathbb{D})$, we have $P_1^*, P_2^* \in \mathbb{D}$ such that $r_1(P_1^*) = a_k$, $r_1(P_2^*) = a_{k+2}$ and $r_2(P_1^*) = r_2(P_2^*) = a_{k+1}$. Thus, $C(P_1^*, P_2^*) = \{a_{k+1}\}$. Then, the tops-only property and the compromise property imply $\beta_{k+1} - \beta_k = \varphi_{a_{k+1}}(a_k, a_{k+2}) = \varphi_{a_{k+1}}(P_1^*, P_2^*) > 0$ as required. This completes the verification of part (i) of the lemma.

Pick a_i, a_j in the sequence $\{a_k\}_{k=1}^s$ such that $i < j$. We will prove part (ii) by induction on the value of $l = j - i$. Observe that part (ii) has already been proved for the cases $l = 1$ (Lemma 3.3.3) and $l = 2$ (in the proof of part (i)). Assume therefore that $3 \leq l \leq s-1$. We impose the following induction hypothesis: for all $1 \leq \underline{i} < \underline{j} \leq s$, we have

$$\left[\underline{j} - \underline{i} < l\right] \Rightarrow \left[\varphi(a_{\underline{i}}, a_{\underline{j}}) = \beta_{\underline{i}} e_{a_{\underline{i}}} + \textstyle\sum_{k=\underline{i}+1}^{\underline{j}-1} (\beta_k - \beta_{k-1}) e_{a_k} + (1 - \beta_{\underline{j}-1}) e_{a_{\underline{j}}}\right].$$

We complete the proof by showing that part (ii) holds for all i, j with $1 \leq i < j \leq s$ and $j - i = l$.

Since $j - i = l \geq 3$, we know $i < i+1 < j-1 < j$. Also $(j-1) - i = l-1 < l$ and $j - (i+1) = l - 1 < l$. The induction hypothesis can then be applied to the profiles (a_i, a_{j-1}) and (a_{i+1}, a_j). Hence

$$\begin{aligned} \varphi(a_i, a_{j-1}) &= \beta_i e_{a_i} + \sum_{k=i+1}^{j-2} (\beta_k - \beta_{k-1}) e_{a_k} + (1 - \beta_{j-2}) e_{a_{j-1}} \quad \text{and} \\ \varphi(a_{i+1}, a_j) &= \beta_{i+1} e_{a_{i+1}} + \sum_{k=i+2}^{j-1} (\beta_k - \beta_{k-1}) e_{a_k} + (1 - \beta_{j-1}) e_{a_j}. \end{aligned}$$

Since $(a_j, a_{j-1}) \in FPT(\mathbb{D})$ and $a_i, \ldots, a_{j-2}$ are distinct from a_{j-1} and a_j, Lemma 3.3.1 implies $\varphi_{a_i}(a_i, a_j) = \varphi_{a_i}(a_i, a_{j-1}) = \beta_i$ and $\varphi_{a_k}(a_i, a_j) = \varphi_{a_k}(a_i, a_{j-1}) = \beta_k - \beta_{k-1}$, $k = i+1, \ldots, j-2$. Similarly, since $(a_i, a_{i+1}) \in FPT(\mathbb{D})$, a_{j-1} and a_j are distinct from a_i and a_{i+1}, Lemma 3.3.1 implies $\varphi_{a_{j-1}}(a_i, a_j) = \varphi_{a_{j-1}}(a_{i+1}, a_j) = \beta_{j-1} - \beta_{j-2}$ and $\varphi_{a_j}(a_i, a_j) = \varphi_{a_j}(a_{i+1}, a_j) = 1 - \beta_{j-1}$. Thus, $\sum_{k=i}^{j} \varphi_{a_k}(a_i, a_j) = 1$ and $\varphi(a_i, a_j) = \beta_i e_{a_i} + \sum_{k=i+1}^{j-1} (\beta_k - \beta_{k-1}) e_{a_k} + (1 - \beta_{j-1}) e_{a_j}$ as required. This completes the verification of the induction hypothesis and hence part (ii) of the lemma. □

To demonstrate that $\mathbb{D}$ is single-peaked, we need to construct a tree $G = \langle A, E \rangle$ and show that $P_i \in \mathbb{D}$ implies P_i is single-peaked on G.

Let $G(\mathbb{D}) = \langle A, FPT(\mathbb{D}) \rangle$ be a graph, i.e., $a, b \in A$ constitute an edge in $G(\mathbb{D})$ only if they satisfy the FPT property. Since $\mathbb{D}$ is path-connected, graph $G(\mathbb{D})$ is connected. The following lemma shows that $G(\mathbb{D})$ is a tree.

Lemma 3.3.5. *The graph $G(\mathbb{D})$ is a tree.*

Proof. Suppose not, i.e., there exists a sequence $\{a_k\}_{k=1}^{s} \subseteq A$, $s \geq 3$, such that

$(a_k, a_{k+1}) \in FPT(\mathbb{D})$, $k = 1, \ldots, s$, where $a_{s+1} = a_1$. Let $\beta_k = \varphi_{a_k}(a_k, a_{k+1})$, $k = 1, \ldots, s-1$. Since $(a_k, a_{k+1}) \in FPT(\mathbb{D})$, $k = 1, \ldots, s-1$, Lemma 3.3.4 implies $\varphi(a_1, a_s) = \beta_1 e_{a_1} + \sum_{k=2}^{s-1}(\beta_k - \beta_{k-1})e_{a_k} + (1 - \beta_{s-1})e_{a_s}$ where $0 \leq \beta_k < \beta_{k+1} \leq 1$, $k = 1, \ldots, s-2$. However, since $(a_1, a_s) \in FPT(\mathbb{D})$, Lemma 3.3.3 implies $\varphi_{a_k}(a_1, a_s) = 0$ for all $a_k \neq a_1, a_s$. We have a contradiction. $\square$

Lemma 3.3.6. *The inclusion* $[P_i \in \mathbb{D}] \Rightarrow [P_i$ *is single-peaked on* $G(\mathbb{D})]$.

Proof. Suppose $x, a, b \in A$ are such that $r_1(P_i) = x$ and $a \in \langle x, b\rangle \backslash \{b\}$. Let $\langle x, b\rangle = \{a_t\}_{t=1}^{T}$ where $a_1 = x$, $a_T = b$ and $a = a_l$ for some $1 \leq l < T$. If $a = x$, aP_ib follows trivially. Assume therefore that $a \neq x$. Thus, $T \geq 3$. Suppose bP_ia. Consider the profile $P = (x, b)$ and $\varphi(P)$. According to Lemma 3.3.4, all alternatives in the sequence $\{a_t\}_{t=2}^{T-1}$ get strictly positive probability. Hence $\varphi_a(x, b) > 0$. Since φ satisfies unanimity, $\varphi_b(b, b) = 1$. Then voter i can obtain a strictly higher probability on the set of alternatives at least as preferred to a under P_i (this set includes b by hypothesis) by putting b on top of her order. This contradicts the strategy-proofness of φ. Therefore, aP_ib as required. $\square$

This completes the verification of the necessity part of the theorem.

In order to demonstrate sufficiency, let $\mathbb{D}$ be a single-peaked domain on a tree $G = \langle A, E\rangle$. We construct a RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ that is strategy-proof, tops-only and ex-post efficient, and satisfies the compromise property. We proceed as follows: in the first step, we use the idea in [17] to construct a specific DSCF (see the proof of the sufficiency part of the Theorem in [17]); in the second step, we consider randomization over such DSCFs.

For any set $B \subseteq A$, let $G(B)$ be the *minimal subgraph* of G that contains B as vertices. More formally, $G(B)$ is the unique graph that satisfies the following properties.

1. The set of vertices in $G(B)$ contains B.

2. Let $a, b \in B$. Graph $G(B)$ has an edge (a, b) only if (a, b) is an edge in G.

3. The graph $G(B)$ is connected.

4. We have $x \in G(B)$ if and only if $x \in \langle a, b \rangle$ where $a, b \in B$.

Fix a profile $P \in \mathbb{D}^N$ and an alternative $a_k \in A$. Consider the graph $G(r_1(P))$. Suppose $a \notin G(r_1(P))$. Since G is a tree and contains no cycles, there exists a *unique* alternative in $G(r_1(P))$ that belongs to *every* path from a to any vertex in $G(r_1(P))$. Let this alternative be denoted by $\beta(a, P)$.[6] Then, define the alternative $\pi(a, P)$ as

$$\pi(a, P) = \begin{cases} a, & \text{if } a \in G(r_1(P)) \\ \beta(a, P), & \text{if } a \notin G(r_1(P)) \end{cases}$$

Consider Example 3.2.1. Suppose $I = \{1, 2, 3\}$. Let a be the alternative a_4 and let P be a profile such that $r_1(P) = \{a_1, a_2, a_3\}$. Then $G(r_1(P))$ is the graph consisting of the vertices $\{a_1, a_2, a_3\}$ and the edges (a_1, a_2) and (a_2, a_3). Then $\pi(a, P) = \beta(a_4, P) = a_2$. Further examples can be found in [17].

For every $a \in A$, the DSCF $\phi^a : \mathbb{D}^N \to A$ is defined as for all $P \in \mathbb{D}^N$, $\phi^a(P) = \pi(a, P)$. Evidently, ϕ^a is a DSCF. Its outcome at profile P is the "projection" of a on the minimal subgraph of G generated by the set of the first-ranked alternatives in P.

In the next step, we construct the RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ as for all $P \in \mathbb{D}^N$, $\varphi(P) = \sum_{a \in A} \lambda^a \phi^a(P)$, where $\lambda^a > 0$ for all $a \in A$ and $\sum_{a \in A} \lambda^a = 1$. The RSCF is obtained by choosing over the DSCFs ϕ^a, $a \in A$, according to a fixed probability distribution where the probability of choosing each such DSCF is strictly positive. We call RSCF φ a *weighted projection rule*. We claim that φ satisfies all the required properties.[7]

Lemma 3.3.7. *The RSCF φ is tops-only and strategy-proof.*

[6] It would have been more appropriate to write $\beta(a, G(r_1(P)))$ but we choose to suppress the dependence of this alternative on G for notational convenience.

[7] Some further properties of weighted projection rules are discussed in Appendix 6.

Proof. According to Proposition 1 in [17], a single-peaked domain is semi-single-peaked where every alternative can be taken to be a threshold in the definition of semi-single-peakedness. The sufficiency part of the Theorem in [17] shows that for any threshold $a \in A$, ϕ^a is strategy-proof, tops-only and satisfies unanimity over a semi-single-peaked domain. Consequently, each ϕ^a is tops-only and strategy-proof. Therefore, φ which is a convex combination of distinct tops-only and strategy-proof RSCFs is also a tops-only and strategy-proof RSCF.[8] $\square$

Lemma 3.3.8. *The RSCF φ is ex-post efficient.*

Proof. Suppose the lemma is false, i.e., there exist $P \in \mathbb{D}^N$ and $a, b \in A$ such that aP_ib for all $i \in I$ and $\varphi_b(P) > 0$. Evidently, $b \notin r_1(P)$. Since φ satisfies unanimity, $\varphi_b(P) > 0$ implies $|r_1(P)| > 1$. Observe that $\pi(x, P) \in G(r_1(P))$ for all $x \in A$. Hence, by construction of φ, if z is not included in the vertex set of $G(r_1(P))$, then $\varphi_z(P) = 0$. Therefore, b belongs to the vertex set of $G(r_1(P))$.

Let $Ext\big(G(r_1(P))\big)$ denote the set of vertices in $G(r_1(P))$ with degree one, i.e., $x \in Ext\big(G(r_1(P))\big)$ if there exists a unique $y \in A$ such that (x, y) is an edge in $G(r_1(P))$. Observe that $Ext\big(G(r_1(P))\big) \subseteq r_1(P)$. (Suppose $x \in Ext\big(G(r_1(P))\big)$ but $x \notin r_1(P)$. Then x can be deleted as a vertex in $G(r_1(P))$ contradicting the assumption that $G(r_1(P))$ is minimal.) In other words, the vertices at the ends of every maximal path in $G(r_1(P))$ must be some elements of $r_1(P)$.

It follows from the arguments in the two previous paragraphs that $b \in G(r_1(P)) \backslash Ext\big(G(r_1(P))\big)$. Consequently, there exist $i, i' \in I$ such that $r_1(P_i) \neq r_1(P_{i'})$, $b \in \langle r_1(P_i), r_1(P_{i'})\rangle$ and $b \neq r_1(P_i), r_1(P_{i'})$. Let x be the projection of a on the interval $\langle r_1(P_i), r_1(P_{i'})\rangle$. By assumption, $x \in \langle r_1(P_i), r_1(P_{i'})\rangle$. Hence, either $b \in \langle r_1(P_i), x\rangle$ or $b \in \langle r_1(P_{i'}), x\rangle$ must hold. Therefore either $b \in \langle r_1(P_i), a\rangle$ or $b \in \langle r_1(P_{i'}), a\rangle$ must hold, i.e., either bP_ia or $bP_{i'}a$ must hold by single-peakedness of $\mathbb{D}$. We have a contradiction to our initial hypothesis that aP_ib for all $i \in I$. Therefore, φ is ex-post efficient. $\square$

[8] These arguments are routine and therefore omitted.

Lemma 3.3.9. *The RSCF φ satisfies the compromise property.*

Proof. Let $P_i, P_j \in \mathbb{D}$ be such that $a = r_1(P_i) \neq r_1(P_j) = b$ and $C(P_i, P_j) = \{c\}$. Let $\hat{I} \subset I$ be such that $|\hat{I}| = N/2$ if N is even and $|\hat{I}| = (N+1)/2$ if N is odd. Let $\bar{P} \in \mathbb{D}^N$ be the profile $\big(\underbrace{P_i, \ldots, P_i}_{\hat{I}}, \underbrace{P_j, \ldots, P_j}_{I \setminus \hat{I}}\big)$. We will show that $\varphi_c(\bar{P}) > 0$.

Since $\mathbb{D}$ is single-peaked on the tree $G = \langle A, E \rangle$, it follows that $(a, c), (b, c) \in E$. Hence $c \in G(r_1(\bar{P}))$ and $\phi^c(\bar{P}) = e_c$. Therefore, $\varphi_c(\bar{P}) \geq \lambda^c > 0$. □

This completes the proof of the sufficiency part of the theorem. □

3.3.1 Discussion: Indispensability of the Axioms and the Richness Condition

This subsection shows that all axioms and richness assumption are indispensable for Theorem 3.3.1. Examples 3.3.1, 3.3.2, 3.3.3 and 3.3.4 drop, respectively, the compromise property, tops-onlyness, unanimity and strategy-proofness in turn, and demonstrate the existence of a non-single-peaked domain that admits RSCFs satisfying the remaining axioms. In addition, Example 3.3.5 shows that the separable domain violates path-connectedness but admits a unanimous, tops-only, strategy-proof RSCF satisfying the compromise property.

Example 3.3.1 (Dropping the compromise property)**.** Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\mathbb{D}^3$ is described in Table 3.3.

P_1	P_2	P_3	P_4	P_5	P_6
a_1	a_2	a_2	a_3	a_3	a_4
a_2	a_1	a_3	a_2	a_4	a_3
a_4	a_4	a_4	a_4	a_2	a_2
a_3	a_3	a_1	a_1	a_1	a_1

Table 3.3: Domain $\mathbb{D}^3$

Since $(a_1, a_2), (a_2, a_3), (a_3, a_4) \in FPT(\mathbb{D}^3)$, domain $\mathbb{D}^3$ is path-connected. In view of the path-connectivity structure, the only candidate for a graph with respect to which $\mathbb{D}^3$ could be single-peaked is the line G^L in Figure 3.2. However, preferences P_1 and P_2 violate single-peakedness in this case. Hence $\mathbb{D}^3$ is not single-peaked.

The domain $\mathbb{D}^3$ is however, semi-single-peaked ([17]) with respect to (G^L, a_2). Consequently, the projection rule ϕ^{a_2} is unanimous, tops-only and strategy-proof. (This can also be verified directly.)

Note that $C(P_i, P_j) = \emptyset$ for all profile pairs with distinct peaks except for the pairs (P_1, P_4) and (P_3, P_6). Accordingly, $C(P_1, P_4) = \{a_2\}$, $C(P_3, P_6) = \{a_3\}$; $\phi^{a_2}_{a_2}(P_1, P_4) = \phi^{a_2}_{a_2}(P_4, P_1) = 1 > 0$, *but* $\phi^{a_2}_{a_3}(P_3, P_6) = \phi^{a_2}_{a_3}(P_6, P_3) = 0$. Therefore, RSCF ϕ^{a_2} violates the compromise property. □

Example 3.3.2 (Dropping the tops-only property)**.** Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\mathbb{D}^4$ is described in Table 3.4.

P_1	P_2	P_3	P_4	P_5	P_6
a_1	a_2	a_2	a_3	a_3	a_4
a_2	a_1	a_3	a_2	a_4	a_3
a_4	a_4	a_4	a_4	a_1	a_1
a_3	a_3	a_1	a_1	a_2	a_2

Table 3.4: Domain $\mathbb{D}^4$

Once again, domain $\mathbb{D}^4$ is path-connected since $(a_1, a_2), (a_2, a_3), (a_3, a_4) \in FPT(\mathbb{D}^4)$. Using the same arguments as in Example 3.3.1, it follows that $\mathbb{D}^4$ is not single-peaked.

Let ϕ^{a_k}, $k = 1, 2, 3, 4$, denote the four projection rules on the line G^L in Figure 3.2. We specify a unanimous RSCF $\varphi : [\mathbb{D}^4]^2 \to \Delta(A)$ as

$$\varphi(P_i, P_j) = \begin{cases} \frac{1}{2}\, e_{r_1(P_i)} + \frac{1}{2} e_{r_1(P_j)}, \\ \qquad \text{if } (P_i, P_j) \in \{P_1, P_2\} \times \{P_5, P_6\} \text{ or } \{P_5, P_6\} \times \{P_1, P_2\}; \\ \frac{1}{3}\phi^{a_1}(P_i, P_j) + \frac{1}{6}\phi^{a_2}(P_i, P_j) + \frac{1}{6}\phi^{a_3}(P_i, P_j) + \frac{1}{3}\phi^{a_4}(P_i, P_j), \\ \qquad \text{otherwise.} \end{cases}$$

The RSCF φ is an equal weight random dictatorship when a preference profile belongs to the subdomain $\big\{\{P_1, P_2\} \times \{P_5, P_6\}\big\} \cup \big\{\{P_5, P_6\} \times \{P_1, P_2\}\big\}$; otherwise it is a specific weighted projection rule on the line G^L. The RSCF φ is also strategy-proof; this can be verified by showing that in every possible manipulation, probabilities are transferred from preferred alternatives to less preferred alternatives in the true preference while probabilities assigned to other alternatives are unchanged. The details of the verification are found in Appendix 7.

Note that $r_1(P_4) = r_1(P_5) = a_3$ and $\varphi_{a_2}(P_1, P_4) = \frac{1}{6} \neq 0 = \varphi_{a_2}(P_1, P_5)$. Therefore φ violates the tops-only property.

Observe that $C(P_i, P_j) = \emptyset$ for all profile pairs with distinct peaks except for pairs (P_1, P_4) and (P_3, P_6). Accordingly, $C(P_1, P_4) = \{a_2\}$, $C(P_3, P_6) = \{a_3\}$; $\varphi_{a_2}(P_4, P_1) = \varphi_{a_2}(P_1, P_4) = \frac{1}{6} > 0$ and $\varphi_{a_3}(P_3, P_6) = \varphi_{a_3}(P_6, P_3) = \frac{1}{6} > 0$. Hence, the compromise property is satisfied. $\square$

Example 3.3.3 (Dropping unanimity)**.** Consider the complete domain $\mathbb{P}$. Fix a collection $[\lambda^a]_{a\in A} \in \mathbb{R}^m_{++}$ with $\sum_{a\in A} \lambda^a = 1$, and construct the RSCF $\varphi : \mathbb{P}^N \to \Delta(A)$ as

$$\varphi(P) = \sum_{a\in A} \lambda^a e_a \text{ for all } P \in \mathbb{P}^N.$$

The RSCF φ is tops-only and strategy-proof, and satisfies the compromise property. Since it is a convex combination of all *constant* DSCFs, it violates unanimity. $\square$

Example 3.3.4 (Dropping strategy-proofness)**.** Consider the complete domain $\mathbb{P}$. Fix a collection $[\lambda^a]_{a\in A} \in \mathbb{R}^m_{++}$ with $\sum_{a\in A} \lambda^a = 1$, and construct the RSCF $\varphi : \mathbb{P}^N \to \Delta(A)$ as

$$\varphi(P) = \begin{cases} e_a, & \text{if } r_1(P) = \{a\} \text{ for some } a \in A; \\ \sum\limits_{a\in A} \lambda^a e_a, & \text{otherwise.} \end{cases}$$

The RSCF φ picks alternative a for sure if a is the peak for all voters in a profile. In all other profiles, it is a convex combination of all constant DSCFs. It is unanimous and tops-only, and satisfies the compromise property but not strategy-proofnot. □

Example 3.3.5 (Dropping path-connectedness)**.** Let $A = \{a_1, a_2, a_3, a_4\}$. The domain $\mathbb{D}^7$ is specified in Table 3.5.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
a_1	a_1	a_2	a_2	a_3	a_3	a_4	a_4
a_2	a_4	a_1	a_3	a_2	a_4	a_1	a_3
a_4	a_2	a_3	a_1	a_4	a_2	a_3	a_1
a_3	a_3	a_4	a_4	a_1	a_1	a_2	a_2

Table 3.5: Domain $\mathbb{D}^7$

A convenient way to represent these preferences is to regard each alternative a_k, as comprising two components (a_k^1, a_k^2). Specifically, $A^1 = \{0, 1\}$, $A^2 = \{0, 1\}$; $a_1 = (0, 0)$, $a_2 = (1, 0)$, $a_3 = (1, 1)$ and $a_4 = (0, 1)$. Then domain $\mathbb{D}^7$ is a separable domain ([5], [13]). Apparently, $FPT(\mathbb{D}^7) = \emptyset$ and hence domain $\mathbb{D}^7$ is not path-connected.

For all $P_i, P_j \in \mathbb{D}^7$, let $r_1(P_i) = a_i \equiv (a_i^1, a_i^2)$ and $r_1(P_j) = a_j \equiv (a_j^1, a_j^2)$. Accordingly, $\mathbb{D}^7$ admits the following four DSCFs: for all $P_i, P_j \in \mathbb{D}^7$,

$$\phi^{a_1}(P_i, P_j) = (\min(a_i^1, a_j^1), \min(a_i^2, a_j^2)), \qquad \phi^{a_2}(P_i, P_j) = (\max(a_i^1, a_j^1), \min(a_i^2, a_j^2)),$$

$$\phi^{a_3}(P_i, P_j) = (\max(a_i^1, a_j^1), \max(a_i^2, a_j^2)), \qquad \phi^{a_4}(P_i, P_j) = (\min(a_i^1, a_j^1), \max(a_i^2, a_j^2)).$$

The DSCFs $\phi^{a_1}, \phi^{a_2}, \phi^{a_3}, \phi^{a_4}$ are unanimous, anonymous, tops-only and strategy-proof.

Pick $\lambda^{a_k} > 0$, $k = 1, 2, 3, 4$ with $\sum_{k=1}^{4} \lambda^{a_k} = 1$, and define RSCF $\varphi : [\mathbb{D}^7]^2 \to \Delta(A)$ as

$$\varphi(P) = \sum_{k=1}^{4} \lambda^{a_k} \phi^{a_k}(P) \text{ for all } P \in [\mathbb{D}^7]^2.$$

Since it is a convex combination of DSCFs satisfying unanimity, anonymity, tops-onlyness and strategy-proofness, φ also satisfies these properties. Finally, observe that $C(P_1, P_5) = \{a_2\}$, $C(P_2, P_6) = \{a_4\}$, $C(P_3, P_7) = \{a_1\}$, $C(P_4, P_8) = \{a_3\}$ and $C(P_i, P_j) = \emptyset$ for all other pairs (P_i, P_j) with $r_1(P_i) \neq r_1(P_j)$. Since $\varphi_{a_2}(P_1, P_5) = \varphi_{a_2}(P_5, P_1) = \lambda^{a_2} > 0$, $\varphi_{a_4}(P_2, P_6) = \varphi_{a_4}(P_6, P_2) = \lambda^{a_4} > 0$, $\varphi_{a_1}(P_3, P_7) = \varphi_{a_1}(P_7, P_3) = \lambda^{a_1} > 0$ and $\varphi_{a_3}(P_4, P_8) = \varphi_{a_3}(P_8, P_4) = \lambda^{a_3} > 0$, RSCF φ satisfies the compromise property. $\square$

Chapter 4 On Random Social Choice Functions with the Tops-only Property

4.1 Main Result

The class of tops-only RSCFs have obvious informational and computational advantages. More importantly, the tops-only property decreases the degree of possible manipulations. For these reasons, they (more accurately, DSCFs) have received a great deal of attention in the literature (see [52] and [15]). This chapter studies the tops-only property in the randomized environment.

This section introduces a condition on domains under which every unanimous and strategy-proof RSCF satisfies the tops-only property. The condition requires two properties which are referred to as the Interior Property and the Exterior Property. The domain first is partitioned into sub-domains where all preferences in a sub-domain have an identical peak. The Interior Property refers to a requirement across any two preferences within a given sub-domain, while the Exterior Property refers to a requirement that applies to two preferences belonging to two distinct sub-domains. To describe the Interior Property, the notion of *adjacent connectedness* (introduced in Sato [44]) is adopted, while to describe the Exterior Property, a more general notion called *isolation* is used.

A pair of distinct preferences $P_i, P_i' \in \mathbb{D}$ is *adjacently connected*, denoted $P_i \sim^A P_i'$, if there exists $1 \leq k \leq m-1$ such that the following two conditions are satisfied

(i) $r_k(P_i) = r_{k+1}(P_i')$ and $r_{k+1}(P_i) = r_k(P_i')$;

(ii) $r_t(P_i) = r_t(P'_i)$ for all $t \neq k, k+1$.

In other words, two preferences are adjacently connected if one pair of alternatives *locally switches* their relative rankings. Given distinct $P_i, P'_i \in \mathbb{D}$, an *AC-path* connecting P_i and P'_i is a sequence $\{P_i^k\}_{k=1}^l$ such that $P_i^1 = P_i$, $P_i^l = P'_i$ and $P_i^k \sim^A P_i^{k+1}$, $k = 1, \ldots, l-1$.

The Interior Property requires that given two distinct preferences with *the same peak*, there is an AC-path connecting them such that every preference on the path *shares that peak*.

Definition 4.1.1. *Domain $\mathbb{D}$ satisfies the Interior Property if for all $a \in A$ and distinct $P_i, P'_i \in \mathbb{D}^a$, there exists an AC-path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}^a$ connecting P_i and P'_i.*

The Interior Property is not enough to ensure that unanimity and strategy-proofness imply the tops-only property (see Example 4.1.1).

Example 4.1.1. Let $A = \{a, b, c\}$ and domain $\mathbb{D}$ of three preferences is specified in Table 4.1.

P_1	P_2	P_3
a	b	b
c	a	c
b	c	a

Table 4.1: Domain $\mathbb{D}$

Evidently, domain $\mathbb{D}$ satisfies the Interior Property, i.e., $P_2 \sim^A P_3$. Moreover, domain $\mathbb{D}$ admits a two-voter unanimous and strategy-proof DSCF: (i) $f(P_1, P_2) = f(P_2, P_1) = e_a$ and $f(P_1, P_3) = f(P_3, P_1) = e_c$, (ii) $f(P_1, P_1) = e_a$ and (iii) $f(P_i, P_j) = e_b$, for all $P_i, P_j \in \{P_2, P_3\}$. Since social lotteries vary at profiles (P_1, P_2) and (P_1, P_3) in favour of the second voter's preference over a and c, DSCF f does not satisfy the tops-only property. $\square$

Example 4.1.1 indicates that in addition to the Interior Property, a condition needs to be imposed on preferences with distinct peaks, which is referred to the Exterior Property. For the description of the Exterior Property, the notion of *isolation* needs to be established. Given distinct $P_i, P_i' \in \mathbb{D}$, a pair of alternatives $x, y \in A$ is *isolated* in (P_i, P_i') if there exists $1 \leq k \leq m$ such that

(i) $B^k(P_i) = B^k(P_i')$,

(ii) either $x \in B^k(P_i)$ and $y \notin B^k(P_i)$, or $x \notin B^k(P_i)$ and $y \in B^k(P_i)$.

In the notion of isolation, two sets of top-k ranked alternatives in P_i and P_i' are identical, include one alternative in $\{x, y\}$ and exclude the other. Note that if x and y are isolated in (P_i, P_i'), the relative rankings of x and y are identical in P_i and P_i', i.e., $[xP_iy] \Leftrightarrow [xP_i'y]$. Given distinct $P_i, P_i' \in \mathbb{D}$ and $x, y \in A$, let $\{P_i^k\}_{k=1}^l$ be a sequence of preferences (not necessarily an AC-path) such that $P_i^1 = P_i$, $P_i^l = P_i'$; and x and y are isolated in (P_i^k, P_i^{k+1}), $k = 1, \ldots, l-1$. Then, $\{P_i^k\}_{k=1}^l$ is referred to as a (x, y)-*Is-path* connecting P_i and P_i'.

Remark 4.1.1. The notion of isolation is independent of adjacent connectedness since preferences in the definition of isolation are not necessarily adjacently connected. Conversely, given $P_i, P_i' \in \mathbb{D}$ with $P_i \sim^A P_i'$, a pair of alternatives $x, y \in A$ is isolated in (P_i, P_i') *if and only if* the relative rankings of x and y are identical in P_i and P_i'.

The Exterior Property says that fixing a pair of preferences with distinct peaks and a pair of alternatives with the same relative ranking across these two preferences, one can construct a sequence of preferences connecting these two fixed preferences such that the fixed pair of alternatives is isolated in every two consecutive preferences of the sequence.

Definition 4.1.2. *Domain $\mathbb{D}$ satisfies the Exterior Property if given $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P_i')$ and $x, y \in A$ with xP_iy and $xP_i'y$, there exists a (x, y)-Is-path connecting P_i and P_i'.*

Note that in the definition of the Exterior Property, x is preferred to y in *every* preference of the (x, y)-Is-path connecting P_i and P_i'. With a small modification of Example 4.1.1, an new example (Example 4.1.1 [continued]) is provided to illustrate how the terminology of isolation in the Exterior Property, in conjunction with the Interior Property, drives a two-voter unanimous and strategy-proof RSCF to satisfy the tops-only property. A general argument can be easily deduced from this example to verify the sufficiency of the Interior Property and the Exterior Property for the tops-only property.

Example. 4.1.1 [continued] Replace preference P_1 in Example 4.1.1 by $\bar{P}_1$: $a\bar{P}_1 b\bar{P}_1 c$, and let $\bar{\mathbb{D}} = \{\bar{P}_1, P_2, P_3\}$. Observe that a and c are not isolated in (P_1, P_2) in Example 4.1.1, but isolated in $(\bar{P}_1, P_2)$, i.e., $B^2(\bar{P}_1) = B^2(P_2) = \{a, b\}$ includes a and excludes c. Correspondingly, domain $\mathbb{D}$ in Example 4.1.1 violates the Exterior Property, i.e., there exists *no* (a, c)-Is-path in $\mathbb{D}$ connecting P_1 and P_2, while it is easy to verify that domain $\bar{\mathbb{D}}$ satisfies both the Interior Property and the Exterior Property. Consequently, for every two-voter unanimous and strategy-proof RSCF $\varphi : \bar{\mathbb{D}}^2 \to \Delta(A)$, we have

$$\begin{aligned}
\varphi_c(\bar{P}_1, P_2) &= 1 - \sum_{x\in\{a,b\}=B^2(\bar{P}_1)} \varphi_x(\bar{P}_1, P_2) \\
&= 1 - \sum_{x\in\{a,b\}=B^2(P_2)} \varphi_x(P_2, P_2) \quad \text{by strategy-proofness on isolation } B^2(\bar{P}_1) = B^2(P_2) = \{a, b\} \\
&= 1 - \sum_{x\in\{a,b\}=B^2(P_2)} \varphi_x(P_2, P_3) \quad \text{by unanimity, } r_1(P_2) = r_1(P_3) = b \\
&= 1 - \sum_{x\in\{a,b\}=B^2(\bar{P}_1)} \varphi_x(\bar{P}_1, P_3) \quad \text{by strategy-proofness on isolation } B^2(\bar{P}_1) = B^2(P_2) = \{a, b\} \\
&= \varphi_c(\bar{P}_1, P_3)
\end{aligned}$$

Therefore, it must be the case that $\varphi(\bar{P}_1, P_2) = \varphi(\bar{P}_1, P_3)$ and moreover, φ satisfies the tops-only property. □

Now, state the main result.

Theorem 4.1.1. *Let domain $\mathbb{D}$ satisfy the Interior Property and the Exterior Property. Every unanimous and strategy-proof RSCF satisfies the tops-only property.*

Proof. We first provide a lemma which is repeatedly applied in the proof of Theorem 4.1.1. Let $\varphi : \mathbb{D}^N \to \Delta(A)$, $N \geq 2$, be a strategy-proof RSCF.

Lemma 4.1.1. *Given $P_i, P_i' \in \mathbb{D}$ with $P_i \sim^A P_i'$, assume $xP_i!y$ and $yP_i'!x$. Given*

$P_j, P'_j \in \mathbb{D}$, *if* x *and* y *are isolated in* (P_j, P'_j), *then for all* $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$,

$$\left[\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})\right] \Rightarrow \left[\varphi(P_i, P'_j, P_{-\{i,j\}}) = \varphi(P'_i, P'_j, P_{-\{i,j\}})\right].$$

The proof of Lemma 4.1.1 is available in Appendix 8.

Now, we prove Theorem 4.1.1. Let domain $\mathbb{D}$ satisfy the Interior Property and the Exterior Property. If $N = 1$, unanimity implies the tops-only property. Now, we provide an induction argument on the number of voters.

Induction hypothesis: Given $N \geq 2$, for all $1 \leq n < N$, every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ satisfies the tops-only property.

Given an unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$, we will show that φ satisfies the tops-only property. It is easy to verify that φ satisfies the tops-only property if for all $i \in I$; $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i)$ and $P_{-i} \in \mathbb{D}^{N-1}$, $\varphi(P_i, P_{-i}) = \varphi(P'_i, P_{-i})$. Given distinct $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i) \equiv a$, the Interior Property implies that there exists an AC-path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}^a$ connecting P_i and P'_i. Then, it suffices to show that for each $1 \leq k \leq l-1$, $\varphi(P_i^k, P_{-i}) = \varphi(P_i^{k+1}, P_{-i})$ for all $P_{-i} \in \mathbb{D}^{N-1}$. Equivalently, we will show that for all $i \in I$; $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i)$ and $P_i \sim^A P'_i$, and $P_{-i} \in \mathbb{D}^{N-1}$, $\varphi(P_i, P_{-i}) = \varphi(P'_i, P_{-i})$.

Fixing two voters $i, j \in I$, we induce a function $\psi : \mathbb{D}^{N-1} \to \Delta(A)$ such that $\psi(P_i, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$ for all $P_i \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. Evidently, ψ is a well-defined RSCF satisfying unanimity and strategy-proofness (please refer to Lemma 3 in [48]). Hence, induction hypothesis implies that ψ satisfies the tops-only property. Accordingly, for all $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i)$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, $\varphi(P_i, P_i, P_{-\{i,j\}}) = \varphi(P'_i, P'_i, P_{-\{i,j\}})$.

Fixing $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i)$ and $P_i \sim^A P'_i$, we assume $xP_i!y$ and $yP'_i!x$. Given $P_j \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, we will prove that $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$.

Claim 1: If $r_1(P_j) = r_1(P_i)$, then $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$.

Firstly, by strategy-proofness, we have that for all $t = 1, \ldots, m$,

$$\sum_{x \in B^t(P_i)} \varphi_x(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{x \in B^t(P_i)} \varphi_x(P_i, P_j, P_{-\{i,j\}}) \leq \sum_{x \in B^t(P_i)} \varphi_x(P_i, P_i, P_{-\{i,j\}}),$$

$$\sum_{x \in B^t(P'_i)} \varphi_x(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{x \in B^t(P'_i)} \varphi_x(P'_i, P_j, P_{-\{i,j\}}) \leq \sum_{x \in B^t(P'_i)} \varphi_x(P'_i, P'_i, P_{-\{i,j\}}).$$

Moreover, since $r_1(P_j) = r_1(P_i) = r_1(P'_i)$, $\varphi(P_j, P_j, P_{-\{i,j\}}) = \psi(P_j, P_{-\{i,j\}}) = \psi(P_i, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$ and $\varphi(P_j, P_j, P_{-\{i,j\}}) = \psi(P_j, P_{-\{i,j\}}) = \psi(P'_i, P_{-\{i,j\}}) = \varphi(P'_i, P'_i, P_{-\{i,j\}})$. Consequently, for all $t = 1, \ldots, m$, $\sum_{x \in B^t(P_i)} \varphi_x(P_i, P_j, P_{-\{i,j\}}) = \sum_{x \in B^t(P_i)} \varphi_x(P_i, P_i, P_{-\{i,j\}})$ and $\sum_{x \in B^t(P'_i)} \varphi_x(P'_i, P_j, P_{-\{i,j\}}) = \sum_{x \in B^t(P'_i)} \varphi_x(P'_i, P'_i, P_{-\{i,j\}})$. Hence, $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$ and $\varphi(P'_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P'_i, P_{-\{i,j\}})$. Then, we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}}) = \psi(P_i, P_{-\{i,j\}}) = \psi(P'_i, P_{-\{i,j\}}) = \varphi(P'_i, P'_i, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$. This completes the verification of the claim.

Next, assume $r_1(P_j) \neq r_1(P_i)$. Evidently, either xP_jy or yP_jx. We assume xP_jy. The argument related to yP_jx is symmetric and we hence omit it. Since xP_iy and xP_jy, the Exterior Property implies that there exists a (x, y)-Is-path $\{P_j^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting P_i and P_j. Firstly, since $P_j^1 = P_i$, Claim 1 implies $\varphi(P_i, P_j^1, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}}) = \varphi(P'_i, P_i, P_{-\{i,j\}}) = \varphi(P'_i, P_j^1, P_{-\{i,j\}})$. Next, following the sequences $\{P_j^k\}_{k=1}^l$, since $P_i \sim^A P'_i$; $xP_i!y$, $yP'_i!x$; and x and y are isolated in (P_j^k, P_j^{k+1}), $k = 1, \ldots, l-1$, we can repeatedly applying Lemma 4.1.1 step by step which eventually implies $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$. This completes the verification of the induction hypothesis and hence Theorem 4.1.1. □

4.2 Applications

This section first studies two important classes of restricted domains in the literature: connected domains ([44]) and the multi-dimensional single-peaked domain ([6]). Both two classes of domains are shown to satisfy the Interior Property and the Exterior Property.

Next, this section slightly modifies both the Interior Property and the Exterior

Property to accord with the Cartesian product setting on the alternative set, and show that unanimity and strategy-proofness imply the tops-only property over the separable domain ([5] and [13]).

After establishing the tops-only property for all unanimous and strategy-proof RSCFs over the multi-dimensional single-peaked domain and the separable domain, this section further characterizes strategy-proof RSCFs over these two domains. The first characterization result generalizes Theorem 4 in [5] to a randomized setting by showing that every ex-post efficient and strategy-proof RSCF over the multi-dimensional single-peaked domain is a random dictatorship. In the second characterization result, every unanimous and strategy-proof RSCF over the separable domain is a generalized random dictatorship. This is a direct extension of Theorem 3 in [16].

Last, this section strengthens the axiom of ex-post efficiency to ex-post efficiency* by enlarging the support of the social lottery under every preference profile and studies a domain implication problem: what must a domain, which admits an ex-post efficient* and strategy-proof RSCF, look like? It is established that in the class of connected domains with minimal richness, single-peakedness is implied by the admission of an ex-post efficient* and strategy-proof RSCF.

4.2.1 Connected Domains

Sato [44] introduces the notion of *weak connectedness* which is a necessary condition (not sufficient) for the equivalence of *adjacent manipulation-proofness* (or AM-proofness) and strategy-proofness in DSCFs.[1]

Definition 4.2.1. *Domain $\mathbb{D}$ is weakly connected if given distinct $P_i, P_i' \in \mathbb{D}$ and $x, y \in A$, there exists an AC-path $\{P_i^k\}_{k=1}^{l} \subseteq \mathbb{D}$ connecting P_i and P_i' such that*

$[xP_i^k y$ *and* $yP_i^{k+1}x$ *for some* $1 \leq k \leq l-1] \Rightarrow [xP_i^t y,\ 1 \leq t \leq k,$ *and* $yP_i^{t'}x,$

[1]A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is *AM-proof* if for all $i \in I$; $P_i, P_i' \in \mathbb{D}$ with $P_i \sim^A P_i'$ and $P_{-i} \in \mathbb{D}^{N-1}$, $\sum_{x \in B^t(P_i)} \varphi_x(P_i, P_{-i}) \geq \sum_{x \in B^t(P_i)} \varphi_x(P_i', P_{-i})$, $t = 1, \ldots, m$. [53] revisits Sato's work and shows that in conjunction with the Interior Property which is also necessary for the equivalence of AM-proofness and strategy-proofness, weak connectedness is sufficient for enhancing AM-proofness to strategy-proofness in the class of unanimous DSCFs.

$k+1 \leq t' \leq l$].

Evidently, weak connectedness implies the Exterior Property. However, the inverse argument does not hold since the Exterior Property only considers two preferences with distinct peaks, and the Is-path in the description of the Exterior Property may not be an AC-path.

In this chapter, a domain satisfying the Interior Property and weak connectedness is referred to as a *connected* domain. Therefore, by Theorem 4.1.1, every unanimous and strategy-proof RSCF over a connected domain must satisfy the tops-only property.

Proposition 4.2.1. *Every unanimous and strategy-proof RSCF over a connected domain satisfies the tops-only property.*

Remark 4.2.1. The complete domain, the single-peaked domain ([34] and [20]), the single-dipped domain ([9]) and maximal single-crossing domains ([42] and [14]) are all connected domains. Therefore, the tops-only property is endogenized in every unanimous and strategy-proof RSCF defined on these domains.

Remark 4.2.2. Every unanimous and AM-proof RSCF over a connected domain also satisfies the tops-only property.[2] Therefore, in a domain satisfying the equivalence of AM-proofness and strategy-proofness, the tops-only property is implied by unanimity and AM-proofness.

4.2.2 The Multi-Dimensional Single-Peaked Domain

In this subsection, a Cartesian product structure is imposed on the alternative set, i.e., $A = \times_{s \in M} A^s$ where M is finite, $|M| \geq 2$; and A^s is finite and $|A^s| \geq 2$ for each $s \in M$. An alternative can be written as $a = (a^s, a^{-s}) = (a^S, a^{-S})$ where

[2] The verification is similar to the proof of Theorem 4.1.1 which makes the following four changes: (i) change strategy-proofness to AM-proofness, (ii) change the sentence "Given $P_j, P_j' \in \mathbb{D}$" in Lemma 4.1.1 to "Given $P_j, P_j' \in \mathbb{D}$ with $P_j \sim^A P_j'$", (iii) change the hypothesis "$r_1(P_j) = r_1(P_i)$" in Claim 1 to "$P_j = P_i$", and (iv) change the first sentence of the last paragraph in the proof of Theorem 4.1.1 to "Next, assume $P_j \neq P_i$.".

$S \subseteq M$ is not empty. For notational convenience, given $s \in M$ and $a^s \in A^s$, let $(a^s, A^{-s}) = \{x \in A | x^s = a^s\}$.

Moreover, for each $s \in M$, assume that all elements in A^s are located on a *tree*, denoted $G(A^s)$. Thus, a product of trees $\times_{s\in M} G(A^s)$ is generated. For each $s \in M$, let $\langle a^s, b^s \rangle$ denote the unique path between a^s and b^s in $G(A^s)$. Given $a, b \in A$, let $\langle a, b \rangle = \{x \in A | x^s \in \langle a^s, b^s \rangle$ for each $s \in M\}$ denote the *minimal box* containing all alternatives located between a and b in each dimension.

Definition 4.2.2. *Given a product of trees $\times_{s\in M} G(A^s)$, a preference P_i is multi-dimensional single-peaked on $\times_{s\in M} G(A^s)$ if for all $a, b \in A$,*

$$\big[a \in \langle r_1(P_i), b \rangle \backslash \{b\}\big] \Rightarrow [aP_ib].$$

Given a product of trees $\times_{s\in M} G(A^s)$, let $\mathbb{D}_{MSP}$ denote the multi-dimensional single-peaked domain on $\times_{s\in M} G(A^s)$ containing all admissible preferences.

Remark 4.2.3. The formulation of multi-dimensional single-peakedness in this subsection is the one where all elements in each component set are located on a tree. This generalizes the earlier notion introduced by [6] where all elements in each component set must be arranged on a line.

The multi-dimensional single-peaked domain satisfies both the Interior Property and the Exterior Property. A simple example is first provided to illustrate (see Example 4.2.1).

Example 4.2.1. Let $A \equiv A^1 \times A^2 = \{0, 1\} \times \{0, 1\}$. The product of lines $G(A^1) \times G(A^2)$ and domain $\mathbb{D}_{MSP}$ are specified in Figure 4.1 and Table 4.2, respectively.

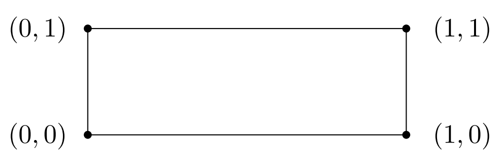

Figure 4.1: The product of lines $G(A^1) \times G(A^2)$

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8
$(0,0)$	$(0,0)$	$(1,0)$	$(1,0)$	$(0,1)$	$(0,1)$	$(1,1)$	$(1,1)$
$(1,0)$	$(0,1)$	$(0,0)$	$(1,1)$	$(0,0)$	$(1,1)$	$(1,0)$	$(0,1)$
$(0,1)$	$(1,0)$	$(1,1)$	$(0,0)$	$(1,1)$	$(0,0)$	$(0,1)$	$(1,0)$
$(1,1)$	$(1,1)$	$(0,1)$	$(0,1)$	$(1,0)$	$(1,0)$	$(0,0)$	$(0,0)$

Table 4.2: Domain $\mathbb{D}_{MSP}$

The Interior Property is satisfied since $P_1 \sim^A P_2$, $P_3 \sim^A P_4$, $P_5 \sim^A P_6$ and $P_7 \sim^A P_8$. An instance is used here to illustrate how the requirement of the Exterior Property is met. Note that $(1,0)P_1(0,1)$ and $(1,0)P_7(0,1)$. Correspondingly, $\{P_1, P_3, P_4, P_7\}$ is a $\big((1,0),(0,1)\big)$-Is-path connecting P_1 and P_7, i.e., $B^2(P_1) = B^2(P_3) = \{(0,0),(1,0)\}$, $B^1(P_3) = B^1(P_4) = \{(1,0)\}$ and $B^2(P_4) = B^2(P_7) = \{(1,0),(1,1)\}$. $\square$

Now, state the formal result.

Proposition 4.2.2. *Domain $\mathbb{D}_{MSP}$ satisfies the Interior Property and the Exterior Property. Therefore, every unanimous and strategy-proof RSCF over $\mathbb{D}_{MSP}$ satisfies the tops-only property.*

The proof of Proposition 4.2.2 is available in Appendix 9.

Remark 4.2.4. Any sub-domain of $\mathbb{D}_{MSP}$ satisfying Lemmas 9.1 - 9.5 in Appendix 9 meets both the Interior Property and the Exterior Property.

4.2.3 Separable Domains

This subsection follows the same Cartesian product setting on the alternative set in Section 4.2.2.

Definition 4.2.3. *A preference P_i is separable if for all $s \in M$ and $a^s, b^s \in A^s$,*

$$\big[(a^s, x^{-s})P_i(b^s, x^{-s}) \text{ for some } x^{-s} \in A^{-s}\big] \Rightarrow \big[(a^s, y^{-s})P_i(b^s, y^{-s}) \text{ for all } y^{-s} \in A^{-s}\big].$$

Let $\mathbb{D}_S$ denote the separable domain containing all separable preferences. For more details and examples, please refer to [5], [13] and [40].

In particular, when $|A^s| = 2$ for all $s \in M$, $\mathbb{D}_S = \mathbb{D}_{MSP}$. Then, Proposition 4.2.2 implies that every unanimous and strategy-proof RSCF over $\mathbb{D}_S$ satisfies the tops-only property. However, if one component set contains more than two elements, Proposition 4.2.2 fails to show the tops-only result over $\mathbb{D}_S$ due to the violation of the Interior Property. For instance, assume $\{a^s, b^s, c^s\} \subseteq A^s$ for some $s \in M$. Given $c \in A$ and $P_i, P_i' \in \mathbb{D}_S^c$, assume $(a^s, z^{-s})P_i(b^s, z^{-s})$ and $(b^s, z^{-s})P_i'(a^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. Transforming P_i to P_i' through a sequence of separable preferences, the relative rankings of (a^s, z^{-s}) and (b^s, z^{-s}) for all $z^{-s} \in A^{-s}$ need to be switched *simultaneously* at some point of the sequence. This indicates the violation of adjacent connectedness in the Interior Property. Therefore, a new notion of connectedness needs to be formulated.

Formally, a pair of distinct preferences P_i, P_i' is *multiple-adjacently connected*, denoted $P_i \sim^{MA} P_i'$, if there exist $s \in M$ and $a^s, b^s \in A^s$ such that

(i) for every $z^{-s} \in A^{-s}$, $(a^s, z^{-s}) = r_k(P_i) = r_{k+1}(P_i')$ and $(b^s, z^{-s}) = r_{k+1}(P_i) = r_k(P_i')$ for some $1 \leq k \leq m$;

(ii) for every $x \notin (a^s, A^{-s}) \cup (b^s, A^{-s})$, $x = r_k(P_i) = r_k(P_i')$ for some $1 \leq k \leq m$.

In other words, in a pair of multiple-adjacently connected preferences, one can identify a component $s \in M$ and two elements $a^s, b^s \in A^s$ such that every pair of alternatives (a^s, z^{-s}) and (b^s, z^{-s}), $z^{-s} \in A^{-s}$, is contiguous in both preferences with opposite relative rankings, while all other alternatives are ranked identically in both preferences. For instance, in Example 4.2.1, $P_1 \sim^{MA} P_3$, $P_2 \sim^{MA} P_5$, $P_4 \sim^{MA} P_7$ and $P_6 \sim^{MA} P_8$.

Remark 4.2.5. The notion of multiple-adjacent connectedness is *independent* of the restriction of separable preferences, and is established to accord with the Cartesian product setting. Similar to Remark 4.1.1, a pair of alternatives is isolated in two multiple-adjacently connected preferences *if and only if* they share the same relative ranking in these two preferences.

Now, a pair of preferences $P_i, P_i' \in \mathbb{D}$ is referred to be *generally connected*, denoted $P_i \sim P_i'$, if either $P_i \sim^A P_i'$, or $P_i \sim^{MA} P_i'$. Given distinct $P_i, P_i' \in \mathbb{D}$, a sequence $\{P_i^k\}_{k=1}^l$ is referred to as a *GC-path* connecting P_i and P_i' if $P_i^1 = P_i$, $P_i^l = P_i'$ and $P_i^k \sim P_i^{k+1}$, $k = 1, \ldots, l-1$. More restrictively, given distinct $P_i, P_i' \in \mathbb{D}$ and $x, y \in A$, a sequence $\{P_i^k\}_{k=1}^l$ is referred to as a (x, y)*-Is-GC-path* connecting P_i and P_i' if it is both a (x, y)-Is-path and a GC-path.

With the notion of general connectedness, the Interior Property and the Exterior Property can be modified. First, *Modified Interior Property* weakens the notion of AC-path in the Interior Property to the GC-path. Similarly, to formulate *Modified Exterior Property*, the Is-path in the Exterior Property is replaced by the Is-GC-path.

Definition 4.2.4. *Domain $\mathbb{D}$ satisfies the Modified Interior Property if for all $a \in A$ and distinct $P_i, P_i' \in \mathbb{D}^a$, there exists a GC-path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}^a$ connecting P_i and P_i'.*

Definition 4.2.5. *Domain $\mathbb{D}$ satisfies the Modified Exterior Property if given $P_i, P_i' \in \mathbb{D}$ with $r_1(P_i) \neq r_1(P_i')$, and $x, y \in A$ with xP_iy and $xP_i'y$, there exists a (x, y)-Is-GC-path connecting P_i and P_i'.*

Remark 4.2.6. The combination of the Modified Interior Property and the Modified Exterior Property is *not* a sufficient condition for the tops-only property in general. However, once embedding them in the separable preference context, they are sufficient (see Proposition 4.2.3).

Example 4.2.2 is used to illustrate the Modified Interior Property and the Modified Exterior Property in the separable domain.

Example 4.2.2. Let $A = \{1, 2, 3\} \times \{1, 2\}$. Six separable preferences are highlighted in Table 4.3, and specify the corresponding general connectedness relations in Figure 4.2.

P_1	P_2	P_3	P_4	P_5	P_6
$(1,1)$	$(1,1)$	$(1,2)$	$(1,2)$	$(1,2)$	$(1,2)$
$(1,2)$	$(1,2)$	$(1,1)$	$(1,1)$	$(1,1)$	$(1,1)$
$(2,1)$	$(2,1)$	$(2,2)$	$(2,2)$	$(3,2)$	$(3,2)$
$(3,1)$	$(2,2)$	$(2,1)$	$(3,2)$	$(2,2)$	$(3,1)$
$(2,2)$	$(3,1)$	$(3,2)$	$(2,1)$	$(3,1)$	$(2,2)$
$(3,2)$	$(3,2)$	$(3,1)$	$(3,1)$	$(2,1)$	$(2,1)$

Table 4.3: Six separable preferences

$$P_1 \;\frac{[(3,1),(2,2)]}{}\; P_2 \;\frac{[(1,1),(1,2)],[(2,1),(2,2)]}{[(3,1),(3,2)]}\; P_3 \;\frac{[(2,1),(3,2)]}{}\; P_4 \;\frac{[(2,2),(3,2)]}{[(2,1),(3,1)]}\; P_5 \;\frac{[(2,2),(3,1)]}{}\; P_6$$

Figure 4.2: General connectedness relations[3]

According to Figure 4.2, the requirement of the Modified Interior Property is satisfied in sub-domains $\{P_1, P_2\} \subseteq \mathbb{D}_S^{(1,1)}$ and $\{P_3, P_4, P_5, P_6\} \subseteq \mathbb{D}_S^{(1,2)}$ respectively.

Observe that P_4 and P_5 share the same peak; but there exists no AC-path in $\mathbb{D}_S$ connecting P_4 and P_5.[4] Therefore, the Interior Property fails in $\mathbb{D}_S$.

An instance is used to illustrate how the requirement of the Modified Exterior Property is satisfied. Observe that $(1,2)P_k(2,2)$ for all $1 \leq k \leq 6$. Thus, the sequence $\{P_k\}_{k=1}^6$ is a $\big((1,2),(2,2)\big)$-Is-GC-path connecting P_1 and P_6. To just meet the requirement of the Exterior Property with respect to (P_1, P_6) and $\big((1,2),(2,2)\big)$, one can refer to a shorter sequence $\{P_1, P_5, P_6\}$, where P_1 and P_5 is *not* generally connected.[5]

[3]For instance, $P_1 \frac{[(3,1),(2,2)]}{} P_2$ denotes $P_1 \sim^A P_2$; $(3,1)P_1!(2,2)$ and $(2,2)P_2!(3,1)$. Similarly, $P_4 \frac{[(2,2),(3,2)]}{[(2,1),(3,1)]} P_5$ denotes $P_4 \sim^{MA} P_5$; $(2,2)P_4!(3,2)$, $(3,2)P_5!(2,2)$; $(2,1)P_4!(3,1)$ and $(3,1)P_5!(2,1)$.

[4]Suppose that there exists an *AC-path* $\{\bar{P}_i^k\}_{k=1}^l \subseteq \mathbb{D}_S$ connecting P_4 and P_5. Since $(2,2)P_4(3,2)$ and $(3,2)P_5(2,2)$, there must exist $1 \leq k \leq l-1$ such that $(2,2)\bar{P}_i^k(3,2)$ and $(3,2)\bar{P}_i^{k+1}(2,2)$. Then, separability implies $(2,1)\bar{P}_i^k(3,1)$ and $(3,1)\bar{P}_i^{k+1}(2,1)$ which contradicts the hypothesis $\bar{P}_i^k \sim^A \bar{P}_i^{k+1}$.

[5]Note that $B^2(P_1) = B^2(P_5) = \{(1,1),(1,2)\}$ and $B^1(P_5) = B^1(P_6) = \{(1,2)\}$.

Although there is no $\big((3,1),(2,2)\big)$-Is-GC-path connecting P_1 and P_6 in Table 4.3. one can identify a $\big((3,1),(2,2)\big)$-Is-GC-path in $\mathbb{D}_S$ connecting P_1 and P_6 according to Proposition 4.2.4. $\square$

Now, here is the result in the restricted environment of separable preferences.

Proposition 4.2.3. *Let domain $\mathbb{D} \subseteq \mathbb{D}_S$ satisfy the Modified Interior Property and the Modified Exterior Property. Every unanimous and strategy-proof RSCF over $\mathbb{D}$ satisfies the tops-only property.*

Proof. We first provide a lemma which is repeatedly applied in the proof of Proposition 4.2.3.

Lemma 4.2.1. *Given $P_i, P_i' \in \mathbb{D}$ with $P_i \sim^{MA} P_i'$, assume that*

(i) *for every $z^{-s} \in A^{-s}$, $(x^s, z^{-s}) = r_k(P_i) = r_{k+1}(P_i')$ and $(y^s, z^{-s}) = r_{k+1}(P_i) = r_k(P_i')$ for some $1 \leq k \leq m$;*

(ii) *for every $z \notin (x^s, A^{-s}) \cup (y^s, A^{-s})$, $z = r_k(P_i) = r_k(P_i')$ for some $1 \leq k \leq m$.*

Given $P_j, P_j' \in \mathbb{D}$ with $P_j \sim P_j'$, assume that for all $z^{-s} \in A^{-s}$, (x^s, z^{-s}) and (y^s, z^{-s}) are isolated in (P_j, P_j'). Then, for all $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$,

$$\big[\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})\big] \Rightarrow \big[\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})\big].$$

The proof of Lemma 4.2.1 is available in Appendix 10. Note that Lemma 4.2.1 holds without restricting preferences to be separable.

Similar to the verification of Theorem 4.1.1, we apply a induction argument to show Proposition 4.2.3. If $N = 1$, unanimity implies the tops-only property. Now, we provide an induction hypothesis on the number of voters.

Induction hypothesis: Given $N \geq 2$, for all $1 \leq n < N$, every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^n \to \Delta(A)$ satisfies the tops-only property.

Given an unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$, we will show that φ satisfies the tops-only property. Similarly to the proof of Theorem 4.1.1, we will show that for all $i \in I$; $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i)$ and $P_i \sim P'_i$ and $P_{-i} \in \mathbb{D}^{N-1}$, $\varphi(P_i, P_{-i}) = \varphi(P'_i, P_{-i})$.

Fixing two voters $i, j \in I$, we induce a function $\psi : \mathbb{D}^{N-1} \to \Delta(A)$ such that $\psi(P_i, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$ for all $P_i \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. Evidently, ψ is a well-defined RSCF satisfying unanimity and strategy-proofness. Therefore, induction hypothesis implies that ψ satisfies the tops-only property. Accordingly, for all $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i)$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, $\varphi(P_i, P_i, P_{-\{i,j\}}) = \psi(P_i, P_{-\{i,j\}}) = \psi(P'_i, P_{-\{i,j\}}) = \varphi(P'_i, P'_i, P_{-\{i,j\}})$.

Fix $P_i, P'_i \in \mathbb{D}$ with $r_1(P_i) = r_1(P'_i)$ and $P_i \sim P'_i$. Given $P_j \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, we will show that $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$.

By a similar argument as Claim 1 in the proof of Theorem 4.1.1, we know that if $r_1(P_j) = r_1(P_i)$, then $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$.

Henceforth, we assume $r_1(P_j) \neq r_1(P_i)$. Since $P_i \sim P'_i$, either $P_i \sim^A P'_i$ or $P_i \sim^{MA} P'_i$. If $P_i \sim^A P'_i$, assume $xP_i!y$ and $yP'_i!x$. Evidently, either xP_jy or yP_jx. Assume xP_jy. The verification related to yP_jx is symmetric and we hence omit it. According to the Modified Exterior Property, we have a (x, y)-Is-GC-path $\{P_j^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting P_i and P_j. Since $\varphi(P_i, P_j^1, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}}) = \varphi(P'_i, P_i, P_{-\{i,j\}}) = \varphi(P'_i, P_j^1, P_{-\{i,j\}})$, following $\{P_j^k\}_{k=1}^l$ and repeatedly applying Lemma 4.1.1 step by step, we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$.

Next, we assume $P_i \sim^{MA} P'_i$, i.e., there exist $s \in M$ and $x^s, y^s \in A^{-s}$ such that

(i) for every $z^{-s} \in A^{-s}$, $(x^s, z^{-s}) = r_k(P_i) = r_{k+1}(P'_i)$ and $(y^s, z^{-s}) = r_{k+1}(P_i) = r_k(P'_i)$ for some $1 \leq k \leq m$;

(ii) for every $z \notin (x^s, A^{-s}) \cup (y^s, A{-}s)$, $z = r_k(P_i) = r_k(P'_i)$ for some $1 \leq k \leq m$.

Separability implies either $(x^s, z^{-s})P_j(y^s, z^{-s})$ for all $z^{-s} \in A^{-s}$, or $(y^s, z^{-s})P_j(x^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. We assume $(x^s, z^{-s})P_j(y^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. The argument related to the other case is symmetric and we hence omit it. Given $z^{-s} \in A^{-s}$, since $(x^s, z^{-s})P_i(y^s, z^{-s})$ and $(x^s, z^{-s})P_j(y^s, z^{-s})$, by the Modified Exterior Property, there exists a $\big((x^s, z^{-s}), (y^s, z^{-s})\big)$-Is-GC-path $\{P_j^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting P_i

and P_j. Evidently, $(x^s, z^{-s})P_j^k(y^s, z^{-s})$ for all $1 \leq k \leq l$. Then, separability implies that for all $z^{-s} \in A^{-s}$, $(x^s, z^{-s})P_j^k(y^s, z^{-s})$, $k = 1, \ldots, l$. Consequently, by Remarks 4.1.1 and 4.2.5, for each $z^{-s} \in A^{-s}$, (x^s, z^{-s}) and (y^s, z^{-s}) are isolated in (P_i^k, P_i^{k+1}), $k = 1, \ldots, l-1$. Now, since $\varphi(P_i, P_j^1, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}}) = \varphi(P_i', P_i, P_{-\{i,j\}}) = \varphi(P_i', P_j^1, P_{-\{i,j\}})$, following $\{P_j^k\}_{k=1}^l$ and repeatedly applying Lemma 4.2.1 step by step, we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P_i', P_j, P_{-\{i,j\}})$. This completes the verification of the induction hypothesis and hence Proposition 4.2.3. □

In Proposition 4.2.3, the Modified Interior Property and the Modified Exterior Property are assumed exogenously to be embedded on separable domains. However, the compatibility of these two properties and separability remains to be established. Proposition 4.2.4 shows that the separable domain satisfies both the Modified Interior Property and the Modified Exterior Property.

Proposition 4.2.4. *Domain $\mathbb{D}_S$ satisfies the Modified Interior Property and the Modified Exterior Property.*

The proof of Proposition 4.2.4 is available in Appendix 11.

Remark 4.2.7. Any sub-domain of $\mathbb{D}_S$ satisfying Lemmas 11.1 - 11.5 in Appendix 11 meets both the Modified Interior Property and the Modified Exterior Property.

Remark 4.2.8. According to the proof of Proposition 4.2.2, the multi-dimensional single-peaked domain $\mathbb{D}_{MSP}$ also satisfies the Modified Interior Property and the Modified Exterior Property. However, Proposition 4.2.3 cannot be simply adapted for the context of multi-dimensional single-peaked preferences due to the violation of separability. An example is provided to illustrate. Let $A \equiv A^1 \times A^2 = \{1, 2, 3\} \times \{0, 1\}$. Both graphs $G(A^1)$ and $G(A^2)$ are lines following the natural number order. Three multi-dimensional single-peaked preferences over $G(A^1) \times G(A^2)$ are highlighted below.

P_i	P'_i	P_j
$(2,0)$	$(2,0)$	$(2,1)$
$(2,1)$	$(2,1)$	$(2,0)$
$(1,0)$	$(3,0)$	$(1,1)$
$(3,0)$	$(1,0)$	$(3,1)$
$(1,1)$	$(3,1)$	$(3,0)$
$(3,1)$	$(1,1)$	$(1,0)$

Note that $P_i \sim^{MA} P'_i$ where (i) $(1,0)P_i!(3,0)$ and $(1,1)P_i!(3,1)$, and (ii) $(3,0)P'_i!(1,0)$ and $(3,1)P'_i!(1,1)$. However, since P_j is not separable, e.g., $(3,0)P_j(1,0)$ and $(1,1)P_j(3,1)$, P_j disagrees with P_i on the relative ranking of $(1,0)$ and $(3,0)$, and disagrees with P'_i on the relative ranking of $(1,1)$ and $(3,1)$. Consequently, the argument in the last paragraph of the proof of Proposition 4.2.3 fails here.

Remark 4.2.9. The lexicographically separable domain (see Definition 1.1 in Appendix 11), where each component set contains at least three elements, violates both the Modified Interior Property and the Modified Exterior Property.[6] However, Chatterji et al. [16] show that the tops-only property is implied by unanimity and strategy-proofness over the lexicographically separable domain.

4.2.4 Characterization of Strategy-proof RSCFs

In characterizing strategy-proof DSCFs and RSCFs, the tops-only property is always established in advance which simplifies the rest of characterization significantly. This subsection provides two characterization results on strategy-proof RSCFs over the multi-dimensional single-peaked domain and the separable domain, respectively.

Theorem 4 in Barberà et al. [5] implies that there exists no *efficient*, strategy-proof and non-dictatorial DSCF over $\mathbb{D}_{MSP}$ where $|M| \geq 3$ and $|A^s| = 2$ for

[6]First, the lexicographically separable domain never includes a pair of adjacently connected preferences. Second, between a pair of lexicographically separable preferences which are multiple-adjacently connected, the lexicographic orders are identical. Consequently, there exists no GC-path in the lexicographically separable domain connecting two lexicographically separable preferences with distinct lexicographic orders.

every $s \in M$. With the tops-only property established in Proposition 4.2.2, their impossibility result can be pushed to the randomized setting.

Proposition 4.2.5. *Assume $|M| \geq 3$. An ex-post efficient RSCF over $\mathbb{D}_{MSP}$ is strategy-proof if and only if it is a random dictatorship.*

The proof of Proposition 4.2.5 is available in Appendix 12.

The verification of Proposition 4.2.5 relies heavily on the tops-only property. For instance, given a preference profile $P \equiv (P_1, P_2) \in \mathbb{D}^2_{MSP}$ where the peaks of two preferences disagree on at least two components, and an arbitrary alternative a distinct from two peaks, one can always construct a tops-equivalent preference profile $\bar{P} \equiv (\bar{P}_1, \bar{P}_2) \in \mathbb{D}^2_{MSP}$, i.e., $r_1(\bar{P}_1) = r_1(P_1)$ and $r_1(\bar{P}_2) = r_1(P_2)$, such that a is Pareto dominated. Then, ex-post efficiency ensures that a gets probability zero under profile $\bar{P}$, and hence the tops-only property implies that a gets probability zero under profile P.

Next, Chatterji et al. [16] show that every unanimous and strategy-proof RSCF over the lexicographically separable domain with $|A^s| \geq 3$ for all $s \in M$ (recall Definition 1.1 in Appendix 1), which is a strict subset of the separable domain, is a *generalized random dictatorship*. Establishing the tops-only property (by Propositions 4.2.3 and 4.2.4) allows one to directly extend their characterization result to the separable domain.

The formal definition of generalized random dictatorship is first presented here. According to the Cartesian product setting, let $\underline{i} = (i^s)_{s \in M} \in I^{|M|}$ denote a $|M|$-tuple of voters. A $|M|$-tuple $\underline{i}$ can be viewed as a combination of $|M|$ dictators where for each $s \in M$, voter i^s is the dictator over A^s. For each $\underline{i} \in I^{|M|}$, a positive real number $\gamma(\underline{i}) \in \mathbb{R}_+$ is associated, and let $\sum_{\underline{i} \in I^{|M|}} \gamma(\underline{i}) = 1$. Given $P \in \mathbb{D}^N$ and $\underline{i} \in I^{|M|}$, according to voter i^s and her peak $r_1(P_{i^s})$, one identifies the component $r_1(P_{i^s})^s$. Then, combine all identified components and assemble an alternative $\big(r_1(P_{i^s})^s\big)_{s \in M}$. In a generalized random dictatorship, the probability assigned to alternative a equals to the sum of weights $\gamma(\underline{i})$ where the assembling of the alternative according to $\underline{i}$ leads to alternative a. Formally, a RSCF $\varphi : \mathbb{D}^N \to$

$\Delta(A)$ is a generalized random dictatorship if there exists a sequence $[\gamma(\underline{i})]_{\underline{i}\in I^{|M|}} \in \mathbb{R}_+^{N^{|M|}}$ with $\sum_{\underline{i}\in I^{|M|}} \gamma(\underline{i}) = 1$ such that for every $P \in \mathbb{D}^N$ and $a \in A$,

$$\varphi_a(P) = \sum_{\underline{i}\in I^{|M|}:\ (r_1(P_{i^s})^s)_{s\in M}=a} \gamma(\underline{i}).$$

Proposition 4.2.6. *Let $|A^s| \geq 3$ for all $s \in M$. An unanimous RSCF over $\mathbb{D}_S$ is strategy-proof if and only if it is a generalized random dictatorship.*

Remark 4.2.10. Both characterization results in Propositions 4.2.5 and 4.2.6 are instances of the extreme point property, i.e., every ex-post efficient (unanimous respectively) and strategy-proof RSCF is a convex combination of the counterpart DSCFs. One may *conjecture* that the extreme point property remains valid over the multi-dimensional single-peaked domain when ex-post efficiency is weakened to unanimity.

4.2.5 A Domain Implication Problem

A random dictatorship obviously satisfies ex-post efficiency since only peaks of preferences receive positive probabilities. However, recall the infirmity of a random dictatorship mentioned in Chapter 3, it is also reasonable to put some positive weight on some *(Pareto) undominated* alternative which is not a peak of any voter, but is "highly ranked" by all voters. To increase the flexibility of the social lottery, all undominated alternatives under a preference profile are allowed to receive strictly positive probabilities. This strengthens ex-post efficiency to a new axiom *ex-post efficiency**.

Given $P \in \mathbb{D}^N$, let $\Omega(P)$ denote the set of undominated alternatives, i.e., $a \in \Omega(P)$ if there exists no $x \in A$ such that xP_ia for all $i \in I$. Given a lottery $\alpha \in \Delta(A)$, the *support* of the lottery α is a set of alternatives with strictly positive probabilities, i.e., $supp\,\alpha = \{a \in A | \alpha_a > 0\}$. The axiom of ex-post efficiency implies $supp\,\varphi(P) \subseteq \Omega(P)$ for all $P \in \mathbb{D}^N$, while ex-post efficiency* requires $supp\,\varphi(P) = \Omega(P)$ for all $P \in \mathbb{D}^N$.

Definition 4.2.6. *A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is ex-post efficient* if for every $P \in \mathbb{D}^N$, $supp\, \varphi(P) = \Omega(P)$.*

According to the random dictatorship result in [26], and Propositions 4.2.5 and 4.2.6, ex-post efficiency* and strategy-proofness are not compatible on the complete domain, the multi-dimensional single-peaked domain and the separable domain. A natural question arises: on what domains, if any, are ex-post efficiency* and strategy-proofness compatible? This subsection shows that in the class of connected domains with minimal richness, single-peakedness (on a tree) is uniquely characterized by the admission of an ex-post efficient* and strategy-proof RSCF.

Here is the domain implication result.

Proposition 4.2.7. *Let domain $\mathbb{D}$ be minimally rich and connected. If it admits an ex-post efficient* and strategy-proof RSCF, it is single-peaked. Conversely, a single-peaked domain admits an ex-post efficient* and strategy-proof RSCF.*

The proof of Proposition 4.2.7 is available in Appendix 13.

Remark 4.2.11. Chapter 3 studies a similar domain implication problem, and shows that on a path-connected domain, the admission of an unanimous, tops-only and strategy-proof RSCF satisfying the compromise property (see Definition 3.1.1) implies single-peakedness. The axiom of ex-post efficiency* implies unanimity and the compromise property. More importantly, the domain condition of connectedness strengthens the richness condition of path-connectedness and helps endogenize the tops-only property so that Theorem 3.3.1 can be adopted to verify Proposition 4.2.7. Moreover, the exogenous combination of the compromise property and the tops-only property might cause inefficiency in some social lottery, e.g., a Pareto dominated alternative is assigned with strictly positive probability. This possibility of inefficiency is precluded here by requiring the RSCF to be ex-post efficient and endogenizing the tops-only property.

4.3 Discussion

This section discusses related literature, and comments on the necessity of the sufficient condition.

4.3.1 Relation to the Literature

Since tops-onlyness imposes an important well-behaved property on social choice functions, it has attracted considerable attention in the literature, especially in the characterization of strategy-proof DSCFs and RSCFs.[7] The model studied in this chapter uses an ordinal formulation of strategy-proofness introduced by [26]. There is an alternative formulation of strategy-proofness which uses cardinal information on preferences (e.g., [28], [21] and [23]). Here too the tops-only property plays an important role in characterizing randomized strategy-proof voting rules. One may conjecture that a version of our richness condition would allow to endogenize the tops-only property in these cardinal models. This is left for future work.

Earlier work has studied the tops-only property for DSCFs. In particular, Weymark [52] initiated the study of the tops-only property in single-peaked preferences on a real line and continuous preferences on a metric space. Subsequent work focuses on the case of finite alternatives and strict preferences. Lemma 3.1 in Nehring and Puppe [35] show that every unanimous and strategy-proof DSCFs over a generalized single-peaked domain satisfying two particular richness conditions must be tops-only, while Chatterji and Sen [15] introduce two conditions: Property T and Property T*, which are sufficient for the tops-only property in DSCFs for the case of two voters and the case of arbitrary number of voters, respectively.[8] However, the sufficient conditions mentioned above imply that the domain must be minimally rich. The sufficient condition in this chapter is independent of minimal richness.

Remark 4.3.1. [15] also study two non-minimally rich domains: the domain of

[7] See for instance [34], [6], [18], [52], [42], [33], [39], [28], [21], [22], [24], [23], [16] and [38].
[8] Property T* implies Property T, and covers generalized single-peaked domains.

in-between preferences ([27]) and Kelly's domain ([31]), and show that the tops-only property is satisfied by every unanimous and strategy-proof DSCF. These two domains do not satisfy the sufficient condition in this chapter directly. However, observe that for instance, in the domain of in-between preferences, the alternative which is never the peak of *any* preference is not included in the range of *any* unanimous and strategy-proof DSCF (a similar argument holds in Kelly's domain).[9] Accordingly, inducing new preferences by removing alternatives excluded from the ranges of all unanimous and strategy-proof DSCFs, the new domains satisfy the Interior Property and the Exterior Property.

Even with minimal richness, there is an example of a domain which meets a sufficient condition in this chapter but violates Property T* (see Example 4.3.1). Moreover, Property T is *not* sufficient for endogenizing the tops-only property in a randomized environment (see Example 4.3.2).

Example 4.3.1. Let $A = \{a_1, a_2, a_3, a_4\}$. Domain $\mathbb{D}$ of seven preferences is specified in Table 4.4.

P_1	P_2	P_3	P_4	P_5	P_6	P_7
a_1	a_2	a_2	a_2	a_3	a_3	a_4
a_2	a_1	a_3	a_3	a_2	a_4	a_3
a_3	a_3	a_1	a_4	a_4	a_2	a_2
a_4	a_4	a_4	a_1	a_1	a_1	a_1

Table 4.4: Domain $\mathbb{D}$

Indeed, domain $\mathbb{D}$ is a maximal single-crossing domain with respect to linear orders $a_1 > a_2 > a_3 > a_4$ and $P_1 \succ P_2 \succ P_3 \succ P_4 \succ P_5 \succ P_6 \succ P_7$. It is easy to verify that domain $\mathbb{D}$ is connected and hence satisfies the Interior Property and the Exterior Property. However, domain $\mathbb{D}$ violates Property T*, e.g., (i) $a_3 P_1 a_4$; (ii) for every $P_i \in \mathbb{D}^{a_1} = \{P_1\}$, $a_1 P_i a_3$, but (iii) there exists *no* preference $P_i' \in \mathbb{D}^{a_3} = \{P_5, P_6\}$ such that $a_1 P_i' a_4$. $\square$

[9]Given a DSCF $f : \mathbb{D}^N \to A$, $Range(f) = \{x \in A | f(P) = x \text{ for some } P \in \mathbb{D}^N\}$.

Example 4.3.2. Let $A = \{a_1, a_2, a_3, a_4, a_5\}$. Domain $\mathbb{D}$ of fourteen preferences is specified in Table 4.5.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}	P_{11}	P_{12}	P_{13}	P_{14}
a_1	a_1	a_1	a_2	a_2	a_2	a_3	a_3	a_3	a_4	a_4	a_4	a_5	a_5
a_2	a_3	a_5	a_1	a_3	a_4	a_1	a_2	a_4	a_2	a_3	a_5	a_1	a_4
a_3	a_2	a_3	a_3	a_1	a_3	a_2	a_1	a_2	a_3	a_2	a_3	a_4	a_1
a_5	a_5	a_4	a_5	a_5	a_5	a_4	a_5	a_1	a_5	a_5	a_2	a_2	a_2
a_4	a_4	a_2	a_4	a_4	a_1	a_5	a_4	a_5	a_1	a_1	a_1	a_3	a_3

Table 4.5: Domain $\mathbb{D}$

It is easy to verify that domain $\mathbb{D}$ satisfies Property T. Therefore, every two-voter unanimous and strategy-proof DSCF satisfies the tops-only property. More specifically, domain $\mathbb{D}$ is *linked* (recall Definition 2.1.1), and hence every unanimous and strategy-proof DSCF is a dictatorship. However, domain $\mathbb{D}$ admits the following unanimous and strategy-proof RSCF which violates the tops-only property:

$$\varphi(P_i, P_j) = \begin{cases} \frac{1}{2}e_{r_1(P_i)} + \frac{1}{2}e_{r_1(P_j)}, & \text{if either } P_i \notin \mathbb{D}^{a_3} \text{ or } P_j \notin \mathbb{D}^{a_5}; \\ \frac{1}{4}e_{a_3} + \frac{1}{4}e_{a_2} + \frac{1}{2}e_{a_5}, & \text{if } P_i = P_8 \text{ and } P_j \in \mathbb{D}^{a_5}; \\ \frac{1}{4}e_{a_3} + \frac{1}{4}e_{a_1} + \frac{1}{4}e_{a_4} + \frac{1}{4}e_{a_5}, & \text{if } P_i \in \{P_7, P_9\} \text{ and } P_j \in \mathbb{D}^{a_5}. \end{cases}$$

The verification of strategy-proofness is put in Appendix 14. □

4.3.2 Necessity

It is easy to observe that the Interior Property and the Exterior Property is *not* necessary for the tops-only property. This is not altogether surprising as every random dictatorship domain (recall Definition 1.1.7) ensures the tops-only property.[10] While the complete domain is an instance of a random dictatorship domain that satisfies the Interior Property and the Exterior Property, one can construct a random dictatorship domain (using Theorem 2.3.1) violating both the Interior Property and the

[10] Characterizing the necessary and sufficient conditions for random dictatorship domains is an important open question in the literature.

Exterior Property, where the tops-only property prevails via a random dictatorship characterization result (see Example 4.3.3).

Example 4.3.3. Let $A = \{a_1, a_2, a_3, a_4\}$. Domain $\mathbb{D}$ of ten preferences is specified in Table 4.6.

P_1	P_2	P_3	P_4	P_5	P_6	P_7	P_8	P_9	P_{10}
a_1	a_1	a_2	a_2	a_2	a_3	a_3	a_3	a_4	a_4
a_2	a_3	a_1	a_3	a_4	a_1	a_2	a_4	a_2	a_3
a_3	a_2	a_3	a_1	a_1	a_2	a_1	a_2	a_1	a_1
a_4	a_4	a_4	a_4	a_3	a_4	a_4	a_1	a_3	a_2

Table 4.6: Domain $\mathbb{D}$

First, domain $\mathbb{D}$ violates the Interior Property, e.g., $\mathbb{D}^{a_2} = \{P_3, P_4, P_5\}$, but P_5 is not adjacently connected to either P_3 or P_4. Second, domain $\mathbb{D}$ violates the Exterior Property, e.g., there exists no (a_1, a_3)-Is-path connecting P_3 and P_9.[11] However, domain $\mathbb{D}$ is *linked* (see Definition 2.1.1) and satisfies *Condition H* (see Definition 2.3.1) which implies that every unanimous and strategy-proof RSCF is a random dictatorship by Theorem 2.3.1, and hence satisfies the tops-only property. □

This chapter is unable to identify a necessary and sufficient condition for the tops-only property in a general setting. Weakening the sufficient condition is one approach to push it closer to necessity. Fortunately, the Exterior Property can be weakened by eliminating some *redundant* Is-paths in the domain, and keeps its sufficiency for the tops-only property in conjunction with the Interior Property. This weakening is referred to as the Exterior Property*. Moreover, this chapter asserts that in some particular circumstance, the combination of the Interior Property and the Exterior Property* is necessary and sufficient for endogenizing the tops-only property.

[11]Note that a_1 is ranked above a_3 in all preferences $\{P_1, P_2, P_3, P_5, P_9\}$, while a_3 is preferred to a_1 in all rest of preferences. One cannot form a (a_1, a_3)-Is-path connecting P_3 and P_9 in $\{P_1, P_2, P_3, P_5, P_9\}$.

Definition 4.3.1. *A domain $\mathbb{D}$ satisfies the Exterior Property* if given $P_i, P_i' \in \mathbb{D}$ and $x, y \in A$ satisfying the following two conditions*

(i) *there exists $\bar{P}_i \in \mathbb{D}$ such that $r_1(P_i) = r_1(\bar{P}_i)$; $P_i \sim^A \bar{P}_i$, $xP_i!y$ and $y\bar{P}_i!x$,*

(ii) *$r_1(P_i) \neq r_1(P_i')$ and $xP_i'y$,*

there exists a (x, y)-Is-path connecting P_i and P_i'.

In Definition 4.3.1, preference $\bar{P}_i$ can be viewed as a *bench-mark* which tests whether P_i and (x, y) are *critical* (refer to condition (i) in Definition 4.3.1). Once the criticality is verified, if P_i and P_i' disagree on peaks but coincide on the relative ranking of x and y (refer to condition (ii) in Definition 4.3.1), the Exterior Property* requires the existence of a (x, y)-Is-path connecting P_i and P_i'. Example 4.3.4 is provided to illustrate the Exterior Property*.

Example 4.3.4. Let $A = \{a_1, a_2, a_3, a_4\}$. Domain $\mathbb{D}$ of five preferences is specified in Table 4.7.

P_1	P_2	P_3	P_4	P_5
a_1	a_2	a_2	a_3	a_4
a_2	a_3	a_3	a_2	a_3
a_3	a_1	a_4	a_4	a_2
a_4	a_4	a_1	a_1	a_1

Table 4.7: Domain $\mathbb{D}$

Domain $\mathbb{D}$ satisfies the Interior Property, i.e., $P_2 \sim^A P_3$; but violates the Exterior Property, i.e., there exists no (a_2, a_3)-Is-path connecting P_1 and P_2. However, since P_2 and (a_2, a_3) are *not* critical, there is no need to construct a (a_2, a_3)-Is-path connecting P_2 and P_1. In domain $\mathbb{D}$, for instance, P_2 and (a_1, a_4) are critical. Correspondingly, sequence $\{P_2, P_1\}$ is a (a_1, a_4)-Is-path connecting P_2 and P_1. Indeed, domain $\mathbb{D}$ satisfies the Exterior Property*. □

The following corollary shows that the combination of the Interior Property and the Exterior Property* is sufficient for the tops-only property.

Corollary 4.3.1. *Let domain $\mathbb{D}$ satisfy the Interior Property and the Exterior Property*. Every unanimous and strategy-proof RSCF satisfies the tops-only property.*

Proof. The verification of Corollary 4.3.1 follows from a slight modification of the proof Theorem 4.1.1: replacing the fifth sentence of the last paragraph in the proof of Theorem 4.1.1 by the following sentence: Since (i) $r_1(P_i) = r_1(P_i')$; $P_i \sim^A P_i'$, $xP_i!y$, $yP_i'!x$; (ii) $r_1(P_i) \neq r_1(P_j)$ and xP_jy, the Exterior Property* implies that there exists a (x, y)-Is-path $\{P_j^k\}_{k=1}^l \subseteq \mathbb{D}$ connecting P_i and P_j. $\square$

The Exterior Property* is *not* necessary for the tops-only property either. For instance, in Example 4.3.3, (i) $r_1(P_3) = r_1(P_4) = a_2$; $P_3 \sim^A P_4$, $a_1P_3!a_3$ and $a_3P_4!a_1$, (ii) $r_1(P_9) = a_4 \neq a_2$ and $a_1P_9a_3$, but (iii) there exists *no* (a_1, a_3)-Is-path connecting P_3 and P_9. In particular, if restrict attention to multi-dimensional single-peaked domains studied in Example 4.2.1, one can show that the Interior Property and the Exterior Property* is a necessary and sufficient condition for the tops-only property, provided a mild richness condition holds.

A multi-dimensional single-peaked domain $\mathbb{D} \subseteq \mathbb{D}_{MSP}$ is *significantly rich* if for all $a \in A$ with $\mathbb{D}^a \neq \emptyset$, $\mathbb{D}^a = \mathbb{D}^a_{MSP}$. In other words, in a significantly rich multi-dimensional single-peaked domain, if an alternative is the peak of *some* preference, then the domain must include *every* multi-dimensional single-peaked preference whose peak is that alternative.

Proposition 4.3.1. *Let $A = \{0, 1\} \times \{0, 1\}$ and $\mathbb{D} \subseteq \mathbb{D}_{MSP}$ be significantly rich. Every unanimous and strategy-proof RSCF over $\mathbb{D}$ satisfies the tops-only property if and only if $\mathbb{D}$ satisfies the Interior Property and the Exterior Property*.*

The proof of Proposition 4.3.1 is available in Appendix 15.

Moreover, observe that the Exterior Property* arises *naturally* in single-peaked domains (on a line), provided the satisfaction of the Interior Property and *the Left-Right Extreme condition* introduced by [36].

Given a graph of line, let $\mathbb{D}$ be a single-peaked domain on the line. For notational convenience, write the line as $a_1 < a_2 < \cdots < a_m$. A single-peaked domain

$\mathbb{D}$ (on the line) satisfies the Left-Right Extreme condition if for all $1 \leq k \leq m$ with $\mathbb{D}^{a_k} \neq \emptyset$, there exist $P_i, P_i' \in \mathbb{D}^{a_k}$ such that the following two conditions are satisfied:

Left-extreme condition: all alternatives at the left side of a_k are preferred to all alternatives at the right side of a_k in P_i, i.e., $[s < k < t] \Rightarrow [a_s P_i a_t]$.

Right-extreme condition: all alternatives at the right side of a_k are preferred to all alternatives at the left side of a_k in P_i', i.e., $[s < k < t] \Rightarrow [a_t P_i' a_s]$.

Proposition 4.3.2. *Given a single-peaked domain on a line, if it satisfies the Interior Property and the Left-Right extreme condition, it satisfies the Exterior Property*.*

The proof of Proposition 4.3.2 is available in Appendix 16.

Chapter 5 Summary of Conclusions

Chapter 2 has shown that dictatorial domains are not necessarily random dictatorship domains. In fact, dictatorial domains may admit "well-behaved" strategy-proof random social choice functions. Chapter 2 has provided additional conditions on a class of dictatorial domains to ensure that they are random dictatorship domains. These additional conditions are quite restrictive, but examples suggest that these conditions are "close" to being necessary.

Chapter 3 has characterized domains of single-peaked preferences as the only domains that admit "well-behaved" strategy-proof random social choice functions. This result provides a justification of the salience of single-peaked preferences and evidence in favor of the Gul conjecture.

Chapter 4 has identified a sufficient condition on domains which ensures that every unanimous and strategy-proof RSCF has the tops-only property. Moreover, Chapter 4 has also provided some applications of this sufficient condition.

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Appendix

1 Some Verifications in Section 2.2.3

This appendix verifies that domain $\mathbb{D}$ in Section 2.2.3 is a subset of the separable domain $\mathbb{D}_S$. Next, we provide the details of the proof of Proposition 2.2.3. We first introduce a particular separable domain: the lexicographic separable domain ([13] and [16]).

Given a separable preference $P_i \in \mathbb{D}_S$, one can induce a marginal preference $[P_i]^s$ on each component set A^s, $s \in M$.

Definition 1.1. *A preference P_i is lexicographically separable if there exists a linear order $\succ$ over M such that for all $x, y \in A$,*

$$\big[x^s [P_i]^s y^s \textit{ and } x^\tau = y^\tau \textit{ for all } \tau \in M \textit{ with } \tau \succ s\big] \Rightarrow [x P_i y].$$

Accordingly, let $\mathbb{D}_{LS}$ denote the lexicographically separable domain containing all admissible preferences. Evidently, $\mathbb{D}_{LS} \subset \mathbb{D}_S$. The linear order $\succ$ in Definition 1.1 is referred to as the lexicographic order. For more details and examples on lexicographically separable preferences, please refer to [16]. We will show that every preference in $\mathbb{D}$ in Section 2.2.3 is either separable or lexicographic separable.

We partition the alternative set A into five parts: $A_0 = \{a_0\}$, $A_1 = \{a_1, a_2, a_3, a_4\}$, $A_2 = \{a_5, a_6, a_7, a_8, a_9, a_{10}\}$, $A_3 = \{a_{11}, a_{12}, a_{13}, a_{14}\}$ and $A_4 = \{a_{15}\}$. Accordingly, A_k, $k = 0, \ldots, 4$, includes all alternatives that k candidates are elected. Moreover, we recall two properties on separable preferences.

Ascending property: Let P_k be a linear order satisfying the following two restrictions: (i) $r_1(P_k) = (0,0,0,0) = a_0$, $r_{16}(P_k) = (1,1,1,1) = a_{15}$; and (ii) xP_kyP_kz for all $x \in A_1$, $y \in A_2$ and $z \in A_3$. Then, $P_k \in \mathbb{D}_S$.

Descending property: Let P_k be a linear order satisfying the following two restrictions: (i) $r_1(P_k) = (1,1,1,1) = a_{15}$, $r_{16}(P_k) = (0,0,0,0) = a_0$; and (ii) xP_kyP_kz for all $x \in A_3$, $y \in A_2$ and $z \in A_1$. Then, $P_k \in \mathbb{D}_S$.

Now, we verify that $\mathbb{D} \subset \mathbb{D}_S$.

1. Let $P_1 \in \mathbb{D}_S$ satisfy the ascending property, $r_2(P_1) = (0,0,1,0) = a_1$, $r_3(P_1) = (0,1,0,0) = a_2$ and $r_{15}(P_1) = (1,1,0,1) = a_{13}$.

2. Let $P_2 \in \mathbb{D}_S$ satisfy the ascending property, $r_2(P_2) = (0,0,1,0) = a_1$, $r_3(P_2) = (1,0,0,0) = a_3$ and $r_{15}(P_2) = (1,1,0,1) = a_{13}$.

3. Let $P_3 \in \mathbb{D}_S$ satisfy the ascending property, $r_2(P_3) = (0,1,0,0) = a_2$, $r_3(P_3) = (0,0,1,0) = a_1$, $a_{13} = (1,1,0,1)P_3(1,1,1,0) = a_{11}$ and $a_{13} = (1,1,0,1)P_3(1,0,1,1) = a_{12}$.

4. Let $P_4 \in \mathbb{D}_S$ satisfy the ascending property, $r_2(P_4) = (0,1,0,0) = a_2$, $r_3(P_4) = (1,0,0,0) = a_3$, $a_{13} = (1,1,0,1)P_4(1,1,1,0) = a_{11}$ and $a_{13} = (1,1,0,1)P_4(1,0,1,1) = a_{12}$.

5. Let $P_6 \in \mathbb{D}_S$ satisfy the ascending property, $r_2(P_6) = (1,0,0,0) = a_3$, $r_3(P_6) = (0,0,1,0) = a_1$, $a_{13} = (1,1,0,1)P_6(1,1,1,0) = a_{11}$ and $a_{13} = (1,1,0,1)P_6(1,0,1,1) = a_{12}$.

6. Let $P_7 \in \mathbb{D}_S$ satisfy the ascending property, $r_2(P_7) = (1,0,0,0) = a_3$, $r_3(P_7) = (0,1,0,0) = a_2$, $a_{13} = (1,1,0,1)P_7(1,1,1,0) = a_{11}$ and $a_{13} = (1,1,0,1)P_7(1,0,1,1) = a_{12}$.

7. Let $P_5 \in \mathbb{D}_{LS}$ with $r_1(P_5) = (1,1,0,0) = a_5$ and the lexicographic order $2 \succ 4 \succ 3 \succ 1$. Then, $r_2(P_5) = (0,1,0,0) = a_2$, $r_3(P_5) = (1,1,1,0) = a_{11}$ and $a_{13} = (1,1,0,1)P_5(0,0,1,0) = a_1$.

8. Let $P_8 \in \mathbb{D}_{LS}$ with $r_1(P_8) = (1,0,1,0) = a_6$ and the lexicographic order $1 \succ 4 \succ 2 \succ 3$. Then, $r_2(P_8) = (1,0,0,0) = a_3$, $r_3(P_8) = (1,1,1,0) = a_{11}$ and $a_{13} = (1,1,0,1)P_8(0,0,1,0) = a_1$.

9. Let $P_9 \in \mathbb{D}_{LS}$ with $r_1(P_9) = (1,0,1,0) = a_6$ and the lexicographic order $1 \succ 2 \succ 4 \succ 3$. Then, $r_2(P_9) = (1,0,0,0) = a_3$, $r_3(P_9) = (1,0,1,1) = a_{12}$ and $a_{13} = (1,1,0,1)P_9(0,0,1,0) = a_1$.

10. Let $P_{10} \in \mathbb{D}_{LS}$ with $r_1(P_{10}) = (1,1,0,0) = a_5$ and the lexicographic order $2 \succ 4 \succ 1 \succ 3$. Then, $r_2(P_{10}) = (1,1,1,0) = a_{11}$, $r_3(P_{10}) = (0,1,0,0) = a_2$ and $a_{13} = (1,1,0,1)P_{10}(0,0,1,0) = a_1$.

11. Let $P_{11} \in \mathbb{D}_{LS}$ with $r_1(P_{11}) = (1,0,1,0) = a_6$ and the lexicographic order $1 \succ 4 \succ 3 \succ 2$. Then, $r_2(P_{11}) = (1,1,1,0) = a_{11}$, $r_3(P_{11}) = (1,0,0,0) = a_3$ and $a_{13} = (1,1,0,1)P_{11}(0,0,1,0) = a_1$.

12. Let $P_{14} \in \mathbb{D}_{LS}$ with $r_1(P_{14}) = (1,0,1,0) = a_6$ and the lexicographic order $1 \succ 2 \succ 3 \succ 4$. Then, $r_2(P_{14}) = (1,0,1,1) = a_{12}$, $r_3(P_{14}) = (1,0,0,0) = a_3$ and $a_{13} = (1,1,0,1)P_{14}(0,0,1,0) = a_1$.

13. For any $P_k \in \{P_{12}, P_{13}, P_{15}, P_{16}, P_{17}, P_{18}\}$, let $P_k \in \mathbb{D}_S$ satisfy the descending property and $a_{13} = (1,1,0,1) \in \{r_2(P_k), r_3(P_k), r_4(P_k)\}$.

The detail proof of Proposition 2.2.3.

Proof. Since domain $\bar{\mathbb{D}}$ satisfies Condition SC, according to Proposition 2.2.1, let $\bar{\varphi} : \bar{\mathbb{D}}^N \to \Delta(X)$, $N \geq 2$, be an anonymous, unanimous and strategy-proof RSCF that is not a random dictatorship.

In particular, for all $P \in \mathbb{D}^N$, let $\bar{P} \in \bar{\mathbb{D}}^N$ denote the induced profile of preferences by P over X, i.e., $\bar{P}_i = (P_i, X)$, $i \in I$. Next, we construct a function $\varphi : \mathbb{D}^N \to \Delta^X(A)$ such that for all $P \in \mathbb{D}^N$, $\varphi_x(P) = \bar{\varphi}_x(\bar{P})$ for all $x \in X$ and $\varphi_y(P) = 0$ for all $y \in A \backslash X$. Evidently, φ is a random constraint voting SCF which is anonymous, unanimous and not a random dictatorship.

We need to verify strategy-proofness of φ. Suppose that φ is not strategy-proof. Then, there exist $i \in I$; $P_i, P_i' \in \mathbb{D}$ and $P_{-i} \in \mathbb{D}^{N-1}$ such that $\sum_{k=1}^{t} \varphi_{r_t(P_i)}(P_i, P_{-i}) < \sum_{k=1}^{t} \varphi_{r_t(P_i)}(P_i', P_{-i})$ for some $1 \leq t \leq m$. Let $T = \{r_k(P_i)\}_{k=1}^{t}$ and $\bar{T} = T \cap X$. Evidently, $\bar{T} \neq \emptyset$. Furthermore, we can assume $\bar{T} = \{r_k(\bar{P}_i)\}_{k=1}^{\bar{t}}$ for some $1 \leq \bar{t} \leq 6$. Then, by construction of φ, we have

$$
\begin{aligned}
& \sum_{k=1}^{\bar{t}} \bar{\varphi}_{r_k(\bar{P}_i)}(\bar{P}_i, \bar{P}_{-i}) - \sum_{k=1}^{\bar{t}} \bar{\varphi}_{r_k(\bar{P}_i)}(\bar{P}_i', \bar{P}_{-i}) \\
= & \sum_{x \in \bar{T}} \varphi_x(P_i, P_{-i}) - \sum_{x \in \bar{T}} \varphi_x(P_i', P_{-i}) \\
= & \Big[\sum_{x \in \bar{T}} \varphi_x(P_i, P_{-i}) + \sum_{y \in T \backslash \bar{T}} \varphi_y(P_i, P_{-i})\Big] - \Big[\sum_{x \in \bar{T}} \varphi_x(P_i', P_{-i}) + \sum_{y \in T \backslash \bar{T}} \varphi_y(P_i', P_{-i})\Big] \\
= & \sum_{x \in T} \varphi_x(P_i, P_{-i}) - \sum_{x \in T} \varphi_x(P_i', P_{-i}) \\
= & \sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i, P_{-i}) - \sum_{k=1}^{t} \varphi_{r_k(P_i)}(P_i', P_{-i}) \\
< & \ 0
\end{aligned}
$$

which contradicts strategy-proofness of $\bar{\varphi}$. $\square$

2 Condition SC *v.s.* Conditions H and TS

Fix a domain $\mathbb{D}$ satisfying Condition SC with respect to B and y in Definition 2.2.1.

2.1 Condition H

Suppose that $\mathbb{D}$ satisfies Condition H. Let a be a hub.

Assume $a = y$. Then, $y \sim z$ for all $z \in A\backslash\{y\}$, which implies $\mathcal{S}(y) = A\backslash\{y\}$. Given $x \in B$, since $x \sim y$, there exists $P_k \in \mathbb{D}^{x,y}$. Now, if $|B| = 1$, then yP_kz for all $z \in A\backslash\{x, y\} \subset \mathcal{S}(y)$, which contradicts part (ii) of Condition SC. If $|B| > 1$, then $r_2(P_k) = y \notin B$, $x \in \mathcal{S}(y)$ and xP_ky, which contradicts part (ii) of Condition SC.

Assume $a \in B$. Now, $a \sim y$ and hence $a \in \mathcal{S}(y)$. If $|B| = 1$, consider some

$z \in A\backslash\{a, y\} = A\backslash[B \cup \{y\}]$ (recall $|A| \geq 3$). Then $a \sim z$ and there exists $P_k \in \mathbb{D}^{z,a}$, which contradicts part (iii) of Condition SC. If $|B| > 1$, we consider $a \sim y$. Hence, there exists $P_k \in \mathbb{D}^{a,y}$, which contradicts part (ii) of Condition SC.

Assume that $a \in A\backslash[B\cup\{y\}]$. Fixing $x \in B$, we know $y \sim a$ and $x \sim a$. Hence, $a \in \mathcal{S}(y)$ and there exists $P_k \in \mathbb{D}^{x,a}$, which contradicts part (iii) of Condition SC.

2.2 Condition TS

Suppose that $\mathbb{D}$ satisfies Condition TS.

Assume $B = \{x\}$. Since $|A| \geq 3$, we pick $z \in A\backslash[B \cup \{y\}]$. Now, between x and z, there are three cases to consider: (i) $x \approx z$, (ii) $x \approx a$ and $z \approx a$ for some $a \in A\backslash\{x, y, z\}$, and (iii) $x \approx y$ and $z \approx y$. According to case (i), we know that there exist $P_k \in \mathbb{D}^{x,z}$ and $P'_k \in \mathbb{D}^{z,x}$ such that $r_t(P_k) = r_t(P'_k)$, $t = 3, \ldots, m$. Firstly, by part (iii) of Condition SC, since $P'_k \in \mathbb{D}^z$ and xP'_ky, it must be the case that yP'_ka for all $a \in \mathcal{S}(y)$. Consequently, yP_ka for all $a \in \mathcal{S}(y)$, which contradicts part (ii) of Condition SC. In case (ii), it is evident that $a \in A\backslash[B \cup \{y\}]$. Since $x \approx a$, we know that there exist $P_k \in \mathbb{D}^{x,a}$ and $P'_k \in \mathbb{D}^{a,x}$ such that $r_t(P_k) = r_t(P'_k)$, $t = 3, \ldots, m$. By part (iii) of Condition SC, since $P'_k \in \mathbb{D}^a$ and xP'_ky, it must be the case that yP'_kb for all $b \in \mathcal{S}(y)$. Consequently, yP_kb for all $b \in \mathcal{S}(y)$, which contradicts part (ii) of Condition SC. Lastly, in case (iii), since $x \approx y$ and $z \approx y$, we know that there exist $P_k \in \mathbb{D}^{x,y}$ and $z \in \mathcal{S}(y)$, which contradicts part (ii) of Condition SC.

Assume $|B| > 1$. Fixing $x \in B$, there are three cases to consider: (i) $x \approx y$, (ii) $x \approx z$ and $y \approx z$ for some $z \in A\backslash\{x, y\}$ where $z \in B$, and (iii) $x \approx z$ and $y \approx z$ for some $z \in A\backslash\{x, y\}$ where $z \notin B$. According to case (i), we know that $x \in \mathcal{S}(y)$ and there exists $P_k \in \mathbb{D}^{x,y}$, which contradicts part (ii) of Condition SC. According to case (ii), we know that $z \in \mathcal{S}(y)$ and there exists $P_k \in \mathbb{D}^{z,y}$, which contradicts part (ii) of Condition SC. In case (iii), we know that $z \in \mathcal{S}(y)$ and there exists $P_k \in \mathbb{D}^{z,x}$, which contradicts part (iii) of Condition SC.

3 Proof of Theorem 2.3.1

Proof. To prove Theorem 2.3.1 we use a Ramification theorem (Theorem 5.1 in Appendix 5) which ensures that if a domain is a random dictatorship domain for two voters, it is also a random dictatorship domain for arbitrary number of voters. In addition to the minimal richness condition, the Ramification theorem requires another richness condition which we specify below.

Given $P_i \in \mathbb{D}$ and $a \in A$, let $B(P_i, a)$ denote the set of alternatives that are better than a according to P_i, i.e., $[x \in B(P_i, a)] \Rightarrow [xP_ia]$, while $W(P_i, a)$ denotes the set of alternatives that are worse than a according to P_i, i.e., $[x \in W(P_i, a)] \Rightarrow [aP_ix]$.

Definition 3.1. *A domain $\mathbb{D}$ satisfies Condition α if there exist three distinct alternatives $a, b, c \in A$; $P_1 \in \mathbb{D}^a$, $P_2 \in \mathbb{D}^b$ and $P_3 \in \mathbb{D}^c$ such that*

(i) bP_1c, cP_2a *and* aP_3b;

(ii) $W(P_1, b) \cup W(P_2, c) \cup W(P_3, a) = A$.

Every linked domain is minimally rich and satisfies Condition α.[1] Therefore, to verify Theorem 2.3.1, it suffices to show that every strategy-proof RSCF of two voters satisfying unanimity is a random dictatorship. Assume $I = \{i, j\}$. We first provide an independent lemma which is repeatedly applied in the following proof.

Lemma 3.1. *Consider a domain $\mathbb{D}$ and $a, b, c \in A$ such that $a \sim b$, $b \sim c$ and $c \sim a$. If $\varphi : \mathbb{D}^2 \to A$ is unanimous and strategy-proof, then there exists $\varepsilon \in [0, 1]$ such that $f(P_i, P_j) = \varepsilon e_{r_1(P_i)} + (1 - \varepsilon)e_{r_1(P_j)}$ for all $P_i, P_j \in \mathbb{D}^a \cup \mathbb{D}^b \cup \mathbb{D}^c$.*

Proof. This lemma follows from Theorem 2 in [48]. □

Now, let $\mathbb{D}$ be a linked domain satisfying Condition H and $\varphi : \mathbb{D}^2 \to A$ be a unanimous and strategy-proof RSCF. For simplicity, we assume that the one to one

[1]Let $\mathbb{D}$ be linked. We know that there exist $a, b, c \in A$ such that $a \sim b$, $b \sim c$ and $c \sim a$. Therefore, there exist $P_1 \in \mathbb{D}^{a,b}$, $P_2 \in \mathbb{D}^{b,c}$ and $P_3 \in \mathbb{D}^{c,a}$ which immediately imply (i) bP_1c, cP_2a and aP_3b; and (ii) $W(P_1, b) \cup W(P_2, c) \cup W(P_3, a) = A$. Thus linked domains satisfy Condition α. Another dictatorial domain is the circular domain studied in [43]. By a similar argument, one can verify that circular domains satisfy Condition α.

function σ in Definition 2.1.1 is the identity function and moreover, a_1 is a hub. Hence, $a_1 \sim x$ for all $x \in A\backslash\{a_1\}$. Next, let $S_l = \{a_1, \ldots, a_l\}$, $l = 3, \ldots, m$. Clearly, $a_1 \in S_l$, $3 \leq l \leq m$. Our proof consists in establishing two steps.

Step 1. There exists $\varepsilon \in [0,1]$ such that for all $P_i, P_j \in \mathbb{D}^{S_3}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.

Step 2. If for all $P_i, P_j \in \mathbb{D}^{S_{l-1}}$, $l > 3$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$, then for all $P_i, P_j \in \mathbb{D}^{S_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.

The following lemma establishes Step 1.

Lemma 3.2. *There exists $\varepsilon \in [0,1]$ such that for all $P_i, P_j \in \mathbb{D}^{S_3}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.*

Proof. Since $a_1 \sim a_2$, $a_2 \sim a_3$ and $a_3 \sim a_1$, Lemma 3.1 applies. □

To verify Step 2, we use the following induction hypothesis: for all $P_i, P_j \in \mathbb{D}^{S_{l-1}}$, $l > 3$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$. We will show that for all $P_i, P_j \in \mathbb{D}^{S_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$. Note that ε is fixed in the induction hypothesis. Since $\mathbb{D}$ is linked and a_1 is a hub, we know that there exists $a_k \in S_{l-1}$ such that $a_l \sim a_k$, $a_k \sim a_1$ and $a_l \sim a_1$. The next 3 lemmas explain the verification of Step 2.

Lemma 3.3. *For all $P_i, P_j \in \mathbb{D}^{a_1} \cup \mathbb{D}^{a_k} \cup \mathbb{D}^{a_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.*

Proof. Since $a_l \sim a_k$, $a_k \sim a_1$ and $a_l \sim a_1$, Lemma 3.1 implies that there exists $\beta \in [0,1]$ such that $\varphi(P_i, P_j) = \beta\, e_{r_1(P_i)} + (1-\beta)e_{r_1(P_j)}$ for all $P_i, P_j \in \mathbb{D}^{a_1} \cup \mathbb{D}^{a_k} \cup \mathbb{D}^{a_l}$. Meanwhile, by the induction hypothesis, we know $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$ for all $P_i, P_j \in \mathbb{D}^{a_1} \cup \mathbb{D}^{a_k}$. Therefore, $\varepsilon = \beta$. □

For the next lemma, pick any $a_j \in S_{l-1}\backslash\{a_1, a_k\}$. Condition H implies $a_j \sim a_1$.

Lemma 3.4. *For all $P_i^* \in \mathbb{D}^{a_l,a_1}$ and $P_j^* \in \mathbb{D}^{a_j,a_1}$, $\varphi_{a_l}(P_i^*, P_j^*) = \varepsilon$ and $\varphi_{a_j}(P_i^*, P_j^*) = 1 - \varepsilon$.*

Proof. We consider two situations.

Firstly, suppose $\varphi_{a_l}(P_i^*, P_j^*) = \beta$ and $\varphi_{a_j}(P_i^*, P_j^*) = 1 - \beta$. Since there exists $P_i' \in \mathbb{D}^{a_1,a_l}$ (recall $a_1 \sim a_l$), strategy-proofness and the induction hypothesis imply $\beta = \varphi_{a_l}(P_i^*, P_j^*) = \varphi_{a_l}(P_i^*, P_j^*) + \varphi_{a_1}(P_i^*, P_j^*) = \varphi_{a_l}(P_i', P_j^*) + \varphi_{a_1}(P_i', P_j^*) = \varphi_{a_1}(P_i', P_j^*) = \varepsilon$.

Secondly, suppose $\varphi_{a_l}(P_i^*, P_j^*) + \varphi_{a_j}(P_i^*, P_j^*) < 1$. Since there exist $P_i' \in \mathbb{D}^{a_1,a_l}$ and $P_j' \in \mathbb{D}^{a_1,a_j}$ (by Condition H), strategy-proofness, the induction hypothesis and Lemma 3.3 imply $\varphi_{a_l}(P_i^*, P_j^*) + \varphi_{a_1}(P_i^*, P_j^*) = \varphi_{a_l}(P_i', P_j^*) + \varphi_{a_1}(P_i', P_j^*) = \varepsilon$ and $\varphi_{a_j}(P_i^*, P_j^*) + \varphi_{a_1}(P_i^*, P_j^*) = \varphi_{a_j}(P_i^*, P_j') + \varphi_{a_1}(P_i^*, P_j') = 1 - \varepsilon$. Therefore, it must be the case that $\varphi_{a_1}(P_i^*, P_j^*) > 0$. Assume $\varphi_{a_1}(P_i^*, P_j^*) = \alpha > 0$. Then, $\varphi_{a_l}(P_i^*, P_j^*) = \varepsilon - \alpha$, $\varphi_{a_j}(P_i^*, P_j^*) = 1 - \varepsilon - \alpha$ and $\sum_{a_t \notin \{a_1, a_j, a_l\}} \varphi_{a_t}(P_i^*, P_j^*) = \alpha$. This implies that there exists $a_i \in A \backslash \{a_1, a_j, a_l\}$ such that $\varphi_{a_i}(P_i^*, P_j^*) > 0$.

By Condition H, there exists $P_k \in \mathbb{D}^{a_1,a_i}$. Let s, s' be such that $a_l = r_s(P_k)$ and $a_j = r_{s'}(P_k)$. We need to consider two cases.

Case 1: $s < s'$.

Let $\bar{P}_i = P_k$. By the induction hypothesis, $\varphi(\bar{P}_i, P_j^*) = \varepsilon\, e_{a_1} + (1 - \varepsilon)e_{a_j}$. Then, $\sum_{k=1}^{s} \varphi_{r_k(\bar{P}_i)}(\bar{P}_i, P_j^*) = \varepsilon < \varphi_{a_1}(P_i^*, P_j^*) + \varphi_{a_l}(P_i^*, P_j^*) + \varphi_{a_i}(P_i^*, P_j^*) \leq \sum_{k=1}^{s} \varphi_{r_k(\bar{P}_i)}(P_i^*, P_j^*)$. Therefore, voter i manipulates at $(\bar{P}_i, P_j^*)$ via P_i^*.

Case 2: $s > s'$.

Let $\bar{P}_j = P_k$. By Lemma 3.3, we have $\varphi(P_i^*, \bar{P}_j) = \varepsilon\, e_{a_l} + (1 - \varepsilon)e_{a_1}$. Then, $\sum_{k=1}^{s'} \varphi_{r_k(\bar{P}_j)}(P_i^*, \bar{P}_j) = 1 - \varepsilon < \varphi_{a_1}(P_i^*, P_j^*) + \varphi_{a_j}(P_i^*, P_j^*) + \varphi_{a_i}(P_i^*, P_j^*) \leq \sum_{k=1}^{s'} \varphi_{r_k(\bar{P}_j)}(P_i^*, P_j^*)$. Therefore, voter j manipulates at $(P_i^*, \bar{P}_j)$ via P_j^*.

Hence, both cases cannot occur. This establishes the lemma. □

Lemma 3.5. *The following two statements hold.*

(i) *For all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{S_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{a_l} + (1 - \varepsilon)e_{r_1(P_j)}$.*

(ii) *For all $P_i \in \mathbb{D}^{S_l}$ and $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1 - \varepsilon)e_{a_l}$.*

Proof. We verify part (i) first. Let $a_j \in S_{l-1}\backslash\{a_1, a_k\}$, $P_j \in \mathbb{D}^{a_j}$, $P_i^* \in \mathbb{D}^{a_l,a_1}$ and $P_j^* \in \mathbb{D}^{a_j,a_1}$. Strategy-proofness and Lemma 3.4 imply $\varphi_{a_j}(P_i^*, P_j) = \varphi_{a_j}(P_i^*, P_j^*) = 1-\varepsilon$ and $1-\varepsilon = \varphi_{a_j}(P_i^*, P_j^*) + \varphi_{a_1}(P_i^*, P_j^*) \geq \varphi_{a_j}(P_i^*, P_j) + \varphi_{a_1}(P_i^*, P_j)$. Therefore, $\varphi_{a_1}(P_i^*, P_j) = 0$.

Next, consider $P_i \in \mathbb{D}^{a_l}$ and $P_i' \in \mathbb{D}^{a_1,a_l}$. By strategy-proofness and the induction hypothesis, $\varphi_{a_l}(P_i, P_j) = \varphi_{a_l}(P_i^*, P_j) = \varphi_{a_l}(P_i^*, P_j) + \varphi_{a_1}(P_i^*, P_j) = \varphi_{a_l}(P_i', P_j) + \varphi_{a_1}(P_i', P_j) = \varphi_{a_1}(P_i', P_j) = \varepsilon$.

Similarly, for all $P_i \in \mathbb{D}^{a_l}$, we have $\varphi_{a_1}(P_i, P_j^*) = 0$. Let $P_j' \in \mathbb{D}^{a_1,a_j}$. Strategy-proofness and Lemma 3.3 imply $\varphi_{a_j}(P_i, P_j) = \varphi_{a_j}(P_i, P_j^*) = \varphi_{a_j}(P_i, P_j^*) + \varphi_{a_1}(P_i, P_j^*) = \varphi_{a_j}(P_i, P_j') + \varphi_{a_1}(P_i, P_j') = \varphi_{a_1}(P_i, P_j') = 1-\varepsilon$.

Therefore, $\varphi(P_i, P_j) = \varepsilon e_{a_l} + (1-\varepsilon)e_{a_j}$ for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_j}$ where $a_j \in S_{l-1}\backslash\{a_1, a_k\}$. By unanimity and Lemma 3.3, we conclude that $\varphi(P_i, P_j) = \varepsilon\, e_{a_l} + (1-\varepsilon)e_{r_1(P_j)}$ for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{S_l}$.

The proof of part (ii) is symmetric to that of part (i) and is therefore omitted. □

Therefore, by the induction hypothesis, we have proved that $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$ for all $P_i, P_j \in \mathbb{D}^{S_l}$. This completes the verification of Step 2 and hence the proof of the theorem. □

4 Proof of Theorem 2.3.2

Proof. In the view of the Ramification Theorem (Theorem 5.1 in Appendix 5), it once again suffices to show that every strategy-proof and unanimous RSCF of two voters is a random dictatorship. Let $\mathbb{D}$ be a strongly linked domain satisfying Condition TS and $I = \{i, j\}$. For notational simplicity, we assume that the function σ in Definition 2.3.3 is the identity function. Let $S_l = \{a_1, a_2, \ldots, a_l\}$, $l = 3, \ldots, m$. Our proof proceeds by establishing the same two steps as those in the proof of Theorem 2.3.1.

Step 1. There exists $\varepsilon \in [0, 1]$ such that for all $P_i, P_j \in \mathbb{D}^{S_3}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.

Step 2. If for all $P_i, P_j \in \mathbb{D}^{S_{l-1}}$, $l > 3$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$, then for all $P_i, P_j \in \mathbb{D}^{S_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.

The following lemma establishes Step 1.

Lemma 4.1. *There exists $\varepsilon \in [0,1]$ such that for all $P_i, P_j \in \mathbb{D}^{S_3}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.*

Proof. Since $a_1 \sim a_2$, $a_2 \sim a_3$ and $a_3 \sim a_1$ (strong connectedness implies the connectedness), Lemma 3.1 applies. $\square$

To verify Step 2, we use the following induction hypothesis.

Level 1 *Induction Hypothesis*: for all $P_i, P_j \in \mathbb{D}^{S_{l-1}}$, $l > 3$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.

We will show that for all $P_i, P_j \in \mathbb{D}^{S_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.

Fix $l > 3$ and $a \in S_{l-1}$. We say that a_l is strongly connected to a by a *chain of length t located in S_l* if there exists a sequence $\{y_k\}_{k=1}^{t+2} \subset S_l$ of length $t+2$ such that $a_l = y_1$, $a = y_{t+2}$ and $y_k \approx y_{k+1}$, $k = 1, \ldots, t+1$. We let $T_t(a_l, S_l)$ denote the set of the alternatives $a \in S_{l-1}$ satisfying the following two properties: (i) a is strongly connected to a_l by a chain of length t located in S_l and (ii) there does not exist a chain of length strictly less than t located in S_l connecting a_l and a.[2]

It is evident that $T_s(a_l, S_l) \cap T_{s'}(a_l, S_l) = \emptyset$ whenever $s \neq s'$. Moreover, it also follows that (i) $T_s(a_l, S_l) = \emptyset$ implies $T_{s'}(a_l, S_l) = \emptyset$ for all $s' > s$, (ii) $\cup_{t \geq 0} T_t(a_l, S_l) = S_{l-1}$ and (iii) if $a \in T_s(a_l, S_l)$ with $s > 0$, there exists $b \in T_{s-1}(a_l, S_l)$ such that $b \approx a$. The next lemma considers $T_0(a_l, S_l)$.

Lemma 4.2. *The following two statements hold.*

(i) *For all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{T_0(a_l, S_l)}$, $\varphi(P_i, P_j) = \varepsilon\, e_{a_l} + (1-\varepsilon)e_{r_1(P_j)}$.*

[2] We use an example to explain the chain. Let Figure 2.2(d) denote a strong connectivity graph of a strongly linked domain. Let the one to one function σ be the identity function. Considering a_1 and a_5, then $\{a_5, a_3, a_2, a_4, a_1\}$, $\{a_5, a_3, a_2, a_1\}$ and $\{a_5, a_3, a_1\}$ are chains of length 3, 2 and 1 located in S_5 respectively. Meanwhile, $T_0(a_5, S_5) = \{a_3, a_4\}$, $T_1(a_5, S_5) = \{a_1, a_2\}$ and $T_t(a_5, S_5) = \emptyset$ for all $t \geq 2$.

(ii) *For all* $P_i \in \mathbb{D}^{T_0(a_l,S_l)}$ *and* $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i,P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{a_l}$.

Proof. We show part (i) first. Let $a_s \in T_0(a_l,S_l)$. Since $a_l \sim a_s$ (recall that $[a_l \approx a_s] \Rightarrow [a_l \sim a_s]$), there must exist $\beta \in [0,1]$ such that for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_s}$, $\varphi(P_i,P_j) = \beta\, e_{a_l} + (1-\beta)e_{a_s}$. Now, pick $a_t \in T_0(a_l,S_l)\backslash\{a_s\}$;$\bar{P}_i \in \mathbb{D}^{a_l,a_t}$, $\bar{P}_i^* \in \mathbb{D}^{a_t,a_l}$ and $P_j \in \mathbb{D}^{a_s}$. Strategy-proofness and Level 1 induction hypothesis imply $\beta = \varphi_{a_l}(\bar{P}_i,P_j) = \varphi_{a_l}(\bar{P}_i,P_j) + \varphi_{a_t}(\bar{P}_i,P_j) = \varphi_{a_l}(\bar{P}_i^*,P_j) + \varphi_{a_t}(\bar{P}_i^*,P_j) = \varphi_{a_t}(\bar{P}_i^*,P_j) = \varepsilon$. By symmetric arguments, part (ii) also holds. □

To exhaust all alternatives in S_{l-1}, we provide another induction hypothesis.

Level 2 *Induction Hypothesis*: Fix $l \leq m$. Suppose that for all $0 \leq t' < t$ and either $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{\cup_{k=0}^{t'} T_k(a_l,S_l)}$; or $P_i \in \mathbb{D}^{\cup_{k=0}^{t'} T_k(a_l,S_l)}$ and $P_j \in \mathbb{D}^{a_l}$, we have $\varphi(P_i,P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.

We will show that for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{T_t(a_l,S_l)}$, or $P_i \in \mathbb{D}^{T_t(a_l,S_l)}$ and $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i,P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$.

Lemma 4.3. *The following two statements hold.*

(i) *For all* $P_i \in \mathbb{D}^{a_l}$ *and* $P_j \in \mathbb{D}^{T_t(a_l,S_l)}$, $\varphi(P_i,P_j) = \varepsilon\, e_{a_l} + (1-\varepsilon)e_{r_1(P_j)}$.

(ii) *For all* $P_i \in \mathbb{D}^{T_t(a_l,S_l)}$ *and* $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i,P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{a_l}$.

Proof. Pick $a_j \in T_t(a_l,S_l)$ with $t > 0$. According to Condition TS, there exists $a_i \in A$ such that $a_i \approx a_l$ and $a_i \approx a_j$. There are two cases to consider: $a_i \in S_{l-1}$ and $a_i \notin S_{l-1}$.[3] The proof of Lemma 4.3 follows the following 6 claims. We verify part (i) first. Claim 1 below consider $a_i \in S_{l-1}$.

Claim 1:

(i) For all $P_i \in \mathbb{D}^{a_l} \cup \mathbb{D}^{a_j}$ and $P_j \in \mathbb{D}^{a_i}$, $\varphi(P_i,P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{a_i}$.

[3] We provide an example to show both cases of $a_i \in S_{l-1}$ and $a_i \notin S_{l-1}$. Let Figure 2.2(e) denote the strong connectivity graph of a strongly linked domain. Then, the domain satisfies Condition TS. Furthermore, it is true that for every one to one function $\sigma : \{1,\dots,7\} \to \{1,\dots,7\}$ satisfied by a domain in Definition 2.3.3, $a_7 = a_{\sigma(7)}$. Let function σ be the identity function. We first consider a_1, a_5 and S_4. We know $a_1 \approx a_3$, $a_3 \approx a_5$ and $a_3 \in S_4$. Next, considering a_1, a_6 and S_5, we know $a_1 \approx a_7$, $a_7 \approx a_6$ and $a_7 \notin S_5$.

(ii) For all $P_i \in \mathbb{D}^{a_i}$ and $P_j \in \mathbb{D}^{a_l} \cup \mathbb{D}^{a_j}$, $\varphi(P_i, P_j) = \varepsilon\, e_{a_i} + (1-\varepsilon)e_{r_1(P_j)}$.

Since $a_i \in S_{l-1}$, it must be the case that $a_i \in T_0(a_l, S_l)$ and $a_j \in T_1(a_l, S_l)$. The claim then follows from Lemma 4.2 and the Level 1 induction hypothesis. This completes the verification of Claim 1.

Next, we will show that the same conclusions hold when $a_i \notin S_{l-1}$. Now, it must be the case that $a_j \in T_t(a_l, S_l)$ where $t > 1$. Since $a_i \notin S_{l-1}$, we can assume $a_l \approx a_s$, where $a_s \in T_0(a_l, S_l)$ (by Definition 2.3.3) and $a_j \approx a_k$, where $a_k \in T_{t-1}(a_l, S_l)$, $t > 1$ (by property (iii) of $T_t(a_l, S_l)$ above). Since $t > 1$, it is evident that $a_s \neq a_k$. The next three claims assume $a_i \notin S_{l-1}$.

Claim 2:

(i) For *some* $\bar{P}_i \in \mathbb{D}^{a_l,a_s}$ and $\bar{P}_j \in \mathbb{D}^{a_j,a_k}$, $\varphi(\bar{P}_i, \bar{P}_j) = \varepsilon\, e_{a_l} + (1-\varepsilon)e_{a_j}$.

(ii) For *some* $\bar{P}_i \in \mathbb{D}^{a_j,a_k}$ and $\bar{P}_j \in \mathbb{D}^{a_l,a_s}$, $\varphi(\bar{P}_i, \bar{P}_j) = \varepsilon\, e_{a_j} + (1-\varepsilon)e_{a_l}$.

We first consider part (i). By strong connectedness, we can assume that there exist $P'_i \in \mathbb{D}^{a_s,a_l}$ and $P'_j \in \mathbb{D}^{a_k,a_j}$ such that $r_\nu(P'_i) = r_\nu(\bar{P}_i)$ and $r_\nu(P'_j) = r_\nu(\bar{P}_j)$, $\nu = 3, \dots, m$. Now, since $a_s, a_j \in S_{l-1}$, by strategy-proofness and the Level 1 induction hypothesis, we have $\varphi_{a_l}(\bar{P}_i, \bar{P}_j) + \varphi_{a_s}(\bar{P}_i, \bar{P}_j) = \varphi_{a_l}(P'_i, \bar{P}_j) + \varphi_{a_s}(P'_i, \bar{P}_j) = \varphi_{a_s}(P'_i, \bar{P}_j) = \varepsilon$. Similarly, since $a_k \in T_{t-1}(a_l, S_l)$, by strategy-proofness and the Level 2 induction hypothesis, $\varphi_{a_j}(\bar{P}_i, \bar{P}_j) + \varphi_{a_k}(\bar{P}_i, \bar{P}_j) = \varphi_{a_j}(\bar{P}_i, P'_j) + \varphi_{a_k}(\bar{P}_i, P'_j) = \varphi_{a_j}(\bar{P}_i, P'_j) = 1-\varepsilon$. Therefore, for all $a \notin \{a_l, a_j, a_s, a_k\}$, $\varphi_a(\bar{P}_i, \bar{P}_j) = 0$.

Suppose $\varphi_{a_s}(\bar{P}_i, \bar{P}_j) = \alpha > 0$. Then, $\varphi_{a_l}(\bar{P}_i, \bar{P}_j) = \varepsilon - \alpha$. Assume $a_l = r_{k_1}(\bar{P}_j)$ and $a_s = r_{k_2}(\bar{P}_j)$. Then, $a_l = r_{k_1}(P'_j)$ and $a_s = r_{k_2}(P'_j)$. We have two cases.

Case 1: $k_1 < k_2$.

Fix $P_j \in \mathbb{D}^{a_l}$. By unanimity, $\varphi_{a_l}(\bar{P}_i, P_j) = 1$. Hence, $\sum_{\nu=1}^{k_1} \varphi_{r_\nu(\bar{P}_j)}(\bar{P}_i, \bar{P}_j) = \varphi_{a_j}(\bar{P}_i, \bar{P}_j) + \varphi_{a_k}(\bar{P}_i, \bar{P}_j) + \varphi_{a_l}(\bar{P}_i, \bar{P}_j) = 1 - \alpha < \sum_{\nu=1}^{k_1} \varphi_{r_\nu(\bar{P}_j)}(\bar{P}_i, P_j)$. Then, voter j would manipulate at $(\bar{P}_i, \bar{P}_j)$ via P_j.

Case 2: $k_1 > k_2$.

By the Level 2 induction hypothesis, $\sum_{\nu=1}^{k_2} \varphi_{r_\nu(P'_j)}(\bar{P}_i, P'_j) = \varphi_{a_k}(\bar{P}_i, P'_j) = 1-\varepsilon < 1-\varepsilon+\alpha = \varphi_{a_j}(\bar{P}_i, \bar{P}_j) + \varphi_{a_k}(\bar{P}_i, \bar{P}_j) + \varphi_{a_s}(\bar{P}_i, \bar{P}_j) = \sum_{\nu=1}^{k_2} \varphi_{r_\nu(P'_j)}(\bar{P}_i, \bar{P}_j)$.

Then, voter j would manipulate at $(\bar{P}_i, P'_j)$ via $\bar{P}_j$.

Now, $\varphi_{a_s}(\bar{P}_i, \bar{P}_j) = 0$. Next, suppose $\varphi_{a_k}(\bar{P}_i, \bar{P}_j) = \alpha > 0$. Then, $\varphi_{a_j}(\bar{P}_i, \bar{P}_j) = 1 - \varepsilon - \alpha$. Assume $a_j = r_{t_1}(\bar{P}_i)$ and $a_k = r_{t_2}(\bar{P}_i)$. Then, $a_j = r_{t_1}(P'_i)$ and $a_k = r_{t_2}(P'_i)$. We have two cases.

Case 1: $t_1 < t_2$.

Fix $P_i \in \mathbb{D}^{a_j}$. By unanimity, $\varphi_{a_j}(P_i, \bar{P}_j) = 1$. Hence, $\sum_{\nu=1}^{t_1} \varphi_{r_\nu(\bar{P}_i)}(\bar{P}_i, \bar{P}_j) = \varphi_{a_l}(\bar{P}_i, \bar{P}_j) + \varphi_{a_s}(\bar{P}_i, \bar{P}_j) + \varphi_{a_j}(\bar{P}_i, \bar{P}_j) = 1 - \alpha < \sum_{\nu=1}^{t_1} \varphi_{r_\nu(\bar{P}_i)}(P_i, \bar{P}_j)$. Then, voter i would manipulate at $(\bar{P}_i, \bar{P}_j)$ via P_i.

Case 2: $t_1 > t_2$.

By the Level 1 induction hypothesis, $\sum_{\nu=1}^{t_2} \varphi_{r_\nu(P'_i)}(P'_i, \bar{P}_j) = \varphi_{a_s}(P'_i, \bar{P}_j) = \varepsilon < \varepsilon + \alpha = \varphi_{a_l}(\bar{P}_i, \bar{P}_j) + \varphi_{a_s}(\bar{P}_i, \bar{P}_j) + \varphi_{a_k}(\bar{P}_i, \bar{P}_j) = \sum_{\nu=1}^{t_2} \varphi_{r_\nu(P'_i)}(\bar{P}_i, \bar{P}_j)$. Then, voter i would manipulate at $(P'_i, \bar{P}_j)$ via $\bar{P}_i$.

Then, $\varphi_{a_k}(\bar{P}_i, \bar{P}_j) = 0$. Therefore, $\varphi(\bar{P}_i, \bar{P}_j) = \varepsilon\, e_{a_l} + (1 - \varepsilon) e_{a_j}$.

By symmetric arguments, part (ii) also holds. This completes the verification of Claim 2.

Claim 3:

(i) For all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_i}$, $\varphi(P_i, P_j) = \varepsilon\, e_{a_l} + (1 - \varepsilon) e_{a_i}$.

(ii) For all $P_i \in \mathbb{D}^{a_i}$ and $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{a_i} + (1 - \varepsilon) e_{a_l}$.

We first consider part (i). Since $a_l \sim a_i$ (recall $[a_l \approx a_i] \Rightarrow [a_l \sim a_i]$), there must exist $\beta \in [0, 1]$ such that for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_i}$, $\varphi(P_i, P_j) = \beta\, e_{a_l} + (1 - \beta) e_i$.

Next, fix $\bar{P}_i \in \mathbb{D}^{a_l, a_s}$, $\bar{P}_j \in \mathbb{D}^{a_j, a_k}$, where profile $(\bar{P}_i, \bar{P}_j)$ satisfies Claim 2(i), $P^*_i \in \mathbb{D}^{a_s, a_l}$ and $P^*_j \in \mathbb{D}^{a_j, a_i}$. Since $a_s, a_j \in S_{l-1}$, by strategy-proofness and the Level 1 induction hypothesis, we have $\varphi_{a_l}(\bar{P}_i, P^*_j) + \varphi_{a_s}(\bar{P}_i, P^*_j) = \varphi_{a_l}(P^*_i, P^*_j) + \varphi_{a_s}(P^*_i, P^*_j) = \varphi_{a_s}(P^*_i, P^*_j) = \varepsilon$. Meanwhile, by strategy-proofness and Claim 2(i), $\varphi_{a_j}(\bar{P}_i, P^*_j) = \varphi_{a_j}(\bar{P}_i, \bar{P}_j) = 1 - \varepsilon$. Therefore, $\varphi_{a_i}(\bar{P}_i, P^*_j) = 0$. Now, fix $\bar{P}^*_j \in \mathbb{D}^{a_i, a_j}$. Strategy-proofness implies $1 - \beta = \varphi_{a_i}(\bar{P}_i, \bar{P}^*_j) = \varphi_{a_i}(\bar{P}_i, \bar{P}^*_j) + \varphi_{a_j}(\bar{P}_i, \bar{P}^*_j) = \varphi_{a_i}(\bar{P}_i, P^*_j) + \varphi_{a_j}(\bar{P}_i, P^*_j) = \varphi_{a_j}(\bar{P}_i, P^*_j) = 1 - \varepsilon$. Therefore, $\beta = \varepsilon$.

In conclusion, for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_i}$, $\varphi(P_i, P_j) = \varepsilon\, e_{a_l} + (1 - \varepsilon) e_{a_i}$.

By symmetric arguments, we have that for all $P_i \in \mathbb{D}^{a_i}$ and $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{a_i} + (1-\varepsilon)e_{a_l}$. This completes the verification of Claim 3.

Claim 4:

(i) For all $P_i \in \mathbb{D}^{a_j}$ and $P_j \in \mathbb{D}^{a_i}$, $\varphi(P_i, P_j) = \varepsilon\, e_{a_j} + (1-\varepsilon)e_{a_i}$.

(ii) For all $P_i \in \mathbb{D}^{a_i}$ and $P_j \in \mathbb{D}^{a_j}$, $\varphi(P_i, P_j) = \varepsilon\, e_{a_i} + (1-\varepsilon)e_{a_j}$.

This Claim is similar to Claim 3 but its proof follows from Claim 2 and the Level 2 induction hypothesis, while the proof for Claim 3 follows from Claim 2 and the Level 1 induction hypothesis. This completes the verification of Claim 4.

We have shown that irrespective of whether $a_i \in S_{l-1}$ or $a_i \notin S_{l-1}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$ holds for all $P_i \in \mathbb{D}^{a_l} \cup \mathbb{D}^{a_j}$ and $P_j \in \mathbb{D}^{a_i}$, or $P_i \in \mathbb{D}^{a_i}$ and $P_j \in \mathbb{D}^{a_l} \cup \mathbb{D}^{a_j}$.

Claim 5: For all $P_i^* \in \mathbb{D}^{a_l, a_i}$ and $P_j^* \in \mathbb{D}^{a_j, a_i}$, $\varphi(P_i^*, P_j^*) = \varepsilon\, e_{a_l} + (1-\varepsilon)e_{a_j}$.

Suppose that the Claim is false. Similar to Lemma 3.4, we can assume $\varphi_{a_i}(P_i^*, P_j^*) = \alpha > 0$. Since $a_l \approx a_i$ and $a_j \approx a_i$, we can assume that there exist $\bar{P}_i^* \in \mathbb{D}^{a_i, a_l}$ and $\bar{P}_j^* \in \mathbb{D}^{a_i, a_j}$ such that $r_k(\bar{P}_j^*) = r_k(P_j^*)$, $k = 3, \ldots, m$. Since Claims 1, 3(i) and 4(ii) imply $\varphi_{a_i}(P_i^*, \bar{P}_j^*) + \varphi_{a_j}(P_i^*, \bar{P}_j^*) = 1 - \varepsilon$ and $\varphi_{a_i}(\bar{P}_i^*, P_j^*) + \varphi_{a_l}(\bar{P}_i^*, P_j^*) = \varepsilon$, by strategy-proofness, we have $\varphi_{a_j}(P_i^*, P_j^*) = 1 - \varepsilon - \alpha$ and $\varphi_{a_l}(P_i^*, P_j^*) = \varepsilon - \alpha$. Assume $a_l = r_s(P_j^*)$. It is evident that $s \geq 3$. Then, strong connectedness implies $a_l = r_s(\bar{P}_j^*)$. According to Claims 1 and 3(i), $\sum_{k=1}^{s-1} \varphi_{r_k(\bar{P}_j^*)}(P_i^*, \bar{P}_j^*) = 1 - \varepsilon$. Next, by strong connectedness, we have $\{r_k(P_j^*)\}_{k=1}^{s-1} = \{r_k(\bar{P}_j^*)\}_{k=1}^{s-1}$. Hence, by strategy-proofness, $\sum_{k=1}^{s-1} \varphi_{r_k(P_j^*)}(P_i^*, P_j^*) = \sum_{k=1}^{s-1} \varphi_{r_k(\bar{P}_j^*)}(P_i^*, \bar{P}_j^*) = 1 - \varepsilon$. Therefore, $\sum_{k=1}^{s} \varphi_{r_k(P_j^*)}(P_i^*, P_j^*) = \sum_{k=1}^{s-1} \varphi_{r_k(P_j^*)}(P_i^*, P_j^*) + \varphi_{a_l}(P_i^*, P_j^*) = 1 - \alpha$. Now, fix $P_j \in \mathbb{D}^{a_l}$. By unanimity, $\sum_{k=1}^{s} \varphi_{r_k(P_j^*)}(P_i^*, P_j^*) = 1 - \alpha < 1 = \varphi_{a_l}(P_i^*, P_j) = \sum_{k=1}^{s} \varphi_{r_k(P_j^*)}(P_i^*, P_j)$. Consequently, voter j manipulates at (P_i^*, P_j^*) via P_j. This completes the verification of Claim 5.

Claim 6: For all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{a_j}$, we have $\varphi(P_i, P_j) = \varepsilon\, e_{a_l} + (1-\varepsilon)e_{a_j}$.

The proof of the claim follows from a symmetric argument in Lemma 3.5.

By symmetric arguments, it follows that $\varphi(P_i, P_j) = \varepsilon\, e_{a_j} + (1-\varepsilon)e_{a_l}$ for all $P_i \in \mathbb{D}^{a_j}$ and $P_j \in \mathbb{D}^{a_l}$. This completes the proof of Lemma 4.3. $\square$

We can now complete the proof of the Theorem. We have shown that under the Level 1 induction hypothesis, the Level 2 induction hypothesis is established. With unanimity, this implies that for all $P_i \in \mathbb{D}^{a_l}$ and $P_j \in \mathbb{D}^{S_l}$, or $P_i \in \mathbb{D}^{S_l}$ and $P_j \in \mathbb{D}^{a_l}$, $\varphi(P_i, P_j) = \varepsilon\, e_{r_1(P_i)} + (1-\varepsilon)e_{r_1(P_j)}$ as required, to complete Step 2. $\square$

5 Ramification Theorem

Theorem 5.1. *Let $\mathbb{D}$ be minimally rich and satisfy Condition α. The following two statements are equivalent:*

(*a*) $\varphi : \mathbb{D}^2 \to \Delta(A)$ *is unanimous and strategy-proof*

$\Rightarrow \varphi$ *is a random dictatorship.*

(*b*) $\varphi : \mathbb{D}^N \to \Delta(A)$*,* $N \geq 2$*, is unanimous and strategy-proof*

$\Rightarrow \varphi$ *is a random dictatorship.*

Technically, we construct the following definition which serves as a critical bridge in the proof of Theorem 5.1.

Definition 5.1. *A unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a quasi-random dictatorship, if there exists $[\varepsilon_i]_{i\in I} \in \mathbb{R}^N_+$ with $\sum_{i\in I} \varepsilon_i = 1$ such that for all $a \in A$ and $P \in \mathbb{D}^N$ with $P_i = P_j$ for some distinct $i, j \in I$, $\varphi_a(P) = \sum\limits_{k\in I: r_1(P_k)=a} \varepsilon_k$.*

The random dictatorship is stronger than quasi-random dictatorship, for quasi-random dictatorship only considers those profiles of preferences with at least two voters sharing a same preference order and the outcome under such a profile of preferences is a convex combination of N (deterministic) dictatorial social choice functions with respect to an N-dimensional sequence $[\varepsilon_i]_{i\in I}$.

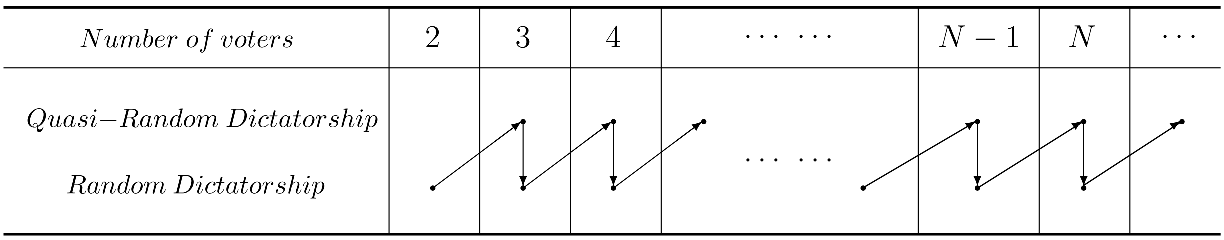
We first provide the outline of the proof of Theorem 5.1. The proof of $(b) \Rightarrow (a)$ in Theorem 5.1 is trivial. We focus on showing $(a) \Rightarrow (b)$. The proof consists in establishing following three steps.

Step 1. Let domain $\mathbb{D}$ satisfy minimal richness condition and Condition α. Every unanimous and strategy-proof RSCF $g : \mathbb{D}^2 \to \Delta(A)$ is a random dictatorship $\Rightarrow$ every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^3 \to \Delta(A)$ is a quasi-random dictatorship. This is shown in Proposition 5.1. □

Step 2. Let domain $\mathbb{D}$ satisfy minimal richness condition. Every unanimous and strategy-proof RSCF $g : \mathbb{D}^{N-1} \to \Delta(A)$, $N > 3$, is a random dictatorship $\Rightarrow$ every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a quasi-random dictatorship. This is shown in Proposition 5.2. □

Step 3. Let domain $\mathbb{D}$ satisfy minimal richness condition. Suppose for all $2 \leq t < N$, every unanimous and strategy-proof RSCF $g : \mathbb{D}^t \to \Delta(A)$, is a random dictatorship. A unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a quasi-random dictatorship $\Rightarrow$ φ is a random dictatorship. This is shown in Proposition 5.3. □

Note that the three steps above are independent and Condition α is only used in extending the random dictatorship result from the case of two voters to the case of three voters.[4] Thus, the three steps together solve the ramification problem in the way shown by the arrows in the diagram below.



[4]As stated in Steps 2 and 3, the Ramification theorem remains valid from the case of *three* voters to the case of $N \geq 3$ voters without Condition α. When $m = 3$, the ramification theorem holds without Condition α. Let $|A| = 3$ and the minimally rich domain $\mathbb{D}$ satisfy part (a) in Theorem 5.1. Suppose that $\mathbb{D}$ violates Condition α. Then, it is true that $|\mathbb{D}^x| = 1$ for some $x \in A$. Consequently, domain $\mathbb{D}$ satisfies the unique seconds property in [1], and hence is not a random dictatorship domain two voters. Contradiction! We conjecture the Ramification theorem is true without Condition α when the cardinality of the set of alternatives is greater than three.

Now, we start the proof. We first introduce some new notation that we shall be using throughout the proof. Given $P_i \in \mathbb{D}$ and a nonempty subset $S \subseteq A$, let $\max(P_i, S)$ and $\min(P_i, S)$ denote the most and worst preferred alternatives respectively in S according to P_i. Given $i \in I$ and $P \in \mathbb{D}^N$, let $\max(P_i, r_1(P_{-i}))$ denote the most preferred alternative in $r_1(P_{-i})$ according to P_i. Given $P \in \mathbb{D}^N$ with $|r_1(P)| = N$, let $\overline{W}(P) = \cup_{i\in I} W\big(P_i, \max(P_i, r_1(P_{-i}))\big)$. Additionally, for all $a, b \in A$, let $I(a, b)$ be the indicator function, where $I(a, b) = 1$ if $a = b$; and $I(a, b) = 0$ if $a \neq b$.

Proposition 5.1. *Let $\mathbb{D}$ be minimally rich and satisfy Condition α. Suppose that every unanimous and strategy-proof RSCF $g : \mathbb{D}^2 \to \Delta(A)$ is a random dictatorship. Then every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^3 \to \Delta(A)$ is a quasi-random dictatorship.*

Proof. Define three RSCFs as follows: $g^{(2,3)}(P_1, P_2) = \varphi(P_1, P_2, P_2)$, $g^{(1,3)}(P_1, P_2) = \varphi(P_1, P_2, P_1)$ and $g^{(1,2)}(P_1, P_3) = \varphi(P_1, P_1, P_3)$ for all $P_1, P_2, P_3 \in \mathbb{D}$. It is easy to verify that $g^{(2,3)}$, $g^{(1,3)}$ and $g^{(1,2)}$ are unanimous and strategy-proof (see Lemma 3 in [48]). Hence, the hypothesis of Proposition 5.1 implies that $g^{(2,3)}$, $g^{(1,3)}$ and $g^{(1,2)}$ are random dictatorships. Then, there exist ε_1, ε_2, $\varepsilon_3 \geq 0$ such that for all $P_1, P_2, P_3 \in \mathbb{D}$,

$$\begin{aligned}
\varphi(P_1, P_2, P_2) &= \varepsilon_1\, e_{r_1(P_1)} + (1 - \varepsilon_1)e_{r_1(P_2)},\\
\varphi(P_1, P_2, P_1) &= (1 - \varepsilon_2)e_{r_1(P_1)} + \varepsilon_2\, e_{r_1(P_2)},\\
\varphi(P_1, P_1, P_3) &= (1 - \varepsilon_3)e_{r_1(P_1)} + \varepsilon_3\, e_{b(r_1(P_3))}.
\end{aligned}$$

To establish that φ is a quasi-random dictatorship, it suffices to show that $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 1$. Since $\mathbb{D}$ satisfies Condition α, we can fix a profile $P^* = (P^*_1, P^*_2, P^*_3)$, where $r_1(P^*_1) = a$, $r_1(P^*_2) = b$, $r_1(P^*_3) = c$; bP^*_1c, cP^*_2a and aP^*_3b; and $W(P^*_1, b) \cup W(P^*_2, c) \cup W(P^*_3, a) = A$. Hence, $\overline{W}(P^*) = A$ and $r_1(P^*) \subset \overline{W}(P^*)$. Furthermore, we assume $b = r_s(P^*_1)$ and $c = r_{s'}(P^*_1)$. Hence, $1 < s < s'$. By strategy-proofness, we know $\sum_{k=1}^{t} \varphi_{r_k(P_1^*)}(P_2^*, P_2^*, P_3^*) \leq \sum_{k=1}^{t} \varphi_{r_k(P_1^*)}(P^*) \leq \sum_{k=1}^{t} \varphi_{r_k(P_1^*)}(P_1^*, P_1^*, P_3^*)$ for all $t \geq 1$. Since $\varphi(P_2^*, P_2^*, P_3^*) = g^{(1,2)}(P_2^*, P_3^*)$ and $\varphi(P_1^*, P_1^*, P_3^*) = g^{(1,2)}(P_1^*, P_3^*)$, we have $\sum_{k=1}^{t} g^{(1,2)}_{r_k(P_1^*)}(P_2^*, P_3^*) \leq \sum_{k=1}^{t} \varphi_{r_k(P_1^*)}(P^*) \leq \sum_{k=1}^{t} g^{(1,2)}_{r_k(P_1^*)}(P_1^*, P_3^*)$ for all $t \geq 1$.

Next, since $g^{(1,2)}$ is a random dictatorship with respect to $\{1-\varepsilon_3, \varepsilon_3\}$, we have

$$\sum_{k=1}^{s} g^{(1,2)}_{r_k(P_1^*)}(P_2^*, P_3^*) = \sum_{k=1}^{s'-1} g^{(1,2)}_{r_k(P_1^*)}(P_2^*, P_3^*) = g^{(1,2)}_{b}(P_2^*, P_3^*) = 1-\varepsilon_3,$$

$$\sum_{k=1}^{s} g^{(1,2)}_{r_k(P_1^*)}(P_1^*, P_3^*) = \sum_{k=1}^{s'-1} g^{(1,2)}_{r_k(P_1^*)}(P_1^*, P_3^*) = g^{(1,2)}_{a}(P_1^*, P_3^*) = 1-\varepsilon_3,$$

$$\sum_{k=1}^{s'} g^{(1,2)}_{r_k(P_1^*)}(P_2^*, P_3^*) = g^{(1,2)}_{b}(P_2^*, P_3^*) + g^{(1,2)}_{c}(P_2^*, P_3^*) = 1,$$

$$\sum_{k=1}^{s'} g^{(1,2)}_{r_k(P_1^*)}(P_1^*, P_3^*) = g^{(1,2)}_{a}(P_1^*, P_3^*) + g^{(1,2)}_{c}(P_1^*, P_3^*) = 1.$$

Therefore, $\sum_{k=1}^{s} \varphi_{r_k(P_1^*)}(P^*) = \sum_{k=1}^{s'-1} \varphi_{r_k(P_1^*)}(P^*) = 1-\varepsilon_3$ and $\sum_{k=1}^{s'} \varphi_{r_k(P_1^*)}(P^*) = 1$. Hence, $\varphi_c(P^*) = \sum_{k=1}^{s'} \varphi_{r_k(P_1^*)}(P^*) - \sum_{k=1}^{s'-1} \varphi_{r_k(P_1^*)}(P^*) = \varepsilon_3$ and $\sum_{k=1}^{s} \varphi_{r_k(P_1^*)}(P^*) + \varphi_c(P^*) = 1$. Then, we know that for all $x \in W(P_1^*, b)\backslash\{c\}$, $\varphi_x(P^*) = 0$. Symmetrically, we can obtain $\varphi_a(P^*) = \varepsilon_1$, $\varphi_x(P^*) = 0$ for all $x \in W(P_2^*, c)\backslash\{a\}$; and $\varphi_b(P^*) = \varepsilon_2$, $\varphi_x(P^*) = 0$ for all $x \in W(P_3^*, a)\backslash\{b\}$. In conclusion, for all $x \in \overline{W}(P^*)\backslash\{a,b,c\}$, $\varphi_x(P^*) = 0$. Furthermore, since $\overline{W}(P^*) = A$, we have $1 = \sum_{x\in A} \varphi_x(P^*) = \sum_{x\in\overline{W}(P^*)} \varphi_x(P^*) = \varphi_a(P^*) + \varphi_b(P^*) + \varphi_c(P^*) = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. This completes the verification of Proposition 5.1. □

Proposition 5.2. *Let $\mathbb{D}$ be a minimally rich domain. Suppose that every unanimous and strategy-proof RSCF $g : \mathbb{D}^{N-1} \to \mathcal{L}(A)$ is a random dictatorship for $N >$ 3. Then every unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^{N} \to \Delta(A)$ is a quasi-random dictatorship.*

Proof. This proposition holds when $m = 3$, since a domain with exactly three alternatives is a random dictatorship domain of $N-1$ voters iff it is the complete domain.[5] We therefore consider $m \geq 4$. The proof of the Proposition follows from Lemmas 5.1 - 5.4.

Let $\varphi : \mathbb{D}^N \to \Delta(A)$ be a unanimous and strategy-proof RSCF. Pick two arbitrary voters, say i and j. Define a RSCF $g^{(i,j)}$ as for all $P_i \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, $g^{(i,j)}(P_i, P_{-\{i,j\}}) = \varphi(P_i, P_i, P_{-\{i,j\}})$.

Lemma 5.1. *The RSCF $g^{(i,j)}$ is a random dictatorship for all $i, j \in I$.*

Proof. Unanimity and strategy-proofness of φ imply that $g^{(i,j)}$ is unanimous and strategy-proof (see Lemma 3 in [48]). Then by the hypothesis of Proposition 5.2, $g^{(i,j)}$ is a random dictatorship. $\square$

Fix $i, j \in I$. It follows from Lemma 5.1 above that there exist $\varepsilon^{(i,j)}$, $\varepsilon_k^{(i,j)} \geq 0$ for all $k \neq i, j$ such that $\varepsilon^{(i,j)} + \sum_{k \neq i,j} \varepsilon_k^{(i,j)} = 1$ and satisfying the following property: $\varphi(P_i, P_i, P_{-\{i,j\}}) = g^{(i,j)}(P_i, P_{-\{i,j\}}) = \varepsilon^{(i,j)} e_{r_1(P_i)} + \sum_{k \neq i,j} \varepsilon_k^{(i,j)} e_{r_1(P_k)}$ for all $P_i \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. The next lemma shows that we can split the probability $\varepsilon^{(i,j)}$ appropriately into two parts and together with all $\varepsilon_k^{(i,j)}$, $k \neq i, j$, construct a new N-dimensional sequence of probabilities, which are able to be applied to all profiles of preferences where voter i and j share a same preference order.

Lemma 5.2. *Pick $i, j \in I$. For all $P \in \mathbb{D}^N$ with $P_i = P_j$ there exists $[\alpha_k^{(i,j;\,s,t)}]_{k=1}^N \geq 0$ with $\sum_{k=1}^N \alpha_k^{(i,j;\,s,t)} = 1$, where $s, t \in I \backslash \{i, j\}$ and $s \neq t$, such that $\varphi(P) = \sum_{k=1}^N \alpha_k^{(i,j;\,s,t)} e_{r_1(P_k)}$.*

Proof. Now, i, j, s, t are mutually distinct. For every $l \neq i, j, s, t$, we consider a profile $P^{(l)} = (P_i, P_i, P_s, P_s, P_l, P_{-\{i,j,s,t,l\}})$ where $r_1(P_i) = a$, $r_1(P_s) = b$, $r_1(P_l) = c$ and $r_1(P_{-\{i,j,s,t,l\}}) \cap \{a, b, c\} = \emptyset$ (recall that $m \geq 4$).[6] Standard properties of $g^{(i,j)}$ imply $\varphi_a(P^{(l)}) = \varepsilon^{(i,j)}$, $\varphi_b(P^{(l)}) = \varepsilon_s^{(i,j)} + \varepsilon_t^{(i,j)}$ and $\varphi_c(P^{(l)}) = \varepsilon_l^{(i,j)}$. Meanwhile, by $g^{(s,t)}$, $\varphi_a(P^{(l)}) = \varepsilon_i^{(s,t)} + \varepsilon_j^{(s,t)}$, $\varphi_b(P^{(l)}) = \varepsilon^{(s,t)}$ and $\varphi_c(P^{(l)}) = \varepsilon_l^{(s,t)}$.

[5]The sufficiency part is shown in [26], [21] and [48]. The unique seconds property in [1] implies the necessity. Let a domain satisfy the unique seconds property. Then this domain is not dictatorial and hence not a random dictatorship domain. Furthermore, when $m = 3$, every domain other than the universal domain satisfies the unique seconds property.

[6]If $N = 4$, let $P = (P_i, P_i, P_s, P_s)$ where $r_1(P_i) = a$ and $r_1(P_s) = b$.

Therefore, $\varepsilon^{(i,j)} = \varepsilon_i^{(s,t)} + \varepsilon_j^{(s,t)}$, $\varepsilon^{(s,t)} = \varepsilon_s^{(i,j)} + \varepsilon_t^{(i,j)}$ and $\varepsilon_l^{(i,j)} = \varepsilon_l^{(s,t)}$ for all $l \neq i,j,s,t$. Since $\varepsilon^{(i,j)} + \sum_{k \neq i,j} \varepsilon_k^{(i,j)} = 1$ and $\varepsilon^{(s,t)} + \sum_{k \neq s,t} \varepsilon_k^{(s,t)} = 1$, we have $\varepsilon_i^{(s,t)} + \varepsilon_j^{(s,t)} + \sum_{k \neq i,j} \varepsilon_k^{(i,j)} = 1$ and $\varepsilon_s^{(i,j)} + \varepsilon_t^{(i,j)} + \sum_{k \neq s,t} \varepsilon_k^{(s,t)} = 1$.

Setting $\alpha_i^{(i,j;\,s,t)} = \varepsilon_i^{(s,t)}$, $\alpha_j^{(i,j;\,s,t)} = \varepsilon_j^{(s,t)}$, $\alpha_s^{(i,j;\,s,t)} = \varepsilon_s^{(i,j)}$, $\alpha_t^{(i,j;\,s,t)} = \varepsilon_t^{(i,j)}$ and $\alpha_l^{(i,j;\,s,t)} = \varepsilon_l^{(s,t)} = \varepsilon_l^{(i,j)}$ for all $l \neq i,j,s,t$, we have $\alpha_k^{(i,j;\,s,t)} \geq 0$, $k = 1, \ldots, N$, and $\sum_{k=1}^{N} \alpha_k^{(i,j;\,s,t)} = 1$.

Fix $P = (P_i, P_j, P_{-\{i,j\}})$ with $P_i = P_j \in \mathbb{D}$ and $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$. It follows from properties of $g^{(i,j)}$ that $\varphi_{r_1(P_i)}(P) = \varepsilon^{(i,j)} + \sum_{k \neq i,j} \varepsilon_k^{(i,j)} I(r_1(P_k), r_1(P_i)) = \sum_{k=1}^{N} \alpha_k^{(i,j;\,s,t)} I(r_1(P_k), r_1(P_i))$ and for all $x \in A \backslash \{r_1(P_i)\}$,
$\varphi_x(P) = \sum_{k \neq i,j} \varepsilon_k^{(i,j)} I(r_1(P_k), x) = \sum_{k=1}^{N} \alpha_k^{(i,j;\,s,t)} I(r_1(P_k), x)$. □

Note that $[\alpha_k^{(i,j;\,s,t)}]_{k=1}^{N} = [\alpha_k^{(s,t;\,i,j)}]_{k=1}^{N}$, where i,j,s,t are distinct. The next lemma shows that sequence $[\alpha_k^{(i,j;\,s,t)}]_{k=1}^{N}$ is independent of $\{s,t\}$ whenever $s,t \in I \backslash \{i,j\}$ and $s \neq t$.

Lemma 5.3. *Fix $i,j \in I$. For all $s,t,\bar{s},\bar{t} \in I \backslash \{i,j\}$, where $s \neq t$ and $\bar{s} \neq \bar{t}$, we have $[\alpha_k^{(i,j;\,s,t)}]_{k=1}^{N} = [\alpha_k^{(i,j;\,\bar{s},\bar{t})}]_{k=1}^{N}$.*

Proof. According to Lemma 5.2, $\alpha_i^{(i,j;\,s,t)} = \varepsilon_i^{(s,t)}$, $\alpha_j^{(i,j;\,s,t)} = \varepsilon_j^{(s,t)}$, $\varepsilon_i^{(s,t)} + \varepsilon_j^{(s,t)} = \varepsilon^{(i,j)}$ and $\alpha_k^{(i,j;\,s,t)} = \varepsilon_k^{(i,j)}$ for all $k \neq i,j$. Meanwhile, $\alpha_i^{(i,j;\,\bar{s},\bar{t})} = \varepsilon_i^{(\bar{s},\bar{t})}$, $\alpha_j^{(i,j;\,\bar{s},\bar{t})} = \varepsilon_j^{(\bar{s},\bar{t})}$, $\varepsilon_i^{(\bar{s},\bar{t})} + \varepsilon_j^{(\bar{s},\bar{t})} = \varepsilon^{(i,j)}$ and $\alpha_k^{(i,j;\,\bar{s},\bar{t})} = \varepsilon_k^{(i,j)}$ for all $k \neq i,j$. Therefore, $\alpha_i^{(i,j;\,s,t)} + \alpha_j^{(i,j;\,s,t)} = \alpha_i^{(i,j;\,\bar{s},\bar{t})} + \alpha_j^{(i,j;\,\bar{s},\bar{t})}$ and $\alpha_k^{(i,j;\,s,t)} = \alpha_k^{(i,j;\,\bar{s},\bar{t})}$ for all $k \neq i,j$.

Next, given a profile $P = (P_i, P_{-i})$ where $r_1(P_i) = a$ and for all $k,l \in I \backslash \{i\}$, $P_k = P_l \notin \mathbb{D}^a$, then by both $g^{(s,t)}$ and $g^{(\bar{s},\bar{t})}$ respectively, we have $\varphi_a(P) = \varepsilon_i^{(s,t)}$ and $\varphi_a(P) = \varepsilon_i^{(\bar{s},\bar{t})}$. Then, $\varepsilon_i^{(s,t)} = \varepsilon_i^{(\bar{s},\bar{t})}$ and hence $\alpha_i^{(i,j;\,s,t)} = \alpha_i^{(i,j;\,\bar{s},\bar{t})}$. Consequently, $\alpha_j^{(i,j;\,s,t)} = \alpha_j^{(i,j;\,\bar{s},\bar{t})}$. □

Fix $i,j \in I$. We have the following: for all $P \in \mathbb{D}^N$ with $P_i = P_j$, there exists $[\alpha_k^{(i,j)}]_{k=1}^{N} \geq 0$ with $\sum_{k=1}^{N} \alpha_k^{(i,j)} = 1$ such that $\varphi(P) = \sum_{k=1}^{N} \alpha_k^{(i,j)} e_{r_1(P_k)}$. In addition, $[\alpha_k^{(i,j)}]_{k=1}^{N} = [\alpha_k^{(j,i)}]_{k=1}^{N}$. We next show that the sequence $[\alpha_k^{(i,j)}]_{k=1}^{N}$ is independent of $\{i,j\}$.

Lemma 5.4. *For all $i, j, s, t \in I$, where $i \neq j$ and $s \neq t$, $[\alpha_k^{(i,j)}]_{k=1}^N = [\alpha_k^{(s,t)}]_{k=1}^N$.*

Proof. It is evident that $|\{i,j\} \cap \{s,t\}| = 0, 1$ or 2. If $|\{i,j\} \cap \{s,t\}| = 0$, then i, j, s, t are mutually distinct. Hence, $[\alpha_k^{(i,j)}]_{k=1}^N = [\alpha_k^{(i,j;\,s,t)}]_{k=1}^N = [\alpha_k^{(s,t;\,i,j)}]_{k=1}^N = [\alpha_k^{(s,t)}]_{k=1}^N$. Next, if $|\{i,j\} \cap \{s,t\}| = 2$, then $\{i,j\} = \{s,t\}$, which implies $[\alpha_k^{(i,j)}]_{k=1}^N = [\alpha_k^{(s,t)}]_{k=1}^N$.

Now, we consider $|\{i,j\} \cap \{s,t\}| = 1$, We can therefore assume without loss of generality that $i = s$. Since $N > 3$, there exists another voter: voter $\bar{s}$ and $\bar{s} \notin \{i, j, t\}$.

For every $k \notin \{i, j, t\}$, we consider a profile $P^{(k)} = (P_k, P_{-k})$ where $P_k \in \mathbb{D}^a$ and for all $l, n \in I \backslash \{k\}$, $P_l = P_n \notin \mathbb{D}^a$. By Lemma 5.3, it follows that $\varphi_a(P^{(k)}) = \alpha_k^{(i,j)}$ and $\varphi_a(P^{(k)}) = \alpha_k^{(i,t)}$. Therefore, $\alpha_k^{(i,j)} = \alpha_k^{(i,t)}$ for all $k \notin \{i, j, t\}$.

From the case where $|\{i,j\} \cap \{\bar{s},t\}| = 0$, we have $\alpha_j^{(i,j)} = \alpha_j^{(\bar{s},t)}$. Consider a profile $P = (P_j, P_{-j})$ where $P_j \in \mathbb{D}^a$ and for all $l, n \in I \backslash \{j\}$, $P_l = P_n \notin \mathbb{D}^a$. By Lemma 5.3, it follows that $\varphi_a(P) = \alpha_j^{(\bar{s},t)}$ and $\varphi_a(P) = \alpha_j^{(i,t)}$. Therefore, $\alpha_j^{(\bar{s},t)} = \alpha_j^{(i,t)}$. Then, $\alpha_j^{(i,j)} = \alpha_j^{(i,t)}$. Similarly, $\alpha_t^{(i,j)} = \alpha_t^{(i,t)}$.

Finally, it is evident that $\alpha_i^{(i,j)} = 1 - \sum_{k \neq i} \alpha_k^{(i,j)} = 1 - \sum_{k \neq i} \alpha_k^{(i,t)} = \alpha_i^{(i,t)}$. We therefore conclude that $[\alpha_k^{(i,j)}]_{k=1}^N = [\alpha_k^{(s,t)}]_{k=1}^N$. $\square$

In conclusion, there exists $\varepsilon_k \in [0, 1]$, $k = 1, \ldots, N$, with $\sum_{k=1}^N \varepsilon_k = 1$ such that for all $P \in \mathbb{D}^N$ with $P_i = P_j$ for some distinct $i, j \in I$, $\varphi(P) = \sum_{k=1}^N \varepsilon_k \, e_{r_1(P_k)}$. Therefore, φ is a quasi-random dictatorship. This completes the verification of Proposition 5.2. $\square$

Proposition 5.3. *Let $\mathbb{D}$ be a minimally rich domain. Suppose that for all $2 \leq t < N$, every unanimous and strategy-proof RSCF $g : \mathbb{D}^t \to \Delta(A)$ is a random dictatorship. If a unanimous and strategy-proof RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is a quasi-random dictatorship, then φ is a random dictatorship.*

Proof. The proof proceeds in a sequence of lemmas. Let $[\varepsilon_k]_{k=1}^N \geq 0$ with $\sum_{k=1}^N \varepsilon_k = 1$ be the sequence for the quasi-random dictatorship that φ satisfies.

Lemma 5.5. *For all $P \in \mathbb{D}^N$, if there exist $i, j \in I$ such that $r_1(P_i) = r_1(P_j)$, then $\varphi(P) = \sum_{k=1}^{N} \varepsilon_k \, e_{r_1(P_k)}$.*

Proof. Fix $P = (P_i, P_j, P_{-\{i,j\}})$. Assume $r_1(P_i) = r_1(P_j) = x_0$. If $r_1(P_{-\{i,j\}})\backslash\{x_0\} = \emptyset$, then $r_1(P) = \{x_0\}$ and unanimity gives the result. We complete the proof by considering $r_1(P_{-\{i,j\}})\backslash\{x_0\} \neq \emptyset$. Now, assume $r_1(P_{-\{i,j\}})\backslash\{x_0\} = \{x_k\}_{k=1}^{l}$, $1 \leq l \leq N-2$ and all elements in $\{x_k\}_{k=1}^{l}$ are distinct. By strategy-proofness and quasi-random dictatorship, we have $\varphi_{x_0}(P) = \varphi_{x_0}(P_i, P_i, P_{-\{i,j\}}) = \varepsilon_i + \varepsilon_j + \sum_{k\neq i,j} \varepsilon_k \, I(r_1(P_k), x_0) = \sum_{k=1}^{N} \varepsilon_k \, I(r_1(P_k), x_0)$.

Next, for the relative rankings of all elements in $\{x_k\}_{k=1}^{l}$ in P_i, we assume without loss of generality that $x_t = r_{k_t}(P_i)$, $t = 1, \ldots, l$ and $k_1 < k_2 < \cdots < k_l$. By strategy-proofness, for all $s \geq 2$, $\sum_{\nu=1}^{s} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{\nu=1}^{s} \varphi_{r_\nu(P_i)}(P) \leq \sum_{\nu=1}^{s} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}})$.

Next, according to quasi-random dictatorship, we have that for $t = 1, \ldots, l$,

$$\begin{aligned}
\sum_{\nu=1}^{k_t-1} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) &= \sum_{\nu=1}^{k_t-1} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}}) \\
&= \varepsilon_i + \varepsilon_j + \sum_{k\neq i,j} \varepsilon_k \Big[\sum_{s=0}^{t-1} I(r_1(P_k), x_s)\Big], \\
\sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) &= \sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}}) \\
&= \varepsilon_i + \varepsilon_j + \sum_{k\neq i,j} \varepsilon_k \Big[\sum_{s=0}^{t} I(r_1(P_k), x_s)\Big].
\end{aligned}$$

Consequently, $\sum_{\nu=1}^{k_t-1} \varphi_{r_\nu(P_i)}(P) = \varepsilon_i + \varepsilon_j + \sum_{k\neq i,j} \varepsilon_k \Big[\sum_{s=0}^{t-1} I(r_1(P_k), x_s)\Big]$ and $\sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P) = \varepsilon_i + \varepsilon_j + \sum_{k\neq i,j} \varepsilon_k \Big[\sum_{s=0}^{t} I(r_1(P_k), x_s)\Big]$ for $t = 1, \ldots, l$. Hence, $\varphi_{x_t}(P) = \sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P) - \sum_{\nu=1}^{k_t-1} \varphi_{r_\nu(P_i)}(P) = \sum_{k\neq i,j} \varepsilon_k \, I(r_1(P_k), x_t) = \sum_{k=1}^{N} \varepsilon_k \, I(r_1(P_k), x_t)$ for $t = 1, \ldots, l$.

Therefore, $\sum_{x\in r_1(P)} \varphi_x(P) \equiv \sum_{i=0}^{l} \varphi_{x_i}(P) = \sum_{k=1}^{N} \varepsilon_k = 1$. Then, for all $x \notin r_1(P)$, $\varphi_x(P) = 0$. In conclusion, $\varphi(P) = \sum_{k=1}^{N} \varepsilon_k e_{r_1(P_k)}$. □

If $|A| = m < N$, then for all $P \in \mathbb{D}^N$, there always exist at least two voters who share a common maximal alternative. Then, Lemma 5.5 implies that φ is a random dictatorship. We complete the proof by considering $|A| = m \geq N$. Given a profile $P \in \mathbb{D}^N$ with $|r_1(P)| = N$, recall $\overline{W}(P) = \cup_{k=1}^{N} W\big(P_k, \max(P_k, r_1(P_{-k}))\big)$.

Lemma 5.6. *For all $P \in \mathbb{D}^N$ with $|r_1(P)| = N$, we have $|r_1(P) \cap \overline{W}(P)| \geq N - 1$.*

Proof. This lemma asserts that for every profile $P \in \mathbb{D}^N$ with $|r_1(P)| = N$, $r_1(P)$ and $\overline{W}(P)$ have at least $N-1$ alternatives in common.

Suppose not. Then there exists $P \in \mathbb{D}^N$ with $|r_1(P)| = N$ such that $|r_1(P) \cap \overline{W}(P)| < N - 1$. Hence, there exist $a, b \in r_1(P) \backslash \overline{W}(P)$. Since $|r_1(P)| = N$ and $N \geq 3$, we know that there exists $P_i \in \mathbb{D}^c$ for some $i \in I$ such that $c \notin \{a, b\}$. Let $\max(P_i, r_1(P_{-i})) = x$. If $x \notin \{a, b\}$, we know $\{a, b\} \subseteq W(P_i, x)$ which implies $\{a, b\} \subseteq \overline{W}(P)$. If $x = a$, then $b \in W(P_i, x)$ which implies $b \in \overline{W}(P)$. If $x = b$, then $a \in W(P_i, x)$ which implies $a \in \overline{W}(P)$. We have a contradiction. $\square$

Lemma 5.7. *For all $P \in \mathbb{D}^N$ with $|r_1(P)| = N$ and $x \in \overline{W}(P)$, we have $\varphi_x(P) = \sum_{k=1}^{N} \varepsilon_k\, I(r_1(P_k), x)$.*

Proof. Fix $i \in I$. Assume without loss of generality that $r_1(P_{-i}) = \{x_k\}_{k=1}^{N-1}$, $x_t = r_{k_t}(P_i)$, $t = 1, \ldots, N-1$, $k_1 < k_2 < \cdots < k_{N-1}$ and $x_1 = r_1(P_j)$ for some $j \in I \backslash \{i\}$. By strategy-proofness, we have $\sum_{\nu=1}^{s} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) \leq \sum_{\nu=1}^{s} \varphi_{r_\nu(P_i)}(P) \leq \sum_{\nu=1}^{s} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}})$ for all $s \geq k_1$.

According to quasi-random dictatorship, we have the following:

$$\sum_{\nu=1}^{k_1} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) = \sum_{\nu=1}^{k_1} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}}) = \varepsilon_i + \varepsilon_j,$$

and for $t = 2, \ldots, N-1$,

$$\begin{aligned}
\sum_{\nu=1}^{k_t-1} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) &= \sum_{\nu=1}^{k_t-1} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}}) \\
&= \varepsilon_i + \varepsilon_j + \sum_{k \neq i,j} \varepsilon_k \Big[\sum_{s=2}^{t-1} I(r_1(P_k), x_s) \Big], \\
\sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P_j, P_j, P_{-\{i,j\}}) &= \sum_{\nu=1}^{k_t} \varphi_{r_\nu(P_i)}(P_i, P_i, P_{-\{i,j\}}) \\
&= \varepsilon_i + \varepsilon_j + \sum_{k \neq i,j} \varepsilon_k \Big[\sum_{s=2}^{t} I(r_1(P_k), x_s) \Big].
\end{aligned}$$

Then, similar to the proof of Lemma 5.5, we have $\sum_{\nu=1}^{k_1} \varphi_{r_\nu(P_i)}(P) = \varepsilon_i + \varepsilon_j$ and $\varphi_{x_t}(P) = \sum_{k=1}^{N} \varepsilon_k \, I(r_1(P_k), x_t)$, for $t = 2, \ldots, N-1$. Since $|r_1(P)| = N$ and $r_1(P_{-\{i,j\}}) = \{x_t\}_{t=2}^{N-1}$, we know $\varphi_{r_1(P_k)}(P) = \varepsilon_k$ for all $k \neq i, j$. Then, $\sum_{\nu=1}^{k_1} \varphi_{r_\nu(P_i)}(P) + \sum_{k \neq i,j} \varphi_{r_1(P_k)}(P) = \sum_{k=1}^{N} \varepsilon_k = 1$. Therefore, for all $x \in W(P_i, x_1) \backslash \{x_t\}_{t=2}^{N-1}$, $\varphi_x(P) = 0$. In conclusion, for all $x \in W(P_i, x_1)$, $\varphi_x(P) = \sum_{k=1}^{N} \varepsilon_k \, I(r_1(P_k), x)$.

Applying the same argument to all other voters, we have $\varphi_x(P) = \sum_{k=1}^{N} \varepsilon_k \, I(r_1(P_k), x)$ for all $x \in \overline{W}(P)$. $\square$

From Lemma 5.7, we can infer that for all $P \in \mathbb{D}^N$ with $|r_1(P)| = N$, if $r_1(P) \subseteq \overline{W}(P)$, then $\varphi(P) = \sum_{k=1}^{N} \varepsilon_k \, e_{r_1(P_k)}$. By Lemmas 5.6 and 5.7, we know that for every $P \in \mathbb{D}^N$ with $|r_1(P)| = N$, the probabilities over at least $N-1$ elements of $r_1(P)$ in $\varphi(P)$ are revealed.

In the next lemma, we will identify properties that a profile P and $\varphi(P)$ must satisfy if $\varphi(P) \neq \sum_{k=1}^{N} \varepsilon_k \, e_{r_1(P_k)}$. Given a profile $P \in \mathbb{D}^N$ with $|r_1(P)| = N$, let $\bar{B}_i(P) = B\big(P_i, \max(P_i, r_1(P_{-i}))\big) \backslash \{r_1(P_i)\}$, $i \in I$ and $\bar{B}(P) = \cap_{i \in I} \bar{B}_i(P)$.

Lemma 5.8. *Given $P \in \mathbb{D}^N$, if $\varphi(P) \neq \sum_{k=1}^{N} \varepsilon_k \, e_{r_1(P_k)}$, the following conditions are satisfied:*

(i) $|r_1(P)| = N$.

(ii) *There exists* $i \in I$ *such that* $\varphi_{r_1(P_i)}(P) < \varepsilon_i$ *and* $\varphi_{r_1(P_k)}(P) = \varepsilon_k$ *for all* $k \neq i$.

(iii) $r_1(P_i) = \max(P_k, r_1(P_{-k}))$ *for all* $k \neq i$.

(iv) $\varphi_{r_1(P_i)}(P) + \sum_{x \in \bar{B}(P)} \varphi_x(P) = \varepsilon_i$.

(v) $\bar{B}(P) \neq \emptyset$. *Furthermore, there exists* $x \in \bar{B}(P)$ *such that* $\varphi_x(P) > 0$.

Proof. (i) Since $\varphi(P) \neq \sum_{k=1}^{N} \varepsilon_k \, e_{r_1(P_k)}$, Lemma 5.5 implies $|r_1(P)| = N$.

(ii) According to Lemmas 5.6 and 5.7 and the hypothesis $\varphi(P) \neq \sum_{k=1}^{N} \varepsilon_k \, e_{r_1(P_k)}$, it must be true that $|r_1(P) \cap \overline{W}(P)| = N - 1$. Assume without loss of generality that $r_1(P_i) \notin \overline{W}(P)$. Then, by Lemma 5.7, we have $\varphi_{r_1(P_k)}(P) = \varepsilon_k$ for all $k \neq i$. Consequently, $\varphi_{r_1(P_i)}(P) \leq 1 - \sum_{k \neq i} \varphi_{r_1(P_k)}(P) = \varepsilon_i$. This implies $\varphi_{r_1(P_i)}(P) < \varepsilon_i$, otherwise $\varphi(P) = \sum_{k=1}^{N} \varepsilon_k e_{r_1(P_k)}$.

(iii) The proof of statement (ii) shows that $r_1(P_i) \notin \overline{W}(P)$, which implies $r_1(P_i) = \max(P_k, r_1(P_{-k}))$ for all $k \neq i$.

(iv) Assume without loss of generality that $\max(P_i, r_1(P_{-i})) = r_1(P_j)$ for some $j \in I \backslash \{i\}$ and let $r_1(P_j) = r_s(P_i)$. Then, as we showed in the proof of Lemma 5.7, $\varepsilon_i + \varepsilon_j = \sum_{k=1}^{s} \varphi_{r_k(P_i)}(P) = \sum_{k=1}^{s-1} \varphi_{r_k(P_i)}(P) + \varphi_{r_1(P_j)}(P) = \varphi_{r_1(P_i)}(P) + \sum_{x \in \bar{B}_i(P)} \varphi_x(P) + \varphi_{r_1(P_j)}(P)$. Furthermore, statement (ii) implies $\varphi_{r_1(P_j)}(P) = \varepsilon_j$. Hence, $\varphi_{r_1(P_i)}(P) + \sum_{x \in \bar{B}_i(P)} \varphi_x(P) = \varepsilon_i$. Next, since $\bar{B}_i(P) \backslash \bar{B}(P) \subset \overline{W}(P)$ and $\bar{B}(P) \subseteq \bar{B}_i(P)$, we have $\varphi_x(P) = 0$ for all $x \in \bar{B}_i(P) \backslash \bar{B}(P)$ by Lemma 5.7 and $\varphi_{r_1(P_i)}(P) + \sum_{x \in \bar{B}(P)} \varphi_x(P) = \varepsilon_i$.

(v) By statements (ii) and (iv), we know $\sum_{x \in \bar{B}(P)} \varphi_x(P) > 0$, which implies $\bar{B}(P) \neq \emptyset$ and furthermore, there exists $x \in \bar{B}(P)$ such that $\varphi_x(P) > 0$. □

The voter i specified in statement (ii) of Lemma 5.8 is called the special voter of P. As we showed in the proof of statement (ii) of Lemma 5.8, we know that the peak of the special voter of P does not belong to $\overline{W}(P)$. It is evident that in a profile P with $\varphi(P) \neq \sum_{k=1}^{N} \varepsilon_k e_{r_1(P_k)}$, there exists a unique special voter.

We next show what property the sequence $[\varepsilon_k]_{k=1}^N$ must satisfy, when there exists a profile P^* such that $\varphi(P^*) \neq \sum_{k=1}^N \varepsilon_k e_{r_1(P_k^*)}$.

Lemma 5.9. *If there exists $P^* \in \mathbb{D}^N$ such that $\varphi(P^*) \neq \sum_{k=1}^N \varepsilon_k\, e_{r_1(P_k^*)}$, then $0 < \varepsilon_k < 1$, $k = 1, \ldots, N$.*

Proof. Suppose there exists $\varepsilon_k = 0$. Fix P_k^* (the k_{th} element of P^*). Define a RSCF: $g(P_{-k}) = \varphi(P_k^*, P_{-k})$ for all $P_{-k} \in \mathbb{D}^{N-1}$. Strategy-proofness of φ implies that g is strategy-proof. Next, Lemma 5.5 implies that g is unanimous. Furthermore, according to Lemma 5.8(v), we know that there exists $x \notin r_1(P^*)$ such that $\varphi_x(P^*) > 0$. Therefore, $g_x(P_{-k}^*) = \varphi_x(P_k^*, P_{-k}^*) > 0$ where $x \notin r_1(P_{-k}^*)$, which implies that RSCF g is not a random dictatorship. This is a contradiction to the hypothesis of Proposition 5.3.

Next, suppose $\varepsilon_k = 1$ for some $k \in I$. Then, there exists $j \neq k$ such that $\varepsilon_j = 0$, which would lead to the same contradiction. $\square$

In the next lemma, we show it is true that for all $P \in \mathbb{D}^N$ with $|r_1(P)| = N$, $\varphi(P) = \sum_{k=1}^N \varepsilon_k e_{r_1(P_k)}$ by contradiction. Suppose φ is not a random dictatorship. Then we construct a RSCF $h : \mathbb{D}^2 \to \Delta(A)$ and show it is unanimous and strategy-proof and not a random dictatorship, which hence contradicts the hypothesis of Proposition 5.3.

Lemma 5.10. *For all $P \in \mathbb{D}^N$ with $|r_1(P)| = N$, we have $\varphi(P) = \sum_{k=1}^N \varepsilon_k\, e_{r_1(P_k)}$.*

Proof. Suppose RSCF φ is not a random dictatorship with respect to $[\varepsilon_k]_{k=1}^N$. Then, there exists $P^* \in \mathbb{D}^N$ such that $\varphi(P^*) \neq \sum_{k=1}^N \varepsilon_k e_{r_1(P_k^*)}$. By Lemma 5.8(ii) and (v), we know that there exist a special voter of P^* and $y \notin r_1(P^*)$ such that $\varphi_y(P^*) > 0$. Assume without loss of generality that voter 1 be the special voter of P^*. Next, pick arbitrarily another voter, i.e., voter 2 and fix $P_{-\{1,2\}}^*$ (elements in P^* other than P_1^* and P_2^*). By Lemma 5.9, we can construct the following function: for all

$P_1, P_2 \in \mathbb{D}$,

$$h(P_1,P_2) = \begin{cases} \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} e_{r_1(P_1)} + \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} e_{r_1(P_2)}, \\ \qquad \text{if } \varphi_{r_1(P_1)}(P_1,P_2,P^*_{-\{1,2\}}) \geq \varepsilon_1, \text{ and } \varphi_{r_1(P_2)}(P_1,P_2,P^*_{-\{1,2\}}) \geq \varepsilon_2; \\ \frac{1}{\varepsilon_1+\varepsilon_2} \Big[\varphi(P_1,P_2,P^*_{-\{1,2\}}) - \sum\limits_{k=3}^{N} \varepsilon_k e_{r_1(P^*_k)} \Big], \\ \qquad \text{otherwise.} \end{cases}$$

Note that Lemma 5.8(ii) implies that it is impossible that $\varphi_{r_1(P_1)}(P_1,P_2,P^*_{-\{1,2\}}) < \varepsilon_1$ and $\varphi_{r_1(P_2)}(P_1,P_2,P^*_{-\{1,2\}}) < \varepsilon_2$ simultaneously. Therefore, given $P = (P_1,P_2,P^*_{-\{1,2\}})$, by Lemma 5.8(ii) and (iv), when either $\varphi_{r_1(P_1)}(P) < \varepsilon_1$ or $\varphi_{r_1(P_2)}(P) < \varepsilon_2$, $h(P_1,P_2)$ must be specified as below:

if $\varphi_{r_1(P_1)}(P) < \varepsilon_1$, then

$$h(P_1,P_2) = \frac{1}{\varepsilon_1+\varepsilon_2} \Big[\varphi_{r_1(P_1)}(P)\, e_{r_1(P_1)} + \sum_{x \in \bar{B}(P)} \varphi_x(P)\, e_x + \varepsilon_2\, e_{r_1(P_2)} \Big] \qquad (5.1)$$

where $\varphi_{r_1(P_1)}(P) + \sum_{x \in \bar{B}(P)} \varphi_x(P) = \varepsilon_1$; and
if $\varphi_{r_1(P_2)}(P) < \varepsilon_2$, then

$$h(P_1,P_2) = \frac{1}{\varepsilon_1+\varepsilon_2} \Big[\varepsilon_1\, e_{r_1(P_1)} + \varphi_{r_1(P_2)}(P)\, e_{r_1(P_2)} + \sum_{x \in \bar{B}(P)} \varphi_x(P)\, e_x \Big] \qquad (5.2)$$

where $\varphi_{r_1(P_2)}(P) + \sum_{x \in \bar{B}(P)} \varphi_x(P) = \varepsilon_2$.

Next, we will show that h is a unanimous and strategy-proof RSCF. Furthermore, to complete the proof of Lemma 5.10, we also show that h is not a random dictatorship which contradicts the hypothesis of Proposition 5.3.

Claim 1: Function h is a RSCF.

Firstly, if $\varphi_{r_1(P_1)}(P_1,P_2,P^*_{-\{1,2\}}) \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P_1,P_2,P^*_{-\{1,2\}}) \geq \varepsilon_2$, it is evident that $h_x(P_1,P_2) \geq 0$ for all $x \in A$, and $\sum_{x \in A} h_x(P_1,P_2) = 1$. Secondly, if either $\varphi_{r_1(P_1)}(P_1,P_2,P^*_{-\{1,2\}}) < \varepsilon_1$ or $\varphi_{r_1(P_2)}(P_1,P_2,P^*_{-\{1,2\}}) < \varepsilon_2$, either Equation (1) or (2) above implies $h_x(P_1,P_2) \geq 0$ for all $x \in A$ and $\sum_{x \in A} h_x(P_1,P_2) = 1$.

This completes the verification of Claim 1.

Claim 2: RSCF h is unanimous.

Let $r_1(P_1) = r_1(P_2) = a$. Then, by Lemma 5.5, we know that $\varphi_a(P_1, P_2, P^*_{-\{1,2\}}) = \varepsilon_1 + \varepsilon_2 + \sum_{k=3}^{N} \varepsilon_k I(a, r_1(P^*_k))$. Hence, $\varphi_{r_1(P_1)}(P_1, P_2, P^*_{-\{1,2\}}) = \varphi_a(P_1, P_2, P^*_{-\{1,2\}}) \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P_1, P_2, P^*_{-\{1,2\}}) = \varphi_a(P_1, P_2, P^*_{-\{1,2\}}) \geq \varepsilon_2$. Consequently, $h_a(P_1, P_2) = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} + \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} = 1$. This completes the verification of Claim 2.

Claim 3: RSCF h is not a random dictatorship.

Since we have assumed that voter 1 is the special voter of P^*, it is true that $\varphi_{r_1(P^*_1)}(P^*) < \varepsilon_1$ by Lemma 5.8(ii). Consequently, $h(P^*_1, P^*_2)$ follows from Equation (1). Next, since we have assumed that $\varphi_y(P^*) > 0$ where $y \notin r_1(P^*)$ in the beginning proof of Lemma 5.10, we have $h_y(P^*_1, P^*_2) > 0$ and $y \notin r_1(P^*_1, P^*_2)$, which implies that h is not a random dictatorship. This completes the verification of Claim 3.

Claim 4: RSCF h is strategy-proof.

Recall that to verify strategy-proofness of a RSCF, it is equivalent to show that a voter's expected utility from truthtelling to be no less than her expected utility from misrepresentation for any cardinal representation of her true preferences independent of other voters' behaviors. Given $P_i \in \mathbb{D}$, let $\mathbb{U}(P_i)$ denote the set of utility functions that represent P_i. Accordingly, given a lottery $\lambda \in \Delta(A)$ and $u_i \in \mathbb{U}(P_i)$, $\sum_{a \in A} \lambda_a u_i(a)$ represents voter i's expected utility. To verify this claim, we adopt this approach, instead of directly showing stochastic dominance.

We consider the possible manipulation of voter 1 in h. Firstly, it is evident that the manipulation only occurs at (P_1, P_2) via P'_1 where either $h(P_1, P_2) = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} e_{r_1(P_1)} + \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} e_{r_1(P_2)}$ and $h(P'_1, P_2) = \frac{1}{\varepsilon_1+\varepsilon_2}\left[\varphi(P'_1, P_2, P^*_{-\{1,2\}}) - \sum_{k=3}^{N} \varepsilon_k e_{r_1(P^*_k)}\right]$, or $h(P_1, P_2) = \frac{1}{\varepsilon_1+\varepsilon_2}\left[\varphi(P_1, P_2, P^*_{-\{1,2\}}) - \sum_{k=3}^{N} \varepsilon_k e_{r_1(P^*_k)}\right]$ and $h(P'_1, P_2) = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} e_{r_1(P'_1)} + \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} e_{r_1(P_2)}$.

Secondly, if $\varphi(P_1, P_2, P^*_{-\{1,2\}}) = \varepsilon_1 e_{r_1(P_1)} + \varepsilon_2 e_{r_1(P_2)} + \sum_{k=3}^{N} \varepsilon_k e_{r_1(P^*_k)}$, then $h(P_1, P_2) = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2} e_{r_1(P_1)} + \frac{\varepsilon_2}{\varepsilon_1+\varepsilon_2} e_{r_1(P_2)} = \frac{1}{\varepsilon_1+\varepsilon_2}\left[\varphi(P_1, P_2, P^*_{-\{1,2\}}) - \sum_{k=3}^{N} \varepsilon_k e_{r_1(P^*_k)}\right]$, which implies that there exists no manipulation at (P_1, P_2) via P'_1 or at (P'_1, P_2) via

P_1.

Therefore, given two profiles $P = (P_1, P_2, P^*_{-\{1,2\}})$ and $P' = (P'_1, P_2, P^*_{-\{1,2\}})$ such that $\varphi(P) \neq \varepsilon_1 e_{r_1(P_1)} + \varepsilon_2 e_{r_1(P_2)} + \sum_{k=3}^{N} \varepsilon_k e_{r_1(P^*_k)}$ and $\varphi(P') \neq \varepsilon_1 e_{r_1(P'_1)} + \varepsilon_2 e_{r_1(P_2)} + \sum_{k=3}^{N} \varepsilon_k e_{r_1(P^*_k)}$, the manipulation at (P_1, P_2) via P'_1 may occur in following 4 cases.[7]

Case 1: (i) $\varphi_{r_1(P_1)}(P) \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P) \geq \varepsilon_2$, and (ii) $\varphi_{r_1(P'_1)}(P') < \varepsilon_1$.

Now, $h(P'_1, P_2)$ follows from Equation (1). Then, given $u_1 \in \mathbb{U}(P_1)$, the loss from misrepresentation in h is $\sum_{x\in A} u_1(x)\varphi_x(P_1, P_2) - \sum_{x\in A} u_1(x)\varphi_x(P'_1, P_2) = \frac{1}{\varepsilon_1+\varepsilon_2}\left[\varepsilon_1 u_1(r_1(P_1)) - \varphi_{r_1(P'_1)}(P')u_1(r_1(P'_1)) - \sum_{x\in \bar{B}(P')} \varphi_x(P')u_1(x)\right] \geq 0$. This completes the verification of Case 1.

Case 2: (i) $\varphi_{r_1(P_1)}(P) \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P) \geq \varepsilon_2$, and (ii) $\varphi_{r_1(P_2)}(P') < \varepsilon_2$.

We first claim that this case only occurs when $N = 3$. Suppose not, i.e., $N \geq 4$. Since $\varphi_{r_1(P_1)}(P) \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P) \geq \varepsilon_2$, by Lemma 5.8(ii), we assume without loss of generality that voter i, where $i \in \{3, \ldots, N\}$, is the special voter of P. Next, since $N \geq 4$, there must exist another voter, i.e., voter j such that $j \notin \{1, 2, i\}$. Furthermore, applying Lemma 5.8(iii) to P, we know $r_1(P_i)\, P_j\, r_1(P_2)$. On the other hand, $\varphi_{r_1(P_2)}(P') < \varepsilon_2$ indicates that voter 2 is the special voter of P'. Therefore, applying Lemma 5.8(iii) to P', we have $r_1(P_2)\, P_j\, r_1(P_i)$. Contradiction!

Now, by Lemma 5.8(i), to simplify the notation, we assume $r_1(P_1) = a$, $r_1(P_2) = b$, $r_1(P^*_3) = c$ and $r_1(P'_1) = d$, where a, b, c are mutually distinct and d, b, c are mutually distinct (it is possible that $a = d$). Furthermore, $h(P'_1, P_2)$ follows from Equation (2). Therefore, given $u_1 \in \mathbb{U}(P_1)$, the loss from misrepresentation in h is

$$\begin{aligned}
& \sum_{x\in A} u_1(x)\varphi_x(P_1, P_2) - \sum_{x\in A} u_1(x)\varphi_x(P'_1, P_2) \\
= \; & \frac{1}{\varepsilon_1+\varepsilon_2}\Big[\varepsilon_1 u_1(a) + \varepsilon_2 u_1(b) - \varepsilon_1 u_1(d) - \varphi_b(P')u_1(b) - \sum_{x\in \bar{B}(P')} \varphi_x(P')u_1(x)\Big]
\end{aligned}$$

[7]Since $\varphi(P) \neq \varepsilon_1 e_{r_1(P_1)} + \varepsilon_2 e_{r_1(P_2)} + \sum_{k=3}^{N} \varepsilon_k e_{r_1(P^*_k)}$ and $\varphi(P') \neq \varepsilon_1 e_{r_1(P'_1)} + \varepsilon_2 e_{r_1(P_2)} + \sum_{k=3}^{N} \varepsilon_k e_{r_1(P^*_k)}$, we can apply Lemma 5.8 to P and P' in the analysis of the following 4 cases.

where $\varepsilon_2 = \varphi_b(P') + \sum_{x\in \bar{B}(P')} \varphi_x(P')$.

To show that $\sum_{x\in A} u_1(x)\varphi_x(P_1, P_2) - \sum_{x\in A} u_1(x)\varphi_x(P_1', P_2) \geq 0$, We will consider the following 2 situations: dP_1b and bP_1d.

Firstly, we claim that if dP_1b, $\sum_{x\in A} u_1(x)\varphi_x(P_1, P_2) - \sum_{x\in A} u_1(x)\varphi_x(P_1', P_2) \geq 0$. Since either $a = d$ or aP_1d, to verify the claim, we only need to show that bP_1x for all $x \in \bar{B}(P')$ with $\varphi_x(P') > 0$. Suppose not, i.e., there exists $x^* \in \bar{B}(P')$ such that $\varphi_{x^*}(P') > 0$ and x^*P_1b. In profile P, since $\varphi_a(P) \geq \varepsilon_1$, $\varphi_b(P) \geq \varepsilon_2$ and $N = 3$, by Lemma 5.8(ii) and (iii), we know that voter 3 is the special voter of P and cP_1b. Let $x' = \min(P_1, \{x^*, d, c\})$. Hence $x'P_1b$. Assume $x' = r_s(P_1)$. As we showed in the proof of Lemma 5.7, $\sum_{k=1}^{s} \varphi_{r_k(P_1)}(P) = \varepsilon_1 + \varepsilon_3$. Meanwhile, Lemma 5.8(ii) implies $\varphi_d(P') = \varepsilon_1$ and $\varphi_c(P') = \varepsilon_3$. Then, $\varphi_{x^*}(P') > 0$ implies that $\sum_{k=1}^{s} \varphi_{r_k(P_1)}(P) < \varepsilon_1 + \varepsilon_3 + \varphi_{x^*}(P') = \varphi_d(P') + \varphi_c(P') + \varphi_{x^*}(P') \leq \sum_{k=1}^{s} \varphi_{r_k(P_1)}(P')$. Therefore, voter 1 manipulates at P via P_1' in φ - a contradiction.

Next, we claim that if bP_1d, then $\sum_{x\in A} u_1(x)\varphi_x(P_1, P_2) - \sum_{x\in A} u_1(x)\varphi_x(P_1', P_2) \geq 0$. Now, it is evident that $a \neq d$. Since $b \notin \bar{B}(P')$, we separate $\bar{B}(P')$ into two parts S and T: for all $x \in S$, xP_1b, and for all $z \in T$, bP_1z. If $S = \emptyset$, then for all $x \in \bar{B}(P')$, bP_1x. Therefore, it is true that $\sum_{x\in A} u_1(x)\varphi_x(P_1, P_2) - \sum_{x\in A} u_1(x)\varphi_x(P_1', P_2) = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}[u_1(a) - u_1(d)] + \frac{1}{\varepsilon_1+\varepsilon_2}\sum_{x\in \bar{B}(P')} \varphi_x(P')[u_1(b) - u_1(x)] \geq 0$.

Next, consider $S \neq \emptyset$. Let $x^* = \max(P_1, S)$. It is true that (i) either aP_1x^* or $a = x^*$, (ii) x^*P_1b, (iii) bP_1d and (iv) bP_1z for all $z \in T$ (if $T \neq \emptyset$). Furthermore, $\sum_{x\in A} u_1(x)\varphi_x(P_1, P_2) - \sum_{x\in A} u_1(x)\varphi_x(P_1', P_2)$ can be modified as

$$
\begin{aligned}
& \sum_{x\in A} u_1(x)\varphi_x(P_1, P_2) - \sum_{x\in A} u_1(x)\varphi_x(P_1', P_2) \\
= \quad & \frac{1}{\varepsilon_1+\varepsilon_2}\Big[\varepsilon_1 u_1(a) + \varepsilon_2 u_1(b) - \varepsilon_1 u_1(d) - \varphi_b(P')u_1(b) - \sum_{x\in S} \varphi_x(P')u_1(x) - \sum_{z\in T} \varphi_z(P')u_1(z)\Big] \\
\geq \quad & \frac{1}{\varepsilon_1+\varepsilon_2}\Big[\varepsilon_1 u_1(a) + \varepsilon_2 u_1(b) - \varepsilon_1 u_1(d) - \varphi_b(P')u_1(b) - u_1(x^*)\sum_{x\in S} \varphi_x(P') - \sum_{z\in T} \varphi_z(P')u_1(z)\Big] \\
= \quad & \frac{\varepsilon_1[u_1(a) - u_1(x^*)]}{\varepsilon_1+\varepsilon_2} + \frac{\varepsilon_1 - \sum\limits_{x\in S}\varphi_x(P')}{\varepsilon_1+\varepsilon_2}[u_1(x^*) - u_1(b)] + \frac{\varepsilon_1[u_1(b) - u_1(d)]}{\varepsilon_1+\varepsilon_2} + \frac{\sum\limits_{z\in T}\varphi_z(P')[u_1(b) - u_1(z)]}{\varepsilon_1+\varepsilon_2}
\end{aligned}
$$

Therefore, according to the relative rankings in P_1 specified above, to show that $\sum_{x\in A} u_1(x)\varphi_x(P_1,P_2) - \sum_{x\in A} u_1(x)\varphi_x(P_1',P_2) \geq 0$, it suffices to show $\varepsilon_1 \geq \sum_{x\in S}\varphi_x(P')$.

Assume $\min(P_1,S) = y^*$ and let $z^* = \min(P_1,\{c,y^*\})$. Assume $z^* = r_s(P_1)$. Hence, $\{r_k(P_1)\}_{k=1}^s = B(P_1,z^*)\cup\{z^*\}$. In profile P, since $\varphi_a(P)\geq\varepsilon_1$, $\varphi_b(P)\geq\varepsilon_2$ and $N=3$, by Lemma 5.8(ii) and (iii), we know that voter 3 is the special voter of P and cP_1b. Hence, z^*P_1b. Therefore, as we showed in the proof of Lemma 5.7, we have $\sum_{k=1}^s \varphi_{r_k(P_1)}(P) = \varepsilon_1+\varepsilon_3$. Next, in profile P', by Lemma 5.8(ii) and (iv), we know that for all $z\notin\{d,b,c\}\cup\bar{B}(P')$, $\varphi_z(P')=0$. Furthermore, since $[B(P_1,z^*)\cup\{z^*\}]\cap\{d,b,c\}=\{c\}$ and $[B(P_1,z^*)\cup\{z^*\}]\cap\bar{B}(P')=S$, we have that $\sum_{k=1}^s\varphi_{r_k(P_1)}(P')\equiv\sum_{x\in B(P_1,z^*)\cup\{z^*\}}\varphi_x(P')=\varphi_c(P')+\sum_{x\in S}\varphi_x(P')=\varepsilon_3+\sum_{x\in S}\varphi_x(P')$ (by Lemma 5.8(ii), $\varphi_c(P')=\varepsilon_3$). Then, strategy-proofness of φ implies $\varepsilon_1\geq\sum_{x\in S}\varphi_x(P')$. This completes the verification of Case 2.

Case 3: (i) $\varphi_{r_1(P_1)}(P)<\varepsilon_1$, and (ii) $\varphi_{r_1(P_1')}(P')\geq\varepsilon_1$ and $\varphi_{r_1(P_2)}(P')\geq\varepsilon_2$.

Now, $h(P_1,P_2)$ follows from Equation (1). Then, given $u_1\in\mathbb{U}(P_1)$, the loss from misrepresentation in h is

$$\begin{aligned}&\sum_{x\in A}u_1(x)\varphi_x(P_1,P_2)-\sum_{x\in A}u_1(x)\varphi_x(P_1',P_2)\\ =\;&\frac{1}{\varepsilon_1+\varepsilon_2}\Big[\varphi_{r_1(P_1)}(P)u_1(r_1(P_1))+\sum_{x\in\bar{B}(P)}\varphi_x(P)u_1(x)-\varepsilon_1u_1(r_1(P_1'))\Big]\end{aligned}$$

where $\varphi_{r_1(P_1)}(P)+\sum_{x\in\bar{B}(P)}\varphi_x(P)=\varepsilon_1$.

Firstly, since $\varphi_{r_1(P_1)}(P)<\varepsilon_1$ and $\varphi_{r_1(P_1')}(P')\geq\varepsilon_1$, strategy-proofness implies that $r_1(P_1)\neq r_1(P_1')$. It is evident that $r_1(P_1)\,P_1\,r_1(P_1')$. Therefore, to show $\sum_{x\in A}u_1(x)\varphi_x(P_1,P_2)-\sum_{x\in A}u_1(x)\varphi_x(P_1',P_2)\geq 0$, it suffices to show that for all $x\in\bar{B}(P)$ with $\varphi_x(P)>0$ and $x\neq r_1(P_1')$, $xP_1r_1(P_1')$.

Now, suppose there exists $z'\in\bar{B}(P)$ such that $\varphi_{z'}(P)>0$ and $r_1(P_1')P_1z'$. Firstly, $\bar{B}(P)\subseteq\bar{B}_1(P)$ implies $z'\in\bar{B}_1(P)$. Let s_1 and s_2 be such that $r_1(P_1')=r_{s_1}(P_1)$ and $z'=r_{s_2}(P_1)$. Hence, $1<s_1<s_2$. As we showed in the proof

of Lemma 5.8(iv), $\varphi_{r_1(P_1)}(P) + \sum_{x\in \bar{B}_1(P)} \varphi_x(P) = \varepsilon_1$. Then, $\varphi_{z'}(P) > 0$ and $z' \in \bar{B}_1(P)$ imply $\sum_{k=1}^{s_1} \varphi_{r_k(P_1)}(P) < \sum_{k=1}^{s_2} \varphi_{r_k(P_1)}(P) \leq \varepsilon_1 = \varphi_{r_1(P_1')}(P') \leq \sum_{k=1}^{s_1} \varphi_{r_k(P_1)}(P')$. Therefore, voter 1 manipulates at P via P_1' in φ - a contradiction. This complete the verification of Case 3.

Case 4: (i) $\varphi_{r_1(P_2)}(P) < \varepsilon_2$, and (ii) $\varphi_{r_1(P_1')}(P') \geq \varepsilon_1$ and $\varphi_{r_1(P_2)}(P') \geq \varepsilon_2$.

As in Case 2, we can claim that this case only occur when $N = 3$. Now, $h(P_1, P_2)$ follows from Equation (2). Since $\varphi_{r_1(P_2)}(P) < \varepsilon_2$, we know that voter 2 is the special voter of P by Lemma 5.8(ii). Hence, for all $x \in \bar{B}(P)$, $xP_1r_1(P_2)$ by Lemma 5.8(iii). Then, given $u_1 \in \mathbb{U}(P_1)$, the loss from manipulation in h is $\sum_{x\in A} u_1(x)\varphi_x(P_1, P_2) - \sum_{x\in A} u_1(x)\varphi_x(P_1', P_2) = \frac{\varepsilon_1}{\varepsilon_1+\varepsilon_2}[u_1(r_1(P_1)) - u_1(r_1(P_1'))] + \frac{1}{\varepsilon_1+\varepsilon_2} \sum_{x\in \bar{B}(P)} \varphi_x(P)[u_1(x) - u_1(r_1(P_2))] \geq 0$. This completes the verification of Case 4.

Finally, using symmetric arguments for voter 2, we conclude that h is strategy-proof. This completes the verification of Claim 4 and the proof of Lemma 5.10. $\square$

This concludes the proof of Proposition 5.3. $\square$

Finally, Proposition 5.1, 5.2 and 5.3 conclude the proof of Theorem 5.1.

6 The Weighted Projection Rule

In the verification of the sufficiency part of Theorem 3.3.1, we constructed a weighted projection rule. Here, we briefly describe some important features of such rules.

A projection rule is a DSCF that is strategy-proof, efficient (deterministic counterpart of ex-post efficiency), tops-only and anonymous. A weighted projection rule is a convex combination of all projection rules and inherits all the properties of projection rules mentioned above and satisfies the compromise property. If the weights are chosen to be $1/|A|$, a weighted projection rule also satisfies neutrality.[8]

[8]A RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ is *neutral* if for every permutation $\sigma : A \to A$ and $P, P' \in \mathbb{D}^N$ with $[aP_ib] \Leftrightarrow [\sigma(a)P_i'\sigma(b)]$ for all $i \in I$ and $a, b \in A$, we have $\varphi_a(P) = \varphi_{\sigma(a)}(P')$ for all $a \in A$.

Weighted projection rules *are not* the only RSCFs that satisfy the required properties in Theorem 3.3.1 on single-peaked domains on a tree. One way to see this is to note that a projection rule on a line is a particular case of a phantom voter rule (see [34], [12] and [46]) where all phantom voters have the same peak.[9] Consider the single-peaked domain on a line (see Example 3.2.2), and let the RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$, $N \geq 3$, be a convex combination of all phantom voter rules on the line where every phantom voter rule has strictly positive weight. It is easy to show that φ is ex-post efficient, anonymous, tops-only and strategy-proof, and satisfies the compromise property. However, RSCF φ is not a weighted projection rule since it includes some phantom voter rules with distinct peaks of phantom voters. In the case of two voters, efficiency reduces the number of phantom voters to one. However even in this case, there exist strategy-proof, ex-post efficient and tops-only RSCFs satisfying the compromise property that are *not* weighted projection rules (see Example 6.1).

Example 6.1. Consider domain $\bar{\mathbb{D}}$ in Example 3.2.1. Note that for all $P_i, P_j \in \bar{\mathbb{D}}$ with $r_1(P_i) \neq r_1(P_j)$, either $C(P_i, P_j) = \{a_2\}$ or $C(P_i, P_j) = \emptyset$. Domain $\bar{\mathbb{D}}$ admits the RSCF $\varphi : \bar{\mathbb{D}}^2 \to \Delta(A)$ specified in Table 1. It is easy to verify that φ is ex-post efficient, anonymous and tops-only and satisfies the compromise property.

$P_j \in \bar{\mathbb{D}}$ \ $P_i \in \bar{\mathbb{D}}$	$r_1(P_i) = a_1$	$r_1(P_i) = a_2$	$r_1(P_i) = a_3$	$r_1(P_i) = a_4$
$r_1(P_j) = a_1$	e_{a_1}	$\frac{1}{3}e_{a_1} + \frac{2}{3}e_{a_2}$	$\frac{1}{3}e_{a_1} + \frac{1}{3}e_{a_2} + \frac{1}{3}e_{a_3}$	$\frac{1}{3}e_{a_1} + \frac{1}{6}e_{a_2} + \frac{1}{2}e_{a_4}$
$r_1(P_j) = a_2$	$\frac{1}{3}e_{a_1} + \frac{2}{3}e_{a_2}$	e_{a_2}	$\frac{2}{3}e_{a_2} + \frac{1}{3}e_{a_3}$	$\frac{1}{2}e_{a_2} + \frac{1}{2}e_{a_4}$
$r_1(P_j) = a_3$	$\frac{1}{3}e_{a_1} + \frac{1}{3}e_{a_2} + \frac{1}{3}e_{a_3}$	$\frac{2}{3}e_{a_2} + \frac{1}{3}e_{a_3}$	e_{a_3}	$\frac{1}{6}e_{a_2} + \frac{1}{3}e_{a_3} + \frac{1}{2}e_{a_4}$
$r_1(P_j) = a_4$	$\frac{1}{3}e_{a_1} + \frac{1}{6}e_{a_2} + \frac{1}{2}e_{a_4}$	$\frac{1}{2}e_{a_2} + \frac{1}{2}e_{a_4}$	$\frac{1}{6}e_{a_2} + \frac{1}{3}e_{a_3} + \frac{1}{2}e_{a_4}$	e_{a_4}

Table 1: RSCF $\varphi : \bar{\mathbb{D}}^2 \to \Delta(A)$

There are three maximal paths in G^T: $\{a_1, a_2, a_3\}$, $\{a_1, a_2, a_4\}$ and $\{a_3, a_2, a_4\}$. Accordingly, we have three subdomains: $\bar{\mathbb{D}}_1 = \big\{P_i \in \bar{\mathbb{D}} \big| r_1(P_i) \in \{a_1, a_2, a_3\}\big\}$,

[9] [50] and [51] show that the family of projection rules is uniquely characterized by Pareto optimality and the axiom of replacement dominance over the single-peaked domain.

$\bar{\mathbb{D}}_2 = \{P_i \in \bar{\mathbb{D}} | r_1(P_i) \in \{a_1, a_2, a_4\}\}$ and $\bar{\mathbb{D}}_3 = \{P_i \in \bar{\mathbb{D}} | r_1(P_i) \in \{a_3, a_2, a_4\}\}$. Observe that for every $P \in \bar{\mathbb{D}}^2$, there exists $k \in \{1, 2, 3\}$ (not necessarily unique) such that $P \in \bar{\mathbb{D}}_k^2$.[10]

The RSCF φ is defined by considering a separate weighted projection rule for each of the subdomains $\bar{\mathbb{D}}_1, \bar{\mathbb{D}}_2$ and $\bar{\mathbb{D}}_3$. Specifically, for all $P^1 \in \bar{\mathbb{D}}_1^2$, $P^2 \in \bar{\mathbb{D}}_2^2$ and $P^3 \in \bar{\mathbb{D}}_3^2$,

$$\begin{aligned}
\varphi(P^1) &= \frac{1}{3}\phi^{a_1}(P^1) + \frac{1}{3}\phi^{a_2}(P^1) + \frac{1}{3}\phi^{a_3}(P^1), \\
\varphi(P^2) &= \frac{1}{3}\phi^{a_1}(P^2) + \frac{1}{6}\phi^{a_2}(P^2) + \frac{1}{2}\phi^{a_4}(P^2), \\
\varphi(P^3) &= \frac{1}{3}\phi^{a_3}(P^3) + \frac{1}{6}\phi^{a_2}(P^3) + \frac{1}{2}\phi^{a_4}(P^3).
\end{aligned}$$

Note that if $P \in \bar{\mathbb{D}}_k^2$ and $P \in \bar{\mathbb{D}}_{k'}^2$, where $k \neq k'$, $\varphi(P)$ is *identically* induced by the two corresponding distinct weighted projection rules. For instance, $(P_1, P_2) \in \bar{\mathbb{D}}_1^2$ and $(P_1, P_2) \in \bar{\mathbb{D}}_2^2$. According to $\bar{\mathbb{D}}_1$, $\varphi(P_1, P_2) = \frac{1}{3}e_{a_1} + \frac{2}{3}e_{a_2}$, while according to $\bar{\mathbb{D}}_2$, we also have $\varphi(P_1, P_2) = \frac{1}{3}e_{a_1} + \frac{2}{3}e_{a_2}$.

Similar to the verification of strategy-proofness in Example 3.3.2, we fix voter i and check all possible manipulations: $(P_i, P_j) \leftrightarrow (P'_i, P_j)$. It follows from standard arguments that manipulation never occurs within any of the subdomains $\bar{\mathbb{D}}_1$, $\bar{\mathbb{D}}_2$ and $\bar{\mathbb{D}}_3$, i.e., if the true preference and the misrepresentation lie within the same subdomain. We will consider every misrepresentation which leads to an outcome according to a different weighted projection rule relative to truth-telling. It covers three situations and we specify the changes of probabilities in each situation which indicate that probabilities are always transferred from the preferred alternatives to less preferred alternatives according to the true preference.

1. In $(P_i, a_3) \leftrightarrow (P'_i, a_3)$ where $r_1(P_i) = a_1$ and $r_1(P'_i) = a_4$, we have

   $$\varphi(P_i, a_3) \xrightarrow[1/3]{a_1 P_i a_4}, \xrightarrow[1/6]{a_2 P_i a_4} \varphi(P'_i, a_3) \text{ and } \varphi(P'_i, a_3) \xrightarrow[1/3]{a_4 P'_i a_1}, \xrightarrow[1/6]{a_4 P'_i a_2} \varphi(P_i, a_3).$$

2. In $(P_i, a_4) \leftrightarrow (P'_i, a_4)$ where $r_1(P_i) = a_1$ and $r_1(P'_i) = a_3$, we have

[10] For instance, $(P_1, P_2) \in \bar{\mathbb{D}}_1^2$ and $(P_1, P_2) \in \bar{\mathbb{D}}_2^2$.

$$\varphi(P_i, a_4) \xrightarrow[1/3]{a_1 P_i a_3} \varphi(P_i', a_4) \text{ and } \varphi(P_i', a_4) \xrightarrow[1/3]{a_3 P_i' a_1} \varphi(P_i, a_4).$$

3. In $(P_i, a_1) \leftrightarrow (P_i', a_1)$ where $r_1(P_i) = a_3$ and $r_1(P_i') = a_4$, we have

$$\varphi(P_i, a_1) \xrightarrow[1/3]{a_3 P_i a_4}, \xrightarrow[1/6]{a_2 P_i a_4} \varphi(P_i', a_1) \text{ and } \varphi(P_i', a_1) \xrightarrow[1/3]{a_4 P_i' a_3}, \xrightarrow[1/6]{a_4 P_i' a_2} \varphi(P_i, a_1).$$

In conclusion, RSCF φ is strategy-proof.

Last, we verify that φ is *not* a weighted projection rule. Suppose it is not true. Then, there exists $\lambda^{a_k} \geq 0$, $k = 1, 2, 3, 4$ with $\sum_{k=1}^{4} \lambda^{a_k} = 1$ such that $\varphi(P) = \sum_{k=1}^{4} \lambda^{a_k} \phi^{a_k}(P)$ for all $P \in \bar{\mathbb{D}}^2$. We must then have (i) $\lambda^{a_1} = \varphi_{a_1}(a_1, a_2) = \frac{1}{3}$, (ii) $\lambda^{a_3} = \varphi_{a_3}(a_3, a_2) = \frac{1}{3}$, and (iii) $\lambda^{a_4} = \varphi_{a_4}(a_4, a_2) = \frac{1}{2}$. Consequently, $\lambda^{a_1} + \lambda^{a_3} + \lambda^{a_4} > 1$ which is a contradiction. Hence, φ is not a weighted projection rule. $\square$

7 Strategy-proofness in Example 3.3.2

To verify that RSCF φ in Example 3.3.2 is strategy-proof, it suffices to show that in every possible manipulation, probabilities are transferred from preferred alternatives to less preferred alternatives in the true preference while probabilities assigned to other alternatives are unchanged. Note that since RSCF φ is anonymous, we can fix a voter, say voter i, and consider all possible manipulations $(P_i, P_j) \leftrightarrow (P_i', P_j)$.

According to the construction of φ, it is evident that manipulation can never occur if both truth-telling and misrepresentation result in a random dictatorship outcome.

Next, voter i would consider a misrepresentation which makes RSCF φ change from random dictatorship to the weighted projection rule or vice versa. Given $P_i \in \{P_1, P_2\}$, $P_j \in \{P_5, P_6\}$ and $P_i' \in \{P_3, P_4, P_5, P_6\}$, we specify the changes of probabilities in all possible manipulations which indicate that probabilities are always transferred from the preferred alternatives to less preferred alternatives in the true preference.

1. In $(P_1, P_5) \leftrightarrow (P_3, P_5)$, we have

$\varphi(P_1,P_5) \xrightarrow[1/2]{a_1P_1a_2} \varphi(P_3,P_5)$ and $\varphi(P_3,P_5) \xrightarrow[1/2]{a_2P_3a_1} \varphi(P_1,P_5)$.

2. In $(P_1,P_5) \leftrightarrow (P_4,P_5)$ or (P_5,P_5), we have

   $\varphi(P_1,P_5) \xrightarrow[1/2]{a_1P_1a_3} \varphi(P_4,P_5)$ or $\varphi(P_5,P_5)$,

   $\varphi(P_4,P_5) \xrightarrow[1/2]{a_3P_4a_1} \varphi(P_1,P_5)$ and $\varphi(P_5,P_5) \xrightarrow[1/2]{a_3P_5a_1} \varphi(P_1,P_5)$.

3. In $(P_1,P_5) \leftrightarrow (P_6,P_5)$, we have

   $\varphi(P_1,P_5) \xrightarrow[1/6]{a_1P_1a_3}, \xrightarrow[1/3]{a_1P_1a_4} \varphi(P_6,P_5)$ and $\varphi(P_6,P_5) \xrightarrow[1/6]{a_3P_6a_1}, \xrightarrow[1/3]{a_4P_6a_1} \varphi(P_1,P_5)$.

4. In $(P_2,P_5) \leftrightarrow (P_3,P_5)$, the lottery does not change.

5. In $(P_2,P_5) \leftrightarrow (P_4,P_5)$ or (P_5,P_5), we have

   $\varphi(P_2,P_5) \xrightarrow[1/2]{a_2P_2a_3} \varphi(P_4,P_5)$ or $\varphi(P_5,P_5)$,

   $\varphi(P_4,P_5) \xrightarrow[1/2]{a_3P_4a_2} \varphi(P_2,P_5)$ and $\varphi(P_5,P_5) \xrightarrow[1/2]{a_3P_5a_2} \varphi(P_2,P_5)$.

6. In $(P_2,P_5) \leftrightarrow (P_6,P_5)$, we have

   $\varphi(P_2,P_5) \xrightarrow[1/6]{a_2P_2a_3}, \xrightarrow[1/3]{a_2P_2a_4} \varphi(P_6,P_5)$ and $\varphi(P_6,P_5) \xrightarrow[1/6]{a_3P_6a_2}, \xrightarrow[1/3]{a_4P_6a_2} \varphi(P_2,P_5)$.

7. In $(P_1,P_6) \leftrightarrow (P_3,P_6)$, we have

   $\varphi(P_1,P_6) \xrightarrow[1/2]{a_1P_1a_2}, \xrightarrow[1/6]{a_4P_1a_3} \varphi(P_3,P_6)$ and $\varphi(P_3,P_6) \xrightarrow[1/2]{a_2P_3a_1}, \xrightarrow[1/6]{a_3P_3a_4} \varphi(P_1,P_6)$.

8. In $(P_1,P_6) \leftrightarrow (P_4,P_6)$ or (P_5,P_6), we have

   $\varphi(P_1,P_6) \xrightarrow[1/2]{a_1P_1a_3}, \xrightarrow[1/6]{a_4P_1a_3} \varphi(P_4,P_6)$ or $\varphi(P_5,P_6)$,

   $\varphi(P_4,P_6) \xrightarrow[1/2]{a_3P_4a_1}, \xrightarrow[1/6]{a_3P_4a_4} \varphi(P_1,P_6)$ and $\varphi(P_5,P_5) \xrightarrow[1/2]{a_3P_5a_1}, \xrightarrow[1/6]{a_3P_5a_4} \varphi(P_1,P_5)$.

9. In $(P_1,P_6) \leftrightarrow (P_6,P_6)$, we have

   $\varphi(P_1,P_6) \xrightarrow[1/2]{a_1P_1a_4} \varphi(P_6,P_6)$ and $\varphi(P_6,P_6) \xrightarrow[1/2]{a_4P_6a_1} \varphi(P_1,P_6)$.

10. In $(P_2,P_6) \leftrightarrow (P_3,P_6)$, we have

    $\varphi(P_2,P_6) \xrightarrow[1/6]{a_4P_2a_3} \varphi(P_3,P_6)$ and $\varphi(P_3,P_6) \xrightarrow[1/6]{a_3P_3a_4} \varphi(P_2,P_6)$.

11. In $(P_2,P_6) \leftrightarrow (P_4,P_6)$ or (P_5,P_6), we have

    $\varphi(P_2,P_6) \xrightarrow[1/2]{a_2P_2a_3}, \xrightarrow[1/6]{a_4P_2a_3} \varphi(P_4,P_6)$ or $\varphi(P_5,P_6)$,

$$\varphi(P_4,P_6)\xrightarrow[1/2]{a_3P_4a_2},\ \xrightarrow[1/6]{a_3P_4a_4}\varphi(P_2,P_6) \text{ and } \varphi(P_5,P_6)\xrightarrow[1/2]{a_3P_5a_2},\ \xrightarrow[1/6]{a_3P_5a_4}\varphi(P_2,P_6).$$

12. In $(P_2,P_6)\leftrightarrow(P_6,P_6)$, we have

$$\varphi(P_2,P_6)\xrightarrow[1/2]{a_2P_2a_4}\varphi(P_6,P_6) \text{ and } \varphi(P_6,P_6)\xrightarrow[1/2]{a_4P_6a_2}\varphi(P_2,P_6).$$

By a symmetric argument, we know that voter i would neither manipulate at $(P_i,P_j)\in\{P_5,P_6\}\times\{P_1,P_2\}$ via $P_i'\in\{P_1,P_2,P_3,P_4\}$, nor manipulate at $(P_i,P_j)\in\{P_1,P_2,P_3,P_4\}\times\{P_1,P_2\}$ via $P_i'\in\{P_5,P_6\}$.

Last, we show that no manipulation occurs within the weighted projection rule. Accordingly, we consider all possible manipulations $(P_i,P_j)\leftrightarrow(P_i',P_j)$ in the following three jointly exhaustive cases:

(i) $P_i,P_j,P_i'\in\{P_1,P_2,P_3,P_4\}$.

(ii) $P_i,P_j,P_i'\in\{P_3,P_4,P_5,P_6\}$.

(iii) $P_i\in\{P_1,P_2\}$, $P_j\in\{P_3,P_4\}$ and $P_i'\in\{P_5,P_6\}$.

Note that in case (i), $\varphi_{a_4}(P_i,P_j)=0$ and $\varphi_{a_4}(P_i',P_j)=0$. Since preferences P_1,P_2,P_3,P_4 are single-peaked on the sub-line $\{a_1,a_2,a_3\}$, possible manipulations via any of the preferences P_1,P_2,P_3,P_4 are not beneficial. Case (ii) is symmetric to case (i).[11] In case (iii), note that $\varphi(P_2,P_j)=\varphi(P_3,P_j)$ and $\varphi(P_5,P_j)=\varphi(P_4,P_j)$. Then, in case (iii), a manipulation of voter i via P_2 or P_5 is identical to a manipulation via P_3 or P_4 respectively, and hence is nonprofitable according to cases (i) and (ii) respectively. Now, we specify the changes of probabilities in the rest of the possible manipulations in case (iii) which also indicate that probabilities are always transferred from the preferred alternatives to less preferred alternatives in the true preference.

1. In $(P_1,P_j)\leftrightarrow(P_6,P_j)$, we have

$$\varphi(P_1,P_j)\xrightarrow[1/3]{a_1P_1a_4},\ \xrightarrow[1/6]{a_2P_1a_3}\varphi(P_6,P_j) \text{ and } \varphi(P_6,P_j)\xrightarrow[1/3]{a_4P_6a_1},\ \xrightarrow[1/6]{a_3P_1a_2}\varphi(P_1,P_j).$$

[11] In case (ii), $\varphi_{a_1}(P_i,P_j)=0$, $\varphi_{a_1}(P_i',P_j)=0$; and preferences P_3,P_4,P_5,P_6 are single-peaked on the sub-line $\{a_2,a_3,a_4\}$.

2. In $(P_2, P_j) \to (P_6, P_j)$, we have $\varphi(P_2, P_j) \xrightarrow[1/6]{a_2 P_2 a_3}, \xrightarrow[1/3]{a_2 P_2 a_4} \varphi(P_6, P_j)$.

3. In $(P_5, P_j) \to (P_1, P_j)$, we have $\varphi(P_5, P_j) \xrightarrow[1/3]{a_3 P_5 a_1}, \xrightarrow[1/6]{a_3 P_5 a_2} \varphi(P_1, P_j)$.

In conclusion, RSCF φ is strategy-proof.

8 Proof of Lemma 4.1.1

Since $P_i \sim^A P_i'$, $xP_i!y$ and $yP_i'!x$, strategy-proofness implies that for all $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$,

Statement (1) $\quad \varphi_z(P_i, P_j', P_{-\{i,j\}}) = \varphi_z(P_i', P_j', P_{-\{i,j\}})$ for all $z \notin \{x, y\}$.[12]

Therefore, to verify $\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})$, it suffices to show either $\varphi_x(P_i, P_j', P_{-\{i,j\}}) = \varphi_x(P_i', P_j', P_{-\{i,j\}})$ or $\varphi_y(P_i, P_j', P_{-\{i,j\}}) = \varphi_y(P_i', P_j', P_{-\{i,j\}})$.

Next, since x and y are isolated in P_j and P_j', there exists $1 \leq t \leq m-1$ such that either $x \in B^t(P_j) = B^t(P_j')$ and $y \notin B^t(P_j) = B^t(P_j')$, or $x \notin B^t(P_j) = B^t(P_j')$ and $y \in B^t(P_j) = B^t(P_j')$. We assume $x \in B^t(P_j) = B^t(P_j')$ and $y \notin B^t(P_j) = B^t(P_j')$. The verification related to the other case is symmetric and we hence omit it. Consequently, strategy-proofness implies that for all $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$,

Statement (2) $\quad \sum_{z \in B^t(P_j')} \varphi_z(P_i, P_j, P_{-\{i,j\}}) = \sum_{z \in B^t(P_j')} \varphi_z(P_i, P_j', P_{-\{i,j\}})$;

Statement (3) $\quad \sum_{z \in B^t(P_j')} \varphi_z(P_i', P_j, P_{-\{i,j\}}) = \sum_{z \in B^t(P_j')} \varphi_z(P_i', P_j', P_{-\{i,j\}})$.

Finally, we have

$$
\begin{aligned}
& \varphi_x(P_i, P_j', P_{-\{i,j\}}) \\
= \; & \sum_{z \in B^t(P_j')} \varphi_z(P_i, P_j', P_{-\{i,j\}}) - \sum_{z \in B^t(P_j') \setminus \{x\}} \varphi_z(P_i, P_j', P_{-\{i,j\}}) \\
= \; & \sum_{z \in B^t(P_j)} \varphi_z(P_i, P_j, P_{-\{i,j\}}) - \sum_{z \in B^t(P_j') \setminus \{x\}} \varphi_z(P_i, P_j', P_{-\{i,j\}}) \quad \text{(by Statement (2))} \\
= \; & \sum_{z \in B^t(P_j)} \varphi_z(P_i', P_j, P_{-\{i,j\}}) - \sum_{z \in B^t(P_j') \setminus \{x\}} \varphi_z(P_i, P_j', P_{-\{i,j\}}) \quad \text{(by the hypothesis of the lemma)} \\
= \; & \sum_{z \in B^t(P_j)} \varphi_z(P_i', P_j, P_{-\{i,j\}}) - \sum_{z \in B^t(P_j') \setminus \{x\}} \varphi_z(P_i', P_j', P_{-\{i,j\}}) \quad \text{(by Statement (1))} \\
= \; & \sum_{z \in B^t(P_j')} \varphi_z(P_i', P_j', P_{-\{i,j\}}) - \sum_{z \in B^t(P_j') \setminus \{x\}} \varphi_z(P_i', P_j', P_{-\{i,j\}}) \quad \text{(by Statement (3))} \\
= \; & \varphi_x(P_i', P_j', P_{-\{i,j\}}).
\end{aligned}
$$

[12] For the detail of verification, please refer to the proof of Lemma 3 in [26].

Therefore, $\varphi(P_i, P'_j, P_{-\{i,j\}}) = \varphi(P'_i, P'_j, P_{-\{i,j\}})$.

9 Proof of Proposition 4.2.2

The proof of Proposition 4.2.2 consists of three steps.

Step 1 includes Lemmas 9.1 - 9.5. Each lemma shows the existence of some multi-dimensional single-peaked preference satisfying some particular properties. Step 1 serves as a preparation for the verifications in Steps 2 and 3.

Step 2 includes Lemmas 9.6 and 9.7. Lemma 9.6 shows that when two distinct multi-dimensional single-peaked preferences P_i and P'_i share the same peak, there exists an AC-path connecting them such that for every pair of alternatives with the same relative rankings across P_i and P'_i, the relative ranking of them is fixed along the whole AC-path. The proof of Lemma 9.6 is a repeated application of Lemma 9.1. We provide a simple example to illustrate before Lemma 9.6. Lemma 9.7 shows that when two multi-dimensional single-peaked preferences P_i and P'_i disagree on peaks in *exactly* one component, and agree on the relative rankings on some pair of alternatives $x, y \in A$, there exists a (x, y)-Is-path connecting them. The construction of the (x, y)-Is-path in the proof of Lemma 9.7 relies completely on the existence of some particular multi-dimensional single-peaked preferences specified in Lemmas 9.3 and 9.4, and the AC-path constructed in Lemma 9.6.

Step 3 shows that $\mathbb{D}_{MSP}$ satisfies the Interior Property and the Exterior Property.

Now, we start Step 1.

Lemma 9.1. *Given $P_i, P'_i \in \mathbb{D}^a_{MSP}$, assume $xP_i!y$ and yP'_ix. There exists $P''_i \in \mathbb{D}^a_{MSP}$ such that $P''_i \sim^A P_i$ and $yP''_i!x$.*

Proof. Since $r_1(P_i) = r_1(P'_i) = a$, it is evident that $a \notin \{x, y\}$. Let P''_i be a preference induced by locally switching x and y in P_i. Thus, $r_1(P''_i) = a$, $P''_i \sim^A P_i$ and $yP''_i!x$. We will show that $P''_i \in \mathbb{D}_{MSP}$.

Suppose not, i.e., there exist $x', y' \in A$ such that $x' \in \langle a, y' \rangle$ and $y'P''_ix'$. Since $x' \in \langle a, y' \rangle$, we know $x'P_iy'$. Since $P_i \sim^A P''_i$, $xP_i!y$ and $yP''_i!x$, it must be the case that $x' = x$ and $y' = y$. Consequently, $x \in \langle a, y\rangle$ and hence $xP_i'y$. Contradiction! Therefore, $P_i'' \in \mathbb{D}_{MSP}$. $\square$

Lemma 9.2. *Given $P_i \in \mathbb{D}_{MSP}^a$, $s \in M$ and $c^s \in A^s$ with $\langle a^s, c^s\rangle = \{a^s, c^s\}$, there exists $P_i' \in \mathbb{D}_{MSP}^a$ satisfying the following two conditions:*

(1) *for all $x, y \notin (c^s, A^{-s})$, $[xP_iy] \Leftrightarrow [xP_i'y]$;*

(2) *for all $z^{-s} \in A^{-s}$, $(a^s, z^{-s})P_i'!(c^s, z^{-s})$.*

Proof. We first construct a preference P_i' satisfying conditions (1) and (2) by the following method. First, we remove all alternatives in (c^s, A^{-s}) from P_i, and thus have an induced preference $\big(P_i, A\backslash(c^s, A^{-s})\big)$. Next, we construct preference P_i' over A by plugging all alternatives in (c^s, A^{-s}) back into the induced preference $\big(P_i, A\backslash(c^s, A^{-s})\big)$ in a particular way: for all $z^{-s} \in A^{-s}$, $(a^s, z^{-s})P_i'!(c^s, z^{-s})$. Evidently, $r_1(P_i') = a$. In the rest of the proof, we show $P_i' \in \mathbb{D}_{MSP}$.

Given $x, y \in A$ with $x \in \langle a, y\rangle\backslash\{y\}$, we will show that $xP_i'y$. Note that xP_iy and $(a^s, z^{-s})P_i(c^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. We consider four cases: (i) $x, y \notin (c^s, A^{-s})$, (ii) $x \notin (c^s, A^{-s})$ and $y \in (c^s, A^{-s})$, (iii) $x \in (c^s, A^{-s})$ and $y \notin (c^s, A^{-s})$ and (iv) $x, y \in (c^s, A^{-s})$.

In case (i), xP_iy implies $xP_i'y$ by condition (1).

In case (ii), $y = (c^s, y^{-s})$. Since $x \in \langle a, y\rangle = \langle a, (c^s, y^{-s})\rangle$ and $\langle a^s, c^s\rangle = \{a^s, c^s\}$, we know $x^s \in \{a^s, c^s\}$ and $x^{-s} \in \langle a^{-s}, y^{-s}\rangle$. Moreover, $x \notin (c^s, A^{-s})$ implies $x^s = a^s$. Hence $x \in \langle a, (a^s, y^{-s})\rangle$. Now, either $x = (a^s, y^{-s})$ or $xP_i(a^s, y^{-s})$. If $x = (a^s, y^{-s})$, then $xP_i'y$ by condition (2). If $xP_i(a^s, y^{-s})$, condition (1) first implies $xP_i'(a^s, y^{-s})$. Next, since $(a^s, y^{-s})P_i'y$ by condition (2), we have $xP_i'y$.

In case (iii), $x = (c^s, x^{-s})$. Evidently, since $(a^s, x^{-s})P_ix$ and xP_iy, we have $(a^s, x^{-s})P_iy$. Then, by condition (1), $(a^s, x^{-s})P_i'y$. Furthermore, since $(a^s, x^{-s})P_i'!x$ by condition (2), it must be the case that $xP_i'y$.

In case (iv), $x = (c^s, x^{-s})$ and $y = (c^s, y^{-s})$ where $x^{-s} \neq y^{-s}$. Since $x \in \langle a, y\rangle$, it is true that $x^{-s} \in \langle a^{-s}, y^{-s}\rangle$ and hence $(a^s, x^{-s}) \in \langle a, (a^s, y^{-s})\rangle$. Consequently,

$(a^s, x^{-s})P_i(a^s, y^{-s})$. Then, condition (1) implies $(a^s, x^{-s})P_i'(a^s, y^{-s})$. Furthermore, since $(a^s, x^{-s})P_i'!x$ and $(a^s, y^{-s})P_i'!y$ by condition (2), we have $xP_i'y$. In conclusion, $P_i' \in \mathbb{D}_{MSP}$. □

Lemma 9.3. *Given $P_i \in \mathbb{D}_{MSP}^a$, $s \in M$ and $c^s \in A^s$, assume $(a^s, z^{-s})P_i!(c^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. There exists $P_i' \in \mathbb{D}_{MSP}$ satisfying the following two conditions*

(1) *for every $z^{-s} \in A^{-s}$,*

$$\left[r_k(P_i) = (a^s, z^{-s}) \text{ and } r_{k+1}(P_i) = (c^s, z^{-s})\right]$$

$$\Rightarrow \left[r_k(P_i') = (c^s, z^{-s}) \text{ and } r_{k+1}(P_i') = (a^s, z^{-s})\right];$$

(2) *for all $x \notin (a^s, A^{-s}) \cup (c^s, A^{-s})$, $[x = r_k(P_i)] \Rightarrow [x = r_k(P_i')]$.*

Proof. We first construct a preference P_i' satisfying conditions (1) and (2) by flipping the relative ranking of (a^s, z^{-s}) and (c^s, z^{-s}) in P_i for each $z^{-s} \in A^{-s}$, and keeping the rankings of all other alternatives fixed. In the rest of the proof, we show $P_i' \in \mathbb{D}_{MSP}$. Note that since $r_1(P_i) = a$ and $aP_i!(c^s, a^{-s})$, it is true that $r_2(P_i) = (c^s, a^{-s})$ and hence $r_1(P_i') = (c^s, a^{-s})$.

Suppose $P_i' \notin \mathbb{D}_{MSP}$. Then, there exist $x, y \in A$ such that $x \in \langle (c^s, a^{-s}), y\rangle$ and $yP_i'x$. We know either xP_iy or yP_ix. If xP_iy, then $yP_i'x$ implies $x = (a^s, z^{-s})$ and $y = (c^s, z^{-s})$ for some $z^{-s} \in A^{-s}$ by conditions (1) and (2). Consequently, $x = (a^s, z^{-s}) \notin \langle (c^s, a^{-s}), (c^s, z^{-s})\rangle = \langle (c^s, a^{-s}), y\rangle$. Contradiction!

Next, assume yP_ix. Then, it is true that $x \notin \langle a, y\rangle$. Since $x \in \langle (c^s, a^{-s}), y\rangle$, we know $x^s \in \langle c^s, y^s\rangle$ and $x^{-s} \in \langle a^{-s}, y^{-s}\rangle$. Furthermore, $x \notin \langle a, y\rangle$ implies $x^s \notin \langle a^s, y^s\rangle$. Since $a = r_1(P_i)$ and $(c^s, a^{-s}) = r_2(P_i)$, it is true that $\langle a^s, c^s\rangle = \{a^s, c^s\}$. Since $x^s \in \langle c^s, y^s\rangle$, $\langle a^s, c^s\rangle = \{a^s, c^s\}$ and $x^s \notin \langle a^s, y^s\rangle$, it must be the case that $a^s \in \langle c^s, y^s\rangle$ and $x^s = c^s$. Thus, $x = (c^s, x^{-s})$. Since $x^{-s} \in \langle a^{-s}, y^{-s}\rangle$, we have $(a^s, x^{-s}) \in \langle a, y\rangle$. Thus, either $(a^s, x^{-s})P_iy$ or $(a^s, x^{-s}) = y$. If $(a^s, x^{-s})P_iy$, then $(a^s, x^{-s})P_i!x$ implies xP_iy. Contradiction! Therefore, $(a^s, x^{-s}) = y$ and hence $yP_i!x$. Consequently, $xP_i'!y$ by condition (1). Contradiction to the hypothesis! Therefore, $P_i' \in \mathbb{D}_{MSP}$. □

Lemma 9.4. *Given $P_i \in \mathbb{D}^a_{MSP}$ and $P_i' \in \mathbb{D}^{(b^s,a^{-s})}_{MSP}$ with $a^s \neq b^s$, assume xP_iy and $xP_i'y$. There exists $P_i'' \in \mathbb{D}^a_{MSP}$ satisfying the following two conditions:*

(1) *for every $z^{-s} \in A^{-s}$, $(a^s, z^{-s})P_i''!(c^s, z^{-s})$ where $c^s \in \langle a^s, b^s\rangle$ and $\langle a^s, c^s\rangle = \{a^s, c^s\}$;*

(2) *$xP_i''y$.*

Proof. We consider two situations: (i) $y \notin (c^s, A^{-s})$ and (ii) $y \in (c^s, A^{-s})$.

Assume that situation (i) occurs. Let $P_i'' \in \mathbb{D}^a_{MSP}$ be a preference induced by P_i satisfying conditions (1) and (2) in Lemma 9.2. Hence, condition (1) of this lemma is satisfied. Evidently, either $x \notin (c^s, A^{-s})$ or $x \in (c^s, A^{-s})$. If $x \notin (c^s, A^{-s})$, by condition (1), xP_iy implies $xP_i''y$. Next, if $x \in (c^s, A^{-s})$, then $x = (c^s, x^{-s})$. Since $(a^s, x^{-s}) \in \langle a, x\rangle$ and xP_iy, we have $(a^s, x^{-s})P_ix$ and hence $(a^s, x^{-s})P_iy$. Then, condition (1) in Lemma 9.2 implies $(a^s, x^{-s})P_i''y$. Furthermore, since $(a^s, x^{-s})P_i''!x$ by condition (1) of this Lemma, it must be the case that $xP_i''y$. This completes the verification of situation (i).

Next, assume that situation (ii) occurs. Thus, $y = (c^s, y^{-s})$. Evidently, either $x \in (c^s, A^{-s})$ or $x \notin (c^s, A^{-s})$. First, assume $x \in (c^s, A^{-s})$. Thus, $x = (c^s, x^{-s})$. Since xP_iy, it is true that $(c^s, y^{-s}) = y \notin \langle a, x\rangle = \langle a, (c^s, x^{-s})\rangle$. Consequently, $y^{-s} \notin \langle a^{-s}, x^{-s}\rangle$ and hence $(a^s, y^{-s}) \notin \langle a, (a^s, x^{-s})\rangle$. Then, there exists $\bar{P}_i \in \mathbb{D}^a_{MSP}$ such that $(a^s, x^{-s})\bar{P}_i(a^s, y^{-s})$. Let $P_i'' \in \mathbb{D}^a_{MSP}$ be a preference induced by $\bar{P}_i$ satisfying conditions (1) and (2) in Lemma 9.2. Hence, condition (1) of this lemma is satisfied. Since $(a^s, x^{-s})\bar{P}_i(a^s, y^{-s})$, condition (1) in Lemma 9.2 implies $(a^s, x^{-s})P_i''(a^s, y^{-s})$. Since $(a^s, x^{-s})P_i''!x$ and $(a^s, y^{-s})P_i''!y$ by condition (1) of this Lemma, we have $xP_i''y$.

Lastly, assume $x \notin (c^s, A^{-s})$. We claim that $(a^s, y^{-s}) \notin \langle a, x\rangle$. Suppose not, i.e., $(a^s, y^{-s}) \in \langle a, x\rangle$. Thus, $y^{-s} \in \langle a^{-s}, x^{-s}\rangle$. Since $c^s \in \langle a^s, b^s\rangle$, it is true that either $c^s \in \langle a^s, x^s\rangle$ or $c^s \in \langle b^s, x^s\rangle$. Consequently, either $y = (c^s, y^{-s}) \in \langle (a^s, a^{-s}), (x^s, x^{-s})\rangle = \langle a, x\rangle$, or $y = (c^s, y^{-s}) \in \langle (b^s, a^{-s}), (x^s, x^{-s})\rangle = \langle (b^s, a^{-s}), x\rangle$, and hence either yP_ix or $yP_i'x$. Contradiction! Therefore, $(a^s, y^{-s}) \notin \langle a, x\rangle$. Accordingly, there exists $\bar{P}_i \in \mathbb{D}_{MSP}$ such that $x\bar{P}_i(a^s, y^{-s})$. Now, let $P_i'' \in \mathbb{D}^a_{MSP}$ be a preference induced by $\bar{P}_i$ satisfying conditions (1) and (2) in Lemma 9.2. Hence, condition (1) of this lemma is satisfied. By condition (1) in Lemma 9.2, $x\bar{P}_i(a^s, y^{-s})$ implies $xP_i''(a^s, y^{-s})$. Next, since $(a^s, y^{-s})P_i''!y$ by condition (1) of this lemma, we have $xP_i''y$. This completes the verification of situation (ii) and hence the lemma. $\square$

Lemma 9.5. *Given $P_i \in \mathbb{D}^a_{MSP}$ and $P_i' \in \mathbb{D}^b_{MSP}$, assume $a^s \neq b^s$ for all $s \in S$ where $S \subseteq M$ and $|S| \geq 2$, and $a^{-S} = b^{-S}$. Given $x, y \in A$, assume xP_iy and $xP_i'y$. There exist $s \in S$ and $\bar{P}_i \in \mathbb{D}^{(b^s, a^{-s})}_{MSP}$ such that $x\bar{P}_iy$.*

Proof. Suppose that it is not true. Then, for all $s \in S$ and $\bar{P}_i \in \mathbb{D}^{(b^s, a^{-s})}_{MSP}$, $y\bar{P}_ix$. Consequently, for every $s \in S$, $y \in \langle (b^s, a^{-s}), x\rangle$. Thus, $y^s \in \langle b^s, x^s\rangle$ for all $s \in S$, and $y^{-S} \in \langle a^{-S}, x^{-S}\rangle$. Consequently, $y \in \langle (b^S, a^{-S}), x\rangle = \langle b, x\rangle$, and hence $yP_i'x$. Contradiction! $\square$

This completes the verification of Step 1. We turn to Step 2.

We first provide a simple example to illustrate Lemma 9.6 below. Given $P_i, P_i' \in \mathbb{D}^a_{MSP}$ specified below, we will construct a *particular* AC-path connecting P_i and P_i' in $\mathbb{D}^a_{MSP}$.

$$P_i: \quad a \succ b \succ c \succ y \succ x_2 \succ x_1 \succ x \succ \cdots$$

$$P_i': \quad a \succ b \succ c \succ x \succ \cdots\cdots \succ y \succ \cdots$$

Observe that P_i and P_i' agree on the top-three alternatives and disagree on the forth-ranked alternatives. There are exactly alternatives x_2 and x_1 ranked between y and x in P_i. Then, by Lemma 9.1, we can identify the following three preferences

$\bar{P}_i, \hat{P}_i, \tilde{P}_i \in \mathbb{D}^a_{MSP}$:

$$\begin{aligned}
\bar{P}_i: &\quad a \succ b \succ c \succ y \succ x_2 \succ x \succ x_1 \succ \cdots \\
\hat{P}_i: &\quad a \succ b \succ c \succ y \succ x \succ x_2 \succ x_1 \succ \cdots \\
\tilde{P}_i: &\quad a \succ b \succ c \succ x \succ y \succ x_2 \succ x_1 \succ \cdots
\end{aligned}$$

where (i) $P_i \sim^A \bar{P}_i$; $x_1 P_i!x$ and $x\bar{P}_i!x_1$; (ii) $\bar{P}_i \sim^A \hat{P}_i$; $x_2\bar{P}_i!x$ and $x\hat{P}_i!x_2$; and (iii) $\hat{P}_i \sim^A \tilde{P}_i$; $y\hat{P}_i!x$ and $x\tilde{P}_i!y$. Now, $\tilde{P}_i$ is "closer" to P'_i than P_i, since $\tilde{P}_i$ and P'_i agree on the top-four ranked alternatives. Next, we identify another pair of distinct alternatives in the same ranking position of $\tilde{P}_i$ and P'_i such that $\tilde{P}_i$ and P'_i agree on all alternatives ranked above, i.e., $r_k(\tilde{P}_i) \neq r_k(P'_i)$ for some $k > 4$ and $r_{k'}(\tilde{P}_i) = r_{k'}(P'_i)$ for all $1 \leq k' < k$. Then, applying the same argument, we can construct another sequence of adjacently connected preferences in $\mathbb{D}^a_{MSP}$ starting from $\tilde{P}_i$ and reaching some preference P''_i "closer" to P'_i. Eventually, we will have an AC-path in $\mathbb{D}^a_{MSP}$ connecting P_i and P'_i.

Lemma 9.6. *Given distinct $P_i, P'_i \in \mathbb{D}_{MSP}$, assume $r_1(P_i) = r_1(P'_i) \equiv a$. There exists an AC-path $\{P^k_i\}^l_{k=1} \subseteq \mathbb{D}^a_{MSP}$ connecting P_i and P'_i such that for all $x, y \in A$, $[xP_iy$ and $xP'_iy] \Rightarrow [xP^k_iy, 1 < k < l]$.*

Proof. Following the algorithm below, we generate an AC-path in $\mathbb{D}^a_{MSP}$ connecting P_i and P'_i.

Algorithm:

Step 1 : Identify the minimal $k \in \{1, \ldots, m\}$ such that $r_k(P_i) \neq r_k(P'_i)$ (evidently, $k > 1$). For notational convenience, let $r_k(P'_i) = x$. Assume $x = r_{\bar{k}}(P_i)$ (evidently, $\bar{k} > k$). Moreover, for notational convenience, let $r_\nu(P_i) = x_{\bar{k}-\nu}$, $k \leq \nu \leq \bar{k} - 1$. By Lemma 9.1, we construct a sequence $\{P^{(1,\nu)}_i\}^{l_1}_{\nu=1} \subseteq \mathbb{D}^a_{MSP}$, where $l_1 = \bar{k} - k$, such that

$$P^{(1,\nu-1)}_i \sim^A P^{(1,\nu)}_i, x_\nu P^{(1,\nu-1)}_i!x \text{ and } xP^{(1,\nu)}_i!x_\nu, \nu = 1, \ldots, l_1, \text{ where } P^{(1,0)}_i = P_i.$$

Step $t \geq 2$: According to $P_i^{(t-1,l_{t-1})}$ generated in Step $t-1$, identify the minimal $k \in \{1, \ldots, m\}$ such that $r_k(P_i^{(t-1,l_{t-1})}) \neq r_k(P_i')$. For notational convenience, let $r_k(P_i') = x$. Assume $x = r_{\bar{k}}(P_i^{(t-1,l_{t-1})})$ (evidently, $\bar{k} > k$). Moreover, for notational convenience, let $r_\nu(P_i^{(t-1,l_{t-1})}) = x_{\bar{k}-\nu}$, $k \leq \nu \leq \bar{k}-1$. By Lemma 9.1, we construct a sequence $\{P_i^{(t,\nu)}\}_{\nu=1}^{l_t} \subseteq \mathbb{D}^a_{MSP}$, where $l_t = \bar{k} - k$, such that

$$P_i^{(t,\nu-1)} \sim^A P_i^{(t,\nu)}, x_\nu P_i^{(t,\nu-1)}!x \text{ and } xP_i^{(t,\nu)}!x_\nu, \nu = 1, \ldots, l_t, \text{ where } P_i^{(t,0)} = P_i^{(t-1,l_{t-1})}.$$

If $r_k(P_i^{(t-1,l_{t-1})}) = r_k(P_i')$, $k = 1, \ldots, m$, (in other words, $P_i^{(t-1,l_{t-1})} = P_i'$), the algorithm terminates.

Evidently, this algorithm will terminate in finite steps. Assume that the algorithm terminates at *Step* $t+1$. Then, we have sequences of preferences $\{P_i\}$, $\{P_i^{(1,\nu)}\}_{\nu=1}^{l_1}, \ldots, \{P_i^{(t,\nu)}\}_{\nu=1}^{l_t}$. Combining these sequences, we have an AC-path

$$\{P_i^k\}_{k=1}^l \equiv \{P_i; P_i^{(1,1)}, \ldots, P_i^{(1,l_1)}; \ldots; P_i^{(t,1)}, \ldots, P_i^{(t,l_t)}\} \subseteq \mathbb{D}^a_{MSP}$$

connecting P_i and P_i'.

Next, given $x, y \in A$ with xP_iy and $xP_i'y$, we will show that xP_i^ky, $1 < k < l$. Suppose not, i.e., there exists $1 < k < l$ such that yP_i^kx. Assume w.l.o.g. that $xP_i^{k'}y$ for all $1 \leq k' < k$. Thus, $xP_i^{k-1}!y$ and $yP_i^k!x$. Moreover, we can assume that P_i^k is generated in *Step* s of the algorithm, i.e., $P_i^k = P_i^{(s,\nu)}$ and $P_i^{k-1} = P_i^{(s,\nu-1)}$ for some $1 \leq s \leq t$ and some $1 \leq \nu \leq l_s$. Thus, $P_i^{(s,\nu-1)} \sim^A P_i^{(s,\nu)}$, $xP_i^{(s,\nu-1)}!y$ and $yP_i^{(s,\nu)}!x$. Then, according to the algorithm, it must be the case that $yP_i'x$. Contradiction! $\square$

Note that according to Remark 4.1.1, in Lemma 9.6, for all $x, y \in A$ with xP_iy and $xP_i'y$, the AC-path $\{P_i^k\}_{k=1}^l$ is also a (x, y)-Is-path connecting P_i and P_i'.

Lemma 9.7. *Given $P_i, P_i' \in \mathbb{D}_{MSP}$, assume $r_1(P_i) = a$ and $r_1(P_i') = (b^s, a^{-s})$ where $b^s \neq a^s$ for some $s \in M$. Given $x, y \in A$, assume xP_iy and $xP_i'y$. There* *exists a (x,y)-Is-path in $\mathbb{D}_{MSP}$ connecting P_i and P'_i.*

Proof. Assume $\langle a^s, b^s\rangle = \{a^s_k\}_{k=1}^t$ where $t \geq 2$, $a^s_1 = a^s$, $a^s_t = b^s$, and $a^s_k \in \langle a^s_1, a^s_{k+1}\rangle$, $k = 1, \ldots, t-1$. Accordingly, $a^s_{k+1} \in \langle a^s_k, a^s_t\rangle$, $k = 1, \ldots, t-1$.

Claim 1: For every $z^{-s} \in A^{-s}$ and $1 \leq k \leq t-1$, $\{x, y\} \neq \{(a^s_k, z^{-s}), (a^s_{k+1}, z^{-s})\}$.

Given $z^{-s} \in A^{-s}$ and $1 \leq k \leq t-1$, since $(a^s_k, z^{-s}) \in \langle (a^s_1, a^{-s}), (a^s_{k+1}, z^{-s})\rangle = \langle a, (a^s_{k+1}, z^{-s})\rangle$ and $(a^s_{k+1}, z^{-s}) \in \langle (a^s_t, a^{-s}), (a^s_k, z^{-s})\rangle = \langle (b^s, a^{-s}), (a^s_k, z^{-s})\rangle$, it is true that $(a^s_k, z^{-s}) P_i (a^s_{k+1}, z^{-s})$ and $(a^s_{k+1}, z^{-s}) P'_i (a^s_k, z^{-s})$. Consequently, xP_iy and xP'_iy imply $\{x, y\} \neq \{(a^s_k, z^{-s}), (a^s_{k+1}, z^{-s})\}$. This completes the verification of the claim.

Now, we identify $t-1$ pairs of multi-dimensional single-peaked preferences $\{(\bar{P}^k_i, \hat{P}^k_i)\}_{k=1}^{t-1}$ specified below by repeated application of Lemmas 9.4 and 9.3:

$$
\begin{array}{lll}
P_i: & (a^s_1, a^{-s}) \succ \cdots \succ x \succ \cdots \succ y \succ \cdots & \\
\vdots & & \\
\bar{P}^1_i: & (a^s_1, a^{-s}) \succ (a^s_2, a^{-s}) \succ \cdots \succ (a^s_1, z^{-s}) \succ (a^s_2, z^{-s}) \succ \cdots & \text{with } x\bar{P}^1_iy \\
\hat{P}^1_i: & (a^s_2, a^{-s}) \succ (a^s_1, a^{-s}) \succ \cdots \succ (a^s_2, z^{-s}) \succ (a^s_1, z^{-s}) \succ \cdots & \text{with } x\hat{P}^1_iy \\
\vdots & & \\
\bar{P}^k_i: & (a^s_k, a^{-s}) \succ (a^s_{k+1}, a^{-s}) \succ \cdots \succ (a^s_k, z^{-s}) \succ (a^s_{k+1}, z^{-s}) \succ \cdots & \text{with } x\bar{P}^k_iy \\
\hat{P}^k_i: & (a^s_{k+1}, a^{-s}) \succ (a^s_k, a^{-s}) \succ \cdots \succ (a^s_{k+1}, z^{-s}) \succ (a^s_k, z^{-s}) \succ \cdots & \text{with } x\hat{P}^k_iy \\
\vdots & & \\
\bar{P}^{t-1}_i: & (a^s_{t-1}, a^{-s}) \succ (a^s_t, a^{-s}) \succ \cdots \succ (a^s_{t-1}, z^{-s}) \succ (a^s_t, z^{-s}) \succ \cdots & \text{with } x\bar{P}^{t-1}_iy \\
\hat{P}^{t-1}_i: & (a^s_t, a^{-s}) \succ (a^s_{t-1}, a^{-s}) \succ \cdots \succ (a^s_t, z^{-s}) \succ (a^s_{t-1}, z^{-s}) \succ \cdots & \text{with } x\hat{P}^{t-1}_iy \\
\vdots & & \\
P'_i: & (a^s_t, a^{-s}) \succ \cdots \succ x \succ \cdots \succ y \succ \cdots &
\end{array}
$$

According to Lemma 9.4, $r_1(\bar{P}^1_i) = r_1(P_i) = a$, $(a^s_1, z^{-s})\bar{P}^1_i!(a^s_2, z^{-s})$ for every $z^{-s} \in A^{-s}$, and $x\bar{P}^1_iy$. Next, according to Lemma 9.3, we can induce $\hat{P}^1_i$ from $\bar{P}^1_i$ by flipping the relative ranking of (a^s_1, z^{-s}) and (a^s_2, z^{-s}) in $\bar{P}^1_i$ for every $z^{-s} \in A^{-s}$, and keeping the ranking of every other alternative fixed. Moreover, by Claim 1 and conditions (1) and (2) in Lemma 9.3, $x\bar{P}_i^1 y$ implies $x\hat{P}_i^1 y$. Then, it is easy to verify that x and y are isolated in $(\bar{P}_i^1, \hat{P}_i^1)$. By a similar argument, for all $k = 2, \ldots, t-1$, we have the pair of multi-dimensional single-peaked preferences $\bar{P}_i^k$ and $\hat{P}_i^k$, where $r_1(\hat{P}_i^{k-1}) = r_1(\bar{P}_i^k)$, $x\bar{P}_i^k y$, $x\hat{P}_i^k y$, x and y are isolated in $\bar{P}_i^k$ and $\hat{P}_i^k$.

For notational convenience, let $\hat{P}_i^0 = P_i$ and $\bar{P}_i^t = P_i'$. For every $1 \leq k \leq t$, since $r_1(\hat{P}_i^{k-1}) = r_1(\bar{P}_i^k) = (a_k^s, a^{-s})$, $x\hat{P}_i^{k-1} y$ and $x\bar{P}_i^k y$, Lemma 9.8 implies that there exists a (x, y)-Is-path in $\mathbb{D}_{MSP}$ connecting $\hat{P}_i^{k-1}$ and $\bar{P}_i^k$. Combining all these t (x, y)-Is-paths, we eventually have a (x, y)-Is-path in $\mathbb{D}_{MSP}$ connecting P_i and P_i'. □

This completes the verification of Step 2. Now, we turn to Step 3.

Lemma 9.8. *Domain $\mathbb{D}_{MSP}$ satisfies the Interior Property.*

Proof. This lemma follows from Lemma 9.6. □

Lemma 9.9. *Domain $\mathbb{D}_{MSP}$ satisfies the Exterior Property.*

Proof. We fix $P_i, P_i \in \mathbb{D}_{MSP}$ with $r_1(P_i) \neq r_1(P_i')$ and $x, y \in A$ with $xP_i y$ and $xP_i' y$. We consider two situations: (i) $r_1(P_i)$ and $r_1(P_i')$ disagree on exactly one component, and (ii) $r_1(P_i)$ and $r_1(P_i')$ disagree on at least two components.

In situation (i), the requirement of the Exterior Property follows from Lemma 9.7.

In situation (ii), we assume $r_1(P_i) = a$ and $r_1(P_i') = (b^S, a^{-S})$ where $a^s \neq b^s$ for all $s \in S$, $S \subseteq M$ and $|S| \geq 2$. Repeatedly applying Lemma 9.5 step by step, we can relabel $S = \{1, \ldots, s\}$ such that for all $1 \leq k \leq s-1$, there exists $\bar{P}_i^k \in \mathbb{D}_{MSP}$ such that $r_1(\bar{P}_i^k) = (b^1, \ldots, b^k, a^{k+1}, \ldots, a^s, a^{-S})$ and $x\bar{P}_i^k y$.

Let $\bar{P}_i^0 = P_i$ and $\bar{P}_i^s = P_i'$. Thus, (i) for all $0 \leq k \leq s$, $x\bar{P}_i^k y$; and (ii) for all $0 \leq k \leq s-1$, $r_1(\bar{P}_i^k)$ and $r_1(\bar{P}_i^{k+1})$ disagree on exactly one component. Now, for each $0 \leq k \leq s-1$, by Lemma 9.7, there exists a (x, y)-Is-path in $\mathbb{D}_{MSP}$ connecting $\bar{P}_i^k$ and $\bar{P}_i^{k+1}$. Finally, combining these (x, y)-Is-paths, we have a (x, y)-Is-path in $\mathbb{D}_{MSP}$ connecting P_i and P_i'. □

This completes the verification of Step 3 and hence Proposition 4.2.2.

10 Proof of Lemma 4.2.1

Fixing $P_{-\{i,j\}} \in \mathbb{D}^{N-2}$, we have $\varphi(P_i, P_j, P_{-\{i,j\}}) = \varphi(P'_i, P_j, P_{-\{i,j\}})$. Evidently, strategy-proofness implies the following two statements:

Statement (1) For all $z^{-s} \in A^{-s}$,

- $\varphi_{(x^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) + \varphi_{(y^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}})$
$$= \varphi_{(x^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}}) + \varphi_{(y^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}});$$
- $\varphi_{(x^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) \geq \varphi_{(x^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$;
- $\varphi_{(y^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) \leq \varphi_{(y^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$.

Statement (2) For all $z \notin (x^s, A^{-s}) \cup (y^s, A^{-s})$, $\varphi_z(P_i, P'_j, P_{-\{i,j\}}) = \varphi_z(P'_i, P'_j, P_{-\{i,j\}})$.

According to the first item in Statement (1) and Statement (2), to show $\varphi(P_i, P'_j, P_{-\{i,j\}}) = \varphi(P'_i, P'_j, P_{-\{i,j\}})$, it suffices to show that for every $z^{-s} \in A^{-s}$, there exists $c^s \in \{x^s, y^s\}$ such that $\varphi_{(c^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) = \varphi_{(c^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$.

Similarly, since $P_j \sim P'_j$, either $P_j \sim^A P'_j$ or $P_j \sim^{MA} P'_j$. If $P_j \sim^A P'_j$, assume $aP_j!b$ and $bP'_j!a$. Thus, $\varphi_z(P_i, P_j, P_{-\{i,j\}}) = \varphi_z(P_i, P'_j, P_{-\{i,j\}})$ and $\varphi_z(P'_i, P_j, P_{-\{i,j\}}) = \varphi_z(P'_i, P'_j, P_{-\{i,j\}})$ for all $z \notin \{a, b\}$ by strategy-proofness. Given $z^{-s} \in A^{-s}$, since $(x^s, z^{-s})P_i!(y^s, z^{-s})$ and $(y^s, z^{-s})P'_i!(x^s, z^{-s})$, the hypothesis implies that (x^s, z^{-s}) and (y^s, z^{-s}) are isolated in (P_j, P'_j). Moreover, by Remark 4.1.1, it is true that either $(x^s, z^{-s})P_j(y^s, z^{-s})$ and $(x^s, z^{-s})P'_j(y^s, z^{-s})$, or $(y^s, z^{-s})P_j(x^s, z^{-s})$ and $(y^s, z^{-s})P'_j(x^s, z^{-s})$ which implies $\{(x^s, z^{-s}), (y^s, z^{-s})\} \neq \{a, b\}$. Hence, there exists $c^s \in \{x^s, y^s\}$ such that $\varphi_{(c^s,z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) = \varphi_{(c^s,z^{-s})}(P_i, P_j, P_{-\{i,j\}}) = \varphi_{(c^s,z^{-s})}(P'_i, P_j, P_{-\{i,j\}}) = \varphi_{(c^s,z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$.

Next, we assume $P_j \sim^{MA} P'_j$, i.e., there exist $\tau \in M$ and $\bar{x}^\tau, \bar{y}^\tau \in A^\tau$ such that

(i) for every $z^{-\tau} \in A^{-\tau}$, $(\bar{x}^\tau, z^{-\tau}) = r_k(P_j) = r_{k+1}(P'_j)$ and $(\bar{y}^\tau, z^{-\tau}) = r_{k+1}(P_j) = r_k(P'_j)$ for some $1 \leq k \leq m$;

(ii) for every $z \notin (\bar{x}^\tau, A^{-s}) \cup (\bar{y}^\tau, A^{-s})$, $z = r_k(P_j) = r_k(P'_j)$ for some $1 \leq k \leq m$.

Evidently, strategy-proofness implies the following two statements:

Statement (3) For all $z^{-\tau} \in A^{-\tau}$,

- $\varphi_{(\bar{x}^\tau, z^{-\tau})}(P_i, P_j, P_{-\{i,j\}}) + \varphi_{(\bar{y}^\tau, z^{-\tau})}(P_i, P_j, P_{-\{i,j\}})$
$$= \varphi_{(\bar{x}^\tau, z^{-\tau})}(P_i, P'_j, P_{-\{i,j\}}) + \varphi_{(\bar{y}^\tau, z^{-\tau})}(P_i, P'_j, P_{-\{i,j\}});$$

- $\varphi_{(\bar{x}^\tau, z^{-\tau})}(P'_i, P_j, P_{-\{i,j\}}) + \varphi_{(\bar{y}^\tau, z^{-\tau})}(P'_i, P_j, P_{-\{i,j\}})$
$$= \varphi_{(\bar{x}^\tau, z^{-\tau})}(P'_i, P'_j, P_{-\{i,j\}}) + \varphi_{(\bar{y}^\tau, z^{-\tau})}(P'_i, P'_j, P_{-\{i,j\}}).$$

Statement (4) For all $z \notin (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$,

- $\varphi_z(P_i, P_j, P_{-\{i,j\}}) = \varphi_z(P_i, P'_j, P_{-\{i,j\}})$;

- $\varphi_z(P'_i, P_j, P_{-\{i,j\}}) = \varphi_z(P'_i, P'_j, P_{-\{i,j\}})$.

Firstly, we consider the situation $\tau = s$. Given $z^{-s} \in A^{-s}$, since $(x^s, z^{-s}) P_i! (y^s, z^{-s})$ and $(y^s, z^{-s}) P'_i! (x^s, z^{-s})$, (x^s, z^{-s}) and (y^s, z^{-s}) are isolated in (P_j, P'_j) by the hypothesis. Then, $P_j \sim^{MA} P'_j$ implies $\{(x^s, z^{-s}), (y^s, z^{-s})\} \neq \{(\bar{x}^s, z^{-s}), (\bar{y}^s, z^{-s})\}$. Hence, there exists $c^s \in \{x^s, y^s\}$ such that $\varphi_{(c^s, z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) = \varphi_{(c^s, z^{-s})}(P_i, P_j, P_{-\{i,j\}}) = \varphi_{(c^s, z^{-s})}(P'_i, P_j, P_{-\{i,j\}}) = \varphi_{(c^s, z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$ where the first and the third equality follow from Statement (4) and the second equality follows from the hypothesis.

Secondly, assume $\tau \neq s$. Given $z^{-s} \in A^{-s}$, we know either $(x^s, z^{-s}) \notin (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$, or $(y^s, z^{-s}) \notin (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$; or $\{(x^s, z^{-s}), (y^s, z^{-s})\} \subseteq (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$. If either $(x^s, z^{-s}) \notin (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$, or $(y^s, z^{-s}) \notin (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$, by statement (4) and the hypothesis, there exists $c^s \in \{x^s, y^s\}$ such that $\varphi_{(c^s, z^{-s})}(P_i, P'_j, P_{-\{i,j\}}) = \varphi_{(c^s, z^{-s})}(P_i, P_j, P_{-\{i,j\}}) = \varphi_{(c^s, z^{-s})}(P'_i, P_j, P_{-\{i,j\}}) = \varphi_{(c^s, z^{-s})}(P'_i, P'_j, P_{-\{i,j\}})$.

If $\{(x^s, z^{-s}), (y^s, z^{-s})\} \subseteq (\bar{x}^\tau, A^{-\tau}) \cup (\bar{y}^\tau, A^{-\tau})$, it must be the case that either $(x^s, z^{-s}) = (x^s, \bar{x}^\tau, z^{-\{s,\tau\}})$ and $(y^s, z^{-s}) = (y^s, \bar{x}^\tau, z^{-\{s,\tau\}})$, or $(x^s, z^{-s}) = (x^s, \bar{y}^\tau, z^{-\{s,\tau\}})$ and $(y^s, z^{-s}) = (y^s, \bar{y}^\tau, z^{-\{s,\tau\}})$. Assume $(x^s, z^{-s}) = (x^s, \bar{x}^\tau, z^{-\{s,\tau\}})$

and $(y^s, z^{-s}) = (y^s, \bar{x}^\tau, z^{-\{s,\tau\}})$, and identify another alternative $(x^s, \bar{y}^\tau, z^{-\{s,\tau\}})$.[13] The verification related to the other case is symmetric and we hence omit it.[14] Then, we have

$$\begin{aligned}
&\varphi_{(x^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P_i, P_j', P_{-\{i,j\}}) + \varphi_{(x^s,\bar{y}^\tau,z^{-\{s,\tau\}})}(P_i, P_j', P_{-\{i,j\}}) \\
= \;& \varphi_{(x^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P_i, P_j, P_{-\{i,j\}}) + \varphi_{(x^s,\bar{y}^\tau,z^{-\{s,\tau\}})}(P_i, P_j, P_{-\{i,j\}}) \quad \text{(by Statement (3))} \\
= \;& \varphi_{(x^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P_i', P_j, P_{-\{i,j\}}) + \varphi_{(x^s,\bar{y}^\tau,z^{-\{s,\tau\}})}(P_i', P_j, P_{-\{i,j\}}) \quad \text{(by hypothesis)} \\
= \;& \varphi_{(x^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P_i', P_j', P_{-\{i,j\}}) + \varphi_{(x^s,\bar{y}^\tau,z^{-\{s,\tau\}})}(P_i', P_j', P_{-\{i,j\}}) \quad \text{(by Statement (3))}.
\end{aligned}$$

Meanwhile, according to the second item in Statement (2), since $\varphi_{(x^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P_i, P_j', P_{-\{i,j\}}) \geq \varphi_{(x^s,\bar{x}^\tau,z^{-\{s,\tau\}})}(P_i', P_j', P_{-\{i,j\}})$ and $\varphi_{(x^s,\bar{y}^\tau,z^{-\{s,\tau\}})}(P_i, P_j', P_{-\{i,j\}}) \geq \varphi_{(x^s,\bar{y}^\tau,z^{-\{s,\tau\}})}(P_i', P_j', P_{-\{i,j\}})$, it is true that $\varphi_{(x^s,\bar{x}^\tau,x^{-\{s,\tau\}})}(P_i, P_j', P_{-\{i,j\}}) = \varphi_{(x^s,\bar{x}^\tau,x^{-\{s,\tau\}})}(P_i', P_j', P_{-\{i,j\}})$. Equivalently, we have $\varphi_{(x^s,z^{-s})}(P_i, P_j', P_{-\{i,j\}}) = \varphi_{(x^s,z^{-s})}(P_i', P_j', P_{-\{i,j\}})$.

In conclusion, for each $z^{-s} \in A^{-s}$, there exists $c^s \in \{x^s, y^s\}$ such that $\varphi_{(c^s,z^{-s})}(P_i, P_j', P_{-\{i,j\}}) = \varphi_{(c^s,z^{-s})}(P_i', P_j', P_{-\{i,j\}})$. Therefore, $\varphi(P_i, P_j', P_{-\{i,j\}}) = \varphi(P_i', P_j', P_{-\{i,j\}})$.

11 Proof of Proposition 4.2.4

We provide some new notion for the proof of Proposition 4.2.4. Given $P_i \in \mathbb{D}_S$ and $s \in M$, let $[P_i]^s$ denote the marginal preference over A^s induced by P_i. Moreover, recall lexicographically separable preferences (see Definition 1.1 in Appendix 1) which are repeatedly used in the proof.

The verification of Proposition 4.2.4 consists of three steps (Lemmas 11.1 - 11.9).

Step 1 includes Lemmas 11.1 - 11.5. Each lemma shows the existence of some

[13] Since $A = \times_{s\in M} A^s$, alternative $(x^s, \bar{y}^\tau, z^{-\{s,\tau\}})$ exists.

[14] If $(x^s, z^{-s}) = (x^s, \bar{y}^\tau, z^{-\{s,\tau\}})$ and $(y^s, z^{-s}) = (y^s, \bar{y}^\tau, z^{-\{s,\tau\}})$, we identify the alternative $(x^s, \bar{x}^\tau, z^{-\{s,\tau\}})$.

separable preference satisfying some particular properties. Note that Lemmas 11.1 - 11.5 are analogous to Lemmas 9.1 - 9.5 respectively. Step 1 serves as a preparation for the verifications in Steps 2 and 3.

Step 2 includes Lemmas 11.6 and 11.7 which provide the construction of GC-path in $\mathbb{D}_S$ connecting two fixed preferences in two distinct situations: (i) the two fixed preferences share the same marginal preferences and (ii) the two fixed preferences have two adjacently connected marginal preferences over some component set and share the same marginal preference over every other component set. Lemmas 11.6 and 11.7 are analogous to Lemmas 9.6 and 9.7 respectively which actually construct GC-path in $\mathbb{D}_{MSP}$ connecting two fixed preferences in two analogous situations: (i) the two fixed preferences share the same peak and (ii) the two fixed preferences have distinct peaks which differ in exactly one component. The verification of Lemma 11.6 is identical to the proof of Lemma 9.6, while the proof of Lemma 11.7 is an application of Lemmas 11.4 and Lemma 11.6.

Step 3 includes Lemmas 11.8 and 11.9 which show that $\mathbb{D}_S$ satisfies the Modified Interior Property and the Modified Exterior Property respectively. In the proof of both Lemmas 11.8 and 11.9, the construction of proper GC-path relies entirely on the existence of separable preferences specified in Lemma 11.5 and the GC-path established in Lemma 11.7.

Now, we start Step 1.

Lemma 11.1 below is analogous to Lemma 9.1.

Lemma 11.1. *Given $P_i, P_i' \in \mathbb{D}_S$ and $x, y \in A$, assume $[P_i]^s = [P_i']^s$ for all $s \in M$, $xP_i!y$ and $yP_i'x$. There exists $P_i'' \in \mathbb{D}_S$ such that $P_i'' \sim^A P_i$ and $yP_i''!x$.*

Proof. Evidently, $x^s \neq y^s$ for some $s \in M$. We first show that there exists $\tau \neq s$ such that $x^\tau \neq y^\tau$. Suppose not, i.e., $y = (y^s, x^{-s})$. Consequently, xP_iy and $yP_i'x$ imply $x^s[P_i]^s y^s$ and $y^s[P_i']^s x^s$ which contradicts $[P_i]^s = [P_i']^s$.

Now, we induce preference P_i'' by locally switching x and y in P_i. Thus, $P_i'' \sim^A P_i$ and $yP_i''!x$. We will show $P_i'' \in \mathbb{D}_S$. Given $\omega \in M$, $a^\omega, b^\omega \in A^\omega$ and $z^{-\omega}, \underline{z}^{-\omega} \in$

$A^{-\omega}$, assume $(a^\omega, z^{-\omega})P_i''(b^\omega, z^{-\omega})$. We will show $(a^\omega, \underline{z}^{-\omega})P_i''(b^\omega, \underline{z}^{-\omega})$. Since $x^s \neq y^s$ and $x^\tau \neq y^\tau$, $\{(a^\omega, \underline{z}^{-\omega}), (b^\omega, \underline{z}^{-\omega})\} \neq \{x, y\}$ and $\{(a^\omega, z^{-\omega}), (b^\omega, z^{-\omega})\} \neq \{x, y\}$. Consequently, $P_i'' \sim^A P_i$ implies $(a^\omega, z^{-\omega})P_i(b^\omega, z^{-\omega})$. By separability, we have $(a^\omega, \underline{z}^{-\omega})P_i(b^\omega, \underline{z}^{-s})$. Then, $P_i \sim^A P_i''$ implies $(a^\omega, \underline{z}^{-\omega})P_i''(b^\omega, \underline{z}^{-\omega})$. $\square$

Lemma 11.2 is analogous to Lemma 9.2.

Lemma 11.2. *Given $P_i \in \mathbb{D}_S$, $s \in M$ and $x^s, y^s \in A^s$ with $x^s[P_i]^s!y^s$, there exists $P_i' \in \mathbb{D}_S$ satisfying the following three conditions:*

(i) *for all $a, b \notin (y^s, A^{-s})$, $[aP_ib] \Leftrightarrow [aP_i'b]$;*

(ii) *for all $z^{-s} \in A^{-s}$, $(x^s, z^{-s})P_i'!(y^{-s}, z^{-s})$;*

(iii) *$[P_i]^\tau = [P_i']^\tau$ for all $\tau \in M$.*

Proof. We first construct a preference P_i' satisfying conditions (i) and (ii) by the following method. First, we remove all alternatives in (y^s, A^{-s}) from P_i, and thus have an induced preference $\big(P_i, A\backslash(y^s, A^{-s})\big)$. Next, we create a new preference P_i' by plugging all alternatives in (y^s, A^{-s}) back into the induced preference $\big(P_i, A\backslash(y^s, A^{-s})\big)$ in a particular way: for all $z^{-s} \in A^{-s}$, $(x^s, z^{-s})P_i'!(y^s, z^{-s})$. We will show $P_i' \in \mathbb{D}_S$.

Given $a^\tau, b^\tau \in A^\tau$ and $z^{-\tau}, \underline{z}^{-\tau} \in A^{-\tau}$, assume $(a^\tau, z^{-\tau})P_i'(b^\tau, z^{-\tau})$. We will show $(a^\tau, \underline{z}^{-\tau})P_i'(b^\tau, \underline{z}^{-\tau})$. Suppose not, i.e., $(b^\tau, \underline{z}^{-\tau})P_i'(a^\tau, \underline{z}^{-\tau})$. We consider five cases: (i) $\tau \neq s$ and $z^s \neq y^s$, (ii) $\tau \neq s$ and $\underline{z}^s \neq y^s$, (iii) $\tau \neq s$, $z^s = \underline{z}^s = y^s$, (iv) $\tau = s$ and $b^s = y^s$, and (v) $\tau = s$ and $b^s \neq y^s$.

In case (i), $(a^\tau, z^{-\tau}), (b^\tau, z^{-\tau}) \notin (y^s, A^{-s})$. By condition (i), $(a^\tau, z^{-\tau})P_i'(b^\tau, z^{-\tau})$ implies $(a^\tau, z^{-\tau})P_i(b^\tau, z^{-\tau})$. Then, separability implies $(a^\tau, x^s, \underline{z}^{-\{\tau,s\}})P_i(b^\tau, x^s, \underline{z}^{-\{\tau,s\}})$ and $(a^\tau, \underline{z}^{-\tau})P_i(b^\tau, \underline{z}^{-\tau})$. Applying condition (i) again, $(a^\tau, x^s, \underline{z}^{-\{\tau,s\}})P_i'(b^\tau, x^s, \underline{z}^{-\{\tau,s\}})$. Moreover, since $(a^\tau, \underline{z}^{-\tau})P_i(b^\tau, \underline{z}^{-\tau})$ and $(b^\tau, \underline{z}^{-\tau})P_i'(a^\tau, \underline{z}^{-\tau})$, it must be the case that $\underline{z}^s = y^s$ by condition (i). Since $(a^\tau, x^s, \underline{z}^{-\{\tau,s\}})P_i'!(a^\tau, y^s, \underline{z}^{-\{\tau,s\}})$ and $(b^\tau, x^s, \underline{z}^{-\{\tau,s\}})P_i'!(b^\tau, y^s, \underline{z}^{-\{\tau,s\}})$ by condition (ii), $(a^\tau, x^s, \underline{z}^{-\{\tau,s\}})P_i'(b^\tau, x^s, \underline{z}^{-\{\tau,s\}})$

implies $(a^\tau, y^s, \underline{z}^{-\{\tau,s\}})P_i'(b^\tau, y^s, \underline{z}^{-\{\tau,s\}})$. Equivalently, $(a^\tau, \underline{z}^{-\tau})P_i'(b^\tau, \underline{z}^{-\tau})$. Contradiction! A similar contradiction occurs in case (ii).

In case (iii), since $(a^\tau, x^s, z^{-\{\tau,s\}})P_i'!(a^\tau, z^{-\tau})$ and $(b^\tau, x^s, z^{-\{\tau,s\}})P_i'!(b^\tau, z^{-\tau})$ by condition (ii), $(a^\tau, z^{-\tau})P_i'(b^\tau, z^{-\tau})$ implies $(a^\tau, x^s, z^{-\{\tau,s\}})P_i'(b^\tau, x^s, z^{-\{\tau,s\}})$. Then, by condition (i), $(a^\tau, x^s, z^{-\tau,s})P_i(b^\tau, x^s, z^{-\tau,s})$. Hence, $a^\tau[P_i]^\tau b^\tau$ by separability. On the other hand, since $(b^\tau, x^s, \underline{z}^{-\{\tau,s\}})P_i'!(b^\tau, \underline{z}^{-\tau})$ and $(a^\tau, x^s, \underline{z}^{-\{\tau,s\}})P_i'!(a^\tau, \underline{z}^{-\tau})$ by condition (ii), $(b^\tau, \underline{z}^{-\tau})P_i'(a^\tau, \underline{z}^{-\tau})$ implies $(b^\tau, x^s, \underline{z}^{-\tau,s})P_i'(a^\tau, x^s, \underline{z}^{-\tau,s})$. Then, $(b^\tau, x^s, \underline{z}^{-\tau,s})P_i(a^\tau, x^s, \underline{z}^{-\tau,s})$ by condition (i), and hence $b^\tau[P_i]^\tau a^\tau$ by separability. Contradiction!

In case (iv), since $(x^s, \underline{z}^{-s})P_i'(y^s, \underline{z}^{-s})$ by condition (ii), and $(y^s, \underline{z}^{-s})P_i'(a^s, \underline{z}^{-s})$, we know $a^s \neq x^s$ and $(x^s, \underline{z}^{-s})P_i'(a^s, \underline{z}^{-s})$. Then, condition (i) implies $(x^s, \underline{z}^{-s})P_i(a^s, \underline{z}^{-s})$. Moreover, separability implies $(x^s, z^{-s})P_i(a^s, z^{-s})$. Hence, $(x^s, z^{-s})P_i'(a^s, z^{-s})$ by condition (i). Lastly, since $(x^s, z^{-s})P_i'!(y^s, z^{-s})$ by condition (ii), $(x^s, z^{-s})P_i'(a^s, z^{-s})$ implies $(y^s, z^{-s})P_i'(a^s, z^{-s})$. Equivalently, $(b^s, z^{-s})P_i'(a^s, z^{-s})$. Contradiction!

In case (v), we first claim $a^s = y^s$. Otherwise, by condition (i), $(a^s, z^{-s})P_i'(b^s, z^{-s})$ and $(b^s, \underline{z}^{-s})P_i'(a^s, \underline{z}^{-s})$ imply $(a^s, z^{-s})P_i(b^s, z^{-s})$ and $(b^s, \underline{z}^{-s})P_i(a^s, \underline{z}^{-s})$ respectively which contradict separability. Since $(x^s, z^{-s})P_i'!(y^s, z^{-s})$ by condition (ii) and $(y^s, z^{-s})P_i'(b^s, z^{-s})$, $(x^s, z^{-s})P_i'(b^s, z^{-s})$. Then, $(x^s, z^{-s})P_i(b^s, z^{-s})$ by condition (i), and hence, $(x^s, \underline{z}^{-s})P_i(b^s, \underline{z}^{-s})$ by separability. Then, $(x^s, \underline{z}^{-s})P_i'(b^s, \underline{z}^{-s})$ by condition (i). Lastly, since $(x^s, \underline{z}^{-s})P_i'!(y^s, \underline{z}^{-s})$ by condition (ii), $(x^s, \underline{z}^{-s})P_i'(b^s, \underline{z}^{-s})$ implies $(y^s, \underline{z}^{-s})P_i'(b^s, \underline{z}^{-s})$. Equivalently, $(a^s, \underline{z}^{-s})P_i'(b^s, \underline{z}^{-s})$. Contradiction! In conclusion, $P_i' \in \mathbb{D}_S$.

Lastly, we show $[P_i]^\tau = [P_i']^\tau$ for all $\tau \in M$. Given $\tau \in M\backslash\{s\}$, suppose $a^\tau[P_i]^\tau b^\tau$ and $b^\tau[P_i']^\tau a^\tau$ for some $a^\tau, b^\tau \in A^\tau$. Thus, $(a^\tau, x^s, z^{-\{\tau,s\}})P_i(b^\tau, x^s, z^{-\{\tau,s\}})$ and $(b^\tau, x^s, z^{-\{\tau,s\}})P_i'(a^\tau, x^s, z^{-\{\tau,s\}})$ which contradict condition (i). Therefore, $[P_i]^\tau = [P_i']^\tau$ for all $\tau \in M\backslash\{s\}$. Next, we show $[P_i]^s = [P_i']^s$. Suppose not, i.e., $a^s[P_i]^s b^s$ and $b^s[P_i']^s a^s$ for some $a^s, b^s \in A^s$. Since $x^s[P_i]^s y^s$ and $x^s[P_i']^s y^s$ by condition (ii) and separability, it must be the case that $\{a^s, b^s\} \neq \{x^s, y^s\}$. Furthermore, we claim that $y^s \in \{a^s, b^s\}$. Otherwise, given $z^{-s} \in A^{-s}$, we have

$(a^s, z^{-s})P_i(b^s, z^{-s})$ and $(b^s, z^{-s})P_i'(a^s, z^{-s})$ which contradict condition (i). Therefore $y^s \in \{a^s, b^s\}$. If $a^s = y^s$, then $b^s \notin \{x^s, y^s\}$. Given $z^{-s} \in A^{-s}$, since $(x^s, z^{-s})P_i'!(a^s, z^{-s})$ by condition (ii), $(b^s, z^{-s})P_i'(a^s, z^{-s})$ implies $(b^s, z^{-s})P_i'(x^s, z^{-s})$. Then, $(b^s, z^{-s})P_i(x^s, z^{-s})$ by condition (i). Since $(x^s, z^{-s})P_i(a^s, z^{-s})$, we have $(b^s, z^{-s})P_i(a^s, z^{-s})$, and hence $b^s[P_i]^s a^s$. Contradiction! If $b^s = y^s$, then $a^s \notin \{x^s, y^s\}$. Given $z^{-s} \in A^{-s}$, since $(x^s, z^{-s})P_i'!(b^s, z^{-s})$ by condition (ii), $(b^s, z^{-s})P_i'(a^s, z^{-s})$ implies $(x^s, z^{-s})P_i'(a^s, z^{-s})$. Then, $(x^s, z^{-s})P_i(a^s, z^{-s})$ by condition (i). Since $(a^s, z^{-s})P_i(y^s, z^{-s})$, $(x^s, z^{-s})P_i(a^s, z^{-s})P_i(y^s, z^{-s})$, and hence $x^s[P_i]^s a^s[P_i]^s y^s$ which contradicts $x^s[P_i]^s!y^s$. Therefore, $[P_i]^s = [P_i']^s$. This completes the verification of condition (iii) and hence the lemma. □

Lemma 11.3 below is analogous to Lemma 9.3.

Lemma 11.3. *Given $P_i \in \mathbb{D}_S$, $s \in M$ and $a^s, b^s \in A^s$, assume $(a^s, z^{-s})P_i!(b^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. There exists $P_i' \in \mathbb{D}_S$ such that $P_i' \sim^{MA} P_i$ and $(b^s, z^{-s})P_i'!(a^s, z^{-s})$ for all $z^{-s} \in A^{-s}$.*

Proof. We first induce preference P_i' by switching (a^s, z^{-s}) and (b^s, z^{-s}) in P_i for each $z^{-s} \in A^{-s}$ and keeping the ranking of every other alternative fixed in P_i. Thus, $P_i' \sim^{MA} P_i$ and $(b^s, z^{-s})P_i'!(a^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. We will show $P_i' \in \mathbb{D}_S$.

Given $\tau \in M$, $x^\tau, y^\tau \in A^\tau$ and $z^{-\tau}, \underline{z}^{-\tau} \in A^{-\tau}$, assume $(x^\tau, z^{-\tau})P_i'(y^\tau, z^{-\tau})$. We will show $(x^\tau, \underline{z}^{-\tau})P_i'(y^\tau, \underline{z}^{-\tau})$. We know

either $\{(x^\tau, z^{-\tau}), (y^\tau, z^{-\tau})\} = \{(a^s, \hat{z}^{-s}), (b^s, \hat{z}^{-s})\}$ for some $\hat{z}^{-s} \in A^{-s}$,

or $\{(x^\tau, z^{-\tau}), (y^\tau, z^{-\tau})\} \neq \{(a^s, \hat{z}^{-s}), (b^s, \hat{z}^{-s})\}$ for all $\hat{z}^{-s} \in A^{-s}$.

If $\{(x^\tau, z^{-\tau}), (y^\tau, z^{-\tau})\} = \{(a^s, \hat{z}^{-s}), (b^s, \hat{z}^{-s})\}$ for some $\hat{z}^{-s} \in A^{-s}$, then $\tau = s$, $x^s = b^s$ and $y^s = a^s$. Since $(y^\tau, \underline{z}^{-\tau})P_i!(x^\tau, \underline{z}^{-\tau})$, the construction of P_i' implies $(x^\tau, \underline{z}^{-\tau})P_i'(y^\tau, \underline{z}^{-\tau})$.

If $\{(x^\tau, z^{-\tau}), (y^\tau, z^{-\tau})\} \neq \{(a^s, \hat{z}^{-s}), (b^s, \hat{z}^{-s})\}$ for all $\hat{z}^{-s} \in A^{-s}$, then $P_i \sim^{MA} P_i'$ and $(x^\tau, z^{-\tau})P_i'(y^\tau, z^{-\tau})$ imply that $(x^\tau, z^{-\tau})P_i(y^\tau, z^{-\tau})$. Then, $(x^\tau, \underline{z}^{-\tau})P_i(y^\tau, \underline{z}^{-\tau})$ by separability. We claim that $\{(x^\tau, \underline{z}^{-\tau}), (y^\tau, \underline{z}^{-\tau})\} \neq \{(a^s, \hat{z}^{-s}), (b^s, \hat{z}^{-s})\}$ for all $\hat{z}^{-s} \in A^{-s}$. Otherwise, $\{(x^\tau, \underline{z}^{-\tau}), (y^\tau, \underline{z}^{-\tau})\} = \{(a^s, \hat{z}^{-s}), (b^s, \hat{z}^{-s})\}$ for some

$\hat{z}^{-s} \in A^{-s}$. Then, $(x^\tau, \underline{z}^{-\tau})P_i(y^\tau, \underline{z}^{-\tau})$ and $(a^s, \hat{z}^{-s})P_i(b^s, \hat{z}^{-s})$ imply $\tau = s$, $x^\tau = a^s$ and $y^\tau = b^s$. Since $(b^\tau, z^{-\tau})P_i'(a^\tau, z^{-\tau})$ by the construction of P_i', $(y^\tau, z^{-\tau})P_i'(x^\tau, z^{-\tau})$. Contradiction! Now, since $\{(x^\tau, \underline{z}^{-\tau}), (y^\tau, \underline{z}^{-\tau})\} \neq \{(a^s, \hat{z}^{-s}), (b^s, \hat{z}^{-s})\}$ for all $\hat{z}^{-s} \in A^{-s}$, $P_i' \sim^{MA} P_i$ and $(x^\tau, \underline{z}^{-\tau})P_i(y^\tau, \underline{z}^{-\tau})$ imply $(x^\tau, \underline{z}^{-\tau})P_i'(y^\tau, \underline{z}^{-\tau})$. In conclusion, $P_i' \in \mathbb{D}_S$. $\square$

Lemma 11.4 below is analogous to Lemma 9.4.

Lemma 11.4. *Given $P_i, P_i' \in \mathbb{D}_S$, $s \in M$ and $a^s, b^s \in A^s$, assume $[P_i]^s \sim^A [P_i']^s$, $a^s[P_i]^s!b^s$, $b^s[P_i']^s!a^s$, and $[P_i]^\tau = [P_i']^\tau$ for all $\tau \neq s$. Given $x, y \in A$, assume xP_iy and $xP_i'y$. There exist $\bar{P}_i, \bar{P}_i' \in \mathbb{D}_S$ satisfying the following five conditions:*

(i) $(a^s, z^{-s})\bar{P}_i!(b^s, z^{-s})$ *for all* $z^{-s} \in A^{-s}$,

(ii) $[P_i]^\tau = [\bar{P}_i]^\tau$ *for all* $\tau \in M$,

(iii) $x\bar{P}_iy$,

(iv) $\bar{P}_i \sim^{MA} \bar{P}_i'$ *and* $(b^s, z^{-s})\bar{P}_i'!(a^s, z^{-s})$ *for all* $z^{-s} \in A^{-s}$,

(v) $x\bar{P}_i'y$.

Proof. We assume $y = (y^S, x^{-S})$ where $x^\tau \neq y^\tau$ for all $\tau \in S$, $S \subseteq M$ and $S \neq \emptyset$. Since xP_iy, there must exist $\omega \in S$ such that $x^\omega[P_i]^\omega y^\omega$. To verify the lemma, we consider the following three jointly exhaustive situations:

(i) $|S| = 1$;

(ii) $|S| \geq 2$ and there exists $\tau^* \in S$ such that $\tau^* \neq s$ and $x^{\tau^*}[P_i]^{\tau^*}y^{\tau^*}$;

(iii) $|S| \geq 2$, $s \in S$, $x^s[P_i]^s y^s$ and $y^\tau[P_i]^\tau x^\tau$ for all $\tau \in S\backslash\{s\}$.

In situation (i), we assume $y = (y^\tau, x^{-\tau})$ and $x^\tau \neq y^\tau$ (either $\tau = s$, or $\tau \neq s$). Since xP_iy and $xP_i'y$, we know $x^\tau[P_i]^\tau y^\tau$ and $x^\tau[P_i']^\tau y^\tau$. Let $\bar{P}_i, \bar{P}_i' \in \mathbb{D}_{LS}$ be such that (1) both lexicographic orders in $\bar{P}_i$ and $\bar{P}_i'$ are identical, (2) component s is lexicographically dominated, i.e., $\omega \succ s$ for all $\omega \neq s$, and (3) $[\bar{P}_i]^\omega = [P_i]^\omega$ and $[\bar{P}_i']^\omega = [P_i']^\omega$ for all $\omega \in M$. Since component s is lexicographically dominated,

$a^s[\bar{P}_i]^s!b^s$ and $b^s[\bar{P}_i]^s!a^s$, it is easy to verify that conditions (i) and (iv) are satisfied. Condition (ii) is satisfied by construction. By lexicographic separability, we know $x\bar{P}_i y$ and $x\bar{P}'_i y$ (conditions (iii) and (v) are met).

In situation (ii), let $\bar{P}_i, \bar{P}'_i \in \mathbb{D}_{LS}$ be such that (1) both lexicographic orders in $\bar{P}_i$ and $\bar{P}'_i$ are identical, (2) component τ^* is lexicographically dominant (i.e., $\tau^* \succ \omega$ for all $\omega \neq \tau^*$), and component s is lexicographically dominated, and (3) $[\bar{P}_i]^\omega = [P_i]^\omega$ and $[\bar{P}'_i]^\omega = [P'_i]^\omega$ for all $\omega \in M$. Then, by a similar argument in the verification of situation (i), we know that conditions (i) - (v) are satisfied.

In situation (iii), we first claim $x^s[P'_i]^s y^s$. Suppose not, i.e., $y^s[P'_i]^s x^s$. Thus, $y^\tau[P'_i]^\tau x^\tau$ for all $\tau \in S$, and hence $yP'_i x$ by separability. Contradiction! Therefore, $x^s[P'_i]^s y^s$. Now, it must be the case that $\{x^s, y^s\} \neq \{a^s, b^s\}$. To verify this lemma, we consider two cases: (1) $y^s \neq b^s$ and (2) $y^s = b^s$.

Case (1): $y^s \neq b^s$.

According to P_i, let $\bar{P}_i \in \mathbb{D}_S$ satisfying conditions (i) - (iii) in Lemma 11.2. Thus, conditions (i) and (ii) in Lemma 11.4 are satisfied. Suppose $y\bar{P}_i x$. We claim $x^s = b^s$. Suppose that it is not true. Thus, $x, y \notin (b^s, A^{-s})$. Then, by condition (i) in Lemma 11.2, $xP_i y$ implies $x\bar{P}_i y$. Contradiction! Since $(a^s, x^{-s})P_i(b^s, x^{-s}) = x$ and $xP_i y$, we have $(a^s, x^{-s})P_i y$. Then, condition (i) in Lemma 11.2 implies $(a^s, x^{-s})\bar{P}_i y$. Moreover, since $(a^s, x^{-s})\bar{P}_i!(b^s, x^{-s}) = x$ by condition (ii) in Lemma 11.2, $(a^s, x^{-s})P_i y$ implies $x\bar{P}_i y$. Contradiction! Therefore, $x\bar{P}_i y$ (condition (iii) in Lemma 11.4 is satisfied).

Furthermore, according to $\bar{P}_i$, by Lemma 11.3, we have $\bar{P}'_i \in \mathbb{D}_S$ such that $\bar{P}_i \sim^{MA} \bar{P}'_i$ and $(b^s, z^{-s})\bar{P}'_i!(a^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. Thus, condition (iv) in Lemma 11.4 is satisfied. If $y^s = a^s$, then $a^s[P_i]^s b^s$ and $x^s[P_i]^s y^s$ imply $x^s \notin \{a^s, b^s\}$. Consequently, $\bar{P}_i \sim^{MA} \bar{P}'_i$ and $x\bar{P}_i y$ imply $x\bar{P}'_i y$. If $y^s \neq a^s$, then $y^s \notin \{a^s, b^s\}$. Symmetrically, $\bar{P}_i \sim^{MA} \bar{P}'_i$ and $x\bar{P}_i y$ imply $x\bar{P}'_i y$. Thus, condition (v) in Lemma 11.4 is satisfied.

Case (2): $y^s = b^s$.

In this case, $\{x^s, y^s\} \neq \{a^s, b^s\}$ implies $x^s \notin \{a^s, b^s\}$. Consequently, $a^s[P_i]^s!b^s$ and $x^s[P_i]^s y^s = b^s$ imply $x^s[P_i]^s a^s$. We first identify $\hat{P}_i \in \mathbb{D}_{LS}$ such that (i) $[\hat{P}_i]^\tau = [P_i]^\tau$ for all $\tau \in M$ and (ii) component s is lexicographically dominant. Thus, $x\hat{P}_i(a^s, y^{-s})\hat{P}_i(b^s, y^{-s}) = y$ by lexicographic separability. Furthermore, according to $\hat{P}_i$, there exists $\bar{P}_i \in \mathbb{D}_S$ satisfying conditions (i) - (iii) in Lemma 11.2. Evidently, condition (i) in Lemma 11.4 is satisfied. Next, by condition (iii) of Lemma 11.2, we have $[\bar{P}_i]^\tau = [\hat{P}_i]^\tau = [P_i]^\tau$ for all $\tau \in M$ (condition (ii) in Lemma 11.4 is met). Next, we show $x\bar{P}_i y$. Suppose not, i.e., $(b^s, y^{-s}) = y\bar{P}_i x$. Since $(a^s, y^{-s})\bar{P}_i!(b^s, y^{-s})$ by condition (ii) of Lemma 11.2, we have $(a^s, y^{-s})\bar{P}_i x$. Then, $(a^s, y^{-s})\hat{P}_i x$ by condition (i) of Lemma 11.2. Contradiction to lexicographic separability! Therefore, $x\bar{P}_i y$ (condition (iii) in Lemma 11.4 is met).

Furthermore, according to $\bar{P}_i$, by Lemma 11.3, we have $\bar{P}_i' \in \mathbb{D}_S$ such that $\bar{P}_i \sim^{MA} \bar{P}_i'$ and $(b^s, z^{-s})\bar{P}_i'!(a^s, z^{-s})$ for all $z^{-s} \in A^{-s}$. Thus, condition (iv) in Lemma 11.4 is satisfied. Since $x^s \notin \{a^s, b^s\}$, $\bar{P}_i \sim^{MA} \bar{P}_i'$ and $x\bar{P}_i y$ imply $x\bar{P}_i' y$. Thus, condition (v) in Lemma 11.4 is satisfied. This completes the verification of situation (iii) and hence the lemma. $\square$

Lemma 11.5 below is analogous to Lemma 9.5.

Lemma 11.5. *Given $P_i, P_i' \in \mathbb{D}_S$, assume $[P_i]^s \neq [P_i']^s$ for all $s \in S$, where $S \subseteq M$ and $S \neq \emptyset$. Given $x, y \in A$, assume $xP_i y$ and $xP_i' y$. There exists $P_i'' \in \mathbb{D}_S$ satisfying the following three conditions:*

(i) $[P_i'']^s \sim^A [P_i]^s$ *for some $s \in S$, and $[P_i'']^\tau = [P_i]^\tau$ for all $\tau \neq s$;*

(ii) $\left[a^s[P_i]^s!b^s \text{ and } b^s[P_i'']^s!a^s\right] \Rightarrow \left[b^s[P_i']^s a^s\right]$;

(iii) $xP_i'' y$.

Proof. Assume $y = (y^T, x^{-T})$ and $y^\tau \neq x^\tau$ for all $\tau \in T$ where $T \subseteq M$ and $T \neq \emptyset$. Since $xP_i y$, there must exist $\tau \in T$ such that $x^\tau[P_i]^\tau y^\tau$.

We consider two cases: (i) there exists $\tau \in T$ such that $x^\tau[P_i]^\tau y^\tau$ and $x^\tau[P_i']^\tau y^\tau$, and (ii) for all $\tau \in T$, $\left[x^\tau[P_i]^\tau y^\tau\right] \Rightarrow \left[y^\tau[P_i']^\tau x^\tau\right]$.

In case (i), we know either $\tau \in S$ or $\tau \notin S$. If $\tau \in S$, there exists a marginal preference $[P_i^*]^\tau$ such that (1) $[P_i^*]^\tau \sim^A [P_i]^\tau$, and (2) $a^\tau[P_i]^\tau!b^\tau$, $b^\tau[P_i^*]^\tau!a^\tau$ and $b^\tau[P_i']^\tau a^\tau$. Thus, $\{a^\tau, b^\tau\} \neq \{x^\tau, y^\tau\}$, and hence $x^\tau[P_i]^\tau y^\tau$ implies $x^\tau[P_i^*]^\tau y^\tau$. Now, let $P_i'' \in \mathbb{D}_{LS}$ be such that $[P_i'']^\tau = [P_i^*]^\tau$, $[P_i'']^\omega = [P_i]^\omega$ for all $\omega \neq \tau$; and component τ is lexicographically dominant. Thus, conditions (i) - (iii) are satisfied by P_i''. If $\tau \notin S$, there exist $s \in S$ and a marginal preference $[P_i^*]^s$ such that (1) $[P_i^*]^s \sim^A [P_i]^s$, and (2) $a^s[P_i]^s!b^s$, $b^s[P_i^*]^s!a^s$ and $b^s[P_i']^\tau a^s$. Then, let $P_i'' \in \mathbb{D}_{LS}$ be such that $[P_i'']^s = [P_i^*]^s$, $[P_i'']^\omega = [P_i]^\omega$ for all $\omega \neq s$; and component τ is lexicographically dominant. Since $[P_i'']^\tau = [P_i]^\tau$, $x^\tau[P_i]^\tau y^\tau$ implies $x^\tau[P_i'']^\tau y^\tau$. Thus, conditions (i) - (iii) are satisfied by P_i''. This completes the verification of case (i).

Next, we consider case (ii). Since $xP_i'y$, there must exist $s \in T$ such that $x^s[P_i']^s y^s$. Then, in case (ii), it must be the case that $y^s[P_i]^s x^s$. Now, we have $\tau, s \in T$ such that $x^\tau[P_i]^\tau y^\tau$, $y^\tau[P_i']^\tau x^\tau$; $y^s[P_i]^s x^s$ and $x^s[P_i']^s y^s$. Thus, $\tau, s \in S$. Now, we construct a marginal preference $[P_i^*]^s$ such that (1) $[P_i^*]^s \sim^A [P_i]^s$, and (2) $a^s[P_i]^s!b^s$, $b^s[P_i^*]^s!a^s$ and $b^s[P_i']^s a^s$. Then, let $P_i'' \in \mathbb{D}_{LS}$ be such that $[P_i'']^s = [P_i^*]^s$, $[P_i'']^q = [P_i]^q$ for all $q \neq s$; and component τ is lexicographically dominant. Since $[P_i'']^\tau = [P_i]^\tau$, $x^\tau[P_i]^\tau y^\tau$ implies $x^\tau[P_i'']^\tau y^\tau$. Thus, conditions (i) - (iii) are satisfied by P_i''. This completes the verification of case (ii) and hence the lemma. □

Note that from the aspect of marginal preferences, in Lemma 11.5, we push P_i one step "closer" to P_i' through P_i'' while still keeping x ranked above y in P_i''.

This completes the verification in Step 1. We turn to Step 2.

Lemma 11.6. *Given $P_i, P_i' \in \mathbb{D}_S$ with $[P_i]^s = [P_i']^s$ for all $s \in M$, there exists an AC-path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}_S$ connecting P_i and P_i' such that for all $x, y \in A$, $[xP_iy$ and $xP_i'y] \Rightarrow [xP_i^ky$, $1 < k < l]$.*

Proof. The construction of the AC-path $\{P_i^k\}_{k=1}^l \subseteq \mathbb{D}_S$ connecting P_i and P_i' is a repeated application of Lemma 11.1, and is symmetric to the verification of Lemma 9.6. □

Note that according to Remark 4.2.5, in Lemma 11.6, for all $x, y \in A$ with xP_iy and xP'_iy, the AC-path $\{P_i^k\}_{k=1}^l$ is also a (x, y)-Is-GC-path connecting P_i and P'_i.

Lemma 11.7. *Given $P_i, P'_i \in \mathbb{D}_S$, assume $[P_i]^s \sim^A [P'_i]^s$ for some $s \in M$ and $[P_i]^\tau = [P'_i]^\tau$ for all $\tau \neq s$. Given $x, y \in A$, assume xP_iy and xP'_iy. There exists a (x, y)-Is-GC-path in $\mathbb{D}_S$ connecting P_i and P'_i.*

Proof. According to P_i and P'_i, we have $\bar{P}_i, \bar{P}'_i \in \mathbb{D}_S$ satisfying conditions (i) - (v) in Lemma 11.4. Since $x\bar{P}_iy$ and $x\bar{P}'_iy$ by conditions (iii) and (v) in Lemma 11.4, $\bar{P}_i \sim^{MA} \bar{P}'_i$ implies that x and y are isolated in $(\bar{P}_i, \bar{P}'_i)$ by Remark 4.2.5. Second, since $[P_i]^\tau = [\bar{P}_i]^\tau$ for all $\tau \in M$, xP_iy and $x\bar{P}_iy$, Lemma 11.6 implies that there exists a (x, y)-Is-GC-path in $\mathbb{D}_S$ connecting P_i and $\bar{P}_i$. Next, by conditions (i) and (iv) in Lemma 11.4, since $[P_i]^s \sim^A [P'_i]^s$ and $[P_i]^\tau = [P'_i]^\tau$ for all $\tau \neq s$, it is true that $[\bar{P}'_i]^\tau = [P'_i]^\tau$ for all $\tau \in M$. Since $x\bar{P}'_iy$ and xP'_iy, Lemma 11.6 implies that there exists a (x, y)-Is-GC-path in $\mathbb{D}_S$ in $\mathbb{D}_S$ connecting $\bar{P}'_i$ and P'_i. Combining both paths, we have a (x, y)-Is-GC-path in $\mathbb{D}_S$ connecting P_i and P'_i. $\square$

This completes the verification in Step 2. We then turn to Step 3.

Lemma 11.8. *Domain $\mathbb{D}_S$ satisfies the Modified Interior Property.*

Proof. Fix $a \in A$ and distinct $P_i, P'_i \in \mathbb{D}_S^a$. If $[P_i]^s = [P'_i]^s$, this lemma follows from Lemma 11.6. Next, we assume $[P_i]^s \neq [P'_i]^s$ for all $s \in S$, where $S \subseteq M$ and $S \neq \emptyset$, and $[P_i]^\tau = [P'_i]^\tau$ for all $\tau \in M\backslash S$. Note that $r_1([P_i]^\tau) = r_1([P'_i]^\tau) = a^\tau$ for all $\tau \in M$.

Fixing $x \in A\backslash\{a\}$, we know aP_ix and aP'_ix. Then, by repeated application of Lemma 11.5, we have a sequence $\{\bar{P}_i^k\}_{k=1}^t \subseteq \mathbb{D}_S$ such that

(i) $\bar{P}_i^1 = P_i$ and $\bar{P}_i^t = P'_i$;

(ii) for every $1 \leq k \leq t-1$, $[\bar{P}_i^k]^s \sim^A [\bar{P}_i^{k+1}]^s$ for some $s \in S$, and $[\bar{P}_i^k]^\tau = [\bar{P}_i^{k+1}]^\tau$ for all $\tau \neq s$;

(iii) $a\bar{P}_i^kx$, $k = 1, \dots, t$.

Moreover, by condition (ii) of Lemma 11.5, since $r_1([P_i]^\tau) = r_1([P'_i]^\tau) = a^\tau$ for all $\tau \in M$, it is true that $r_1(\bar{P}_1^k) = a$ for all $1 < k < t$.

For every $1 \leq k \leq t-1$, Lemma 11.7 implies that there exists a (a, x)-Is-GC-path in $\mathbb{D}_S$ connecting $\bar{P}_i^k$ and $\bar{P}_i^{k+1}$. Moreover, according to the construction of (a, x)-Is-GC-path connecting $\bar{P}_i^k$ and $\bar{P}_i^{k+1}$ in the proof of Lemma 11.7, it is easy to verify that the peak of every preference in the (a, x)-Is-GC-path is a. Combining all these paths, we have a GC-path in $\mathbb{D}_S^a$ connecting P_i and P'_i. $\square$

Lemma 11.9. *Domain $\mathbb{D}_S$ satisfies the Modified Exterior Property.*

Proof. Fix $P_i, P'_i \in \mathbb{D}_S$ with $r_1(P_i) \neq r_1(P'_i)$, and $x, y \in A$ with xP_iy and xP'_iy. Evidently, there exists $s \in M$ such that $[P_i]^s \neq [P'_i]^s$. We assume $[P_i]^s \neq [P'_i]^s$ for all $s \in S$, where $S \subseteq M$ and $S \neq \emptyset$, and $[P_i]^\tau = [P'_i]^\tau$ for all $\tau \in M \backslash S$. Then, by repeated application of Lemma 11.5, we have a sequence $\{\bar{P}_i^k\}_{k=1}^t \subseteq \mathbb{D}_S$ such that

(i) $\bar{P}_i^1 = P_i$ and $\bar{P}_i^t = P'_i$;

(ii) for every $1 \leq k \leq t-1$, $[\bar{P}_i^k]^s \sim^A [\bar{P}_i^{k+1}]^s$ for some $s \in S$, and $[\bar{P}_i^k]^\tau = [\bar{P}_i^{k+1}]^\tau$ for all $\tau \neq s$;

(iii) $x\bar{P}_i^k y$, $k = 1, \ldots, t$.

For every $1 \leq k \leq t-1$, Lemma 11.7 implies that there exists a (x, y)-Is-GC-path in $\mathbb{D}_S$ connecting $\bar{P}_i^k$ and $\bar{P}_i^{k+1}$. Combining all these paths, we have a (x, y)-Is-GC-path in $\mathbb{D}_S$ connecting P_i and P'_i. $\square$

We have completed the verification of Step 3 and hence Proposition 4.2.4.

12 Proof of Proposition 4.2.5

It is evident that a random dictatorship is ex-post efficient and strategy-proof since it is a convex combination of dictatorial DSCFs. We focus on showing the necessity part of Proposition 4.2.5. We first show that every two-voter ex-post efficient and strategy-proof RSCF $\varphi : \mathbb{D}^2_{MSP} \rightarrow \Delta(A)$ is a random dictatorship.[15] Proposition 4.2.2 implies that φ satisfies the tops-only property.

Lemma 12.1. *For all $a, b \in A$ with $a \neq b$, $\varphi_a(a, b) + \varphi_b(a, b) = 1$.*

Proof. Claim 1: Given $a, b \in A$ with $a \neq b$, and $x \notin \langle a, b\rangle$, $\varphi_x(a, b) = 0$.

Since $x \notin \langle a, b\rangle$, there exists unique $x' \in \langle a, b\rangle$ such that $\langle a, x\rangle \cap \langle b, x\rangle = \langle x', x\rangle$. Accordingly, there exist $P_1 \in \mathbb{D}^a_{MSP}$ and $P_2 \in \mathbb{D}^b_{MSP}$ such that $x' P_1 x$ and $x' P_2 x$. Hence, $x \notin \Omega(P_1, P_2)$ and $x \notin supp\, \varphi(P_1, P_2)$ by ex-post efficiency. By tops-onlyness, we have $\varphi_x(a, b) = \varphi_x(P_1, P_2) = 0$. This completes the verification of the claim.

Claim 2: Given $a, b \in A$, assume $a^s \neq b^s$ and $a^\tau \neq b^\tau$ for some $s, \tau \in M$. Given $x \in \langle a, b\rangle \backslash \{a, b\}$, $\varphi_x(a, b) = 0$.

Since a and b disagree on at least two components, and $x \in \langle a, b\rangle \backslash \{a, b\}$, it is true that $\langle a, b\rangle \backslash [\langle a, x\rangle \cup \langle b, x\rangle] \neq \emptyset$. Fixing $x' \in \langle a, b\rangle \backslash [\langle a, x\rangle \cup \langle b, x\rangle]$, we know $x \notin \langle a, x'\rangle$ and $x \notin \langle b, x'\rangle$. Accordingly, there exist $P_1 \in \mathbb{D}^a_{MSP}$ and $P_2 \in \mathbb{D}^b_{MSP}$ such that $x' P_1 x$ and $x' P_2 x$. Then, by tops-onlyness and ex-post efficiency, $\varphi_x(a, b) = \varphi_x(P_1, P_2) = 0$. This completes the verification of the claim.

According to Claims 1 and 2, we know that for all $a, b \in A$ with $a^s \neq b^s$, $a^\tau \neq b^\tau$ and $a^{-\{s,\tau\}} = b^{-\{s,\tau\}}$ for some $s, \tau \in M$, $\varphi_a(a, b) + \varphi_b(a, b) = 1$.

Claim 3: Given $a, b \in A$, assume $a^s \neq b^s$ and $a^{-s} = b^{-s}$ for some $s \in M$. Given $x \in \langle a, b\rangle \backslash \{a, b\}$, $\varphi_x(a, b) = 0$.

We assume $a = (a^s, x^{-s})$, $b = (b^s, x^{-s})$ and $x = (x^s, x^{-s})$ where $x^s \in \langle a^s, b^s\rangle \backslash \{a^s, b^s\}$. We identify two other alternatives $\bar{b} = (b^s, y^\tau, x^{-s,\tau})$ and $\bar{x} = (x^s, y^\tau, x^{-s,\tau})$ where (x^τ, y^τ) is an edge in $G(A^\tau)$.[16] Since $a^s \neq b^s = \bar{b}^s$ and $a^\tau = x^\tau \neq y^\tau = \bar{b}^\tau$, Claims 1 and 2 imply $\varphi_x(a, \bar{b}) = 0$ and $\varphi_{\bar{x}}(a, \bar{b}) = 0$. According to b and $\bar{b}$, we have $P_2 \in \mathbb{D}^b_{MSP}$ and $P_2' \in \mathbb{D}^{\bar{b}}_{MSP}$ such that $P_2 \sim^{MA}$

[15] In the case of two voters, we *do not* impose the restriction $|M| \geq 3$.

[16] Alternatives $\bar{b}$ and $\bar{x}$ exist since $A = \times_{q \in M} A^q$ and $|M| \geq 2$.

P_2'; $(x^\tau, z^{-\tau})P_2!(y^\tau, z^{-\tau})$ and $(y^\tau, z^{-\tau})P_2'!(x^\tau, z^{-\tau})$ for all $z^{-\tau} \in A^{-\tau}$. By tops-onlyness and strategy-proofness, $\varphi_x(a, b) + \varphi_{\bar{x}}(a, b) = \varphi_x(a, P_2) + \varphi_{\bar{x}}(a, P_2) = \varphi_x(a, P_2') + \varphi_{\bar{x}}(a, P_2') = \varphi_x(a, \bar{b}) + \varphi_{\bar{x}}(a, \bar{b}) = 0$. Hence, $\varphi_x(a, b) = 0$. This completes the verification of the claim.

According to Claims 1 and 3, we know that for all $a, b \in A$ with $a^s \neq b^s$ and $a^{-s} = b^{-s}$ for some $s \in M$, $\varphi_a(a, b) + \varphi_b(a, b) = 1$. Therefore, by Claims 1, 2 and 3, $\varphi_a(a, b) + \varphi_b(a, b) = 1$ for all $a, b \in A$ with $a \neq b$. $\square$

Lemma 12.2. *Given $a, b; x, y \in A$ with $a \neq b$ and $x \neq y$, $\varphi_a(a, b) = \varphi_x(x, y)$.*

Proof. Assume $\varphi_a(a, b) = \lambda$. We consider two cases: (i) either $x \notin \{a, b\}$ or $y \notin \{a, b\}$, and (ii) $x \in \{a, b\}$ and $y \in \{a, b\}$.

In case (i), we assume w.l.o.g. that $x \notin \{a, b\}$. The verification related to $y \notin \{a, b\}$ is symmetric and we hence omit it. Since $|M| \geq 2$, there exists a sequence $\{a_k\}_{k=1}^t \subseteq A$ such that $a_1 = a$, $a_t = x$, (a_k, a_{k+1}) is an edge in $\times_{s \in M} G(A^s)$, $k = 1, \ldots, t-1$, and $b \notin \{a_k\}_{k=1}^t$.[17] According to a_1 and a_2, we have $P_1 \in \mathbb{D}_{MSP}^{a_1}$ and $P_1' \in \mathbb{D}_{MSP}^{a_2}$ such that $r_2(P_1) = a_2$ and $r_2(P_1') = a_1$. By tops-onlyness and strategy-proofness, $\varphi_{a_1}(a_1, b) + \varphi_{a_2}(a_1, b) = \varphi_{a_1}(P_1, b) + \varphi_{a_2}(P_1, b) = \varphi_{a_1}(P_1', b) + \varphi_{a_2}(P_1', b) = \varphi_{a_1}(a_2, b) + \varphi_{a_2}(a_2, b)$. Then, Lemma 12.1 implies $\varphi_{a_2}(a_2, b) = \varphi_{a_1}(a_1, b) = \lambda$. Following the sequence $\{a_k\}_{k=1}^t$ and repeatedly applying the symmetric argument step by step, we have $\varphi_x(x, b) = \varphi_{a_t}(a_t, b) = \lambda$. Hence, $\varphi_b(x, b) = 1 - \lambda$ by Lemma 12.1. If $y = b$, the verification is completed. We assume $y \neq b$. Then, there exists a sequence $\{b_k\}_{k=1}^{t'} \subseteq A$ such that $b_1 = b$, $b_{t'} = y$, (b_k, b_{k+1}) is an edge in $\times_{s \in M} G(A^s)$, $k = 1, \ldots, t'-1$, and $x \notin \{b_k\}_{k=1}^{t'}$. Following the sequence $\{b_k\}_{k=1}^{t'}$, by a symmetric argument, we have $\varphi_y(x, y) = 1 - \lambda$. Then, by Lemma 12.1, $\varphi_x(x, y) = \lambda = \varphi_a(a, b)$.

In case (ii), since $x \neq y$, it must be either $(x, y) = (a, b)$ or $(x, y) = (b, a)$. The lemma holds evidently if $(x, y) = (a, b)$. Assume $(x, y) = (b, a)$. Fix $x' \notin \{a, b\}$. Between (a, b) and (x', b), since $x' \notin \{a, b\}$, the verification of case (i)

[17] A pair of alternatives $c, d \in A$ forms an edge in $\times_{s \in M} G(A^s)$, if $c^{-s} = d^{-s}$ and (c^s, d^s) is an edge in $G(A^s)$ for some $s \in M$.

implies $\varphi_{x'}(x', b) = \lambda$. Similarly, between (b, a) and (x', b), since $x' \notin \{b, a\}$, the verification of case (i) implies $\varphi_b(b, a) = \varphi_{x'}(x', b) = \lambda$. Thus, $\varphi_a(a, b) = \varphi_b(b, a)$. This completes the verification of the lemma. $\square$

Fixing arbitrary $a, b \in A$ with $a \neq b$, let $\varphi_a(a, b) = \lambda$. We will show that for all $x, y \in A$, $\varphi(x, y) = \lambda e_x + (1 - \lambda)e_y$. If $x = y$, it holds evidently. If $x \neq y$, it holds by Lemmas 12.1 and 12.2. Therefore, φ is a random dictatorship.

Next, we modify the Ramification Theorem in Appendix 5 so that the random dictatorship result over $\mathbb{D}_{MSP}$ (henceforth, assume $|M| \geq 3$) can be extended to the case of arbitrary number of voters. We first introduce the primary induction hypothesis.

The Primary Induction Hypothesis: Given $N > 2$, for all $2 \leq n < N$, we have

$$[\varphi : \mathbb{D}^n_{MSP} \to \Delta(A) \text{ is ex-post efficient and strategy-proof}] \Rightarrow [\varphi \text{ is a random dictatorship}].$$

Fixing an ex-post efficient and strategy-proof RSCF $\varphi : \mathbb{D}^N_{MSP} \to \Delta(A)$, we will show that φ is a random dictatorship. By Proposition 4.2.2, RSCF φ satisfies the tops-only property.

Lemma 12.3. *RSCF φ is a quasi random dictatorship (recall Definition 5.1 in Appendix 5).*

Proof. We consider two cases: $N > 3$ and $N = 3$. If $N > 3$, the verification is exactly identical to the verification of Proposition 5.2 by simply changing "unanimity" to "ex-post efficiency". Thus, we focus on the case $N = 3$.[18]

According to RSCF $\varphi : \mathbb{D}^3 \to \Delta(A)$, we define three RSCFs: $g^{(2,3)}(P_1, P_2) = \varphi(P_1, P_2, P_2)$, $g^{(1,3)}(P_1, P_2) = \varphi(P_1, P_2, P_1)$ and $g^{(1,2)}(P_1, P_3) = \varphi(P_1, P_1, P_3)$ for all $P_1, P_2, P_3 \in \mathbb{D}$. Evidently, $g^{(2,3)}$, $g^{(1,3)}$ and $g^{(1,2)}$ are ex-post efficient and strategy-proof. Then, $g^{(2,3)}$, $g^{(1,3)}$ and $g^{(1,2)}$ are random dictatorships by the primary

[18]The proof of Proposition 5.1 relies on Condition α (see Definition 3.1). However, $\mathbb{D}_{MSP}$ violates Condition α. Instead, the proof of Lemma 12.3 relies on the restriction of multi-dimensional single-peakedness and the tops-only property.

induction hypothesis. Thus, there exist $\varepsilon_1, \varepsilon_2, \varepsilon_3 \geq 0$ such that for all $P_1, P_2, P_3 \in \mathbb{D}$,

$$\begin{aligned}
\varphi(P_1, P_2, P_2) &= g^{(2,3)}(P_1, P_2) = \varepsilon_1\, e_{r_1(P_1)} + (1-\varepsilon_1)e_{r_1(P_2)},\\
\varphi(P_1, P_2, P_1) &= g^{(1,3)}(P_1, P_2) = (1-\varepsilon_2)e_{r_1(P_1)} + \varepsilon_2\, e_{r_1(P_2)},\\
\varphi(P_1, P_1, P_3) &= g^{(1,2)}(P_1, P_3) = (1-\varepsilon_3)e_{r_1(P_1)} + \varepsilon_3\, e_{r_1(P_3)}.
\end{aligned}$$

To establish that φ is a quasi random dictatorship, it suffices to show $\varepsilon_1+\varepsilon_2+\varepsilon_3 = 1$. Fixing $\{1,2,3\} \subseteq M$; $\{x^s, y^s\} \subseteq A^s$ where (x^s, y^s) is an edge in $G(A^s)$, $s = 1, 2, 3$, and $z^{-\{1,2,3\}} \in A^{-\{1,2,3\}}$, we identify the following eight alternatives (see Figure 1):

$$\begin{aligned}
&a = (x^1, x^2, x^3, z^{-\{1,2,3\}}),\ b = (y^1, y^2, x^3, z^{-\{1,2,3\}}),\ c = (y^1, x^2, y^3, z^{-\{1,2,3\}});\\
&\bar{a} = (x^1, y^2, x^3, z^{-\{1,2,3\}}),\ \bar{b} = (y^1, y^2, y^3, z^{-\{1,2,3\}}),\ \bar{c} = (x^1, x^2, y^3, z^{-\{1,2,3\}});\\
&\bar{x} = (y^1, x^2, x^3, z^{-\{1,2,3\}}),\ \bar{y} = (x^1, y^2, y^3, z^{-\{1,2,3\}}).
\end{aligned}$$

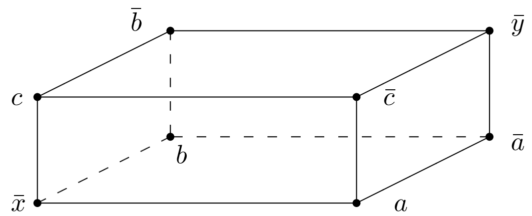


Figure 1: The geometric relations among $a, b, c, \bar{a}, \bar{b}, \bar{c}, \bar{x}$ and $\bar{y}$

Now, we can construct two preference profiles: $P = (P_1, P_2, P_3) \in \mathbb{D}^3_{MSP}$ and $P' = (P'_1, P'_2, P'_3) \in \mathbb{D}^3_{MSP}$ such that the following five conditions are satisfied

(i) $r_1(P_1) = r_1(P'_1) = a$, $r_1(P_2) = r_1(P'_2) = b$ and $r_1(P_3) = r_1(P'_3) = c$;

(ii) $r_2(P_1) = \bar{x}$, $r_3(P_1) = \bar{a}$ and $r_4(P_1) = b$;

(iii) $r_2(P_2) = \bar{x}$, $r_3(P_2) = \bar{b}$ and $r_4(P_2) = c$;

(iv) $r_2(P_3) = \bar{x}$, $r_3(P_3) = \bar{c}$ and $r_4(P_3) = a$;

(v) $\bar{y} P'_i \bar{x}$, $i = 1, 2, 3$

By a similar argument in the proof of Proposition 5.1, we first have $\varphi_a(P) = \varepsilon_1$, $\varphi_b(P) = \varepsilon_2$, $\varphi_c(P) = \varepsilon_3$ and $\varphi_x(P) = 0$ for all $x \notin \{a, b, c, \bar{x}\}$. Moreover, since $\varphi_{\bar{x}}(P) = \varphi_{\bar{x}}(P') = 0$ by tops-onlyness and ex-post efficiency, we have $\varepsilon_1+\varepsilon_2+\varepsilon_3 = \varphi_a(P) + \varphi_b(P) + \varphi_c(P) = \sum_{x\in A} \varphi_x(P) = 1$ as required. $\square$

Furthermore, by a same argument of Lemma 5.5, we know that for all $P \in \mathbb{D}^N_{MSP}$ with $r_1(P_i) = r_1(P_j)$ for some $i, j \in I$, $\varphi(P) = \sum_{i\in I} \varepsilon_i e_{r_1(P_i)}$. Therefore, to complete the verification of the Primary induction hypothesis, we show in Lemmas 12.4 and 12.5 below that for all $P \in \mathbb{D}^N$ with $r_1(P_i) \neq r_1(P_j)$ for all $i, j \in I$, $\varphi(P) = \sum_{i\in I} \varepsilon_i e_{r_1(P_i)}$.

We first recall one notation used in Appendix 5. Given $P \in \mathbb{D}^N$ with $|r_(P)| = N$, let $\overline{W}(P) = \cup_{i\in I} W\big(P_i, \max(P_i, r_1(P_{-i}))\big)$.

Lemma 12.4. *For all $P \in \mathbb{D}^N_{MSP}$ with $|r_1(P)| = N$ and $x \in \overline{W}(P)$, $\varphi_x(P) = \sum\limits_{i\in I: r_1(P_i)=x} \varepsilon_i$.*

Proof. This lemma follows from Lemma 5.7. $\square$

Lemma 12.5. *For all $P \in \mathbb{D}^N_{MSP}$ with $|r_1(P)| = N$, $\varphi(P) = \sum_{i\in I} \varepsilon_i e_{r_1(P_i)}$.*

Proof. Fix $P \in \mathbb{D}^N_{MSP}$ with $|r_1(P)| = N$. For notational convenience, let $a_i = r_1(P_i)$ for all $i \in I$. We can identify two voters $i, j \in I$ such that the minimal box between a_i and a_j contains no other voter's peak, i.e., $\langle a_i, a_j\rangle \cap r_1(P_{-\{i,j\}}) = \emptyset$.

We construct a sequence $\{x_k\}_{k=1}^t \subseteq \langle a_i, a_j\rangle$ such that $x_1 = a_i$, $x_t = a_j$; $x_k \in \langle x_1, x_{k+1}\rangle$ and (x_k, x_{k+1}) is an edge in $\times_{s\in M} G(A^s)$, $k = 1, \ldots, t-1$. Evidently, we know that for all $1 \leq k' \leq k \leq t$,

(i) $\langle x_{k'}, x_k\rangle \cap r_1(P_{-\{i,j\}}) = \emptyset$;

(ii) $\big[k' \leq k'' \leq k\big] \Rightarrow \big[x_{k''} \in \langle x_{k'}, x_k\rangle\big]$, and $\big[k'' < k'$ or $k'' > k\big] \Rightarrow \big[x_{k''} \notin \langle x_{k'}, x_k\rangle\big]$.

Note that given $1 \leq k' \leq k \leq t$, there exist $\bar{P}_i^{(k',k)} \in \mathbb{D}^{x_{k'}}_{MSP}$ and $\bar{P}_j^{(k,k')} \in \mathbb{D}^{x_k}_{MSP}$ such that all alternatives in $\langle x_{k'}, x_k\rangle$ are ranked above all alternatives out of $\langle x_{k'}, x_k\rangle$, i.e.,

$$\big[x \in \langle x_{k'}, x_k\rangle \text{ and } y \notin \langle x_{k'}, x_k\rangle\big] \Rightarrow \big[x\bar{P}_i^{(k',k)}y \text{ and } x\bar{P}_j^{(k,k')}y\big].$$

Hence, for all $l \neq i, j$, $x_k\bar{P}_i^{(k',k)}a_l$ and $x_{k'}\bar{P}_j^{(k,k')}a_l$.

We will show $\varphi(P) = \varphi(x_1, x_t, P_{-\{i,j\}}) = \varepsilon_i e_{x_1} + \varepsilon_j e_{x_t} + \sum_{l\neq i,j}\varepsilon_l e_{a_l}$. Firstly, by quasi random dictatorship and tops-onlyness, we know $\varphi(x_1, x_1, P_{-\{i,j\}}) = (\varepsilon_i + \varepsilon_j)e_{x_1} + \sum_{l\neq i,j}\varepsilon_l e_{a_l}$. Next, we provide an induction argument.

Induction Hypothesis: Given $1 < k \leq t$, for all $1 \leq \underline{k} \leq \bar{k} < k$, $\varphi(x_{\underline{k}}, x_{\bar{k}}, P_{-\{i,j\}}) = \varepsilon_i e_{x_{\underline{k}}} + \varepsilon_j e_{x_{\bar{k}}} + \sum_{l\neq i,j}\varepsilon_l e_{a_l}$.

We will show that for all $1 \leq \underline{k} \leq \bar{k} \leq k$, $\varphi(x_{\underline{k}}, x_{\bar{k}}, P_{-\{i,j\}}) = \varepsilon_i e_{x_{\underline{k}}} + \varepsilon_j e_{x_{\bar{k}}} + \sum_{l\neq i,j}\varepsilon_l e_{a_l}$.

Given $1 \leq \underline{k} \leq \bar{k} \leq k$, if $\bar{k} < k$, the induction hypothesis gives the result. Next, assume $\bar{k} = k$. If $\underline{k} = k$, then quasi random dictatorship and tops-onlyness imply $\varphi(x_k, x_k, P_{-\{i,j\}}) = (\varepsilon_i + \varepsilon_j)e_{x_k} + \sum_{l\neq i,j}\varepsilon_l e_{a_l}$. Moreover, we provide another induction argument.

The Secondary Induction Hypothesis: Given $1 \leq \hat{k} < k$, for all $\hat{k} < \underline{k} \leq k$, $\varphi(x_{\underline{k}}, x_k, P_{-\{i,j\}}) = \varepsilon_i e_{x_{\underline{k}}} + \varepsilon_j e_{x_k} + \sum_{l\neq i,j}\varepsilon_l e_{a_l}$.

We will show $\varphi(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varepsilon_i e_{x_{\hat{k}}} + \varepsilon_j e_{x_k} + \sum_{l\neq i,j}\varepsilon_l e_{a_l}$.

Claim 1: For all $l \neq i, j$, $\varphi_{a_l}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varepsilon_l$.

Given preferences $\bar{P}_i^{(\hat{k},k)}$, $\bar{P}_j^{(k,\hat{k})}$, since $\langle x_{\hat{k}}, x_k\rangle \cap r_1(P_{-\{i,j\}}) = \emptyset$, $|r_1(\bar{P}_i^{(\hat{k},k)}, \bar{P}_i^{(k,\hat{k})}, P_{-\{i,j\}})| = N$. Fixing $l \neq i, j$, since $x_k\bar{P}_i^{(\hat{k},k)}a_l$, it is true that $a_l \in W\big(\bar{P}_i^{(\hat{k},k)}, \max\big(\bar{P}_i^{(\hat{k},k)}, r_1(\bar{P}_j^{(k,\hat{k})}, P_{-\{i,j\}})\big)\big) \subseteq \overline{W}(\bar{P}_i^{(\hat{k},k)}, \bar{P}_j^{(k,\hat{k})}, P_{-\{i,j\}})$. Then, by tops-onlyness and Lemma 12.4, we have $\varphi_{a_l}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varphi_{a_l}(\bar{P}_i^{(\hat{k},k)}, \bar{P}_j^{(k,\hat{k})}, P_{-\{i,j\}}) = \varepsilon_l$. This completes the verification of the claim.

Claim 2: $\varphi_{x_{k-1}}(x_{k-1}, x_k, P_{-\{i,j\}}) = \varepsilon_i$ and $\varphi_{x_k}(x_{k-1}, x_k, P_{-\{i,j\}}) = \varepsilon_j$.

Since (x_{k-1}, x_k) is an edge in $\times_{s\in M}G(A^s)$, we have $\bar{P}_i \in \mathbb{D}_{MSP}^{x_{k-1}}$ and $\bar{P}'_i \in \mathbb{D}_{MSP}^{x_k}$ such that $r_2(\bar{P}_i) = x_k$ and $r_2(\bar{P}'_i) = x_{k-1}$. Firstly, by quasi random dictatorship and tops-onlyness, $\varphi_{x_{k-1}}(\bar{P}'_i, x_k, P_{-\{i,j\}}) = 0$ and $\varphi_{x_k}(\bar{P}'_i, x_k, P_{-\{i,j\}}) =$

$\varepsilon_i + \varepsilon_j$. Next, by tops-onlyness and strategy-proofness, we have $\varphi_{x_{k-1}}(x_{k-1}, x_k, P_{-\{i,j\}}) + \varphi_{x_k}(x_{k-1}, x_k, P_{-\{i,j\}}) = \varphi_{x_{k-1}}(\bar{P}_i, x_k, P_{-\{i,j\}}) + \varphi_{x_k}(\bar{P}_i, x_k, P_{-\{i,j\}}) = \varphi_{x_{k-1}}(\bar{P}'_i, x_k, P_{-\{i,j\}}) + \varphi_{x_k}(\bar{P}'_i, x_k, P_{-\{i,j\}}) = \varepsilon_i + \varepsilon_j$.

Given $\bar{P}_i \in \mathbb{D}^{x_{k-1}}_{MSP}$ and $\bar{P}_j \in \mathbb{D}^{x_k}_{MSP}$, it is evident that $|r_1(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})| = N$. Fixing $l \neq i, j$, since either $x_{k-1} P_l x_k$ or $x_k P_l x_{k-1}$, we have either $x_k \in W\big(P_l, \max(P_l, r_1(\bar{P}_i, \bar{P}_j, P_{-i,j,l}))\big) \subseteq \overline{W}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})$, or $x_{k-1} \in W\big(P_l, \max(P_l, r_1(\bar{P}_i, \bar{P}_j, P_{-i,j,l}))\big) \subseteq \overline{W}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})$. Then, by tops-onlyness and Lemma 12.4, we have either $\varphi_{x_k}(x_{k-1}, x_k, P_{-\{i,j\}}) = \varphi_{x_k}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}}) = \varepsilon_j$ or $\varphi_{x_{k-1}}(x_{k-1}, x_k, P_{-\{i,j\}}) = \varphi_{x_{k-1}}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}}) = \varepsilon_i$. Consequently, we have $\varphi_{x_{k-1}}(x_{k-1}, x_k, P_{-\{i,j\}}) = \varepsilon_i$ and $\varphi_{x_k}(x_{k-1}, x_k, P_{-\{i,j\}}) = \varepsilon_j$. This completes the verification of the claim.

Claim 3: If $\hat{k} < k-1$, the following two equalities hold:

$$\begin{aligned} \varphi_{x_{\hat{k}}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) + \varphi_{x_{\hat{k}+1}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) &= \varepsilon_i \text{ and} \\ \varphi_{x_k}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) + \varphi_{x_{k-1}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) &= \varepsilon_j. \end{aligned}$$

Since $(x_{\hat{k}}, x_{\hat{k}+1})$ is an edge in $\times_{s\in M} G(A^s)$, we have $\bar{P}_i \in \mathbb{D}^{x_{\hat{k}}}_{MSP}$ and $\bar{P}'_i \in \mathbb{D}^{x_{\hat{k}+1}}_{MSP}$ such that $r_2(\bar{P}_i) = x_{\hat{k}+1}$ and $r_2(\bar{P}'_i) = x_{\hat{k}}$. Firstly, the secondary induction hypothesis implies $\varphi_{x_{\hat{k}}}(\bar{P}'_i, x_k, P_{-\{i,j\}}) = 0$ and $\varphi_{x_{\hat{k}+1}}(\bar{P}'_i, x_k, P_{-\{i,j\}}) = \varepsilon_i$. Then, by tops-onlyness and strategy-proofness, $\varphi_{x_{\hat{k}}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) + \varphi_{x_{\hat{k}+1}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varphi_{x_{\hat{k}}}(\bar{P}_i, x_k, P_{-\{i,j\}}) + \varphi_{x_{\hat{k}+1}}(\bar{P}_i, x_k, P_{-\{i,j\}}) = \varphi_{x_{\hat{k}}}(\bar{P}'_i, x_k, P_{-\{i,j\}}) + \varphi_{x_{\hat{k}+1}}(\bar{P}'_i, x_k, P_{-\{i,j\}}) = \varepsilon_i$.

Since (x_k, x_{k-1}) is an edge in $\times_{s\in M} G(A^s)$, we have $\bar{P}_j \in \mathbb{D}^{x_k}_{MSP}$ and $\bar{P}'_j \in \mathbb{D}^{x_{k-1}}_{MSP}$ such that $r_2(\bar{P}_j) = x_{k-1}$ and $r_2(\bar{P}'_j) = x_k$. Firstly, induction hypothesis implies $\varphi_{x_k}(x_{\hat{k}}, \bar{P}'_j, P_{-\{i,j\}}) = 0$ and $\varphi_{x_{k-1}}(x_{\hat{k}}, \bar{P}'_j, P_{-\{i,j\}}) = \varepsilon_j$. Then, by tops-onlyness and strategy-proofness, it is true that $\varphi_{x_k}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) + \varphi_{x_{k-1}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varphi_{x_k}(x_{\hat{k}}, \bar{P}_j, P_{-\{i,j\}}) + \varphi_{x_{k-1}}(x_{\hat{k}}, \bar{P}_j, P_{-\{i,j\}}) = \varphi_{x_k}(x_{\hat{k}}, \bar{P}'_j, P_{-\{i,j\}}) + \varphi_{x_{k-1}}(x_{\hat{k}}, \bar{P}'_j, P_{-\{i,j\}}) = \varepsilon_j$. This completes the verification of the claim.

Claim 4: If $\hat{k} < k-1$, $\varphi_{x_{\hat{k}}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varepsilon_i$ and $\varphi_{x_k}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varepsilon_j$.

Fixing $l \neq i, j$, we know that either one of the following three cases occurs: (i) $x_{\hat{k}} \in \langle a_l, x_k \rangle$, (ii) $x_k \in \langle a_l, x_{\hat{k}} \rangle$ and (iii) $x_{\hat{k}} \notin \langle a_l, x_k \rangle$ and $x_k \notin \langle a_l, x_{\hat{k}} \rangle$.

In case (i), since $x_{\hat{k}+1} \in \langle x_{\hat{k}}, x_k \rangle$, it must be the case that $x_{\hat{k}} \in \langle a_l, x_{\hat{k}+1} \rangle$. Thus, $x_{\hat{k}} P_l x_k$ and $x_{\hat{k}} P_l x_{\hat{k}+1}$. Given $\bar{P}_i \in \mathbb{D}_{MSP}^{x_{\hat{k}}}$ and $\bar{P}_j \in \mathbb{D}_{MSP}^{x_k}$, it is evident that $|r_1(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})| = N$. Moreover, $x_{\hat{k}} P_l x_{\hat{k}+1}$ and $x_{\hat{k}} P_l x_k$ imply $x_{\hat{k}+1}, x_k \in W\big(P_l, \max(P_l, r_1(\bar{P}_i, \bar{P}_j, P_{-i,j,l}))\big) \subseteq \overline{W}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})$. Consequently, by tops-onlyness and Lemma 12.4, we have $\varphi_{x_{\hat{k}+1}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varphi_{x_{\hat{k}+1}}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}}) = 0$ and $\varphi_{x_k}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varphi_{x_k}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}}) = \varepsilon_j$. Furthermore, by Claim 3, $\varphi_{x_{\hat{k}}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varepsilon_i$.

In case (ii), since $x_{k-1} \in \langle x_{\hat{k}}, x_k \rangle$, it must be the case that $x_k \in \langle a_l, x_{k-1} \rangle$. Thus, $x_k P_l x_{\hat{k}}$ and $x_k P_l x_{k-1}$. Given $\bar{P}_i \in \mathbb{D}_{MSP}^{x_{\hat{k}}}$ and $\bar{P}_j \in \mathbb{D}_{MSP}^{x_k}$, it is evident that $|r_1(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})| = N$. Moreover, $x_k P_l x_{\hat{k}}$ and $x_k P_l x_{k-1}$ imply $x_{\hat{k}}, x_{k-1} \in W\big(P_l, \max(P_l, r_1(\bar{P}_i, \bar{P}_j, P_{-i,j,l}))\big) \subseteq \overline{W}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}})$. Consequently, by tops-onlyness and Lemma 12.4, we have $\varphi_{x_{\hat{k}}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varphi_{x_{\hat{k}}}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}}) = \varepsilon_i$ and $\varphi_{x_{k-1}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varphi_{x_{k-1}}(\bar{P}_i, \bar{P}_j, P_{-\{i,j\}}) = 0$. Then, by Claim 3, $\varphi_{x_k}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varepsilon_j$.

In case (iii), we have $\bar{P}_l, \hat{P}_l \in \mathbb{D}_{MSP}^{a_l}$ such that $x_k \bar{P}_l x_{\hat{k}}$ and $x_{\hat{k}} \hat{P}_l x_k$. Given $\bar{P}_i \in \mathbb{D}_{MSP}^{x_{\hat{k}}}$ and $\bar{P}_j \in \mathbb{D}_{MSP}^{x_k}$, it is evident that $|r_1(\bar{P}_i, \bar{P}_j, \bar{P}_l, P_{-i,j,l})| = N$ and $|r_1(\bar{P}_i, \bar{P}_j, \hat{P}_l, P_{-i,j,l})| = N$. Moreover, $x_k \bar{P}_l x_{\hat{k}}$ and $x_{\hat{k}} \hat{P}_l x_k$ imply
$x_{\hat{k}} \in W\big(\bar{P}_l, \max(\bar{P}_l, r_1(\bar{P}_i, \bar{P}_j, P_{-i,j,l}))\big) \subseteq \overline{W}(\bar{P}_i, \bar{P}_j, \bar{P}_l, P_{-i,j,l})$ and
$x_k \in W\big(\hat{P}_l, \max(\hat{P}_l, r_1(\bar{P}_i, \bar{P}_j, P_{-i,j,l}))\big) \subseteq \overline{W}(\bar{P}_i, \bar{P}_j, \hat{P}_l, P_{-i,j,l})$ respectively. Hence, by tops-onlyness and Lemma 12.4, we have $\varphi_{x_{\hat{k}}}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varphi_{x_{\hat{k}}}(\bar{P}_i, \bar{P}_j, \hat{P}_l, P_{-i,j,l}) = \varepsilon_i$ and $\varphi_{x_k}(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varphi_{x_k}(\bar{P}_i, \bar{P}_j, \hat{P}_l, P_{-i,j,l}) = \varepsilon_j$. This completes the verification of the claim.

In conclusion, according to Claims 1, 2 and 4, $\varphi(x_{\hat{k}}, x_k, P_{-\{i,j\}}) = \varepsilon_i e_{x_{\hat{k}}} + \varepsilon_j e_{x_k} + \sum_{l \neq i,j} \varepsilon_l e_{a_l}$. This completes the verification of the secondary induction hypothesis. Then, for all $1 \leq \underline{k} \leq k$, $\varphi(x_{\underline{k}}, x_k, P_{-\{i,j\}}) = \varepsilon_i e_{x_{\underline{k}}} + \varepsilon_j e_{x_k} + \sum_{l \neq i,j} \varepsilon_l e_{a_l}$. This completes the verification of induction hypothesis. Therefore, for all $1 \leq \underline{k} \leq \bar{k} \leq t$, $\varphi(x_{\underline{k}}, x_{\bar{k}}, P_{-\{i,j\}}) = \varepsilon_i e_{x_{\underline{k}}} + \varepsilon_j e_{x_{\bar{k}}} + \sum_{l \neq i,j} \varepsilon_l e_{a_l}$. Therefore, $\varphi(P) =$

$\varphi(x_1, x_t, P_{-\{i,j\}}) = \varepsilon_i e_{x_1} + \varepsilon_j e_{x_t} + \sum_{l \neq i,j} \varepsilon_l e_{a_l} = \sum_{l \in I} \varepsilon_l e_{r_1(P_l)}$. □

13 Proof of Proposition 4.2.7

We first show the necessity part of Proposition 4.2.7. Let $\varphi : \mathbb{D}^N \to \Delta(A)$ be an ex-post efficient* and strategy-proof RSCF. First, since $\mathbb{D}$ is minimally rich and connected, it is easy to verify that $\mathbb{D}$ is path-connected (recall Section 3.2 in Chapter 3). Since ex-post efficiency* implies unanimity, Proposition 4.2.1 implies that φ satisfies the tops-only property.

Claim 1: RSCF φ satisfies the compromise property (recall Definition 3.1.1).

We simply let $\hat{I} = \{1, \ldots, \frac{N}{2}\}$ if N is even, and $\hat{I} = \{1, \ldots, \frac{N+1}{2}\}$ if N is odd. Given $P_i, P_j \in \mathbb{D}$ with $r_1(P_i) \equiv x \neq y \equiv r_1(P_j)$ and $r_2(P_i) = r_2(P_j) \equiv a$, it is evident that $\Omega\big(\underbrace{P_i, \ldots, P_i}_{\hat{I}}, \underbrace{P_j, \ldots, P_j}_{I \backslash \hat{I}}\big) = \{x, y, a\}$. Then, ex-post efficiency* implies $\varphi_a\big(\underbrace{P_i, \ldots, P_i}_{\hat{I}}, \underbrace{P_j, \ldots, P_j}_{I \backslash \hat{I}}\big) > 0$. This completes the verification of the claim.

Now, domain $\mathbb{D}$ is path-connected; and φ satisfies unanimity, the tops-only property, strategy-proofness and the compromise property. Then, Theorem 3.3.1 implies that $\mathbb{D}$ is single-peaked. This completes the verification of necessity part.

Now, we move to the verification of the sufficiency part. Let $\mathbb{D}$ be a single-peaked domain on a tree G. We the RSCF $\varphi : \mathbb{D}^N \to \Delta(A)$ constructed in the verification of the sufficiency part of Theorem 3.3.1. Then, we know that φ is strategy-proof, and moreover, $supp\, \varphi(P) = G(r_1(P))$ for all $P \in \mathbb{D}^N$. Therefore, to show that φ is ex-post efficient*, it suffices to show $\Omega(P) = G(r_1(P))$ for all $P \in \mathbb{D}^N$. Evidently, If $|r_1(P)| = 1$, then $\Omega(P) = G(r_1(P))$.

Claim 2: Given $P \in \mathbb{D}^N$ with $|r_1(P)| > 1$, $\Omega(P) \subseteq G(r_1(P))$.

Suppose that there exists $a \in \Omega(P) \backslash G(r_1(P))$. Thus, there exists an unique $b \in G(r_1(P))$ such that for all $i \in I$, $b \in \langle r_1(P_i), a \rangle$. Then, single-peakedness implies bP_ia for all $i \in I$. Hence, $a \notin \Omega(P)$. Contradiction! This completes the verification of the claim.

Claim 3: Given $P \in \mathbb{D}^N$ with $|r_1(P)| > 1$, $\Omega(P) \supseteq G(r_1(P))$.

Suppose that there exists $a \in G(r_1(P))\backslash\Omega(P)$. Thus, there exists $b \in A$ such that bP_ia for all $i \in I$. Evidently, $a \notin r_1(P)$. Since $a \in G(r_1(P))$ and $a \notin r_1(P)$, there exist $i, j \in I$ such that $r_1(P_i) \equiv x \neq y \equiv r_1(P_j)$ and $a \in \langle x, y\rangle\backslash\{x, y\}$. Since G is a tree, it is true that either $a \in \langle x, b\rangle$ or $a \in \langle y, b\rangle$. Consequently, either aP_ib or aP_jb. Contradiction! This completes the verification of the claim.

In conclusion, $\Omega(P) = G(r_1(P))$ for all $P \in \mathbb{D}^N$. Hence, $supp\,\varphi(P) = \Omega(P)$ for all $P \in \mathbb{D}^N$, and φ is ex-post efficient*. This completes the verification of the sufficiency part.

14 Strategy-proofness in Example 4.3.2

RSCF φ follows three distinct function forms according to preference profiles. Evidently, if both voters share the same peak of preferences, no one has the incentive to deviate. Next, it is easy to show that if two social lotteries, which are induced by truthtelling and misrepresentation of some voter, are generated by the same function form, the one under truthtelling always stochastically dominates the one under misrepresentation according to the true preference. Therefore, we only need to consider the possible manipulation where the corresponding social lotteries are generated by distinct function forms. In these possible manipulations (16 situations specified below), we assert that probabilities are always transferred systematically from the preferred alternatives to less preferred alternatives according to the true preference which equivalently indicates stochastic dominance.

1. In $(P_i, P_j) \leftrightarrow (P_i', P_j)$, where $P_i \in \mathbb{D}^{a_1}$, $P_i' \in \{P_7, P_9\}$ and $P_j \in \mathbb{D}^{a_5}$, we have

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_1P_ia_3}, \xrightarrow[1/4]{a_5P_ia_4} \varphi(P_i', P_j), \text{ and } \varphi(P_i', P_j) \xrightarrow[1/4]{a_3P_i'a_1}, \xrightarrow[1/4]{a_4P_i'a_5} \varphi(P_i, P_j).$$

2. In $(P_i, P_j) \leftrightarrow (P_i', P_j)$, where $P_i \in \mathbb{D}^{a_2} = \{P_4, P_5, P_6\}$, $P_i' \in \{P_7, P_9\}$ and $P_j \in \mathbb{D}^{a_5}$, we have

$$\begin{aligned}
&\varphi(P_i, P_j) \xrightarrow[1/4]{a_2 P_i a_3}, \xrightarrow[1/4]{a_2 P_i a_1}, \xrightarrow[1/4]{a_5 P_i a_4} \varphi(P_i', P_j), && \text{if } P_i \in \{P_4, P_5\}.\\
&\varphi(P_i, P_j) \xrightarrow[1/4]{a_2 P_i a_3}, \xrightarrow[1/4]{a_2 P_i a_4}, \xrightarrow[1/4]{a_5 P_i a_1} \varphi(P_i', P_j), && \text{if } P_i = P_6.\\
&\varphi(P_i', P_j) \xrightarrow[1/4]{a_3 P_i' a_2}, \xrightarrow[1/4]{a_1 P_i' a_2}, \xrightarrow[1/4]{a_4 P_i' a_5} \varphi(P_i, P_j), && \text{if } P_i' = P_7.\\
&\varphi(P_i', P_j) \xrightarrow[1/4]{a_3 P_i' a_2}, \xrightarrow[1/4]{a_4 P_i' a_2}, \xrightarrow[1/4]{a_1 P_i' a_5} \varphi(P_i, P_j), && \text{if } P_i' = P_9.
\end{aligned}$$

3. In $(P_i, P_j) \leftrightarrow (P_i', P_j)$, where $P_i \in \mathbb{D}^{a_4}$, $P_i' \in \{P_7, P_9\}$ and $P_j \in \mathbb{D}^{a_5}$, we have

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_4 P_i a_3}, \xrightarrow[1/4]{a_5 P_i a_1} \varphi(P_i', P_j), \text{ and } \varphi(P_i', P_j) \xrightarrow[1/4]{a_3 P_i' a_4}, \xrightarrow[1/4]{a_1 P_i' a_5} \varphi(P_i, P_j).$$

4. In $(P_i, P_j) \leftrightarrow (P_i, P_j')$, where $P_i \in \{P_7, P_9\}$, $P_j \in \mathbb{D}^{a_1}$ and $P_j' \in \mathbb{D}^{a_5}$, we have

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_1 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_4} \varphi(P_i, P_j'), \text{ and } \varphi(P_i, P_j') \xrightarrow[1/4]{a_5 P_j' a_1}, \xrightarrow[1/4]{a_4 P_j' a_3} \varphi(P_i, P_j).$$

5. In $(P_i, P_j) \leftrightarrow (P_i, P_j')$, where $P_i \in \{P_7, P_9\}$, $P_j \in \mathbb{D}^{a_2}$ and $P_j' \in \mathbb{D}^{a_5}$, we have

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_2 P_j a_1}, \xrightarrow[1/4]{a_2 P_j a_4}, \xrightarrow[1/4]{a_3 P_j a_5} \varphi(P_i, P_j'), \text{ and } \varphi(P_i, P_j') \xrightarrow[1/4]{a_5 P_j' a_3}, \xrightarrow[1/4]{a_1 P_j' a_2}, \xrightarrow[1/4]{a_4 P_j' a_2} \varphi(P_i, P_j).$$

6. In $(P_i, P_j) \leftrightarrow (P_i, P_j')$, where $P_i \in \{P_7, P_9\}$, $P_j \in \mathbb{D}^{a_4}$ and $P_j' \in \mathbb{D}^{a_5}$, we have

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_4 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_1} \varphi(P_i, P_j'), \text{ and } \varphi(P_i, P_j') \xrightarrow[1/4]{a_5 P_j' a_4}, \xrightarrow[1/4]{a_1 P_j' a_3} \varphi(P_i, P_j).$$

7. In $(P_i, P_j) \leftrightarrow (P_i', P_j)$ where $P_i \in \mathbb{D}^{a_1} \cup \mathbb{D}^{a_4}$, $P_i' = P_8$ and $P_j \in \mathbb{D}^{a_5}$, we have

$$\varphi(P_i, P_j) \xrightarrow[1/4]{r_1(P_i) P_i a_3}, \xrightarrow[1/4]{r_1(P_i) P_i a_2} \varphi(P_i', P_j), \text{ and } \varphi(P_i', P_j) \xrightarrow[1/4]{a_3 P_i' r_1(P_i)}, \xrightarrow[1/4]{a_2 P_i' r_1(P_i)} \varphi(P_i, P_j).$$

8. In $(P_i, P_j) \leftrightarrow (P_i', P_j)$ where $P_i \in \mathbb{D}^{a_2}$, $P_i' = P_8$ and $P_j \in \mathbb{D}^{a_5}$, we have

$$\varphi(P_i, P_j) \xrightarrow[1/4]{a_2 P_i a_3} \varphi(P_i', P_j), \text{ and } \varphi(P_i', P_j) \xrightarrow[1/4]{a_3 P_i' a_2} \varphi(P_i, P_j).$$

9. In $(P_i, P_j) \leftrightarrow (P_i, P'_j)$ where $P_i = P_8$, $P_j \in \mathbb{D}^{a_1} = \{P_1, P_2, P_3\}$ and $P'_j \in \mathbb{D}^{a_5}$, we have

$$\begin{aligned}
\varphi(P_i, P_j) \xrightarrow[1/4]{a_1 P_j a_2}, \xrightarrow[1/4]{a_1 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_5} \varphi(P_i, P'_j), &\qquad \text{if } P_j = P_1. \\
\varphi(P_i, P_j) \xrightarrow[1/2]{a_1 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_2} \varphi(P_i, P'_j), &\qquad \text{if } P_j \in \{P_2, P_3\}. \\
\varphi(P_i, P'_j) \xrightarrow[1/2]{a_5 P'_j a_1}, \xrightarrow[1/4]{a_2 P'_j a_3} \varphi(P_i, P_j). &
\end{aligned}$$

10. In $(P_i, P_j) \leftrightarrow (P_i, P'_j)$ where $P_i = P_8$, $P_j \in \mathbb{D}^{a_2}$ and $P'_j \in \mathbb{D}^{a_5}$, we have $\varphi(P_i, P_j) \xrightarrow[1/4]{a_2 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_5} \varphi(P_i, P'_j)$, and $\varphi(P_i, P'_j) \xrightarrow[1/4]{a_5 P'_j a_2}, \xrightarrow[1/4]{a_5 P'_j a_3} \varphi(P_i, P_j)$.

11. In $(P_i, P_j) \leftrightarrow (P_i, P'_j)$ where $P_i = P_8$, $P_j \in \mathbb{D}^{a_4} = \{P_{10}, P_{11}, P_{12}\}$ and $P'_j \in \mathbb{D}^{a_5}$, we have

$$\begin{aligned}
\varphi(P_i, P_j) \xrightarrow[1/4]{a_4 P_j a_2}, \xrightarrow[1/4]{a_4 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_5} \varphi(P_i, P'_j), &\qquad \text{if } P_j = P_{10}. \\
\varphi(P_i, P_j) \xrightarrow[1/2]{a_4 P_j a_5}, \xrightarrow[1/4]{a_3 P_j a_2} \varphi(P_i, P'_j), &\qquad \text{if } P_j \in \{P_{11}, P_{12}\}. \\
\varphi(P_i, P'_j) \xrightarrow[1/2]{a_5 P'_j a_4}, \xrightarrow[1/4]{a_2 P'_j a_3} \varphi(P_i, P_j). &
\end{aligned}$$

12. In $(P_i, P_j) \leftrightarrow (P'_i, P_j)$ where $P_i \in \{P_7, P_9\}$, $P'_i = P_8$ and $P_j \in \mathbb{D}^{a_5}$, we have

$$\begin{aligned}
\varphi(P_i, P_j) \xrightarrow[1/4]{a_1 P_i a_2}, \xrightarrow[1/4]{a_4 P_i a_5} \varphi(P'_i, P_j), &\qquad \text{if } P_i = P_7. \\
\varphi(P_i, P_j) \xrightarrow[1/4]{a_4 P_i a_2}, \xrightarrow[1/4]{a_1 P_i a_5} \varphi(P'_i, P_j), &\qquad \text{if } P_i = P_9. \\
\varphi(P'_i, P_j) \xrightarrow[1/4]{a_2 P'_i a_1}, \xrightarrow[1/4]{a_5 P'_i a_4} \varphi(P_i, P_j). &
\end{aligned}$$

13. In $(P_i, P_j) \rightarrow (P'_i, P_j)$ where $P_i \in \{P_7, P_9\}$, $P'_i \in \mathbb{D}^{a_5}$ and $P_j \in \mathbb{D}^{a_5}$, we have $\varphi(P_i, P_j) \xrightarrow[1/4]{a_3 P_i a_5}, \xrightarrow[1/4]{a_1 P_i a_5}, \xrightarrow[1/4]{a_4 P_i a_5} \varphi(P'_i, P_j)$.

14. In $(P_i, P_j) \rightarrow (P'_i, P_j)$ where $P_i = P_8$, $P'_i \in \mathbb{D}^{a_5}$ and $P_j \in \mathbb{D}^{a_5}$, we have $\varphi(P_i, P_j) \xrightarrow[1/4]{a_3 P_i a_5}, \xrightarrow[1/4]{a_3 P_i a_5} \varphi(P'_i, P_j)$.

15. In $(P_i, P_j) \rightarrow (P_i, P'_j)$ where $P_i \in \{P_7, P_9\}$, $P_j \in \mathbb{D}^{a_5}$ and $P'_j \in \mathbb{D}^{a_3}$, we have $\varphi(P_i, P_j) \xrightarrow[1/4]{a_5 P_j a_3}, \xrightarrow[1/4]{a_1 P_j a_3}, \xrightarrow[1/4]{a_4 P_j a_3} \varphi(P_i, P'_j)$.

16. In $(P_i, P_j) \to (P_i, P'_j)$ where $P_i = P_8$, $P_j \in \mathbb{D}^{a_5}$ and $P'_j \in \mathbb{D}^{a_3}$, we have $\varphi(P_i, P_j) \xrightarrow[1/2]{a_5 P_j a_3}, \xrightarrow[1/4]{a_2 P_j a_3} \varphi(P_i, P'_j)$.

15 Proof of Proposition 4.3.1

The sufficiency part is implied by Theorem 4.1.1. We focus on the necessity part. Since $A = \{0,1\} \times \{0,1\}$, the multi-dimensional single-peaked domain $\mathbb{D}_{MSP}$ is specified in Table 4.2 of Example 4.2.1. Since $\mathbb{D}$ is significantly rich, it is evidently that $\mathbb{D}$ satisfies the Interior Property. Now, suppose that $\mathbb{D}$ violates the Exterior Property*. Accordingly, we have $P_i, \bar{P}_i, P'_i \in \mathbb{D}$ satisfying the following three conditions:

(i) $r_1(P_i) = r_1(\bar{P}_i)$; $P_i \sim^A \bar{P}_i$, $xP_i!y$ and $y\bar{P}_i!x$;

(ii) $r_1(P'_i) \neq r_1(P_i)$ and xP'_iy;

(iii) there exists no (x, y)-Is-path connecting P_i and P'_i.

According to Table 4.2, condition (i) implies $\{P_i, \bar{P}_i\} = \{P_k, P_{k+1}\}$ for some $k \in \{1, 3, 5, 7\}$. We assume w.l.o.g. that $P_i = P_1$ and $\bar{P}_i = P_2$. Consequently, $x = (1, 0)$ and $y = (0, 1)$. Since $(1, 0)P'_i(0, 1)$ in condition (ii), we know $P'_i \notin \{P_1, P_2, P_5, P_6, P_8\}$. Thus, either $P'_i \in \{P_3, P_4\}$ or $P'_i = P_7$. Furthermore, if $P'_i \in \{P_3, P_4\}$, significant richness condition implies $\{P_3, P_4\} \subseteq \mathbb{D}$. Consequently, $\{P_1, P_3\}$ is a (x, y)-Is-path connecting P_i and P'_i if $P'_i = P_3$, or $\{P_1, P_3, P_4\}$ is a (x, y)-Is-path connecting P_i and P'_i if $P'_i = P_4$. This contradicts condition (iii) above. Therefore, $P'_i = P_7$. Then significant richness condition implies $\{P_7, P_8\} \subseteq \mathbb{D}$. Thus, $\{P_1, P_2, P_7, P_8\} \subseteq \mathbb{D}$. We show that $P_3, P_4 \notin \mathbb{D}$. Suppose not, i.e., either $P_3 \in \mathbb{D}$ or $P_4 \in \mathbb{D}$. Then, significant richness implies $P_3, P_4 \in \mathbb{D}$. Consequently, $\{P_1, P_3, P_4, P_7\}$ is a (x, y)-Is-path connecting P_i and P'_i. Contradiction! Therefore, either $\mathbb{D} = \{P_1, P_2, P_7, P_8\}$ or $\mathbb{D} = \{P_1, P_2, P_5, P_6, P_7, P_8\}$.

Assume $\mathbb{D} = \{P_1, P_2, P_5, P_6, P_7, P_8\}$. We will show domain $\mathbb{D}$ admits an unanimous and strategy-proof RSCF violating the tops-only property. We first provide four unanimous and strategy-proof DSCFs below: for all $P_i, P_j \in \mathbb{D}$, assuming $r_1(P_i) = a$ and $r_1(P_j) = b$,

$$f^{(0,0)}(P_i, P_j) = \big(med(a^1, b^1, 0), med(a^2, b^2, 0)\big), \qquad f^{(1,0)}(P_i, P_j) = \big(med(a^1, b^1, 1), med(a^2, b^2, 0)\big),$$
$$f^{(0,1)}(P_i, P_j) = \big(med(a^1, b^1, 0), med(a^2, b^2, 1)\big), \qquad f^{(1,1)}(P_i, P_j) = \big(med(a^1, b^1, 1), med(a^2, b^2, 1)\big).$$

According to [6], all four DSCFs above are unanimous and strategy-proof. Next, we specify two RSCFs.

$$\bar{\varphi}(P_i, P_j) = \frac{1}{4}\big[f^{(0,0)}(P_i, P_j) + f^{(1,0)}(P_i, P_j) + f^{(0,1)}(P_i, P_j) + f^{(1,1)}(P_i, P_j)\big]$$
$$\text{for all } P_i, P_j \in \mathbb{D}.$$

$$\varphi(P_i, P_j) = \begin{cases} \frac{1}{4}f^{(0,0)}(P_i, P_j) + \frac{1}{2}f^{(1,0)}(P_i, P_j) + \frac{1}{4}f^{(1,1)}(P_i, P_j), \\ \qquad \text{if } (P_i, P_j) = (P_1, P_7); \\ \\ \bar{\varphi}(P_i, P_j), \\ \qquad \text{Otherwise.} \end{cases}$$

RSCF $\bar{\varphi}$ follows from a convex combination of all *four* DSCFs above. Therefore, $\bar{\varphi}$ is unanimous and strategy-proof. Observe that RSCF φ is identical to $\bar{\varphi}$ if the preference profile is not (P_1, P_7), and otherwise follows from a convex combination of *three* DSCFs, where $f^{(0,1)}$ is removed and corresponding the weight is transferred to $f^{(1,0)}$. Evidently, RSCF φ is unanimous and violates the tops-only property, e.g., $\varphi(P_1, P_7) = \frac{1}{4}e_{(0,0)} + \frac{1}{2}e_{(1,0)} + \frac{1}{4}e_{(1,1)} \neq \frac{1}{4}e_{(0,0)} + \frac{1}{4}e_{(1,0)} + \frac{1}{4}e_{(0,1)} + \frac{1}{4}e_{(1,1)} = \varphi(P_1, P_8)$. We claim that φ is strategy-proof. There are following four possible manipulations:

1. $(P_1, P_7) \to (P_i', P_7)$ where $P_i' \neq P_1$, and $(P_1, P_7) \to (P_1, P_j')$ where $P_j' \neq P_7$.

2. $(P_i, P_7) \to (P_1, P_7)$ where $P_i \neq P_1$.

3. $(P_1, P_j) \to (P_1, P_7)$ where $P_j \neq P_7$.

In situation 1, we first observe that

$$\varphi(P_1, P_7) \xrightarrow[1/4]{(1,0)P_1(0,1)} \bar{\varphi}(P_1, P_7), \text{ and } \varphi(P_1, P_7) \xrightarrow[1/4]{(1,0)P_7(0,1)} \bar{\varphi}(P_1, P_7)$$

Therefore, $\varphi(P_1, P_7)$ stochastically dominates $\bar{\varphi}(P_1, P_7)$ according to P_1 and P_7 respectively. Next, since $\bar{\varphi}$ is strategy-proof, we know that $\bar{\varphi}(P_1, P_7)$ stochastically dominates $\bar{\varphi}(P'_i, P_7) = \varphi(P'_i, P_7)$ and $\bar{\varphi}(P_1, P'_j) = \varphi(P_1, P'_j)$ according to P_1 and P_7 respectively. Therefore, $\varphi(P_1, P_7)$ stochastically dominates $\varphi(P'_i, P_7)$ and $\varphi(P_1, P'_j)$ according to P_1 and P_7 respectively.

In situation 2, we have

$$\begin{aligned}
\varphi(P_i, P_7) &\xrightarrow[1/4]{(0,1)P_i(1,0)} \varphi(P_1, P_7), && \text{if } P_i = P_2.\\
\varphi(P_i, P_7) \xrightarrow[1/4]{(0,0)P_i(0,1)}, \xrightarrow[1/4]{(1,0)P_i(0,1)}, &\xrightarrow[1/4]{(1,0)P_i(1,1)} \varphi(P_1, P_7), && \text{if } P_i = P_5.\\
\varphi(P_i, P_7) \xrightarrow[1/2]{(1,0)P_i(0,1)}, &\xrightarrow[1/4]{(0,0)P_i(1,1)} \varphi(P_1, P_7), && \text{if } P_i = P_6.\\
\varphi(P_i, P_7) \xrightarrow[1/2]{(1,1)P_i(1,0)}, &\xrightarrow[1/4]{(1,1)P_i(0,0)} \varphi(P_1, P_7), && \text{if } P_i \in \{P_7, P_8\}.
\end{aligned}$$

Therefore, $\varphi(P_i, P_7)$ stochastically dominates $\varphi(P_1, P_7)$ according to $P_i \in \{P_2, P_5, P_6, P_7, P_8\}$ respectively. By a symmetric argument, in situation 3, $\varphi(P_1, P_j)$ stochastically dominates $\varphi(P_1, P_7)$ according to $P_j \in \{P_1, P_2, P_5, P_6, P_8\}$ respectively. Therefore, φ is strategy-proof.

Lastly, assume $\mathbb{D} = \{P_1, P_2, P_7, P_8\}$. Since $\mathbb{D} \subseteq \{P_1, P_2, P_5, P_6, P_7, P_8\}$ and $P_1, P_7, P_8 \in \mathbb{D}$, φ specified above is also unanimous, strategy-proof and violates the tops-only property under domain $\mathbb{D}$. This contradicts the hypothesis of the proposition. Therefore, domain $\mathbb{D}$ must satisfy the Exterior Property*.

16 Proof of Proposition 4.3.2

Lemma 16.1. *Given distinct $P_i, P'_i \in \mathbb{D}$, assume $r_1(P_i) = r_1(P'_i) = a_k$, $a_s P_i a_t$ and $a_s P'_i a_t$, where $k \neq s$. There exists a (a_s, a_t)-Is-path connecting P_i and P'_i.*

Proof. By the Interior Property, we have an AC-path $\{P_i^l\}_{l=1}^K \subseteq \mathbb{D}^a$ connecting P_i

and P'_i. If $a_s P_i^l a_t$ for all $1 \leq l \leq K$, it is evident that $\{P_i^l\}_{l=1}^K$ is a (a_s, a_t)-Is-path connecting P_i and P'_i. For the rest of the proof, we assume $a_t P_i^l a_s$ for some $1 < l < K$. Since $a_s P_i^1 a_t$ and $a_t P_i^l a_s$, it must be the case that either $s < k < t$, or $t < k < s$. We assume $s < k < t$. The verification related to $t < k < s$ is symmetric and we hence omit it. Evidently, if a_s is ranked above a_t in two consecutive preferences of the path $\{P_i^l\}_{l=1}^K$, then a_s and a_t are isolated in these two preferences by Remark 4.1.1. Furthermore, according to Claim 1 below, we can show that after removing all preferences in the path $\{P_i^l\}_{l=1}^K$ where a_t is ranked above a_s, the rest of preferences in the path construct a (a_s, a_t)-Is-path connecting P_i and P'_i.

Claim 1: Assume $a_s P_i^{l_1} a_t$, $a_s P_i^{l_2} a_t$ and $a_t P_i^l a_s$ for all $l_1 < l < l_2$. Thus, a_s and a_t are isolated in $(P_i^{l_1}, P_i^{l_2})$.

Since $P_i^{l_1} \sim^A P_i^{l_1+1}$, $a_s P_i^{l_1} a_t$ and $a_t P_i^{l_1+1} a_s$, we know $a_s P_i^{l_1} ! a_t$. Symmetrically, $a_s P_i^{l_2} ! a_t$. Given $a_q \in A$ with $a_q P_i^{l_1} a_s$, we have $a_q P_i^{l_1} a_t$ and moreover, single-peakedness implies $s < q < t$. Thus, $\{a_q \in A | a_q P_i^{l_1} a_s\} \subseteq \{a_q\}_{q=s+1}^{t-1}$. Conversely, given $s < q < t$, it is either $s < q \leq k$, or $k \geq q > t$. Correspondingly, single-peakedness implies either $a_q P_i^{l_1} a_s$ or $a_q P_i^{l_1} a_t$. Furthermore, since $a_s P_i^{l_1} ! a_t$, it is true that $a_q P_i^{l_1} a_s$. Therefore, $\{a_q \in A | a_q P_i^{l_1} a_s\} \supseteq \{a_q\}_{q=s+1}^{t-1}$. Thus, $\{a_q \in A | a_q P_i^{l_1} a_s\} = \{a_q\}_{q=s+1}^{t-1}$. Symmetrically, $\{a_q \in A | a_q P_i^{l_2} a_s\} = \{a_q\}_{q=s+1}^{t-1}$. Now, since $\{a_q \in A | a_q P_i^{l_1} a_s\} = \{a_q \in A | a_q P_i^{l_2} a_s\}$, $a_s P_i^{l_1} ! a_t$ and $a_s P_i^{l_2} ! a_t$ imply that a_s and a_t are isolated in $(P_i^{l_1}, P_i^{l_2})$. This completes the verification of the claim and hence the lemma. $\square$

Lemma 16.2. *Domain $\mathbb{D}$ satisfies the Exterior Property*.*

Proof. Given $P_i, P'_i \in \mathbb{D}$ satisfying the following two conditions:

(i) there exists $\bar{P}_i \in \mathbb{D}$ such that $r_1(P_i) = r_1(\bar{P}_i) \equiv a_k$; $P_i \sim^A \bar{P}_i$, $a_s P_i ! a_t$ and $a_t \bar{P}_i ! a_s$;

(ii) $r_1(P'_i) \neq a_k$ and $a_s P'_i a_t$.

we will show that there exists a (a_s, a_t)-Is-path connecting P_i and P'_i.

Since $r_1(P_i) = r_1(\bar{P}_i) = a_k$, $a_s P_i ! a_t$ and $a_t \bar{P}_i ! a_s$, it is true that either $s < k < t$, or $t < k < s$. We assume $s < k < t$. The verification related to $t < k < s$ is symmetric and hence we omit it. We assume $r_1(P'_i) = a_q$. Since $\mathbb{D}^{a_k} \neq \emptyset$ and $\mathbb{D}^{a_q} \neq \emptyset$, let $\bar{P}_i \in \mathbb{D}^{a_k}$ and $\hat{P}_i \in \mathbb{D}^{a_q}$ be two preferences satisfying the left-extreme condition. Accordingly, $a_s \bar{P}_i a_t$.

We first assume $q > k$. Since $a_s P'_i a_t$, it must be the case $k < q < t$. Hence, $a_s \hat{P}_i a_t$ by left-extreme condition. Moreover, since $B^q(\bar{P}_i) = \{a_1, \ldots, a_s, \ldots, a_q\} = B^q(\hat{P}_i)$, a_s and a_t are isolated in $(\bar{P}_i, \hat{P}_i)$. Next, assume $q < k$. If $q \leq s$, single-peakedness implies $a_s \hat{P}_i a_t$. If $s < q$, left-extreme condition implies $a_s \hat{P}_i a_t$. Now, since $B^k(\bar{P}_i) = \{a_1, \ldots, a_s, \ldots, a_k\} = B^k(\hat{P}_i)$, a_s and a_t are isolated in $(\bar{P}_i, \hat{P}_i)$.

Lastly, by Lemma 16.1, we have a (a_s, a_t)-Is-path $\{\bar{P}_i^l\}_{l=1}^{l_1}$ connecting P_i and $\bar{P}_i$, and a (a_s, a_t)-Is-path $\{\hat{P}_i^l\}_{l=1}^{l_2}$ connecting $\hat{P}_i$ and P'_i. Combining these two paths, the sequence $\{\bar{P}_i^1, \ldots, \bar{P}_i^{l_1}; \hat{P}_i^1, \ldots, \hat{P}_i^{l_2}\}$ is a (a_s, a_t)-Is-path connecting P_i and P'_i. $\square$