DISSERTATION

K-SIMPLEX VOLUME OPTIMIZING PROJECTION ALGORITHMS FOR HIGH-DIMENSIONAL DATA SETS

Submitted by

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ABSTRACT

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Many applications produce data sets that contain hundreds or thousands of features, and consequently sit in very high dimensional space. It is desirable for purposes of analysis to reduce the dimension in a way that preserves certain important properties. Previous work has established conditions necessary for projecting data into lower dimensions while preserving pairwise distances up to some tolerance threshold, and algorithms have been developed to do so optimally. However, although similar criteria for projecting data into lower dimensions while preserving k-simplex volumes has been established, there are currently no algorithms seeking to optimally preserve such embedded volumes. In this work, two new algorithms are developed and tested: one which seeks to optimize the smallest projected k-simplex volume, and another which optimizes the average projected k-simplex volume.

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DEDICATION

For Husband. Thank you for everything.

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Chapter 1

Introduction

1.1 Motivation

Many applications produce data sets that contain hundreds or thousands of features, and consequently sit in very high dimensional space. Very often, the dimension of the space exceeds the number of data points. This is seen in the gene expression data generated by [1], which contains a total of 2,886 microarray samples, each with information on 12,023 different genes. Newer gene expression analyses involve RNA sequencing data, which has thousands of features associated with each sample corresponding to the genes of an organism. Another example of high dimensional data is the AVIRIS Indian Pines hyperspectral data set [2]. When the data sits sparsely in the space it occupies, it can be difficult to analyze. Machine learning methods require a large number of data points to build a reliable model. However, collecting the number of samples needed for such analysis is typically infeasible due to the cost and time required. One method to deal with this problem is to reduce the dimension of the ambient space prior to analysis.

It is desirable to reduce the dimension in a way that preserves important properties of the data. For example, principal component analysis preserves variance within the data set, and many spectral embedding methods focus on preserving the relative positioning of data points. A theorem by Johnson and Lindenstrauss [3] proved the existence of projections into lower dimensions that preserve pairwise distances up to some tolerance threshold. This has been expanded on by many authors, and multiple algorithms have been developed to find the optimal way to project data and preserve pairwise distances. More recently, this theorem was extended to find tolerances on projecting volumes enclosed by the convex hull of $k + 1$ sets of points, or k-simplices, in [4–6]. However, there are currently no algorithms that attempt to find optimal k-volume preserving embeddings.

In this work, two algorithms for finding the optimal k-simplex volume preserving projection for a data set are developed and tested.

1.2 Overview

Chapter 2 provides an overview of previous work in the field of dimensionality reduction. It also presents background information integral to the understanding of the algorithms developed in this work. Chapter 3 narrows the focus to the two specific optimization problems used for the algorithms being developed, and Chapter 4 provides in-depth coverage of mathematical concepts utilized by each algorithm.

The remainder of the work is dedicated to developing and testing both algorithms. Chapter 5 outlines other algorithms from the literature that will be used for validation and comparison with the new algorithms. Chapter 6 focuses on the development of the algorithm that optimizes the smallest embedded volume and includes extensive testing with various parameters. The algorithm that optimizes average volume is developed in Chapter 7. Chapter 8 compares both new algorithms with previously developed algorithms in terms of both optimal embeddings and computational complexity. Chapter 9 focuses on a specific application of the algorithm from Chapter 6 in which it is utilized as a supervised method for embedding labeled data.

To aid the reader's understanding, Appendix A provides a reference for the notation conventions used throughout this work. Additionally, because of the number of tests performed, the majority of the results generated are not included in the main body of this dissertation, and can instead be found in Appendix B.

Chapter 2

Review of Literature

2.1 Metric Embedding Theory

An embedding is, in the most general sense, an injective, structure preserving map of one object into another. Thus, an embedding is a one-to-one map that can be viewed as transporting an object, together with its underlying structure, into a new setting. High dimensional data projected into a lower dimensional space is a specific case of this, where the projection acts as the mapping that embeds the data into the lower dimensional setting.

The problem of finding volume preserving embeddings for dimensionality reduction is related to the more general problem of embedding metric spaces. A metric space is a set X and an associated metric d_X that gives some notion of distance between points in X. Metric spaces are denoted by (X, d_X). The goal of a metric space embedding is to take a metric space (X, d_X) and embed it into another metric space (Y, d_Y) in a manner that pairwise distances are preserved. In other words, we would like an injective map $F : X \to Y$ satisfying $d_Y(F(x_1), F(x_2)) = d_X(x_1, x_2)$ for every pair of points $x_1, x_2 \in X$, see [7–9]. In this dissertation, we will soften the requirement of strict preservation of distances. Instead, the quality of the map between metric spaces will be measured by various criteria expressing how much the relative distances between points are changed after the mapping. We will typically require the map on data to be injective but will optimize, subject to this constraint, the preservation of pairwise distance relationships.

The primary method for evaluating the quality of an embedding is through the use of a distortion measure that relates an embedded configuration of points to its original configuration. The precise formulation of the distortion measure varies based on the geometric properties we are looking to conserve. Let (X, d_X) and (Y, d_Y) be metric spaces, let f be an embedding $f : X \to Y$, and define points $u, v \in X$. One possible distortion measure $dist_f$ given in [8] uses the maximum pairwise distortion to evaluate an embedding f. Though the metric in this paper is not directly related to the problems considered in this work, it can be rewritten for the relevant case of contractive embeddings, which require $d_Y(f(u), f(v)) \leq d_X(u, v)$ for all $u, v \in X$, as

$$dist_f = \inf_{u \neq v \in X} \frac{d_Y(f(u), f(v))}{d_X(u, v)}. \tag{2.1}$$

Here, an optimal embedding f will minimize the worst distortion by maximizing $dist_f \in (0, 1]$. An equivalent formulation is

$$dist_f = \sup_{u \neq v \in X} \frac{d_X(u, v)}{d_Y(f(u), f(v))} \tag{2.2}$$

which optimized by instead minimizing $dist_f$. Another example of a distortion measure in seen in [4], which seeks to minimize

$$dist_f = \sup_{u,v \in X} \frac{d_Y(f(u), f(v))}{d_X(u, v)} \times \sup_{u,v \in X} \frac{d_X(u, v)}{d_Y(f(u), f(v))}. \tag{2.3}$$

Both of these measures focus on improving the worst pairwise distortion between points of an embedding as much as possible. Alternatively, one can consider optimizing the average pairwise distortion between points of an embedding. Abraham et al. give this general notion of average distortion for a set of k data points in [8]:

$$dist_{avg} = \frac{1}{\binom{k}{2}} \sum_{u \neq v \in X} \text{dist}_f(u, v) \tag{2.4}$$

Average distortion is more commonly used in practical applications, though it is often still desirable to have bounds on the worst case distortion [8].

This work focuses on a specific subset of problems in metric embedding related to reducing the dimension of data. In practical applications, data often sits in a high-dimensional ambient space. The goal of dimensionality reduction is to find some embedding $f : X \to Y$ with $dim(Y) \ll dim(X)$ that preserves relationships between points in the data set as much as possible. In particular, the focus will be on the case where both X and Y are Euclidean spaces and the embedding map is an orthogonal projection map P. This linear mapping is guaranteed to be a contraction mapping, so the relevant distortion measures are found in Equations (2.1) and (2.4). The following sections cover important properties of such embeddings and relevant applications from the existing literature.

2.2 Johnson-Lindenstrauss Lemma

When projecting data into a lower dimension, it is helpful to to have some notion of the expected range of distortions in the projection. The Johnson-Lindenstrauss (JL) lemma [3] provides bounds on the distortion of pairwise distances when embedding a set of points in a high dimensional space into a lower dimensional space. Let X be a set of k points in $\mathbb{R}^n$, u and v be arbitrary points in X, and let m be an integer greater than k. Then for $\epsilon > 0$, there exists a function $f : \mathbb{R}^n \to \mathbb{R}^m$ such that

$$(1-\epsilon)||u-v||^2 \leq ||f(u)-f(v)||^2 \leq (1+\epsilon)||u-v||^2 \tag{2.5}$$

as long as $m \geq \mathcal{O}(\epsilon^{-2}\log(k))$. In other words, there exists an embedding map of the points in $\mathbb{R}^n$ into $\mathbb{R}^m$ which preserves the pairwise distances to within a tolerance of ϵ as long as m is not too small. In particular, when f is a Gaussian random projection onto an m-dimensional subspace, these bounds are achieved with high probability. Dasgupta and Gupta provide an alternate proof of the Johnson-Lindenstrauss result in [10].

The JL lemma has seen wide use in developing methods to reduce the ambient dimension of data. It is applied in [11] to achieve dimensionality reduction for the purpose of computational and memory efficiency in databases. Linial et al. apply it to graph embedding and approximating nearest neighbors in [12]. An application to finding sphericity of graphs is found in [13]. Ailon and Chazelle formulate a fast version of the JL transform for finding nearest neighbors in [14]. Engebretsen et al. construct a deterministic mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ that achieves JL tolerance much faster than random projections [15]. The JL lemma has also been extended from a finite set of points to embeddings of surfaces and manifolds. Baraniuk and Wakin apply random projections to embed manifolds in Euclidean space [16], and their results are built on in [17] to provide JL-type bounds for these projections. Broomhead and Kirby proposed an optimization strategy for computing bilipschitz mappings by maximizing the minimum projected secant norm [18]. More recent work has built upon the JL lemma to preserve more properties than just pairwise distortion, as discussed in the next section.

2.3 Volume Preserving Embeddings

Volume preserving embeddings extend the idea behind the Johnson-Lindenstrauss lemma to higher dimensions, i.e., bounding the distortion of volumes enclosed by sets of 3 or more points instead of pairwise distances. For instance, suppose we start with a metric space (X, d_x) embedded in $\mathbb{R}^n$. For a given value of k, we would like to find an injective map of X into $\mathbb{R}^m$ such that for any $k+1$ element subset of points $A \subset X$, the volume of the convex hull of A is as close as possible to the volume of the convex hull of the image of A in $\mathbb{R}^m$. Although this work will focus on volume preserving embeddings under orthogonal projection, much of the current work in this area was done considering general metric spaces.

In [19], Feige addresses the idea of volume respecting embeddings for finite metric spaces, including those where S is a finite set of points in $\mathbb{R}^n$. However, Feige addresses the question using a more generalized definition of volume for arbitrary metric spaces that is not equivalent to the standard Euclidean volume. In what follows, if A is a $k+1$ element subset of a Euclidean space, then $vol(A)$ will denote the standard k-dimensional Euclidean volume of the space enclosed by the convex hull of the points in A. Now consider a finite metric space (S, d_S) with S consisting of $k+1$ points. Let ϕ be a contraction mapping $\phi : S \to \mathbb{R}^k$. We then have for any $u, v \in S$ that $d_{\mathbb{R}^k}(\phi(u), \phi(v)) \leq d_S(u, v)$. If vol represents the standard Euclidean definition of volume, Feige's volume for S is then defined by

$$vol_F(S) = \max_{\phi} vol(\phi(S)) \tag{2.6}$$

where ϕ is taken from the set of all possible contraction mappings from S to $\mathbb{R}^k$. Note that this is not equivalent to Euclidean volume, since a set of $k+1$ points with zero k-volume can be mapped by ϕ to a configuration with nonzero k-volume. For example, three co-linear points in $\mathbb{R}^2$ define an area (or 2-volume) of size 0, but can easily be shifted to define a triangle with positive area whose edges have length less than the original pairwise distance between their vertices.

Due to the inherent computational difficulty of calculating vol_F, Feige introduces a second notion of volume, called tree volume, which uses a minimum spanning tree of the points in S to derive a volume for S. Let $\{v_0, ..., v_k\}$ be the vertices of a minimum spanning tree for S, let v_0 be the root, and let δ_i be the length of the edge connecting v_i to its parent vertex. Tree volume is then given by

$$vol_T(S) = \frac{1}{k!}\prod_{i=1}^{k} \delta_i \tag{2.7}$$

Thus, the tree volume is found by taking the product of the edge lengths in a minimal spanning tree for the $k+1$ points then dividing by $k!$. Feige goes on to prove that

$$vol_F(S) \leq vol_T(S) \leq 2^{(k-1)/2} vol_F(S) \tag{2.8}$$

for any S with $k+1$ points [19]. Feige uses this property to develop algorithms for embedding sets of points while preserving vol_T as an approximation for vol_F.

Feige's work inspired many other advances in this problem. Feige's notion of tree volume is shown to be robust to ordering of the vertices in [20]. Rao derived improved bounds for tree volume respecting embeddings in [7], and Lee derived an upper bound independent of the embedding dimension [21]. In [22], Krauthgamer et al. provide a slight improvement of the Euclidean bounds discovered by Rao, and in [23] they develop a novel method for embedding metrics while preserving tree volumes.

Magen and Zouzias produced a series of works further developing the theory for projections that preserve volumes, focusing on the case where both the original and projection spaces are Euclidean. Magen initially showed that given a set of N points in $\mathbb{R}^n$ and some $d \leq N$, there exists an embedding into $O(d\epsilon^{-2}\log(n))$ dimensions such that the volume of any subset of $k \leq d$ points was distorted by no more than $(1+\epsilon)^{k-1}$ [4]. In other words, we can bound the distortion of volumes up to $d-1$ dimensions in our embedding. Magen also developed a method for preserving angles between data points by adding a constraint to preserve the distance from points to affine subspaces. This allows for angles to be preserved using the same bounds as Johnson-Lindenstrauss [4], which again bounds the volume of k points by $(1+\epsilon)^{k-1}$. Magen and Zouzias co-authored a work that improved on the requirements for achieving this bound on volume distortion for any $k \leq d$ points, showing that the projection dimension need be only $O(\max\{\frac{d}{\epsilon}, \epsilon^{-2}\log(n)\})$. They additionally calculated the probability that using a random Gaussian projection for the embedding achieves the desired bounds on volume [5]. They also raise an open question regarding whether more sophisticated methods would outperform Gaussian projections for preserving volumes. Zouzias further expands on their results in [6] with a bound on the distortion for embedding any set of N points into a fixed dimension m. Zouzias proves the existence of a linear mapping that preserves the volume of any subset of up to $\frac{m}{2}$ or fewer points within a factor of $O(N^{2/m}\sqrt{\log(N)\log(\log(N))})$ as long as $m \geq 3$.

Chapter 3

The Optimization Problem

The goal of the algorithms developed in this work is to project data into a lower dimension while optimally preserving the volumes enclosed by subsets of the data points. As seen in Section 2.1, there are many ways to define the "optimal" embedding depending on the structures we want to preserve. This chapter outlines two different distortion measures for embedded volumes, as well as the specific optimization problems that we seek to solve.

3.1 Simplices and Their Volumes

Before formally defining the problem, it is important to be familiar with the notion of k-dimensional volumes. Two distinct points in $\mathbb{R}^n$ can be connected by a line segment which has some positive length. Adding a third point that is not co-linear and connecting it to the previous two defines a triangle with positive area. Adding a fourth point that is not in the affine span of the previous points yields a tetrahedron whose convex hull defines a 3-dimensional volume (see Figure 3.1). Higher dimensional regions are enclosed by taking the convex hull of affinely independent sets of points. The simplex is the name given to such convex hulls of affinely independent points. Specifically, a k-dimensional simplex, or k-simplex, in $\mathbb{R}^n$ is the convex hull of $k+1$ affinely independent points. The points determining the simplex are called the vertices [24]. Note that $k+1$ must be less than or equal to n, or the simplex will be degenerate and have zero k-dimensional volume. Suppose $\{x_0, ..., x_k\}$ is a set of vertices. The simplex determined by these vertices is

$$S = \{\sum_{i=0}^{k} a_i x_i \mid a_i \geq 0 \text{ and } \sum_{i=0}^{k} a_i = 1\} \tag{3.1}$$

The volume of a simplex describes the size of the space enclosed by the convex hull of the vertices. In the case $k = 1$, this is simply the length of a line segment. For $k = 2$, the "volume" of the simplex is the area of the triangle enclosed by the three vertices. When $k = 3$ we have the

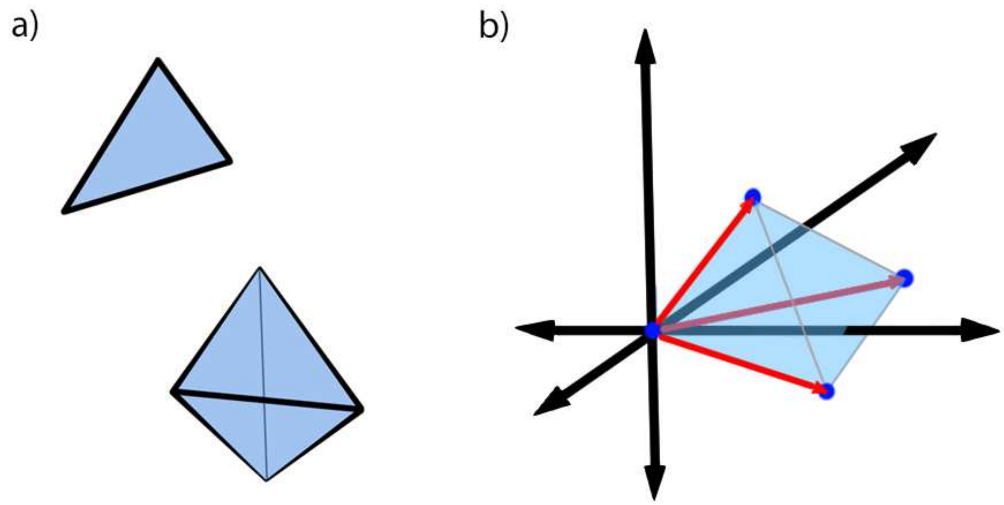


Figure 3.1: Left: A 2-simplex (triangle) and a 3-simplex (tetrahedron). Right: A simplex shifted to the origin.

familiar notion of three-dimensional volume. In general, the k-dimensional space enclosed by a k-simplex is referred to as having k-dimensional volume.

The volume of a simplex can be calculated using matrix determinants. Given $k+1$ points in $\mathbb{R}^k$, Stein proves the determinant definition of the k volume enclosed by the $k+1$ points in [25]. If $x_0, x_1, ..., x_k$ are the vertices of the simplex S, define

$$\hat{\mathbf{X}} = \begin{bmatrix} x_0 & x_1 & ... & x_k \\ 1 & 1 & ... & 1 \end{bmatrix}. \tag{3.2}$$

Then the volume of S is given by

$$vol(S) = \frac{1}{k!} |\det(\hat{\mathbf{X}})| \tag{3.3}$$

Alternatively, S can be translated so one vertex lies on the origin as in Figure 3.1 without changing the configuration of the points. Let $\mathbf{X} = [x_1 - x_0, ..., x_k - x_0]$ be the matrix of the remaining non-zero vertices after translation. The volume of S is then given by

$$vol(S) = \frac{1}{k!}|\det(\mathbf{X})|. \tag{3.4}$$

If we have $k+1$ vertices in $\mathbb{R}^n$ with $n > k+1$ then we can again form $X = [x_1 - x_0, ..., x_k - x_0]$ as the matrix of the non-zero vertices after translation. In this case the volume of S is given by

$$vol(S) = \frac{1}{k!}\sqrt{\det(\mathbf{X}^T\mathbf{X})}. \tag{3.5}$$

This version of the volume function will be used in all computations performed in this work.

3.2 Problem Formulation

Let $n > m \geq k \geq 1$, where $n, m,$ and k are all integers. Our goal is to project data from $\mathbb{R}^n$ into an m-dimensional subspaces in a way that optimally preserves the volumes of all k-dimensional simplices within the data. This section focuses on setting up the problem, explores the reasoning behind the algorithm, and discusses potential pitfalls.

For the sake of this algorithm, projection mappings are required to be orthogonal. Let $P : \mathbb{R}^n \to \mathbb{R}^m$ be an orthogonal mapping of the data. Because P is an orthogonal mapping, P is determined by an orthonormal matrix $\mathbf{P}$, and for any two points $x, y \in \mathbb{R}^n$ we have $|x - y| \geq |P(x) - P(y)|$. Furthermore, by extension, for any set S of $k+1$ points we have $vol(S) \geq vol(P(S))$ (where $vol(S)$ means the volume of the convex hull of S). Because of this, our goal is to find the mapping P that preserves, as much as is possible, the volumes of the k-simplices determined by the data.

Let $S \in \mathbb{R}^n$ be a simplex, $\mathbf{S}$ be a matrix with the vertices of S as its columns, and vol be the volume function. We want to find an orthonormal matrix $\mathbf{P}^*$ that maximizes the volume of the projected simplex.

$$\mathbf{P}^* = \arg\underset{\mathbf{P}}{\text{maximize}}\, vol(\mathbf{PS}) \tag{3.6}$$

where $\mathbf{PS} \in \mathbb{R}^m$ is the projected simplex. For a single simplex, this is fairly straightforward; however, when considering multiple simplices the problem becomes more complex.

Given a set of M k-simplices $S_i \in \mathbb{R}^n$, $i \in \{1, .., M\}$, the goal is to find the orthogonal matrix $\mathbf{P}$ that best preserves the volumes of all the simplices after projection into $\mathbb{R}^m$. In order to prevent the largest simplices from carrying the most weight in the optimization problem, all simplex volumes will be treated as unit volumes. This is accomplished by dividing projected volume by initial volume to yield a normalized projected volume. The optimal normalized embedded volume for a single simplex is then given by the objective function

$$\mathbf{P}^* = \arg\underset{\mathbf{P}}{\text{maximize}} \frac{vol(\mathbf{PS})}{vol(\mathbf{S})}. \tag{3.7}$$

When optimizing over multiple simplices, there are multiple ways to define the "best" projection $\mathbf{P}$. One method is to look at the simplex with the worst post-projection distortion, i.e., the smallest ratio $\frac{vol(\mathbf{PS}_i)}{vol(\mathbf{S}_i)}$ over all i, and adjust the projection map $\mathbf{P}$ to improve it as much as possible. This optimization problem can be formulated as

$$\mathbf{P}^* = \arg\underset{\mathbf{P}}{\text{maximize}} \min_i \frac{vol(\mathbf{PS}_i)}{vol(\mathbf{S}_i)} \tag{3.8}$$

where $vol(\mathbf{S}_i)$ is the simplex volume before projection and $vol(\mathbf{PS}_i)$ is the volume after projection. This optimization algorithm for this function is developed and tested in Chapter 6.

One of the major pitfalls of this optimization problem is that the objective function is not necessarily continuous, and is decidedly not convex. Consider the normalized volumes of 2-simplices in a fixed set of randomly generated data in $\mathbb{R}^3$ projected onto a plane. Figure 3.2 shows the evaluation of

$$vol_{\min} = \min_i \frac{vol(\mathbf{PS}_i)}{vol(\mathbf{S}_i)} \tag{3.9}$$

as the data is projected onto a plane in $\mathbb{R}^3$ being rotated $180°$ about the x-axis one degree at a time. Steep drops and jumps correspond to points where the simplex with the smallest normalized projection volume changed. Methods used to address this issue are discussed in Section 6.3.1.

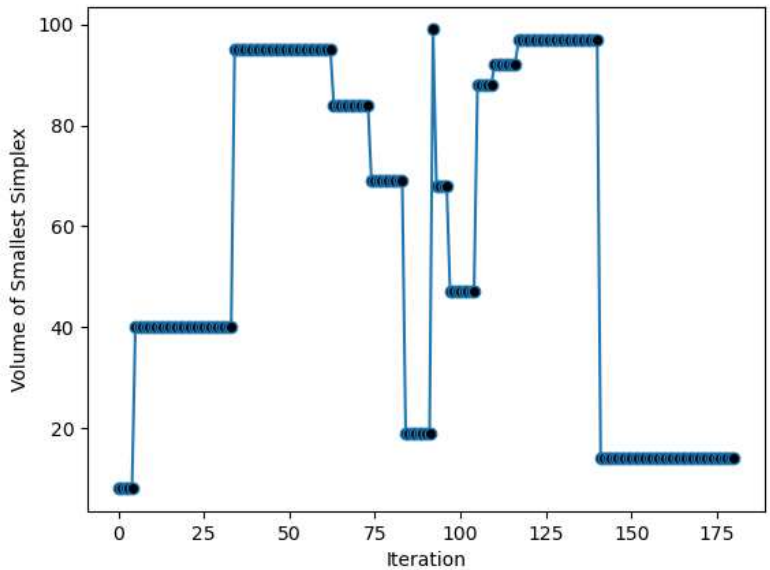


Figure 3.2: Change in smallest simplex volume as the projection plane is rotated $180°$ about the x-axis in one degree increments.

A second method is to find an orthonormal matrix $\mathbf{P}$ that maximizes the average of the normalized embedded volumes. This can be accomplished by solving

$$\mathbf{P}^* = \arg \underset{\mathbf{P}}{\text{maximize}} \sum_{i=1}^{k} \frac{vol(\mathbf{PS}_i)}{vol(\mathbf{S}_i)}. \tag{3.10}$$

Unlike the previous objective function, this function is convex and can be solved using gradient descent. However, a more elegant solution utilizing the flag mean of subspaces is developed in Chapter 7.

Chapter 4

Mathematical Background

This chapter provides an in-depth discussion of mathematical concepts used in the development of the algorithms in this paper. The algorithm in Chapter 6 primarily utilizes concepts in Section 4.1, while the second algorithm developed in Chapter 7 relies heavily on concepts in Section 4.2. Additionally, Section 4.2.2 presents a novel proof regarding the average of multiple Schubert varieties.

4.1 The Grassmann Manifold

Given positive integers, $d < n$, the Grassmann manifold, $Gr(d, n)$, is a manifold whose points parameterize the d-dimensional subspaces of $\mathbb{R}^n$. Thus, if V is a d-dimensional subspace of $\mathbb{R}^n$ then there exists a point $P_V \in Gr(d, n)$ corresponding to V. Furthermore, distinct points on $Gr(d, n)$ correspond to distinct d-dimensional subspaces of $\mathbb{R}^n$. A d-dimensional subspace of $\mathbb{R}^n$ can be represented as an $n \times d$ matrix whose columns span the subspace. Denote the column space of a matrix $\mathbf{X}$ by $[\mathbf{X}]$. If we restrict $\mathbf{X}$ to have orthonormal columns, then we can identify a point on the Grassmannian with the set of all orthonormal $n \times d$ matrices with a given column space. The collection of all $n \times d$ matrices with orthonormal columns has the structure of a manifold. This is known as the Stiefel manifold and is typically denoted $St(d, n)$. From this perspective, we can consider $Gr(d, n)$ as a quotient manifold of $St(d, n)$ determined by an equivalence relation. The equivalence relation for the quotient $Gr(d, n) = St(d, n)/\sim$ is defined by

$$\mathbf{X} \sim \mathbf{Y} \iff [\mathbf{X}] = [\mathbf{Y}] \tag{4.1}$$

where $\mathbf{X}$ and $\mathbf{Y}$ are both $n \times d$ matrices with orthonormal columns. Note that $\mathbf{X}$ and $\mathbf{Y}$ have the same column space if and only if there exists an orthogonal $d \times d$ matrix $\mathbf{A}$ such that $\mathbf{XA} = \mathbf{Y}$. The Grassmannian is thus the set of equivalence classes of the points in $St(d, n)$ under the action of the orthogonal group $O(d)$. This relationship is used to represent points on the Grassmannian using $n \times d$ matrices for the purposes of computation. To calculate the dimension of a Grassmannian, we can use the fact that the column space of an orthonormal $n \times d$ matrix is invariant under right multiplication by a $d \times d$ orthonormal matrix. This means that the column space of an $n \times d$ matrix is uniquely determined by $d(n-d)$ parameters, therefore the dimension of $Gr(d,n)$ is $d(n-d)$ [26].

A function defined on $St(d,n)$ is invariant under $\sim$ if

$$[\mathbf{X}] = [\mathbf{Y}] \implies f(\mathbf{X}) = f(\mathbf{Y}) \tag{4.2}$$

for all $\mathbf{X}, \mathbf{Y} \in St(d,n)$ [26]. A function that is invariant under $\sim$ can be viewed as a function on $Gr(d,n)$. Thus $\mathbf{X}$ can serve as a representation for $[\mathbf{X}]$ for the purposes of numerical computations on the Grassmannian. In particular, we will restrict matrix representations of points on $Gr(d,n)$ to the set of orthonormal $n \times d$ matrices.

4.1.1 Metrics

Many computations on a Grassmannian can be reduced to the computation of pairwise distances on the Grassmannian and the ability to move one point closer to another point. Distances between two points on the Grassmannian can be defined using the principal angles between the two subspaces the points represent.

Denote the principal angles between two subspaces $[\mathbf{X}]$ and $[\mathbf{Y}]$ generated by the column space of matrices $\mathbf{X}, \mathbf{Y} \in \mathbb{R}^{n \times d}$ by θ_i for $1 \le i \le d$. These angles are defined recursively. We start with:

$$\theta_1 = \min\{\arccos(x^T y) |\, x \in [\mathbf{X}],\ y \in [\mathbf{Y}], \|x\| = \|y\| = 1\}. \tag{4.3}$$

Letting $x_j \in [\mathbf{X}]$ and $y_j \in [\mathbf{Y}]$ be elements that satisfy $\theta_j = \arccos(x_j^T y_j)$ we define

$$\theta_i = \min\{\arccos(x^T y) |\, x \in [\mathbf{X}],\ y \in [\mathbf{Y}], \|x\| = \|y\| = 1,\ x^T x_j = y^T y_j = 0\ \forall\, j < i\}. \tag{4.4}$$

So θ_1 is the smallest possible angle between the two subspaces, and $\theta_i \geq \theta_j$ for all $i > j$. The corresponding unit vectors $x_i \in [\mathbf{X}]$ and $y_i \in [\mathbf{Y}]$ are called the principal vectors [27].

For two orthonormal matrices $\mathbf{X}$ and $\mathbf{Y}$, the cosines of the principal angles between $[\mathbf{X}]$ and $[\mathbf{Y}]$ are easily obtained from the singular value decomposition (SVD) of $\mathbf{X}^T\mathbf{Y}$. Given the SVD decomposition $\mathbf{X}^T\mathbf{Y} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$, the singular values σ_i are by definition the diagonal elements of $\mathbf{\Sigma}$ and $\theta_i = \arccos(\sigma_i)$. Furthermore, the columns of $\mathbf{X}$ and $\mathbf{Y}$ are the corresponding principle vectors [27]. Thus, the SVD provides a method for quickly obtaining principal angles when given orthonormal bases for two subspaces. Note that all principal angles θ_i must exist in the interval $[0, \frac{\pi}{2}]$.

The Grassmannian can be endowed with many distinct orthogonally invariant functions provided these functions are written in terms of principal angles; see [28] for details. For this work, we will use the chordal distance metric. This metric is is orthogonally invariant, is based on the projection Frobenius norm, and can be expressed in terms of the sines of principal angles. If $[\mathbf{X}], [\mathbf{Y}] \in Gr(d, n)$ then

$$d_c([\mathbf{X}], [\mathbf{Y}]) = \left(\sum_{i=1}^{d} (\sin\theta_i)^2 \right)^{1/2} = \|\sin\theta\|_c. \tag{4.5}$$

Using chordal distances, the Grassmannian can be embedded into Euclidean space using multi-dimensional scaling (MDS) [28].

4.1.2 Geodesics

The shortest path distance between two points on a manifold is known as a geodesic curve. We would like to move between two points $[\mathbf{X}]$ and $[\mathbf{Y}]$ at constant speed along a geodesic curve on a Grassmann manifold $Gr(d, n)$. This can be normalized so as to be accomplished with a function $G(t)$, $t \in [0, 1]$ such that $G(0) = [\mathbf{X}]$ and $G(1) = [\mathbf{Y}]$. The explicit form of $G(t)$ is accomplished by using the quotient geometry of the Grassmannian to derive a formula for this geodesic curve [28].

Consider the set of all $n \times n$ orthonormal matrices, denoted by $O(n)$. $O(n)$ is a manifold in Euclidean space. In what follows, we describe the tangent space to $O(n)$ at the identity matrix $\mathbf{I}$. Each tangent vector to $O(n)$ at $\mathbf{I}$ arises from differentiating (with respect to t) some smooth parametrized curve $\mathbf{X}(t)$, defined for $t \in (-\epsilon, \epsilon)$, with $\mathbf{X}(0) = \mathbf{I}$. Since $\mathbf{X}(t) \in O(n)$, $\mathbf{X}(t)^T\mathbf{X}(t) = \mathbf{I}$ for all t. Differentiating with respect to t yields

$$\dot{\mathbf{X}}(t)^T\mathbf{X}(t) + \mathbf{X}(t)^T\dot{\mathbf{X}}(t) = 0. \tag{4.6}$$

Evaluating at $t = 0$ and noting that $\mathbf{X}(0) = \mathbf{I}$, we get

$$\dot{\mathbf{X}}(0)^T + \dot{\mathbf{X}}(0) = 0. \tag{4.7}$$

This means that for any smooth curve $\mathbf{X}(t)$ that passes through $\mathbf{I}$ at $t = 0$, $\dot{\mathbf{X}}(0)$ is a skew symmetric matrix [29]. Therefore the tangent space to $O(n)$ at $\mathbf{I}$ is given by

$$T_{\mathbf{I}}\, O(n) = \{\mathbf{A} | \mathbf{A} = -\mathbf{A}^T\}. \tag{4.8}$$

Each element $\mathbf{A} \in T_{\mathbf{I}}\, O(n)$ can be used to construct a geodesic, $\mathbf{G}(t) = e^{\mathbf{A}t}$, that passes through $\mathbf{I}$ with $\mathbf{A}$ as its tangent vector at $t = 0$. Now consider an element $\mathbf{Q} \in O(n)$. We can use $\mathbf{Q}$ to shift this geodesic so that it passes through $\mathbf{Q}$ at time $t = 0$ as $\mathbf{H}(t) = \mathbf{Q}e^{\mathbf{A}t}$ [28].

To find the geodesic formula on $Gr(d, n)$, we consider the Grassmannian as a quotient of $O(n)$ based on the equivalence relation

$$[\tilde{\mathbf{Q}}] = \left\{ \tilde{\mathbf{Q}} \begin{bmatrix} \mathbf{Q}_d & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{(n-d)} \end{bmatrix}, \quad \mathbf{Q}_d \in O(d), \mathbf{Q}_{(n-d)} \in O(n-d) \right\}. \tag{4.9}$$

Thus, two points on $O(n)$ are considered to be equivalent if they differ by multiplication by a block diagonal orthogonal matrix of the form above. Let

$$\mathbf{C} = \begin{bmatrix} \mathbf{I}_d \\ \mathbf{0} \end{bmatrix}.$$

where $\mathbf{I}_d$ is a $d \times d$ identity matrix and $\mathbf{0}$ is an $(n-d) \times d$ matrix of zeros. For any element $\mathbf{X}$ in $[\tilde{\mathbf{Q}}]$, the column space of the first d columns of $\mathbf{X}$ is the same. In other words, the column space of $\mathbf{XC}$ is constant for any $\mathbf{X}$ in the equivalence class $[\tilde{\mathbf{Q}}]$.

From the relationship $O(n)/\sim \cong Gr(d,n)$, the Grassmannian has a quotient geometry inherited from $O(n)$. In particular, the tangent space at a point $\mathbf{Q} \in O(n)$ can be decomposed into orthogonal horizontal and vertical spaces $H_{\mathbf{Q}}$ and $V_{\mathbf{Q}}$, respectively, with $H_{\mathbf{Q}} = T_{[\mathbf{Q}]}Gr(d,n)$ [28]. The formula for the Grassmannian geodesic will therefore be developed on $O(n)/\sim$ and subsequently mapped to $Gr(d,n)$.

To derive a formula for the geodesic on $H_{\mathbf{Q}}$, begin by defining $V_{\mathbf{Q}}$. Let

$$\mathbf{V}(t) = \tilde{\mathbf{Q}} \begin{bmatrix} \mathbf{Q}_d(t) & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{(n-d)}(t) \end{bmatrix} \tag{4.10}$$

be a curve in $O(n)$ along $[\tilde{\mathbf{Q}}]$ with $\mathbf{V}(0) = \tilde{\mathbf{Q}}$. As before, we must have $\mathbf{V}(t)^T\mathbf{V}(t) = \mathbf{I}$ for all t [29]. Taking the derivative with respect to t and plugging in the initial conditions yields

$$\dot{\mathbf{Q}}_d^T \mathbf{Q}_d + \mathbf{Q}_d^T \dot{\mathbf{Q}}_d = 0$$

and

$$\dot{\mathbf{Q}}_{(n-d)}^T \mathbf{Q}_{(n-d)} + \mathbf{Q}_{(n-d)}^T \dot{\mathbf{Q}}_{(n-d)} = 0.$$

The same reasoning as before yields

$$V_{\mathbf{Q}} = \left\{ \tilde{\mathbf{Q}} \begin{bmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \right\}$$

where $\mathbf{C}$ is $d \times d$ skew symmetric and $\mathbf{D}$ is $(n-d) \times (n-d)$ skew symmetric [**?**]. Since $H_{\mathbf{Q}}$ must be orthogonal to $V_{\mathbf{Q}}$ with all elements of $H_{\mathbf{Q}}$ skew symmetric, it follows that

$$H_{\mathbf{Q}} = \left\{ \tilde{\mathbf{Q}}\tilde{\mathbf{A}} \right\}$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{0} & -\mathbf{B} \\ \mathbf{B} & \mathbf{0} \end{bmatrix}$$

The geodesic on $H_{\mathbf{Q}}$ is then

$$\tilde{\mathbf{Q}}(t) = \tilde{\mathbf{Q}} e^{\tilde{\mathbf{A}}t},$$

and therefore the matrix representation for a geodesic on $Gr(d, n)$ through $[\mathbf{Q}]$ is

$$\Phi(t) = \tilde{\mathbf{Q}} e^{\tilde{\mathbf{A}}t} \begin{bmatrix} \mathbf{I}_d \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{0}$ is an $(n-d) \times d$ matrix of zeroes.

From Theorem 1 in [29], given $\Phi(0) = \mathbf{X}$ and $\dot{\Phi}(0) = \mathbf{H}$, write the geodesic curve using the compact SVD of the velocity matrix $\mathbf{H} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$ as

$$\Phi(t) = \mathbf{X}\mathbf{V}\cos(\boldsymbol{\Sigma}t)\mathbf{V}^T + \mathbf{U}\sin(\boldsymbol{\Sigma}t)\mathbf{V}^T. \tag{4.11}$$

This provides a formula for a geodesic through $[\mathbf{X}] \in Gr(d, n)$ given a known velocity matrix $\mathbf{H}$; however, a method to find $\mathbf{H}$ given two points $[\mathbf{X}]$ and $[\mathbf{Y}]$ in $Gr(d, n)$ such that $\Phi(0) = \mathbf{X}$ and $\Phi(1) = \mathbf{Y}\mathbf{D} \in [\mathbf{Y}]$ ($\mathbf{D}$ being any $d \times d$ orthogonal matrix) is still needed. This computation is found in [30], and it is reiterated here for the purpose of thoroughness.

At $t = 1$, the above requires that

$$\mathbf{Y}\mathbf{D} = \mathbf{X}\mathbf{V}\cos(\boldsymbol{\Sigma}t)\mathbf{V}^T + \mathbf{U}\sin(\boldsymbol{\Sigma}t)\mathbf{V}^T. \tag{4.12}$$

Left multiplication by $\mathbf{X}^T$ yields

$$\mathbf{X}^T\mathbf{Y}\mathbf{D} = \mathbf{V}\cos(\mathbf{\Sigma})\mathbf{V}^T \tag{4.13}$$

since $\mathbf{X}^T\mathbf{U} = 0$. Substituting $\mathbf{X}^T\mathbf{Y}\mathbf{D}$ back into Equation (4.12) yields

$$\mathbf{Y}\mathbf{D} = \mathbf{X}\mathbf{X}^T\mathbf{Y}\mathbf{D} + \mathbf{U}\sin(\mathbf{\Sigma})\mathbf{V}^T, \tag{4.14}$$

and combining Equations (4.13) and (4.14) yields

$$\mathbf{U}\sin(\mathbf{\Sigma})\mathbf{V}^T(\mathbf{V}\cos(\mathbf{\Sigma})\mathbf{V}^T)^{-1} = \mathbf{U}\tan(\mathbf{\Sigma})\mathbf{V}^T = (\mathbf{I} - \mathbf{X}\mathbf{X}^T)\mathbf{Y}(\mathbf{X}^T\mathbf{Y})^{-1}. \tag{4.15}$$

Hence $\mathbf{U}\mathbf{\Theta}\mathbf{V}^T$ is the SVD of $\mathbf{H}$, with $\mathbf{\Theta} = \arctan(\mathbf{\Sigma})$ and

$$\mathbf{H} = (\mathbf{I} - \mathbf{X}\mathbf{X}^T)\mathbf{Y}(\mathbf{X}^T\mathbf{Y})^{-1}. \tag{4.16}$$

Using this formula for $\mathbf{H}$ and the SVD, Equation (4.11) yields

$$\Phi(t) = \mathbf{X}\mathbf{V}\cos(\mathbf{\Theta}t) + \mathbf{U}\sin(\mathbf{\Theta}t). \tag{4.17}$$

Together, Equations (4.16) and (4.17) parameterize a curve between any two points $[\mathbf{X}]$ and $[\mathbf{Y}]$ in $Gr(d, n)$.

4.1.3 Averaging Subspaces

The flag mean is an algorithm for computing averages of points on Grassmann manifolds [31–33]. The flag mean can be used to identify common attributes of a set of points on the Grassmannian through a set of nested subspaces, called a flag [31].

A flag is a defined as nested sequence of subspaces. The flag mean algorithm computes the best flag representation of a given finite collection of subspaces with respect to a particular criterion.

Let $\{[\mathbf{X}_i]\}_{i=1}^m$ be a set of points in $Gr(d,n)$ with $\{\mathbf{X}_i\}_{i=1}^m$ corresponding matrix representations. Denote the flag mean of the $[\mathbf{X}_i]$ by $\{[u_1],[u_2],...,[u_r]\}$, where $r \leq d$ and the u_i are orthogonal unit vectors. To construct the flag mean, iteratively solve the following optimization problem:

$$[u_j] = \operatorname*{arg\,min}_{[u]\in Gr(1,n)} \sum_{i=1}^m d_c([u],[\mathbf{X}_i])^2, \quad \text{subject to } [u] \perp [u_l] \text{ for all } l < j \tag{4.18}$$

for $[u_1],...,[u_r]$ [33]. From Equation (4.5),

$$\operatorname*{arg\,min}_{[u]} \sum_{i=1}^m d_c([u],[\mathbf{X}_i])^2 = \operatorname*{arg\,min}_{[u]} \sum_{i=1}^m (\sin(\theta_i))^2, \tag{4.19}$$

and the optimization problem is then equivalent to

$$[u_j] = \operatorname*{arg\,max}_{[u]\in Gr(1,n)} \sum_{i=1}^m (\cos(\theta_i))^2 \tag{4.20}$$

with the same constraints on the $[u_i]$ [33]. Consider the thin SVD of $u^T\mathbf{X}_i$. Using the SVD,

$$u^T\mathbf{X}_i\mathbf{X}_i^Tu = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{V}\mathbf{\Sigma}^T\mathbf{U}^T = \cos^2(\theta_i). \tag{4.21}$$

Combining Equations (4.20) and (4.21) yields

$$[u_j] = \operatorname*{arg\,max}_{[u]\in Gr(1,n)} u^T\left(\sum_{i=1}^m \mathbf{X}_i\mathbf{X}_i^T\right)u, \quad \text{subject to } [u] \perp [u_l] \text{ for all } l < j. \tag{4.22}$$

For simplicity, let $\mathbf{A} = \sum_{i=1}^m \mathbf{X}_i\mathbf{X}_i^T$. To find an optimal u_j, take the Lagrangian

$$\mathcal{L}(u,\lambda,\lambda_1,...,\lambda_{j-1}) = u^T\mathbf{A}u - \lambda(u^Tu-1) - \sum_{l=1}^{j-1}\lambda_l(u^Tu_l) \tag{4.23}$$

and its partial derivatives. Setting the partials equal to zero yields the optimality conditions

$$\mathbf{A}u = \lambda u, \quad u^Tu = 1, \quad u^Tu_l = 0, \tag{4.24}$$

and the problem is reduced to an eigenvector computation [33]. Given that in most cases $d \ll n$, it is desirable to find a more efficient method than standard eigenvector computations that have complexity $\mathcal{O}(n^3)$. Define the concatenation of the matrix representations of the $\{[\mathbf{X}_i]\}$ by $\mathcal{X} = [\mathbf{X}_1, ..., \mathbf{X}_m]$. Then $\mathcal{X}$ is an $n \times (md)$ matrix, and $\mathcal{X}\mathcal{X}^T = \sum_{i=1}^m \mathbf{X}_i \mathbf{X}_i^T = \mathbf{A}$. The thin SVD of $\mathcal{X} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ then yields $\mathcal{X}\mathcal{X}^T = \mathbf{U}\mathbf{\Sigma}\mathbf{\Sigma}^T\mathbf{U}^T$, and the columns of $\mathbf{U}$ are therefore the eigenvalues of $\mathcal{X}\mathcal{X}^T$ with corresponding eigenvectors σ_i^2 [33]. Using the SVD for this calculation reduces the order to $\mathcal{O}(nm^2d^2)$ and is therefore more efficient whenever $md < n$, which is quite often the case.

4.1.4 Generating Points on a Grassmannian from Data

The first step in analyzing data on a Grassmannian is converting data from Euclidean points to points on the Grassmann manifold. In many cases, data is represented as a feature vector in $\mathbb{R}^n$. To construct points on the Grassmannian from Euclidean data in $\mathbb{R}^n$, take d data vectors and concatenate them into an $n \times d$ matrix $\mathbf{X} = [x_1, x_2, ..., x_d]$. Performing QR factorization on $\mathbf{X}$ yields $\mathbf{X} = \mathbf{QR}$, where $\mathbf{Q}$ is an orthonormal matrix with column space equal to that of $\mathbf{X}$. Thus $\mathbf{Q}$ is an orthonormal matrix representation for a point in $Gr(d, n)$ defined by the d data points. Each subspace is assigned the same label as the points used to construct it. Given m total data points for a class, there are $\lfloor m/d \rfloor$ total subspaces, discarding m modulo d points. Alternatively, the remaining points may be used to generate a subspace of lower dimension to be included in computations, provided that the algorithm does not require all subspaces to have equal dimensions.

4.2 Schubert Varieties

In order to develop the algorithm for preserving average volumes (Chapter 7), the idea of averaging subspaces must be extended more generally to finding the average of a set of Schubert varieties.

Given the real n-dimensional vector space $\mathbb{R}^n$, a complete flag $V : [V_1] \subset [V_2] \subset ... \subset [V_n] = \mathbb{R}^n$ where $dim([V_i]) = i$, and an integer a with $1 \leq a \leq d$, there are associated Schubert varieties on $Gr(d, n)$ given by

$$\Omega_{[V_i],a} = \{[X] \ \in Gr(d, n) \,|\, dim([X] \cap [V_i]) \geq a\}. \tag{4.25}$$

Additional background on Schubert varieties is found in [34] and [35], the latter of which contains the full details of the proofs discussed in the next section.

4.2.1 Point to Schubert Variety Distance

In [35], Schwickerath used Schubert varieties as a Grassmanian analog for linear signal models. To accomplish this, he formulated a point-to-set distance function on $Gr(d, n)$ that gives the minimum chordal distance from any point $[P] \in Gr(d, n)$ to a single Schubert Variety $\Omega_{[S],a} \in Gr(d, n)$.

Define a vector-valued function on the principal angles between $[X]$ and $[Y]$ by

$$\Theta([X], [Y]) = [\, \sin^2(\theta_1),\ \sin^2(\theta_2),\ ...,\ \sin^2(\theta_d)\,]^T \tag{4.26}$$

and similarly denote

$$\hat{\Theta}([X], [Y]) = [\, \cos^2(\theta_1),\ \cos^2(\theta_2),\ ...,\ \cos^2(\theta_d)\,]^T. \tag{4.27}$$

Denote a second equation on the first a singular values by

$$\Theta_a([X], [Y]) = [\, \sin^2(\theta_1),\ \sin^2(\theta_2),\ ...,\ \sin^2(\theta_a),\ 0, \dots,\ 0]^T \tag{4.28}$$

and similarly

$$\hat{\Theta}_a([X], [Y]) = [\, \cos^2(\theta_1),\ \cos^2(\theta_2),\ ...,\ \cos^2(\theta_a),\ 0, \dots,\ 0]^T. \tag{4.29}$$

Schwickerath proves that the minimum chordal distance from $[P]$ to $\Omega_{[S],a}$ is given by

$$d_c(\Omega_{[S],a},[P]) = d_c^*([S],[P]) = \left(\sum_{j=1}^{a}(\sin\theta_j)^2 + \sum_{j=a+1}^{d}(\sin(0))^2\right)^{1/2} = \left(\sum_{j=1}^{a}(\sin\theta_j)^2\right)^{1/2} \tag{4.30}$$

where θ_j is the jth principal angle between $[S]$ and $[P]$. Note that this is equivalent to summing over the components of $\Theta_a([S],[P])$.

4.2.2 Averaging Schubert Varieties

Theorem 4.2.1. *Let $[S_1],...,[S_k]$ be a collection of fixed subspaces of dimension d in $\mathbb{R}^n$, i.e., points in $Gr(d,n)$, with corresponding Schubert varieties $\Omega_{[S_i],a}$ for some integer a with $1 \leq a \leq d$. Then the flag mean with dimension a of the subspaces $[S_1],...,[S_k]$ minimizes*

$$\underset{[P]}{\arg\text{minimize}} \sum_{i=1}^{k} d_c(\Omega_{[S_i],a},[P])^2. \tag{4.31}$$

In other words, the flag mean of a set of subspaces is also the mean of the associated Schubert Varieties of these subspaces when using the chordal metric.

Proof. Let $\{\mathbf{S}_i\}_{i=1}^k$ be orthonormal matrices whose columns define the k subspaces $[S_1],...,[S_k]$.

Given k Schubert varieties in $Gr(d,n)$, the goal is to find a point $[P] \in Gr(d,n)$ that minimizes the average distance to each Schubert variety according to

$$\underset{[P]}{\arg\text{minimize}} \sum_{i=1}^{k} d_c(\Omega_{S_i,a},[P])^2. \tag{4.32}$$

This is equivalent to minimizing

$$\underset{[P]}{\arg\text{minimize}} \sum_{i=1}^{k} \mathbf{1}^T\Theta_a([S_i],[P]) \tag{4.33}$$

(where $\mathbf{1}^T$ is a vector of ones) which is again equivalent to

$$\underset{[P]}{\arg\text{maximize}} \sum_{i=1}^{k} \mathbf{1}^T\hat{\Theta}_a([S_i],[P]). \tag{4.34}$$

Let $\mathbf{P}$ be an orthonormal matrix representation of $[P]$. Then since the $\mathbf{S}_i$ are also orthonormal, the principal angles and corresponding principal vectors between $[P]$ and each subspace $[S_i]$ are obtained from the SVD

$$\mathbf{P}^T\mathbf{S}_i = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T. \tag{4.35}$$

where the diagonal entries of $\mathbf{\Sigma}$ are the cosines of the principal angles between $[P]$ and $[S_i]$. Taking the thin SVD of $\mathbf{P}^T\mathbf{S}_i\mathbf{S}_i^T\mathbf{P}$ yields

$$\mathbf{P}^T\mathbf{S}_i\mathbf{S}_i^T\mathbf{P} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T\mathbf{V}\mathbf{\Sigma}\mathbf{U}^T = \mathbf{U}\mathbf{U}^T\mathbf{\Sigma}^2 = \mathbf{\Sigma}^2 \tag{4.36}$$

since $\mathbf{U}$ is a square orthonormal matrix. Note that

$$tr(\mathbf{\Sigma}^2) = \mathbf{1}^T\cos(\hat{\theta})^2 = \mathbf{1}^T\hat{\Theta}([S_i],[P]) \tag{4.37}$$

for a single subspace $[S_i]$ (where $\hat{\theta}$ is the vector of ordered principal angles).

Thus

$$\sum_{i=1}^{k}\mathbf{1}^T\hat{\Theta}([S_i],[P]) = \sum_{i=1}^{k} tr(\mathbf{P}^T\mathbf{S}_i\mathbf{S}_i^T\mathbf{P}) = tr(\mathbf{P}^T\left(\sum_{i=1}^{k}\mathbf{S}_i\mathbf{S}_i^T\right)\mathbf{P}), \tag{4.38}$$

and therefore $\sum_{i=1}^{k}\mathbf{1}^T\hat{\Theta}_a([S_i],[P])$ is given by the trace of first a columns of the $\mathbf{P}^T\left(\sum_{i=1}^{k}\mathbf{S}_i\mathbf{S}_i^T\right)\mathbf{P}$. Since the singular values are ordered from largest to smallest, the first a columns of the matrix $\mathbf{P}$ that maximizes

$$\sum_{i=1}^{k}\mathbf{1}^T\hat{\Theta}([S_i],[P]) = tr(\mathbf{P}^T\left(\sum_{i=1}^{k}\mathbf{S}_i\mathbf{S}_i^T\right)\mathbf{P})$$

will also maximize $\sum_{i=1}^{k}\mathbf{1}^T\hat{\Theta}_a([S_i],[P])$. Thus we can restate 4.34 as

$$\arg\underset{P}{\text{maximize}}\, tr(\mathbf{P}^T\left(\sum_{i=1}^{k}\mathbf{S}_i\mathbf{S}_i^T\right)\mathbf{P}) \tag{4.39}$$

or equivalently

$$\arg\underset{P}{\text{maximize}}\, \mathbf{P}^T\left(\sum_{i=1}^{k}\mathbf{S}_i\mathbf{S}_i^T\right)\mathbf{P} \tag{4.40}$$

which is solved by the flag mean of the subspaces $[S_i]$.

The flag mean optimization problem iteratively solves for the optimal set of orthonormal principal vectors u_i, $i \in 1, ..., d$ where $d \leq dim(\sum_{i=1}^{k} \mathbf{S}_i \mathbf{S}_i^T)$,

$$\arg\underset{u_i}{\text{maximize}}\, u_i^T \sum_{i=1}^{k} \mathbf{S}_i \mathbf{S}_i^T u_i \tag{4.41}$$

subject to $u_i^T u_i = 1$, $u_i^T u_j = 0$, $i \neq j$. This yields the principal vectors u_i associated with each principal angle in order from smallest to largest.

Find the optimal first principal vector u_1 by

$$\arg\underset{u_1}{\text{maximize}} \sum_{i=1}^{k} u_1^T \mathbf{S}_i \mathbf{S}_i^T u_1 = \arg\underset{u_1}{\text{maximize}}\, u_1^T \left(\sum_{i=1}^{k} \mathbf{S}_i \mathbf{S}_i^T \right) u_1. \tag{4.42}$$

Let $\mathbf{A} = \sum_{i=1}^{k} \mathbf{S}_i \mathbf{S}_i^T$. Then the Lagrangian for the optimization problem is

$$\mathcal{L}(u_1, \lambda) = u_1^T \mathbf{A} u_1 - \lambda(1 - u_1^T u_1) \tag{4.43}$$

and the partials are

$$\frac{\partial \mathcal{L}}{\partial u_1} = 2\mathbf{A}u_1 - 2\lambda u_1 \tag{4.44}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = 1 - u_1^T u_1 \tag{4.45}$$

which yields the eigenvalue problem

$$\mathbf{A}u_1 = \lambda u_1 \tag{4.46}$$

and yields the first flag vector for $[S_1], ..., [S_k]$. Similarly, solve for the remaining principal vectors by adding orthogonality constraints. Let u_i be the ith principal vector ($1 < i \leq d$) and suppose $u_1, ..., u_{i-1}$ are previously computed principal vectors. Then the Lagrangian for optimizing u_i is

$$\mathcal{L}(u_i, \lambda_1, ..., \lambda_i) = u_i^T \mathbf{A} u_i - \lambda_i(1 - u_i^T u_i) - \lambda_1 u_1^T u_i - ... - \lambda_{i-1} u_{i-1}^T u_i \tag{4.47}$$

and the partial derivatives are

$$\frac{\partial \mathcal{L}}{\partial u_i} = 2\mathbf{A}u_i - 2\lambda_i u_i - \sum_{j=1}^{i-1} \lambda_j u_j \tag{4.48}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda_i} = 1 - u_i^T u_i \tag{4.49}$$

and for $j \in 1, ..., i-1$

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = u_i^T u_j \tag{4.50}$$

which again yields an eigenvector problem. Therefore u_i is the ith eigenvector of $\mathbf{A}$.

Set $\mathbf{P} = [u_1, ..., u_a]$. Then $\mathbf{P}$ maximizes $\sum_{i=1}^{k} \mathbf{1}^T \hat{\Theta}_a([S_i], [P])$ and consequently also minimizes Equation (4.31). Thus the a-dimensional flag mean of the $[S_i]$ is the average of their associated Schubert varieties $\Omega_{[S_i],a}$.

$\square$

Chapter 5

Comparison Algorithms

Multiple previously developed algorithms are used for validation of and comparison with the algorithms developed in this work. This chapter provides an overview of these algorithms.

5.1 Principal Component Analysis

Principal component analysis (PCA) will be used as a baseline for comparing embeddings. Data reduced using PCA is projected onto the orthonormal basis that captures the greatest variance in the data set. For the sake of simplicity, the Scikit-Learn [36] Python package implementation of PCA is utilized for all calculations. This implementation first mean centers the data and then uses the LAPACK implementation [37] of the singular value decomposition to calculate the PCA projection basis.

5.2 Random Orthogonal Projections

In [5], the authors advocate the use of Gaussian random projections as a reliable way to reduce dimension while keeping volume distortion within the given tolerance.

The matrix normal distribution for an $n \times d$ matrix is denoted $\mathcal{N}_{n,d}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{\Psi})$, where $\mathbf{M}$ is the mean matrix and $\mathbf{\Sigma}_{n\times n}$ and $\mathbf{\Psi}_{d\times d}$ are positive definite matrices such that $\mathbf{\Sigma} \otimes \mathbf{\Psi}$ is the covariance matrix ($\otimes$ being the tensor product of the two matrices). A random matrix $\mathbf{X}$ in this distribution is denoted by $\mathbf{X} \sim \mathcal{N}_{n,d}(\mathbf{M}, \mathbf{\Sigma}, \mathbf{\Psi})$. Note that this is equivalent to the multivariate distribution $vec(\mathbf{X}^T) \sim \mathcal{N}_{n,d}(vec(\mathbf{M}^T), \mathbf{\Sigma} \otimes \mathbf{\Psi})$.

Data is projected onto the space spanned by the columns of $\mathbf{X}$. In order for this projection to be orthogonal, QR-decomposition is used to find an orthonormal matrix representation for the column span of $\mathbf{X}$ as discussed in Section 4.1.4.

5.3 Grassmannian Gradient Descent

A gradient descent algorithm on the Grassmann manifold is used to validate the use of the flag mean algorithm for averaging Schubert varieties.

$$\underset{[\mathbf{P}]}{\arg\text{minimize}}\, \mathbf{1}^T \sum_{i=1}^{p} \Theta([\mathbf{S}_i], [\mathbf{P}]). \tag{5.1}$$

where Θ is a function on the principal angles between subspaces with

$$\Theta([\mathbf{S}_i], [\mathbf{P}]) = \cos^2(\hat{\theta}) = diag(\mathbf{\Sigma}) \tag{5.2}$$

Here, $\hat{\theta}$ is the vector of principal angles between $[\mathbf{S}_i]$ and $[\mathbf{P}]$ and $\mathbf{\Sigma}$ comes from the singular value decomposition $\mathbf{P}^T\mathbf{S}_i\mathbf{S}_i^T\mathbf{P} = \mathbf{U\Sigma U}^T$. This can be rewritten as

$$\sum_{i=1}^{p} \mathbf{1}^T\Theta([\mathbf{S}_i], [\mathbf{P}]) = \sum_{i=1}^{p} tr(\mathbf{P}^T\mathbf{S}_i\mathbf{S}_i^T) = tr(\mathbf{P}^T(\sum_{i=1}^{p} \mathbf{S}_i\mathbf{S}_i^T)\mathbf{P}). \tag{5.3}$$

Then the optimization problem becomes

$$\underset{\mathbf{P}}{\arg\text{minimize}}\, tr(\mathbf{P}^T(\sum_{i=1}^{p} \mathbf{S}_i\mathbf{S}_i^T)\mathbf{P}) \tag{5.4}$$

Let $\mathbf{A} = \sum_{i=1}^{p} \mathbf{S}_i\mathbf{S}_i^T$.

$$\underset{\mathbf{P}}{\arg\text{minimize}}\, tr(\mathbf{P}^T\mathbf{AP}) \tag{5.5}$$

where $\mathbf{P}^T\mathbf{P} = \mathbf{I}$. Differentiating with respect to $\mathbf{P}$ gives us the velocity matrix $\mathbf{H}(\mathbf{P}) = 2\mathbf{AP}$. Using Equation (4.11), parameterize a curve from $[\mathbf{P}]$ in the direction of $\mathbf{H}$. Given a current projection space $[\mathbf{P}^{(i)}]$, a step size $t \in (0, 1)$, and $\mathbf{H} = \mathbf{U\Sigma V}^T$, the point $[\mathbf{P}]$ is updated in the direction of steepest descent according to

$$\mathbf{P}^{(i+1)} = \mathbf{P}^{(i)}\mathbf{V}\cos(\mathbf{\Sigma}t)\mathbf{V}^T + \mathbf{U}\sin(\mathbf{\Sigma}t)\mathbf{V}^T. \tag{5.6}$$

The algorithm is terminated when the change in value of the objective function is sufficiently small. For purposes of this work, the threshold for termination is set to 10^{-12}.

5.4 Secant-Avoiding Projections

This algorithm developed in [18] looks to optimally preserve pairwise distances between points when projecting into a lower dimension. The intuition for this algorithm is based on Whitney's Embedding Theorem [38].

Given a set of M data points in $\mathbb{R}^n$, the algorithm proceeds as follows: Calculate all secants between all data pairs of data points. This yields a total of $k(k-1)/2$ secants. Let $\mathbf{K}$ be the $n \times k(k-1)/2$ matrix whose columns contain all normalized (unit) secant vectors. Let N be the desired number of total iterations, and $\alpha \in (0,1)$ be the desired step size. Details of the algorithm are given in Algorithm 1.

Algorithm 1: Secant Avoiding Projections

Initialize projection matrix $\mathbf{P}^{(0)}$ to be the first m vectors of $\mathbf{U}$, where $\mathbf{K} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$.
Project secants onto $\mathbf{P}^{(0)}$
while $\underline{i \leq N}$ **do**
    Find smallest projected secant $s^{(i)} = \arg\min_{s \in \mathbf{K}} ||\mathbf{P}^{(i)} s||$
    Find column of $\mathbf{P}^{(i)}$ nearest $s^{(i)}$, $p_1^{(i)} = \arg\max_{p_j \in P} |\langle p_j, s^* \rangle|$
    Beginning with $p_1^{(i)}$, generate a new orthonormal basis

$$\frac{\mathbf{P}^{(i)}\mathbf{P}^{(i)T}s^{(i)}}{||\mathbf{P}^{(i)}\mathbf{P}^{(i)T}s^{(i)}||}, p_2^{(i+1)}, ..., p_m^{(i+1)}$$

    Set $p^{(i+1)} = (1-\alpha)P^{(i)}P^{(i)T}s^{(i)} + \alpha(s^{(i)} - P^{(i)}P^{(i)T}s^{(i)})$

A more complete algorithm description is found in [39].

5.5 Modified Secant Avoiding Projections

An alternate idea for preserving volumes is proposed in [4]. The author suggests preserving volumes and angles by "supporting" simplices with rods as seen in Figure 5.1 and performs some analysis based on this idea, but no actual algorithm is implemented.

Building upon this idea, the Secant Avoiding Projection algorithm is modified to include unit vectors in the direction of the supporting rods to the set of unit vectors Σ. Given a set of simplices $\{S_i\}_{i=1}^p$, for each S_i find the vector that passes through a single vertex and is orthogonal to the span of the remaining vertices. The unit secant defined by this vector is then added to the set of unit secants for the SAP algorithm, which is otherwise performed the same way as in Algorithm 1. It is worth noting, however, that since the number of simplices increases factorially with the number of data points, this method adds a very large number of unit vectors to the initial data set.

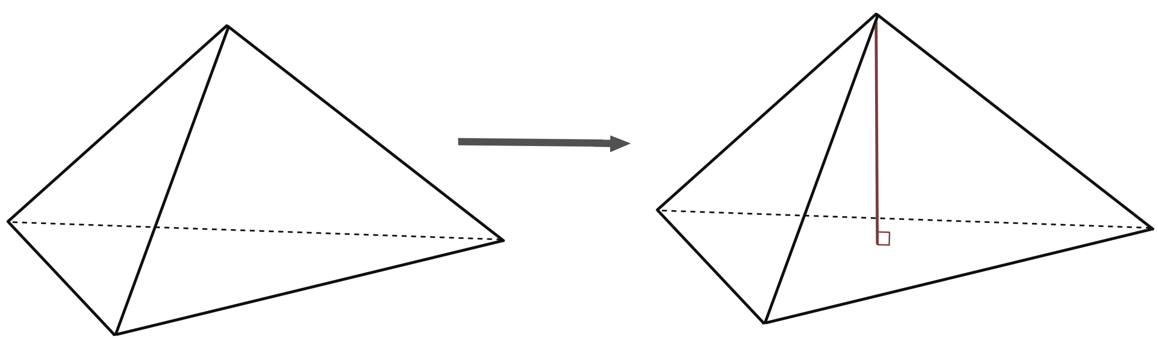


Figure 5.1: A supporting "rod" vector added to a 3-simplex. The red line is added to the set of secant vectors to prevent SAP from collapsing the tetrahedron.

Chapter 6

Smallest Volume Optimizing Projections

To begin, consider the problem of embedding a set of points into a lower dimension in a way that optimally preserves the areas enclosed by all sets of 3 points. Given a data set in $\mathbb{R}^n$, let $\mathbf{X} \in \mathbb{R}^{n \times N}$ be the matrix representation of the N data points. Let $\{S_i : i \in 1, ..., k\}$ be the set of 2-simplices in our data set. Each simplex S_i can be represented by $\mathbf{X}_{S_i}$, where $\mathbf{X}_{S_i}$ contains the columns of $\mathbf{X}$ corresponding to the vertices of S_i.

Consider the plane defined by the vertices of a single simplex S_i. This plane is a 2-dimensional affine subspace. The simplex is translated to the origin by subtracting one of the vertices; then the vertices define a standard 2-dimensional subspace $[\mathbf{S}_i]$, represented by orthonormal matrix $\mathbf{S}_i$. Now consider projecting the data onto some m-dimensional subspace $[\mathbf{P}]$ with $m < n$. If $[\mathbf{S}_i] \subseteq [\mathbf{P}]$, then the orthogonal projection of $\mathbf{X}$ onto $[\mathbf{P}]$ completely preserves the area of the simplex used to define $[\mathbf{S}_i]$. Conversely, the greater the principal angles between $[\mathbf{P}]$ and $[\mathbf{S}_i]$, the smaller the area of $\mathbf{P}\mathbf{X}_{S_i}$ will become. Given these facts, it stands to reason that optimally embedding all simplices S_i requires finding a projection space $[\mathbf{W}]$ that is as close as possible in terms of principal angles to each of the $[\mathbf{S}_i]$. This concept is illustrated for 1-simplices in Figure 6.1. In particular, the goal is to improve the smallest projected simplex area by moving $[\mathbf{P}]$ incrementally closer to the subspace spanned by its corresponding simplex in the data.

The next section considers the special case where $dim([\mathbf{P}]) = 2$ and all simplices are 2-dimensional. The simplest version of the algorithm is developed to solve the problem for this case. It is then extended to the case where $dim([\mathbf{P}]) = 3$, and then more generally to $dim([\mathbf{P}]) = m$. Section 6.3 extends the results for 2-simplices to more general algorithms for embedding k-simplices into m dimensions while preserving the appropriate notion of volume.

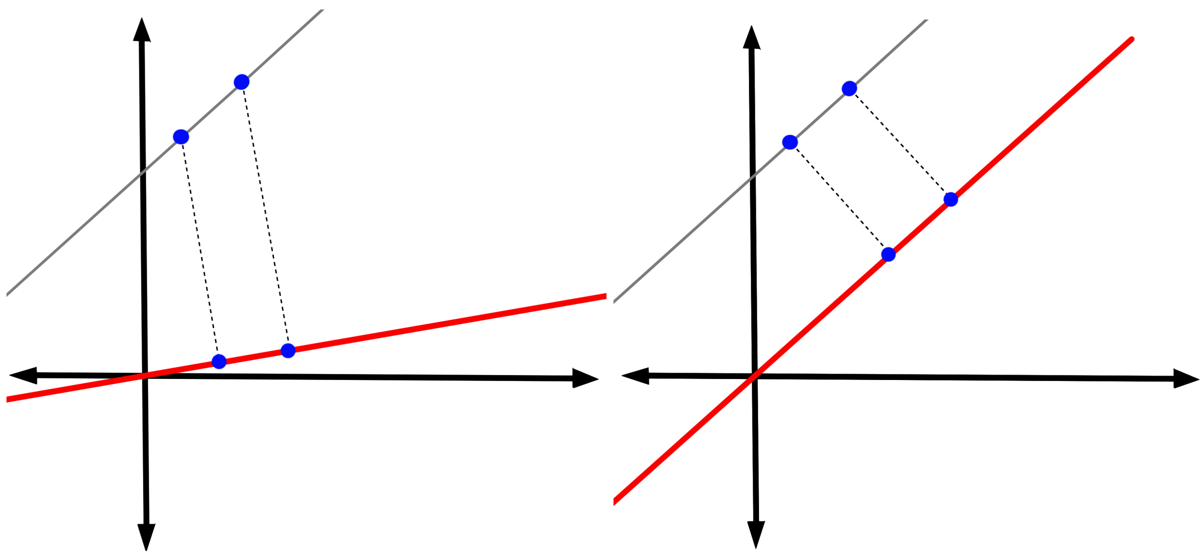


Figure 6.1: Orthogonal projections of two data points (blue) defining a 1-simplex onto different lines. The grey line represents the affine subspace spanned by this simplex. **Left:** If the grey line were translated to the origin, it would be distinct from the red line. The size of the 1-simplex is decreased by orthogonal projection onto the red line. **Right:** The grey line is now parallel to the red line, and the size of the 1-simplex remains the same after projection.

6.1 Embedding Areas in 2D

Given a set of 2-simplices in $\mathbb{R}^n$, the goal is to project the simplices onto a 2-dimensional subspace while attempting to preserve simplex areas. To accomplish this, the data should be projected onto a plane that is as near as possible, in terms of principal angles, to the 2-dimensional subspaces spanned by the set of translated simplices.

Consider all the simplex subspaces as points on $Gr(2, n)$. The matrix $\mathbf{P}$ is an orthonormal $n \times 2$ matrix that projects data onto plane spanned columns of $\mathbf{P}$. Thus $[\mathbf{P}]$ also defines a point on $Gr(2, n)$. Calculations can then be performed on the Grassmannian to find a projection $[\mathbf{P}]$ that is optimally close to the set of points defined by the other planes.

This algorithm focuses on improving the worst simplex projection to solve the optimization problem from Equation (3.8). Given a projection into 2 dimensions, find the smallest normalized projected area and update $\mathbf{P}$ in a way that improves it. In terms of Grassmannians, this means moving $[\mathbf{P}]$ in the direction of the point corresponding to the simplex. This is accomplished by parameterizing a geodesic between the points as described in Section 4.1.2 and updating the projection along the geodesic curve.

Let $\{S_1, ...S_d\}$ be the set of 2-simplices in $\mathbb{R}^n$. The algorithm is initialized with a projection matrix $\mathbf{P}_0$ and updated iteratively to improve the worst embedded simplex area. At each iteration, the projection subspace is moved along a geodesic curve in the direction of the point in $Gr(2, n)$ corresponding to the current worst simplex. The step size at each iteration is given by some function $f(t)$, where t is the current iteration. This function can be heuristically adjusted based on the problem and is discussed more thoroughly later. Iterations are terminated when the difference in simplex area ratios falls below a given threshold θ or when the maximum number of iterations T is reached. Details are given in Algorithm 2.

Algorithm 2: Smallest Area Optimizer in 2D

Set $t = 0$
Define set of vertices of all 2-dimensional simplices $\mathcal{S} = \{x_i, x_j, x_k | i, j, k \in I, i \neq j \neq d\}$
Calculate areas of all 2-dimensional simplices defined by $\mathcal{S}$
Select initial 2-dimensional subspace $[\mathbf{P}]^{(0)}$
Project data onto $[\mathbf{P}]^{(0)}$
while $t < T$ **do**
    Calculate areas of all 2-dimensional simplices after projection onto $[\mathbf{P}]^{(t)}$
    Normalize projected areas by $|S_{proj}|/|S|$ for all $S \in \mathcal{S}$
    Find simplex $S^{(t)} \in \mathcal{S}$ with smallest normalized projected area
    Find subspace $[\mathbf{V}]$ spanned by vertices of $S^{(t)}$
    Parameterize a geodesic curve $G(s)$ between $[\mathbf{P}]^{(t)}$ and $[\mathbf{V}]$, with $G(0) = [\mathbf{P}]^{(t)}$ and $G(1) = [\mathbf{V}]$
    Update $[\mathbf{P}]^{(t+1)} = G(f(t))$
    Project data onto $[\mathbf{P}]^{(t+1)}$
    Terminate if $t + 1 \geq T$ or difference in normalized projected areas of $S^{(t+1)}$ and $S^{(t)}$ is less than δ
    Set $t = t + 1$

6.1.1 Examples

To test the efficacy of this algorithm, it was applied to 20 randomly chosen points from the MNIST data set, containing 10 data points each from the 0 and 1 classes. The ratios between the areas of simplices before and after projection were calculated for both the Smallest Area Optimizing Projection algorithm and a simple PCA projection into 2 dimensions. The Smallest Area Optimizer was terminated when the difference in smallest projected area between consecutive steps fell below $\delta = 10^{-12}$ or after a maximum of $T = 1000$ iterations. Results for all trials are found in Table 6.1. The projected data from Trial 5 for both methods is shown in Figure 6.2. Note that although all projected points are colored by class for clarity, the label information was not used to inform the result in any way. For this trial, the smallest normalized embedded simplex area for PCA was 1.39E-07 and the smallest normalized embedded area found by the Smallest Area Optimizing Projection algorithm was 4.71E-06. The Smallest Area Optimizing Projector shows a ten-fold improvement over PCA. The other trials on various sets of randomly selected data points yielded similar and frequently better results (see Table 6.1).

Figure 6.3 shows the changes in both smallest normalized simplex area and average normalized simplex area for the Smallest Area Optimizing Projector at each iteration. Although the smallest area initially demonstrates oscillatory behavior, it quickly converges to a local maximum value. This is aided in part by utilizing an annealing scheme for the step size function $f(t)$, which is

Table 6.1: Smallest normalized areas from 10 sets of randomly selected data points embedded from $\mathbb{R}^{784}$ into 2 dimensions.

Trial	Area Optimizing	PCA
1	4.68E-07	9.32E-11
2	7.26E-07	4.76E-08
3	1.04E-06	3.91E-10
4	8.03E-07	3.76E-08
5	4.71E-06	1.39E-07
6	3.22E-13	1.34E-08
7	5.64E-07	9.87E-09
8	5.76E-06	7.61E-12
9	1.02E-06	1.34E-08
10	1.36E-06	1.21E-08

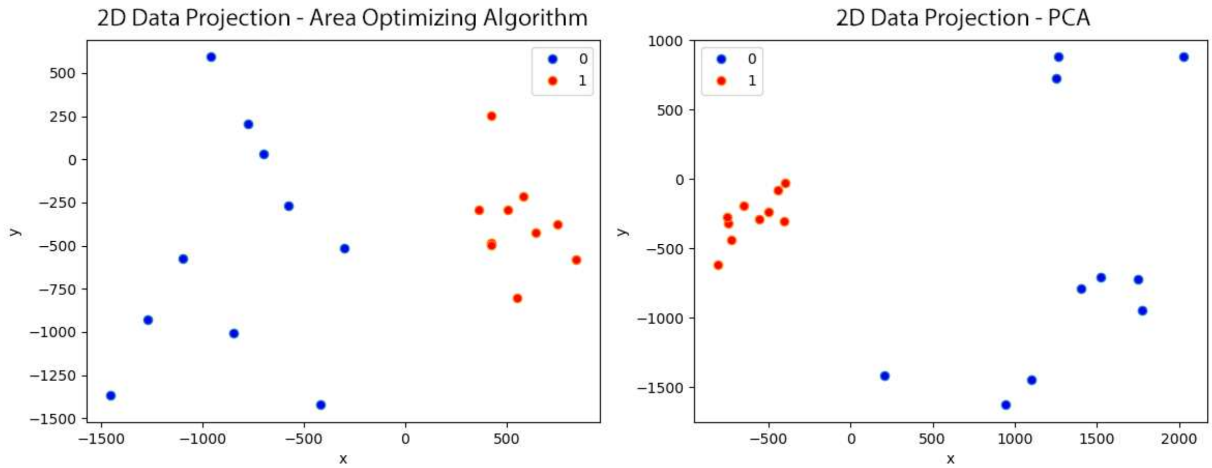


Figure 6.2: Embeddings of data from $\mathbb{R}^{784}$ into 2 dimensions using Smallest Area Optimizing Projections and PCA, results from Trial 5 in Table 6.1.

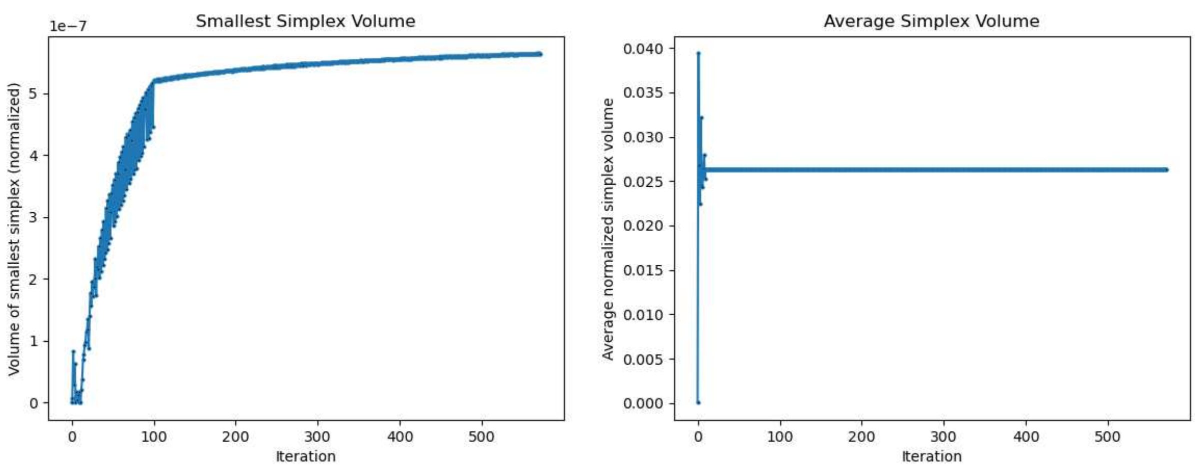


Figure 6.3: Change in smallest normalized 2-simplex volume (area) and average normalized 2-simplex volume at each iteration of the Smallest Area Optimizing Projection algorithm for results pictured in Figure 6.2

further discussed in Section 6.3.1. However, in this case the algorithm clearly misses the maximum average volume, demonstrating the difference in the two optimization problems presented in Section 3.2.

As an additional test, a set of 50 points were randomly generated in a single plane embedded in $\mathbb{R}^3$, and slight perturbations were added to generate a point cloud that lies near a single plane. Both the Smallest Area Optimizing Projector and PCA were used to project these points onto a 2-dimensional plane. To visually compare the differences between the two projections and the initial data, both 2-dimensional projections were again embedded back into $\mathbb{R}^3$ and plotted alongside the initial data. Figure 6.4 shows the three sets of points from various angles. The PCA projection plane is extremely close to the plane initially used to generate the data, whereas the plane found by the Smallest Area Optimizing Projector is nearly orthogonal to the initial plane. However, the smallest normalized simplex areas were 4.361e-06 for PCA and 4.78e-06 for the Smallest Area Optimizing Projector. Even though variance in the data is lost by the Smallest Area Optimizing Projector, it succeeds in improving the worst projected area. It is worth noting that the optimal projection of the smallest area yields a projection plane quite far from the plane initially used to generate the data points.

6.2 Embedding Areas in 3D

Optimizing areas of 2-simplices projected into 3 dimensions is an extension of the case discussed in the previous section, with the added complication that the projection subspace is no longer on the same Grassmannian as the simplex subspaces. Let $[\mathbf{S}_i]$ be the subspace spanned by the current worst simplex and $[\mathbf{P}]$ be the current projection subspace. In order to update $[\mathbf{P}]$ to be closer to $[\mathbf{S}_i]$, consider the singular value decomposition (SVD) of $\mathbf{P}^T\mathbf{S}_i = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$. The two principal vectors in $[\mathbf{P}]$ corresponding to the two principal angles between $[\mathbf{P}]$ and $[\mathbf{S}_i]$ are the first two columns of $\mathbf{PU}$. These vectors span the plane in $[\mathbf{P}]$ that is closest to the subspace $[\mathbf{S}_i]$. Denote this plane by $[\mathbf{P}]'$. The remaining column of $\mathbf{PU}$ completes the decomposition of $[\mathbf{P}]$, call this column $[\mathbf{P}]^*$. The plane nearest to $[\mathbf{S}_i]$ can now be represented as a point on $Gr(2, n)$. To update $[\mathbf{P}]$, parameterize a geodesic on $Gr(2, n)$ between $[\mathbf{P}]'$ and $[\mathbf{S}_i]$ and update $[\mathbf{P}]'$ by some step size according to $f(t)$. Once the update is completed, the new projection space is the span of the columns of the updated $[\mathbf{P}]'$ and $[\mathbf{P}]^*$. As before, iterations are terminated when the difference in worst simplex volume ratios falls below a given threshold δ or when the maximum number of iterations T is reached. Details are given in Algorithm 3.

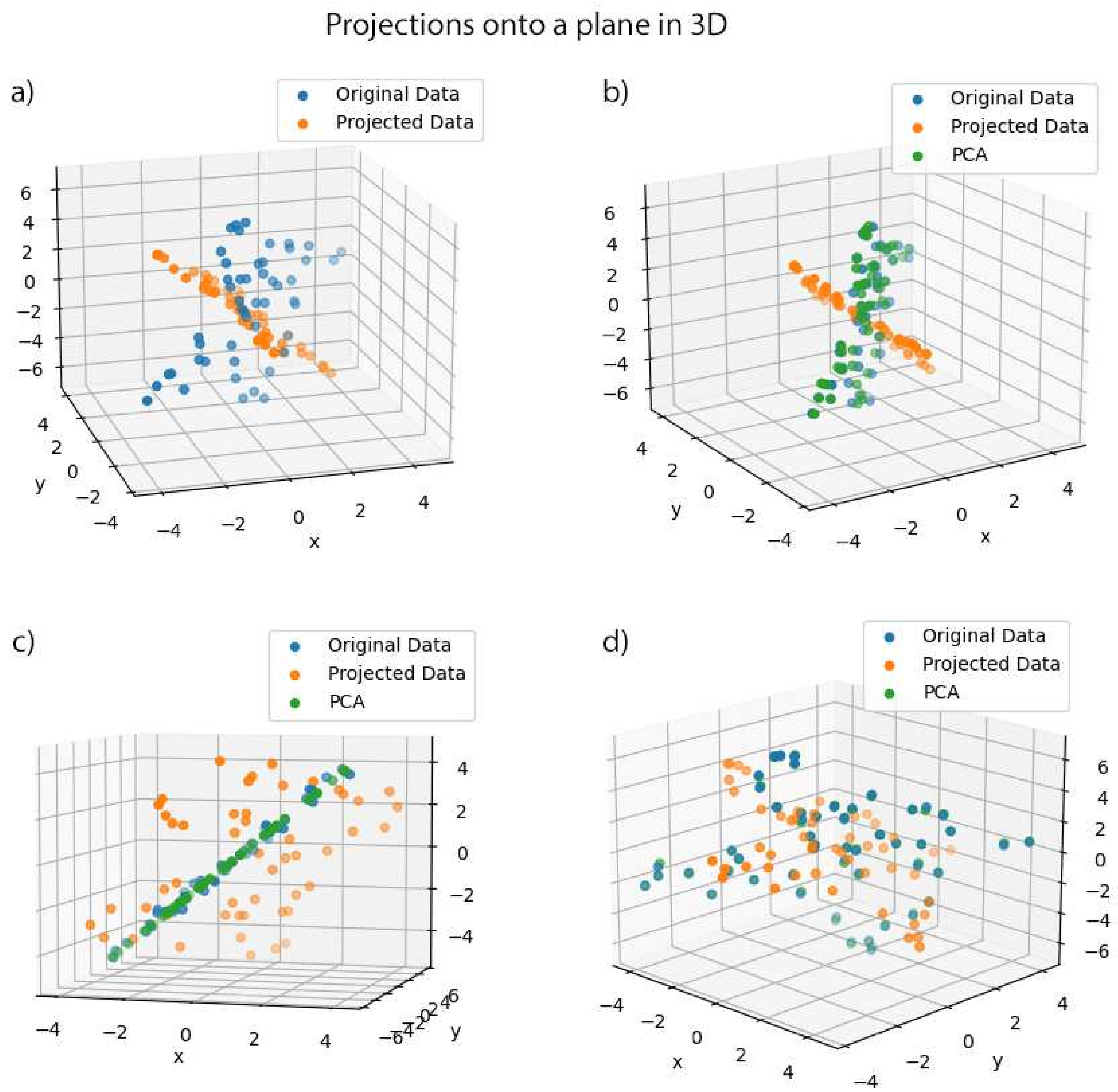


Figure 6.4: Projected data plotted on the projection plane in $\mathbb{R}^3$. Fig. a) Original data and projection found by the Smallest Area Optimizing Projector. Figs. b)-d) Several angles of original and projected data along with 2D PCA projection.

6.2.1 Examples

Once again, 20 randomly selected MNIST data points (10 from class 0, 10 from class 1) were projected into lower dimensions using both the Smallest Area Optimizing Projector and PCA. This time, the data was projected into 3 dimensions and normalized volumes were compared. In the trial pictured in Figure 6.5, PCA achieved a smallest normalized area of 3.08E-05 and the Smallest

Algorithm 3: Smallest Area Optimizer in 3D

Set $t = 0$
Define set of vertices of all 2-simplices $\mathcal{S} = \{x_i, x_j, x_k | i, j, k \in I, x_i \neq x_j \neq x_k\}$
Calculate areas of all 2-simplices defined by $\mathcal{S}$
Select initial 3-dimensional subspace $[\mathbf{P}]^{(0)}$
while t < T **do**
    Project data onto $[\mathbf{P}]^{(t)}$
    Calculate areas of all 2-simplices after projection onto $[\mathbf{P}]^{(t)}$
    Normalize projected areas by $|S_{proj}|/|S|$ for all $S \in \mathcal{S}$
    Find simplex $S^{(t)} \in \mathcal{S}$ with smallest normalized projected area
    Find subspace $[\mathbf{V}]$ spanned by the vertices of $S^{(t)}$
    Find subspace $[\mathbf{P}]' \subset [\mathbf{P}]^{(t)}$ nearest to $[V]$ and orthogonal space $[\mathbf{P}]^* \subset [\mathbf{P}]^{(t)}$
    Parameterize a geodesic curve $G(s)$ between $[\mathbf{V}]$ and $[\mathbf{P}]'$, with $G(0) = [\mathbf{P}]'$ and $G(1) = [\mathbf{V}]$
    Update $[\mathbf{P}]'_{new} = G(f(t))$
    Concatenate $[\mathbf{P}]'_{new}$ and $[\mathbf{P}]^*$ and orthogonalize the resulting matrix to make $[\mathbf{P}]^{(t+1)}$
    Terminate if $t \geq T$ or difference in normalized projected areas of $S^{(t)}$ and $S^{(t-1)}$ is less than δ
    Set $t = t + 1$

Area Optimizing Projector achieved a smallest area of 4.85E-04. Note that although Figure 6.5 colors the projected data based on its initial class label, these labels do not in any way inform the projection. The changes in both smallest area and average area for the Smallest Area Optimizing Projector are shown in Figure 6.6. More results from different sets of randomly selected data points are included in Table 6.2. Once again, the Smallest Area Optimizing Projector consistently yielded results an order of magnitude better than those produced by PCA.

6.3 General Case: Optimal k-Volumes in m Dimensions

The algorithm from Section 6.2 can be again extended to apply to the general case of embedding data into m dimensions while maximizing the smallest embedded k-simplex volume. Details are given in Algorithm 4.

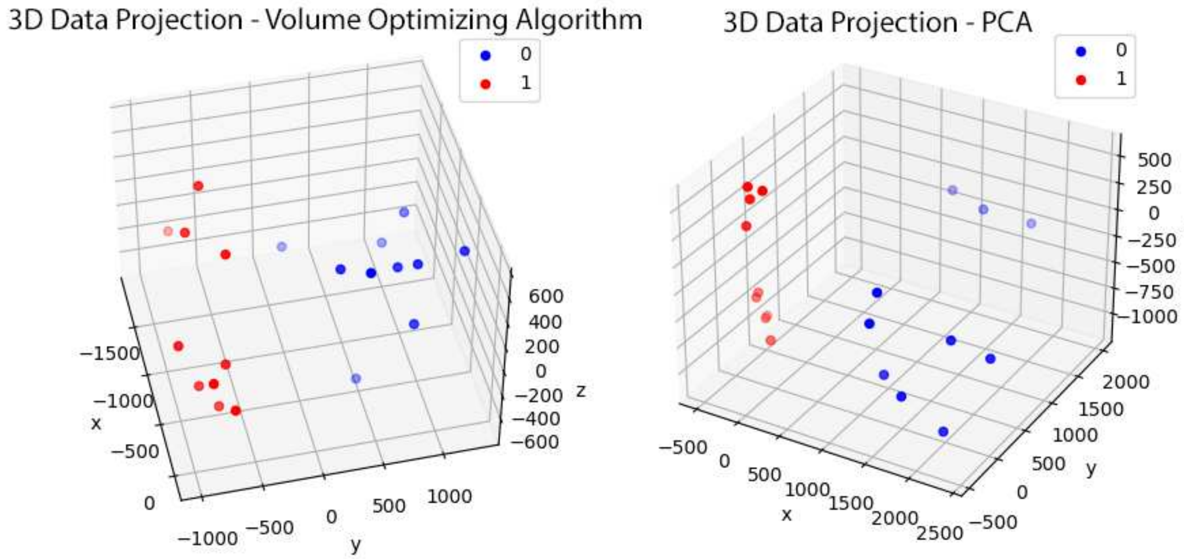


Figure 6.5: Embeddings from each algorithm produced in Trial 3 from Table 6.2. Points are colored by class for clarity, but no labels are used to inform the embedding.

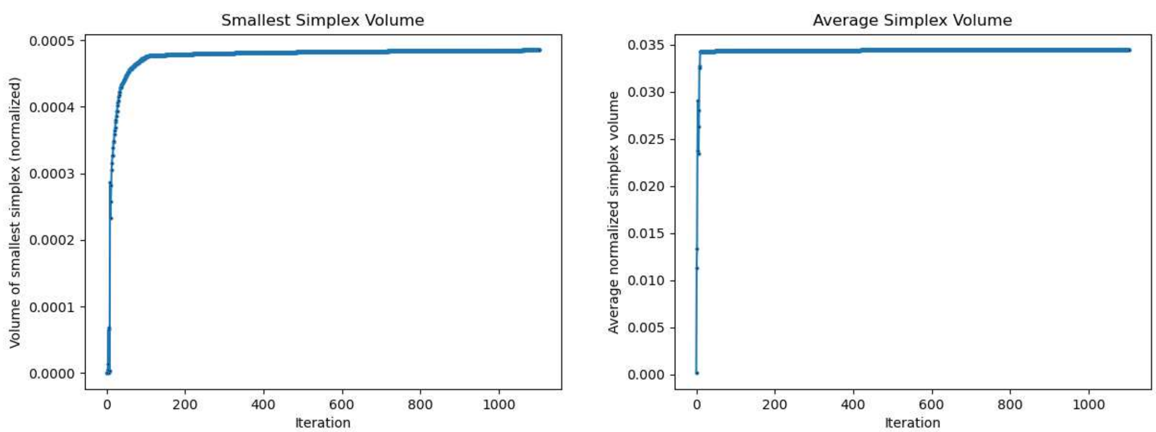


Figure 6.6: Change in smallest embedded 2-simplex area (left) and average embedded 2-simplex area with each iteration of the Smallest Area Optimizing Projection algorithm.

6.3.1 Computational Considerations

Reducing Simplex Lists

The algorithm as written requires all projected simplex volumes to be re-calculated at each iteration, and the computational complexity scales linearly with the total number of simplices. However, it may be desirable to look at the set of all possible k-simplices with vertices defined by points in a given data set, in which case the complexity will scale factorially with the number of

Table 6.2: Comparison of smallest normalized embedded areas in 3 dimensions over 10 trials on randomly generated data using PCA and the Smallest Area Optimizing Projector.

Smallest Normalized Areas in 3D

Trial	Vol. Optimizing	PCA
1	6.67E-04	4.53E-06
2	1.22E-03	6.76E-06
3	4.85E-04	3.08E-05
4	1.78E-04	2.82E-06
5	3.67E-04	3.52E-05
6	3.14E-04	3.81E-05
7	2.32E-04	1.06E-05
8	5.22E-04	6.88E-05
9	1.00E-04	1.51E-06
10	4.52E-04	8.86E-06

Algorithm 4: Smallest k-Volume Optimizer in m Dimensions

Set $t = 0$
Define set of vertices of all k-dimensional simplices $\mathcal{S} = \{x_0, x_1, ..., x_k | x_i \in X \;\forall\, i, x_i \neq x_j \;\forall\, i \neq j\}$
Calculate volumes of all k-dimensional simplices defined by $\mathcal{S}$
Select initial m-dimensional subspace $[\mathbf{P}]^{(0)}$
while $t < T$ **do**
  Project data onto $[\mathbf{P}^{(t)}]$
  Calculate volumes of all k-dimensional simplices after projection
  Normalize projected volumes by $|S_{proj}|/|S|$ for all $S \in \mathcal{S}$
  Find simplex $S^{(t)} \in \mathcal{S}$ with smallest normalized projected area
  Find subspace $[\mathbf{W}]$ spanned by vertices of $S^{(t)}$
  Take the SVD $\mathbf{P}^{(t)T}\mathbf{W} = \mathbf{U\Sigma V}^T$ and find the left principal vectors corresponding to the smallest d singular vectors
  Set $[\mathbf{P}]'$ to be the subspace spanned by these principal vectors and $[\mathbf{P}]^*$ to be the subspace spanned by the remaining $m - k$ singular vectors
  Parameterize a geodesic curve $G(s)$ between $[\mathbf{W}]$ and $[\mathbf{P}]'$, with $G(0) = [\mathbf{P}]'$ and $G(1) = [\mathbf{W}]$
  Update $[\mathbf{P}]'_{new} = G(f(t))$
  Concatenate $[\mathbf{P}]'_{new}$ and $[\mathbf{P}]^*$ and orthogonalize the resulting matrix to make $[\mathbf{P}]^{(t+1)}$
  Terminate if $t \geq T$ or difference in normalized projected volumes of $S^{(t)}$ and $S^{(t-1)}$ is less than δ
  Set $t = t + 1$

data points. This is highly undesirable for an iterative algorithm. With the proper choice of step function, however, it is almost always the case that after the the first few iterations the Smallest

Volume Optimizer will oscillate between two or three simplices with the smallest volumes for the remainder of the run. To reduce overall computational complexity, only a small subset of simplices need to be tracked beyond the first 100 iterations.

Table 6.3 shows the results of optimizing the normalized embedded volumes of 3-simplices in a set of 30 randomly generated points projected from $\mathbb{R}^{10}$ into 5 dimensions. The first trial utilized the full set of 27,405 simplices for all iterations of the algorithm, and the second trial reduced the set after the 100th iteration to only the 5 smallest simplices. In both cases, the algorithm was started with the same initial condition by using the first five singular vectors of the SVD of the initial data as the initial projection subspace. Both trials also utilized identical step size functions $f(t)$. Not only did reducing the simplex list result in a significantly faster run time with fewer iterations, it also converged to a better result. Because the improvement in computation time and complexity is so significant, this strategy is used for the rest of the tests included in this work.

Table 6.3: Testing the effect of reducing the number of simplices after a set number of iterations on algorithm results and run time.

Simplex Set	Smallest Volume	Run Time (s)	Iterations
Full	0.000545	3343.555	2127
Reduced	0.000647	157.683	1364

Initialization

The success of this iterative algorithm appears sensitive to the choice of initial condition. Because the problem is non-convex, randomly initialized trials must be repeated many times to obtain the best possible result. This greatly increases the resources and time required to complete even a single trial, and still does not guarantee that the best result is obtained. Combating this problem requires finding a good starting point for the iterative algorithm that is near the best solution.

In the Secant Avoiding Projection (SAP) algorithm [18, 39], the singular value decomposition of the data is used as a starting point for the iterative algorithm because it should be reasonably close to the optimal solution. Since the Smallest Volume Optimizing algorithm functions identically to SAP when $k = 1$, the same reasoning suggests that the SVD projection basis will provide a good starting point for optimizing higher dimensional volumes as well.

Table 6.4 shows two trials performed on a set of 20 MNIST data points. In the first trial, the data was projected from $\mathbb{R}^{784}$ to three dimensions while optimizing the volumes of 3-simplices. A single trial with SVD initialization was compared with the best result from 30 randomly initialized trials. The smallest projected volume found using SVD initialization was an order of magnitude larger than the smallest projected volumes found during any of the randomized initialization trials. Additionally, since a fixed starting point makes the algorithm deterministic, only one trial was required to get the best possible result using SVD initialization.

In the second trial, the same set of 20 data points was embedded into 20-dimensional subspaces. Since there are only 20 data points, there is guaranteed to be a projection into 20 dimensions that exactly preserves the configuration of the original data points, in which case all normalized projected volumes should be exactly 1. Previous testing has shown that random initialization is not guaranteed to find this projection in the cases where it exists. For this specific comparison, 30 trials with random initialization were compared with a single run using SVD initialization. Since the SVD basis provides an isometric embedding of the 20-dimensional data into a 20-dimensional subspace, all normalized volumes are already maximized upon initialization, and the algorithm terminates immediately as expected. Random initialization, however, did not find any projection that came close to the optimal solution.

Table 6.4: Comparison of results produced using random initialization with those produced from SVD initialization.

Simplex Dim	Proj Dim	Initialization	Best Smallest Vol	Trials
3	5	SVD	0.001012	1
3	5	Random	0.000694	30
3	20	SVD	1.000000	1
3	20	Random	0.031285	30

Based on these results, using the SVD of the data points to initialize the Smallest Volume Optimizing algorithm gives a much better approximate of the best possible solution than a randomly chosen starting condition. Additionally, it greatly reduces the computations required to find a good solution.

Annealing Experiments

The choice of step size function $f(t)$ is integral to the performance of the Smallest Volume Optimizing algorithm. Initial testing showed that when $f(t)$ decreases linearly with step size, the algorithm frequently fails to converge even after as many as 10,000 iterations. This is due to an oscillatory behavior that occurs when the step size is too large relative to the size of the simplices, resulting in the algorithm bouncing between the smallest two or three simplices indefinitely and ultimately failing to converge. Examples of this oscillatory behavior can be seen in the first 100 iterations of the algorithm in Figures 6.3 and 6.6. This issue is addressed by using an annealing scheme for $f(t)$ to reduce the step size at certain points during iterations. In all previous trials in this work, the function

$$f(t) = \begin{cases} \frac{1}{t}, & t \leq 10 \\ \frac{1}{10t}, & t \in (10, 100] \\ \frac{1}{1000t}, & t > 100 \end{cases} \tag{6.1}$$

was used based on repeated testing that showed that it worked well in tandem with random initialization. However, preliminary experimentation with SVD initialization suggested that having a good starting condition drastically changed the effectiveness of this step size function, and it quite frequently failed to reach an optimum even though the starting condition should set it relatively close to one. To address this new issue, several variations of annealing functions for step size were compared on the same sets of data using SVD initialization. As a baseline, a step size function with no annealing was also compared.

The constant step size function with no annealing is

$$f_0(t) = 0.1. \tag{6.2}$$

The first function with annealing is a variation of the step size function originally used, scaling down the step size with each iteration at varying rates:

$$f_1(t) = \begin{cases} \frac{1}{t} & t \leq 1000 \\ \frac{1}{10t} & t \in (1000, 5000] \\ \frac{1}{100t} & t > 5000 \end{cases} \tag{6.3}$$

The second step size function scales the step size down with each iteration t:

$$f_2(t) = \frac{1}{t} \tag{6.4}$$

and the third function operates similarly, but scales step size down more quickly:

$$f_3(t) = \frac{1}{10t} \tag{6.5}$$

The fourth function utilizes both a fixed step size and annealing at certain iterations:

$$f_4(t) = \begin{cases} 0.1 & t \leq 1000 \\ 0.01 & t \in (1000, 5000] \\ 0.0001 & t > 5000 \end{cases} \tag{6.6}$$

The fifth function is the the one used in the previous sections:

$$f_5(t) = \begin{cases} \frac{1}{t}, & t \leq 10 \\ \frac{1}{10t}, & t \in (10, 100] \\ \frac{1}{1000t}, & t > 100 \end{cases} \tag{6.7}$$

and the last function is one initially used for the Secant Avoiding Projection algorithm:

$$f_6(t) = \begin{cases} 0.01 & t \leq 1000 \\ 0.001 & t \in (1000, 5000] \\ 0.0001 & t > 5000 \end{cases} \tag{6.8}$$

Equation (6.8) is nearly identical to Equation (6.6), but is included separately for the purposes of directly comparing the Smallest Volume Optimizing algorithm with SAP.

The first test was performed on a fixed set of 20 points from the MNIST handwritten digit data set. These points were embedded into 5 dimensions while optimizing the volume of the smallest embedded 3-simplex and using SVD to initialize the algorithm. Each algorithm was terminated either when the difference in smallest volumes between consecutive steps fell below $\delta = 10^{-12}$ or after 10,000 iterations. Table 6.5 lists the smallest volume found using each of the seven step size functions, as well as the total run time in seconds and the number of iterations performed. Figures 6.7 and 6.8 show how the smallest embedded volume changed in each trial.

Table 6.5: Comparison of step size functions on a fixed set of 20 MNIST data points, optimizing volumes of 3-simplices embedded into 5 dimensions.

Step Size Function	Smallest Volume	Run Time (s)	Iterations
$f_0(t)$	0.00437982	1464.96	10000
$f_1(t)$	0.00854070	823.78	5405
$f_2(t)$	0.00909129	1466.27	10000
$f_3(t)$	0.00216232	858.84	5850
$f_4(t)$	0.01200914	1486.41	10000
$f_5(t)$	0.00271254	73.95	504
$f_6(t)$	0.01119615	1501.49	10000

The best embedding was found by using $f_4(t)$ as the step size function, and although it did not terminate prior to the maximum of $10,000$ iterations, the smallest volume plot at the top left of Figure 6.8 shows the curve flattening out after approximately 6000 iterations, suggesting that it does in fact converge. The second best result was found using the step size function $f_6(t)$. The remaining five step size functions all resulted in projections with significantly worse smallest embedded volumes. The top left of Figure 6.7 shows how the smallest simplex volume changed at each step using $f_0(t)$. Even though the smallest volume in the final projection was larger than that found by some other step size functions, using $f_0(t)$ results in highly oscillatory non-convergent behavior, and the end result is likely random and changes drastically based on the number of iterations performed. Using step size function $f_3(t)$ appears to result in premature termination before the algorithm has actually converged. The remaining functions all show convergent behavior but find a sub-optimal solution.

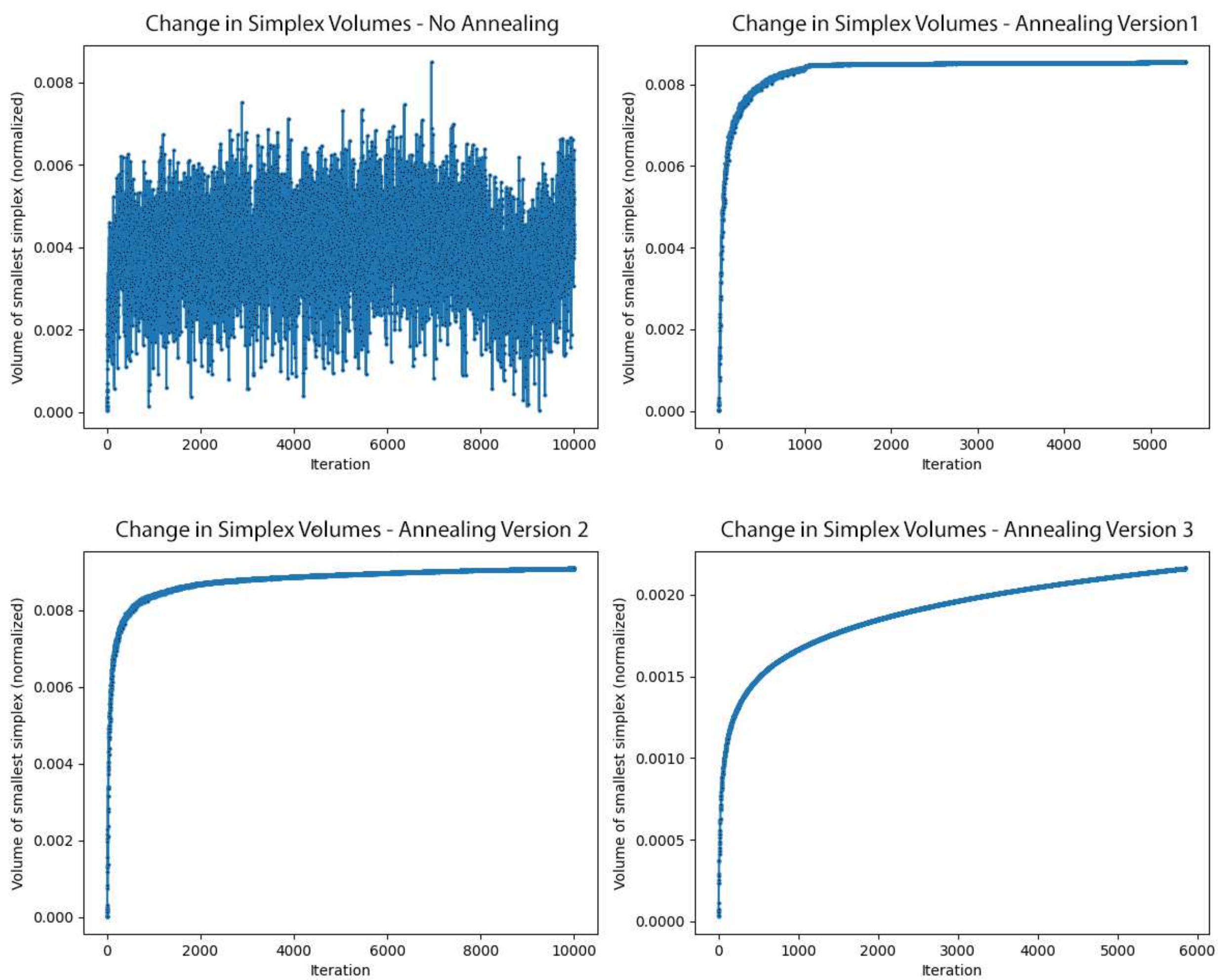


Figure 6.7: Changes in smallest 3-simplex volume for four different choices of step size function.

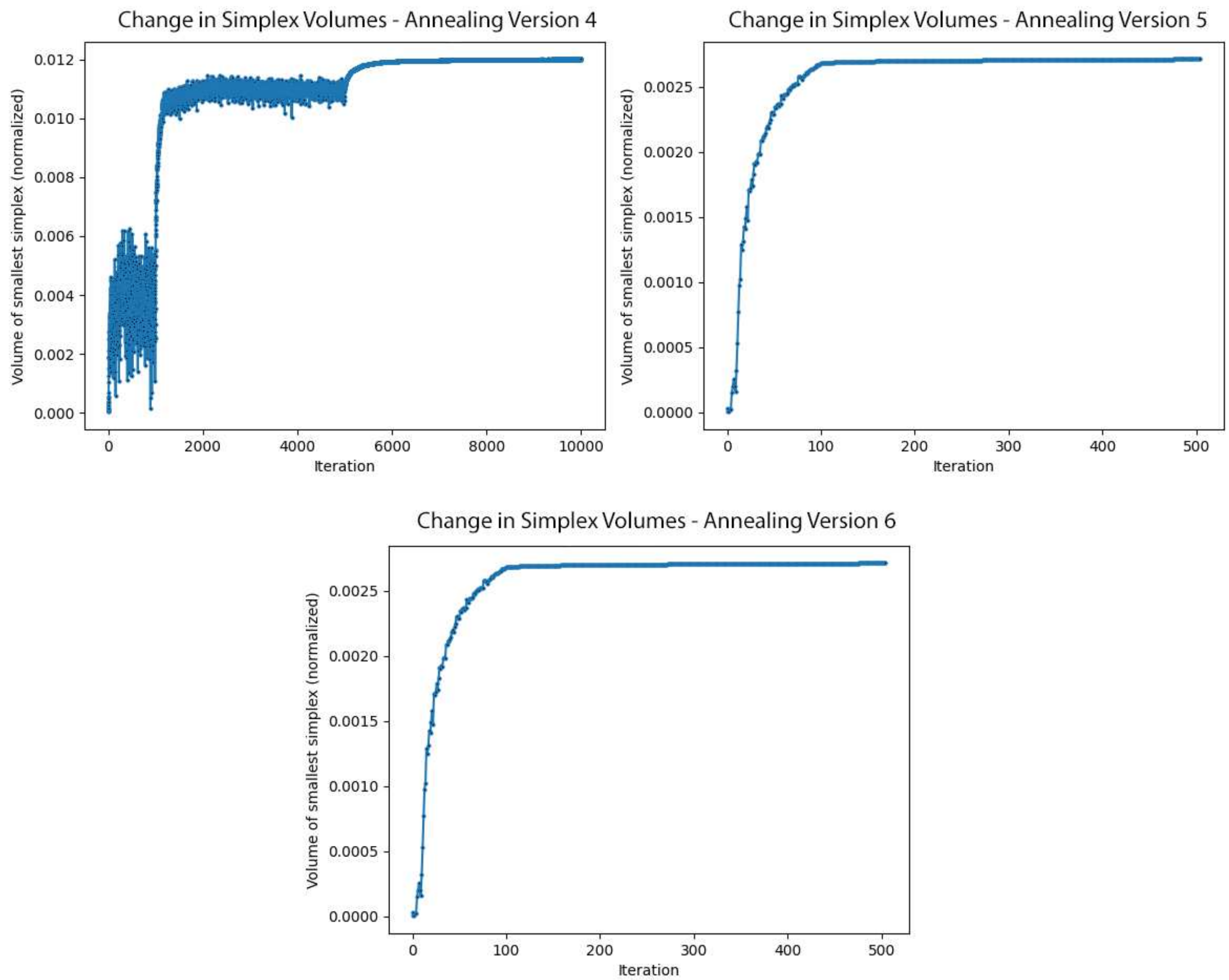


Figure 6.8: Changes in smallest 3-simplex volume for three different choices of step size function.

Additional trials were performed on 10 sets of randomly generated data points in $\mathbb{R}^{10}$. For each trial, 30 new data points were randomly generated and embedded into 5 dimensions while optimizing volumes of 3-simplices. All seven step size functions were used along with SVD initialization on each set of data points, with the maximum number of iterations set to 20,000. The results for these trials are found in Table 6.6.

Unlike the previous trials, step size function $f_5(t)$ found the best solution for every single trial. Additionally, it successfully converged well before the maximum number of iterations was reached. Function $f_1(t)$ was only able to converge early three times, and $f_3(t)$ converged only once across all ten trials. Other than $f_5(t)$, no choice of step size function stood out as better or worse than the other choices. The total run time required for each step size function varied only with the total number of iterations performed.

Table 6.6: Results of optimizing the smallest embedded 3-simplex volume on randomly generated data points in $\mathbb{R}^{10}$ projected into 5 dimensions. The best result for each trial is highlighted in blue.

Trial	Step Size Function	Smallest Volume	Runtime (s)	Iterations
1	$f_0(t)$	1.7446E-05	90.7127	20000
	$f_1(t)$	1.3681E-04	91.2947	20000
	$f_2(t)$	6.3186E-05	92.5815	20000
	$f_3(t)$	3.7354E-04	83.0228	7052
	$f_4(t)$	3.8203E-05	91.0885	20000
	$f_5(t)$	1.7761E-03	79.8789	2849
	$f_6(t)$	1.3761E-05	92.5833	20000
2	$f_0(t)$	4.0247E-05	92.7566	20000
	$f_1(t)$	1.3995E-04	92.1234	20000
	$f_2(t)$	1.8438E-04	91.9336	20000
	$f_3(t)$	4.8282E-04	93.3776	20000
	$f_4(t)$	5.6218E-05	92.9046	20000
	$f_5(t)$	1.1447E-03	78.7144	785
	$f_6(t)$	2.8877E-05	92.7852	20000
3	$f_0(t)$	1.7402E-05	92.4601	20000
	$f_1(t)$	1.2145E-04	92.2271	20000
	$f_2(t)$	4.0625E-05	92.3978	20000
	$f_3(t)$	1.2044E-05	93.1695	20000
	$f_4(t)$	1.1300E-05	91.8556	20000
	$f_5(t)$	1.1602E-03	80.4410	1859
	$f_6(t)$	9.5844E-05	93.1214	20000
	$f_0(t)$	2.6754E-04	92.2074	20000

Trial	Step Size Function	Smallest Volume	Runtime (s)	Iterations
4	$f_1(t)$	1.5953E-04	92.7812	20000
	$f_2(t)$	8.9510E-05	93.2414	20000
	$f_3(t)$	1.7233E-04	92.6516	20000
	$f_4(t)$	2.7101E-04	91.6425	20000
	$f_5(t)$	1.2923E-03	79.5611	1163
	$f_6(t)$	7.6945E-05	91.6012	20000
5	$f_0(t)$	5.9132E-07	92.1399	20000
	$f_1(t)$	1.0566E-04	91.1065	20000
	$f_2(t)$	3.2684E-05	93.1065	20000
	$f_3(t)$	6.0456E-04	93.6256	20000
	$f_4(t)$	3.7997E-05	93.8258	20000
	$f_5(t)$	1.2948E-03	79.7474	2284
	$f_6(t)$	7.6628E-05	92.8107	20000
6	$f_0(t)$	5.4410E-05	93.0469	20000
	$f_1(t)$	6.9327E-05	93.8346	20000
	$f_2(t)$	1.0308E-04	93.3522	20000
	$f_3(t)$	4.1421E-04	93.6518	20000
	$f_4(t)$	3.3990E-05	92.8925	20000
	$f_5(t)$	1.7498E-03	79.9350	1422
	$f_6(t)$	1.3579E-05	93.5483	20000
7	$f_0(t)$	6.6867E-05	92.3821	20000
	$f_1(t)$	4.3941E-05	86.7210	10434
	$f_2(t)$	6.3109E-05	91.6682	20000
	$f_3(t)$	3.7153E-04	92.2503	20000
	$f_4(t)$	8.0410E-05	91.5303	20000
	$f_5(t)$	2.1672E-03	79.7740	1683

Trial	Step Size Function	Smallest Volume	Runtime (s)	Iterations
	$f_6(t)$	2.7114E-05	92.5498	20000
8	$f_0(t)$	5.0373E-05	91.3104	20000
	$f_1(t)$	2.8725E-04	85.2015	9948
	$f_2(t)$	1.5181E-04	91.4983	20000
	$f_3(t)$	2.6691E-04	92.1184	20000
	$f_4(t)$	1.6109E-05	91.6459	20000
	$f_5(t)$	6.5996E-04	78.5657	1020
	$f_6(t)$	2.7609E-04	93.0953	20000
9	$f_0(t)$	1.9136E-04	93.2194	20000
	$f_1(t)$	1.9982E-04	86.2590	12868
	$f_2(t)$	1.5776E-04	94.1523	20000
	$f_3(t)$	3.5379E-04	93.7588	20000
	$f_4(t)$	3.3964E-04	93.1239	20000
	$f_5(t)$	7.1673E-04	79.0646	851
	$f_6(t)$	3.6110E-05	93.2637	20000
10	$f_0(t)$	9.6712E-05	94.0770	20000
	$f_1(t)$	9.8519E-05	93.2205	20000
	$f_2(t)$	1.6186E-04	91.9525	20000
	$f_3(t)$	1.4281E-04	91.4732	20000
	$f_4(t)$	2.5027E-04	91.0292	20000
	$f_5(t)$	1.2473E-03	79.1721	2647
	$f_6(t)$	4.2094E-05	91.3836	20000

Computational Complexity

Initializing the Smallest Volume Optimizing algorithm requires five main computation steps: finding the initial volumes of all simplices, taking the SVD of all data points/simplex vertices, projecting the data into m dimensions, calculating the projected volumes, and finding the projected volume ratios for each simplex. After initialization, each iteration requires updating the projection matrix along a geodesic curve, projecting the data again, and recalculating the new simplex volumes and ratios. Let p be the number of data points, M be the total number of k-simplices defined by those data points, N be the total number of iterations, and $\mathbf{P} : \mathbb{R}^n \to \mathbb{R}^m$ be a mapping. Here, each step is addressed individually to obtain an estimate of computational complexity for the algorithm in terms of total number of required floating point operations (FLOPS).

Calculating the volume a single k-simplex S in $\mathbb{R}^n$ requires translating its matrix representation $\mathbf{S}$ to the origin, multiplication of $\mathbf{S}^T\mathbf{S}$, finding the determinant of the $k \times k$ matrix, computing the square root of the determinant, and dividing by a constant. The most computationally intensive steps are the matrix multiplication and determinant. Multiplying $\mathbf{S}^T\mathbf{S}$ requires $\mathcal{O}(k^2n)$ FLOPS. To calculate the determinant of a matrix, python utilizes the LAPACK routine based on LU factorization [37], which requires approximately $\mathcal{O}(\frac{2}{3}k^3)$ computations. Ignoring the lower order terms from the other steps, calculating the volume of a single simplex has complexity $\mathcal{O}(k^2n + \frac{2}{3}k^3)$. Given M total simplices, the total cost of calculating all volumes becomes $\mathcal{O}(Mk^2n + M\frac{2}{3}k^3)$.

The remaining steps of initialization are comparatively straightforward. Calculating the SVD of a $n \times p$ matrix requires $\mathcal{O}(np^2 + p^3)$ computations. Projecting the data requires the multiplication of a $m \times n$ matrix and an $n \times p$ which requires $\mathcal{O}(mnp)$ FLOPS. Recalculating projected volumes requires the same steps outlined in the previous paragraph, and thus adds $\mathcal{O}(Mk^2m + M\frac{2}{3}k^3)$ FLOPS. Calculating volume ratios is simply M total division operations, which is a low order term that is ultimately dropped.

Combining all these steps and dropping low order terms results in a complexity of $\mathcal{O}(Mk^2(n + m + \frac{4}{3}k) + p^2(n + p) + mnp)$ FLOPS. However, the number of simplices M is dependent on both p and k. Taking M to be the set of all possible combinations of $k+1$ vertices taken from the p data points yields

$$M = \binom{p}{k+1} = \frac{p!}{(k+1)!(p-k)!}.$$

This ratio increases factorially as $k+1$ approaches $\frac{p}{2}$. Replacing M with this formula gives a total of $\mathcal{O}(\frac{p!}{(k+1)!(p-k)!}k^2(n+m+\frac{4}{3}k)+p^2(n+p)+mnp)$ FLOPS required for initialization of the algorithm.

Each subsequent iteration of the algorithm requires updating k columns of the $m \times n$ matrix $\mathbf{P}$ along a geodesic, which was previously shown to require $\mathcal{O}(k^3+nk^2)$ in [40]. Projecting and recalculating the simplex volumes are identical to the steps discussed in the previous paragraph, requiring $\mathcal{O}(mnp)$ and $\mathcal{O}(Mk^2m+M\frac{2}{3}k^3)$ FLOPS, respectively. Combining all steps and substituting for M shows the total number of FLOPS required to be $\mathcal{O}(nk^2+mnp+\frac{p!}{(k+1)!(p-k)!}(k^2m+\frac{2}{3}k^3)$ for each iteration of the algorithm. However, since the full set of simplex volumes is only calculated for the first 100 iterations, the total cost of all iterations is‘ $\mathcal{O}(100[\frac{p!}{(k+1)!(p-k)!}(k^2m+\frac{2}{3}k)+k^3+nk+mnp]+(N-100)[5k^2(m+\frac{2}{3}k)+k^3+nk+mnp])$.

Combining the initialization steps with the iteration steps and eliminating lower order terms, the total number of FLOPS required for the volume preserving algorithm is

$$\mathcal{O}\left(\frac{p!}{(k+1)!(p-k)!}(k^2(n+m)+k^3)+M(k^3+k^2m+kn+mnp)\right).$$

The algorithm scales linearly in terms of M, m, and n. The biggest complicating factor is the total number of points p, which scales the complexity factorially, with the severity depending on the value of k. As k approaches $\frac{p}{2}-1$, significantly more simplices are generated from the data and the factorial complexity is even more severe. However, for a fixed number of simplices M, the cost of each iteration of the algorithm increases cubically with k itself.

6.3.2 Testing

To thoroughly test the efficacy of the Smallest Volume Optimizing algorithm and the preliminary conclusions drawn in Section 6.3.1, the algorithm was tested on a set of 40 randomly generated data points in $\mathbb{R}^{100}$. These data points were projected down into dimensions from 2 to 30 while optimizing areas of 1-,2-,3-, and 4-simplices. Additional trials were performed using 5-simplices, but as these raised additional complications that had to be addressed, they are discussed separately in Section 6.3.3. The total number of simplices used in optimization is 190 for $k = 1$, 1140 for $k = 2$, 4845 for $k = 3$, 1,5504 for $k = 4$, and 38,760 for $k = 5$.

All trials for $k \in \{1, 2, 3\}$ were tested using both random initialization and SVD initialization. For random initialization, each trial is run 30 times with different starting conditions, and the best result is kept from each (average results are also tracked for sake of thoroughness). Each combination of embedding dimension, simplex dimension, and initialization was tested with the seven previously discussed step size functions. Because SVD initialization was ultimately significantly more efficient than random initialization and the total number of simplices increased drastically for $k \in \{4, 5\}$, the random initialization tests were omitted for the 4- and 5-simplex trials, and only SVD was tested. The maximum number of iterations for each run is set at $30,000$. Iterations terminate when the change in smallest embedded volume between two consecutive iterations is less than $\delta = 10^{-12}$.

This section presents a summary of the outcome of this experiment with highlights that provide a good representation of the overall results. A table with the complete results from this experiment can be found in Appendix B.

Performance and Convergence

As before, random initialization consistently under-performed compared to SVD initialization. Using either $f_0(t)$ or $f_4(t)$ for for the step size function consistently yielded the best results in tandem with random initialization. These methods yielded the highest volumes for 1- and 2-simplices in embedding dimensions greater than 16 and for 3-simplices in dimensions greater than 21. These functions were comparable to other randomly initialized step size functions for lower dimensions for. The overall best function for random initialization when projecting into lower dimensions, however, appears to be $f_1(t)$. Figures 6.9, 6.11, and 6.13 show how the best embedded volume for each choice of k increases with projection dimension for random initialization and each choice of step size function.

By contrast, SVD initialization yielded significantly better results regardless of simplex dimension. The optimal step size function for SVD initialization was clearly $f_5(t)$, with $f_1(t)$ and $f_3(t)$

trailing behind slightly. Figure 6.10 shows that SVD initialization using $f_5(t)$ was able to embed all 1-simplices with normalized lengths above 0.8 in 29 and 30, and Figure 6.12 shows a similar result for 2-simplices, with all normalized areas above 0.6 for embeddings in 27 or greater. This function with SVD initialization was even able to break the 0.6 threshold for $k = 3$, shown in Figure 6.14. The step size functions $f_0(t)$, $f_4(t)$ and $f_6(t)$ consistently under-performed compared to the others. Functions $f_0(t)$ and $f_4(t)$ appear to function nearly identically under these initial conditions.

Appendix B contains the complete results of these tests, including greatest smallest volumes, average smallest volumes, and run times for each variation of the experiment.

Run Time

One major advantage of SVD initialization over random initialization is that the fixed starting point eliminates the need for repeated runs of the same trial. Because of this, the SVD trials for these experiments should be approximately 30 times faster than the randomly initialized runs. These experiments confirmed that not only did SVD initialization achieve this level of speed-up, it quite frequently surpassed it several times over. Table 6.7 displays some of the total required run times for a set of specific trials. The selected trials provide a good representation for the overall results when comparing times required for each initialization method. The average SVD trial is close to 30 times faster than the average randomly initialized trial. In some cases this ratio drops somewhat, as seen in the trial embedding 3-simplices into 4 dimensions where the SVD trial was in once case only 14 times faster than the random trial. However, SVD runs can also be hundreds of times faster than repeated random trials, which is seen several times in Table 6.7. In particular, when embedding 3-simplices into 3 dimensions SVD initialization consistently resulted in run times over 700 times faster than using random initialization.

6.3.3 Extremely Small Volumes

Initial attempts to test the algorithm on 5-simplices presented an unexpected problem: the smallest embedded volume ratios in the initial projection were on the order of 10^{-13} or even

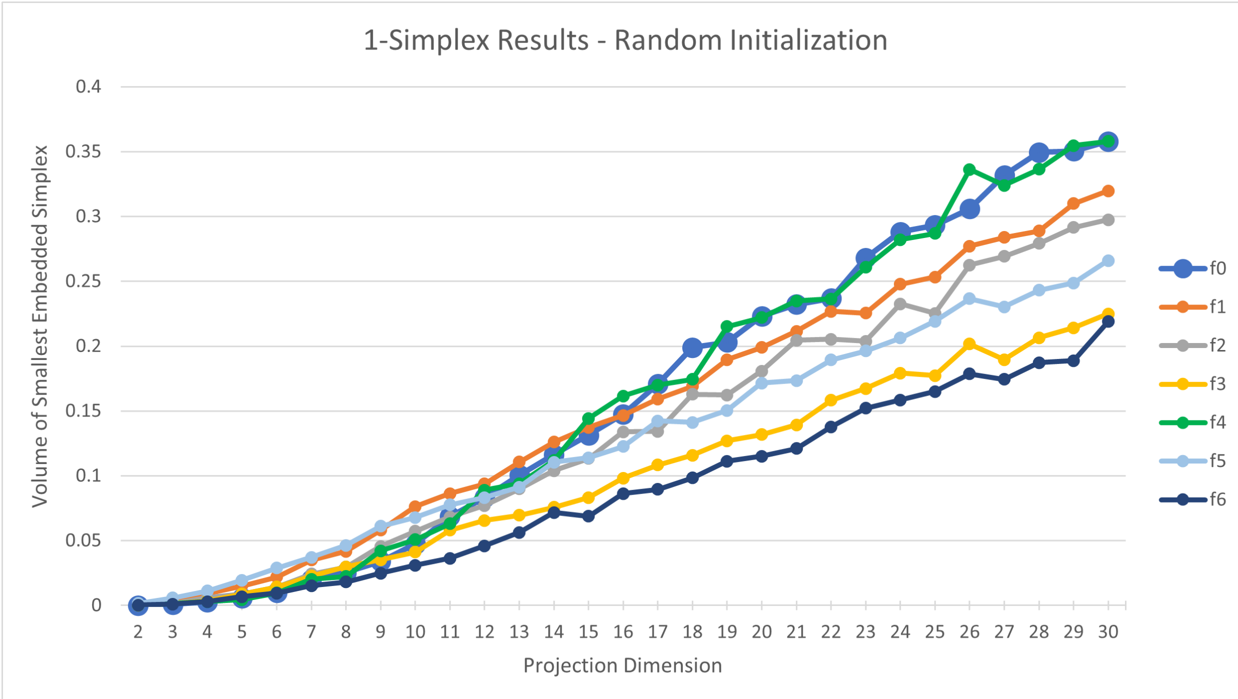


Figure 6.9: Greatest minimum 1-simplex lengths achieved at each projection dimension using random initialization for all seven versions of $f(t)$.

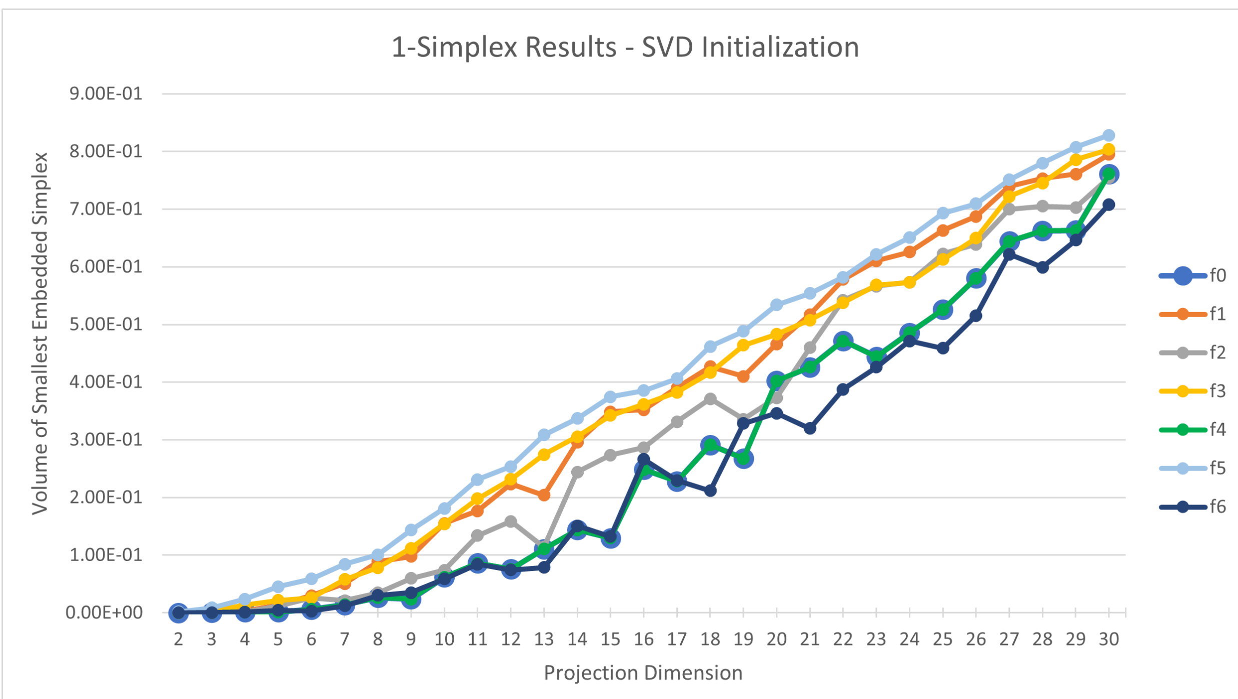


Figure 6.10: Greatest minimum 1-simplex lengths achieved at each projection dimension using SVD initialization for all seven versions of $f(t)$.

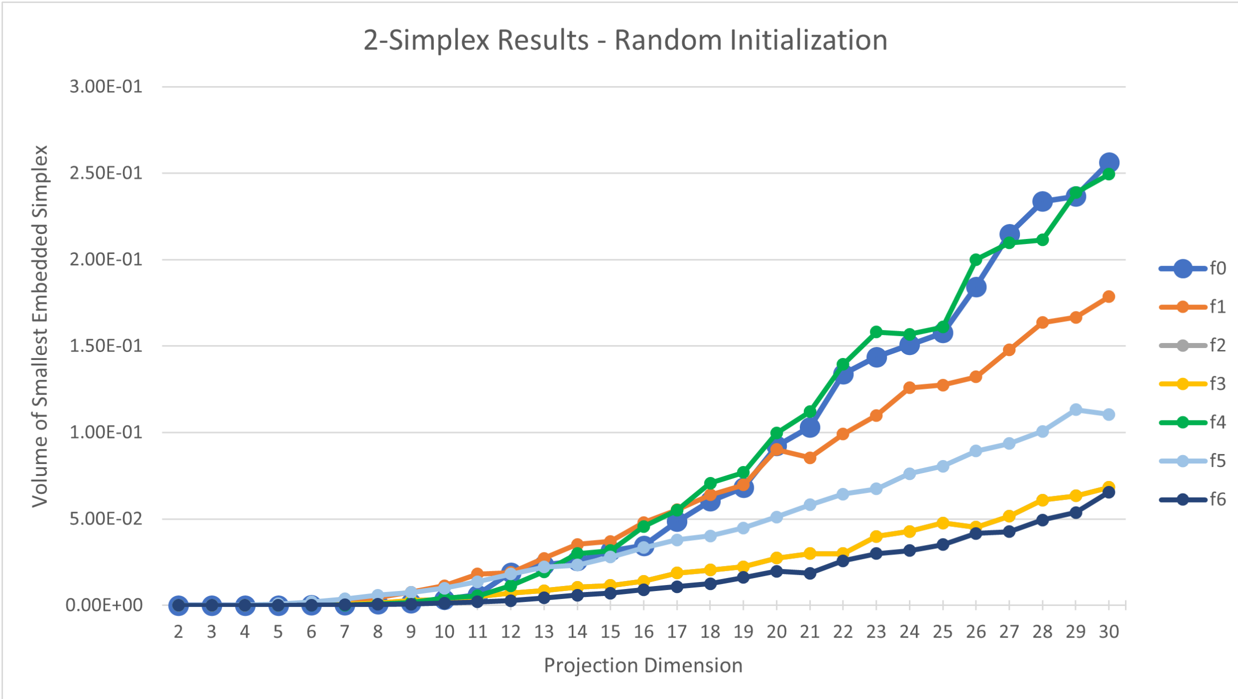


Figure 6.11: Greatest minimum 2-simplex areas achieved at each projection dimension using random initialization for all seven versions of $f(t)$.

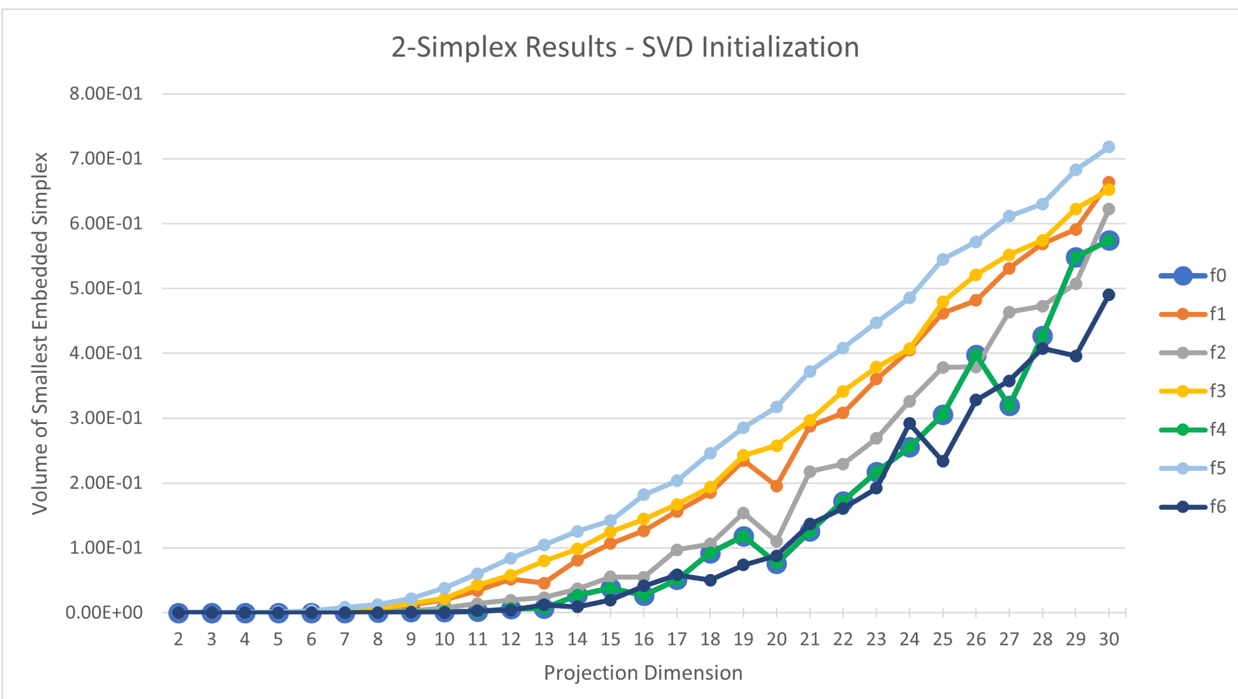


Figure 6.12: Greatest minimum 2-simplex areas achieved at each projection dimension using SVD initialization for all seven versions of $f(t)$.

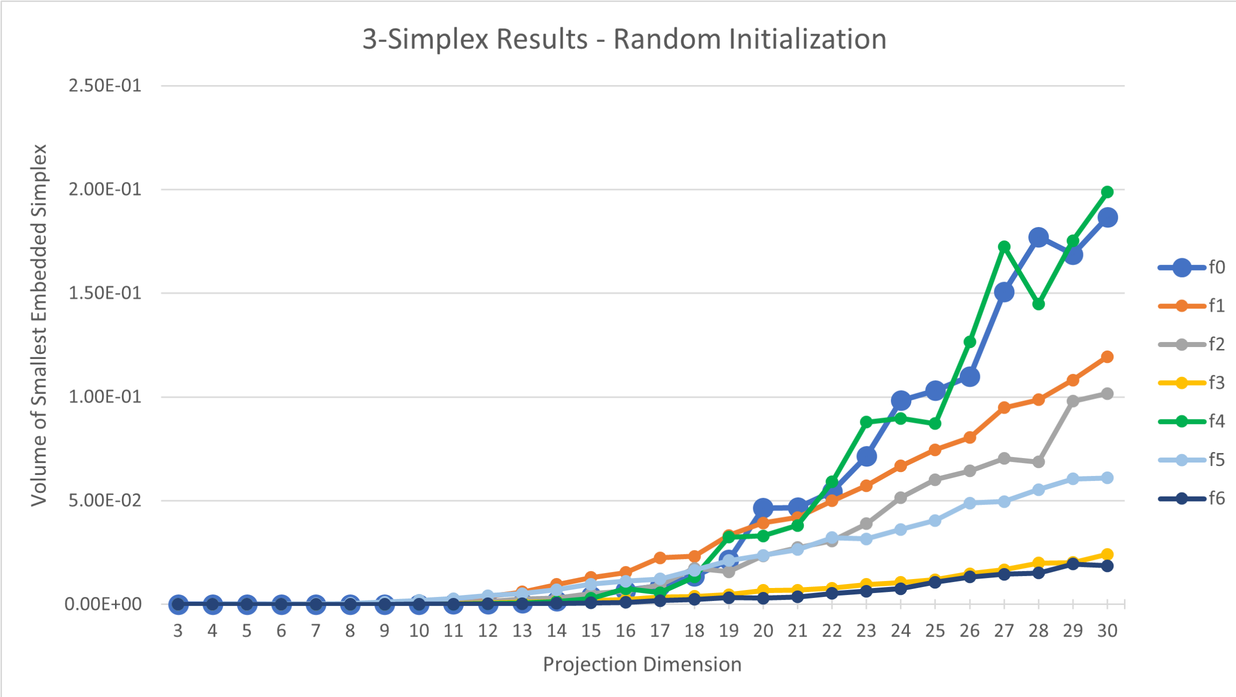


Figure 6.13: Greatest minimum 3-simplex volumes achieved at each projection dimension using random initialization for all seven versions of $f(t)$.

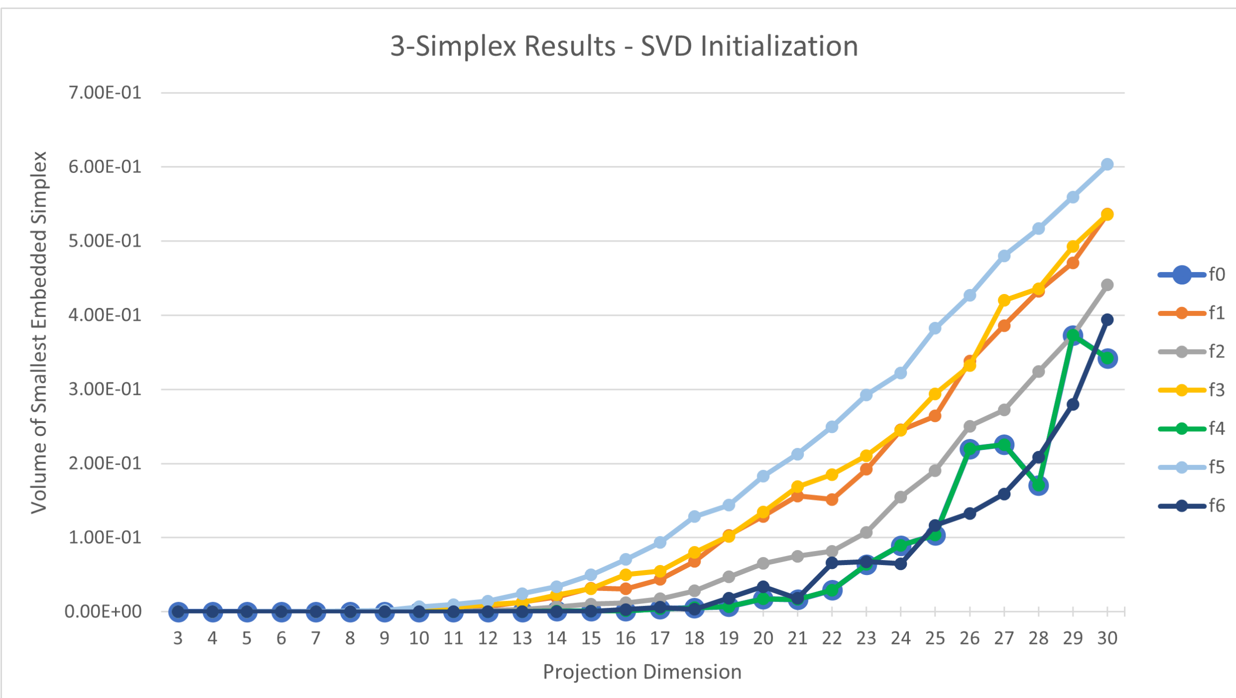


Figure 6.14: Greatest minimum 3-simplex volumes achieved at each projection dimension using SVD initialization for all seven versions of $f(t)$.

Table 6.7: Comparisons from selected trials of total run time in seconds of the volume optimizing algorithm using SVD and random initialization. Ratios in the far right column are the result of dividing the random run time by the SVD run time.

Simplex Dim	Projection Dim	$f(t)$	Random Run Time	SVD Run Time	Ratio
1	2	$f_0(t)$	1001.6531	34.9220	28.68
1	2	$f_1(t)$	685.8873	24.1438	28.41
1	2	$f_2(t)$	999.4515	33.2486	30.06
1	2	$f_3(t)$	972.7760	33.9604	28.64
1	2	$f_4(t)$	1006.7869	34.3436	29.32
1	19	$f_0(t)$	152.9216	5.2078	29.36
1	19	$f_1(t)$	598.9444	8.1451	73.53
1	19	$f_2(t)$	1036.6565	30.2088	34.32
1	19	$f_3(t)$	1059.0903	38.7061	27.36
1	19	$f_4(t)$	151.6034	5.2399	28.93
1	20	$f_0(t)$	147.4203	4.7892	30.78
1	20	$f_1(t)$	632.7700	15.2381	41.53
1	20	$f_2(t)$	1095.7552	38.3262	28.59
1	20	$f_3(t)$	1105.0453	38.0793	29.02
1	20	$f_4(t)$	159.3606	4.7541	33.52
2	2	$f_0(t)$	735.1851	4.7487	154.82
2	2	$f_1(t)$	732.9174	3.5241	207.97
2	2	$f_2(t)$	621.4996	3.4725	178.98
2	2	$f_3(t)$	780.0632	3.5024	222.72
2	2	$f_4(t)$	715.8448	4.8202	148.51
2	3	$f_0(t)$	3061.6525	105.2616	29.09
2	3	$f_1(t)$	2518.9251	78.3159	32.16
2	3	$f_2(t)$	3037.9897	104.0751	29.19
2	3	$f_3(t)$	2738.9458	106.4202	25.74
2	3	$f_4(t)$	3045.0340	106.9270	28.48
3	3	$f_0(t)$	19541.4350	25.2999	772.39
3	3	$f_1(t)$	18797.3812	25.8863	726.15
3	3	$f_2(t)$	19559.7836	25.4435	768.75
3	3	$f_3(t)$	18632.8862	25.4529	732.05
3	3	$f_4(t)$	19696.2348	26.0545	755.96
3	4	$f_0(t)$	14127.1275	675.0088	20.93
3	4	$f_1(t)$	15352.3442	669.0272	22.95
3	4	$f_2(t)$	15679.0812	683.5192	22.94
3	4	$f_3(t)$	7690.1252	176.2776	43.63
3	4	$f_4(t)$	9527.6128	673.6261	14.14
3	5	$f_0(t)$	19535.6036	672.7750	29.04
3	5	$f_1(t)$	19097.5546	662.6919	28.82
3	5	$f_2(t)$	19483.9576	672.0290	28.99
3	5	$f_3(t)$	18396.4891	646.8787	28.44
3	5	$f_4(t)$	19860.8580	676.6430	29.35

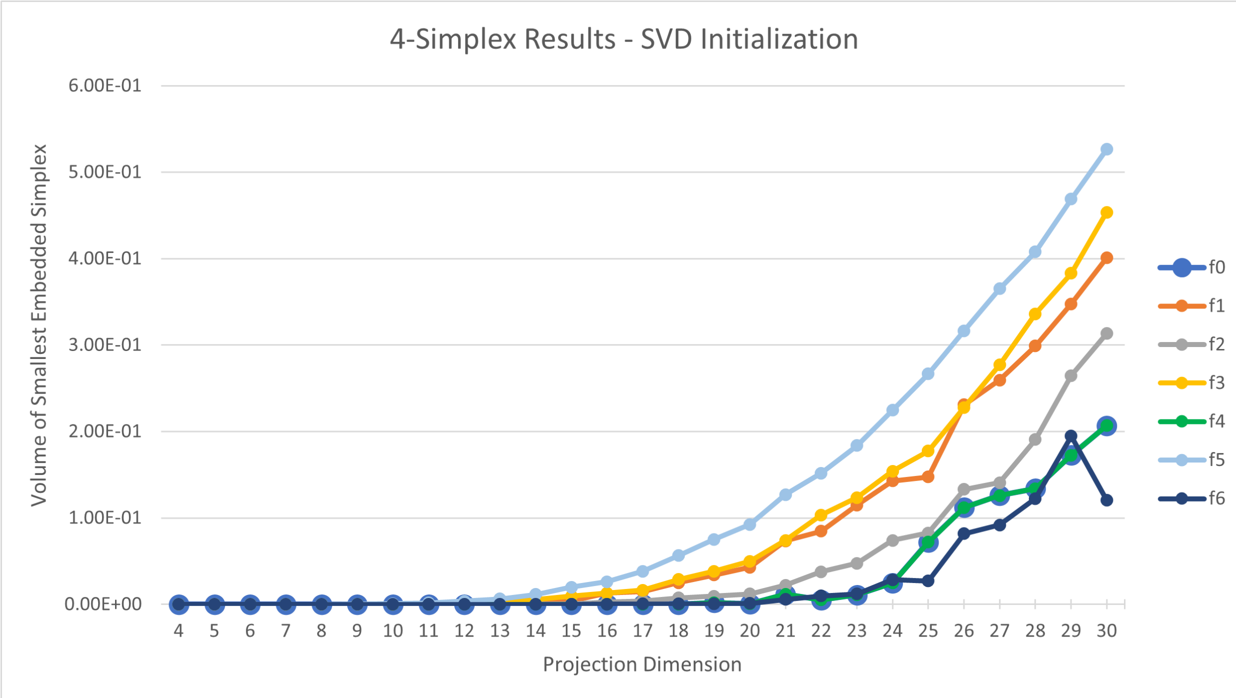


Figure 6.15: Greatest minimum 4-simplex volumes achieved at each projection dimension using SVD initialization for all seven versions of $f(t)$.

smaller. Since these volumes were smaller than the specified termination threshold, the algorithm terminated immediately. This occurred using both random and SVD initialization. An example of this is shown in Table 6.8.

Table 6.8: Some preliminary results from embedding 5-simplices into 5 dimensions. In all cases, the initial embedded volume ratios were less than 10^{-12} and the algorithm terminated after a single iteration.

Initialization	$f(t)$	**Best Smallest Vol**	**Avg Smallest Vol**	**Avg Iterations**
Random	$f_0(t)$	6.000450E-21	1.005870E-21	1.00 ± 0.00
Random	$f_1(t)$	4.864636E-21	6.024118E-22	1.00 ± 0.00
Random	$f_2(t)$	1.410167E-20	1.359919E-21	1.00 ± 0.00
Random	$f_3(t)$	1.479819E-20	7.447136E-22	1.00 ± 0.00
Random	$f_4(t)$	2.453245E-21	5.511043E-22	1.00 ± 0.00
SVD	$f_0(t)$	1.061767E-18	1.061767E-18	1.00 ± 0.00
SVD	$f_1(t)$	1.048675E-22	1.048675E-22	1.00 ± 0.00
SVD	$f_2(t)$	1.048675E-22	1.048675E-22	1.00 ± 0.00
SVD	$f_3(t)$	1.048675E-22	1.048675E-22	1.00 ± 0.00
SVD	$f_4(t)$	1.061767E-18	1.061767E-18	1.00 ± 0.00

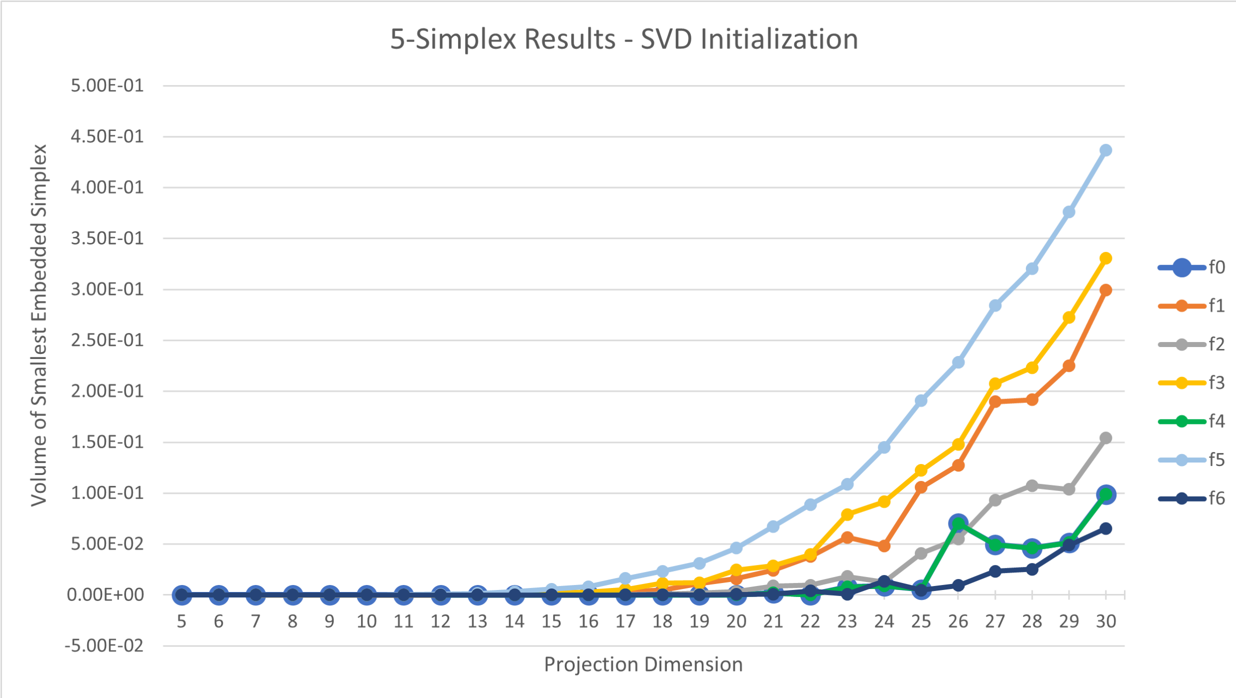


Figure 6.16: Greatest minimum 5-simplex volumes achieved at each projection dimension using SVD initialization for all five versions of $f(t)$.

Table 6.9: Results from embedding 5-simplices into 5 dimensions using the modified termination conditions (SVD only due to run times). Each trial performed better under the new conditions, and the algorithm no longer terminated prematurely due to small starting volumes.

Initialization	$f(t)$	Best Smallest Vol	Iterations
SVD	$f_0(t)$	2.91E-19	30000
SVD	$f_1(t)$	6.63E-20	4382
SVD	$f_2(t)$	5.37E-20	30000
SVD	$f_3(t)$	3.06E-21	675
SVD	$f_4(t)$	6.10E-21	30000

Although in theory this could happen for any simplex dimension given an extremely poor choice of starting condition, it likely did not happen for lower simplex dimensions because the probability of finding a projection bad enough was much smaller.

It is not really ideal to try circumventing this by making the termination threshold smaller without knowing ahead of time whether the embedded volume ratios will be small enough to cause this issue. Instead, the algorithm was updated to ignore the termination conditions if either of the current or previous smallest embedded volume ratios are smaller than 100δ, where δ is the selected termination threshold, in this case $\delta = 10^{-12}$. This forces the algorithm to continue until the smallest volumes are at least 10^{-10} and prevents it from getting stuck due to a poor starting conditions.

Figure 6.16 shows the results of repeating the SVD initialization trials for 5-simplices with the adjusted termination conditions. Overall performance results were comparable to those for other simplices. Additionally, Table 6.9 shows the new outcomes for embedding 5-simplices in 5 dimensions. SVD initialization was specifically used to exactly replicate the starting conditions for the same trials from Table 6.8 and directly compare outcomes. Regardless of the step size function used, applying the new termination conditions produced better optimization of smallest volumes and prevented the algorithm from terminating after only one step, although it is unclear how accurate these volumes are given how close they are to machine precision. Even so, making this adjustment prevents the algorithm from getting stuck solely due to extremely poor starting conditions, which becomes more notable as the embedding dimension increases.

6.3.4 Conclusions

Projecting data based on optimizing the smallest normalized volume yields very different results than those from PCA and will emphasize different aspects of the data in the projection. The primary drawback of this method is the inherent computational complexity that comes with generating k-simplices from a data set. Fortunately, this can be offset by taking advantage of the fact that after the first few iterations, the algorithm will bounce between only handful of the smallest simplices, meaning that the majority of the simplices can be ignored once this point is reached.

Based on the trials in this chapter, it seems the ideal approach to Smallest Volume Optimizing projections is to use the SVD of the data points as an initial starting condition. This consistently yielded better optimizations along with faster results than random initialization. The best choices of step size function for this algorithm was overwhelmingly $f_5(t)$, although there were several cases where it did not perform as well. The best step size function likely depends on the data being projected as well as the chosen initial conditions. The changes in smallest normalized simplex volumes in Figure 6.7 suggest that both $f_2(t)$ and $f_3(t)$ can have issues with convergence due to reducing the step size too quickly, and the oscillations displayed by $f_0(t)$ mark it as a very poor choice. Fortunately, since none of the choices of step size function significantly change the computational cost, all functions can easily be tested for the remaining experiments in this work.

6.4 Optimizing Volumes vs. Distances

One important question about this problem is whether optimally embedding volumes or areas is truly any different than optimally embedding pairwise distances. Intuitively, it seems like preserving volumes should be different than preserving lengths. This section seeks to concretely demonstrate the differences using the Secant Avoiding Projections (SAP) algorithm and the volume optimizing algorithm (VOL).

In this experiment, a subset of the MNIST data points used for testing are embedded into 2 dimensions using both the Secant Avoiding Projector and the Smallest Volume Optimizing Projector. This is done to allow clear visualization of the differences in embeddings based on the algorithm used. The volumes optimized by the Smallest Volume Optimizing Projector are those of 2-simplices, i.e. areas. Both functions are run for 1000 iterations using step size function $f_6(t)$.

For this trial, three points are initially selected from the data and embedded into 2 dimensions using both algorithms. A fourth point is then added, and the embeddings are performed again. This is repeated, adding one point at a time until there are 11 points total.

Figure 6.17 shows the embeddings using four (top row), nine (middle row), and ten (bottom row) points. Each data point is assigned the same color label in every plot. Note that even with only four points - embedding a single tetrahedron into 2 dimensions - there is a significant difference in the configurations found by the two algorithms. Here, the Smallest Volume Optimizing Projector places point 1 inside the convex hull of the other three points, but the Secant Avoiding Projector places it outside, suggesting that the two algorithms projected completely different orientations of same the tetrahedron. These differences become more and more noticeable as additional points are added. This clearly demonstrates that optimizing embedded areas is quite different from optimizing embedded lengths.

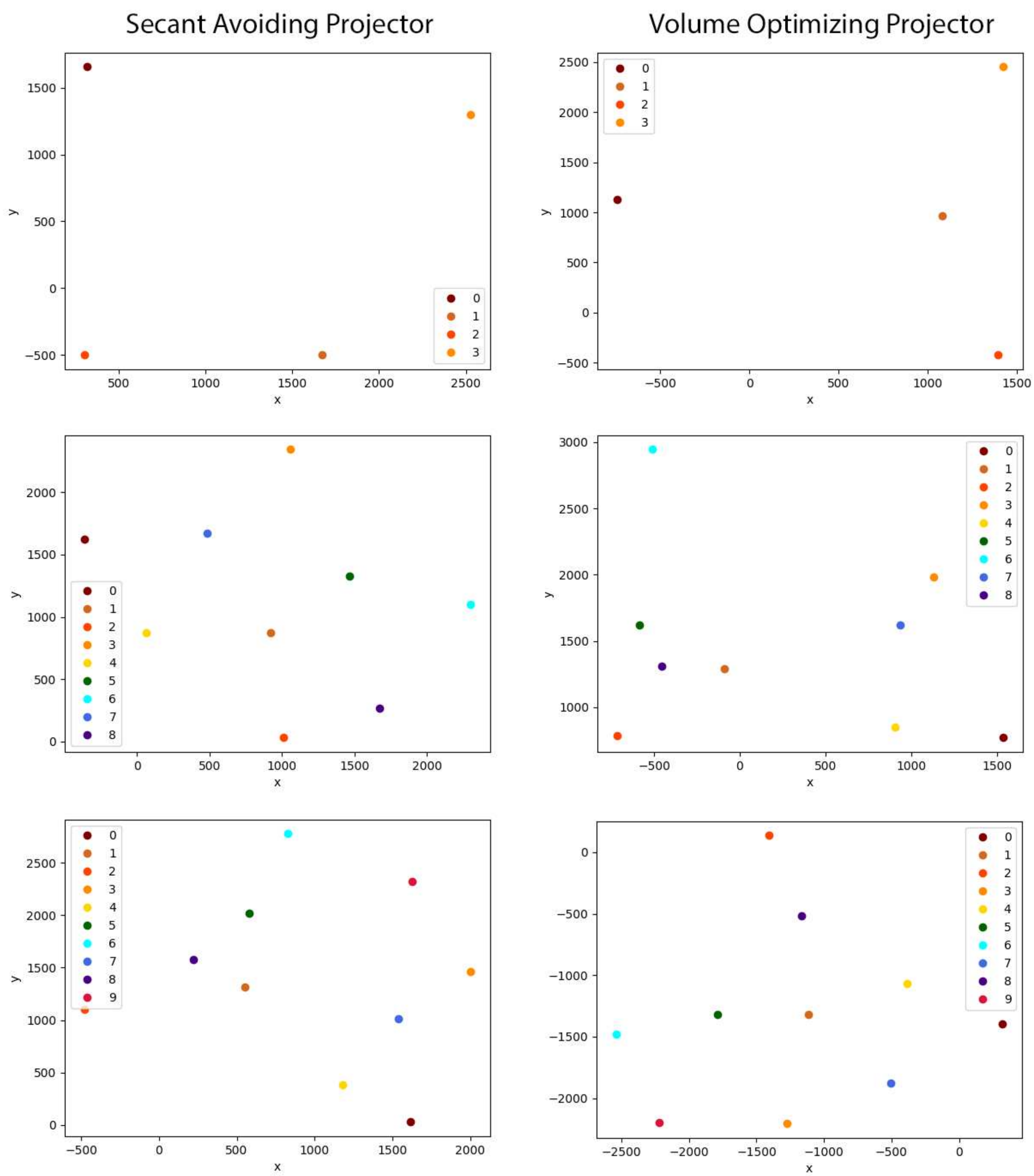


Figure 6.17: Points embedded into 2 dimensions optimizing pairwise distances with the Secant Avoiding Projector (left) and areas with the Smallest Volume Optimizing Projector (right).

Numerical results from this trial are shown in Table 6.10. In all but one case, the Smallest Volume Optimizing Projector (VOL) found a better embedding than the Secant Avoiding Projector (SAP) in terms of smallest embedded volumes. Additionally, as the number of data points increased, the performance of SAP dropped drastically compared to that of VOL. Even in the case where SAP performed better, the margin between the two algorithms was quite small. Clearly, when seeking to optimally embed areas, it is better to optimize over areas of 2-simplices than to only consider pairwise lengths between data points.

Table 6.10: Comparison of smallest embedded areas using the Smallest Volume Optimizing algorithm and the Secant Avoiding Projection algorithm on varying numbers of data points.

Num Data Pts	Algorithm	Best Small Area	Best Avg Area	Run Time (s)
3	VOL	1.00000000	1.00000000	22.18
	SAP	0.99999147	0.99999147	4.31
4	VOL	0.13830144	0.34892340	11.25
	SAP	0.07639997	0.36517700	0.68
5	VOL	0.02960945	0.21338982	11.39
	SAP	0.03151442	0.23587668	0.76
6	VOL	0.02587124	0.15191625	11.63
	SAP	0.01618232	0.16437337	0.73
7	VOL	0.01376501	0.10448343	11.70
	SAP	0.00000590	0.13422930	0.66
8	VOL	0.00682119	0.10540343	12.32
	SAP	0.00040259	0.09916718	0.68
9	VOL	0.00145510	0.08563916	13.16
	SAP	0.00000261	0.06700051	0.78
10	VOL	0.00052030	0.07749009	14.21
	SAP	0.00010461	0.08650237	0.73
11	VOL	0.00082291	0.08027737	15.47
	SAP	0.00000694	0.07708892	0.68

Chapter 7

Maximizing Average Volume

The second version of the algorithm seeks to maximize the average simplex volume distortion. To reiterate, given a set $\mathcal{S} = \{S_1, ..., S_N\} \in \mathbb{R}^n$ of N k-simplices and their matrix representations $\mathbf{S}_1, ..., \mathbf{S}_N$, the goal is to find the m-dimensional subspace $[\mathbf{P}]$ and corresponding orthogonal matrix $\mathbf{P}$ that solves the objective function

$$\mathbf{P}^* = \arg\underset{\mathbf{P}}{\text{maximize}} \sum_{i=1}^{N} \frac{vol(\mathbf{P}\mathbf{S}_i)}{vol(\mathbf{S}_i)}. \tag{7.1}$$

However, this objective function is not directly used for the algorithm developed in this chapter. Instead, an alternate method of normalizing volumes is presented.

Rather than normalizing volumes after projection, the k-simplices are instead normalized by considering the subspace spanned by their vertices after translation to the origin. Let $\bar{\mathbf{S}}_i$ denote the S_i after applying Gram-Schmidt orthonormalization. The optimization problem then becomes

$$\mathbf{P}^* = \arg\underset{\mathbf{P}}{\text{maximize}} \sum_{i=1}^{M} vol(\mathbf{P}\bar{\mathbf{S}}_i) \tag{7.2}$$

As previously stated, the volume of a single k-simplex S is best preserved by projecting onto a subspace $[\mathbf{P}]$ such that $[\mathbf{S}] \subseteq [\mathbf{P}]$. It follows that when considering multiple simplices, the projection space $[\mathbf{P}]$ that best preserves the normalized volumes of all the simplices S_i for $i \in \{1, ..., M\}$ will be as near as possible in terms of principal angles to all of the simplex subspaces $[\mathbf{S}_i]$.

Because the chordal distance metric from Equation (4.5) measures distances between subspaces in terms of principal angles, we can formulate a new optimization problem in terms of minimizing the chordal distances between $[\mathbf{P}]$ and all $[\mathbf{S}_i]$ as follows:

$$[\mathbf{P}^*] = \arg\underset{[\mathbf{P}]}{\text{minimize}} \sum_{i=1}^{M} d_c([\mathbf{P}], [\mathbf{S}_i])^2. \tag{7.3}$$

or equivalently

$$\mathbf{P}^* = \arg\underset{\mathbf{P}}{\text{maximize}} \sum_{i=1}^{M} tr(\mathbf{P}^T \mathbf{S}_i \mathbf{S}_i^T \mathbf{P}). \tag{7.4}$$

This is the same optimization problem solved by the m-dimensional flag mean (refer to Sections 4.1.3 and 4.2.2). By this reasoning, choosing $[\mathbf{P}]$ to be the m-dimensional flag mean of the simplex subspaces $[\mathbf{S}_i]$ will provide the optimal solution to Equation (7.2).

This idea is first developed for the special case where $m = k$, and then extended to the general case of optimally projecting k-simplices into an m-dimensional subspace for any $m \geq k$.

7.1 Maximum Average k-Volume Projected Into k Dimensions

When optimizing volumes of k-simplices embedded into k dimensions, we want to find the k-dimensional subspace $[\mathbf{P}]$ that averages the set of M k-dimensional subspaces $[\mathbf{S}_i]$. This is done by taking $[\mathbf{P}]$ to be the k-dimensional flag mean. Algorithm 5 outlines the process.

Algorithm 5: Optimizing Average k-Volumes in k Dimensions

1. Define set of vertices of all k-dimensional simplices $\mathcal{S} = \{x_0, x_1, ..., x_k | x_i \in X \; \forall \, i, x_i \neq x_j \; \forall \, i \neq j\}$
2. Generate the orthonormal k-dimensional matrix representations $\mathbf{S}_i$ spanned by the vertices of each simplex S_i
3. Calculate the k-dimensional flag mean $[P]$ of all subspaces $[S_i]$
4. Embed data into k dimensions using the orthonormal matrix representation $\mathbf{P}$ of $[P]$

Note in this particular case, both $[\mathbf{P}]$ and all subspaces $[\mathbf{S}_i]$ can be treated as points in $Gr(k, n)$. Using Equation (4.26), rewrite the optimization problem in Equation (7.3) as

$$[\mathbf{P}^*] = \arg\underset{[\mathbf{P}]}{\text{minimize}} \sum_{i=1}^{M} \Theta([\mathbf{P}], [\mathbf{S}_i]) \tag{7.5}$$

which is identical to the objective function for the gradient descent algorithm described in Section 5.3. This allows for numerical verification that the flag mean solves 7.3 by comparing it with the results of the gradient descent algorithm.

7.1.1 Numerical Testing

To numerically test the hypothesis that Equation (7.3) is equivalent to 7.2 and that the k-dimensional flag is in fact the optimal projection space for maximizing the average of volumes of k-simplices in k dimensions, data was projected onto random perturbations of the flag mean subspace $[\mathbf{P}]$ as well as onto $[\mathbf{P}]$ itself and the average normalized projected volumes were compared. For this test, sets of 10 randomly generated points in $\mathbb{R}^n$ for each $n \in \{10, 20, 50, 100\}$ are used to generate sets of k-simplices for each $k \in \{1, 2, 3, 4, 5\}$. These points are projected onto the k-dimensional flag mean of the simplex subspaces as well as onto 10 random perturbations of the k-dimensional flag mean. Perturbations $[\mathbf{P}_j]$ of $[\mathbf{P}]$ are generated using its matrix representation $\mathbf{P}$ according to

$$\mathbf{P}_j = \mathbf{P} + \epsilon \mathbf{Y} \tag{7.6}$$

where $\mathbf{Y}$ is a randomly generated $n \times k$ matrix and $\epsilon = 0.01$. The resulting matrix $\mathbf{P}_i$ is then orthonormalized for use in projection. Normalized k-simplex volumes are calculated for each projection.

Table 7.1 shows the results of these trials. In each case, the average normalized embedded volume of all simplices was greatest using the flag mean projection, supporting the hypothesis that taking the average of all subspaces generated using the k-simplices gives the optimal solution to Equation (7.2).

Table 7.1: Comparison of average normalized projected volumes of k-simplices embedded from $\mathbb{R}^n$ into $\mathbb{R}^k$ using the k-dimensional flag mean of all simplex subspaces and 10 k-dimensional random perturbations of the flag mean subspace. Best average volume for each trial is in blue.

n	k	**Flag Mean**	**Proj 1**	**Proj 2**	**Proj 3**	**Proj 4**	**Proj 5**	**Proj 6**	**Proj 7**	**Proj 8**	**Proj 9**	**Proj 10**
10	1	0.27875	0.27868	0.27870	0.27865	0.27871	0.27867	0.27870	0.27867	0.27869	0.27870	0.27872
10	2	0.13654	0.13648	0.13646	0.13647	0.13643	0.13650	0.13650	0.13648	0.13647	0.13649	0.13648
10	3	0.12101	0.12088	0.12092	0.12093	0.12091	0.12088	0.12089	0.12090	0.12093	0.12093	0.12088
10	4	0.11877	0.11867	0.11868	0.11869	0.11867	0.11870	0.11863	0.11874	0.11863	0.11870	0.11866
10	5	0.09953	0.09948	0.09949	0.09949	0.09946	0.09949	0.09951	0.09949	0.09950	0.09951	0.09947
20	1	0.20131	0.20119	0.20121	0.20122	0.20122	0.20123	0.20126	0.20125	0.20119	0.20118	0.20124
20	2	0.11098	0.11085	0.11086	0.11087	0.11088	0.11086	0.11086	0.11080	0.11083	0.11087	0.11083
20	3	0.05797	0.05789	0.05788	0.05789	0.05789	0.05790	0.05787	0.05788	0.05788	0.05789	0.05791
20	4	0.06610	0.06590	0.06591	0.06591	0.06594	0.06593	0.06593	0.06595	0.06594	0.06594	0.06592
20	5	0.08837	0.08822	0.08821	0.08822	0.08822	0.08821	0.08820	0.08821	0.08820	0.08817	0.08820
50	1	0.18559	0.18537	0.18538	0.18536	0.18535	0.18539	0.18535	0.18529	0.18534	0.18532	0.18535
50	2	0.06608	0.06593	0.06591	0.06589	0.06591	0.06590	0.06589	0.06588	0.06591	0.06590	0.06589
50	3	0.03797	0.03781	0.03777	0.03780	0.03782	0.03781	0.03780	0.03780	0.03780	0.03778	0.03782
50	4	0.02240	0.02227	0.02228	0.02228	0.02229	0.02227	0.02228	0.02227	0.02228	0.02228	0.02227
50	5	0.03113	0.03095	0.03092	0.03095	0.03092	0.03095	0.03096	0.03095	0.03097	0.03094	0.03096
100	1	0.16570	0.16518	0.16514	0.16520	0.16518	0.16518	0.16520	0.16516	0.16523	0.16515	0.16506

n	k	**Flag Mean**	**Proj 1**	**Proj 2**	**Proj 3**	**Proj 4**	**Proj 5**	**Proj 6**	**Proj 7**	**Proj 8**	**Proj 9**	**Proj 10**
100	2	0.05442	0.05410	0.05409	0.05408	0.05409	0.05408	0.05410	0.05410	0.05409	0.05410	0.05410
100	3	0.02865	0.02842	0.02841	0.02842	0.02841	0.02840	0.02838	0.02838	0.02839	0.02840	0.02839
100	4	0.01939	0.01915	0.01915	0.01915	0.01915	0.01912	0.01915	0.01911	0.01915	0.01916	0.01912
100	5	0.02124	0.02095	0.02095	0.02091	0.02095	0.02092	0.02093	0.02096	0.02093	0.02092	0.02093

Additionally, comparison with the gradient descent algorithm showed that both algorithms found the same optimum for Equation (7.2). Figure 7.1 shows the same set of points projected by each algorithm into 2 dimensions while maximizing the area of 2-simplices. Since the optimal projection subspace has infinitely many possible orthonormal matrix representations, the configuration of the projected points can be rotated or flipped, but their relative positions remain the same. This result is part of further similar tests, shown in Table 7.3. Overall, numerical evidence supports the conclusion that the flag mean of subspaces yields optimal normalized embedded volumes.

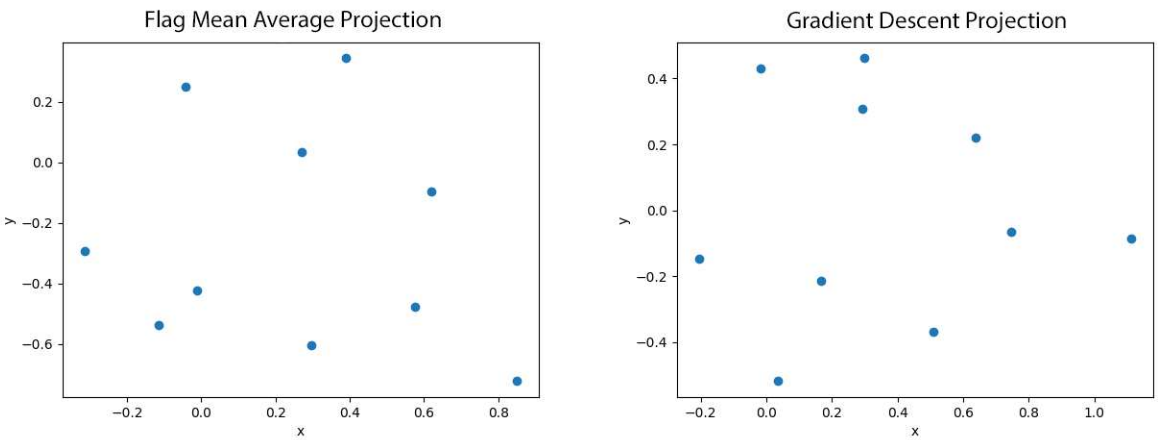


Figure 7.1: Projections maximizing the average area of 2-simplices embedded in 2 dimensions. Note that both algorithms produce the same configuration of points up to reflective/rotational symmetry.

7.2 Maximum Average k-Volume Projected Into m Dimensions

The general case becomes slightly more complicated. Optimally embedding k-simplices into an m-dimensional space when $m > k$ requires the m-dimensional subspace that best describes a set of k dimensional subspaces.

It is reasonable to assume that such a subspace would contain the k-dimensional average of the simplex subspaces. If $[\mathbf{V}]$ is the k-dimensional flag mean of $[\mathbf{S}_1], ..., [\mathbf{S}_M]$, the subspace $[\mathbf{V}]$ must be extended by $m - k$ dimensions while still minimizing the principal angles between $[\mathbf{V}]$ and the $[\mathbf{S}_i]$. This requires somehow measuring the distance between these subspaces in $Gr(m, n)$.

To consider distances between subspaces in $Gr(m,n)$, take each k-dimensional $[\mathbf{S}_i] \subset \mathbb{R}^n$ and consider all possible m-dimensional subspaces in $\mathbb{R}^n$ containing them. This is the set of Schubert varieties $\Omega_{[\mathbf{S}_i],k} \in Gr(m,n)$ with $i = 1, ..., N$. Thus $[\mathbf{V}]$ must be extended to some $[\mathbf{P}] \in Gr(m,n)$ that minimizes the distances to each Schubert variety. The objective function for this optimization problem then becomes

$$\arg\min_{[\mathbf{P}]} \sum_{i=1}^{k} d_c(\Omega_{[\mathbf{S}_i],a}, [\mathbf{P}])^2. \tag{7.7}$$

It was previously demonstrated that the flag mean solves this minimization problem in Section 4.2.2. Because of this, the algorithm for the general case is nearly identical to Algorithm 5, with the only difference being the dimension of the flag mean subspace used for projection. For the sake of completeness, this version of the algorithm is outlined in Algorithm 6 below.

Algorithm 6: Average k-Volume Optimizer in m Dimensions

1. Define set of vertices of all k-dimensional simplices $\mathcal{S} = \{x_0, x_1, ..., x_k | x_i \in X \ \forall\, i, x_i \neq x_j \ \forall\, i \neq j\}$
2. Generate the orthonormal k-dimensional matrix representations $\mathbf{S}_i$ spanned by the vertices of each simplex S_i
3. Calculate the m-dimensional flag mean $[\mathbf{P}]$ of all subspaces $[S_i]$
4. Project data into m dimensions using the orthonormal matrix representation $\mathbf{P}$

It is worth noting, however, that the maximum possible dimension of the flag mean is equal to the dimension of the span of all subspaces being averaged. It cannot therefore be used to project data into a dimension higher than what the affine spaces associated with each simplex inherently span, though it is unlikely that anyone would ever want to do that anyway.

7.2.1 Numerical Testing

As before, sets of 10 randomly generated points in $\mathbb{R}^n$ for $n \in \{10, 20, 50, 100\}$ were used to generate sets of k-simplices for $k \in \{1, 2, 3, 4, 5\}$. However, in this case the data was embedded into m-dimensional subspaces for $m \in \{2, 3, 5, 10\}$ (provided $n > m$) while optimizing average normalized k simplex volumes for $k \geq m$. Results of this trial are shown in Table 7.2. There were only two cases where the flag mean projection did not outperform the epsilon shifted projections. These trials had $n = 10$ and either $m = k = 2$ or $m = k = 3$. However, both of these trials were performed on different sets of randomly generated points in the previous section, where the flag mean again outperformed all other projections, so it is very likely this is due to some numerical issue. Overall, the flag mean consistently found the best projection, and in particular was able to find a perfect configuration of the data when the embedding dimension was large enough, which none of the perturbations were able to accomplish.

Table 7.2: Comparison of average projected volume ratios of k-simplices projected from $\mathbb{R}^n$ down to m dimensions using the m-dimensional flag mean of all simplex subspaces and 10 k-dimensional random perturbations of the flag mean subspace. Best average volume for each trial is in blue.

n	m	k	**Flag Mean**	**Proj 1**	**Proj 2**	**Proj 3**	**Proj 4**	**Proj 5**	**Proj 6**	**Proj 7**	**Proj 8**	**Proj 9**	**Proj 10**
10	2	2	0.12463	0.12474	0.12471	0.12471	0.12470	0.12471	0.12473	0.12469	0.12474	0.12467	0.12471
10	3	2	0.38807	0.38786	0.38782	0.38789	0.38781	0.38792	0.38781	0.38789	0.38793	0.38784	0.38789
10	3	3	0.08746	0.08750	0.08746	0.08745	0.08742	0.08742	0.08736	0.08741	0.08737	0.08745	0.08740
10	5	2	0.80711	0.80684	0.80692	0.80688	0.80691	0.80690	0.80679	0.80688	0.80671	0.80682	0.80680
10	5	3	0.64737	0.64701	0.64683	0.64698	0.64693	0.64686	0.64697	0.64684	0.64693	0.64698	0.64688
10	5	4	0.35346	0.35327	0.35332	0.35324	0.35330	0.35330	0.35332	0.35332	0.35336	0.35335	0.35331
10	5	5	0.17490	0.17463	0.17467	0.17463	0.17475	0.17464	0.17465	0.17463	0.17468	0.17464	0.17469
20	2	2	0.12197	0.12178	0.12182	0.12182	0.12185	0.12176	0.12175	0.12182	0.12181	0.12183	0.12177
20	3	2	0.28558	0.28530	0.28531	0.28534	0.28533	0.28529	0.28531	0.28534	0.28536	0.28531	0.28527
20	3	3	0.06286	0.06278	0.06278	0.06280	0.06280	0.06280	0.06278	0.06284	0.06280	0.06281	0.06279
20	5	2	0.64991	0.64939	0.64934	0.64941	0.64938	0.64939	0.64931	0.64935	0.64937	0.64933	0.64937
20	5	3	0.37236	0.37189	0.37177	0.37168	0.37163	0.37178	0.37172	0.37183	0.37175	0.37177	0.37175
20	5	4	0.25813	0.25775	0.25766	0.25754	0.25776	0.25763	0.25757	0.25770	0.25770	0.25764	0.25771
20	5	5	0.06152	0.06142	0.06142	0.06141	0.06142	0.06141	0.06139	0.06140	0.06144	0.06143	0.06141
20	10	2	1.00000	0.99916	0.99925	0.99914	0.99919	0.99922	0.99918	0.99921	0.99912	0.99914	0.99921
20	10	3	1.00000	0.99877	0.99868	0.99858	0.99879	0.99869	0.99889	0.99870	0.99860	0.99878	0.99856

n	m	k	**Flag Mean**	**Proj 1**	**Proj 2**	**Proj 3**	**Proj 4**	**Proj 5**	**Proj 6**	**Proj 7**	**Proj 8**	**Proj 9**	**Proj 10**
20	10	4	1.00000	0.99893	0.99871	0.99882	0.99864	0.99882	0.99883	0.99907	0.99905	0.99888	0.99890
20	10	5	1.00000	0.99800	0.99811	0.99761	0.99809	0.99764	0.99773	0.99792	0.99828	0.99797	0.99800
50	2	2	0.07548	0.07526	0.07519	0.07526	0.07526	0.07526	0.07525	0.07527	0.07524	0.07525	0.07528
50	3	2	0.17810	0.17763	0.17764	0.17758	0.17771	0.17761	0.17766	0.17762	0.17759	0.17756	0.17762
50	3	3	0.03256	0.03237	0.03239	0.03237	0.03237	0.03239	0.03235	0.03239	0.03238	0.03238	0.03240
50	5	2	0.45668	0.45531	0.45535	0.45536	0.45543	0.45522	0.45532	0.45537	0.45537	0.45518	0.45535
50	5	3	0.27411	0.27309	0.27286	0.27289	0.27300	0.27312	0.27303	0.27300	0.27309	0.27297	0.27298
50	5	4	0.11562	0.11498	0.11498	0.11501	0.11491	0.11497	0.11494	0.11495	0.11496	0.11494	0.11492
50	5	5	0.02625	0.02608	0.02607	0.02606	0.02606	0.02605	0.02605	0.02606	0.02606	0.02607	0.02607
50	10	2	1.00000	0.99739	0.99774	0.99746	0.99744	0.99745	0.99743	0.99738	0.99739	0.99751	0.99752
50	10	3	1.00000	0.99653	0.99659	0.99633	0.99618	0.99630	0.99601	0.99614	0.99595	0.99601	0.99621
50	10	4	1.00000	0.99471	0.99437	0.99483	0.99428	0.99490	0.99434	0.99482	0.99446	0.99433	0.99497
50	10	5	1.00000	0.99386	0.99335	0.99305	0.99337	0.99346	0.99382	0.99300	0.99369	0.99308	0.99317
100	2	2	0.05151	0.05125	0.05120	0.05122	0.05118	0.05119	0.05123	0.05118	0.05120	0.05121	0.05124
100	3	2	0.13973	0.13879	0.13883	0.13880	0.13878	0.13887	0.13882	0.13878	0.13880	0.13876	0.13874
100	3	3	0.02840	0.02813	0.02813	0.02814	0.02814	0.02813	0.02816	0.02812	0.02812	0.02814	0.02815
100	5	2	0.39523	0.39285	0.39279	0.39280	0.39273	0.39301	0.39286	0.39291	0.39285	0.39293	0.39272
100	5	3	0.19491	0.19321	0.19325	0.19312	0.19309	0.19311	0.19315	0.19317	0.19314	0.19312	0.19321

n	m	k	**Flag Mean**	**Proj 1**	**Proj 2**	**Proj 3**	**Proj 4**	**Proj 5**	**Proj 6**	**Proj 7**	**Proj 8**	**Proj 9**	**Proj 10**
100	5	4	0.08356	0.08252	0.08255	0.08255	0.08258	0.08257	0.08256	0.08259	0.08253	0.08261	0.08246
100	5	5	0.02094	0.02061	0.02059	0.02062	0.02059	0.02056	0.02060	0.02062	0.02059	0.02062	0.02060
100	10	2	1.00000	0.99349	0.99402	0.99400	0.99395	0.99374	0.99409	0.99346	0.99371	0.99374	0.99420
100	10	3	1.00000	0.99160	0.99137	0.99138	0.99148	0.99142	0.99171	0.99151	0.99146	0.99122	0.99156
100	10	4	1.00000	0.98843	0.98832	0.98814	0.98818	0.98818	0.98914	0.98901	0.98846	0.98844	0.98857
100	10	5	1.00000	0.98671	0.98652	0.98624	0.98690	0.98647	0.98618	0.98634	0.98619	0.98635	0.98608

To verify that the flag mean does in fact find the optimal solution to Equation (7.2), it was compared with the gradient descent algorithm described in Section 5.3. Sets of 10 data points were generated in $\mathbb{R}^n$ for $n \in \{5, 10, 20\}$ and embedded into m dimensions for $m \in \{2, 3, 5\}$ while optimizing areas of k-simplices for $k \in \{1, 2, 5\}$. Results are displayed in Table 7.3. Regardless of starting dimension, embedding dimension, or simplex size, both algorithms found the same optimal projection for the data. In addition to demonstrating the functionality of both algorithms, it numerically supports the conclusion of the proof in 4.2.2. As before, all projections are the same up to rotation and multiplication by -1; however, the time required to achieve that result is much lower using the flag mean.

Table 7.3: Comparison of Flag Mean Projections and Gradient Descent on sets of 10 randomly generated data points in $\mathbb{R}^5$, $\mathbb{R}^{10}$, and $\mathbb{R}^{20}$. The data points were projected into $\mathbb{R}^2$, $\mathbb{R}^3$, or $\mathbb{R}^5$ while optimizing average normalized volume of 1-,2-, and 5-simplices.

Algorithm	n	m	k	**Avg Vol**	**Min Vol**	**Iterations**	**Run Time (s)**
Flag Mean Projection	5	2	1	0.61465	0.11022	0	0.0070
Gradient Descent	5	2	1	0.61465	0.11022	2650	7.1867
Flag Mean Projection	5	2	2	0.61325	0.07239	0	0.0291
Gradient Descent	5	2	2	0.61325	0.07239	775	4.4904
Flag Mean Projection	5	3	1	0.80733	0.15039	0	0.0102
Gradient Descent	5	3	1	0.80733	0.15039	120	1.3188
Flag Mean Projection	5	3	2	0.80525	0.07242	0	0.0222
Gradient Descent	5	3	2	0.80525	0.07242	220	1.9943
Flag Mean Projection	10	2	1	0.51481	0.07529	0	0.0136
Gradient Descent	10	2	1	0.51481	0.07529	2731	7.3495
Flag Mean Projection	10	2	2	0.51416	0.04500	0	0.0164
Gradient Descent	10	2	2	0.51416	0.04500	1111	5.7784
Flag Mean Projection	10	3	1	0.70455	0.11344	0	0.0109
Gradient Descent	10	3	1	0.70455	0.11344	60	1.1405

Algorithm	n	m	k	**Avg Vol**	**Min Vol**	**Iterations**	**Run Time (s)**
Flag Mean Projection	10	3	2	0.70371	0.11166	0	0.0251
Gradient Descent	10	3	2	0.70371	0.11166	66	1.2690
Flag Mean Projection	10	5	1	0.90463	0.46278	0	0.0062
Gradient Descent	10	5	1	0.90463	0.46278	68	1.2752
Flag Mean Projection	10	5	2	0.90414	0.45765	0	0.0269
Gradient Descent	10	5	2	0.90414	0.45765	50	1.3811
Flag Mean Projection	10	5	5	0.90140	0.44735	0	0.0509
Gradient Descent	10	5	5	0.90140	0.44735	53	1.6571
Flag Mean Projection	20	2	1	0.47161	0.00626	0	0.0210
Gradient Descent	20	2	1	0.47161	0.00626	2816	7.5485
Flag Mean Projection	20	2	2	0.47137	0.00666	0	0.0261
Gradient Descent	20	2	2	0.47137	0.00666	615	3.5356
Flag Mean Projection	20	3	1	0.60995	0.16373	0	0.0133
Gradient Descent	20	3	1	0.60995	0.16373	273	1.7936
Flag Mean Projection	20	3	2	0.60942	0.17273	0	0.0323
Gradient Descent	20	3	2	0.60942	0.17273	316	2.6446
Flag Mean Projection	20	5	1	0.82431	0.38689	0	0.0147
Gradient Descent	20	5	1	0.82431	0.38689	93	1.2417
Flag Mean Projection	20	5	2	0.82393	0.42627	0	0.0316
Gradient Descent	20	5	2	0.82393	0.42627	103	1.5171
Flag Mean Projection	20	5	5	0.82316	0.42303	0	0.0534
Gradient Descent	20	5	5	0.82316	0.42303	149	2.4968
Flag Mean Projection	20	10	1	1.00000	1.00000	0	0.0100
Gradient Descent	20	10	1	1.00000	1.00000	3	1.0681
Flag Mean Projection	20	10	2	1.00000	1.00000	0	0.0406

Algorithm	n	m	k	Avg Vol	Min Vol	Iterations	Run Time (s)
Gradient Descent	20	10	2	1.00000	1.00000	4	1.3468
Flag Mean Projection	20	10	5	1.00000	1.00000	0	0.0447
Gradient Descent	20	10	5	1.00000	1.00000	7	1.1734

7.2.2 Computational Complexity

Let n be the starting ambient dimension of the data, m be the desired embedding dimension, p be the number of data points, and M be the total number of k-simplices defined by those data points. Generating the matrix representations of the subspace spanned each simplex requires a QR-factorization, resulting in a total of $\mathcal{O}(Mn^2k)$ computations. The matrices are concatenated into one large $n \times Mk$ matrix. Taking the SVD of this matrix to compute the flag mean requires $\mathcal{O}(n(Mk)^2 + (Mk)^3)$ FLOPS. From before, calculating the normalized simplex volumes comes out to approximately $\mathcal{O}(Mk^2n + M\frac{2}{3}k^3)$ FLOPS for initial volumes and $\mathcal{O}(Mk^2m + M\frac{2}{3}k^3)$ for projected volumes.

Combining and dropping lower order terms yields a complexity of $\mathcal{O}(Mn^2k + M^2nk^2 + M^3k^3 + Mk^2m)$. Recall from previously that M is equal to $\frac{p!}{(k+1)!(p-k-1)!}$. Substituting this in for M and simplifying yields

$$\mathcal{O}\left(n^2\frac{k(p!)}{(k+1)!(p-k-1)!} + (n+m)\left(\frac{k(p!)}{(k+1)!(p-k-1)!}\right)^2 + \left(\frac{k(p!)}{(k+1)!(p-k-1)!}\right)^3\right)$$

total FLOPS required to run the algorithm. The complexity of the algorithm scales linearly with m and quadratically with n. Complexity scales cubically with the number of actual simplices, but increases factorially with the actual number of data points p. This is slightly offset the farther k is from $\frac{p}{2} - 1$. Once again, the main factor in determining complexity is the number of points used to generate simplices.

Chapter 8

Algorithm Comparisons

In order to evaluate the performance of the algorithms outlined in Chapters 6 and 7, both were tested on a fixed set of data points and compared with other algorithms for dimensionality reduction. In this chapter, the algorithms are compared in terms of smallest embedded volume and average embedded volume. Total required run time is also included for the iterative algorithms as well as the flag mean averaging algorithm. In addition to the flag mean averaging algorithm (FLAG) and the Smallest Volume Optimizing Projection algorithm (VOL), four algorithms are used: principal component analysis (PCA), random Gaussian projections (RGP), Secant-Avoiding Projections (SAP), and the modified Secant-Avoiding Projections (mSAP). The details of these algorithms are outlined in Chapter 5.

8.1 Experiment Set-Up

The data used for algorithm evaluation comes from the MNIST handwritten digit data set [41]. Ten data points were taken from both the "0" and the "1" classes, for a total of 20 data points in $\mathbb{R}^{784}$ used in evaluation. The same set of 20 points is used for every single trial to allow for comparisons between different embedding dimensions and simplex dimensions. All algorithms were tested over all possible combinations of k-simplices for $k \in \{1, 2, 3, 5, 10\}$ and embedding dimensions $m \in \{2, 3, 5, 10, 20, 30\}$. Generating all possible simplices from this set of points results in 190 1-simplices, 1140 2-simplices, 4845 3-simplices, of 38,760 5-simplices, and 167,960 10-simplices.

Note that since there are only 20 data points, the data is at most 20-dimensional and therefore there is an isometric embedding of the data that exactly preserves geometric structure in both 20 and 30 dimensions. These projection dimensions are included to evaluate how well each algorithm handles the case where the embedding dimension is equal to the data dimension and the case where the embedding dimension is greater than the data dimension.

8.1.1 Baseline Comparison Embeddings

Two embedding methods were included to serve as a baseline comparison for the more complex algorithms. These are PCA and RGP.

Because PCA is a deterministic algorithm and is primarily included as a baseline, only one embedding of the data into each projection dimension is required, and all k-simplices for each dimension can be calculated from a single projection. Consequently, the run time for PCA is not recorded. Additionally, PCA will not yield an embedding in a $\mathbb{R}^{30}$ for only 20 data points, so it is omitted from these cases.

For RGP, 30 random projection matrices are created for each trial, and the best embedded volume ratios from across the 30 projections are kept as the final result. Additionally, the average performance across all 30 runs is reported for both smallest embedded volume and average embedded volume. Because this is another baseline performance indicator, total run time was omitted for this.

8.1.2 SAP and mSAP

The SAP and mSAP algorithms were tested using all the same parameters, with the only difference being the generation of additional supporting secants in mSAP prior to starting iterations. Note however that since an additional secant must be found for each simplex, the complexity of this step increases approximately factorially as simplex dimension is increased. In order to generate this secant, one must solve for the vector through one simplex vertex that is orthogonal to the affine space spanned by the remaining vertices, and do so for every single simplex. This massively increases the startup cost of this algorithm, especially for $k = 10$ when there are 167,960 total simplices.

For each distinct trial, both algorithms were tested using both SVD initialization and random initialization. When SVD was used, only one run was performed, and when random initialization was used, 30 runs were performed with different random starting points. Both the average results and the best results overall results for both average and smallest volume are reported. The maximum number of iterations allowed is 30,000, and the step size used at each iteration t is given by

$$f(t) = \begin{cases} 0.01 & t < 1000 \\ 0.001 & t \in [1000, 5000) \\ 0.0001 & t \in [5000, 30000] \end{cases} \tag{8.1}$$

based on preliminary experimentation with the algorithms on the data. Iterations are terminated when the difference between the smallest secant lengths at consecutive steps is less than $\delta = 10^{-12}$.

8.1.3 Flag Mean Averaging

The FLAG algorithm is applied as described in Section 7.2. Since it is a deterministic algorithm, only one trial is necessary. Both smallest embedded volume ratio and average embedded volume ratio are recorded. Total run time is also tracked to compare its efficiency with the other algorithms.

8.1.4 Volume Optimizing Projections

For the VOL algorithm, both random and SVD initialization method were utilized, along with all seven step size functions outlined in Section 6.3.1. In the case of random initialization, each trial consisted of 30 runs with different starting conditions and results were compiled from the average performance and the best overall performance. SVD initialization, as usual, only required a single trial and both best smallest volume and average volume are reported. Average time per run as well as overall required time is reported for each trial. In all cases, the list of considered simplices was reduced to only the smallest five after 100 iterations to drastically reduce computation time. The termination threshold for this algorithm was set to $\delta 10^{-12}$, and iterations were terminated when the difference in smallest volume between two consecutive steps fell below the threshold. The exception to this is when either the previous smallest volume or the current smallest volume is smaller than $100\delta = 10^{-10}$, in which case this condition is ignored and iterations continue. This is done to prevent the algorithm from terminating immediately due to an unfortunate choice of initial condition.

8.2 Results

This section summarizes the overall results of the algorithm comparison, focusing on the highlights and interesting questions raised by the outcome of the experiment. Due to the size of the experiment, the complete table of results is located in Appendix B. Because some algorithms attempt to maximize the smallest normalized embedded volume while others maximize the average embedded volume, the discussion of results is split to focus on each problem individually. However, results for both average and smallest volumes are included for all algorithms.

8.2.1 Smallest Embedded k-Volumes

Figures 8.1 and 8.2 show best smallest 1-simplex length achieved by each algorithm for all projection dimensions. Overall, SAP with SVD initialization performed the best in terms of smallest embedded 1-simplex length across all tested embedding dimensions.

All algorithms except for mSAP that used SVD initialization successfully found the isometric embedding of the data in both 20 and 30 dimensions. In lower embedding dimensions, SAP performed best overall, followed by the VOL algorithm using step size function $f_5(t)$ and then the remaining SVD initialized VOL trials. Otherwise, VOL using random initialization performed comparably to the randomly initialized SVD for projecting into 2 and 3 dimensions, and performed only somewhat worse in higher dimensions. Both versions of mSAP performed incredibly poorly across all dimensions, and RGP was consistently the worst method by a comparatively large margin.

Optimizing over 2-simplex volumes (areas), VOL using step size function $f_5(t)$ consistently outperformed almost all other algorithms. The only exceptions were embedding into 5 dimensions, where randomly initialized SAP found a better solution, and embedding into 10 dimensions, where SVD initialized SAP found a very slightly better solution solution than SVD initialized VOL.

However, in both cases the SAP algorithm failed to terminate prior to 30,000 iterations, while VOL using $f_5(t)$ terminated in roughly one-third as many iterations. It is possible that in both cases, iterating further may have resulted in an equivalent solution from SVD initialized VOL. In the case of $m = 5$, randomly initialized SAP found the best solution, but SVD initialized VOL with $f_5(t)$ outperformed SVD initialized SAP. Relative performance of the remaining algorithms was functionally the same as it was for 1-simplices. Results for each projection dimension are shown in Figures 8.3 and 8.4.

Optimizing over volumes of 3-simplices yielded slightly different results, seen in Figures 8.5 and 8.6. SVD initialized VOL using $f_5(t)$ as the step size function performed decidedly better than all other algorithms except when embedding into 3 dimensions, where it was beaten by randomly initialized VOL with $f_5(t)$. Otherwise, in lower embedding dimensions, the randomly initialized tests for the VOL algorithm frequently performed as well as or better than their SVD initialized counterparts, and additionally outperformed the randomly initialized SAP. Algorithm behavior when embedding into 20 and 30 dimensions remained largely the same, other than a notable improvement in randomly initialized VOL using $f_0(t)$ or $f_4(t)$.

Figures 8.7 and 8.8 shows the results from optimizing volumes of 5-simplices. Results for embedding dimensions 10 and higher are largely the same as for 3-simplices. For $m = 5$, however, the best embedding was found by randomly initialized VOL using $f_1(t)$ as the step size function. VOL with $f_4(t)$ also performed surprisingly well. Notably, in this dimension all the randomly initialized algorithms outperformed their SVD initialized counterparts, generally by an order of magnitude or more.

Attempting to maximize volumes of 10-simplices introduced several complications. First of all, both the randomly initialized mSAP and the FLAG algorithms were omitted for all $k = 10$ trials due to unreasonably long run times. For embeddings into 10 dimensions, SVD initialized VOL using $f_5(t)$ found an embedding with a minimum volume that was an order of magnitude larger than that found by any of the other algorithms. For $m = 20$ and $m = 30$, there was a significant disparity between the algorithms that performed well and those that performed poorly.

Interestingly, random initialization VOL using $f_1(t)$, $f_2(t)$, and $f_6(t)$ performed very well, with $f_2(t)$ in particular finding embeddings that were very nearly isometric. Results for $k = 10$ are shown in Figure 8.9. Once again, the overall best performance came from VOL with step size function $f_5(t)$.

The performance of the mSAP algorithm was intriguing, because it performed comparably to and occasionally better than SAP when embedding into low dimensions, but completely failed to find an isometric embedding in 20 and 30 dimensions, even with SVD initialization. It is possible that the added secants change the starting point chosen by the SVD. There was generally very little difference between the algorithm's performance when starting conditions were changed. Additionally, the computational complexity and run time increased dramatically with the dimension of simplices, and it was still significantly slower than SAP when producing similar results. Due to the inconsistent results and factorially increasing run time, it is better in all cases to just use SAP to embed data.

Unlike the tests done in Chapter 6, initializing the VOL algorithm using SVD did not always result in larger minimum normalized volumes. Table 8.1 compares the best minimum volume achieved by VOL for each choice of step size function. The best smallest values found using random initialization as well as average smallest values over all random initialization tests are compared to the smallest volume from the SVD tests for each combination of k and embedding dimension m. Dimensions where an isometric embedding of the data exists are excluded, since SVD is already shown to perform much better. Random initialization yields the largest overall embedded volume when $\{k = 1, m = 2\}$, $\{k = 1, m = 3\}$, $\{k = 2, m = 2\}$, $\{k = 2, m = 3\}$, $\{k = 3, m = 3\}$, $\{k = 5, m = 5\}$ and $\{k = 10, m = 10\}$, accounting for 7 out of the 14 total combinations shown in Table 8.1. In particular, it finds a solution an order of magnitude better than SVD initialization for $\{k = 1, m = 2\}$, $\{k = 2, m = 3\}$, and $\{k = 10, m = 10\}$. However, the average smallest embedded volume is only larger than the SVD smallest embedded volume for $\{k = 1, m = 2\}$, $\{k = 2, m = 3\}$, and $\{k = 3, m = 3\}$, suggesting that the random initialization does not in general find a better solution than SVD for the remaining cases. Additionally, SVD initialization results in a smallest embedded volume that is an order of magnitude larger than any found using random initialization for $\{k = 1, m = 5\}$, $\{k = 2, m = 10\}$, $\{k = 3, m = 10\}$, and $\{k = 5, m = 10\}$. Overall, there appears to be a trend of SVD performing better in higher ambient dimensions and random initialization improving with lower ambient dimensions. Regardless, the difference in performance between the two methods is frequently negligible, and VOL can be run significantly faster using SVD initialization. Because of this, it is likely worthwhile in most cases to sacrifice some accuracy for the sake of efficiency and just run VOL using SVD initialization.

Table 8.1: Comparison of performance of initialization methods for the Smallest Volume Optimizing algorithm.

k	**Proj Dim**	$f(t)$	**Rand Best Small Vol**	**Rand Avg Small Vol**	**SVD Smallest Vol**
1	2	$f_0(t)$	2.6226E-03	4.9609E-04	3.6250E-04
1	2	$f_1(t)$	6.3412E-03	1.4600E-03	5.9164E-04
1	2	$f_2(t)$	2.7246E-03	7.6278E-04	8.6226E-04
1	2	$f_3(t)$	4.1813E-03	1.0335E-03	9.8569E-05
1	2	$f_4(t)$	2.5371E-03	4.8608E-04	2.0775E-04
1	2	$f_5(t)$	3.0635E-02	1.4582E-02	7.5506E-03
1	2	$f_6(t)$	3.2569E-03	7.1419E-04	3.2225E-04
1	3	$f_0(t)$	1.4558E-02	3.5622E-03	9.9073E-03
1	3	$f_1(t)$	2.2712E-02	8.1397E-03	9.4698E-03
1	3	$f_2(t)$	1.5104E-02	6.2652E-03	7.9640E-03
1	3	$f_3(t)$	1.6898E-02	5.8395E-03	1.5989E-02
1	3	$f_4(t)$	1.6317E-02	2.9448E-03	8.9821E-03
1	3	$f_5(t)$	6.3620E-02	3.6705E-02	5.1053E-02
1	3	$f_6(t)$	1.3120E-02	3.4084E-03	2.3132E-03
1	5	$f_0(t)$	4.8805E-02	1.2457E-02	9.0344E-03
1	5	$f_1(t)$	9.8295E-02	4.9679E-02	9.5927E-02
1	5	$f_2(t)$	7.2690E-02	3.3954E-02	5.1838E-02

k	**Proj Dim**	$f(t)$	**Rand Best Small Vol**	**Rand Avg Small Vol**	**SVD Smallest Vol**
1	5	$f_3(t)$	3.0216E-02	1.9571E-02	1.0434E-01
1	5	$f_4(t)$	3.7500E-02	1.2813E-02	9.0327E-03
1	5	$f_5(t)$	9.0739E-02	6.1852E-02	1.7532E-01
1	5	$f_6(t)$	3.3571E-02	1.4692E-02	2.2723E-02
1	10	$f_0(t)$	2.6044E-01	1.4886E-01	2.8870E-01
1	10	$f_1(t)$	1.9444E-01	1.3368E-01	4.5853E-01
1	10	$f_2(t)$	1.7224E-01	1.0580E-01	3.8328E-01
1	10	$f_3(t)$	5.2202E-02	3.6868E-02	4.0456E-01
1	10	$f_4(t)$	1.9738E-01	1.3718E-01	2.8870E-01
1	10	$f_5(t)$	1.1218E-01	8.6143E-02	5.0693E-01
1	10	$f_6(t)$	6.1123E-02	3.4801E-02	2.8716E-01
2	2	$f_0(t)$	8.9158E-08	1.1543E-08	4.5521E-09
2	2	$f_1(t)$	9.0718E-08	1.6152E-08	3.0951E-09
2	2	$f_2(t)$	1.0636E-07	1.4424E-08	1.0119E-10
2	2	$f_3(t)$	3.5591E-07	3.4948E-08	5.3866E-11
2	2	$f_4(t)$	2.0761E-07	2.7486E-08	9.6340E-09
2	2	$f_5(t)$	2.5501E-05	8.9384E-06	1.3717E-05
2	2	$f_6(t)$	2.2465E-07	3.5724E-08	9.1049E-09
2	3	$f_0(t)$	1.6184E-04	1.7149E-05	3.5824E-06
2	3	$f_1(t)$	2.2213E-04	4.0610E-05	5.6097E-05
2	3	$f_2(t)$	1.4417E-04	2.9225E-05	1.9085E-05
2	3	$f_3(t)$	1.2028E-04	3.0043E-05	2.9250E-07
2	3	$f_4(t)$	6.5178E-05	7.7453E-06	5.8646E-07
2	3	$f_5(t)$	2.2748E-03	1.0705E-03	5.1980E-04
2	3	$f_6(t)$	7.3023E-05	1.2200E-05	8.9110E-06
2	5	$f_0(t)$	1.4781E-03	3.8424E-04	2.5478E-04

k	**Proj Dim**	$f(t)$	**Rand Best Small Vol**	**Rand Avg Small Vol**	**SVD Smallest Vol**
2	5	$f_1(t)$	9.4806E-03	4.8000E-03	9.3345E-03
2	5	$f_2(t)$	4.7883E-03	1.6449E-03	2.6189E-03
2	5	$f_3(t)$	3.9180E-03	1.2641E-03	1.0109E-02
2	5	$f_4(t)$	1.0661E-03	3.6203E-04	2.6846E-04
2	5	$f_5(t)$	2.3597E-02	1.4194E-02	3.0943E-02
2	5	$f_6(t)$	1.7592E-03	4.3625E-04	1.9657E-04
2	10	$f_0(t)$	7.1140E-02	2.4629E-02	6.8127E-02
2	10	$f_1(t)$	9.4693E-02	6.0722E-02	1.4443E-01
2	10	$f_2(t)$	4.9803E-02	2.9897E-02	6.8482E-02
2	10	$f_3(t)$	9.8148E-03	5.5681E-03	1.7823E-01
2	10	$f_4(t)$	5.9249E-02	2.3756E-02	6.8127E-02
2	10	$f_5(t)$	4.9627E-02	3.9016E-02	3.0003E-01
2	10	$f_6(t)$	1.7013E-02	4.9827E-03	1.5732E-02
3	3	$f_0(t)$	9.2797E-10	1.0020E-10	1.1777E-12
3	3	$f_1(t)$	7.0359E-09	6.2972E-10	2.4501E-11
3	3	$f_2(t)$	1.7811E-09	2.1526E-10	2.4501E-11
3	3	$f_3(t)$	4.6376E-10	7.3485E-11	1.4035E-11
3	3	$f_4(t)$	1.0470E-09	1.1999E-10	6.3419E-12
3	3	$f_5(t)$	5.1292E-07	1.4773E-07	1.2192E-07
3	3	$f_6(t)$	1.9009E-09	1.6005E-10	9.9718E-18
3	5	$f_0(t)$	6.6037E-05	1.4704E-05	4.7016E-08
3	5	$f_1(t)$	5.2403E-04	9.7828E-05	1.3292E-04
3	5	$f_2(t)$	3.4028E-04	4.0867E-05	5.1027E-05
3	5	$f_3(t)$	1.0496E-04	2.8038E-05	6.3176E-05
3	5	$f_4(t)$	6.2092E-05	1.1533E-05	2.8990E-08
3	5	$f_5(t)$	2.1067E-03	1.1787E-03	2.7125E-03

k	**Proj Dim**	$f(t)$	**Rand Best Small Vol**	**Rand Avg Small Vol**	**SVD Smallest Vol**
3	5	$f_6(t)$	2.0888E-04	1.9351E-05	6.9682E-06
3	10	$f_0(t)$	5.1239E-03	1.1921E-03	5.4139E-03
3	10	$f_1(t)$	4.3507E-02	2.3409E-02	3.8276E-02
3	10	$f_2(t)$	1.9816E-02	7.7587E-03	1.3592E-02
3	10	$f_3(t)$	2.2958E-03	1.1799E-03	7.1081E-02
3	10	$f_4(t)$	5.1583E-03	1.2045E-03	5.4139E-03
3	10	$f_5(t)$	2.3878E-02	1.9763E-02	1.3753E-01
3	10	$f_6(t)$	1.8267E-03	6.2050E-04	3.1820E-03
5	5	$f_0(t)$	2.7121E-11	4.3871E-12	1.2930E-11
5	5	$f_1(t)$	3.3327E-11	7.1564E-12	1.0009E-13
5	5	$f_2(t)$	2.5098E-11	5.9697E-12	4.0350E-14
5	5	$f_3(t)$	2.7238E-12	5.7367E-13	1.5768E-12
5	5	$f_4(t)$	3.2305E-11	7.7737E-12	1.2930E-11
5	5	$f_5(t)$	2.8443E-11	5.8263E-12	7.3670E-13
5	5	$f_6(t)$	1.5607E-12	2.0181E-13	1.3488E-15
5	10	$f_0(t)$	1.1893E-04	3.0047E-05	4.1707E-05
5	10	$f_1(t)$	5.0940E-03	1.7196E-03	1.5475E-03
5	10	$f_2(t)$	1.2703E-03	2.4630E-04	1.2637E-04
5	10	$f_3(t)$	2.5795E-04	1.0549E-04	2.4478E-03
5	10	$f_4(t)$	2.7148E-04	3.3105E-05	4.8169E-05
5	10	$f_5(t)$	7.2023E-03	4.7932E-03	1.5656E-02
5	10	$f_6(t)$	6.8132E-04	6.7797E-05	5.6185E-05
10	10	$f_0(t)$	7.4869E-15	1.1651E-15	5.4480E-15
10	10	$f_1(t)$	5.5220E-14	3.9929E-15	1.9456E-15
10	10	$f_2(t)$	1.3642E-14	9.4854E-16	9.5891E-17
10	10	$f_3(t)$	3.1948E-14	1.8533E-15	5.5460E-16

k	**Proj Dim**	$f(t)$	**Rand Best Small Vol**	**Rand Avg Small Vol**	**SVD Smallest Vol**
10	10	$f_4(t)$	6.1286E-14	3.5582E-15	8.8557E-16
10	10	$f_5(t)$	1.0506E-13	1.4271E-14	2.3826E-14
10	10	$f_6(t)$	2.1726E-14	1.5681E-15	4.0850E-16

8.2.2 Average Embedded k-Volumes

The maximum average embedded normalized volumes for each algorithm are plotted in Figures 8.10, 8.11, 8.12, 8.13, and 8.14 for $k = 1, 2, 3, 5$, and 10, respectively. Once again, RGP consistently had the worst performance in every single trial. For $k < 5$ and $m < 20$, VOL, SAP, and mSAP all performed moderately well. When $k = 5$, both versions of mSAP as well as randomly initialized SAP and random VOL using $f_3(t)$ show a comparatively drastic drop in performance.

Once again, for $m = 20$ and $m - 30$, all SVD initialization except for mSAP found the isometric embedding, as did FLAG and PCA. When $k = 10$, random initialization VOL using $f_1(t)$, $f_2(t)$, and $f_6(t)$ again performed surprisingly well, this time in terms of average embedded volume. Further testing with different step size functions could be applied in future work.

In all trials where it was tested, the FLAG algorithm found the best result out of all the other algorithms. It is likely it would have also achieved the best result for $k = 10$ if the computations were completed successfully. Unfortunately, the factorial increase in computational cost as k increases proved to be prohibitively large for trials run on CPUs. The limiting step is likely the SVD computation needed for the flag mean. For $k = 10$ and any ambient dimension n, this experiment requires the algorithm to take the SVD of an $n \times 1,679,600$ matrix. Adding a GPU implementation of the SVD step of this algorithm could potentially provide enough of a speed up to make it viable, and is worth exploring in future work.

PCA was consistently a close second in terms of average normalized embedded volume. Although the FLAG algorithm performance seems to pull further ahead of PCA as simplex dimension increases, its computational cost also increases drastically compared to that of PCA. Table 8.2 displays the total run times for each algorithm from all trials where both were tested. Clearly, the time required to run the FLAG algorithm displays a severe increase with each change in simplex dimension. Because PCA performs very nearly as well as FLAG, it is probably worthwhile to sacrifice the extra accuracy in favor of computational efficiency, at least for computations that require large numbers of simplices.

Table 8.2: Comparison of required run times for PCA and FLAG for each trial.

Simplex Dim	Projection Dim	PCA Time (s)	FLAG Time (s)
1	2	0.0198	1.3619
1	3	0.0089	1.2407
1	5	0.0092	1.3742
1	10	0.0291	1.8079
1	20	0.0475	1.7611
2	2	0.0550	240.7221
2	3	0.0529	158.9926
2	5	0.0552	248.9655
2	10	0.0545	107.1937
2	20	0.1089	236.0877
3	3	0.2216	3440.6373
3	5	0.2626	4105.4920
3	10	0.2187	2842.6276
3	20	0.2765	1691.3432
5	5	2.0739	193456.3911
5	10	1.6988	162076.7908
5	20	1.8497	175778.6741
10	10	8.1806	incomplete
10	20	8.5680	incomplete

8.2.3 Conclusions

The results of this experiment suggest that using VOL with SVD initialization and step size $f(t)$ is the overall best choice for maximizing the smallest embedded volume. Although SAP performed the best when optimizing embedded secant length ($k = 1$) and remained competitive in some cases when optimizing embedded areas ($k = 2$), for optimizing volumes of dimension 3 or greater VOL consistently outperformed all other algorithms by at least an order of magnitude. This suggests that optimizing over lengths is, of course, effective when $k = 1$, but becomes significantly less effective than optimizing over volumes as simplex dimensions increase. Even though the best choice of step size function for VOL varies some based on embedding dimension and k, $f_5(t)$ is the best choice the majority of the time, and even when it does not provide the best result, it still provides results that are comparable to the best results found using other step size functions. SVD initialization for VOL is overall much more consistent in performance, and even in cases where random initialization may be better, the reduced computational cost of using the SVD makes it generally worth some loss of accuracy.

For optimizing average embedded volume, FLAG was the clear winner in terms of finding the best solution but suffered greatly from a high computational cost, to the point where it could not even be tested for $k = 10$. The embeddings found using PCA had a slight drop in average volume compared to FLAG but were significantly easier to compute. Overall, it is likely better to sacrifice some accuracy and use the PCA embedding, especially as simplex dimensions increase and the complexity of FLAG increases factorially. However, it would be interesting to see if this still holds for particularly oddly shaped data.

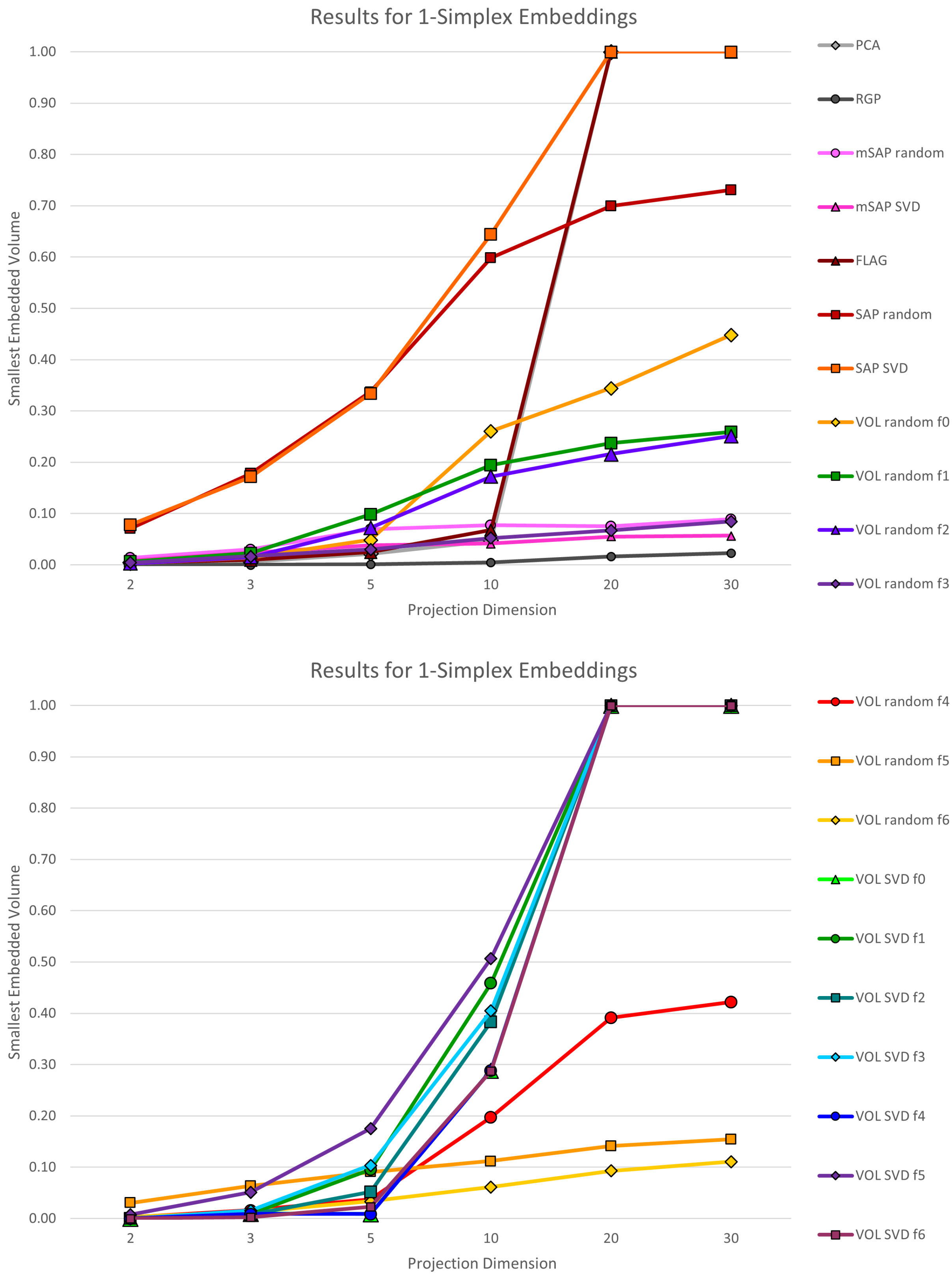
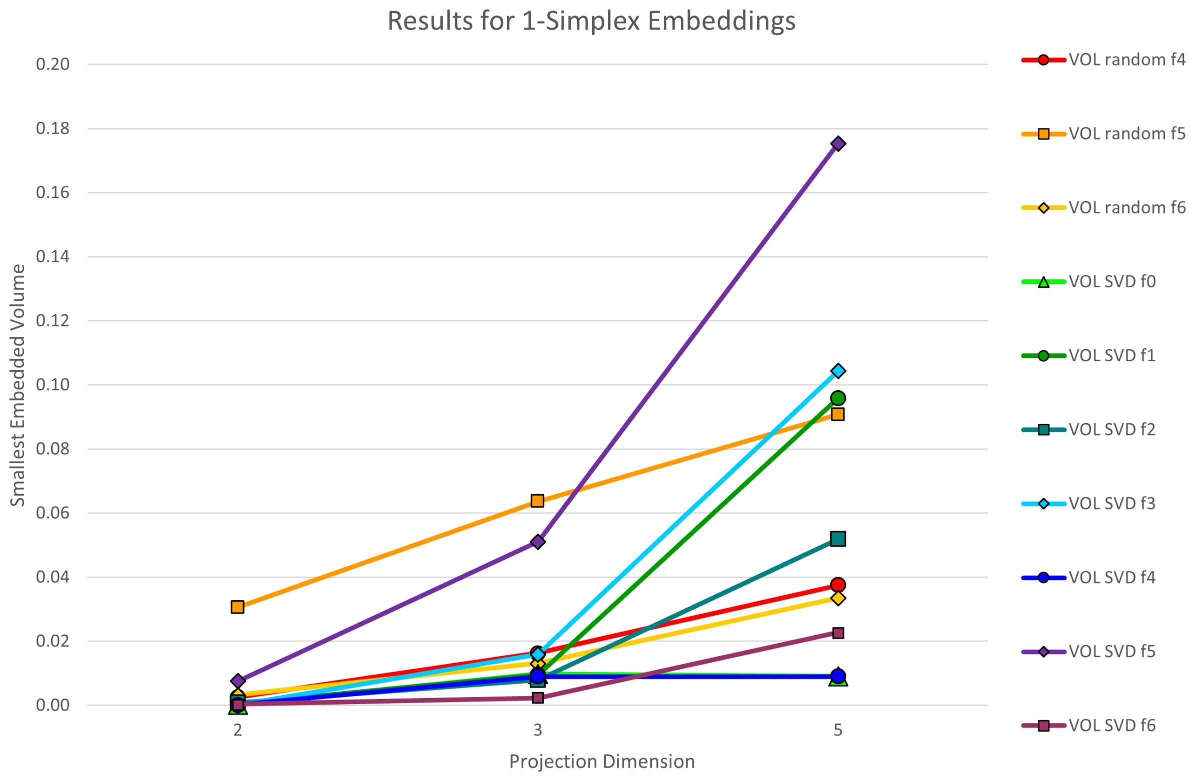


Figure 8.1: Comparison of smallest embedded 1-simplex lengths for all algorithms.



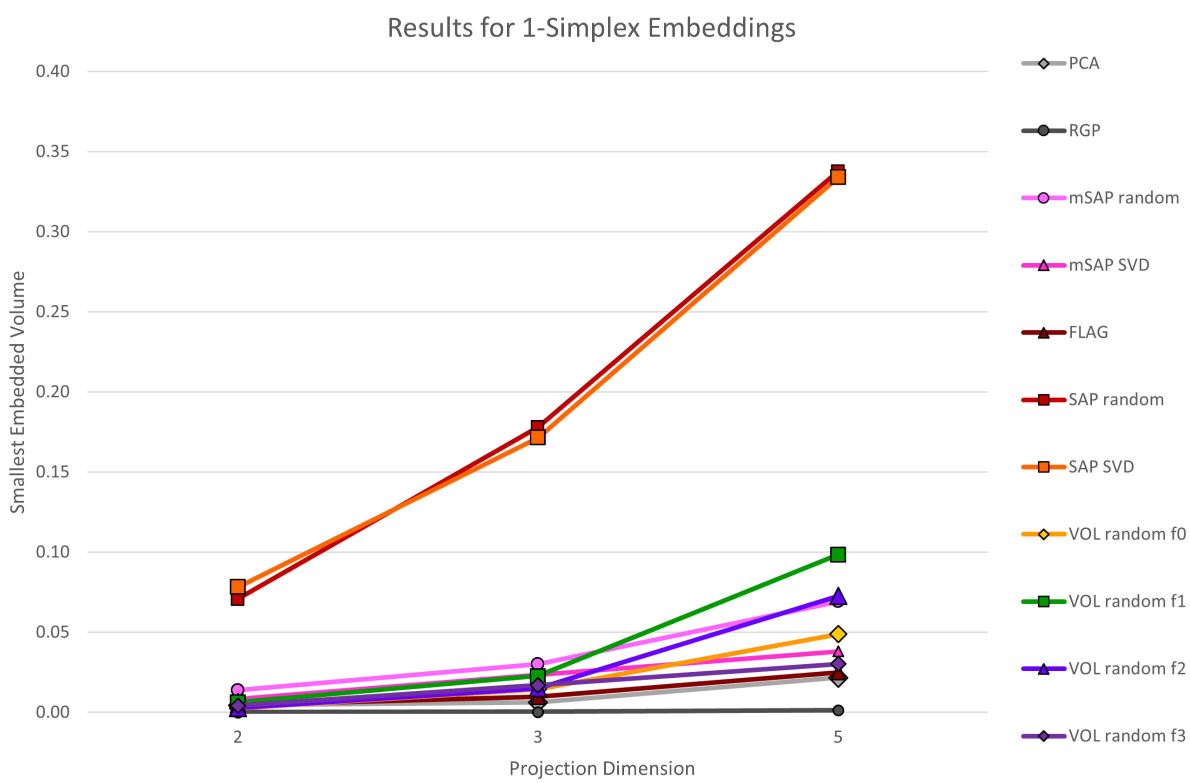
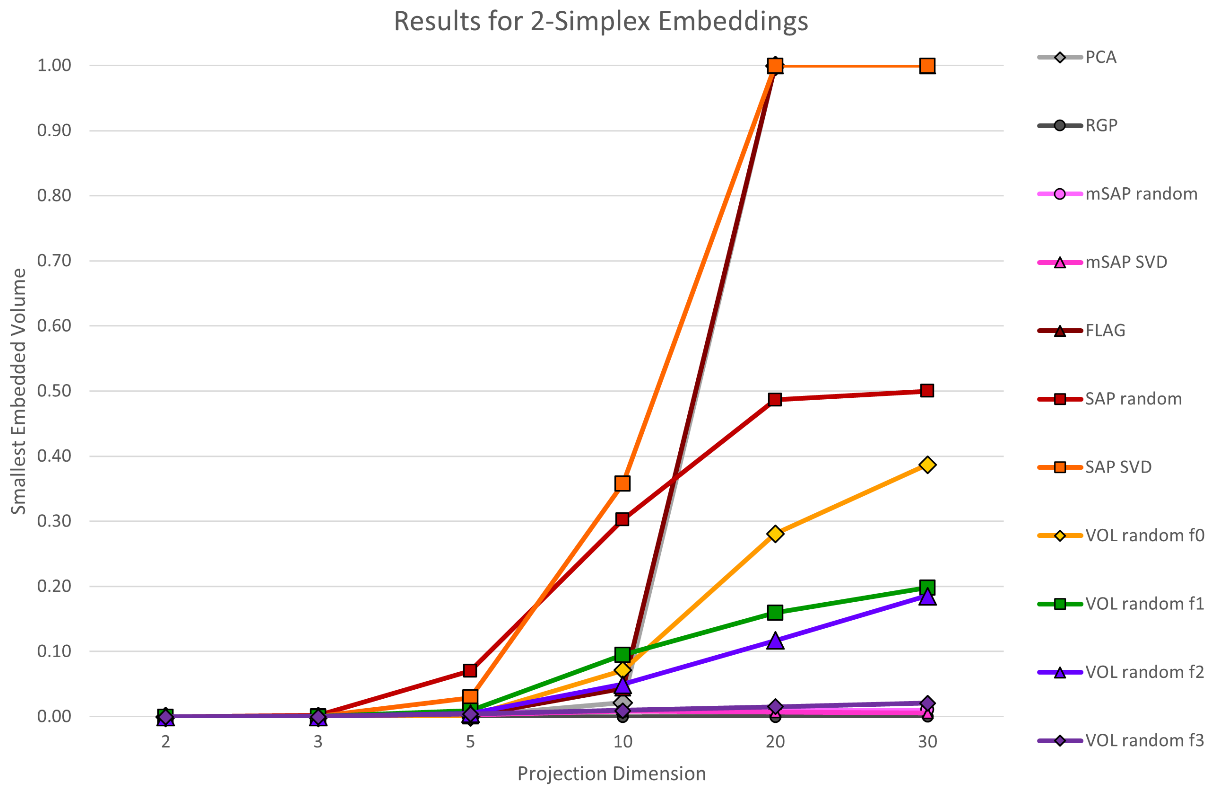


Figure 8.2: Comparison of smallest embedded 1-simplex lengths for all algorithms, zoomed in to more clearly show results.



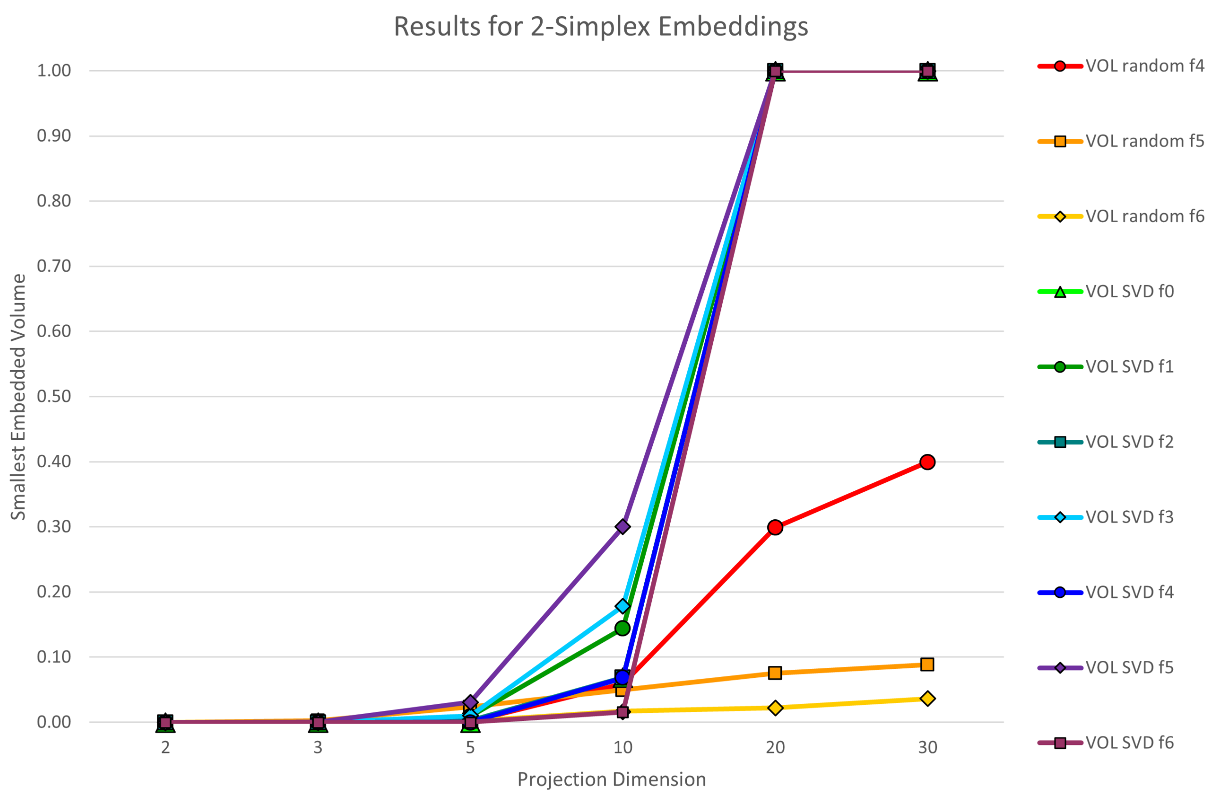


Figure 8.3: Comparison of smallest embedded 2-simplex areas for all algorithms.

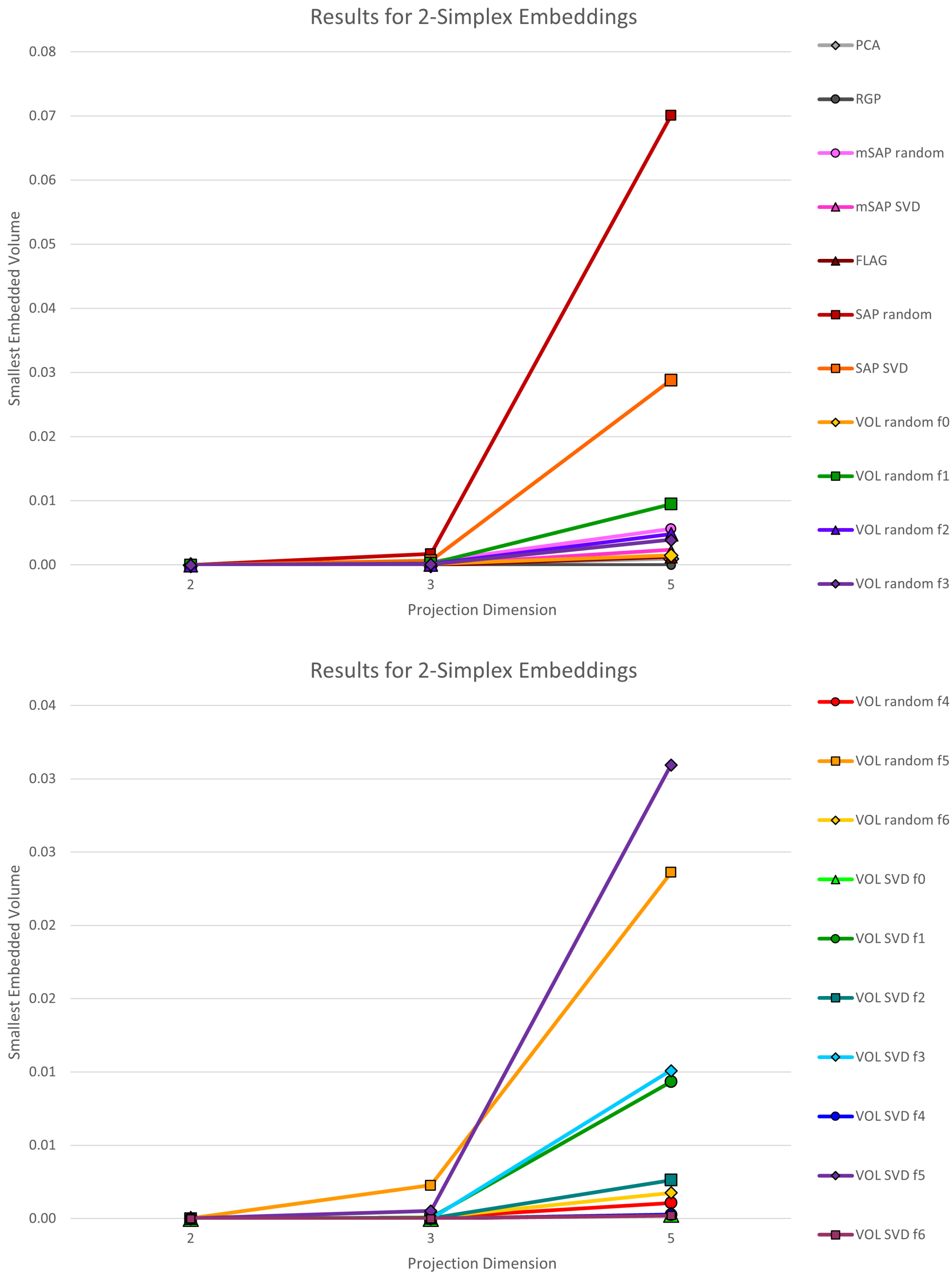


Figure 8.4: Comparison of smallest embedded 2-simplex areas for all algorithms, zoomed in to more clearly show results.

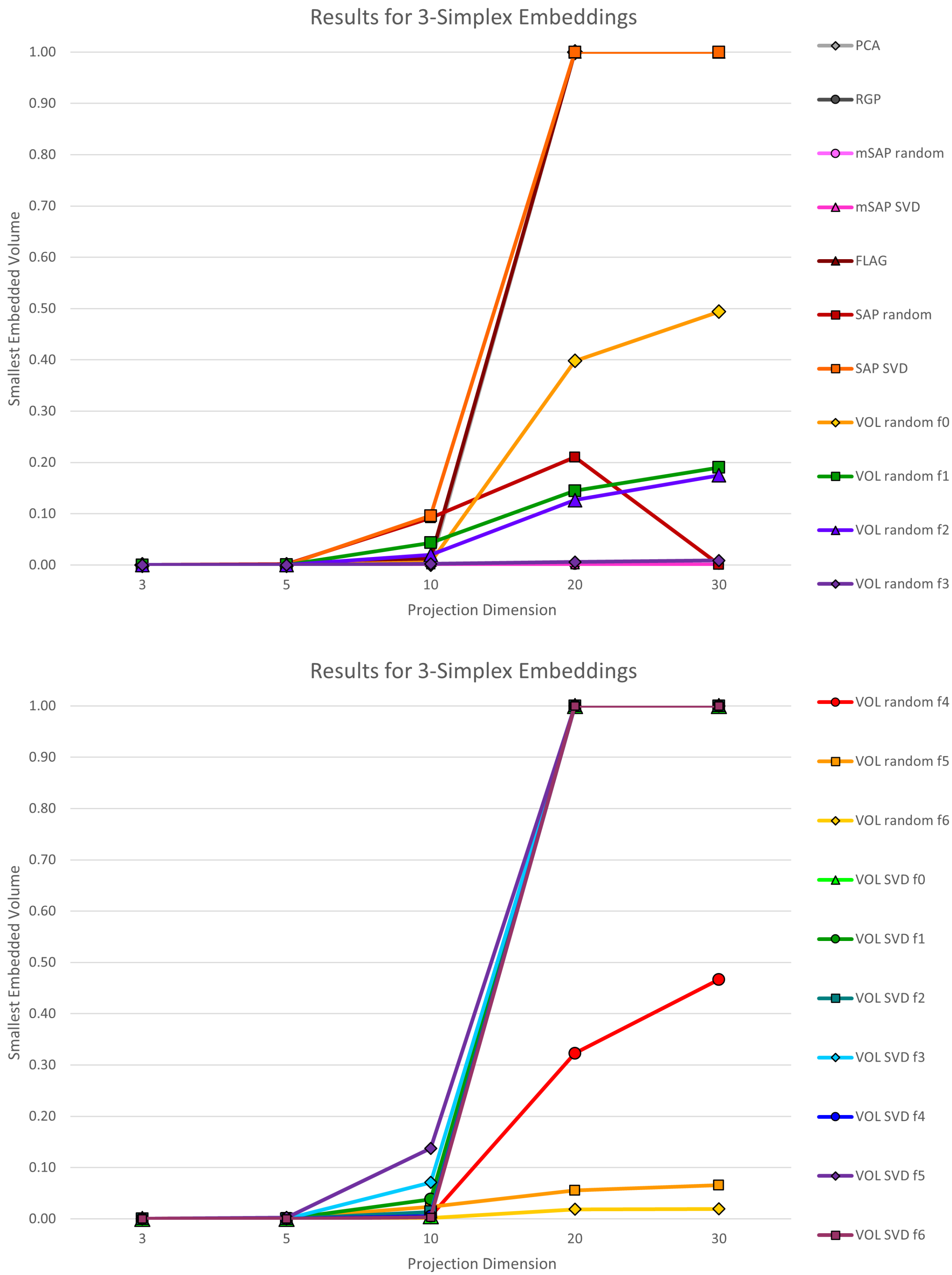


Figure 8.5: Comparison of smallest embedded 3-simplex volumes for all algorithms.

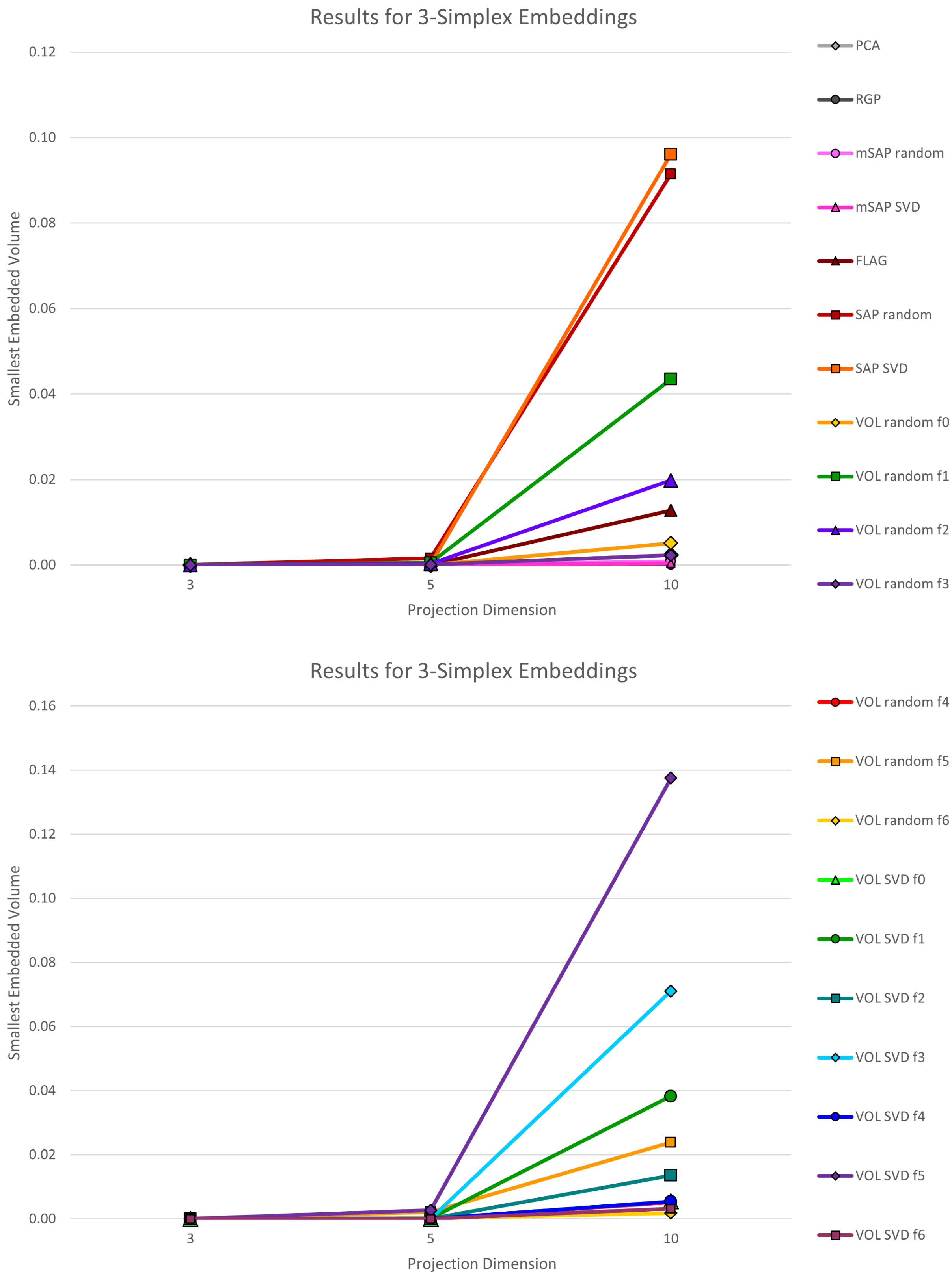


Figure 8.6: Comparison of smallest embedded 3-simplex volumes for all algorithms, zoomed in to more clearly show results.

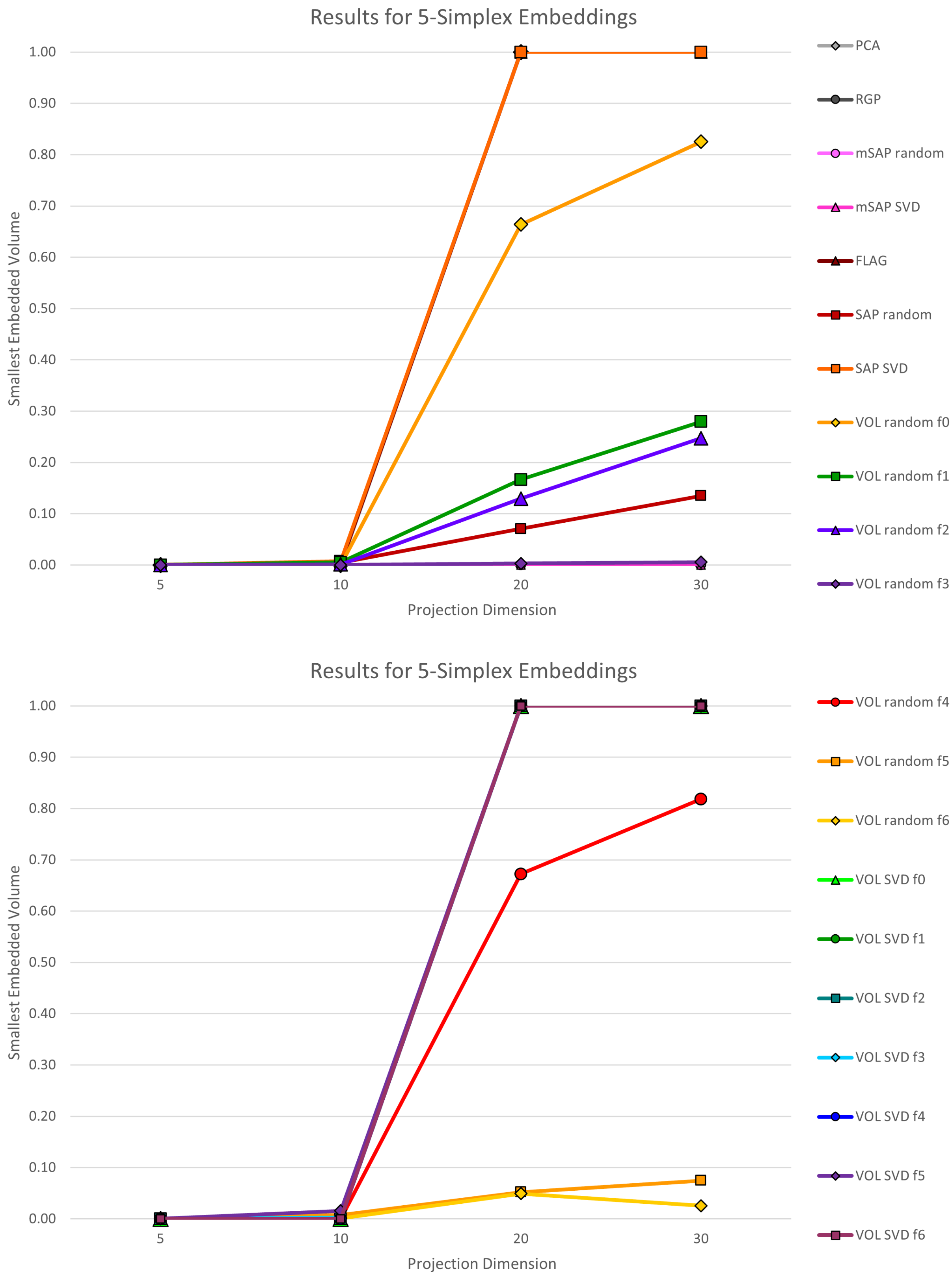


Figure 8.7: Comparison of smallest embedded 5-simplex volumes for all algorithms.

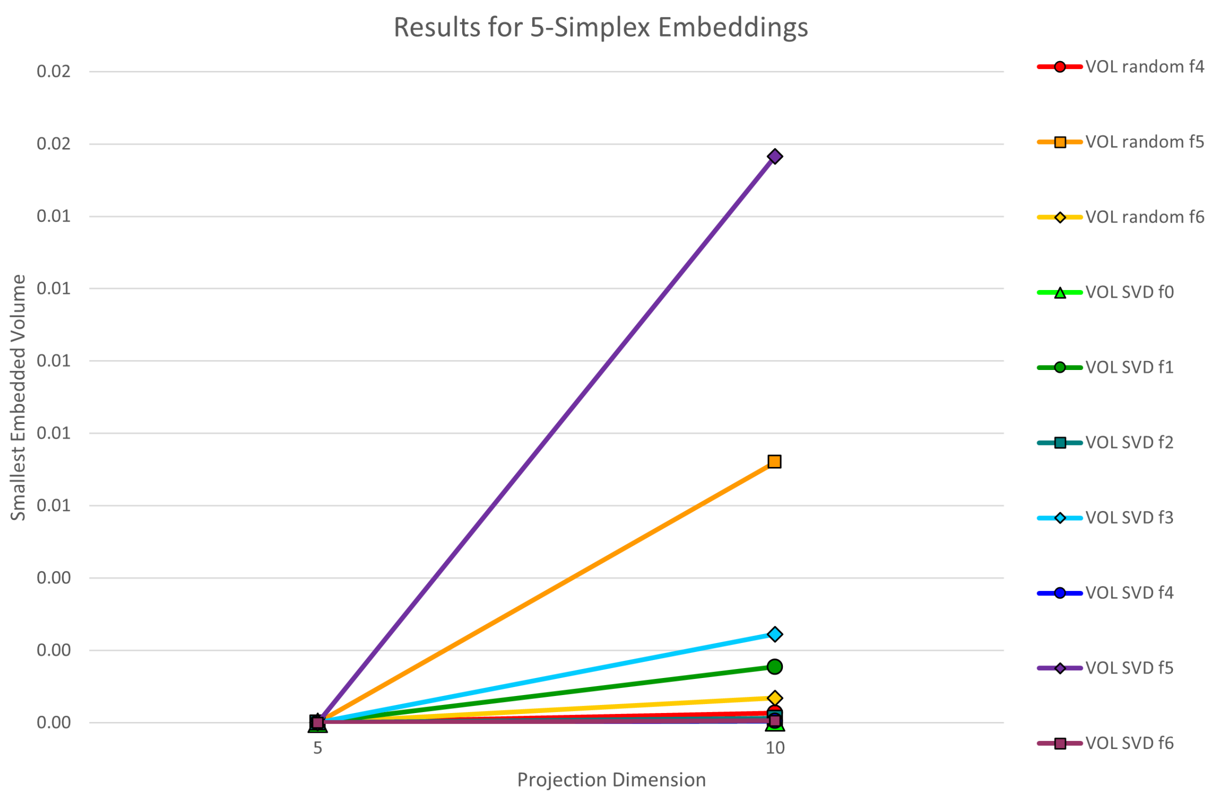


Figure 8.8: Comparison of smallest embedded 5-simplex volumes for all algorithms, zoomed in to more clearly show results.

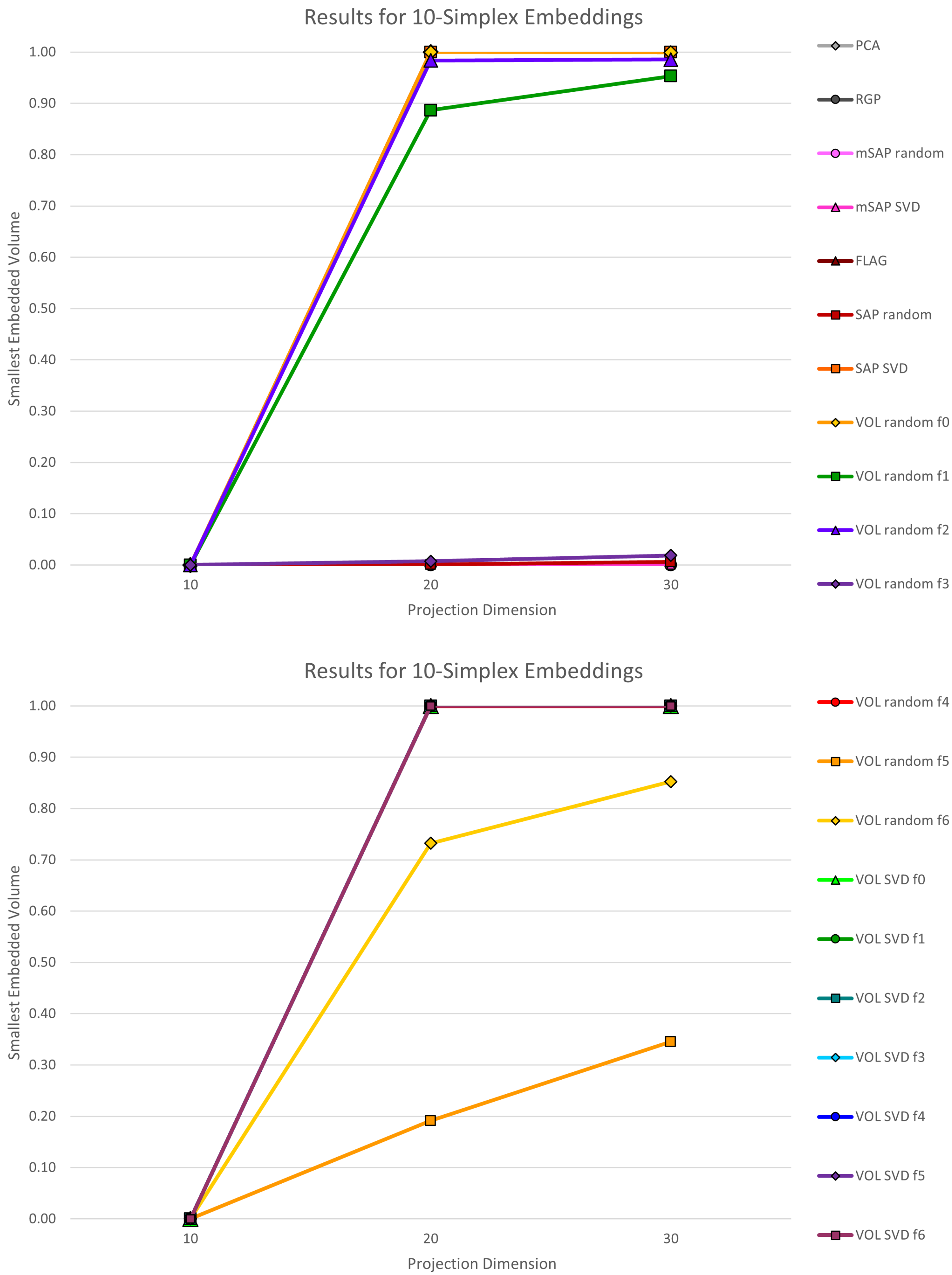
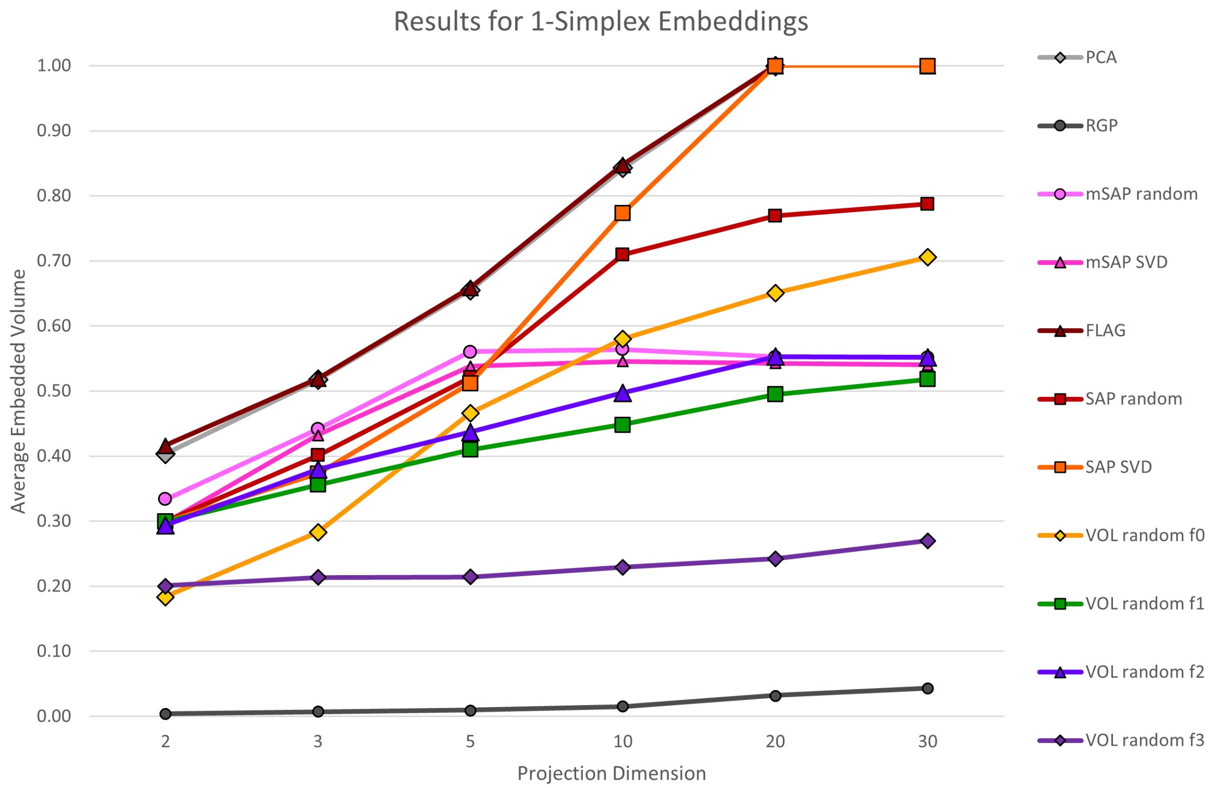


Figure 8.9: Comparison of smallest embedded 10-simplex volumes for all algorithms.



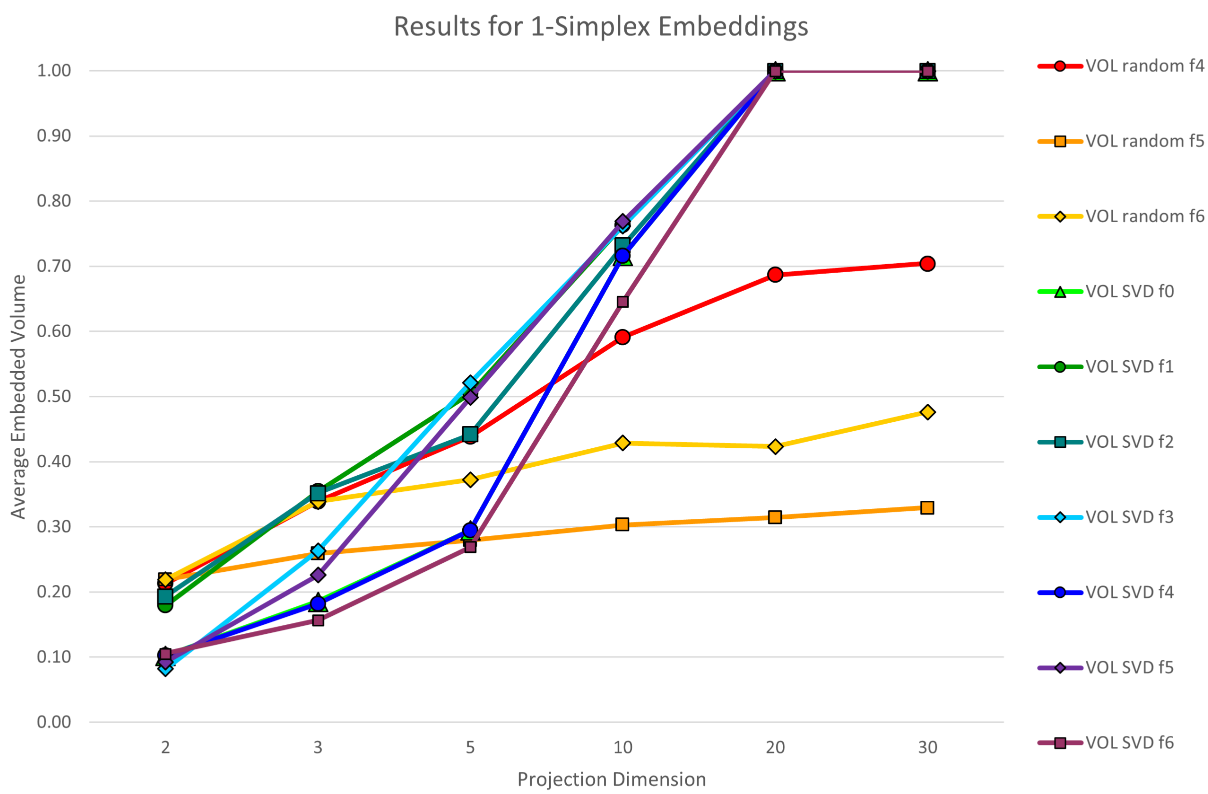


Figure 8.10: Comparison of average embedded 1-simplex lengths for all algorithms.

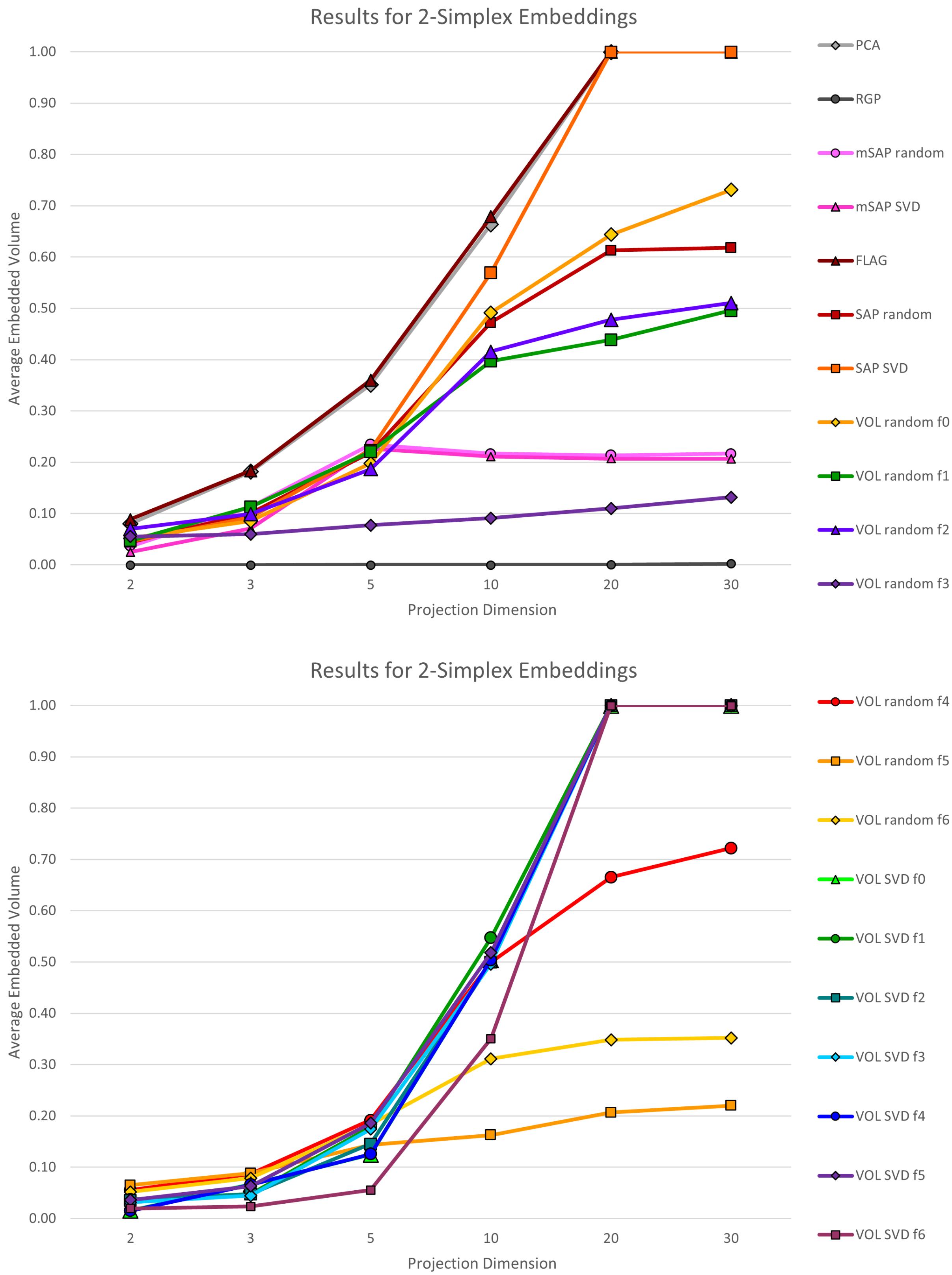


Figure 8.11: Comparison of average embedded 2-simplex areas for all algorithms.

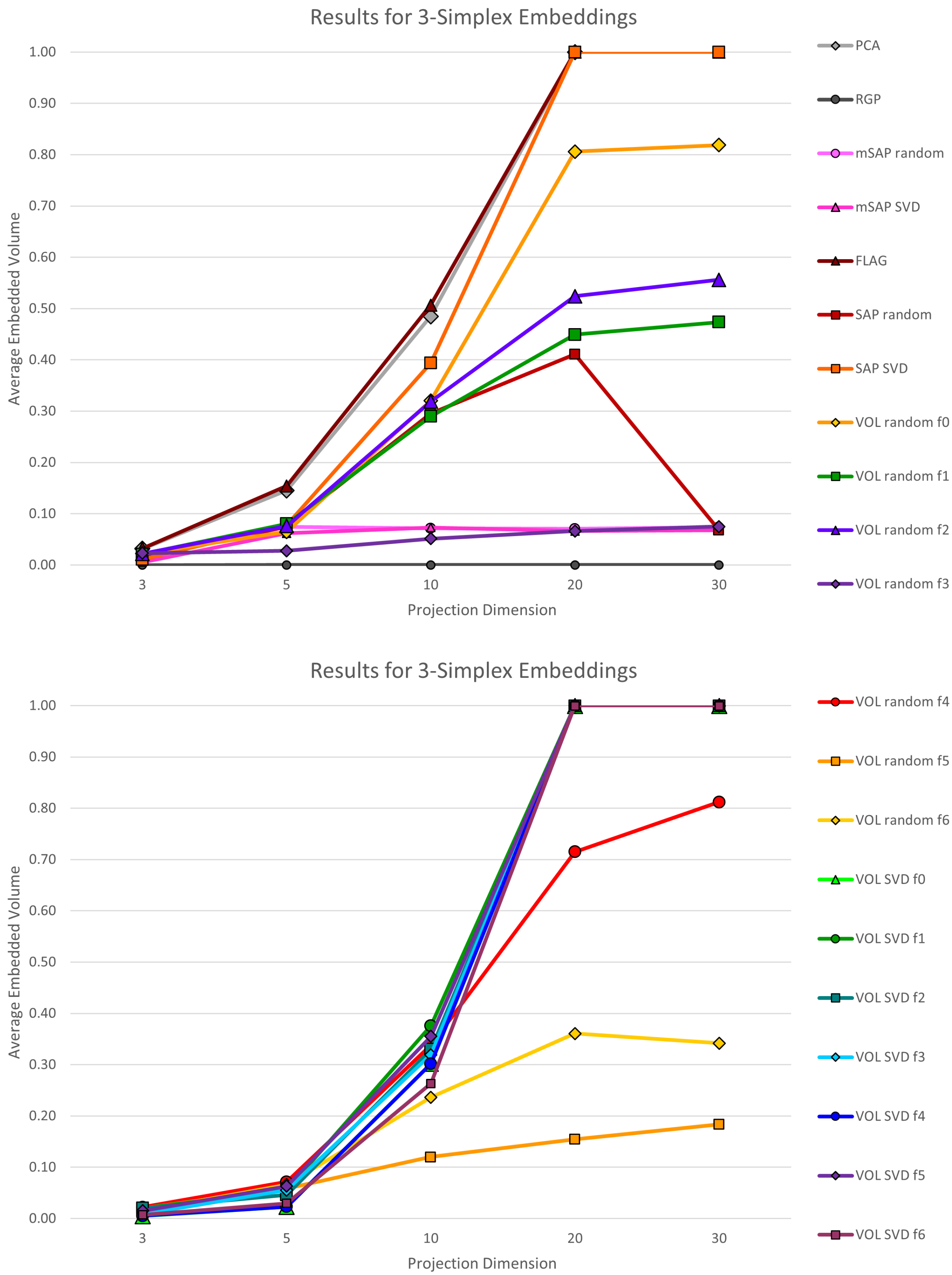


Figure 8.12: Comparison of average embedded 3-simplex volumes for all algorithms.

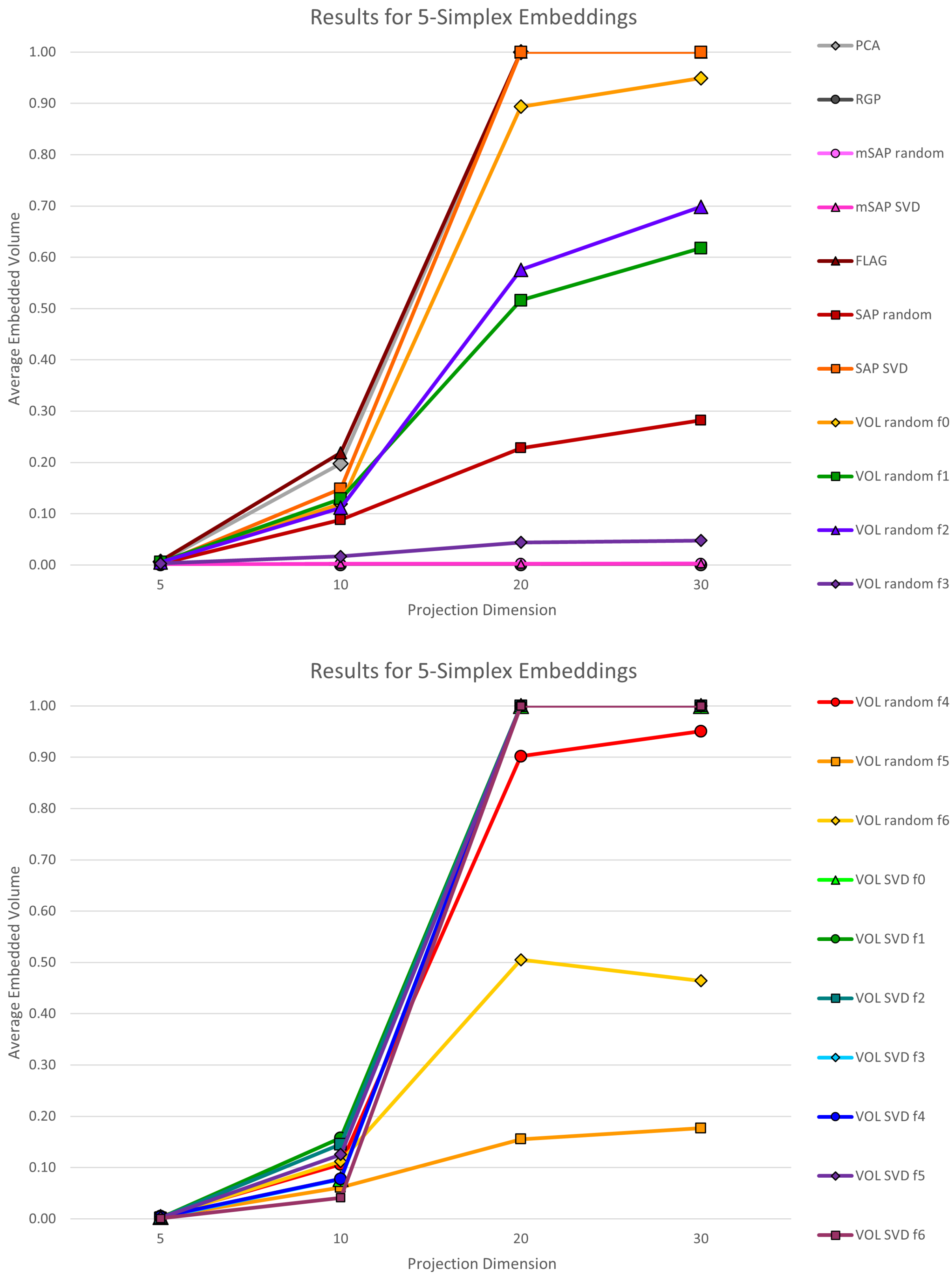
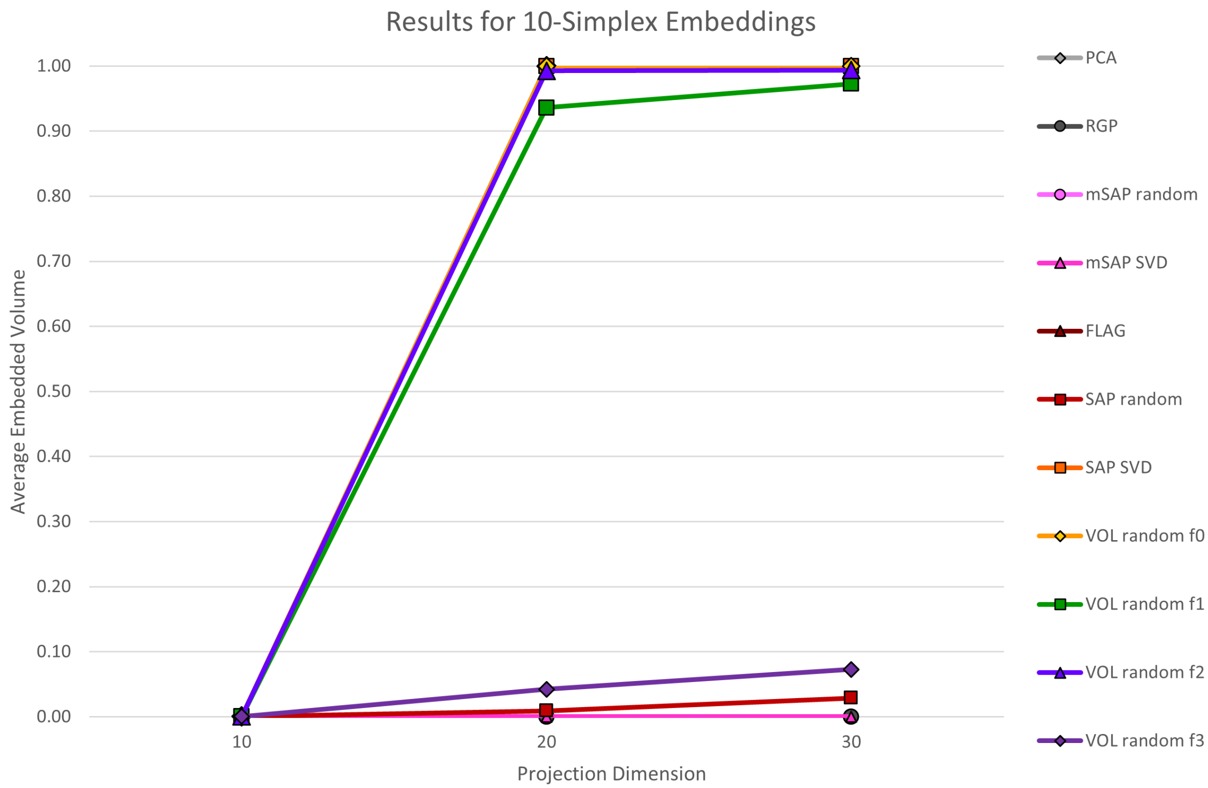


Figure 8.13: Comparison of average embedded 5-simplex volumes for all algorithms.



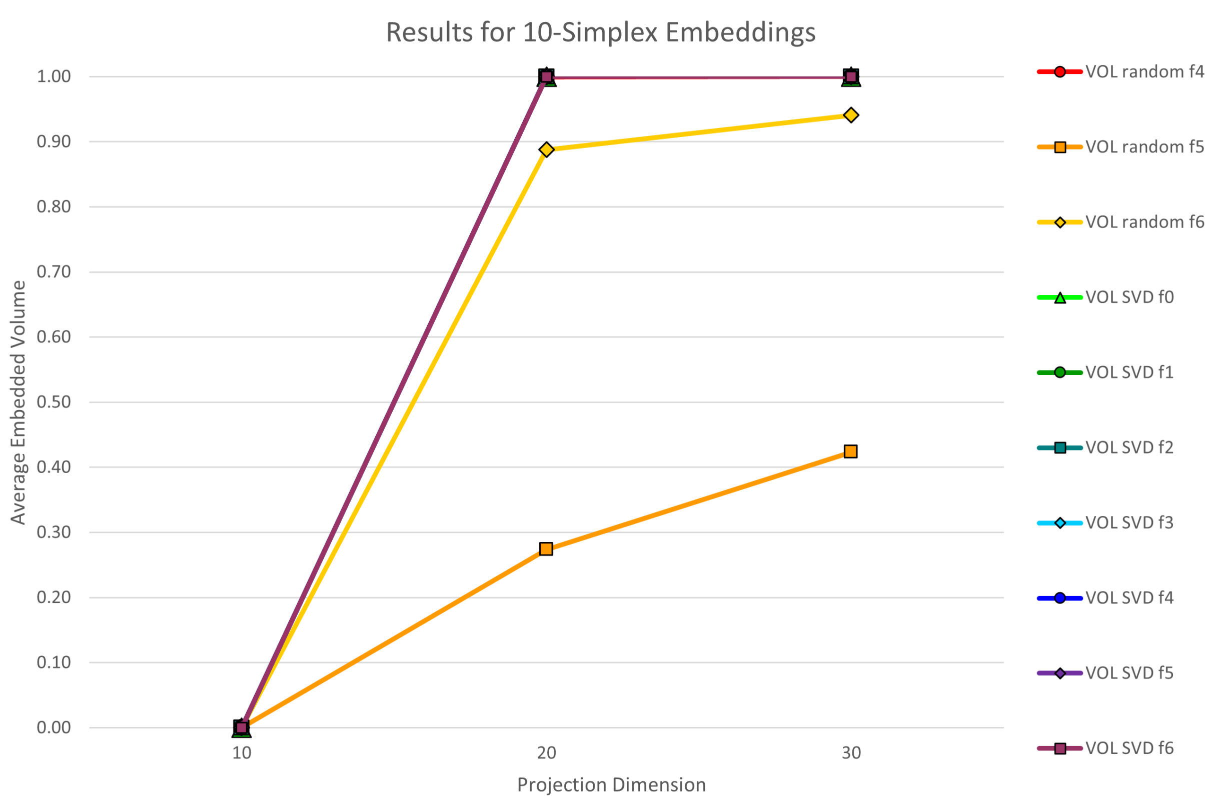


Figure 8.14: Comparison of average embedded 10-simplex volumes for all algorithms.

Chapter 9

Supervised Embeddings

A benefit of the Smallest Volume Optimizing Projection algorithm is that it can be turned into a supervised embedding method for data with multiple labeled classes. To accomplish this, the list of k simplices generated from the data is restricted to those that contain vertices from a minimum of $c \leq k+1$ distinct classes. This reduced simplex list is then given to the algorithm for optimization. This encourages embeddings that separate points from different classes over those that separate points within the same class. It has the additional benefit of reducing the overall number of simplices required for calculations.

Because it consistently out-performed all tests using random initialization, SVD initialization is used for all trials in this chapter. Because of this, each test is only run once. All five options for step size function $f(t)$ are used for each test, and the maximum number of allowed iterations is 30,000.

9.1 Testing

For initial testing, three points in $\mathbb{R}^3$ are projected into 2 dimensions to observe the differences in supervised and unsupervised embedding. Each point cloud is created by randomly generating a single point x_0, and then adding nine perturbations of the initial point using another randomly generated point. Thus we have $x_i = x_0 + 0.01y_i$ for $i \in \{1, ..., 9\}$. This is repeated to generate three such epsilon balls in $\mathbb{R}^3$, with the points from each ball labeled as classes $0, 1,$ or 2, with 30 data points total.

Embeddings into 2 dimensions are performed by optimizing over 1-simplices and 2-simplices using the standard unsupervised algorithm outlined in Chapter 6. These trials require 435 and 4060 simplices, respectively.

The same embeddings are performed using supervised embeddings. For the 1-simplex embedding, each simplex must contain $c = 2$ distinct labels among its vertices, which leaves a total of

362 simplices for optimization. For 2-simplices, two embeddings are performed requiring each simplex to contain $c = 2$ distinct labels (3860 total simplices) and then $c = 3$ distinct class labels (2507 total simplices).

9.1.1 Results

Table 9.1 shows the result of each embedding for each choice of step size function. Note that requiring only one label per simplex $c = 1$ corresponds to the unsupervised version of the algorithm. The best results in terms of smallest embedded volume ratio and average embedded volume ratio are highlighted in blue. Since this algorithms optimizes the smallest embedded volume, all embedding images in this section correspond to the step size function that yielded the greatest smallest volume for each trial, i.e. the highlighted result in the table, unless otherwise stated.

Figure 9.1 shows the results of embedding the three epsilon balls into 2 dimensions while optimizing 1-simplex lengths. The unsupervised embedding (right) considers all 435 simplices while optimizing, whereas the supervised version (left) requires each simplex to contain points from two distinct classes, which reduces the set to 362 simplices. Although the three classes can be easily embedded into 2 dimensions with clear separation, the supervised embedding provides slightly more separation between classes 1 and 2 than the unsupervised version does. The supervised embedding clearly emphasizes the distinction between the three groups of points.

Embeddings optimizing over 2-simplices are shown in Figure 9.2. Here, the unsupervised embedding (top left) is compared with supervised trials requiring each 2-simplex to contain vertices in either at least two classes (top right, bottom left) or all three classes (bottom right). Both versions of the supervised embedding placed the three classes farther from each other than in the unsupervised version. Interestingly, although the bottom right plot corresponds to the best result for the $c = 2$ trial, the top left plot shows better separation of the three classes, suggesting that the best embedding for separation may not always be the optimal one in terms of volumes. Over all, the clustering of points in classes 1 and 2 appears to be slightly tighter in the supervised embeddings. This makes

Table 9.1: Results of embedding three artificially generated point clouds from $\mathbb{R}^3$ into 2 dimensions, both supervised and unsupervised.

k	c	$f(t)$	**Smallest Volume**	**Average Volume**	**Iterations**	**Run Time**
1	2	$f_0(t)$	9.294657E-02	0.81432402	30000	45.15
1	2	$f_1(t)$	6.739071E-02	0.72348746	30000	43.63
1	2	$f_2(t)$	6.738618E-02	0.72348912	30000	43.53
1	2	$f_3(t)$	1.375407E-03	0.37294818	16094	25.14
1	2	$f_4(t)$	1.308839E-01	0.82882846	30000	43.47
1	2	$f_5(t)$	6.728548E-02	0.72334018	30000	19.29
1	2	$f_6(t)$	1.308899E-01	0.82881732	30000	19.38
2	2	$f_0(t)$	2.890930E-07	0.31023610	323	39.54
2	2	$f_1(t)$	2.622353E-07	0.36558653	30000	83.08
2	2	$f_2(t)$	7.229020E-08	0.30331679	30000	83.44
2	2	$f_3(t)$	1.865178E-05	0.44321236	30000	83.33
2	2	$f_4(t)$	2.890930E-07	0.31023610	323	39.00
2	2	$f_5(t)$	9.832475E-06	0.46304050	1765	16.33
2	2	$f_6(t)$	1.031149E-07	0.46002368	30000	41.02
2	3	$f_0(t)$	5.377483E-10	0.48084928	309	26.02
2	3	$f_1(t)$	2.151902E-09	0.20221222	30000	69.77
2	3	$f_2(t)$	6.783570E-08	0.19582346	30000	69.98
2	3	$f_3(t)$	2.360202E-05	0.52028171	21301	56.90
2	3	$f_4(t)$	5.377483E-10	0.48084928	309	26.08
2	3	$f_5(t)$	5.461152E-05	0.32241754	878	10.03
2	3	$f_6(t)$	2.446060E-07	0.53948899	30000	36.73
1	1	$f_0(t)$	4.003761E-02	0.81733921	30000	46.47
1	1	$f_1(t)$	5.421952E-02	0.76693338	10231	17.65
1	1	$f_2(t)$	5.421570E-02	0.76693301	30000	43.40
1	1	$f_3(t)$	1.375407E-03	0.42516726	16094	26.13
1	1	$f_4(t)$	6.683083E-02	0.81791036	30000	44.70
1	1	$f_5(t)$	3.294499E-02	0.74828210	30000	16.37
1	1	$f_6(t)$	6.516151E-02	0.80500117	30000	17.02
2	1	$f_0(t)$	8.564592E-10	0.24405820	322	41.64
2	1	$f_1(t)$	2.147723E-07	0.43004918	30000	85.26
2	1	$f_2(t)$	1.720757E-09	0.46824363	30000	84.62
2	1	$f_3(t)$	2.926922E-08	0.49994640	30000	85.27
2	1	$f_4(t)$	8.564592E-10	0.24405820	322	41.21
2	1	$f_5(t)$	1.990971E-07	0.36464681	30000	33.30
2	1	$f_6(t)$	4.033583E-08	0.51825539	30000	30.81

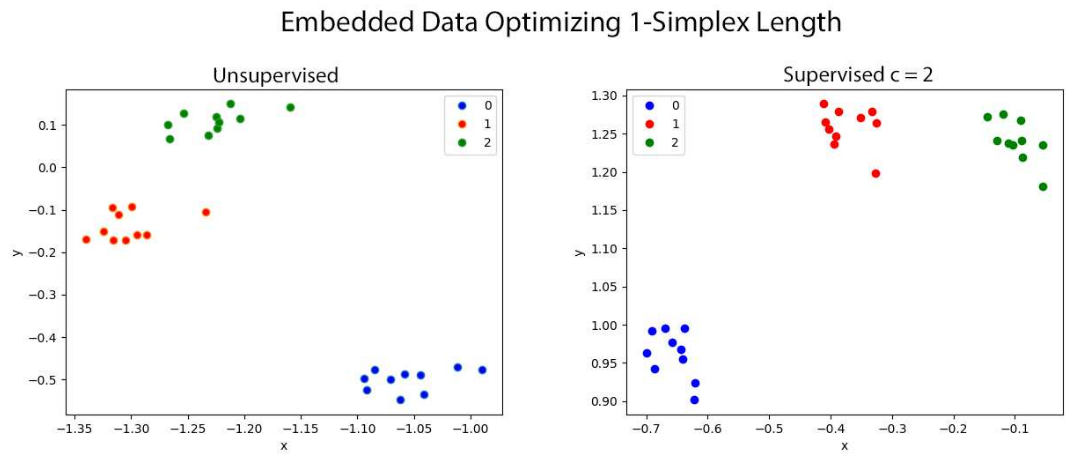


Figure 9.1: Embedding of point clouds into 2 dimensions while optimizing over 1-simplices, both unsupervised (left) and supervised (right).

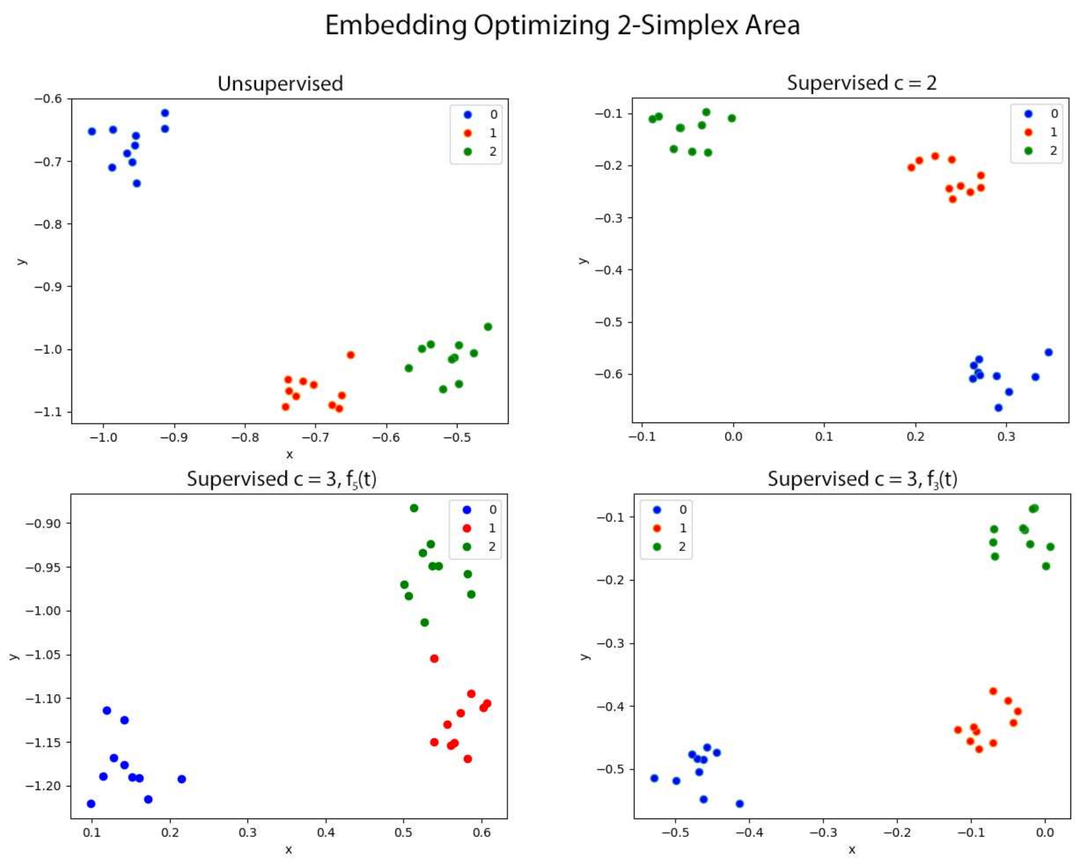


Figure 9.2: Embedding of point clouds in 2 dimensions while optimizing over 2-simplices. The unsupervised embedding (top) is contrasted with two versions of supervised embedding (bottom).

sense, since the setup of the supervised versions should de-emphasize intra-class variation in the projections.

Based on this simple example, the supervised version of the algorithm works as expected and can be tested on more complex problems.

9.2 MNIST Trials

The supervised Smallest Volume Optimizing Projection algorithm was tested on different subsets of points taken from the MNIST handwritten data set. The first trial utilizes the same data from classes 0 and 1 featured in Chapter 6 and compares the results of performing a supervised embedding with the unsupervised results from the previous chapter. The second trial compares embeddings for three different classes. Ten points are taken from each of the 0, 1, and 5 classes to test performance on a harder problem that allows for more variation in the choice of c. These results are again compared to the unsupervised version of the algorithm.

9.2.1 Two Class Trial

The supervised Smallest Volume Optimizing Projection algorithm was applied to the same set of 20 data points used for comparisons in Chapter 8. This set consisted of 10 points from class 0 and 10 from class 1. The best results from the unsupervised trials performed in Chapter 8 are compared with various supervised embeddings into 2 and 3 dimensions. The supervised embeddings are performed using $k = \{1, 2, 3\}$ with $c = \{2\}$.

Recall that in the case of the unsupervised embedding, optimizing over volumes for $k = 1$ requires 190 simplices, optimizing using $k = 2$ generates 1140 simplices, and setting $k = 3$ yields 4845 total simplices. For the supervised version of the algorithm, each simplex is required to contain $c = 2$ distinct class labels among its vertices, and any simplex that does not meet the criteria is eliminated. This reduces the simplex count to 142 for $k = 1$, 1010 for $k = 2$, and 4610 for $k = 3$.

9.2.2 Results

Results from both the unsupervised version of the algorithm and the supervised version are in Table 9.2. The best result for each combination of embedding dimension, simplex dimension k, and labels per simplex c is highlighted in blue, and all figures in this section correspond to the best result for that particular trial.

Table 9.2: Results of both supervised ($c = 2$) and unsupervised ($c = 1$) embeddings of the 20 MNIST points for varying embedding dimensions and simplex dimensions.

m	k	c	$f(t)$	**Smallest Vol**	**Average Vol**	**Iterations**	**Run Time**
2	1	1	$f_0(t)$	3.624961E-04	0.10220347	30000	152.62
2	1	1	$f_1(t)$	5.796176E-04	0.17955810	29759	141.91
2	1	1	$f_2(t)$	8.622637E-04	0.19365210	30000	144.87
2	1	1	$f_3(t)$	9.856875E-05	0.08207915	30000	155.38
2	1	1	$f_4(t)$	2.077548E-04	0.10246746	30000	147.85
2	1	1	$f_5(t)$	7.551618E-03	0.09252580	2418	25.86
2	1	1	$f_6(t)$	3.222483E-04	0.10584590	30000	307.97
2	2	1	$f_0(t)$	4.552132E-09	0.01545648	30000	125.44
2	2	1	$f_1(t)$	3.095084E-09	0.03726954	30000	96.64
2	2	1	$f_2(t)$	1.011944E-10	0.03531113	30000	96.95
2	2	1	$f_3(t)$	5.386613E-11	0.03171653	7385	27.03
2	2	1	$f_4(t)$	9.634004E-09	0.01530622	30000	96.83
2	2	1	$f_5(t)$	1.371697E-05	0.03617139	702	16.15
2	2	1	$f_6(t)$	9.104938E-09	0.01957450	30000	398.15
3	1	1	$f_0(t)$	9.907344E-03	0.18560545	30000	145.99
3	1	1	$f_1(t)$	9.469826E-03	0.35437305	7201	37.71
3	1	1	$f_2(t)$	7.963975E-03	0.35178111	30000	145.34
3	1	1	$f_3(t)$	1.598877E-02	0.26380230	30000	145.69
3	1	1	$f_4(t)$	8.982057E-03	0.18187704	30000	148.97

m	k	c	$f(t)$	**Smallest Vol**	**Average Vol**	**Iterations**	**Run Time**
3	1	1	$f_5(t)$	5.105273E-02	0.22586481	7713	81.28
3	1	1	$f_6(t)$	2.313210E-03	0.15643573	30000	322.06
3	2	1	$f_0(t)$	1.447951E-05	0.03856000	30000	89.95
3	2	1	$f_1(t)$	2.511632E-05	0.07631110	15277	46.02
3	2	1	$f_2(t)$	4.716414E-07	0.06559333	30000	86.63
3	2	1	$f_3(t)$	3.008270E-05	0.04460860	30000	87.13
3	2	1	$f_4(t)$	2.271325E-05	0.03797086	30000	85.40
3	2	1	$f_5(t)$	5.197991E-04	0.06305436	1604	25.53
3	2	1	$f_6(t)$	8.910990E-06	0.02350864	30000	304.88
3	3	1	$f_0(t)$	1.177711E-12	0.00481294	30000	586.27
3	3	1	$f_1(t)$	8.995338E-11	0.00959238	15540	267.19
3	3	1	$f_2(t)$	3.950434E-10	0.00832624	30000	474.85
3	3	1	$f_3(t)$	2.729385E-09	0.01119360	30000	476.46
3	3	1	$f_4(t)$	6.341945E-12	0.00524168	30000	473.52
3	3	1	$f_5(t)$	1.219157E-07	0.01519736	186	16.71
3	3	1	$f_6(t)$	1.096202E-17	0.00711418	30000	384.93
2	1	2	$f_0(t)$	1.797549E-03	0.15512295	30000	442.21
2	1	2	$f_1(t)$	4.807618E-05	0.18629039	30000	427.00
2	1	2	$f_2(t)$	7.629448E-04	0.13399496	30000	417.19
2	1	2	$f_3(t)$	8.979762E-03	0.34013498	30000	411.03
2	1	2	$f_4(t)$	1.018599E-03	0.16083959	30000	423.48
2	1	2	$f_5(t)$	4.668371E-02	0.30485893	8680	89.71
2	1	2	$f_6(t)$	1.245289E-03	0.12240143	30000	305.90
2	2	2	$f_0(t)$	1.510570E-08	0.03595974	30000	515.74
2	2	2	$f_1(t)$	6.902145E-09	0.03269160	10534	162.80
2	2	2	$f_2(t)$	2.099387E-07	0.03715404	30000	441.77

m	k	c	$f(t)$	**Smallest Vol**	**Average Vol**	**Iterations**	**Run Time**
2	2	2	$f_3(t)$	9.933433E-08	0.04501827	11378	169.11
2	2	2	$f_4(t)$	8.638230E-09	0.03441449	30000	436.28
2	2	2	$f_5(t)$	3.147475E-06	0.04055314	351	8.18
2	2	2	$f_6(t)$	1.721830E-07	0.01963042	30000	312.97
3	1	2	$f_0(t)$	3.477058E-03	0.18032301	30000	411.06
3	1	2	$f_1(t)$	2.198637E-02	0.28387554	23299	329.39
3	1	2	$f_2(t)$	1.343321E-02	0.20320344	30000	426.78
3	1	2	$f_3(t)$	9.493991E-03	0.34910161	30000	425.02
3	1	2	$f_4(t)$	3.581782E-03	0.18187014	30000	416.79
3	1	2	$f_5(t)$	8.974415E-02	0.36457510	4216	44.41
3	1	2	$f_6(t)$	1.226825E-03	0.14161667	30000	307.47
3	2	2	$f_0(t)$	5.300666E-06	0.02785668	30000	442.29
3	2	2	$f_1(t)$	5.303215E-05	0.06210515	11917	173.35
3	2	2	$f_2(t)$	4.746911E-06	0.04745892	30000	411.98
3	2	2	$f_3(t)$	9.309785E-07	0.06008471	28414	400.34
3	2	2	$f_4(t)$	4.223326E-06	0.02786620	30000	421.62
3	2	2	$f_5(t)$	1.313320E-03	0.08150846	492	9.62
3	2	2	$f_6(t)$	9.343714E-06	0.03523860	30000	305.42
3	3	2	$f_0(t)$	5.354935E-10	0.00438678	30000	574.33
3	3	2	$f_1(t)$	4.336872E-12	0.01289064	11198	199.61
3	3	2	$f_2(t)$	1.004742E-12	0.01083940	30000	444.67
3	3	2	$f_3(t)$	1.578441E-11	0.01300664	26968	425.97
3	3	2	$f_4(t)$	2.935376E-11	0.00444241	30000	450.70
3	3	2	$f_5(t)$	1.399825E-07	0.01472342	246	19.92
3	3	2	$f_6(t)$	4.221074E-12	0.01032571	30000	374.66

Figure 9.3 compares unsupervised and supervised embeddings into 2 dimensions. Both supervised and unsupervised versions of the algorithm were tested while optimizing over simplices of dimension $k = 1$, followed by $k = 2$. The plots of the unsupervised embedded data (top) shows points from the two classes intermixed with no clear separation when $k = 1$ and only slight separation when $k = 2$. Conversely, the supervised plots (bottom) have very distinctly separated classes

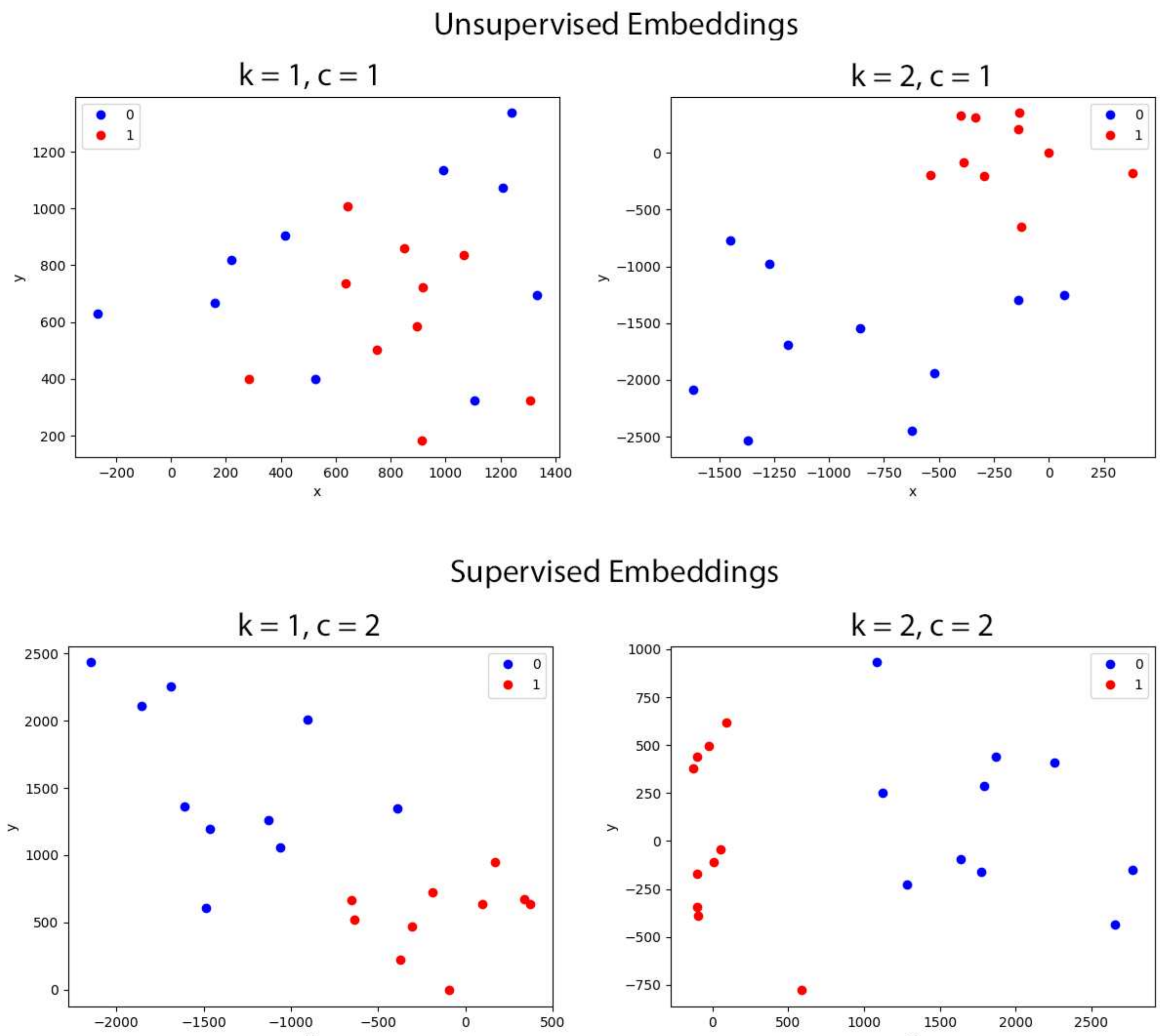


Figure 9.3: Embeddings of 20 points from the MNIST data set into 2 dimensions, both supervised and unsupervised.

for both choices of k, and the plot for $k = 2$ shows significantly more separation using $c = 2$ than in the unsupervised $c = 1$ version.

Figure 9.4 compares supervised and unsupervised embeddings into 3 dimensions while optimizing simplices for $k = 1, 2$ or 3. In this case, the unsupervised embedding for $k = 1$ (top left) already has the two classes grouped reasonably well, but the remaining unsupervised embeddings using $k = 2$ and $k = 3$ have points from both classes intermixed. The supervised version of the algorithm shows some difference from the unsupervised embedding for $k = 1$, particularly in how tightly each class is clustered. For $k = 2$, the data is now clearly separable in the supervised version. When $k = 3$, the supervised embedding very clearly emphasizes the distinction between classes by spreading the two classes much further apart and putting the points into tighter groupings.

9.2.3 Three Class Trial

A three-class trial using digits 0,1, and 5 from the MNIST data set was also performed to test the algorithm on data that is slightly harder to separate, as well as to compare embeddings using different choices of c. Again, ten points were selected from each of the classes 0, 1, and 5 for a total of 30 total data points (note that the points for classes 0 and 1 are different than those used in the previous section). The points were embedded in 2 dimensions optimizing over 1-simplices or 2-simplices, and into 3 dimensions optimizing over $1, 2,$ and 3-simplices. Unsupervised versions of each embedding are compared to embeddings using all possible values of c from $c \in \{2, 3\}$.

The unsupervised Smallest Volume Optimizing Projection algorithm requires a total of 435 simplices when $k = 1$, 4060 simplices when $k = 2$, and 27,405 simplices when $k = 3$. Requiring each simplex to contain vertices in at least $c = 3$ classes reduces these numbers to 362 for $k = 1$, 3860 for $k = 2$, and 27,040 for $k = 3$. For $k \geq 2$, simplices can be required to contain a vertex in all three possible classes. Doing so reduces the simplex count even further to 2507 for $k = 2$ and $20,248$ for $k = 3$.

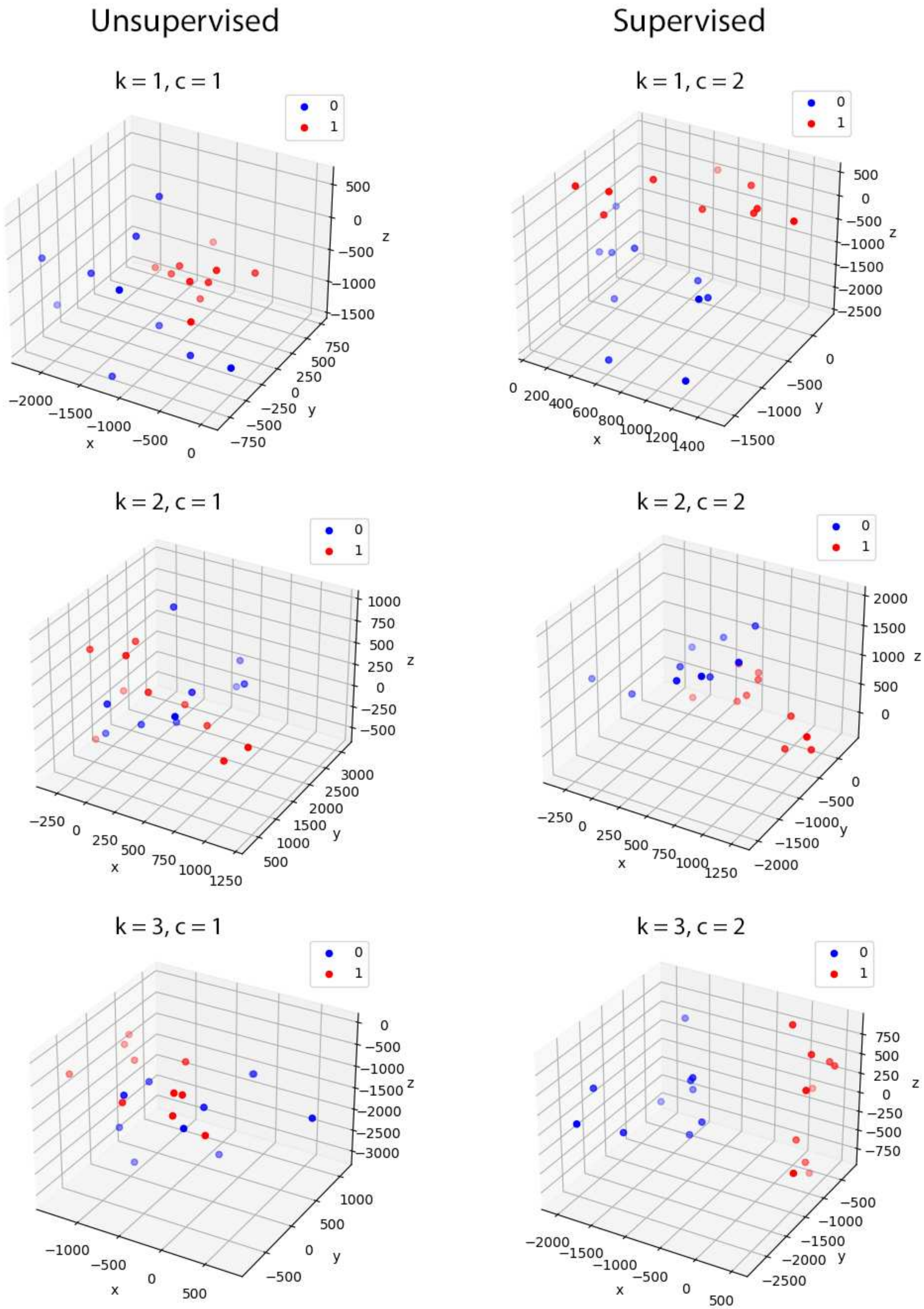


Figure 9.4: Embeddings of 20 points from the MNIST data set into 3 dimensions, both supervised and unsupervised

9.2.4 Results

Table 9.3 contains the results from all trials on this data set. Once again, the best result for each trial is marked in blue, and all plots shown in this section correspond to the best result for that particular trial unless otherwise stated.

Table 9.3: Results of unsupervised and supervised embeddings of the 30 points from MNIST classes 0, 1, and 5 for varying embedding dimensions and simplex dimensions (k).

m	k	c	$f(t)$	**Smallest Vol**	**Average Vol**	**Iterations**	**Run Time**
2	1	1	$f_0(t)$	1.365702E-04	0.09496191	30000	154.23
2	1	1	$f_1(t)$	6.950126E-04	0.17521008	10043	51.21
2	1	1	$f_2(t)$	3.913354E-04	0.12617333	30000	147.18
2	1	1	$f_3(t)$	7.837724E-04	0.23255910	30000	149.94
2	1	1	$f_4(t)$	2.458717E-04	0.09573350	30000	152.40
2	1	1	$f_5(t)$	1.009469E-02	0.23728943	6148	60.36
2	1	1	$f_6(t)$	1.396374E-04	0.10093911	50000	536.54
2	2	1	$f_0(t)$	1.260266E-10	0.02755172	18	3.47
2	2	1	$f_1(t)$	1.512057E-08	0.03266167	16461	62.73
2	2	1	$f_2(t)$	1.201293E-08	0.02362877	30000	106.10
2	2	1	$f_3(t)$	2.597938E-11	0.01350953	17006	64.66
2	2	1	$f_4(t)$	1.260266E-10	0.02755172	18	2.87
2	2	1	$f_5(t)$	8.129552E-07	0.02656961	202	14.88
2	2	1	$f_6(t)$	1.573032E-10	0.01277880	50000	631.62
3	1	1	$f_0(t)$	1.408074E-03	0.11966545	30000	154.50
3	1	1	$f_1(t)$	1.703915E-02	0.23592629	27033	139.45
3	1	1	$f_2(t)$	4.091383E-03	0.19761735	30000	154.89
3	1	1	$f_3(t)$	1.110317E-02	0.29911803	30000	155.57
3	1	1	$f_4(t)$	8.990254E-04	0.11960082	30000	143.86
3	1	1	$f_5(t)$	3.818807E-02	0.26662484	1914	22.35

m	k	c	$f(t)$	**Smallest Vol**	**Average Vol**	**Iterations**	**Run Time**
3	1	1	$f_6(t)$	3.350564E-04	0.11132891	50000	525.08
3	2	1	$f_0(t)$	2.816856E-07	0.03206005	30000	106.95
3	2	1	$f_1(t)$	1.076540E-05	0.02911222	26689	85.86
3	2	1	$f_2(t)$	1.253618E-06	0.01751952	30000	97.11
3	2	1	$f_3(t)$	2.466753E-06	0.02251359	10471	40.57
3	2	1	$f_4(t)$	3.111526E-06	0.03228157	30000	95.86
3	2	1	$f_5(t)$	2.812777E-04	0.04829052	895	28.79
3	2	1	$f_6(t)$	3.613473E-06	0.03231692	50000	596.15
3	3	1	$f_0(t)$	3.205152E-14	0.00315610	50000	389.28
3	3	1	$f_1(t)$	1.957225E-13	0.00407649	6704	92.39
3	3	1	$f_2(t)$	1.587562E-13	0.00303270	50000	228.83
3	3	1	$f_3(t)$	2.445088E-11	0.00625344	50000	230.92
3	3	1	$f_4(t)$	1.706637E-13	0.00320958	50000	228.40
3	3	1	$f_5(t)$	3.482063E-11	0.00587508	146	98.97
3	3	1	$f_6(t)$	1.892154E-14	0.00364278	50000	668.08
2	1	2	$f_0(t)$	1.552665E-04	0.11244123	30000	150.75
2	1	2	$f_1(t)$	1.396273E-04	0.11863081	14043	70.18
2	1	2	$f_2(t)$	2.271079E-04	0.09262317	30000	146.21
2	1	2	$f_3(t)$	1.618868E-03	0.24595421	30000	149.15
2	1	2	$f_4(t)$	1.371930E-04	0.11177959	30000	142.20
2	1	2	$f_5(t)$	1.357563E-02	0.18785105	6748	73.86
2	1	2	$f_6(t)$	1.591661E-05	0.11510475	50000	529.05
2	2	2	$f_0(t)$	1.731110E-09	0.02752316	30000	155.92
2	2	2	$f_1(t)$	1.067587E-10	0.02517558	30000	103.44
2	2	2	$f_2(t)$	7.820271E-09	0.02161990	30000	104.81
2	2	2	$f_3(t)$	2.220925E-11	0.01579988	14967	57.27

m	k	c	$f(t)$	**Smallest Vol**	**Average Vol**	**Iterations**	**Run Time**
2	2	2	$f_4(t)$	4.787848E-09	0.02743483	30000	103.61
2	2	2	$f_5(t)$	8.094851E-07	0.02813830	381	16.78
2	2	2	$f_6(t)$	1.658441E-08	0.02387604	50000	533.28
3	1	2	$f_0(t)$	2.016108E-03	0.12966487	30000	159.75
3	1	2	$f_1(t)$	9.358855E-03	0.22231384	8057	43.60
3	1	2	$f_2(t)$	6.336362E-03	0.17011378	30000	146.85
3	1	2	$f_3(t)$	1.457191E-02	0.32050444	22540	111.77
3	1	2	$f_4(t)$	2.041109E-03	0.12893883	30000	154.23
3	1	2	$f_5(t)$	3.786746E-02	0.29276755	8628	94.57
3	1	2	$f_6(t)$	5.441073E-04	0.10737648	50000	505.71
3	2	2	$f_0(t)$	3.558250E-06	0.04517100	30000	98.00
3	2	2	$f_1(t)$	1.754342E-06	0.04547945	30000	94.63
3	2	2	$f_2(t)$	4.182305E-06	0.03363245	30000	92.82
3	2	2	$f_3(t)$	8.657660E-06	0.03207286	30000	94.86
3	2	2	$f_4(t)$	1.991349E-06	0.04478855	30000	93.91
3	2	2	$f_5(t)$	2.020848E-04	0.06160782	1173	25.19
3	2	2	$f_6(t)$	3.665140E-06	0.02531917	50000	562.53
3	3	2	$f_0(t)$	2.869578E-13	0.00263493	50000	322.07
3	3	2	$f_1(t)$	7.457418E-14	0.00453351	17966	128.73
3	3	2	$f_2(t)$	4.034804E-13	0.00332737	50000	230.37
3	3	2	$f_3(t)$	2.461085E-13	0.00648767	22534	140.92
3	3	2	$f_4(t)$	5.910459E-13	0.00263535	50000	234.30
3	3	2	$f_5(t)$	1.145921E-10	0.00644859	145	97.46
3	3	2	$f_6(t)$	1.244754E-12	0.00541394	50000	669.44
2	2	3	$f_0(t)$	1.304818E-10	0.01462672	30000	152.16
2	2	3	$f_1(t)$	1.103313E-08	0.02865061	22605	76.04

m	k	c	$f(t)$	**Smallest Vol**	**Average Vol**	**Iterations**	**Run Time**
2	2	3	$f_2(t)$	2.950412E-09	0.02312254	30000	99.41
2	2	3	$f_3(t)$	2.440945E-10	0.01341196	30000	97.42
2	2	3	$f_4(t)$	1.201420E-11	0.01497418	30000	102.81
2	2	3	$f_5(t)$	1.147505E-06	0.01985284	556	14.69
2	2	3	$f_6(t)$	4.845641E-11	0.01707172	50000	526.57
3	2	3	$f_0(t)$	6.835985E-07	0.02559370	30000	100.27
3	2	3	$f_1(t)$	7.176923E-06	0.04667282	17193	56.04
3	2	3	$f_2(t)$	1.102981E-05	0.04981084	30000	88.56
3	2	3	$f_3(t)$	4.196500E-07	0.02381687	17835	56.29
3	2	3	$f_4(t)$	2.471351E-07	0.02610086	30000	91.26
3	2	3	$f_5(t)$	3.083831E-04	0.03832089	1096	23.41
3	2	3	$f_6(t)$	6.946655E-06	0.02291259	50000	560.58
3	3	3	$f_0(t)$	2.938427E-14	0.00651361	50000	289.38
3	3	3	$f_1(t)$	8.461471E-11	0.00863617	13034	95.70
3	3	3	$f_2(t)$	2.100534E-12	0.00743500	48238	205.42
3	3	3	$f_3(t)$	5.129498E-12	0.00953001	15463	105.71
3	3	3	$f_4(t)$	1.667141E-11	0.00646684	50000	212.07
3	3	3	$f_5(t)$	2.214302E-09	0.00688635	114	77.01
3	3	3	$f_6(t)$	2.717263E-12	0.00546523	50000	624.00

Embeddings of the data in 2 dimensions are plotted in Figure 9.5. The unsupervised embeddings separate the three classes surprisingly well for both choices of k, though there is still distinct mixing between the groups. For the supervised algorithm, only optimizing using $k = 2$ and $c = 2$ yields completely separable data in 2D. Additionally, the plot where the three classes are at all separable (bottom left) does not correspond to the best result from that trial, whereas the plot from the best run (top right) looks very similar to the unsupervised embedding. When $k = 1$, the supervised algorithm performs comparably to the unsupervised version, but for $k = 2$ and $c = 3$ the supervised embedding is remarkably worse.

Figure 9.6 shows the unsupervised embeddings of the data into 3 dimensions for $k = 1, 2$, and 3. The three classes are not at all separated except when using $k = 3$ (bottom), which shows some tendency towards grouping by class. Figure 9.7 displays the five different supervised embeddings in 3 dimensions. Optimizing over 1-simplices yields an embedding similar to the unsupervised embedding using $k = 3$. Optimizing over 2-simplices with $c = 2$ yielded relatively good separation between classes, but using $c = 3$ resulted in a much worse embedding. Using $k = 3$ and $c = 2$ did not really separate the three classes either. The best separation overall occurred using 3-simplices with $c = 3$.

9.3 Conclusions

Applying the supervised version of the Smallest Volume Optimizing Projection algorithm provides distinct separation between classes in many cases. However, there are some cases where it did not improve results over the unsupervised version of the projection. Additionally, the choice of c for the supervised embedding significantly impacts the resulting configuration of data points, and the best choice for c varies both with the projection dimension and the data itself. Interestingly, there were also cases where the best separation between classes was found using a step size function that did not produce the best results for that specific trial. This may again be related to the choices of k and c, since a poor choice of parameters can make the optimal solution to the algorithm a poor solution to the supervised problem.

In all the MNIST trials, the best smallest embedded volume was found using $f_5(t)$ as the step size function, regardless of whether or not the trial was supervised. Embeddings using randomly generated epsilon balls did not display this preference for $f_5(t)$, with the best results for every trial coming from a different step size function depending on the choices of c and k. This, combined

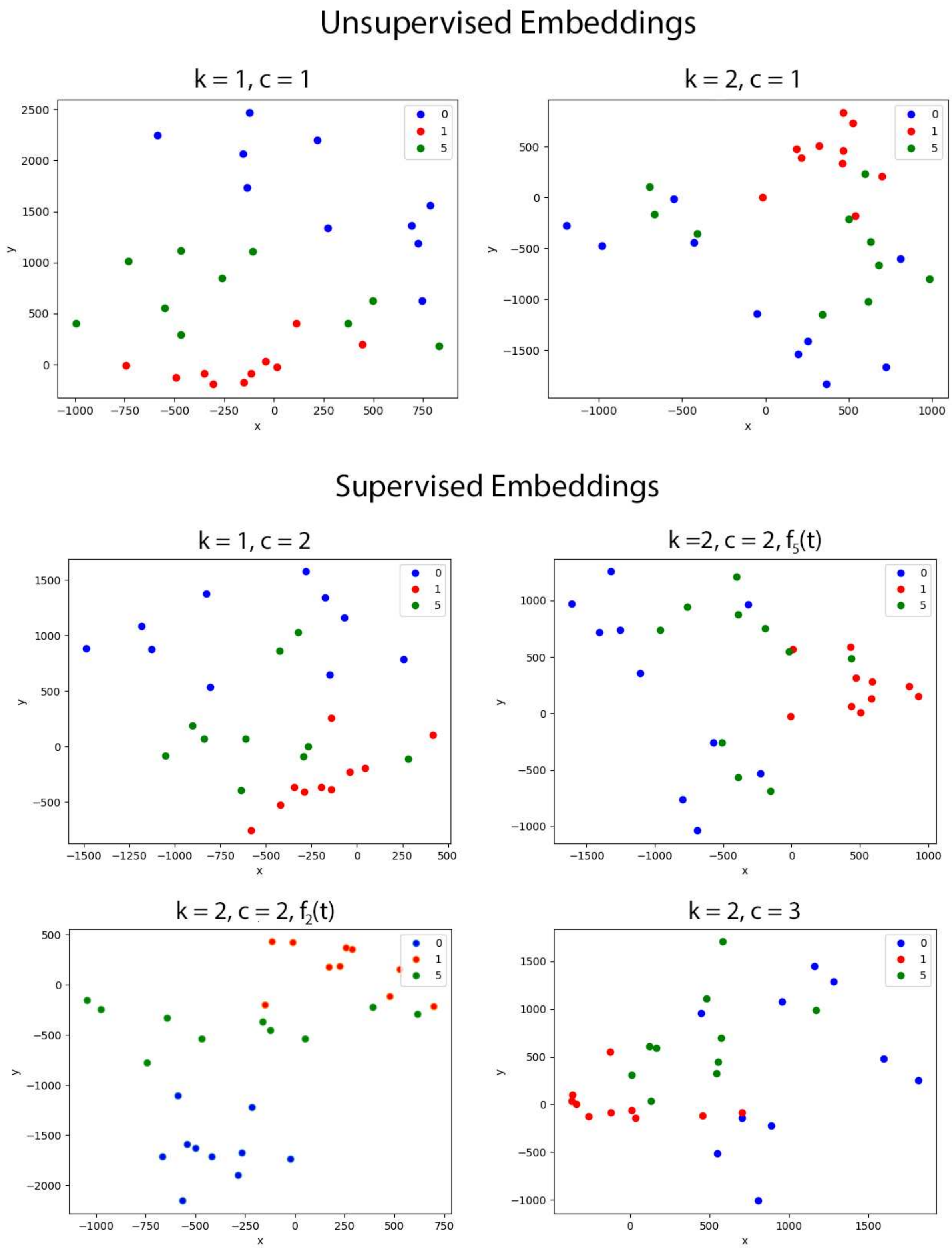


Figure 9.5: Projections of MNIST data from classes 0, 1, and 5 into 2 dimensions, both supervised and unsupervised.

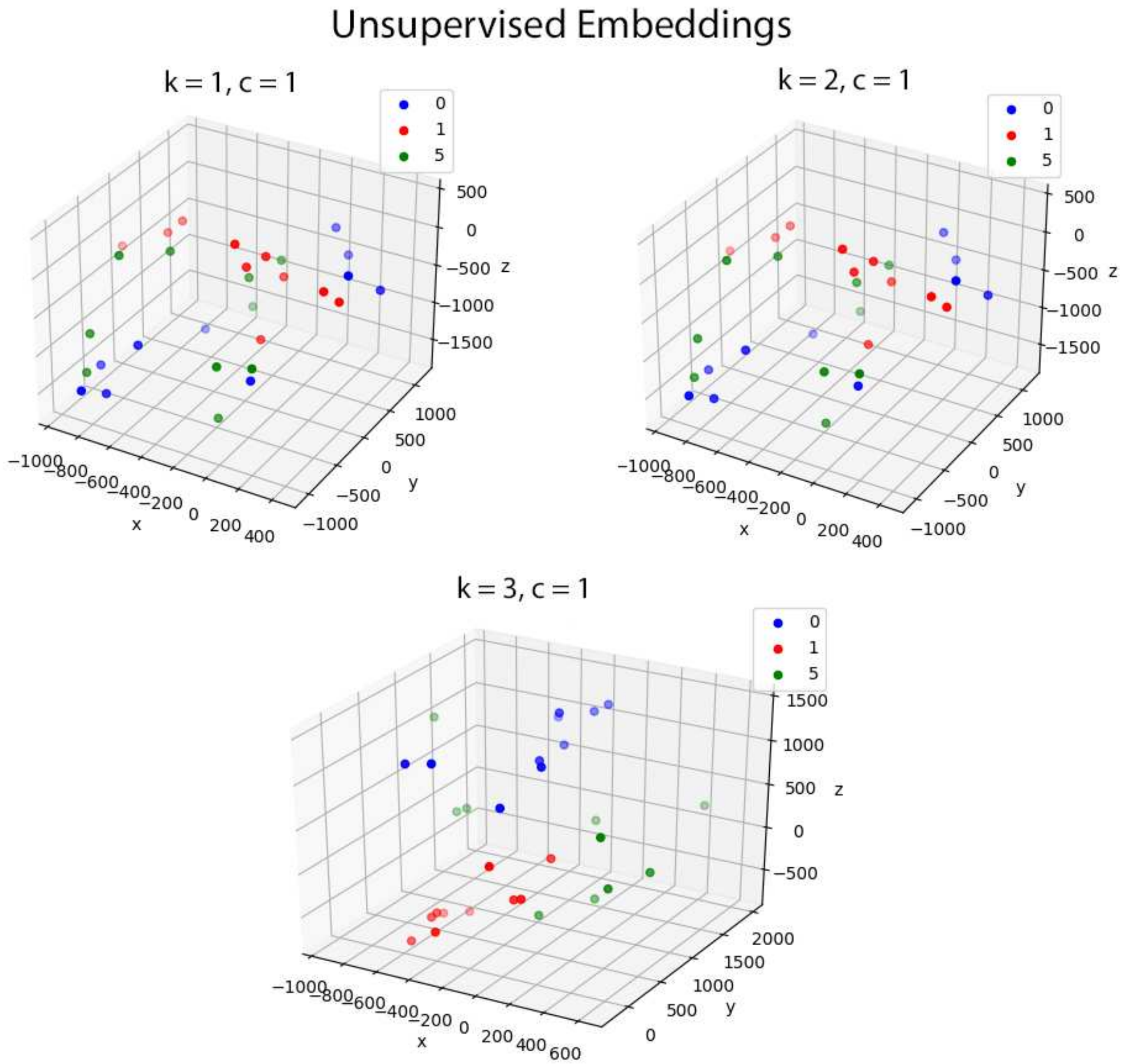


Figure 9.6: Unsupervised embeddings of MNIST data from classes 0, 1, and 5 into 3 dimensions.

with the results in the previous chapters, suggests that $f_5(t)$ may be particularly well-suited for use with the MNIST data set but is not necessarily a universally good choice for all data sets.

Overall, the supervised version of the Smallest Volume Optimizing Projection algorithm shows promising results, but more thorough testing is required.

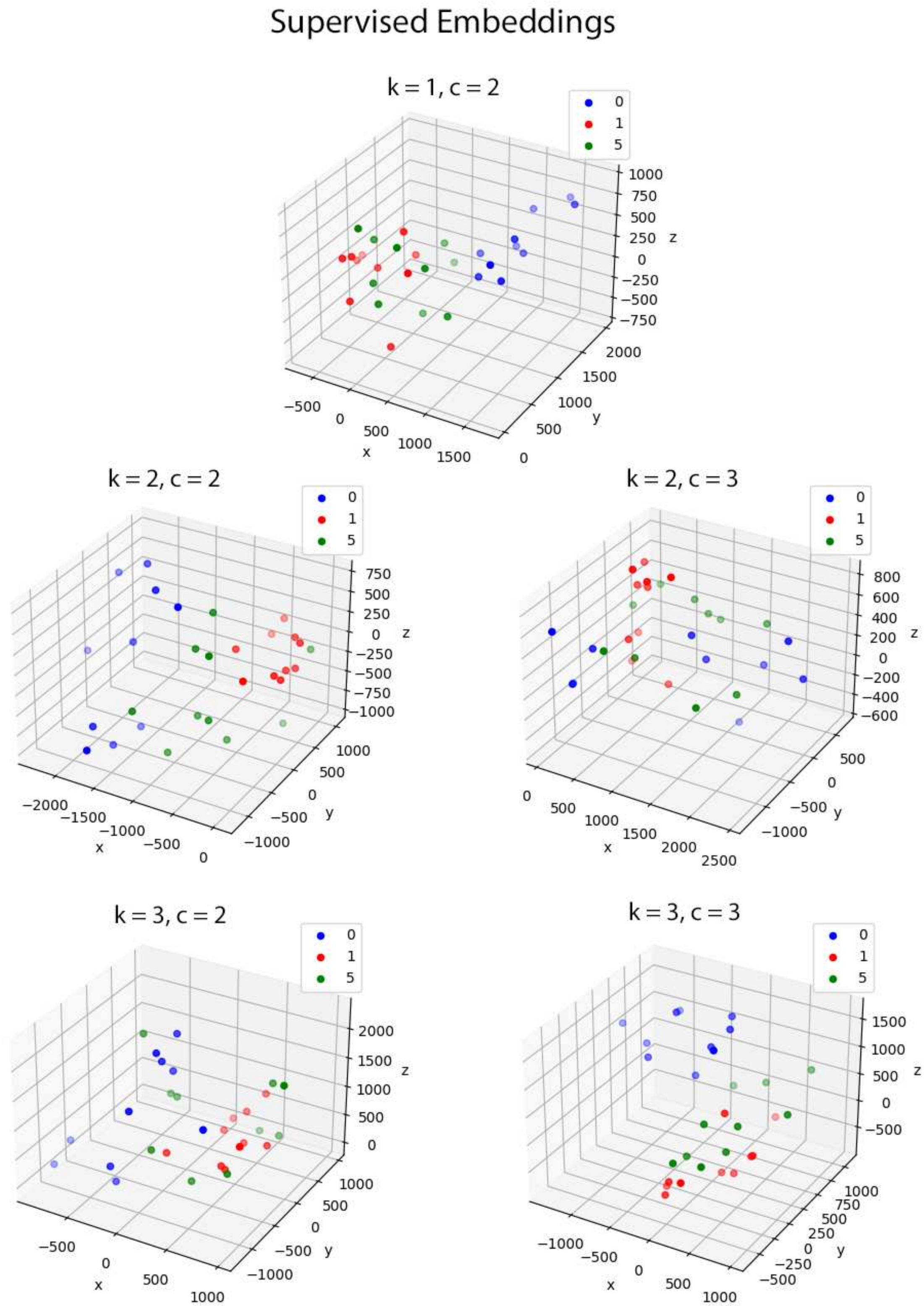


Figure 9.7: Supervised embeddings of MNIST data from classes 0, 1, and 5 into 3 dimensions.

Chapter 10

Conclusions and Further Work

In this work, two novel algorithms were developed for projecting data into lower dimensions while preserving the volumes enclosed by the data points. Performance of both algorithms was rigorously tested for a variety of embedding dimensions and volumes. The new algorithms were then compared to existing algorithms both in terms of optimizing volumes and overall computational complexity. Finally, the algorithm for optimizing the smallest embedded volume was tested as a method for supervised embedding of labeled data sets.

Optimizing over embedded volumes was demonstrated to be a distinct problem from optimizing embedded lengths. This is shown in Section 6.4, as well as consistently throughout the results in Chapter 9. Additionally, Section 4.2.2 demonstrated mathematically that the flag mean is the average of multiple Schubert varieties, which is confirmed to correspond to the projection space yielding the largest average simplex volume in Chapter 7.

The Smallest Volume Optimizing Projection algorithm shows promise in terms of performance, particularly as simplex dimensions increase. Its biggest drawback is the large computational cost associated with generating all possible simplices from a data set. Its performance varies with the choice of step size function $f(t)$, as expected, although the function $f_5(t)$ performed unusually well throughout the testing. It may be the case that $f_5(t)$ is most suited to the MNIST data, although results from Chapter 6 contradict this notion. Further testing is required to determine whether $f_5(t)$ is a universally good choice of step size function and how significantly its performance changes for different data sets.

The flag volume averaging algorithm performed excellently in terms of optimizing the average embedded k-volumes in the data, with the only real drawback being the large computational cost for higher dimensional simplices. A GPU implementation for the SVD step in the flag averaging algorithm would make it much more viable for large problems.

Going forward, additional methods for reducing the number of simplices without losing data structure should be explored to improve the performance of both algorithms. Possibilities include applying clustering algorithms to the data and utilizing the centers to generate simplices, or restricting simplex generation to only nearby points (i.e. generate all d-simplices from the k-nearest neighbors of each data point).

More exploration using the supervised version of the Smallest Volume Optimizing Projection may yield interesting results, as the algorithm favors projections that cluster data points in the same class together. This would be a good method to test on real-world data applications, such as the biological data mentioned in Chapter 1.

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Appendix A

Notation Dictionary

This work attempts to, as much as possible, maintain consistency in naming and notation conventions across all chapters. This section provides a reference for such conventions.

- n - dimension of ambient space
- m - dimension of embedding space
- k - dimension of simplex ($k + 1$ vertices)
- d - other non-simplex based dimension
- N - total number of points in a set
- M - total number of simplices in a set
- $\mathbf{X}$ - matrix
- $x_0, x_1, ..., x_d$ - columns of $\mathbf{X}$
- $[X]$ - subspace spanned by columns of $\mathbf{X}$
- $\mathcal{S}$ - set of all simplices
- S - a single simplex
- $\mathbf{S}$ - matrix representation of single simplex
- $\{s_0, ..., s_k\}$ - vertices of a simplex
- X, Y - arbitrary sets or spaces
- m - an arbitrary measure on a space
- $d_\Box$ - a distance measure, differentiated by subscripts

- $f : X \to Y$ - arbitrary mapping/embedding
- $P : \mathbb{R}^n \to \mathbb{R}^m$ - orthogonal projection mapping
- $\mathbf{P}$ - orthogonal matrix representation of P
- i, j - indices
- $\square_i$ - list index
- $\square_{i_j}$ - jth column (or other indexed item) of the ith item in an indexed list
- $\square^{(i)}$ - iteration index
- s, t - parameters
- ϕ - contraction mapping
- θ - angle
- σ_i = ith singular value from SVD
- θ_i - ith principal angle between given subspaces
- $\Theta(X, Y)$ - a function of the principal angles between X and Y

Appendix B

Tables

This section contains tables with the complete results from the two largest experiments contained in this work. Table B.1 presents all results from testing done on the Smallest Volume Optimizing Projection algorithm in Chapter 6. Table B.2 contains the complete results of the algorithm comparison testing performed in Chapter 8.

Table B.1: Complete results from testing the Smallest Volume Optimizing Projector as described in 6.3.2. Here, k is the simplex dimension and m is the projection dimension.

k	m	Init	$f(t)$	Best Min Vol	Avg Min Vol	Avg Iterations		Avg Run Time (s)		Total Time
1	2	rand	$f_0(t)$	1.14689751E-04	4.09956517E-05	30000.00	±0.00	33.39	±0.39	1001.65
1	2	rand	$f_1(t)$	2.81335959E-04	6.68245999E-05	19046.93	± 7266.34	22.86	±7.19	685.89
1	2	rand	$f_2(t)$	1.92346819E-04	4.96464389E-05	30000.00	± 0.00	33.32	±0.29	999.45
1	2	rand	$f_3(t)$	2.74088625E-04	4.83241290E-05	28792.27	± 3805.87	32.43	±3.78	972.78
1	2 hubv	rand	$f_4(t)$	7.98471667E-05	2.39147884E-05	30000.00	± 0.00	33.56	±0.34	1006.79
1	2	rand	$f_5(t)$	1.63713019E-03	1.25306379E-03	2256.50	± 1225.62	6.20	±1.22	185.90
1	2	rand	$f_6(t)$	1.85508672E-04	5.66533737E-05	30000.00	± 0.00	33.01	±0.31	990.30
1	2	SVD	$f_0(t)$	1.71414666E-05	1.71414666E-05	30000.00	± 0.00	34.92	±0.00	34.92
1	2	SVD	$f_1(t)$	7.14730146E-05	7.14730146E-05	20578.00	± 0.00	24.14	±0.00	24.14
1	2	SVD	$f_2(t)$	4.23667190E-05	4.23667190E-05	30000.00	± 0.00	33.25	±0.00	33.25
1	2	SVD	$f_3(t)$	2.19806379E-05	2.19806379E-05	30000.00	± 0.00	33.96	±0.00	33.96
1	2	SVD	$f_4(t)$	2.57136192E-06	2.57136192E-06	30000.00	± 0.00	34.34	±0.00	34.34
1	2	SVD	$f_5(t)$	1.52937766E-03	1.52937766E-03	1284.00	± 0.00	5.16	±0.00	5.16
1	2	SVD	$f_6(t)$	1.62563225E-04	1.62563225E-04	30000.00	± 0.00	33.59	±0.00	33.59
1	3	rand	$f_0(t)$	9.11379539E-04	3.61520848E-04	30000.00	± 0.00	34.38	±0.33	1031.39
1	3	rand	$f_1(t)$	2.85951597E-03	8.76000025E-04	18705.60	± 9572.16	22.32	±9.40	669.52

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	3	rand	$f_2(t)$	1.59038079E-03	5.49401665E-04	30000.00	± 0.00	34.24	±0.51	1027.33
1	3	rand	$f_3(t)$	1.41385560E-03	5.57582012E-04	28898.00	± 3699.88	33.10	±3.73	993.10
1	3	rand	$f_4(t)$	1.49548025E-03	3.62974187E-04	30000.00	± 0.00	33.89	±0.47	1016.83
1	3	rand	$f_5(t)$	5.82681257E-03	4.39055673E-03	3382.83	± 1865.76	7.52	±1.88	225.70
1	3	rand	$f_6(t)$	1.01340793E-03	3.31521614E-04	29212.97	± 4238.30	33.21	±4.32	996.40
1	3	SVD	$f_0(t)$	4.05657321E-04	4.05657321E-04	30000.00	± 0.00	34.33	±0.00	34.33
1	3	SVD	$f_1(t)$	4.48625130E-04	4.48625130E-04	17366.00	± 0.00	21.38	±0.00	21.38
1	3	SVD	$f_2(t)$	5.30121147E-04	5.30121147E-04	30000.00	± 0.00	33.40	±0.00	33.40
1	3	SVD	$f_3(t)$	2.25284161E-03	2.25284161E-03	30000.00	± 0.00	34.83	±0.00	34.83
1	3	SVD	$f_4(t)$	4.23629745E-04	4.23629745E-04	30000.00	± 0.00	34.27	±0.00	34.27
1	3	SVD	$f_5(t)$	8.75749498E-03	8.75749498E-03	7110.00	± 0.00	11.06	±0.00	11.06
1	3	SVD	$f_6(t)$	2.28382811E-04	2.28382811E-04	30000.00	± 0.00	33.64	±0.00	33.64
1	4	rand	$f_0(t)$	2.75305185E-03	1.28673292E-03	30000.00	± 0.00	34.21	±0.31	1026.43
1	4	rand	$f_1(t)$	9.00289024E-03	3.10761052E-03	17327.13	± 8023.07	20.98	±7.94	629.45
1	4	rand	$f_2(t)$	6.22366952E-03	1.95581611E-03	30000.00	± 0.00	34.27	±0.26	1028.24
1	4	rand	$f_3(t)$	3.44589987E-03	1.88601911E-03	26609.30	± 6633.78	30.53	±6.67	915.90
1	4	rand	$f_4(t)$	2.88013077E-03	1.35549214E-03	29473.30	± 2222.39	33.53	±2.22	1005.93
1	4	rand	$f_5(t)$	1.12132844E-02	9.58702387E-03	4476.07	± 2332.30	8.39	±2.32	251.67

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	4	rand	$f_6(t)$	2.94330510E-03	1.18405304E-03	28130.90	± 5872.05	31.36	±5.73	940.71
1	4	SVD	$f_0(t)$	1.15613719E-03	1.15613719E-03	30000.00	± 0.00	34.53	±0.00	34.53
1	4	SVD	$f_1(t)$	3.57024057E-03	3.57024057E-03	25806.00	± 0.00	30.15	±0.00	30.15
1	4	SVD	$f_2(t)$	1.80808284E-03	1.80808284E-03	30000.00	± 0.00	34.15	±0.00	34.15
1	4	SVD	$f_3(t)$	1.32307339E-02	1.32307339E-02	17173.00	± 0.00	21.22	±0.00	21.22
1	4	SVD	$f_4(t)$	1.09985397E-03	1.09985397E-03	30000.00	± 0.00	34.51	±0.00	34.51
1	4	SVD	$f_5(t)$	2.35792624E-02	2.35792624E-02	2781.00	± 0.00	6.94	±0.00	6.94
1	4	SVD	$f_6(t)$	1.44306973E-03	1.44306973E-03	30000.00	± 0.00	33.35	±0.00	33.35
1	5	rand	$f_0(t)$	5.63556628E-03	2.87674423E-03	956.50	± 289.04	4.72	±0.31	141.63
1	5	rand	$f_1(t)$	1.50803034E-02	7.76203770E-03	20253.57	± 6750.70	24.62	±6.97	738.53
1	5	rand	$f_2(t)$	8.89734332E-03	4.88772780E-03	28730.00	± 4963.57	33.06	±5.07	991.85
1	5	rand	$f_3(t)$	8.89998348E-03	5.10840291E-03	28889.00	± 4200.88	33.16	±4.24	994.91
1	5	rand	$f_4(t)$	4.39565519E-03	2.31558960E-03	1149.73	± 1065.85	5.01	±1.04	150.17
1	5	rand	$f_5(t)$	1.94924486E-02	1.61831694E-02	4018.53	± 1847.87	8.00	±1.89	240.05
1	5	rand	$f_6(t)$	6.69197535E-03	3.34911123E-03	28640.40	± 5129.69	32.36	±5.07	970.91
1	5	SVD	$f_0(t)$	1.44935021E-03	1.44935021E-03	673.00	± 0.00	4.62	±0.00	4.62
1	5	SVD	$f_1(t)$	1.39842634E-02	1.39842634E-02	15421.00	± 0.00	19.58	±0.00	19.58
1	5	SVD	$f_2(t)$	1.20455028E-02	1.20455028E-02	30000.00	± 0.00	34.97	±0.00	34.97

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	5	SVD	$f_3(t)$	2.14156682E-02	2.14156682E-02	30000.00	± 0.00	34.82	±0.00	34.82
1	5	SVD	$f_4(t)$	1.44935021E-03	1.44935021E-03	673.00	± 0.00	4.62	±0.00	4.62
1	5	SVD	$f_5(t)$	4.53741649E-02	4.53741649E-02	4997.00	± 0.00	8.79	±0.00	8.79
1	5	SVD	$f_6(t)$	4.11565848E-03	4.11565848E-03	30000.00	± 0.00	34.29	±0.00	34.29
1	6	rand	$f_0(t)$	9.95167484E-03	4.77879699E-03	986.30	± 267.16	4.90	±0.31	146.95
1	6	rand	$f_1(t)$	2.18787348E-02	1.44174791E-02	17553.53	± 7551.45	21.80	±7.70	653.97
1	6	rand	$f_2(t)$	1.41976359E-02	8.01556200E-03	29687.57	± 1563.58	34.41	±1.79	1032.17
1	6	rand	$f_3(t)$	1.42545642E-02	9.57032544E-03	26642.63	± 8082.61	30.98	±8.22	929.49
1	6	rand	$f_4(t)$	9.58422068E-03	5.46596714E-03	956.47	± 411.25	4.84	±0.44	145.13
1	6	rand	$f_5(t)$	2.88841008E-02	2.30230467E-02	3978.20	± 2023.50	8.15	±2.12	244.44
1	6	rand	$f_6(t)$	9.54966022E-03	5.41573924E-03	28027.63	± 5825.72	31.71	±5.78	951.41
1	6	SVD	$f_0(t)$	6.12650515E-03	6.12650515E-03	759.00	± 0.00	5.24	±0.00	5.24
1	6	SVD	$f_1(t)$	2.99690526E-02	2.99690526E-02	10636.00	± 0.00	14.80	±0.00	14.80
1	6	SVD	$f_2(t)$	2.59923232E-02	2.59923232E-02	30000.00	± 0.00	34.66	±0.00	34.66
1	6	SVD	$f_3(t)$	2.59291668E-02	2.59291668E-02	18649.00	± 0.00	23.44	±0.00	23.44
1	6	SVD	$f_4(t)$	6.12650515E-03	6.12650515E-03	759.00	± 0.00	4.87	±0.00	4.87
1	6	SVD	$f_5(t)$	5.86428670E-02	5.86428670E-02	10517.00	± 0.00	14.96	±0.00	14.96
1	6	SVD	$f_6(t)$	3.14058203E-03	3.14058203E-03	30000.00	± 0.00	33.78	±0.00	33.78

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	7	rand	$f_0(t)$	1.97573728E-02	9.71484598E-03	920.10	± 222.90	4.76	±0.25	142.74
1	7	rand	$f_1(t)$	3.50923948E-02	2.15599316E-02	18586.70	± 8287.18	23.06	±8.53	691.67
1	7	rand	$f_2(t)$	2.42966116E-02	1.41944473E-02	28771.30	± 4689.51	33.09	±4.78	992.79
1	7	rand	$f_3(t)$	2.27583764E-02	1.54211151E-02	28693.67	± 3933.73	33.61	±4.08	1008.24
1	7	rand	$f_4(t)$	2.01277147E-02	9.38046464E-03	1001.17	± 470.30	4.87	±0.48	145.99
1	7	rand	$f_5(t)$	3.70935633E-02	3.14163439E-02	6169.97	± 2985.10	10.88	±3.22	326.46
1	7	rand	$f_6(t)$	1.53126395E-02	8.28979458E-03	26750.53	± 6433.85	31.55	±6.60	946.43
1	7	SVD	$f_0(t)$	1.38495355E-02	1.38495355E-02	895.00	± 0.00	5.27	±0.00	5.27
1	7	SVD	$f_1(t)$	5.06183177E-02	5.06183177E-02	10161.00	± 0.00	14.58	±0.00	14.58
1	7	SVD	$f_2(t)$	2.10131971E-02	2.10131971E-02	30000.00	± 0.00	34.77	±0.00	34.77
1	7	SVD	$f_3(t)$	5.80010659E-02	5.80010659E-02	30000.00	± 0.00	34.85	±0.00	34.85
1	7	SVD	$f_4(t)$	1.38495355E-02	1.38495355E-02	895.00	± 0.00	5.49	±0.00	5.49
1	7	SVD	$f_5(t)$	8.41136568E-02	8.41136568E-02	9450.00	± 0.00	13.57	±0.00	13.57
1	7	SVD	$f_6(t)$	1.20855815E-02	1.20855815E-02	30000.00	± 0.00	33.90	±0.00	33.90
1	8	rand	$f_0(t)$	2.54742013E-02	1.65038619E-02	939.03	± 359.75	4.80	±0.35	144.06
1	8	rand	$f_1(t)$	4.17979233E-02	2.99773980E-02	18168.23	± 6877.92	23.04	±7.23	691.07
1	8	rand	$f_2(t)$	2.97009109E-02	1.89504801E-02	28147.50	± 5611.13	32.48	±5.66	974.32
1	8	rand	$f_3(t)$	2.97696784E-02	2.17154442E-02	29367.23	± 2780.07	33.76	±2.89	1012.92

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	8	rand	$f_4(t)$	2.22757404E-02	1.51096763E-02	961.20	± 616.33	4.99	±0.67	149.58
1	8	rand	$f_5(t)$	4.64009168E-02	3.94434584E-02	4899.37	± 2485.56	8.91	±2.54	267.15
1	8	rand	$f_6(t)$	1.80194217E-02	1.11251324E-02	28798.97	± 3788.89	33.93	±4.00	1017.87
1	8	SVD	$f_0(t)$	2.64694211E-02	2.64694211E-02	1184.00	± 0.00	5.18	±0.00	5.18
1	8	SVD	$f_1(t)$	8.89571901E-02	8.89571901E-02	22399.00	± 0.00	27.20	±0.00	27.20
1	8	SVD	$f_2(t)$	3.46310944E-02	3.46310944E-02	30000.00	± 0.00	35.18	±0.00	35.18
1	8	SVD	$f_3(t)$	7.82623490E-02	7.82623490E-02	30000.00	± 0.00	35.79	±0.00	35.79
1	8	SVD	$f_4(t)$	2.64736232E-02	2.64736232E-02	1053.00	± 0.00	5.31	±0.00	5.31
1	8	SVD	$f_5(t)$	1.00525483E-01	1.00525483E-01	7979.00	± 0.00	11.95	±0.00	11.95
1	8	SVD	$f_6(t)$	3.02226175E-02	3.02226175E-02	30000.00	± 0.00	33.99	±0.00	33.99
1	9	rand	$f_0(t)$	3.39679763E-02	2.25115155E-02	864.37	± 243.70	5.24	±0.34	157.20
1	9	rand	$f_1(t)$	5.79755704E-02	4.07895638E-02	15813.63	± 7375.70	20.16	±7.56	604.84
1	9	rand	$f_2(t)$	4.55586730E-02	3.06740405E-02	29869.97	± 700.25	34.78	±0.79	1043.54
1	9	rand	$f_3(t)$	3.52071312E-02	2.75892044E-02	28074.93	± 4295.12	32.81	±4.39	984.35
1	9	rand	$f_4(t)$	4.21564029E-02	2.33191526E-02	1011.00	± 594.57	5.02	±0.60	150.67
1	9	rand	$f_5(t)$	6.10582569E-02	4.86405894E-02	5692.83	± 2828.45	9.67	±2.88	289.99
1	9	rand	$f_6(t)$	2.48926193E-02	1.43260858E-02	28232.87	± 5609.69	33.55	±5.85	1006.62
1	9	SVD	$f_0(t)$	2.38123183E-02	2.38123183E-02	650.00	± 0.00	4.60	±0.00	4.60

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	9	SVD	$f_1(t)$	9.77783754E-02	9.77783754E-02	18740.00	± 0.00	23.28	±0.00	23.28
1	9	SVD	$f_2(t)$	5.95821451E-02	5.95821451E-02	30000.00	± 0.00	34.94	±0.00	34.94
1	9	SVD	$f_3(t)$	1.11793409E-01	1.11793409E-01	30000.00	± 0.00	35.58	±0.00	35.58
1	9	SVD	$f_4(t)$	2.38123183E-02	2.38123183E-02	650.00	± 0.00	4.72	±0.00	4.72
1	9	SVD	$f_5(t)$	1.43266221E-01	1.43266221E-01	1303.00	± 0.00	5.40	±0.00	5.40
1	9	SVD	$f_6(t)$	3.47691687E-02	3.47691687E-02	29188.00	± 0.00	35.06	±0.00	35.06
1	10	rand	$f_0(t)$	4.72339067E-02	3.22469262E-02	824.20	± 276.88	4.75	±0.34	142.52
1	10	rand	$f_1(t)$	7.62421716E-02	5.54524280E-02	17451.70	± 6367.40	22.04	±6.65	661.33
1	10	rand	$f_2(t)$	5.73104158E-02	3.70417761E-02	29313.10	± 3118.95	34.25	±3.22	1027.40
1	10	rand	$f_3(t)$	4.11717653E-02	3.36867275E-02	29406.90	± 2416.11	34.34	±2.57	1030.29
1	10	rand	$f_4(t)$	5.07252835E-02	3.17607310E-02	824.37	± 186.76	4.62	±0.21	138.68
1	10	rand	$f_5(t)$	6.78373092E-02	5.66092798E-02	4733.90	± 2162.88	8.91	±2.27	267.15
1	10	rand	$f_6(t)$	3.10101361E-02	2.14834825E-02	28694.10	± 4837.26	33.99	±5.09	1019.66
1	10	SVD	$f_0(t)$	6.16779113E-02	6.16779113E-02	638.00	± 0.00	4.78	±0.00	4.78
1	10	SVD	$f_1(t)$	1.55096261E-01	1.55096261E-01	11288.00	± 0.00	15.85	±0.00	15.85
1	10	SVD	$f_2(t)$	7.40318320E-02	7.40318320E-02	30000.00	± 0.00	36.48	±0.00	36.48
1	10	SVD	$f_3(t)$	1.54431316E-01	1.54431316E-01	30000.00	± 0.00	36.78	±0.00	36.78
1	10	SVD	$f_4(t)$	6.16779113E-02	6.16779113E-02	638.00	± 0.00	4.58	±0.00	4.58

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	10	SVD	$f_5(t)$	1.81104522E-01	1.81104522E-01	10147.00	± 0.00	14.83	±0.00	14.83
1	10	SVD	$f_6(t)$	5.89021676E-02	5.89021676E-02	30000.00	± 0.00	34.95	±0.00	34.95
1	11	rand	$f_0(t)$	6.86361217E-02	4.30477878E-02	886.70	± 245.11	4.93	±0.27	147.89
1	11	rand	$f_1(t)$	8.61878042E-02	6.50206582E-02	16747.70	± 7141.11	21.38	±7.54	641.37
1	11	rand	$f_2(t)$	6.81949599E-02	4.72968176E-02	29156.43	± 3212.38	34.83	±3.43	1045.02
1	11	rand	$f_3(t)$	5.80447839E-02	4.32285065E-02	28544.50	± 3950.05	34.09	±4.18	1022.62
1	11	rand	$f_4(t)$	6.30057874E-02	4.60181299E-02	829.80	± 180.85	4.83	±0.21	144.94
1	11	rand	$f_5(t)$	7.76018568E-02	6.43971807E-02	6132.67	± 3547.65	10.45	±3.75	313.58
1	11	rand	$f_6(t)$	3.64406206E-02	2.40537919E-02	25981.03	± 6728.05	31.03	±7.08	930.92
1	11	SVD	$f_0(t)$	8.61080574E-02	8.61080574E-02	782.00	± 0.00	4.62	±0.00	4.62
1	11	SVD	$f_1(t)$	1.76789161E-01	1.76789161E-01	29054.00	± 0.00	35.11	±0.00	35.11
1	11	SVD	$f_2(t)$	1.33809632E-01	1.33809632E-01	18246.00	± 0.00	23.59	±0.00	23.59
1	11	SVD	$f_3(t)$	1.97797200E-01	1.97797200E-01	30000.00	± 0.00	36.31	±0.00	36.31
1	11	SVD	$f_4(t)$	8.61080574E-02	8.61080574E-02	782.00	± 0.00	4.85	±0.00	4.85
1	11	SVD	$f_5(t)$	2.30963549E-01	2.30963549E-01	6196.00	± 0.00	10.79	±0.00	10.79
1	11	SVD	$f_6(t)$	8.41741106E-02	8.41741106E-02	30000.00	± 0.00	34.24	±0.00	34.24
1	12	rand	$f_0(t)$	8.31894929E-02	5.50185864E-02	857.43	± 204.25	4.67	±0.24	139.99
1	12	rand	$f_1(t)$	9.37800165E-02	7.68349138E-02	16650.43	± 7549.20	21.33	±7.91	640.00

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	12	rand	$f_2(t)$	7.71404627E-02	5.68303234E-02	29611.53	± 2091.96	34.66	±2.18	1039.74
1	12	rand	$f_3(t)$	6.55000884E-02	5.11357232E-02	28635.93	± 4342.11	34.77	±4.65	1042.97
1	12	rand	$f_4(t)$	8.90156171E-02	5.62714254E-02	1011.43	± 672.99	4.78	±0.69	143.34
1	12	rand	$f_5(t)$	8.31532174E-02	7.32929182E-02	5515.40	± 2691.95	9.86	±2.85	295.65
1	12	rand	$f_6(t)$	4.59044698E-02	3.24015940E-02	26950.80	± 5605.37	31.87	±5.81	956.03
1	12	SVD	$f_0(t)$	7.57727601E-02	7.57727601E-02	806.00	± 0.00	4.93	±0.00	4.93
1	12	SVD	$f_1(t)$	2.23408003E-01	2.23408003E-01	15318.00	± 0.00	20.53	±0.00	20.53
1	12	SVD	$f_2(t)$	1.58594314E-01	1.58594314E-01	24708.00	± 0.00	30.00	±0.00	30.00
1	12	SVD	$f_3(t)$	2.31343519E-01	2.31343519E-01	30000.00	± 0.00	36.30	±0.00	36.30
1	12	SVD	$f_4(t)$	7.57727601E-02	7.57727601E-02	806.00	± 0.00	4.91	±0.00	4.91
1	12	SVD	$f_5(t)$	2.53400134E-01	2.53400134E-01	9151.00	± 0.00	13.07	±0.00	13.07
1	12	SVD	$f_6(t)$	7.44168630E-02	7.44168630E-02	30000.00	± 0.00	35.28	±0.00	35.28
1	13	rand	$f_0(t)$	1.00429919E-01	6.90443642E-02	928.30	± 250.71	4.81	±0.26	144.32
1	13	rand	$f_1(t)$	1.10700654E-01	9.08975242E-02	17058.27	± 8139.86	22.07	±8.64	662.05
1	13	rand	$f_2(t)$	8.99873226E-02	6.90911301E-02	30000.00	± 0.00	35.96	±0.39	1078.89
1	13	rand	$f_3(t)$	6.95303205E-02	5.93007825E-02	29262.97	± 2767.75	35.21	±2.98	1056.31
1	13	rand	$f_4(t)$	9.38998549E-02	7.31703475E-02	973.73	± 487.92	5.02	±0.51	150.55
1	13	rand	$f_5(t)$	9.13376941E-02	8.33741810E-02	6410.80	± 3594.84	10.57	±3.78	316.95

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	13	rand	$f_6(t)$	5.64163898E-02	3.84933531E-02	28355.73	± 3851.87	34.97	±4.10	1049.09
1	13	SVD	$f_0(t)$	1.10514579E-01	1.10514579E-01	862.00	± 0.00	4.98	±0.00	4.98
1	13	SVD	$f_1(t)$	2.04079442E-01	2.04079442E-01	11571.00	± 0.00	16.42	±0.00	16.42
1	13	SVD	$f_2(t)$	1.14072409E-01	1.14072409E-01	30000.00	± 0.00	35.92	±0.00	35.92
1	13	SVD	$f_3(t)$	2.74450675E-01	2.74450675E-01	30000.00	± 0.00	36.53	±0.00	36.53
1	13	SVD	$f_4(t)$	1.10514579E-01	1.10514579E-01	862.00	± 0.00	4.94	±0.00	4.94
1	13	SVD	$f_5(t)$	3.08528680E-01	3.08528680E-01	7371.00	± 0.00	12.03	±0.00	12.03
1	13	SVD	$f_6(t)$	7.84783815E-02	7.84783815E-02	30000.00	± 0.00	35.40	±0.00	35.40
1	14	rand	$f_0(t)$	1.16386507E-01	8.54068247E-02	902.33	± 355.30	4.89	±0.40	146.62
1	14	rand	$f_1(t)$	1.26028897E-01	1.00553969E-01	18917.57	± 6746.35	24.50	±7.34	735.07
1	14	rand	$f_2(t)$	1.03930994E-01	7.87072621E-02	29910.23	± 483.41	35.62	±0.58	1068.55
1	14	rand	$f_3(t)$	7.57561972E-02	6.66094550E-02	28778.40	± 4578.93	35.81	±4.99	1074.23
1	14	rand	$f_4(t)$	1.11738929E-01	7.67123480E-02	1041.47	± 735.38	5.01	±0.77	150.19
1	14	rand	$f_5(t)$	1.10415584E-01	9.46200265E-02	5870.17	± 2240.52	10.24	±2.42	307.15
1	14	rand	$f_6(t)$	7.16776072E-02	4.38695472E-02	26392.43	± 6647.82	31.49	±6.93	944.83
1	14	SVD	$f_0(t)$	1.43949145E-01	1.43949145E-01	806.00	± 0.00	4.99	±0.00	4.99
1	14	SVD	$f_1(t)$	2.96114477E-01	2.96114477E-01	23082.00	± 0.00	28.60	±0.00	28.60
1	14	SVD	$f_2(t)$	2.43955503E-01	2.43955503E-01	30000.00	± 0.00	36.72	±0.00	36.72

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	14	SVD	$f_3(t)$	3.05481411E-01	3.05481411E-01	30000.00	± 0.00	36.59	±0.00	36.59
1	14	SVD	$f_4(t)$	1.43949145E-01	1.43949145E-01	806.00	± 0.00	4.97	±0.00	4.97
1	14	SVD	$f_5(t)$	3.37120847E-01	3.37120847E-01	7784.00	± 0.00	12.30	±0.00	12.30
1	14	SVD	$f_6(t)$	1.50712395E-01	1.50712395E-01	8616.00	± 0.00	13.20	±0.00	13.20
1	15	rand	$f_0(t)$	1.31306420E-01	9.65382843E-02	818.20	± 168.66	4.81	±0.18	144.38
1	15	rand	$f_1(t)$	1.37305236E-01	1.14876953E-01	15014.63	± 6376.41	20.35	±6.95	610.63
1	15	rand	$f_2(t)$	1.13335049E-01	9.23758301E-02	28301.90	± 5487.01	34.58	±5.99	1037.47
1	15	rand	$f_3(t)$	8.32383853E-02	7.35326533E-02	29406.73	± 2417.43	36.29	±2.58	1088.74
1	15	rand	$f_4(t)$	1.44291993E-01	9.65826083E-02	807.27	± 168.08	4.75	±0.19	142.63
1	15	rand	$f_5(t)$	1.13873174E-01	1.02981137E-01	5522.00	± 3523.53	9.63	±3.71	289.04
1	15	rand	$f_6(t)$	6.88490756E-02	4.95624905E-02	27520.13	± 5956.06	33.11	±6.27	993.45
1	15	SVD	$f_0(t)$	1.29671099E-01	1.29671099E-01	1182.00	± 0.00	5.34	±0.00	5.34
1	15	SVD	$f_1(t)$	3.48781785E-01	3.48781785E-01	10418.00	± 0.00	15.49	±0.00	15.49
1	15	SVD	$f_2(t)$	2.73342899E-01	2.73342899E-01	30000.00	± 0.00	37.08	±0.00	37.08
1	15	SVD	$f_3(t)$	3.42204745E-01	3.42204745E-01	30000.00	± 0.00	37.94	±0.00	37.94
1	15	SVD	$f_4(t)$	1.29731118E-01	1.29731118E-01	1097.00	± 0.00	5.25	±0.00	5.25
1	15	SVD	$f_5(t)$	3.74611694E-01	3.74611694E-01	3841.00	± 0.00	8.03	±0.00	8.03
1	15	SVD	$f_6(t)$	1.32195238E-01	1.32195238E-01	30000.00	± 0.00	35.76	±0.00	35.76

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	16	rand	$f_0(t)$	1.47543804E-01	1.11621422E-01	957.83	± 340.77	4.97	±0.37	149.24
1	16	rand	$f_1(t)$	1.46480196E-01	1.22955425E-01	17858.63	± 6213.04	23.25	±6.70	697.38
1	16	rand	$f_2(t)$	1.33838172E-01	1.07552304E-01	30000.00	± 0.00	36.07	±0.13	1081.95
1	16	rand	$f_3(t)$	9.81049932E-02	8.23349557E-02	29021.70	± 3208.49	35.93	±3.70	1077.88
1	16	rand	$f_4(t)$	1.61466099E-01	1.17525706E-01	892.47	± 262.94	4.96	±0.30	148.89
1	16	rand	$f_5(t)$	1.22503773E-01	1.09696597E-01	7622.33	± 6532.24	11.96	±6.92	358.73
1	16	rand	$f_6(t)$	8.62295080E-02	5.95662103E-02	24961.30	± 8479.69	30.46	±9.01	913.94
1	16	SVD	$f_0(t)$	2.48305193E-01	2.48305193E-01	729.00	± 0.00	4.97	±0.00	4.97
1	16	SVD	$f_1(t)$	3.52006951E-01	3.52006951E-01	6877.00	± 0.00	11.57	±0.00	11.57
1	16	SVD	$f_2(t)$	2.86294834E-01	2.86294834E-01	30000.00	± 0.00	37.43	±0.00	37.43
1	16	SVD	$f_3(t)$	3.61547222E-01	3.61547222E-01	30000.00	± 0.00	36.97	±0.00	36.97
1	16	SVD	$f_4(t)$	2.48305193E-01	2.48305193E-01	729.00	± 0.00	4.77	±0.00	4.77
1	16	SVD	$f_5(t)$	3.85510060E-01	3.85510060E-01	8665.00	± 0.00	13.66	±0.00	13.66
1	16	SVD	$f_6(t)$	2.66062970E-01	2.66062970E-01	30000.00	± 0.00	36.50	±0.00	36.50
1	17	rand	$f_0(t)$	1.70723970E-01	1.29772011E-01	814.10	± 185.49	4.93	±0.23	148.00
1	17	rand	$f_1(t)$	1.59218665E-01	1.36861164E-01	16208.53	± 6757.52	21.93	±7.47	658.02
1	17	rand	$f_2(t)$	1.34210105E-01	1.16398652E-01	28361.10	± 4538.58	35.73	±5.17	1072.01
1	17	rand	$f_3(t)$	1.08389258E-01	9.10923269E-02	29293.50	± 1656.46	36.61	±1.56	1098.43

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	17	rand	$f_4(t)$	1.69962763E-01	1.32360311E-01	887.10	± 330.54	4.93	±0.37	147.91
1	17	rand	$f_5(t)$	1.42436204E-01	1.21327490E-01	5791.47	± 3109.76	10.17	±3.35	304.98
1	17	rand	$f_6(t)$	8.94877613E-02	6.57021295E-02	27743.90	± 5340.28	34.34	±5.79	1030.15
1	17	SVD	$f_0(t)$	2.28160584E-01	2.28160584E-01	884.00	± 0.00	5.07	±0.00	5.07
1	17	SVD	$f_1(t)$	3.89939592E-01	3.89939592E-01	20655.00	± 0.00	28.58	±0.00	28.58
1	17	SVD	$f_2(t)$	3.31296157E-01	3.31296157E-01	30000.00	± 0.00	37.50	±0.00	37.50
1	17	SVD	$f_3(t)$	3.82467402E-01	3.82467402E-01	30000.00	± 0.00	37.38	±0.00	37.38
1	17	SVD	$f_4(t)$	2.28160584E-01	2.28160584E-01	884.00	± 0.00	5.13	±0.00	5.13
1	17	SVD	$f_5(t)$	4.05950531E-01	4.05950531E-01	2895.00	± 0.00	7.55	±0.00	7.55
1	17	SVD	$f_6(t)$	2.29406533E-01	2.29406533E-01	30000.00	± 0.00	36.69	±0.00	36.69
1	18	rand	$f_0(t)$	1.99087046E-01	1.41221688E-01	847.50	± 170.82	4.88	±0.19	146.46
1	18	rand	$f_1(t)$	1.69054502E-01	1.48724295E-01	16978.60	± 7311.32	23.10	±8.21	693.15
1	18	rand	$f_2(t)$	1.62964363E-01	1.28266309E-01	28715.30	± 5138.15	35.48	±5.63	1064.29
1	18	rand	$f_3(t)$	1.15944856E-01	9.61457268E-02	28945.63	± 3922.86	36.32	±4.39	1089.51
1	18	rand	$f_4(t)$	1.74430024E-01	1.41160144E-01	811.10	± 199.85	4.88	±0.23	146.47
1	18	rand	$f_5(t)$	1.41200347E-01	1.29324459E-01	5941.57	± 2736.56	10.36	±2.96	310.69
1	18	rand	$f_6(t)$	9.84226539E-02	7.63479482E-02	27329.20	± 6128.82	33.96	±6.80	1018.69
1	18	SVD	$f_0(t)$	2.91111543E-01	2.91111543E-01	902.00	± 0.00	5.38	±0.00	5.38

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	18	SVD	$f_1(t)$	4.26855102E-01	4.26855102E-01	6980.00	$\pm$ 0.00	11.96	$\pm$0.00	11.96
1	18	SVD	$f_2(t)$	3.71136901E-01	3.71136901E-01	30000.00	$\pm$ 0.00	38.01	$\pm$0.00	38.01
1	18	SVD	$f_3(t)$	4.16515466E-01	4.16515466E-01	30000.00	$\pm$ 0.00	37.30	$\pm$0.00	37.30
1	18	SVD	$f_4(t)$	2.91111543E-01	2.91111543E-01	902.00	$\pm$ 0.00	5.04	$\pm$0.00	5.04
1	18	SVD	$f_5(t)$	4.61657014E-01	4.61657014E-01	5259.00	$\pm$ 0.00	9.82	$\pm$0.00	9.82
1	18	SVD	$f_6(t)$	2.11814537E-01	2.11814537E-01	30000.00	$\pm$ 0.00	36.26	$\pm$0.00	36.26
1	19	rand	$f_0(t)$	2.03030279E-01	1.61843253E-01	874.60	$\pm$ 174.70	5.10	$\pm$0.32	152.92
1	19	rand	$f_1(t)$	1.89655669E-01	1.64211928E-01	14482.30	$\pm$ 6212.43	19.96	$\pm$6.85	598.94
1	19	rand	$f_2(t)$	1.62328414E-01	1.36784066E-01	27255.00	$\pm$ 6687.99	34.56	$\pm$7.54	1036.66
1	19	rand	$f_3(t)$	1.26954659E-01	1.07614483E-01	27912.53	$\pm$ 5506.09	35.30	$\pm$6.23	1059.09
1	19	rand	$f_4(t)$	2.15168654E-01	1.61779473E-01	974.83	$\pm$ 513.48	5.05	$\pm$0.57	151.60
1	19	rand	$f_5(t)$	1.50420129E-01	1.38183125E-01	6872.37	$\pm$ 3315.23	11.83	$\pm$3.73	354.79
1	19	rand	$f_6(t)$	1.11334512E-01	8.10887145E-02	28665.97	$\pm$ 4701.09	35.31	$\pm$5.20	1059.18
1	19	SVD	$f_0(t)$	2.67618872E-01	2.67618872E-01	923.00	$\pm$ 0.00	5.21	$\pm$0.00	5.21
1	19	SVD	$f_1(t)$	4.10009030E-01	4.10009030E-01	3490.00	$\pm$ 0.00	8.15	$\pm$0.00	8.15
1	19	SVD	$f_2(t)$	3.35850964E-01	3.35850964E-01	23071.00	$\pm$ 0.00	30.21	$\pm$0.00	30.21
1	19	SVD	$f_3(t)$	4.63824378E-01	4.63824378E-01	30000.00	$\pm$ 0.00	38.71	$\pm$0.00	38.71
1	19	SVD	$f_4(t)$	2.67618872E-01	2.67618872E-01	923.00	$\pm$ 0.00	5.24	$\pm$0.00	5.24

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	19	SVD	$f_5(t)$	4.88088441E-01	4.88088441E-01	4596.00	± 0.00	9.35	±0.00	9.35
1	19	SVD	$f_6(t)$	3.28503303E-01	3.28503303E-01	19965.00	± 0.00	26.48	±0.00	26.48
1	20	rand	$f_0(t)$	2.23091578E-01	1.77330893E-01	843.90	± 206.46	4.91	±0.23	147.42
1	20	rand	$f_1(t)$	1.99019423E-01	1.71255776E-01	15312.57	± 5630.44	21.09	±6.27	632.77
1	20	rand	$f_2(t)$	1.80658716E-01	1.52188038E-01	29537.37	± 2491.36	36.53	±2.79	1095.76
1	20	rand	$f_3(t)$	1.31733982E-01	1.16305553E-01	29771.00	± 1007.42	36.83	±1.26	1105.05
1	20	rand	$f_4(t)$	2.22095363E-01	1.81295459E-01	899.23	± 381.18	5.31	±0.47	159.36
1	20	rand	$f_5(t)$	1.71430976E-01	1.51274537E-01	7286.63	± 3733.97	11.79	±4.04	353.68
1	20	rand	$f_6(t)$	1.14999956E-01	8.76699691E-02	27788.73	± 5613.07	34.53	±6.16	1035.94
1	20	SVD	$f_0(t)$	4.02058773E-01	4.02058773E-01	679.00	± 0.00	4.79	±0.00	4.79
1	20	SVD	$f_1(t)$	4.66301336E-01	4.66301336E-01	9852.00	± 0.00	15.24	±0.00	15.24
1	20	SVD	$f_2(t)$	3.72652294E-01	3.72652294E-01	30000.00	± 0.00	38.33	±0.00	38.33
1	20	SVD	$f_3(t)$	4.82916425E-01	4.82916425E-01	30000.00	± 0.00	38.08	±0.00	38.08
1	20	SVD	$f_4(t)$	4.02058773E-01	4.02058773E-01	679.00	± 0.00	4.75	±0.00	4.75
1	20	SVD	$f_5(t)$	5.34080411E-01	5.34080411E-01	2012.00	± 0.00	5.11	±0.00	5.11
1	20	SVD	$f_6(t)$	3.45863398E-01	3.45863398E-01	9433.00	± 0.00	14.63	±0.00	14.63
1	21	rand	$f_0(t)$	2.32256703E-01	1.84742717E-01	870.33	± 270.47	4.94	±0.31	148.18
1	21	rand	$f_1(t)$	2.11509742E-01	1.82525449E-01	15376.33	± 7139.15	21.17	±8.04	635.12

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	21	rand	$f_2(t)$	2.04584962E-01	1.59173351E-01	27819.77	± 5396.30	34.78	±6.00	1043.29
1	21	rand	$f_3(t)$	1.39285570E-01	1.25756605E-01	28732.33	± 3879.96	36.49	±4.41	1094.65
1	21	rand	$f_4(t)$	2.34933227E-01	1.91994426E-01	833.00	± 154.86	4.92	±0.17	147.73
1	21	rand	$f_5(t)$	1.73390344E-01	1.60989977E-01	7472.97	± 4435.68	12.45	±4.98	373.50
1	21	rand	$f_6(t)$	1.20983212E-01	1.00767741E-01	26458.87	± 6464.02	33.08	±7.14	992.26
1	21	SVD	$f_0(t)$	4.26128919E-01	4.26128919E-01	656.00	± 0.00	4.89	±0.00	4.89
1	21	SVD	$f_1(t)$	5.17267458E-01	5.17267458E-01	6727.00	± 0.00	11.82	±0.00	11.82
1	21	SVD	$f_2(t)$	4.59726840E-01	4.59726840E-01	21310.00	± 0.00	29.07	±0.00	29.07
1	21	SVD	$f_3(t)$	5.07758521E-01	5.07758521E-01	8620.00	± 0.00	14.49	±0.00	14.49
1	21	SVD	$f_4(t)$	4.26128919E-01	4.26128919E-01	656.00	± 0.00	4.94	±0.00	4.94
1	21	SVD	$f_5(t)$	5.53738768E-01	5.53738768E-01	4264.00	± 0.00	9.06	±0.00	9.06
1	21	SVD	$f_6(t)$	3.19633021E-01	3.19633021E-01	20591.00	± 0.00	26.60	±0.00	26.60
1	22	rand	$f_0(t)$	2.36884337E-01	2.02589778E-01	906.60	± 349.44	5.12	±0.40	153.74
1	22	rand	$f_1(t)$	2.26907342E-01	1.97446298E-01	16388.90	± 5509.64	22.55	±6.21	676.51
1	22	rand	$f_2(t)$	2.05312417E-01	1.68887541E-01	29327.60	± 3620.98	37.94	±4.21	1138.05
1	22	rand	$f_3(t)$	1.58246027E-01	1.32608569E-01	29200.33	± 3639.49	36.86	±4.12	1105.77
1	22	rand	$f_4(t)$	2.36334963E-01	2.05974028E-01	922.60	± 721.34	5.03	±0.81	150.97
1	22	rand	$f_5(t)$	1.89336536E-01	1.65386915E-01	6153.83	± 3059.07	10.80	±3.43	323.95

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	22	rand	$f_6(t)$	1.37756877E-01	1.06389230E-01	29299.80	± 2341.23	37.29	±3.04	1118.77
1	22	SVD	$f_0(t)$	4.71530113E-01	4.71530113E-01	700.00	± 0.00	4.82	±0.00	4.82
1	22	SVD	$f_1(t)$	5.78216292E-01	5.78216292E-01	8796.00	± 0.00	14.53	±0.00	14.53
1	22	SVD	$f_2(t)$	5.42239857E-01	5.42239857E-01	30000.00	± 0.00	39.47	±0.00	39.47
1	22	SVD	$f_3(t)$	5.38075864E-01	5.38075864E-01	24440.00	± 0.00	32.67	±0.00	32.67
1	22	SVD	$f_4(t)$	4.71530113E-01	4.71530113E-01	700.00	± 0.00	4.98	±0.00	4.98
1	22	SVD	$f_5(t)$	5.81791393E-01	5.81791393E-01	3624.00	± 0.00	8.10	±0.00	8.10
1	22	SVD	$f_6(t)$	3.87660075E-01	3.87660075E-01	13496.00	± 0.00	18.74	±0.00	18.74
1	23	rand	$f_0(t)$	2.68196371E-01	2.25952584E-01	851.67	± 212.36	4.94	±0.23	148.09
1	23	rand	$f_1(t)$	2.25446924E-01	2.06203487E-01	14654.07	± 6837.16	21.06	±7.95	631.82
1	23	rand	$f_2(t)$	2.03831096E-01	1.82028361E-01	29096.47	± 3497.57	38.62	±4.27	1158.59
1	23	rand	$f_3(t)$	1.67388429E-01	1.42312940E-01	28311.10	± 4477.85	36.56	±5.20	1096.88
1	23	rand	$f_4(t)$	2.60769584E-01	2.21376049E-01	815.70	± 152.87	5.14	±0.37	154.07
1	23	rand	$f_5(t)$	1.96333901E-01	1.78965285E-01	6644.97	± 3650.35	11.52	±4.17	345.72
1	23	rand	$f_6(t)$	1.52151372E-01	1.13084117E-01	25334.10	± 7531.54	33.21	±8.68	996.26
1	23	SVD	$f_0(t)$	4.43901561E-01	4.43901561E-01	755.00	± 0.00	5.59	±0.00	5.59
1	23	SVD	$f_1(t)$	6.10227161E-01	6.10227161E-01	6951.00	± 0.00	12.23	±0.00	12.23
1	23	SVD	$f_2(t)$	5.66157540E-01	5.66157540E-01	27964.00	± 0.00	36.84	±0.00	36.84

k	m	Init	$f(t)$	Best Min Vol	Avg Min Vol	Avg Iterations		Avg Run Time (s)		Total Time
1	23	SVD	$f_3(t)$	5.68482326E-01	5.68482326E-01	30000.00	± 0.00	39.82	±0.00	39.82
1	23	SVD	$f_4(t)$	4.43901561E-01	4.43901561E-01	755.00	± 0.00	5.12	±0.00	5.12
1	23	SVD	$f_5(t)$	6.21538457E-01	6.21538457E-01	8067.00	± 0.00	13.71	±0.00	13.71
1	23	SVD	$f_6(t)$	4.26054594E-01	4.26054594E-01	21500.00	± 0.00	29.11	±0.00	29.11
1	24	rand	$f_0(t)$	2.88009067E-01	2.32584795E-01	855.50	± 197.29	4.96	±0.25	148.93
1	24	rand	$f_1(t)$	2.47724549E-01	2.24958265E-01	16011.93	± 8771.80	22.13	±9.95	663.93
1	24	rand	$f_2(t)$	2.32526286E-01	1.99452682E-01	29442.60	± 2745.46	37.61	±3.31	1128.21
1	24	rand	$f_3(t)$	1.79123798E-01	1.50864252E-01	27949.50	± 6286.72	35.81	±.12	1074.25
1	24	rand	$f_4(t)$	2.82198738E-01	2.37216596E-01	830.63	± 148.54	4.92	±0.17	147.72
1	24	rand	$f_5(t)$	2.06339820E-01	1.89081804E-01	8720.50	± 5307.60	13.77	±5.95	413.13
1	24	rand	$f_6(t)$	1.58438127E-01	1.22693102E-01	27311.80	± 6499.80	34.56	±7.27	1036.80
1	24	SVD	$f_0(t)$	4.85784095E-01	4.85784095E-01	656.00	± 0.00	4.92	±0.00	4.92
1	24	SVD	$f_1(t)$	6.25611785E-01	6.25611785E-01	9400.00	± 0.00	15.11	±0.00	15.11
1	24	SVD	$f_2(t)$	5.73108196E-01	5.73108196E-01	20943.00	± 0.00	27.92	±0.00	27.92
1	24	SVD	$f_3(t)$	5.72728700E-01	5.72728700E-01	27624.00	± 0.00	35.44	±0.00	35.44
1	24	SVD	$f_4(t)$	4.85784095E-01	4.85784095E-01	656.00	± 0.00	4.77	±0.00	4.77
1	24	SVD	$f_5(t)$	6.50437015E-01	6.50437015E-01	5294.00	± 0.00	11.04	±0.00	11.04
1	24	SVD	$f_6(t)$	4.71100574E-01	4.71100574E-01	30000.00	± 0.00	37.15	±0.00	37.15

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	25	rand	$f_0(t)$	2.93203650E-01	2.51056761E-01	826.73	± 159.44	4.84	±0.19	145.06
1	25	rand	$f_1(t)$	2.53284470E-01	2.31810311E-01	16775.60	± 7429.53	23.53	±8.67	705.80
1	25	rand	$f_2(t)$	2.25209192E-01	2.01630981E-01	29248.83	± 2706.47	37.84	±2.91	1135.29
1	25	rand	$f_3(t)$	1.77304653E-01	1.59871572E-01	28336.03	± 5067.33	36.30	±5.86	1088.91
1	25	rand	$f_4(t)$	2.87005757E-01	2.49346290E-01	910.23	± 345.81	5.05	±0.42	151.39
1	25	rand	$f_5(t)$	2.19220002E-01	1.96450199E-01	6992.90	± 2971.66	12.27	±3.67	368.20
1	25	rand	$f_6(t)$	1.64958201E-01	1.31735532E-01	27921.10	± 5473.64	35.57	±6.24	1067.02
1	25	SVD	$f_0(t)$	5.25727379E-01	5.25727379E-01	746.00	± 0.00	4.86	±0.00	4.86
1	25	SVD	$f_1(t)$	6.63286409E-01	6.63286409E-01	15912.00	± 0.00	23.11	±0.00	23.11
1	25	SVD	$f_2(t)$	6.22695935E-01	6.22695935E-01	18728.00	± 0.00	26.39	±0.00	26.39
1	25	SVD	$f_3(t)$	6.12483431E-01	6.12483431E-01	30000.00	± 0.00	38.79	±0.00	38.79
1	25	SVD	$f_4(t)$	5.25727379E-01	5.25727379E-01	746.00	± 0.00	5.01	±0.00	5.01
1	25	SVD	$f_5(t)$	6.93038415E-01	6.93038415E-01	6304.00	± 0.00	11.95	±0.00	11.95
1	25	SVD	$f_6(t)$	4.58683880E-01	4.58683880E-01	12270.00	± 0.00	18.38	±0.00	18.38
1	26	rand	$f_0(t)$	3.05994420E-01	2.66324469E-01	868.00	± 232.45	4.85	±0.27	145.50
1	26	rand	$f_1(t)$	2.77125365E-01	2.41900283E-01	14407.27	± 7799.70	20.84	±9.07	625.16
1	26	rand	$f_2(t)$	2.62619776E-01	2.18653693E-01	29158.07	± 1990.35	37.75	±2.50	1132.44
1	26	rand	$f_3(t)$	2.01623883E-01	1.64941291E-01	29820.03	± 969.15	37.42	±1.20	1122.58

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	26	rand	$f_4(t)$	3.36303181E-01	2.71291561E-01	971.83	± 617.05	5.09	±0.82	152.57
1	26	rand	$f_5(t)$	2.36675098E-01	2.04768177E-01	7085.30	± 4478.38	12.31	±5.23	369.21
1	26	rand	$f_6(t)$	1.78568797E-01	1.35734634E-01	25508.20	± 6605.95	33.21	±7.63	996.30
1	26	SVD	$f_0(t)$	5.80371992E-01	5.80371992E-01	700.00	± 0.00	4.76	±0.00	4.76
1	26	SVD	$f_1(t)$	6.87426894E-01	6.87426894E-01	8716.00	± 0.00	14.54	±0.00	14.54
1	26	SVD	$f_2(t)$	6.39033489E-01	6.39033489E-01	25649.00	± 0.00	34.74	±0.00	34.74
1	26	SVD	$f_3(t)$	6.49689620E-01	6.49689620E-01	30000.00	± 0.00	40.18	±0.00	40.18
1	26	SVD	$f_4(t)$	5.80371992E-01	5.80371992E-01	700.00	± 0.00	4.82	±0.00	4.82
1	26	SVD	$f_5(t)$	7.09561028E-01	7.09561028E-01	5045.00	± 0.00	10.15	±0.00	10.15
1	26	SVD	$f_6(t)$	5.15238071E-01	5.15238071E-01	9405.00	± 0.00	15.22	±0.00	15.22
1	27	rand	$f_0(t)$	3.31826460E-01	2.80783871E-01	893.97	± 259.68	5.09	±0.32	152.71
1	27	rand	$f_1(t)$	2.83859440E-01	2.56387056E-01	15183.60	± 6911.59	21.73	±8.13	652.01
1	27	rand	$f_2(t)$	2.69307789E-01	2.33107707E-01	29569.23	± 2202.31	38.28	±2.53	1148.51
1	27	rand	$f_3(t)$	1.89545995E-01	1.73979740E-01	29051.23	± 3829.97	37.37	±4.39	1121.20
1	27	rand	$f_4(t)$	3.24026397E-01	2.84505974E-01	880.80	± 256.59	5.04	±0.31	151.18
1	27	rand	$f_5(t)$	2.30189518E-01	2.12911199E-01	7861.90	± 6149.71	13.29	±7.25	398.71
1	27	rand	$f_6(t)$	1.74341976E-01	1.50450714E-01	28771.93	± 4209.75	37.57	±4.95	1126.96
1	27	SVD	$f_0(t)$	6.44211465E-01	6.44211465E-01	805.00	± 0.00	5.11	±0.00	5.11

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	27	SVD	$f_1(t)$	7.39151142E-01	7.39151142E-01	8832.00	± 0.00	14.90	±0.00	14.90
1	27	SVD	$f_2(t)$	6.99923700E-01	6.99923700E-01	30000.00	± 0.00	40.82	±0.00	40.82
1	27	SVD	$f_3(t)$	7.22016806E-01	7.22016806E-01	30000.00	± 0.00	40.46	±0.00	40.46
1	27	SVD	$f_4(t)$	6.44211465E-01	6.44211465E-01	805.00	± 0.00	5.11	±0.00	5.11
1	27	SVD	$f_5(t)$	7.50705138E-01	7.50705138E-01	1487.00	± 0.00	5.95	±0.00	5.95
1	27	SVD	$f_6(t)$	6.21472580E-01	6.21472580E-01	9371.00	± 0.00	15.45	±0.00	15.45
1	28	rand	$f_0(t)$	3.49513550E-01	2.90057517E-01	818.80	± 192.91	4.97	±0.23	148.96
1	28	rand	$f_1(t)$	2.88861784E-01	2.66391069E-01	15515.03	± 7271.03	22.31	±8.48	669.32
1	28	rand	$f_2(t)$	2.79275180E-01	2.38164904E-01	29774.77	± 863.03	37.94	±1.00	1138.19
1	28	rand	$f_3(t)$	2.06506500E-01	1.86922812E-01	29097.30	± 3841.73	37.63	±4.39	1128.78
1	28	rand	$f_4(t)$	3.36478753E-01	2.95929337E-01	953.77	± 373.33	5.01	±0.44	150.21
1	28	rand	$f_5(t)$	2.43093187E-01	2.26784782E-01	7887.00	± 6356.14	13.27	±7.55	398.14
1	28	rand	$f_6(t)$	1.87191466E-01	1.54887561E-01	25590.63	± 6307.89	33.83	±7.25	1014.96
1	28	SVD	$f_0(t)$	6.62120348E-01	6.62120348E-01	827.00	± 0.00	5.13	±0.00	5.13
1	28	SVD	$f_1(t)$	7.53149804E-01	7.53149804E-01	4744.00	± 0.00	9.87	±0.00	9.87
1	28	SVD	$f_2(t)$	7.05037487E-01	7.05037487E-01	30000.00	± 0.00	40.38	±0.00	40.38
1	28	SVD	$f_3(t)$	7.45092593E-01	7.45092593E-01	30000.00	± 0.00	39.75	±0.00	39.75
1	28	SVD	$f_4(t)$	6.62120348E-01	6.62120348E-01	827.00	± 0.00	5.13	±0.00	5.13

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	28	SVD	$f_5(t)$	7.79600099E-01	7.79600099E-01	2498.00	± 0.00	6.42	±0.00	6.42
1	28	SVD	$f_6(t)$	5.99298307E-01	5.99298307E-01	17678.00	± 0.00	24.93	±0.00	24.93
1	29	rand	$f_0(t)$	3.50575613E-01	3.07592395E-01	831.50	± 249.65	4.96	±0.30	148.90
1	29	rand	$f_1(t)$	3.10035975E-01	2.77422250E-01	15952.43	± 6705.41	22.89	±7.92	686.77
1	29	rand	$f_2(t)$	2.91538346E-01	2.50518972E-01	28326.73	± 5007.45	37.35	±5.90	1120.55
1	29	rand	$f_3(t)$	2.14224399E-01	1.93530252E-01	28529.20	± 4353.73	37.62	±5.17	1128.55
1	29	rand	$f_4(t)$	3.54672993E-01	3.00524996E-01	797.83	± 147.18	5.06	±0.21	151.70
1	29	rand	$f_5(t)$	2.48677995E-01	2.33743589E-01	7388.03	± 4476.68	12.46	±5.25	373.93
1	29	rand	$f_6(t)$	1.88879194E-01	1.70130773E-01	25961.93	± 7160.13	34.56	±8.45	1036.77
1	29	SVD	$f_0(t)$	6.63303505E-01	6.63303505E-01	676.00	± 0.00	4.92	±0.00	4.92
1	29	SVD	$f_1(t)$	7.60660747E-01	7.60660747E-01	6478.00	± 0.00	11.83	±0.00	11.83
1	29	SVD	$f_2(t)$	7.03027050E-01	7.03027050E-01	30000.00	± 0.00	40.29	±0.00	40.29
1	29	SVD	$f_3(t)$	7.85901666E-01	7.85901666E-01	30000.00	± 0.00	40.87	±0.00	40.87
1	29	SVD	$f_4(t)$	6.63303505E-01	6.63303505E-01	676.00	± 0.00	4.75	±0.00	4.75
1	29	SVD	$f_5(t)$	8.07484002E-01	8.07484002E-01	7017.00	± 0.00	13.11	±0.00	13.11
1	29	SVD	$f_6(t)$	6.46445491E-01	6.46445491E-01	15315.00	± 0.00	22.56	±0.00	22.56
1	30	rand	$f_0(t)$	3.57974937E-01	3.18278712E-01	909.10	± 219.48	5.25	±0.31	157.49
1	30	rand	$f_1(t)$	3.19855709E-01	2.88158965E-01	12709.73	± 6442.09	19.77	±7.87	593.02

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
1	30	rand	$f_2(t)$	2.97571934E-01	2.62099103E-01	29552.27	± 1474.10	38.31	±1.89	1149.32
1	30	rand	$f_3(t)$	2.24884405E-01	2.07170723E-01	29963.30	± 197.64	39.78	±0.77	1193.33
1	30	rand	$f_4(t)$	3.57985217E-01	3.21405330E-01	799.13	± 148.22	4.83	±0.18	144.76
1	30	rand	$f_5(t)$	2.65922462E-01	2.45777835E-01	8684.93	± 4598.84	14.19	±5.50	425.77
1	30	rand	$f_6(t)$	2.19033966E-01	1.75257042E-01	25502.73	± 7037.05	34.12	±8.32	1023.58
1	30	SVD	$f_0(t)$	7.61241679E-01	7.61241679E-01	905.00	± 0.00	5.32	±0.00	5.32
1	30	SVD	$f_1(t)$	7.94754340E-01	7.94754340E-01	8571.00	± 0.00	14.36	±0.00	14.36
1	30	SVD	$f_2(t)$	7.54665915E-01	7.54665915E-01	30000.00	± 0.00	39.83	±0.00	39.83
1	30	SVD	$f_3(t)$	8.03548165E-01	8.03548165E-01	27344.00	± 0.00	37.23	±0.00	37.23
1	30	SVD	$f_4(t)$	7.61241679E-01	7.61241679E-01	905.00	± 0.00	5.07	±0.00	5.07
1	30	SVD	$f_5(t)$	8.28267973E-01	8.28267973E-01	7082.00	± 0.00	12.46	±0.00	12.46
1	30	SVD	$f_6(t)$	7.08076815E-01	7.08076815E-01	16964.00	± 0.00	24.63	±0.00	24.63
2	2	rand	$f_0(t)$	1.58002463E-10	2.91884566E-11	34.10	± 20.11	24.51	±13.72	735.19
2	2	rand	$f_1(t)$	8.26007471E-11	2.55980942E-11	817.60	± 4236.04	24.43	±20.70	732.92
2	2	rand	$f_2(t)$	1.06159593E-10	3.09572382E-11	29.50	± 22.21	20.72	±14.54	621.50
2	2	rand	$f_3(t)$	5.99029174E-11	2.23988039E-11	669.03	± 3317.55	26.00	±21.66	780.06
2	2	rand	$f_4(t)$	6.26858915E-11	2.41034525E-11	1029.10	± 5379.80	23.86	±19.80	715.84
2	2	rand	$f_5(t)$	6.98016846E-09	1.85260430E-09	131.37	± 20.66	66.76	±1.32	2002.69

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	2	rand	$f_6(t)$	6.42738220E-11	1.10224483E-11	30000.00	± 0.00	100.97	±2.11	3029.04
2	2	SVD	$f_0(t)$	2.33521575E-11	2.33521575E-11	4.00	± 0.00	4.75	±0.00	4.75
2	2	SVD	$f_1(t)$	2.78281936E-12	2.78281936E-12	2.00	± 0.00	3.52	±0.00	3.52
2	2	SVD	$f_2(t)$	2.78281936E-12	2.78281936E-12	2.00	± 0.00	3.47	±0.00	3.47
2	2	SVD	$f_3(t)$	2.97651663E-12	2.97651663E-12	2.00	± 0.00	3.50	±0.00	3.50
2	2	SVD	$f_4(t)$	2.33521575E-11	2.33521575E-11	4.00	± 0.00	4.82	±0.00	4.82
2	2	SVD	$f_5(t)$	8.19958459E-10	8.19958459E-10	130.00	± 0.00	67.23	±0.00	67.23
2	2	SVD	$f_6(t)$	2.17390775E-11	2.17390775E-11	30000.00	± 0.00	102.18	±0.00	102.18
2	3	rand	$f_0(t)$	3.36956971E-07	8.42531368E-08	30000.00	± 0.00	102.06	±1.15	3061.65
2	3	rand	$f_1(t)$	1.13733083E-06	2.34967240E-07	15109.73	± 6508.33	83.96	±7.42	2518.93
2	3	rand	$f_2(t)$	3.58285243E-07	9.33047908E-08	29829.33	± 919.07	101.27	±1.46	3037.99
2	3	rand	$f_3(t)$	5.95398706E-07	1.36084946E-07	19603.47	± 9242.49	91.30	±11.54	2738.95
2	3	rand	$f_4(t)$	3.46512308E-07	8.96136286E-08	30000.00	± 0.00	101.50	±2.10	3045.03
2	3	rand	$f_5(t)$	2.18506434E-05	1.39652497E-05	456.80	± 217.68	67.22	±2.30	2016.47
2	3	rand	$f_6(t)$	1.21362375E-06	1.31668053E-07	30000.00	± 0.00	103.85	±0.96	3115.40
2	3	SVD	$f_0(t)$	5.43854845E-08	5.43854845E-08	30000.00	± 0.00	105.26	±0.00	105.26
2	3	SVD	$f_1(t)$	1.19780893E-06	1.19780893E-06	7184.00	± 0.00	78.32	±0.00	78.32
2	3	SVD	$f_2(t)$	3.68442974E-07	3.68442974E-07	30000.00	± 0.00	104.08	±0.00	104.08

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	3	SVD	$f_3(t)$	5.00408539E-08	5.00408539E-08	30000.00	± 0.00	106.42	±0.00	106.42
2	3	SVD	$f_4(t)$	2.85504805E-09	2.85504805E-09	30000.00	± 0.00	106.93	±0.00	106.93
2	3	SVD	$f_5(t)$	1.68129106E-05	1.68129106E-05	240.00	± 0.00	67.73	±0.00	67.73
2	3	SVD	$f_6(t)$	7.54460213E-08	7.54460213E-08	30000.00	± 0.00	103.94	±0.00	103.94
2	4	rand	$f_0(t)$	6.57033866E-06	1.89907474E-06	30000.00	± 0.00	102.32	±2.06	3069.68
2	4	rand	$f_1(t)$	2.94910049E-05	1.17124858E-05	14548.53	± 5980.74	85.48	±7.65	2564.41
2	4	rand	$f_2(t)$	9.51186788E-06	4.03548093E-06	30000.00	± 0.00	101.75	±1.68	3052.65
2	4	rand	$f_3(t)$	1.41627432E-05	4.58943875E-06	21708.53	± 8322.88	91.38	±9.39	2741.33
2	4	rand	$f_4(t)$	7.84895825E-06	2.00782078E-06	30000.00	± 0.00	100.87	±0.46	3026.01
2	4	rand	$f_5(t)$	2.01459855E-04	1.27623326E-04	721.00	± 372.80	67.09	±0.64	2012.72
2	4	rand	$f_6(t)$	8.58391019E-06	2.64685884E-06	29983.87	± 86.88	103.41	±2.54	3102.42
2	4	SVD	$f_0(t)$	1.45081024E-06	1.45081024E-06	30000.00	± 0.00	105.09	±0.00	105.09
2	4	SVD	$f_1(t)$	1.94271224E-05	1.94271224E-05	23591.00	± 0.00	95.81	±0.00	95.81
2	4	SVD	$f_2(t)$	5.47978745E-06	5.47978745E-06	30000.00	± 0.00	106.54	±0.00	106.54
2	4	SVD	$f_3(t)$	4.98872265E-06	4.98872265E-06	6975.00	± 0.00	79.76	±0.00	79.76
2	4	SVD	$f_4(t)$	1.14339972E-06	1.14339972E-06	30000.00	± 0.00	104.06	±0.00	104.06
2	4	SVD	$f_5(t)$	1.90776722E-04	1.90776722E-04	837.00	± 0.00	69.29	±0.00	69.29
2	4	SVD	$f_6(t)$	2.41762869E-06	2.41762869E-06	30000.00	± 0.00	103.66	±0.00	103.66

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	5	rand	$f_0(t)$	3.05895596E-05	1.07863284E-05	30000.00	± 0.00	101.33	±0.74	3039.84
2	5	rand	$f_1(t)$	2.63785034E-04	7.20490577E-05	19548.40	± 7616.51	90.83	±8.95	2724.80
2	5	rand	$f_2(t)$	6.02852709E-05	2.56343450E-05	29656.03	± 1852.32	102.77	±2.53	3083.20
2	5	rand	$f_3(t)$	1.12156601E-04	4.57928817E-05	21908.00	± 8429.02	92.92	±10.20	2787.46
2	5	rand	$f_4(t)$	4.52731387E-05	1.22776885E-05	30000.00	± 0.00	102.85	±0.83	3085.62
2	5	rand	$f_5(t)$	8.37179834E-04	5.48617114E-04	1252.80	± 747.25	68.35	±1.56	2050.38
2	5	rand	$f_6(t)$	3.38278781E-05	1.44524509E-05	29754.57	± 1321.70	102.94	±2.56	3088.17
2	5	SVD	$f_0(t)$	1.15917067E-05	1.15917067E-05	30000.00	± 0.00	108.65	±0.00	108.65
2	5	SVD	$f_1(t)$	8.87731712E-05	8.87731712E-05	15735.00	± 0.00	88.21	±0.00	88.21
2	5	SVD	$f_2(t)$	7.77565190E-05	7.77565190E-05	30000.00	± 0.00	103.99	±0.00	103.99
2	5	SVD	$f_3(t)$	1.60260014E-04	1.60260014E-04	30000.00	± 0.00	104.85	±0.00	104.85
2	5	SVD	$f_4(t)$	1.22595414E-05	1.22595414E-05	30000.00	± 0.00	106.23	±0.00	106.23
2	5	SVD	$f_5(t)$	1.46894868E-03	1.46894868E-03	717.00	± 0.00	68.66	±0.00	68.66
2	5	SVD	$f_6(t)$	5.88968744E-05	5.88968744E-05	30000.00	± 0.00	102.21	±0.00	102.21
2	6	rand	$f_0(t)$	1.01614787E-04	3.91147227E-05	27104.27	± 8687.34	99.58	±10.30	2987.47
2	6	rand	$f_1(t)$	8.01688623E-04	3.99261474E-04	17133.03	± 7187.62	86.33	±8.49	2589.84
2	6	rand	$f_2(t)$	3.42827547E-04	1.11445618E-04	30000.00	± 0.00	101.70	±1.43	3050.89
2	6	rand	$f_3(t)$	3.20949922E-04	1.86942416E-04	22605.83	± 8377.43	95.75	±9.93	2872.35

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	6	rand	$f_4(t)$	7.89380820E-05	3.38054633E-05	28002.53	± 6846.65	99.92	±8.15	2997.47
2	6	rand	$f_5(t)$	2.03722275E-03	1.31660234E-03	1667.77	± 1165.42	68.29	±1.74	2048.83
2	6	rand	$f_6(t)$	1.13413925E-04	4.28342043E-05	29480.83	± 2795.80	102.94	±3.73	3088.23
2	6	SVD	$f_0(t)$	2.73864733E-05	2.73864733E-05	30000.00	± 0.00	106.78	±0.00	106.78
2	6	SVD	$f_1(t)$	4.81938880E-04	4.81938880E-04	22762.00	± 0.00	99.24	±0.00	99.24
2	6	SVD	$f_2(t)$	1.13146232E-04	1.13146232E-04	30000.00	± 0.00	107.15	±0.00	107.15
2	6	SVD	$f_3(t)$	1.42959814E-03	1.42959814E-03	14720.00	± 0.00	86.48	±0.00	86.48
2	6	SVD	$f_4(t)$	2.18620132E-05	2.18620132E-05	30000.00	± 0.00	107.49	±0.00	107.49
2	6	SVD	$f_5(t)$	3.18368568E-03	3.18368568E-03	440.00	± 0.00	69.13	±0.00	69.13
2	6	SVD	$f_6(t)$	4.33627905E-05	4.33627905E-05	30000.00	± 0.00	102.43	±0.00	102.43
2	7	rand	$f_0(t)$	2.04694190E-04	9.62171141E-05	23248.43	± 12238.69	97.77	±15.11	2933.03
2	7	rand	$f_1(t)$	2.20030431E-03	1.08691417E-03	17958.13	± 6859.32	88.47	±7.96	2654.10
2	7	rand	$f_2(t)$	7.94228820E-04	2.49142514E-04	30000.00	± 0.00	102.10	±0.66	3063.06
2	7	rand	$f_3(t)$	8.06574874E-04	4.65829859E-04	23118.53	± 9290.91	95.40	±11.25	2861.90
2	7	rand	$f_4(t)$	2.08446483E-04	8.81148194E-05	19196.27	± 13684.32	89.73	±16.51	2691.83
2	7	rand	$f_5(t)$	3.66953613E-03	2.45314048E-03	3608.53	± 7112.95	72.01	±8.22	2160.44
2	7	rand	$f_6(t)$	2.75842255E-04	1.05756548E-04	29854.17	± 785.34	102.35	±1.17	3070.54
2	7	SVD	$f_0(t)$	1.61877577E-04	1.61877577E-04	832.00	± 0.00	70.83	±0.00	70.83

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	7	SVD	$f_1(t)$	3.20176446E-03	3.20176446E-03	7439.00	± 0.00	77.51	±0.00	77.51
2	7	SVD	$f_2(t)$	6.45564200E-04	6.45564200E-04	30000.00	± 0.00	108.10	±0.00	108.10
2	7	SVD	$f_3(t)$	1.34404348E-03	1.34404348E-03	26740.00	± 0.00	103.08	±0.00	103.08
2	7	SVD	$f_4(t)$	1.61877577E-04	1.61877577E-04	832.00	± 0.00	71.55	±0.00	71.55
2	7	SVD	$f_5(t)$	8.24676242E-03	8.24676242E-03	3395.00	± 0.00	71.97	±0.00	71.97
2	7	SVD	$f_6(t)$	4.16103846E-05	4.16103846E-05	30000.00	± 0.00	104.24	±0.00	104.24
2	8	rand	$f_0(t)$	7.75890904E-04	2.40059570E-04	9862.87	± 13187.28	78.14	±15.68	2344.22
2	8	rand	$f_1(t)$	4.59858372E-03	2.26132599E-03	15941.87	± 7434.92	87.05	±9.28	2611.49
2	8	rand	$f_2(t)$	1.29050506E-03	6.58548245E-04	29390.53	± 3282.08	102.63	±4.28	3079.02
2	8	rand	$f_3(t)$	1.48389431E-03	8.59301655E-04	27389.53	± 4384.62	98.96	±5.03	2968.65
2	8	rand	$f_4(t)$	8.47873939E-04	2.65197574E-04	11222.10	± 13342.44	80.11	±16.01	2403.34
2	8	rand	$f_5(t)$	5.89993111E-03	4.19159040E-03	2443.67	± 1688.69	71.84	±2.41	2155.12
2	8	rand	$f_6(t)$	5.66463513E-04	2.18162021E-04	29491.40	± 2043.89	103.11	±2.54	3093.33
2	8	SVD	$f_0(t)$	2.61663597E-04	2.61663597E-04	30000.00	± 0.00	105.80	±0.00	105.80
2	8	SVD	$f_1(t)$	6.67749161E-03	6.67749161E-03	24185.00	± 0.00	99.91	±0.00	99.91
2	8	SVD	$f_2(t)$	1.98546722E-03	1.98546722E-03	30000.00	± 0.00	107.64	±0.00	107.64
2	8	SVD	$f_3(t)$	5.93049129E-03	5.93049129E-03	28970.00	± 0.00	107.61	±0.00	107.61
2	8	SVD	$f_4(t)$	2.73411650E-04	2.73411650E-04	30000.00	± 0.00	106.44	±0.00	106.44

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	8	SVD	$f_5(t)$	1.26377057E-02	1.26377057E-02	4449.00	± 0.00	71.41	±0.00	71.41
2	8	SVD	$f_6(t)$	2.16352667E-04	2.16352667E-04	30000.00	± 0.00	102.14	±0.00	102.14
2	9	rand	$f_0(t)$	1.41717163E-03	5.45159014E-04	2021.07	± 5206.49	68.89	±6.13	2066.76
2	9	rand	$f_1(t)$	7.66819934E-03	3.85681485E-03	18362.50	± 7538.73	89.10	±9.12	2673.02
2	9	rand	$f_2(t)$	2.71080212E-03	1.41972569E-03	29930.43	± 374.63	103.87	±1.36	3116.07
2	9	rand	$f_3(t)$	2.85423229E-03	1.63814962E-03	25659.67	± 7196.40	97.88	±8.78	2936.39
2	9	rand	$f_4(t)$	1.57213691E-03	5.86753718E-04	2011.17	± 1597.39	69.65	±2.07	2089.60
2	9	rand	$f_5(t)$	7.45423071E-03	5.97596431E-03	2582.40	± 1400.97	69.03	±2.05	2070.93
2	9	rand	$f_6(t)$	7.70245625E-04	4.48384142E-04	28350.97	± 5824.30	101.03	±7.36	3030.84
2	9	SVD	$f_0(t)$	1.00518074E-03	1.00518074E-03	862.00	± 0.00	72.33	±0.00	72.33
2	9	SVD	$f_1(t)$	1.20400464E-02	1.20400464E-02	15367.00	± 0.00	89.88	±0.00	89.88
2	9	SVD	$f_2(t)$	2.59792984E-03	2.59792984E-03	30000.00	± 0.00	107.45	±0.00	107.45
2	9	SVD	$f_3(t)$	1.41359282E-02	1.41359282E-02	20387.00	± 0.00	94.48	±0.00	94.48
2	9	SVD	$f_4(t)$	1.00518074E-03	1.00518074E-03	862.00	± 0.00	72.51	±0.00	72.51
2	9	SVD	$f_5(t)$	2.19726183E-02	2.19726183E-02	3807.00	± 0.00	72.60	±0.00	72.60
2	9	SVD	$f_6(t)$	6.05703594E-04	6.05703594E-04	30000.00	± 0.00	103.91	±0.00	103.91
2	10	rand	$f_0(t)$	3.40841735E-03	1.26587778E-03	1161.93	± 377.93	68.13	±0.67	2044.02
2	10	rand	$f_1(t)$	1.13719511E-02	7.05117284E-03	18004.13	± 7419.27	89.96	±8.74	2698.89

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	10	rand	$f_2(t)$	4.84477200E-03	2.52857802E-03	30000.00	± 0.00	102.75	±0.89	3082.40
2	10	rand	$f_3(t)$	3.37086907E-03	2.55177950E-03	22813.63	± 8085.29	94.23	±9.60	2826.86
2	10	rand	$f_4(t)$	4.34611570E-03	1.32669717E-03	1642.40	± 1719.80	69.33	±2.04	2079.96
2	10	rand	$f_5(t)$	9.97631319E-03	8.39779733E-03	4670.40	± 5523.15	71.85	±6.44	2155.57
2	10	rand	$f_6(t)$	1.23240865E-03	7.14049549E-04	28468.83	± 5491.60	102.26	±6.88	3067.84
2	10	SVD	$f_0(t)$	1.42288623E-03	1.42288623E-03	652.00	± 0.00	70.82	±0.00	70.82
2	10	SVD	$f_1(t)$	1.95658540E-02	1.95658540E-02	20222.00	± 0.00	97.87	±0.00	97.87
2	10	SVD	$f_2(t)$	7.47081725E-03	7.47081725E-03	30000.00	± 0.00	111.79	±0.00	111.79
2	10	SVD	$f_3(t)$	2.16947334E-02	2.16947334E-02	30000.00	± 0.00	111.13	±0.00	111.13
2	10	SVD	$f_4(t)$	1.42288623E-03	1.42288623E-03	652.00	± 0.00	72.75	±0.00	72.75
2	10	SVD	$f_5(t)$	3.80769365E-02	3.80769365E-02	4125.00	± 0.00	72.73	±0.00	72.73
2	10	SVD	$f_6(t)$	5.34683854E-04	5.34683854E-04	30000.00	± 0.00	104.53	±0.00	104.53
2	11	rand	$f_0(t)$	6.06195963E-03	2.25909130E-03	1150.20	± 513.76	68.43	±0.90	2052.91
2	11	rand	$f_1(t)$	1.80741074E-02	1.11782148E-02	17530.23	± 7145.28	88.46	±8.75	2653.73
2	11	rand	$f_2(t)$	8.71107225E-03	4.44545076E-03	29579.30	± 1735.82	104.57	±2.61	3137.16
2	11	rand	$f_3(t)$	5.02656462E-03	3.47814202E-03	28072.50	± 4073.21	100.70	±5.01	3020.96
2	11	rand	$f_4(t)$	5.48800762E-03	2.25234834E-03	1441.77	± 1051.03	68.66	±1.45	2059.73
2	11	rand	$f_5(t)$	1.36029801E-02	1.03433354E-02	4295.07	± 5031.57	72.14	±5.96	2164.25

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	11	rand	$f_6(t)$	2.05655411E-03	1.20184314E-03	29499.10	± 2069.01	103.69	±2.76	3110.65
2	11	SVD	$f_0(t)$	2.01904672E-03	2.01904672E-03	842.00	± 0.00	73.34	±0.00	73.34
2	11	SVD	$f_1(t)$	3.41159757E-02	3.41159757E-02	19786.00	± 0.00	96.53	±0.00	96.53
2	11	SVD	$f_2(t)$	1.43053667E-02	1.43053667E-02	30000.00	± 0.00	108.78	±0.00	108.78
2	11	SVD	$f_3(t)$	4.29546934E-02	4.29546934E-02	6293.00	± 0.00	80.55	±0.00	80.55
2	11	SVD	$f_4(t)$	2.01904672E-03	2.01904672E-03	842.00	± 0.00	70.92	±0.00	70.92
2	11	SVD	$f_5(t)$	6.01965956E-02	6.01965956E-02	3247.00	± 0.00	71.58	±0.00	71.58
2	11	SVD	$f_6(t)$	3.05753011E-03	3.05753011E-03	30000.00	± 0.00	104.61	±0.00	104.61
2	12	rand	$f_0(t)$	1.90665374E-02	4.54610528E-03	1139.37	± 507.72	69.05	±1.74	2071.49
2	12	rand	$f_1(t)$	1.90297532E-02	1.45008922E-02	14960.67	± 7649.39	84.98	±9.26	2549.30
2	12	rand	$f_2(t)$	1.14193642E-02	7.12810698E-03	30000.00	± 0.00	104.12	±1.52	3123.61
2	12	rand	$f_3(t)$	7.21421793E-03	4.91833308E-03	24806.97	± 8635.56	97.11	±10.58	2913.40
2	12	rand	$f_4(t)$	1.13713148E-02	4.26388740E-03	1595.17	± 1509.52	68.67	±1.94	2059.97
2	12	rand	$f_5(t)$	1.81355823E-02	1.31273041E-02	4345.00	± 5155.63	72.88	±6.22	2186.26
2	12	rand	$f_6(t)$	2.73850676E-03	1.77561150E-03	28268.37	± 5270.60	99.86	±6.61	2995.85
2	12	SVD	$f_0(t)$	6.27981886E-03	6.27981886E-03	2087.00	± 0.00	72.15	±0.00	72.15
2	12	SVD	$f_1(t)$	5.18618136E-02	5.18618136E-02	22396.00	± 0.00	98.15	±0.00	98.15
2	12	SVD	$f_2(t)$	1.97525789E-02	1.97525789E-02	30000.00	± 0.00	108.86	±0.00	108.86

k	m	Init	$f(t)$	Best Min Vol	Avg Min Vol	Avg Iterations		Avg Run Time (s)		Total Time
2	12	SVD	$f_3(t)$	5.79177272E-02	5.79177272E-02	30000.00	$\pm$ 0.00	106.46	$\pm$0.00	106.46
2	12	SVD	$f_4(t)$	6.28875774E-03	6.28875774E-03	5112.00	$\pm$ 0.00	78.94	$\pm$0.00	78.94
2	12	SVD	$f_5(t)$	8.40167907E-02	8.40167907E-02	3617.00	$\pm$ 0.00	72.79	$\pm$0.00	72.79
2	12	SVD	$f_6(t)$	4.38093482E-03	4.38093482E-03	30000.00	$\pm$ 0.00	103.53	$\pm$0.00	103.53
2	13	rand	$f_0(t)$	2.32465099E-02	7.01971898E-03	1145.83	$\pm$ 514.65	67.76	$\pm$0.71	2032.77
2	13	rand	$f_1(t)$	2.71832163E-02	1.92981264E-02	18079.20	$\pm$ 6518.94	90.70	$\pm$7.79	2721.14
2	13	rand	$f_2(t)$	1.40134027E-02	9.40663160E-03	29256.73	$\pm$ 3126.76	102.53	$\pm$3.91	3075.81
2	13	rand	$f_3(t)$	8.44149892E-03	6.23180335E-03	28238.50	$\pm$ 5043.67	101.82	$\pm$6.32	3054.68
2	13	rand	$f_4(t)$	1.94226611E-02	7.31198127E-03	1729.87	$\pm$ 1381.71	69.56	$\pm$2.25	2086.88
2	13	rand	$f_5(t)$	2.21865436E-02	1.68221489E-02	3593.03	$\pm$ 1930.60	70.68	$\pm$2.56	2120.42
2	13	rand	$f_6(t)$	4.34247564E-03	2.37751923E-03	29476.70	$\pm$ 2818.06	103.47	$\pm$4.08	3103.99
2	13	SVD	$f_0(t)$	7.03517277E-03	7.03517277E-03	950.00	$\pm$ 0.00	79.63	$\pm$0.00	79.63
2	13	SVD	$f_1(t)$	4.57993100E-02	4.57993100E-02	15020.00	$\pm$ 0.00	91.46	$\pm$0.00	91.46
2	13	SVD	$f_2(t)$	2.32394467E-02	2.32394467E-02	30000.00	$\pm$ 0.00	108.08	$\pm$0.00	108.08
2	13	SVD	$f_3(t)$	7.99019378E-02	7.99019378E-02	30000.00	$\pm$ 0.00	109.27	$\pm$0.00	109.27
2	13	SVD	$f_4(t)$	7.03517277E-03	7.03517277E-03	950.00	$\pm$ 0.00	70.56	$\pm$0.00	70.56
2	13	SVD	$f_5(t)$	1.04565112E-01	1.04565112E-01	1951.00	$\pm$ 0.00	69.91	$\pm$0.00	69.91
2	13	SVD	$f_6(t)$	1.22347256E-02	1.22347256E-02	30000.00	$\pm$ 0.00	103.62	$\pm$0.00	103.62

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	14	rand	$f_0(t)$	2.57866102E-02	1.21902301E-02	1122.23	± 422.13	67.05	±0.78	2011.54
2	14	rand	$f_1(t)$	3.52435826E-02	2.42353602E-02	16499.87	± 7169.16	86.62	±8.70	2598.69
2	14	rand	$f_2(t)$	1.71067159E-02	1.11735603E-02	29979.03	± 112.91	103.81	±1.20	3114.20
2	14	rand	$f_3(t)$	1.05648631E-02	7.80890160E-03	27989.47	± 5295.88	103.95	±8.43	3118.55
2	14	rand	$f_4(t)$	2.99908771E-02	1.19714083E-02	1031.53	± 543.23	67.70	±0.72	2031.06
2	14	rand	$f_5(t)$	2.33464895E-02	1.94404526E-02	5943.33	± 6405.86	73.49	±7.91	2204.79
2	14	rand	$f_6(t)$	5.89201875E-03	3.58179864E-03	27766.27	± 5830.86	101.87	±7.60	3056.10
2	14	SVD	$f_0(t)$	2.71896976E-02	2.71896976E-02	815.00	± 0.00	71.19	±0.00	71.19
2	14	SVD	$f_1(t)$	8.11180685E-02	8.11180685E-02	17192.00	± 0.00	91.85	±0.00	91.85
2	14	SVD	$f_2(t)$	3.69507004E-02	3.69507004E-02	30000.00	± 0.00	110.20	±0.00	110.20
2	14	SVD	$f_3(t)$	9.83216690E-02	9.83216690E-02	30000.00	± 0.00	112.16	±0.00	112.16
2	14	SVD	$f_4(t)$	2.71896976E-02	2.71896976E-02	815.00	± 0.00	72.23	±0.00	72.23
2	14	SVD	$f_5(t)$	1.25518267E-01	1.25518267E-01	3350.00	± 0.00	72.72	±0.00	72.72
2	14	SVD	$f_6(t)$	9.03353709E-03	9.03353709E-03	30000.00	± 0.00	104.72	±0.00	104.72
2	15	rand	$f_0(t)$	3.15592759E-02	1.68445764E-02	1005.40	± 349.44	68.25	±1.18	2047.64
2	15	rand	$f_1(t)$	3.69907084E-02	3.02689512E-02	18133.50	± 8143.18	88.94	±10.06	2668.12
2	15	rand	$f_2(t)$	2.69481781E-02	1.58738264E-02	29595.50	± 2178.30	103.26	±2.77	3097.87
2	15	rand	$f_3(t)$	1.14601636E-02	9.25722546E-03	28046.20	± 5418.02	101.01	±6.76	3030.44

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	15	rand	$f_4(t)$	3.16188342E-02	1.65996576E-02	1371.17	± 1147.47	69.64	±1.92	2089.22
2	15	rand	$f_5(t)$	2.79193838E-02	2.27873337E-02	4660.47	± 5197.40	73.11	±6.72	2193.16
2	15	rand	$f_6(t)$	7.04734225E-03	4.08136292E-03	29625.50	± 1681.88	104.00	±2.71	3119.95
2	15	SVD	$f_0(t)$	3.80289567E-02	3.80289567E-02	1301.00	± 0.00	72.75	±0.00	72.75
2	15	SVD	$f_1(t)$	1.06807359E-01	1.06807359E-01	26017.00	± 0.00	108.59	±0.00	108.59
2	15	SVD	$f_2(t)$	5.51024029E-02	5.51024029E-02	30000.00	± 0.00	110.14	±0.00	110.14
2	15	SVD	$f_3(t)$	1.24717046E-01	1.24717046E-01	6811.00	± 0.00	80.16	±0.00	80.16
2	15	SVD	$f_4(t)$	3.80517837E-02	3.80517837E-02	1048.00	± 0.00	71.44	±0.00	71.44
2	15	SVD	$f_5(t)$	1.42213712E-01	1.42213712E-01	6846.00	± 0.00	75.43	±0.00	75.43
2	15	SVD	$f_6(t)$	1.93742783E-02	1.93742783E-02	30000.00	± 0.00	107.83	±0.00	107.83
2	16	rand	$f_0(t)$	3.45769252E-02	2.08623262E-02	1160.00	± 453.15	68.32	±1.59	2049.64
2	16	rand	$f_1(t)$	4.79459981E-02	3.67785780E-02	16416.60	± 7059.67	87.08	±8.88	2612.50
2	16	rand	$f_2(t)$	3.53168508E-02	2.20178170E-02	29519.53	± 2587.39	103.93	±3.19	3118.00
2	16	rand	$f_3(t)$	1.40142335E-02	1.12666528E-02	28163.73	± 4092.76	103.57	±5.01	3107.15
2	16	rand	$f_4(t)$	4.56603876E-02	2.47442211E-02	1548.80	± 1406.51	68.26	±1.78	2047.89
2	16	rand	$f_5(t)$	3.34382023E-02	2.69269889E-02	5769.23	± 5301.06	74.84	±6.74	2245.34
2	16	rand	$f_6(t)$	9.02378796E-03	5.88532060E-03	28333.33	± 5223.40	102.34	±7.04	3070.15
2	16	SVD	$f_0(t)$	2.68813458E-02	2.68813458E-02	1195.00	± 0.00	72.19	±0.00	72.19

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	16	SVD	$f_1(t)$	1.26404468E-01	1.26404468E-01	30000.00	$\pm$ 0.00	107.78	$\pm$0.00	107.78
2	16	SVD	$f_2(t)$	5.50631851E-02	5.50631851E-02	30000.00	$\pm$ 0.00	111.99	$\pm$0.00	111.99
2	16	SVD	$f_3(t)$	1.44465435E-01	1.44465435E-01	30000.00	$\pm$ 0.00	110.37	$\pm$0.00	110.37
2	16	SVD	$f_4(t)$	2.68816976E-02	2.68816976E-02	1556.00	$\pm$ 0.00	73.75	$\pm$0.00	73.75
2	16	SVD	$f_5(t)$	1.82138143E-01	1.82138143E-01	6561.00	$\pm$ 0.00	75.03	$\pm$0.00	75.03
2	16	SVD	$f_6(t)$	4.15774550E-02	4.15774550E-02	22680.00	$\pm$ 0.00	95.33	$\pm$0.00	95.33
2	17	rand	$f_0(t)$	4.85263793E-02	3.01018888E-02	1112.33	$\pm$ 535.86	68.51	$\pm$0.96	2055.31
2	17	rand	$f_1(t)$	5.53210208E-02	4.30589507E-02	16707.27	$\pm$ 8104.04	88.29	$\pm$10.21	2648.55
2	17	rand	$f_2(t)$	3.55775264E-02	2.55390475E-02	29470.53	$\pm$ 2832.38	104.05	$\pm$3.84	3121.45
2	17	rand	$f_3(t)$	1.87666490E-02	1.39948056E-02	24778.57	$\pm$ 7770.33	97.63	$\pm$9.76	2928.80
2	17	rand	$f_4(t)$	5.50556313E-02	2.80357170E-02	1471.07	$\pm$ 1108.17	69.95	$\pm$1.90	2098.47
2	17	rand	$f_5(t)$	3.78778389E-02	3.09745271E-02	6598.70	$\pm$ 8155.83	75.72	$\pm$10.41	2271.68
2	17	rand	$f_6(t)$	1.08160367E-02	6.90629682E-03	29053.23	$\pm$ 3002.51	104.09	$\pm$4.48	3122.77
2	17	SVD	$f_0(t)$	5.10748473E-02	5.10748473E-02	655.00	$\pm$ 0.00	70.56	$\pm$0.00	70.56
2	17	SVD	$f_1(t)$	1.56144510E-01	1.56144510E-01	17082.00	$\pm$ 0.00	93.30	$\pm$0.00	93.30
2	17	SVD	$f_2(t)$	9.73149971E-02	9.73149971E-02	30000.00	$\pm$ 0.00	108.52	$\pm$0.00	108.52
2	17	SVD	$f_3(t)$	1.66935967E-01	1.66935967E-01	26555.00	$\pm$ 0.00	104.68	$\pm$0.00	104.68
2	17	SVD	$f_4(t)$	5.10748473E-02	5.10748473E-02	655.00	$\pm$ 0.00	72.43	$\pm$0.00	72.43

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	17	SVD	$f_5(t)$	2.03554982E-01	2.03554982E-01	3684.00	± 0.00	71.80	±0.00	71.80
2	17	SVD	$f_6(t)$	5.83368674E-02	5.83368674E-02	30000.00	± 0.00	105.18	±0.00	105.18
2	18	rand	$f_0(t)$	6.03446207E-02	3.71664438E-02	1271.50	± 578.89	68.59	±0.89	2057.72
2	18	rand	$f_1(t)$	6.38772764E-02	5.11121352E-02	17704.07	± 7263.02	88.94	±9.19	2668.08
2	18	rand	$f_2(t)$	4.81750405E-02	3.28403276E-02	29589.83	± 2055.09	105.60	±2.75	3168.01
2	18	rand	$f_3(t)$	2.04306001E-02	1.55670570E-02	27407.13	± 6440.90	101.41	±8.27	3042.28
2	18	rand	$f_4(t)$	7.06279927E-02	4.13732924E-02	1367.20	± 1176.86	69.73	±1.68	2091.90
2	18	rand	$f_5(t)$	4.02185606E-02	3.42775986E-02	6547.37	± 5165.55	75.84	±7.01	2275.22
2	18	rand	$f_6(t)$	1.25789691E-02	8.52786433E-03	29280.07	± 3137.88	103.75	±4.21	3112.37
2	18	SVD	$f_0(t)$	9.22337140E-02	9.22337140E-02	724.00	± 0.00	70.82	±0.00	70.82
2	18	SVD	$f_1(t)$	1.85075577E-01	1.85075577E-01	30000.00	± 0.00	112.53	±0.00	112.53
2	18	SVD	$f_2(t)$	1.06048303E-01	1.06048303E-01	30000.00	± 0.00	108.84	±0.00	108.84
2	18	SVD	$f_3(t)$	1.93498594E-01	1.93498594E-01	30000.00	± 0.00	111.61	±0.00	111.61
2	18	SVD	$f_4(t)$	9.22337140E-02	9.22337140E-02	724.00	± 0.00	72.37	±0.00	72.37
2	18	SVD	$f_5(t)$	2.45940035E-01	2.45940035E-01	5653.00	± 0.00	75.27	±0.00	75.27
2	18	SVD	$f_6(t)$	5.03238482E-02	5.03238482E-02	30000.00	± 0.00	105.88	±0.00	105.88
2	19	rand	$f_0(t)$	6.81658778E-02	5.05914844E-02	1121.77	± 444.76	70.23	±2.33	2106.77
2	19	rand	$f_1(t)$	6.96935678E-02	5.61622672E-02	17604.37	± 7482.06	91.44	±9.19	2743.23

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	19	rand	$f_2(t)$	6.00252165E-02	3.90269830E-02	29050.83	± 3427.74	103.22	±4.27	3096.73
2	19	rand	$f_3(t)$	2.23292678E-02	1.75725618E-02	28035.87	± 5663.86	102.26	±7.35	3067.92
2	19	rand	$f_4(t)$	7.68371558E-02	4.61864394E-02	1422.60	± 1117.94	70.73	±2.33	2121.91
2	19	rand	$f_5(t)$	4.47824240E-02	3.84328503E-02	6542.77	± 4976.18	75.04	±6.26	2251.08
2	19	rand	$f_6(t)$	1.61122525E-02	1.12396628E-02	28581.67	± 4362.32	103.12	±5.58	3093.65
2	19	SVD	$f_0(t)$	1.18200197E-01	1.18200197E-01	651.00	± 0.00	71.21	±0.00	71.21
2	19	SVD	$f_1(t)$	2.34758163E-01	2.34758163E-01	7758.00	± 0.00	85.45	±0.00	85.45
2	19	SVD	$f_2(t)$	1.54004809E-01	1.54004809E-01	30000.00	± 0.00	108.76	±0.00	108.76
2	19	SVD	$f_3(t)$	2.42785478E-01	2.42785478E-01	22957.00	± 0.00	101.94	±0.00	101.94
2	19	SVD	$f_4(t)$	1.18200197E-01	1.18200197E-01	651.00	± 0.00	70.27	±0.00	70.27
2	19	SVD	$f_5(t)$	2.85298432E-01	2.85298432E-01	7921.00	± 0.00	77.58	±0.00	77.58
2	19	SVD	$f_6(t)$	7.35818713E-02	7.35818713E-02	30000.00	± 0.00	107.38	±0.00	107.38
2	20	rand	$f_0(t)$	9.23272634E-02	6.26261507E-02	1025.90	± 392.91	67.57	±0.85	2027.01
2	20	rand	$f_1(t)$	9.01360358E-02	6.55235696E-02	16398.97	± 7048.66	89.52	±9.24	2685.54
2	20	rand	$f_2(t)$	6.53970259E-02	4.56690238E-02	29400.53	± 2938.79	103.67	±3.69	3110.17
2	20	rand	$f_3(t)$	2.73438531E-02	2.09154293E-02	28049.67	± 3653.35	102.96	±4.72	3088.78
2	20	rand	$f_4(t)$	9.95993726E-02	6.08409493E-02	1788.50	± 1528.74	69.21	±2.04	2076.16
2	20	rand	$f_5(t)$	5.11347429E-02	4.46886317E-02	6757.03	± 6748.53	75.51	±8.59	2265.22

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	20	rand	$f_6(t)$	1.97391179E-02	1.35415135E-02	28260.47	± 4783.55	101.79	±5.96	3053.67
2	20	SVD	$f_0(t)$	7.58802546E-02	7.58802546E-02	2087.00	± 0.00	74.40	±0.00	74.40
2	20	SVD	$f_1(t)$	1.95584886E-01	1.95584886E-01	8005.00	± 0.00	80.80	±0.00	80.80
2	20	SVD	$f_2(t)$	1.10210218E-01	1.10210218E-01	30000.00	± 0.00	109.07	±0.00	109.07
2	20	SVD	$f_3(t)$	2.57753009E-01	2.57753009E-01	26166.00	± 0.00	105.38	±0.00	105.38
2	20	SVD	$f_4(t)$	7.62014979E-02	7.62014979E-02	5087.00	± 0.00	77.72	±0.00	77.72
2	20	SVD	$f_5(t)$	3.17029541E-01	3.17029541E-01	12298.00	± 0.00	83.66	±0.00	83.66
2	20	SVD	$f_6(t)$	8.78387715E-02	8.78387715E-02	30000.00	± 0.00	105.76	±0.00	105.76
2	21	rand	$f_0(t)$	1.03162840E-01	7.22661645E-02	1155.00	± 373.79	67.71	±0.88	2031.27
2	21	rand	$f_1(t)$	8.54229560E-02	6.99446506E-02	15752.03	± 6148.79	87.33	±7.95	2619.97
2	21	rand	$f_2(t)$	7.15114464E-02	5.47503308E-02	27623.07	± 6456.28	102.55	±8.27	3076.51
2	21	rand	$f_3(t)$	2.98906952E-02	2.43813192E-02	26976.33	± 5952.69	101.87	±7.64	3056.17
2	21	rand	$f_4(t)$	1.12032646E-01	7.08923765E-02	1189.67	± 1136.48	67.70	±1.60	2030.89
2	21	rand	$f_5(t)$	5.81360354E-02	4.85175596E-02	4945.93	± 3370.78	72.99	±4.43	2189.72
2	21	rand	$f_6(t)$	1.86268674E-02	1.42270548E-02	28183.40	± 5135.40	103.96	±6.89	3118.81
2	21	SVD	$f_0(t)$	1.25540750E-01	1.25540750E-01	981.00	± 0.00	73.25	±0.00	73.25
2	21	SVD	$f_1(t)$	2.87513682E-01	2.87513682E-01	15441.00	± 0.00	90.99	±0.00	90.99
2	21	SVD	$f_2(t)$	2.17747242E-01	2.17747242E-01	30000.00	± 0.00	109.42	±0.00	109.42

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	21	SVD	$f_3(t)$	2.97029294E-01	2.97029294E-01	30000.00	± 0.00	109.66	±0.00	109.66
2	21	SVD	$f_4(t)$	1.25540750E-01	1.25540750E-01	981.00	± 0.00	71.97	±0.00	71.97
2	21	SVD	$f_5(t)$	3.72140644E-01	3.72140644E-01	12267.00	± 0.00	84.19	±0.00	84.19
2	21	SVD	$f_6(t)$	1.36537119E-01	1.36537119E-01	30000.00	± 0.00	107.27	±0.00	107.27
2	22	rand	$f_0(t)$	1.33798741E-01	8.96741162E-02	1102.60	± 442.27	68.09	±0.62	2042.83
2	22	rand	$f_1(t)$	9.91234247E-02	8.31901773E-02	14319.40	± 6545.63	87.01	±8.69	2610.26
2	22	rand	$f_2(t)$	7.84051679E-02	6.07998892E-02	29390.57	± 3157.63	105.74	±3.95	3172.32
2	22	rand	$f_3(t)$	2.98957697E-02	2.65069858E-02	26299.13	± 7621.37	101.58	±9.64	3047.41
2	22	rand	$f_4(t)$	1.39325475E-01	8.56500419E-02	1422.20	± 1197.62	68.76	±1.59	2062.84
2	22	rand	$f_5(t)$	6.43271842E-02	5.37707364E-02	7320.93	± 5815.34	75.91	±7.42	2277.29
2	22	rand	$f_6(t)$	2.57865907E-02	1.81823231E-02	26481.70	± 6774.18	101.78	±8.73	3053.45
2	22	SVD	$f_0(t)$	1.71939131E-01	1.71939131E-01	1032.00	± 0.00	71.20	±0.00	71.20
2	22	SVD	$f_1(t)$	3.07983004E-01	3.07983004E-01	9814.00	± 0.00	85.03	±0.00	85.03
2	22	SVD	$f_2(t)$	2.29134661E-01	2.29134661E-01	30000.00	± 0.00	113.06	±0.00	113.06
2	22	SVD	$f_3(t)$	3.41498940E-01	3.41498940E-01	30000.00	± 0.00	110.69	±0.00	110.69
2	22	SVD	$f_4(t)$	1.71955036E-01	1.71955036E-01	1004.00	± 0.00	70.99	±0.00	70.99
2	22	SVD	$f_5(t)$	4.08331746E-01	4.08331746E-01	3557.00	± 0.00	71.76	±0.00	71.76
2	22	SVD	$f_6(t)$	1.60943780E-01	1.60943780E-01	26932.00	± 0.00	103.53	±0.00	103.53

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	23	rand	$f_0(t)$	1.43680108E-01	9.98677713E-02	1005.87	± 339.35	68.52	±1.27	2055.50
2	23	rand	$f_1(t)$	1.09898264E-01	8.75913507E-02	15625.73	± 6731.64	87.20	±8.86	2616.08
2	23	rand	$f_2(t)$	9.61138024E-02	6.94938786E-02	29561.60	± 2360.86	105.09	±3.16	3152.70
2	23	rand	$f_3(t)$	3.98726353E-02	3.11182761E-02	28648.60	± 4508.63	105.05	±6.01	3151.63
2	23	rand	$f_4(t)$	1.58098283E-01	9.89481306E-02	1363.83	± 1183.44	68.78	±1.66	2063.27
2	23	rand	$f_5(t)$	6.73689602E-02	5.82109745E-02	8515.33	± 7307.01	77.52	±9.50	2325.61
2	23	rand	$f_6(t)$	2.99073513E-02	2.05693307E-02	28661.57	± 3397.44	105.13	±4.53	3153.80
2	23	SVD	$f_0(t)$	2.16976424E-01	2.16976424E-01	1132.00	± 0.00	73.17	±0.00	73.17
2	23	SVD	$f_1(t)$	3.60170181E-01	3.60170181E-01	8808.00	± 0.00	81.09	±0.00	81.09
2	23	SVD	$f_2(t)$	2.68645002E-01	2.68645002E-01	30000.00	± 0.00	109.53	±0.00	109.53
2	23	SVD	$f_3(t)$	3.78532778E-01	3.78532778E-01	30000.00	± 0.00	109.06	±0.00	109.06
2	23	SVD	$f_4(t)$	2.17021945E-01	2.17021945E-01	1058.00	± 0.00	72.68	±0.00	72.68
2	23	SVD	$f_5(t)$	4.47469869E-01	4.47469869E-01	12440.00	± 0.00	83.68	±0.00	83.68
2	23	SVD	$f_6(t)$	1.92154981E-01	1.92154981E-01	27600.00	± 0.00	104.20	±0.00	104.20
2	24	rand	$f_0(t)$	1.50742005E-01	1.13948813E-01	941.17	± 269.41	69.93	±1.72	2097.93
2	24	rand	$f_1(t)$	1.25924534E-01	9.43865501E-02	18086.77	± 7552.45	90.43	±10.36	2712.82
2	24	rand	$f_2(t)$	9.67259280E-02	7.34269378E-02	30000.00	± 0.00	106.86	±0.96	3205.94
2	24	rand	$f_3(t)$	4.28329500E-02	3.36151543E-02	28895.57	± 4201.11	106.08	±5.48	3182.34

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	24	rand	$f_4(t)$	1.56783429E-01	1.15248802E-01	1360.80	± 984.74	69.65	±1.57	2089.42
2	24	rand	$f_5(t)$	7.61850070E-02	6.48189632E-02	7043.20	± 4829.57	76.11	±6.35	2283.32
2	24	rand	$f_6(t)$	3.16293018E-02	2.43233952E-02	28586.97	± 3448.74	103.87	±4.70	3116.25
2	24	SVD	$f_0(t)$	2.55526357E-01	2.55526357E-01	1087.00	± 0.00	72.08	±0.00	72.08
2	24	SVD	$f_1(t)$	4.04895038E-01	4.04895038E-01	23926.00	± 0.00	102.65	±0.00	102.65
2	24	SVD	$f_2(t)$	3.25820105E-01	3.25820105E-01	30000.00	± 0.00	109.71	±0.00	109.71
2	24	SVD	$f_3(t)$	4.07025722E-01	4.07025722E-01	30000.00	± 0.00	111.19	±0.00	111.19
2	24	SVD	$f_4(t)$	2.55569271E-01	2.55569271E-01	1042.00	± 0.00	73.60	±0.00	73.60
2	24	SVD	$f_5(t)$	4.85272335E-01	4.85272335E-01	3593.00	± 0.00	72.70	±0.00	72.70
2	24	SVD	$f_6(t)$	2.91857541E-01	2.91857541E-01	5546.00	± 0.00	73.72	±0.00	73.72
2	25	rand	$f_0(t)$	1.57646103E-01	1.19698170E-01	945.53	± 272.36	67.72	±0.55	2031.60
2	25	rand	$f_1(t)$	1.27405835E-01	1.08050749E-01	17315.17	± 6657.03	90.06	±8.59	2701.91
2	25	rand	$f_2(t)$	9.88335327E-02	8.32426945E-02	29477.17	± 2112.46	106.59	±2.80	3197.72
2	25	rand	$f_3(t)$	4.76336484E-02	3.77346133E-02	27684.10	± 4830.26	102.09	±6.18	3062.55
2	25	rand	$f_4(t)$	1.61008477E-01	1.20849562E-01	1194.50	± 915.23	70.53	±1.32	2115.76
2	25	rand	$f_5(t)$	8.04214981E-02	6.94287077E-02	7979.57	± 6720.70	77.64	±8.94	2329.22
2	25	rand	$f_6(t)$	3.51004101E-02	2.64344670E-02	27477.03	± 5778.43	103.63	±7.65	3108.82
2	25	SVD	$f_0(t)$	3.05562636E-01	3.05562636E-01	743.00	± 0.00	71.16	±0.00	71.16

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	25	SVD	$f_1(t)$	4.62141661E-01	4.62141661E-01	18055.00	± 0.00	97.40	±0.00	97.40
2	25	SVD	$f_2(t)$	3.78110941E-01	3.78110941E-01	30000.00	± 0.00	110.41	±0.00	110.41
2	25	SVD	$f_3(t)$	4.79345897E-01	4.79345897E-01	21661.00	± 0.00	101.67	±0.00	101.67
2	25	SVD	$f_4(t)$	3.05562636E-01	3.05562636E-01	743.00	± 0.00	72.46	±0.00	72.46
2	25	SVD	$f_5(t)$	5.44990324E-01	5.44990324E-01	5370.00	± 0.00	75.93	±0.00	75.93
2	25	SVD	$f_6(t)$	2.33498327E-01	2.33498327E-01	30000.00	± 0.00	107.43	±0.00	107.43
2	26	rand	$f_0(t)$	1.84327428E-01	1.52466491E-01	956.90	± 315.20	68.77	±0.88	2063.23
2	26	rand	$f_1(t)$	1.32147656E-01	1.15398273E-01	17199.00	± 7021.75	89.85	±8.98	2695.36
2	26	rand	$f_2(t)$	1.12096319E-01	9.14706829E-02	29292.93	± 3141.22	108.14	±4.53	3244.32
2	26	rand	$f_3(t)$	4.52693140E-02	4.08793706E-02	28367.77	± 5590.47	105.61	±7.52	3168.19
2	26	rand	$f_4(t)$	2.00031945E-01	1.44927075E-01	1425.93	± 1142.54	68.51	±1.42	2055.28
2	26	rand	$f_5(t)$	8.93445157E-02	7.51113395E-02	5894.97	± 4386.92	75.05	±5.92	2251.54
2	26	rand	$f_6(t)$	4.16001971E-02	3.15175578E-02	28137.23	± 5291.61	104.18	±7.37	3125.34
2	26	SVD	$f_0(t)$	3.97980052E-01	3.97980052E-01	477.00	± 0.00	70.09	±0.00	70.09
2	26	SVD	$f_1(t)$	4.81519240E-01	4.81519240E-01	10887.00	± 0.00	85.41	±0.00	85.41
2	26	SVD	$f_2(t)$	3.78845005E-01	3.78845005E-01	30000.00	± 0.00	113.71	±0.00	113.71
2	26	SVD	$f_3(t)$	5.21300125E-01	5.21300125E-01	22406.00	± 0.00	101.34	±0.00	101.34
2	26	SVD	$f_4(t)$	3.97980052E-01	3.97980052E-01	477.00	± 0.00	72.30	±0.00	72.30

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	26	SVD	$f_5(t)$	5.71863102E-01	5.71863102E-01	6894.00	± 0.00	78.85	±0.00	78.85
2	26	SVD	$f_6(t)$	3.27770558E-01	3.27770558E-01	26100.00	± 0.00	102.39	±0.00	102.39
2	27	rand	$f_0(t)$	2.14804030E-01	1.52987666E-01	1046.13	± 379.43	68.41	±0.92	2052.19
2	27	rand	$f_1(t)$	1.47803500E-01	1.24536870E-01	15348.07	± 6753.46	87.83	±9.07	2635.02
2	27	rand	$f_2(t)$	1.15194825E-01	1.00589860E-01	28514.80	± 4405.23	107.08	±6.29	3212.28
2	27	rand	$f_3(t)$	5.15705570E-02	4.48072381E-02	28001.20	± 5863.09	104.37	±7.90	3131.21
2	27	rand	$f_4(t)$	2.09687992E-01	1.53560601E-01	1437.73	± 1050.82	69.69	±2.18	2090.75
2	27	rand	$f_5(t)$	9.35623262E-02	8.16038596E-02	8022.40	± 6872.52	77.74	±9.10	2332.33
2	27	rand	$f_6(t)$	4.25797003E-02	3.38612071E-02	28092.50	± 5782.87	107.09	±7.90	3212.63
2	27	SVD	$f_0(t)$	3.19026858E-01	3.19026858E-01	1148.00	± 0.00	73.46	±0.00	73.46
2	27	SVD	$f_1(t)$	5.31023659E-01	5.31023659E-01	18381.00	± 0.00	98.17	±0.00	98.17
2	27	SVD	$f_2(t)$	4.63410852E-01	4.63410852E-01	29210.00	± 0.00	111.69	±0.00	111.69
2	27	SVD	$f_3(t)$	5.51932136E-01	5.51932136E-01	30000.00	± 0.00	112.67	±0.00	112.67
2	27	SVD	$f_4(t)$	3.19079878E-01	3.19079878E-01	1044.00	± 0.00	70.90	±0.00	70.90
2	27	SVD	$f_5(t)$	6.11326460E-01	6.11326460E-01	5172.00	± 0.00	74.75	±0.00	74.75
2	27	SVD	$f_6(t)$	3.57537519E-01	3.57537519E-01	7554.00	± 0.00	78.13	±0.00	78.13
2	28	rand	$f_0(t)$	2.33665625E-01	1.65938754E-01	1064.53	± 385.66	68.29	±1.33	2048.69
2	28	rand	$f_1(t)$	1.63522104E-01	1.34873572E-01	14625.60	± 6448.56	86.54	±8.49	2596.22

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	28	rand	$f_2(t)$	1.32669109E-01	1.08652628E-01	29113.60	± 2726.64	105.93	±3.67	3177.84
2	28	rand	$f_3(t)$	6.08388726E-02	4.99985554E-02	25524.90	± 8495.92	101.53	±11.04	3045.96
2	28	rand	$f_4(t)$	2.11465265E-01	1.74245742E-01	1305.73	± 1271.53	69.01	±1.94	2070.18
2	28	rand	$f_5(t)$	1.00624454E-01	8.78119232E-02	7871.37	± 7059.90	78.64	±9.51	2359.19
2	28	rand	$f_6(t)$	4.93440189E-02	3.81586459E-02	27613.57	± 5251.01	103.21	±6.87	3096.17
2	28	SVD	$f_0(t)$	4.26951409E-01	4.26951409E-01	1298.00	± 0.00	72.23	±0.00	72.23
2	28	SVD	$f_1(t)$	5.68545832E-01	5.68545832E-01	8523.00	± 0.00	82.72	±0.00	82.72
2	28	SVD	$f_2(t)$	4.72588135E-01	4.72588135E-01	30000.00	± 0.00	111.32	±0.00	111.32
2	28	SVD	$f_3(t)$	5.74071096E-01	5.74071096E-01	30000.00	± 0.00	112.82	±0.00	112.82
2	28	SVD	$f_4(t)$	4.26996094E-01	4.26996094E-01	1194.00	± 0.00	70.74	±0.00	70.74
2	28	SVD	$f_5(t)$	6.30152548E-01	6.30152548E-01	4136.00	± 0.00	74.05	±0.00	74.05
2	28	SVD	$f_6(t)$	4.07378051E-01	4.07378051E-01	27133.00	± 0.00	105.10	±0.00	105.10
2	29	rand	$f_0(t)$	2.36563924E-01	1.82721854E-01	1017.60	± 380.26	68.24	±1.02	2047.08
2	29	rand	$f_1(t)$	1.66587404E-01	1.41787139E-01	15032.37	± 8103.36	86.88	±10.82	2606.27
2	29	rand	$f_2(t)$	1.37371654E-01	1.13919649E-01	28771.33	± 3444.04	106.26	±4.58	3187.88
2	29	rand	$f_3(t)$	6.33502623E-02	5.33931017E-02	28203.93	± 4819.66	106.09	±6.75	3182.56
2	29	rand	$f_4(t)$	2.38712016E-01	1.81088012E-01	1166.23	± 741.22	67.87	±1.05	2036.10
2	29	rand	$f_5(t)$	1.13195108E-01	9.33797327E-02	5873.60	± 3446.70	75.24	±4.60	2257.11

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	29	rand	$f_6(t)$	5.36508534E-02	4.08747828E-02	26781.50	± 6615.05	105.02	±9.20	3150.56
2	29	SVD	$f_0(t)$	5.48319865E-01	5.48319865E-01	1403.00	± 0.00	72.59	±0.00	72.59
2	29	SVD	$f_1(t)$	5.91043110E-01	5.91043110E-01	5835.00	± 0.00	80.40	±0.00	80.40
2	29	SVD	$f_2(t)$	5.07289886E-01	5.07289886E-01	30000.00	± 0.00	116.00	±0.00	116.00
2	29	SVD	$f_3(t)$	6.22644240E-01	6.22644240E-01	25854.00	± 0.00	105.80	±0.00	105.80
2	29	SVD	$f_4(t)$	5.48396463E-01	5.48396463E-01	1518.00	± 0.00	73.03	±0.00	73.03
2	29	SVD	$f_5(t)$	6.82683101E-01	6.82683101E-01	3631.00	± 0.00	72.47	±0.00	72.47
2	29	SVD	$f_6(t)$	3.96027822E-01	3.96027822E-01	24210.00	± 0.00	100.95	±0.00	100.95
2	30	rand	$f_0(t)$	2.56260586E-01	2.02730757E-01	1130.20	± 469.82	67.81	±1.43	2034.39
2	30	rand	$f_1(t)$	1.78503440E-01	1.52365490E-01	15027.30	± 7817.26	88.27	±10.83	2648.08
2	30	rand	$f_2(t)$	1.45928056E-01	1.25139667E-01	28859.00	± 3699.51	107.04	±5.73	3211.21
2	30	rand	$f_3(t)$	6.82747902E-02	5.89641282E-02	29081.93	± 3464.65	107.34	±4.99	3220.23
2	30	rand	$f_4(t)$	2.49360585E-01	2.02992864E-01	1182.10	± 902.51	68.97	±1.66	2069.11
2	30	rand	$f_5(t)$	1.10373799E-01	9.99447714E-02	7546.70	± 6139.16	77.98	±8.47	2339.42
2	30	rand	$f_6(t)$	6.54649745E-02	4.56382820E-02	27940.13	± 4476.26	104.72	±6.32	3141.46
2	30	SVD	$f_0(t)$	5.73955261E-01	5.73955261E-01	663.00	± 0.00	70.81	±0.00	70.81
2	30	SVD	$f_1(t)$	6.64020642E-01	6.64020642E-01	9532.00	± 0.00	85.38	±0.00	85.38
2	30	SVD	$f_2(t)$	6.22639933E-01	6.22639933E-01	17800.00	± 0.00	97.82	±0.00	97.82

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
2	30	SVD	$f_3(t)$	6.52401749E-01	6.52401749E-01	30000.00	$\pm$ 0.00	113.86	$\pm$0.00	113.86
2	30	SVD	$f_4(t)$	5.73955261E-01	5.73955261E-01	663.00	$\pm$ 0.00	71.56	$\pm$0.00	71.56
2	30	SVD	$f_5(t)$	7.18293271E-01	7.18293271E-01	4233.00	$\pm$ 0.00	72.97	$\pm$0.00	72.97
2	30	SVD	$f_6(t)$	4.90348616E-01	4.90348616E-01	8102.00	$\pm$ 0.00	79.73	$\pm$0.00	79.73
3	3	rand	$f_0(t)$	1.21658450E-13	1.02169819E-14	30000.00	$\pm$ 0.00	651.38	$\pm$5.86	19541.44
3	3	rand	$f_1(t)$	3.74260778E-13	3.32639878E-14	10919.60	$\pm$ 4619.54	626.58	$\pm$6.74	18797.38
3	3	rand	$f_2(t)$	6.83624464E-14	9.45328660E-15	29003.43	$\pm$ 5366.68	651.99	$\pm$8.13	19559.78
3	3	rand	$f_3(t)$	1.07707156E-13	1.12845927E-14	6367.30	$\pm$ 6437.99	621.10	$\pm$8.84	18632.89
3	3	rand	$f_4(t)$	2.60822415E-14	3.95443479E-15	29113.33	$\pm$ 3728.62	656.54	$\pm$9.34	19696.23
3	3	rand	$f_5(t)$	3.66839117E-13	4.25932554E-14	1056.77	$\pm$ 1432.50	613.07	$\pm$5.34	18392.10
3	3	rand	$f_6(t)$	1.08506927E-13	9.83291063E-15	30000.00	$\pm$ 0.00	664.84	$\pm$11.00	19945.06
3	3	SVD	$f_0(t)$	7.09384657E-16	7.09384657E-16	1.00	$\pm$ 0.00	25.30	$\pm$0.00	25.30
3	3	SVD	$f_1(t)$	3.76530221E-18	3.76530221E-18	1.00	$\pm$ 0.00	25.89	$\pm$0.00	25.89
3	3	SVD	$f_2(t)$	3.76530221E-18	3.76530221E-18	1.00	$\pm$ 0.00	25.44	$\pm$0.00	25.44
3	3	SVD	$f_3(t)$	3.76530221E-18	3.76530221E-18	1.00	$\pm$ 0.00	25.45	$\pm$0.00	25.45
3	3	SVD	$f_4(t)$	7.09384657E-16	7.09384657E-16	1.00	$\pm$ 0.00	26.05	$\pm$0.00	26.05
3	3	SVD	$f_5(t)$	1.54125094E-15	1.54125094E-15	266.00	$\pm$ 0.00	625.33	$\pm$0.00	625.33
3	3	SVD	$f_6(t)$	5.35655639E-15	5.35655639E-15	30000.00	$\pm$ 0.00	654.55	$\pm$0.00	654.55

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	4	rand	$f_0(t)$	2.30754086E-09	3.65551553E-10	21000.90	± 13746.35	470.90	±288.76	14127.13
3	4	rand	$f_1(t)$	3.11154574E-08	3.68853301E-09	11887.97	± 7974.40	511.74	±241.83	15352.34
3	4	rand	$f_2(t)$	2.34509035E-08	1.15409287E-09	22227.83	± 12940.02	522.64	±251.62	15679.08
3	4	rand	$f_3(t)$	2.59539698E-08	1.75419659E-09	2562.23	± 5418.55	256.34	±248.09	7690.13
3	4	rand	$f_4(t)$	1.70556246E-09	2.89117241E-10	12504.13	± 14530.44	317.59	±298.80	9527.61
3	4	rand	$f_5(t)$	3.08273052E-07	1.77371444E-07	161.77	± 73.63	585.22	±112.78	17556.61
3	4	rand	$f_6(t)$	3.24368400E-09	6.23100756E-10	26010.17	± 10172.12	626.13	±63.61	18784.03
3	4	SVD	$f_0(t)$	9.60116806E-12	9.60116806E-12	30000.00	± 0.00	675.01	±0.00	675.01
3	4	SVD	$f_1(t)$	2.21406944E-09	2.21406944E-09	15960.00	± 0.00	669.03	±0.00	669.03
3	4	SVD	$f_2(t)$	1.33599042E-09	1.33599042E-09	30000.00	± 0.00	683.52	±0.00	683.52
3	4	SVD	$f_3(t)$	1.44786926E-09	1.44786926E-09	25.00	± 0.00	176.28	±0.00	176.28
3	4	SVD	$f_4(t)$	2.25330253E-10	2.25330253E-10	30000.00	± 0.00	673.63	±0.00	673.63
3	4	SVD	$f_5(t)$	2.11824449E-07	2.11824449E-07	148.00	± 0.00	654.35	±0.00	654.35
3	4	SVD	$f_6(t)$	5.42031887E-10	5.42031887E-10	30000.00	± 0.00	660.74	±0.00	660.74
3	5	rand	$f_0(t)$	1.05006898E-07	2.54109969E-08	30000.00	± 0.00	651.19	±5.57	19535.60
3	5	rand	$f_1(t)$	1.07518264E-06	3.39970958E-07	12937.47	± 6695.43	636.59	±10.18	19097.55
3	5	rand	$f_2(t)$	2.56930067E-07	7.03983381E-08	29388.03	± 3295.54	649.47	±4.86	19483.96
3	5	rand	$f_3(t)$	4.47888702E-07	1.23327199E-07	9793.57	± 7952.82	613.22	±59.79	18396.49

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	5	rand	$f_4(t)$	7.42457630E-08	2.79246148E-08	29682.33	± 1710.69	662.03	±9.59	19860.86
3	5	rand	$f_5(t)$	9.63909856E-06	6.50211902E-06	250.20	± 80.67	615.12	±4.78	18453.56
3	5	rand	$f_6(t)$	2.55403712E-07	4.23136884E-08	28002.83	± 7472.72	625.54	±108.51	18766.24
3	5	SVD	$f_0(t)$	3.67423283E-08	3.67423283E-08	30000.00	± 0.00	672.77	±0.00	672.77
3	5	SVD	$f_1(t)$	7.84271724E-07	7.84271724E-07	22392.00	± 0.00	662.69	±0.00	662.69
3	5	SVD	$f_2(t)$	1.10619455E-07	1.10619455E-07	30000.00	± 0.00	672.03	±0.00	672.03
3	5	SVD	$f_3(t)$	1.24209096E-07	1.24209096E-07	4539.00	± 0.00	646.88	±0.00	646.88
3	5	SVD	$f_4(t)$	3.53485365E-08	3.53485365E-08	30000.00	± 0.00	676.64	±0.00	676.64
3	5	SVD	$f_5(t)$	1.05914582E-05	1.05914582E-05	332.00	± 0.00	621.09	±0.00	621.09
3	5	SVD	$f_6(t)$	7.97427563E-09	7.97427563E-09	30000.00	± 0.00	665.21	±0.00	665.21
3	6	rand	$f_0(t)$	5.72911687E-07	2.36215678E-07	30000.00	± 0.00	650.96	±5.60	19528.76
3	6	rand	$f_1(t)$	9.86020854E-06	4.37245221E-06	17019.73	± 6389.98	643.14	±9.73	19294.21
3	6	rand	$f_2(t)$	7.26253841E-06	1.15937065E-06	29000.60	± 5381.93	650.18	±98.59	19505.54
3	6	rand	$f_3(t)$	3.80858155E-06	1.38560693E-06	14896.87	± 9091.44	636.81	±11.26	19104.29
3	6	rand	$f_4(t)$	1.62350908E-06	2.73920029E-07	30000.00	± 0.00	662.04	±11.46	19861.34
3	6	rand	$f_5(t)$	7.68573708E-05	4.80314474E-05	447.70	± 155.94	624.36	±9.27	18730.79
3	6	rand	$f_6(t)$	7.91385173E-07	2.67335712E-07	30000.00	± 0.00	658.18	±6.00	19745.36
3	6	SVD	$f_0(t)$	1.51548314E-07	1.51548314E-07	30000.00	± 0.00	664.28	±0.00	664.28

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	6	SVD	$f_1(t)$	6.86148975E-06	6.86148975E-06	14473.00	± 0.00	657.82	±0.00	657.82
3	6	SVD	$f_2(t)$	4.06768307E-07	4.06768307E-07	30000.00	± 0.00	670.83	±0.00	670.83
3	6	SVD	$f_3(t)$	3.80372967E-06	3.80372967E-06	12644.00	± 0.00	649.86	±0.00	649.86
3	6	SVD	$f_4(t)$	1.93342564E-07	1.93342564E-07	30000.00	± 0.00	673.66	±0.00	673.66
3	6	SVD	$f_5(t)$	8.34031961E-05	8.34031961E-05	418.00	± 0.00	622.93	±0.00	622.93
3	6	SVD	$f_6(t)$	1.25126414E-07	1.25126414E-07	30000.00	± 0.00	655.33	±0.00	655.33
3	7	rand	$f_0(t)$	6.56648810E-06	1.09181892E-06	30000.00	± 0.00	650.98	±5.37	19529.46
3	7	rand	$f_1(t)$	6.15577212E-05	2.70637225E-05	14751.30	± 6342.61	637.48	±13.40	19124.46
3	7	rand	$f_2(t)$	1.25756591E-05	4.06340928E-06	30000.00	± 0.00	654.53	±4.65	19636.05
3	7	rand	$f_3(t)$	2.21401004E-05	7.60700963E-06	17692.47	± 9662.57	637.81	±15.36	19134.19
3	7	rand	$f_4(t)$	3.33629146E-06	1.28457720E-06	29356.90	± 3463.20	659.01	±10.63	19770.20
3	7	rand	$f_5(t)$	2.55054323E-04	1.63735652E-04	618.83	± 296.12	632.85	±10.95	18985.57
3	7	rand	$f_6(t)$	4.29917687E-06	1.45651104E-06	30000.00	± 0.00	669.30	±11.49	20079.06
3	7	SVD	$f_0(t)$	2.27684809E-06	2.27684809E-06	30000.00	± 0.00	646.56	±0.00	646.56
3	7	SVD	$f_1(t)$	1.91153611E-05	1.91153611E-05	17440.00	± 0.00	633.50	±0.00	633.50
3	7	SVD	$f_2(t)$	1.69857630E-06	1.69857630E-06	30000.00	± 0.00	669.11	±0.00	669.11
3	7	SVD	$f_3(t)$	3.89919217E-05	3.89919217E-05	24762.00	± 0.00	660.90	±0.00	660.90
3	7	SVD	$f_4(t)$	4.03444700E-07	4.03444700E-07	30000.00	± 0.00	678.84	±0.00	678.84

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	7	SVD	$f_5(t)$	2.97885611E-04	2.97885611E-04	925.00	± 0.00	621.06	±0.00	621.06
3	7	SVD	$f_6(t)$	1.32244132E-06	1.32244132E-06	30000.00	± 0.00	661.12	±0.00	661.12
3	8	rand	$f_0(t)$	1.33404019E-05	4.13158882E-06	28060.60	± 7257.12	659.42	±10.99	19782.56
3	8	rand	$f_1(t)$	1.86714280E-04	1.04040478E-04	17670.97	± 6952.92	644.78	±12.14	19343.53
3	8	rand	$f_2(t)$	5.57808865E-05	1.68988326E-05	29983.43	± 89.21	661.61	±7.02	19848.17
3	8	rand	$f_3(t)$	5.29480349E-05	2.71404547E-05	17607.30	± 9683.37	637.57	±12.93	19127.11
3	8	rand	$f_4(t)$	1.13113504E-05	3.32888106E-06	30000.00	± 0.00	651.59	±5.22	19547.56
3	8	rand	$f_5(t)$	5.22825910E-04	3.70616531E-04	954.93	± 430.75	617.29	±4.53	18518.67
3	8	rand	$f_6(t)$	1.15165174E-05	4.80660043E-06	30000.00	± 0.00	647.44	±3.32	19423.27
3	8	SVD	$f_0(t)$	1.23041973E-06	1.23041973E-06	30000.00	± 0.00	673.48	±0.00	673.48
3	8	SVD	$f_1(t)$	1.38961829E-04	1.38961829E-04	5198.00	± 0.00	634.23	±0.00	634.23
3	8	SVD	$f_2(t)$	1.68148100E-05	1.68148100E-05	30000.00	± 0.00	674.87	±0.00	674.87
3	8	SVD	$f_3(t)$	1.03036968E-04	1.03036968E-04	28268.00	± 0.00	658.22	±0.00	658.22
3	8	SVD	$f_4(t)$	1.22994227E-06	1.22994227E-06	30000.00	± 0.00	658.13	±0.00	658.13
3	8	SVD	$f_5(t)$	8.38524183E-04	8.38524183E-04	1126.00	± 0.00	619.18	±0.00	619.18
3	8	SVD	$f_6(t)$	8.50937839E-07	8.50937839E-07	30000.00	± 0.00	671.59	±0.00	671.59
3	9	rand	$f_0(t)$	2.85090934E-05	7.36532255E-06	30000.00	± 0.00	658.30	±6.38	19749.00
3	9	rand	$f_1(t)$	6.03686660E-04	3.08825610E-04	17445.10	± 6238.14	644.94	±13.99	19348.23

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	9	rand	$f_2(t)$	8.52018870E-05	4.36769965E-05	28678.80	± 4424.41	656.06	±10.03	19681.72
3	9	rand	$f_3(t)$	1.28387394E-04	6.93945323E-05	18885.43	± 9007.32	653.70	±16.64	19610.94
3	9	rand	$f_4(t)$	1.92865340E-05	8.12365993E-06	30000.00	± 0.00	654.83	±3.21	19644.86
3	9	rand	$f_5(t)$	1.06397231E-03	8.04822610E-04	1255.50	± 661.12	625.91	±5.57	18777.31
3	9	rand	$f_6(t)$	2.98917904E-05	1.08040252E-05	30000.00	± 0.00	659.23	±4.80	19776.94
3	9	SVD	$f_0(t)$	7.01494723E-06	7.01494723E-06	30000.00	± 0.00	671.48	±0.00	671.48
3	9	SVD	$f_1(t)$	9.63883324E-04	9.63883324E-04	17905.00	± 0.00	650.19	±0.00	650.19
3	9	SVD	$f_2(t)$	1.74153786E-04	1.74153786E-04	30000.00	± 0.00	659.60	±0.00	659.60
3	9	SVD	$f_3(t)$	7.08393946E-04	7.08393946E-04	17641.00	± 0.00	637.30	±0.00	637.30
3	9	SVD	$f_4(t)$	6.16856013E-06	6.16856013E-06	30000.00	± 0.00	671.93	±0.00	671.93
3	9	SVD	$f_5(t)$	2.12128573E-03	2.12128573E-03	1041.00	± 0.00	630.40	±0.00	630.40
3	9	SVD	$f_6(t)$	1.26210246E-05	1.26210246E-05	30000.00	± 0.00	658.88	±0.00	658.88
3	10	rand	$f_0(t)$	3.71140966E-05	1.72280634E-05	28092.87	± 7136.07	654.17	±10.33	19625.16
3	10	rand	$f_1(t)$	1.17207517E-03	6.03010398E-04	18172.43	± 7603.10	638.99	±10.55	19169.66
3	10	rand	$f_2(t)$	3.31609355E-04	1.49147828E-04	29753.43	± 1327.80	665.10	±11.38	19952.92
3	10	rand	$f_3(t)$	2.78322420E-04	1.54481139E-04	20678.63	± 9066.67	643.45	±13.89	19303.57
3	10	rand	$f_4(t)$	3.17410288E-05	1.64547920E-05	26476.57	± 8879.97	658.82	±19.46	19764.68
3	10	rand	$f_5(t)$	1.85908037E-03	1.38079415E-03	1513.40	± 935.68	625.54	±8.37	18766.07

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	10	rand	$f_6(t)$	4.94873938E-05	2.20431737E-05	29625.30	± 2017.82	651.08	±6.84	19532.31
3	10	SVD	$f_0(t)$	1.14149772E-06	1.14149772E-06	30000.00	± 0.00	667.29	±0.00	667.29
3	10	SVD	$f_1(t)$	8.10682910E-04	8.10682910E-04	19583.00	± 0.00	652.31	±0.00	652.31
3	10	SVD	$f_2(t)$	1.72979780E-04	1.72979780E-04	30000.00	± 0.00	662.85	±0.00	662.85
3	10	SVD	$f_3(t)$	3.00942249E-03	3.00942249E-03	30000.00	± 0.00	670.13	±0.00	670.13
3	10	SVD	$f_4(t)$	1.16712517E-06	1.16712517E-06	30000.00	± 0.00	658.80	±0.00	658.80
3	10	SVD	$f_5(t)$	6.54740290E-03	6.54740290E-03	2618.00	± 0.00	634.07	±0.00	634.07
3	10	SVD	$f_6(t)$	2.99374906E-05	2.99374906E-05	30000.00	± 0.00	662.43	±0.00	662.43
3	11	rand	$f_0(t)$	5.72196144E-05	3.13905615E-05	25269.27	± 10581.96	655.54	±14.59	19666.26
3	11	rand	$f_1(t)$	2.47257832E-03	1.44041698E-03	19893.73	± 8144.43	646.66	±12.69	19399.93
3	11	rand	$f_2(t)$	8.63052664E-04	3.10802106E-04	29225.70	± 4169.73	653.12	±7.59	19593.54
3	11	rand	$f_3(t)$	3.88733690E-04	2.58701244E-04	21627.77	± 9734.32	653.05	±13.45	19591.48
3	11	rand	$f_4(t)$	4.91713966E-05	2.78852532E-05	26114.53	± 9343.77	662.46	±19.54	19873.83
3	11	rand	$f_5(t)$	2.75143989E-03	2.10554176E-03	1701.57	± 1423.98	619.64	±4.74	18589.16
3	11	rand	$f_6(t)$	8.77071718E-05	3.94216205E-05	29078.17	± 4504.63	659.31	±8.86	19779.38
3	11	SVD	$f_0(t)$	2.38277646E-05	2.38277646E-05	30000.00	± 0.00	665.87	±0.00	665.87
3	11	SVD	$f_1(t)$	3.52014807E-03	3.52014807E-03	15638.00	± 0.00	646.64	±0.00	646.64
3	11	SVD	$f_2(t)$	4.02472555E-04	4.02472555E-04	30000.00	± 0.00	677.84	±0.00	677.84

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	11	SVD	$f_3(t)$	5.93238977E-03	5.93238977E-03	8771.00	± 0.00	655.33	±0.00	655.33
3	11	SVD	$f_4(t)$	2.35145796E-05	2.35145796E-05	30000.00	± 0.00	669.63	±0.00	669.63
3	11	SVD	$f_5(t)$	9.62718511E-03	9.62718511E-03	3858.00	± 0.00	638.30	±0.00	638.30
3	11	SVD	$f_6(t)$	4.40246644E-05	4.40246644E-05	30000.00	± 0.00	672.96	±0.00	672.96
3	12	rand	$f_0(t)$	2.08716808E-04	7.22035473E-05	17730.63	± 14039.97	644.02	±19.79	19320.67
3	12	rand	$f_1(t)$	3.73507352E-03	2.28495049E-03	19955.93	± 8075.71	646.98	±10.71	19409.37
3	12	rand	$f_2(t)$	1.37642440E-03	5.48555923E-04	29663.43	± 1812.47	655.00	±6.20	19650.06
3	12	rand	$f_3(t)$	7.07881981E-04	4.36349680E-04	20770.00	± 10493.58	646.53	±13.05	19395.83
3	12	rand	$f_4(t)$	2.92276333E-04	8.23222725E-05	14852.33	± 12679.29	654.81	±20.90	19644.21
3	12	rand	$f_5(t)$	4.18760854E-03	2.99039853E-03	2214.03	± 1070.94	622.89	±3.24	18686.74
3	12	rand	$f_6(t)$	1.96992939E-04	8.69570545E-05	28190.17	± 4917.13	651.16	±7.55	19534.73
3	12	SVD	$f_0(t)$	2.59985804E-05	2.59985804E-05	1838.00	± 0.00	648.34	±0.00	648.34
3	12	SVD	$f_1(t)$	7.36774265E-03	7.36774265E-03	16781.00	± 0.00	648.57	±0.00	648.57
3	12	SVD	$f_2(t)$	1.66780153E-03	1.66780153E-03	30000.00	± 0.00	688.10	±0.00	688.10
3	12	SVD	$f_3(t)$	1.00111807E-02	1.00111807E-02	30000.00	± 0.00	670.37	±0.00	670.37
3	12	SVD	$f_4(t)$	2.64671802E-05	2.64671802E-05	2243.00	± 0.00	631.53	±0.00	631.53
3	12	SVD	$f_5(t)$	1.45252907E-02	1.45252907E-02	1399.00	± 0.00	638.10	±0.00	638.10
3	12	SVD	$f_6(t)$	1.50410960E-04	1.50410960E-04	30000.00	± 0.00	656.56	±0.00	656.56

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	13	rand	$f_0(t)$	6.00123588E-04	1.48883531E-04	8613.40	± 11829.85	622.55	±15.67	18676.44
3	13	rand	$f_1(t)$	6.02920252E-03	3.95401149E-03	19400.53	± 6361.78	648.96	±7.58	19468.91
3	13	rand	$f_2(t)$	2.57791700E-03	1.12169659E-03	30000.00	± 0.00	666.43	±6.36	19992.80
3	13	rand	$f_3(t)$	1.13161479E-03	7.30311334E-04	25277.53	± 6807.42	655.51	±9.96	19665.36
3	13	rand	$f_4(t)$	5.37137022E-04	1.50627852E-04	13553.10	± 12420.30	636.81	±16.49	19104.20
3	13	rand	$f_5(t)$	5.19299536E-03	4.18334543E-03	3849.53	± 5131.05	628.45	±7.74	18853.44
3	13	rand	$f_6(t)$	2.58761502E-04	1.30225869E-04	29828.23	± 656.65	652.64	±4.66	19579.22
3	13	SVD	$f_0(t)$	1.56709907E-04	1.56709907E-04	1271.00	± 0.00	646.22	±0.00	646.22
3	13	SVD	$f_1(t)$	1.29584983E-02	1.29584983E-02	16056.00	± 0.00	665.91	±0.00	665.91
3	13	SVD	$f_2(t)$	3.08172386E-03	3.08172386E-03	30000.00	± 0.00	668.07	±0.00	668.07
3	13	SVD	$f_3(t)$	1.25860054E-02	1.25860054E-02	30000.00	± 0.00	668.83	±0.00	668.83
3	13	SVD	$f_4(t)$	1.57007175E-04	1.57007175E-04	1624.00	± 0.00	649.24	±0.00	649.24
3	13	SVD	$f_5(t)$	2.44664243E-02	2.44664243E-02	4902.00	± 0.00	633.26	±0.00	633.26
3	13	SVD	$f_6(t)$	2.00978120E-04	2.00978120E-04	30000.00	± 0.00	666.51	±0.00	666.51
3	14	rand	$f_0(t)$	1.71980175E-03	4.91881813E-04	3696.63	± 7129.61	633.38	±10.94	19001.43
3	14	rand	$f_1(t)$	9.60891638E-03	5.53369249E-03	16629.43	± 5845.19	640.46	±8.49	19213.81
3	14	rand	$f_2(t)$	3.14816391E-03	1.93535667E-03	29797.67	± 1089.60	668.60	±10.30	20058.11
3	14	rand	$f_3(t)$	1.39328601E-03	9.97006449E-04	24979.00	± 7410.67	653.07	±12.70	19592.24

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	14	rand	$f_4(t)$	1.32345641E-03	3.05055838E-04	9043.30	$\pm$ 9849.51	631.57	$\pm$12.83	18947.17
3	14	rand	$f_5(t)$	7.12637345E-03	5.55681450E-03	2671.03	$\pm$ 2085.99	625.10	$\pm$4.29	18752.88
3	14	rand	$f_6(t)$	5.21287203E-04	2.23568257E-04	29889.17	$\pm$ 596.86	660.21	$\pm$5.54	19806.26
3	14	SVD	$f_0(t)$	9.19219377E-04	9.19219377E-04	1259.00	$\pm$ 0.00	648.57	$\pm$0.00	648.57
3	14	SVD	$f_1(t)$	1.96784639E-02	1.96784639E-02	13730.00	$\pm$ 0.00	654.68	$\pm$0.00	654.68
3	14	SVD	$f_2(t)$	6.62902521E-03	6.62902521E-03	30000.00	$\pm$ 0.00	674.17	$\pm$0.00	674.17
3	14	SVD	$f_3(t)$	2.28538148E-02	2.28538148E-02	23770.00	$\pm$ 0.00	674.27	$\pm$0.00	674.27
3	14	SVD	$f_4(t)$	9.21356543E-04	9.21356543E-04	1473.00	$\pm$ 0.00	645.39	$\pm$0.00	645.39
3	14	SVD	$f_5(t)$	3.40051507E-02	3.40051507E-02	4531.00	$\pm$ 0.00	634.29	$\pm$0.00	634.29
3	14	SVD	$f_6(t)$	2.18812856E-04	2.18812856E-04	30000.00	$\pm$ 0.00	662.56	$\pm$0.00	662.56
3	15	rand	$f_0(t)$	4.13154900E-03	8.13628482E-04	2031.87	$\pm$ 1108.89	624.78	$\pm$6.31	18743.33
3	15	rand	$f_1(t)$	1.29193632E-02	7.55490339E-03	17611.77	$\pm$ 7519.51	646.96	$\pm$9.60	19408.77
3	15	rand	$f_2(t)$	5.13694837E-03	2.95391917E-03	30000.00	$\pm$ 0.00	663.67	$\pm$5.12	19910.18
3	15	rand	$f_3(t)$	1.87737917E-03	1.30862549E-03	24295.07	$\pm$ 8201.99	651.16	$\pm$13.54	19534.95
3	15	rand	$f_4(t)$	2.84107429E-03	8.31076054E-04	5181.00	$\pm$ 6580.02	636.45	$\pm$9.42	19093.58
3	15	rand	$f_5(t)$	9.76020631E-03	7.42462324E-03	2757.53	$\pm$ 1418.47	622.48	$\pm$4.73	18674.37
3	15	rand	$f_6(t)$	6.69765994E-04	3.12473335E-04	29370.23	$\pm$ 3391.40	663.04	$\pm$6.66	19891.15
3	15	SVD	$f_0(t)$	1.06370406E-03	1.06370406E-03	1154.00	$\pm$ 0.00	629.09	$\pm$0.00	629.09

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	15	SVD	$f_1(t)$	3.20055887E-02	3.20055887E-02	9918.00	± 0.00	643.60	±0.00	643.60
3	15	SVD	$f_2(t)$	1.04282236E-02	1.04282236E-02	30000.00	± 0.00	679.70	±0.00	679.70
3	15	SVD	$f_3(t)$	3.13398320E-02	3.13398320E-02	30000.00	± 0.00	676.95	±0.00	676.95
3	15	SVD	$f_4(t)$	1.06415166E-03	1.06415166E-03	1049.00	± 0.00	640.42	±0.00	640.42
3	15	SVD	$f_5(t)$	4.96338348E-02	4.96338348E-02	2257.00	± 0.00	624.82	±0.00	624.82
3	15	SVD	$f_6(t)$	7.98273070E-04	7.98273070E-04	30000.00	± 0.00	685.96	±0.00	685.96
3	16	rand	$f_0(t)$	6.74434352E-03	1.90802990E-03	1994.63	± 957.93	630.54	±12.36	18916.06
3	16	rand	$f_1(t)$	1.53513319E-02	1.07489731E-02	18906.67	± 7903.66	644.55	±12.19	19336.55
3	16	rand	$f_2(t)$	6.41721137E-03	4.43264793E-03	29455.07	± 2934.56	670.38	±10.93	20111.41
3	16	rand	$f_3(t)$	2.27455846E-03	1.76889109E-03	22378.53	± 9159.17	646.95	±12.64	19408.38
3	16	rand	$f_4(t)$	7.51858746E-03	1.68869871E-03	3797.47	± 2425.19	627.84	±8.17	18835.26
3	16	rand	$f_5(t)$	1.12122100E-02	8.78716352E-03	4763.63	± 5140.21	633.88	±12.32	19016.43
3	16	rand	$f_6(t)$	9.18912974E-04	5.36097669E-04	29406.33	± 3196.99	663.61	±12.85	19908.44
3	16	SVD	$f_0(t)$	1.33390021E-03	1.33390021E-03	1013.00	± 0.00	639.25	±0.00	639.25
3	16	SVD	$f_1(t)$	3.08072964E-02	3.08072964E-02	20857.00	± 0.00	666.73	±0.00	666.73
3	16	SVD	$f_2(t)$	1.20421858E-02	1.20421858E-02	20372.00	± 0.00	656.22	±0.00	656.22
3	16	SVD	$f_3(t)$	5.01628120E-02	5.01628120E-02	30000.00	± 0.00	668.55	±0.00	668.55
3	16	SVD	$f_4(t)$	1.33394704E-03	1.33394704E-03	1017.00	± 0.00	628.09	±0.00	628.09

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	16	SVD	$f_5(t)$	7.05453756E-02	7.05453756E-02	2976.00	± 0.00	629.05	±0.00	629.05
3	16	SVD	$f_6(t)$	3.06606301E-03	3.06606301E-03	30000.00	± 0.00	664.65	±0.00	664.65
3	17	rand	$f_0(t)$	8.87252486E-03	3.35837699E-03	1801.63	± 713.10	639.31	±4.56	19179.20
3	17	rand	$f_1(t)$	2.24010073E-02	1.60279154E-02	18244.27	± 8204.79	639.11	±10.56	19173.44
3	17	rand	$f_2(t)$	9.19621364E-03	5.85328211E-03	29813.80	± 1002.72	663.63	±4.60	19908.85
3	17	rand	$f_3(t)$	3.50228249E-03	2.35397991E-03	24516.23	± 8198.94	653.68	±12.37	19610.49
3	17	rand	$f_4(t)$	5.68511021E-03	2.65988729E-03	3473.13	± 2091.68	626.62	±4.55	18798.49
3	17	rand	$f_5(t)$	1.22637543E-02	1.08148321E-02	4790.47	± 5306.65	630.59	±8.14	18917.57
3	17	rand	$f_6(t)$	1.76622176E-03	9.13857786E-04	27992.70	± 5407.91	653.98	±7.98	19619.43
3	17	SVD	$f_0(t)$	3.84282193E-03	3.84282193E-03	1970.00	± 0.00	636.19	±0.00	636.19
3	17	SVD	$f_1(t)$	4.37233111E-02	4.37233111E-02	14856.00	± 0.00	657.05	±0.00	657.05
3	17	SVD	$f_2(t)$	1.73785842E-02	1.73785842E-02	30000.00	± 0.00	679.02	±0.00	679.02
3	17	SVD	$f_3(t)$	5.49290323E-02	5.49290323E-02	30000.00	± 0.00	675.11	±0.00	675.11
3	17	SVD	$f_4(t)$	3.86055104E-03	3.86055104E-03	3488.00	± 0.00	641.67	±0.00	641.67
3	17	SVD	$f_5(t)$	9.32585375E-02	9.32585375E-02	2057.00	± 0.00	626.92	±0.00	626.92
3	17	SVD	$f_6(t)$	5.97078199E-03	5.97078199E-03	30000.00	± 0.00	690.62	±0.00	690.62
3	18	rand	$f_0(t)$	1.35630286E-02	6.20205408E-03	1705.10	± 866.88	632.70	±2.73	18981.04
3	18	rand	$f_1(t)$	2.30903053E-02	1.91124378E-02	19370.27	± 7638.65	654.46	±12.57	19633.70

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	18	rand	$f_2(t)$	1.73875382E-02	9.45262849E-03	30000.00	± 0.00	662.20	±5.72	19866.04
3	18	rand	$f_3(t)$	3.80580304E-03	2.83864561E-03	26253.60	± 7826.57	655.26	±10.41	19657.88
3	18	rand	$f_4(t)$	1.32420402E-02	5.57280830E-03	3083.20	± 2055.65	621.48	±3.94	18644.31
3	18	rand	$f_5(t)$	1.64642507E-02	1.36529422E-02	6865.10	± 9050.84	630.07	±12.06	18902.23
3	18	rand	$f_6(t)$	2.36016565E-03	1.11656219E-03	28491.33	± 5175.97	664.63	±8.74	19938.95
3	18	SVD	$f_0(t)$	5.33157299E-03	5.33157299E-03	2545.00	± 0.00	634.86	±0.00	634.86
3	18	SVD	$f_1(t)$	6.78945532E-02	6.78945532E-02	11260.00	± 0.00	651.05	±0.00	651.05
3	18	SVD	$f_2(t)$	2.80690280E-02	2.80690280E-02	30000.00	± 0.00	671.19	±0.00	671.19
3	18	SVD	$f_3(t)$	7.99633612E-02	7.99633612E-02	30000.00	± 0.00	675.60	±0.00	675.60
3	18	SVD	$f_4(t)$	5.48744866E-03	5.48744866E-03	5428.00	± 0.00	639.32	±0.00	639.32
3	18	SVD	$f_5(t)$	1.28454442E-01	1.28454442E-01	3779.00	± 0.00	637.40	±0.00	637.40
3	18	SVD	$f_6(t)$	3.46159065E-03	3.46159065E-03	30000.00	± 0.00	676.50	±0.00	676.50
3	19	rand	$f_0(t)$	2.15281670E-02	8.24861956E-03	1825.37	± 846.62	623.79	±5.72	18713.69
3	19	rand	$f_1(t)$	3.32153505E-02	2.43375806E-02	17491.63	± 7608.00	647.81	±10.96	19434.22
3	19	rand	$f_2(t)$	1.57582159E-02	1.13070551E-02	29263.03	± 3968.69	665.57	±7.30	19966.99
3	19	rand	$f_3(t)$	4.69056096E-03	3.63441913E-03	23537.20	± 9050.32	659.27	±14.82	19778.03
3	19	rand	$f_4(t)$	3.23419146E-02	9.53462845E-03	2938.63	± 3430.85	626.67	±5.28	18800.15
3	19	rand	$f_5(t)$	2.11016471E-02	1.58396969E-02	5255.23	± 6849.12	626.42	±10.31	18792.66

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	19	rand	$f_6(t)$	3.24569337E-03	1.55522413E-03	29966.80	± 178.79	664.83	±7.77	19944.79
3	19	SVD	$f_0(t)$	6.96368084E-03	6.96368084E-03	2024.00	± 0.00	634.49	±0.00	634.49
3	19	SVD	$f_1(t)$	1.03271063E-01	1.03271063E-01	10088.00	± 0.00	645.61	±0.00	645.61
3	19	SVD	$f_2(t)$	4.71386104E-02	4.71386104E-02	30000.00	± 0.00	659.65	±0.00	659.65
3	19	SVD	$f_3(t)$	1.01805889E-01	1.01805889E-01	30000.00	± 0.00	687.80	±0.00	687.80
3	19	SVD	$f_4(t)$	7.15039724E-03	7.15039724E-03	4217.00	± 0.00	642.49	±0.00	642.49
3	19	SVD	$f_5(t)$	1.43749877E-01	1.43749877E-01	3705.00	± 0.00	641.11	±0.00	641.11
3	19	SVD	$f_6(t)$	1.85593346E-02	1.85593346E-02	30000.00	± 0.00	687.23	±0.00	687.23
3	20	rand	$f_0(t)$	4.64708690E-02	1.50834322E-02	1669.87	± 670.77	623.77	±4.23	18713.22
3	20	rand	$f_1(t)$	3.91973327E-02	2.79663974E-02	18161.10	± 6429.49	647.78	±9.43	19433.25
3	20	rand	$f_2(t)$	2.34232960E-02	1.45331864E-02	29809.10	± 1028.03	661.03	±4.85	19830.92
3	20	rand	$f_3(t)$	6.69463659E-03	4.57522624E-03	25557.00	± 7544.98	665.58	±12.56	19967.29
3	20	rand	$f_4(t)$	3.30369874E-02	1.33883637E-02	2887.83	± 1938.83	630.77	±5.29	18923.18
3	20	rand	$f_5(t)$	2.36144575E-02	1.82104966E-02	4389.30	± 2599.86	636.77	±9.07	19103.11
3	20	rand	$f_6(t)$	2.97273646E-03	1.98789877E-03	28289.63	± 5168.11	658.77	±9.03	19762.98
3	20	SVD	$f_0(t)$	1.75188621E-02	1.75188621E-02	2213.00	± 0.00	624.31	±0.00	624.31
3	20	SVD	$f_1(t)$	1.28593016E-01	1.28593016E-01	10824.00	± 0.00	639.48	±0.00	639.48
3	20	SVD	$f_2(t)$	6.51922457E-02	6.51922457E-02	30000.00	± 0.00	672.70	±0.00	672.70

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	20	SVD	$f_3(t)$	1.34365042E-01	1.34365042E-01	30000.00	± 0.00	671.74	±0.00	671.74
3	20	SVD	$f_4(t)$	1.75392034E-02	1.75392034E-02	4970.00	± 0.00	633.32	±0.00	633.32
3	20	SVD	$f_5(t)$	1.82906393E-01	1.82906393E-01	8249.00	± 0.00	642.43	±0.00	642.43
3	20	SVD	$f_6(t)$	3.37809563E-02	3.37809563E-02	17248.00	± 0.00	655.43	±0.00	655.43
3	21	rand	$f_0(t)$	4.66867426E-02	2.17620792E-02	1598.93	± 726.63	624.33	±2.42	18729.92
3	21	rand	$f_1(t)$	4.18965626E-02	3.22479168E-02	18710.20	± 7502.89	662.39	±14.53	19871.65
3	21	rand	$f_2(t)$	2.73797202E-02	1.89637010E-02	29647.37	± 1898.99	663.45	±7.22	19903.60
3	21	rand	$f_3(t)$	6.77076743E-03	5.28652307E-03	23530.57	± 8985.08	657.11	±11.49	19713.22
3	21	rand	$f_4(t)$	3.80555285E-02	1.85644129E-02	2487.53	± 1974.52	631.93	±7.84	18957.95
3	21	rand	$f_5(t)$	2.64107949E-02	2.15290238E-02	6163.13	± 6799.30	650.12	±11.95	19503.59
3	21	rand	$f_6(t)$	3.62362661E-03	2.46431883E-03	27089.57	± 6739.82	663.29	±9.58	19898.67
3	21	SVD	$f_0(t)$	1.64111027E-02	1.64111027E-02	2049.00	± 0.00	640.04	±0.00	640.04
3	21	SVD	$f_1(t)$	1.56318729E-01	1.56318729E-01	6051.00	± 0.00	638.60	±0.00	638.60
3	21	SVD	$f_2(t)$	7.50693135E-02	7.50693135E-02	30000.00	± 0.00	668.93	±0.00	668.93
3	21	SVD	$f_3(t)$	1.68797592E-01	1.68797592E-01	30000.00	± 0.00	675.09	±0.00	675.09
3	21	SVD	$f_4(t)$	1.68814195E-02	1.68814195E-02	5232.00	± 0.00	636.60	±0.00	636.60
3	21	SVD	$f_5(t)$	2.12745113E-01	2.12745113E-01	7198.00	± 0.00	636.96	±0.00	636.96
3	21	SVD	$f_6(t)$	1.80423445E-02	1.80423445E-02	30000.00	± 0.00	660.57	±0.00	660.57

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	22	rand	$f_0(t)$	5.43037954E-02	2.75500250E-02	1752.07	± 931.49	638.63	±5.86	19158.91
3	22	rand	$f_1(t)$	4.98956910E-02	4.07991688E-02	19207.43	± 7661.18	651.15	±12.00	19534.51
3	22	rand	$f_2(t)$	3.05785145E-02	2.32976394E-02	28799.10	± 4720.39	678.92	±7.73	20367.58
3	22	rand	$f_3(t)$	7.80847954E-03	6.41248756E-03	26757.47	± 6918.01	679.28	±11.74	20378.50
3	22	rand	$f_4(t)$	5.89937998E-02	2.93239613E-02	2786.57	± 2000.58	640.60	±6.37	19217.98
3	22	rand	$f_5(t)$	3.21207259E-02	2.54031628E-02	6612.33	± 6569.21	636.86	±11.73	19105.67
3	22	rand	$f_6(t)$	5.26083909E-03	3.22401842E-03	27497.63	± 6546.84	656.03	±11.71	19680.92
3	22	SVD	$f_0(t)$	2.91151743E-02	2.91151743E-02	2183.00	± 0.00	633.21	±0.00	633.21
3	22	SVD	$f_1(t)$	1.51412841E-01	1.51412841E-01	15513.00	± 0.00	656.91	±0.00	656.91
3	22	SVD	$f_2(t)$	8.14712262E-02	8.14712262E-02	30000.00	± 0.00	678.83	±0.00	678.83
3	22	SVD	$f_3(t)$	1.84974612E-01	1.84974612E-01	30000.00	± 0.00	679.70	±0.00	679.70
3	22	SVD	$f_4(t)$	2.91568079E-02	2.91568079E-02	5303.00	± 0.00	665.44	±0.00	665.44
3	22	SVD	$f_5(t)$	2.49601614E-01	2.49601614E-01	4207.00	± 0.00	644.57	±0.00	644.57
3	22	SVD	$f_6(t)$	6.58319323E-02	6.58319323E-02	30000.00	± 0.00	668.40	±0.00	668.40
3	23	rand	$f_0(t)$	7.14327294E-02	3.88428846E-02	1673.63	± 961.27	635.37	±13.82	19061.00
3	23	rand	$f_1(t)$	5.72221463E-02	4.67649818E-02	17410.00	± 7715.77	659.70	±15.92	19791.15
3	23	rand	$f_2(t)$	3.89199291E-02	2.94510876E-02	29141.90	± 2688.69	663.82	±6.87	19914.46
3	23	rand	$f_3(t)$	9.58684189E-03	7.75162207E-03	29353.40	± 2279.94	669.44	±4.67	20083.33

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	23	rand	$f_4(t)$	8.78810218E-02	4.12056542E-02	2101.70	± 2056.93	636.07	±12.97	19081.95
3	23	rand	$f_5(t)$	3.15509688E-02	2.74539687E-02	4406.33	± 1938.68	627.35	±4.80	18820.42
3	23	rand	$f_6(t)$	6.37416488E-03	4.30573861E-03	28306.57	± 4988.17	663.15	±8.83	19894.36
3	23	SVD	$f_0(t)$	6.35957510E-02	6.35957510E-02	2521.00	± 0.00	634.35	±0.00	634.35
3	23	SVD	$f_1(t)$	1.92148629E-01	1.92148629E-01	19796.00	± 0.00	673.27	±0.00	673.27
3	23	SVD	$f_2(t)$	1.07241738E-01	1.07241738E-01	30000.00	± 0.00	677.91	±0.00	677.91
3	23	SVD	$f_3(t)$	2.10675414E-01	2.10675414E-01	30000.00	± 0.00	685.35	±0.00	685.35
3	23	SVD	$f_4(t)$	6.37490440E-02	6.37490440E-02	5126.00	± 0.00	638.60	±0.00	638.60
3	23	SVD	$f_5(t)$	2.92536334E-01	2.92536334E-01	3741.00	± 0.00	628.74	±0.00	628.74
3	23	SVD	$f_6(t)$	6.73374131E-02	6.73374131E-02	30000.00	± 0.00	679.48	±0.00	679.48
3	24	rand	$f_0(t)$	9.82822078E-02	5.11237614E-02	1391.67	± 666.15	624.74	±3.41	18742.27
3	24	rand	$f_1(t)$	6.66880946E-02	5.45109185E-02	18467.37	± 6725.19	647.04	±10.95	19411.06
3	24	rand	$f_2(t)$	5.14538276E-02	3.58234248E-02	29506.10	± 2137.03	670.84	±7.71	20125.12
3	24	rand	$f_3(t)$	1.05943034E-02	8.86218362E-03	27177.57	± 5556.63	665.00	±10.08	19950.12
3	24	rand	$f_4(t)$	8.96242063E-02	4.85860119E-02	2743.13	± 1983.39	638.84	±11.52	19165.12
3	24	rand	$f_5(t)$	3.61359977E-02	3.02148606E-02	6526.90	± 6717.56	637.43	±11.56	19122.75
3	24	rand	$f_6(t)$	7.59005915E-03	5.07251239E-03	28995.17	± 2775.11	664.27	±7.69	19927.97
3	24	SVD	$f_0(t)$	8.93567615E-02	8.93567615E-02	1831.00	± 0.00	636.53	±0.00	636.53

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	24	SVD	$f_1(t)$	2.45415570E-01	2.45415570E-01	19221.00	$\pm$ 0.00	657.23	$\pm$0.00	657.23
3	24	SVD	$f_2(t)$	1.54989572E-01	1.54989572E-01	30000.00	$\pm$ 0.00	677.98	$\pm$0.00	677.98
3	24	SVD	$f_3(t)$	2.44977054E-01	2.44977054E-01	30000.00	$\pm$ 0.00	676.27	$\pm$0.00	676.27
3	24	SVD	$f_4(t)$	8.96151220E-02	8.96151220E-02	3893.00	$\pm$ 0.00	631.74	$\pm$0.00	631.74
3	24	SVD	$f_5(t)$	3.22375738E-01	3.22375738E-01	9567.00	$\pm$ 0.00	632.12	$\pm$0.00	632.12
3	24	SVD	$f_6(t)$	6.48224048E-02	6.48224048E-02	30000.00	$\pm$ 0.00	675.15	$\pm$0.00	675.15
3	25	rand	$f_0(t)$	1.03064084E-01	6.40749087E-02	1406.10	$\pm$ 701.35	629.49	$\pm$4.74	18884.70
3	25	rand	$f_1(t)$	7.44853912E-02	5.91655314E-02	15748.43	$\pm$ 5647.97	647.19	$\pm$7.51	19415.81
3	25	rand	$f_2(t)$	6.00546925E-02	3.97029415E-02	29854.23	$\pm$ 784.98	674.60	$\pm$9.39	20237.99
3	25	rand	$f_3(t)$	1.18770939E-02	1.00307963E-02	26669.93	$\pm$ 6142.52	681.34	$\pm$10.51	20440.31
3	25	rand	$f_4(t)$	8.71132189E-02	5.30373211E-02	2899.67	$\pm$ 2111.19	639.83	$\pm$11.49	19194.78
3	25	rand	$f_5(t)$	4.04753292E-02	3.35727962E-02	5328.30	$\pm$ 5239.84	632.47	$\pm$7.67	18974.03
3	25	rand	$f_6(t)$	1.06950488E-02	5.74140199E-03	26807.07	$\pm$ 7494.78	670.63	$\pm$9.73	20118.98
3	25	SVD	$f_0(t)$	1.03405026E-01	1.03405026E-01	1304.00	$\pm$ 0.00	640.64	$\pm$0.00	640.64
3	25	SVD	$f_1(t)$	2.64199343E-01	2.64199343E-01	14784.00	$\pm$ 0.00	657.24	$\pm$0.00	657.24
3	25	SVD	$f_2(t)$	1.90384447E-01	1.90384447E-01	30000.00	$\pm$ 0.00	675.46	$\pm$0.00	675.46
3	25	SVD	$f_3(t)$	2.94050070E-01	2.94050070E-01	30000.00	$\pm$ 0.00	681.46	$\pm$0.00	681.46
3	25	SVD	$f_4(t)$	1.03613236E-01	1.03613236E-01	2393.00	$\pm$ 0.00	642.89	$\pm$0.00	642.89

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	25	SVD	$f_5(t)$	3.82387016E-01	3.82387016E-01	6147.00	± 0.00	645.77	±0.00	645.77
3	25	SVD	$f_6(t)$	1.16721615E-01	1.16721615E-01	30000.00	± 0.00	672.21	±0.00	672.21
3	26	rand	$f_0(t)$	1.09880648E-01	7.83852471E-02	1291.60	± 583.72	631.51	±8.30	18945.20
3	26	rand	$f_1(t)$	8.03433653E-02	6.76839350E-02	16885.00	± 6991.65	647.63	±9.71	19429.04
3	26	rand	$f_2(t)$	6.43162234E-02	4.99056571E-02	29333.23	± 3212.02	664.65	±5.36	19939.54
3	26	rand	$f_3(t)$	1.47034116E-02	1.17002955E-02	28148.30	± 4578.06	663.82	±10.87	19914.48
3	26	rand	$f_4(t)$	1.26492251E-01	7.87774946E-02	2072.00	± 1877.65	636.04	±6.66	19081.21
3	26	rand	$f_5(t)$	4.87828365E-02	3.78097732E-02	8777.97	± 8785.51	642.58	±12.88	19277.49
3	26	rand	$f_6(t)$	1.32399332E-02	7.16064728E-03	28514.47	± 3823.24	666.44	±8.18	19993.33
3	26	SVD	$f_0(t)$	2.19660490E-01	2.19660490E-01	618.00	± 0.00	648.35	±0.00	648.35
3	26	SVD	$f_1(t)$	3.38024008E-01	3.38024008E-01	30000.00	± 0.00	687.10	±0.00	687.10
3	26	SVD	$f_2(t)$	2.50261844E-01	2.50261844E-01	30000.00	± 0.00	670.91	±0.00	670.91
3	26	SVD	$f_3(t)$	3.31930114E-01	3.31930114E-01	30000.00	± 0.00	676.09	±0.00	676.09
3	26	SVD	$f_4(t)$	2.19660490E-01	2.19660490E-01	618.00	± 0.00	633.37	±0.00	633.37
3	26	SVD	$f_5(t)$	4.26805527E-01	4.26805527E-01	7078.00	± 0.00	644.51	±0.00	644.51
3	26	SVD	$f_6(t)$	1.32758038E-01	1.32758038E-01	30000.00	± 0.00	666.36	±0.00	666.36
3	27	rand	$f_0(t)$	1.50760271E-01	8.82320114E-02	1187.13	± 458.60	628.12	±5.34	18843.52
3	27	rand	$f_1(t)$	9.47880343E-02	7.81251769E-02	16523.67	± 7312.69	656.24	±10.69	19687.21

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	27	rand	$f_2(t)$	7.03107613E-02	5.50729315E-02	29182.53	$\pm$ 2826.61	667.46	$\pm$5.83	20023.72
3	27	rand	$f_3(t)$	1.66441388E-02	1.36469735E-02	28461.13	$\pm$ 3606.66	673.36	$\pm$9.11	20200.88
3	27	rand	$f_4(t)$	1.72482219E-01	9.51987888E-02	1895.93	$\pm$ 1407.04	638.52	$\pm$8.28	19155.56
3	27	rand	$f_5(t)$	4.96141676E-02	4.19277853E-02	7594.03	$\pm$ 8154.29	639.47	$\pm$12.09	19183.97
3	27	rand	$f_6(t)$	1.44864255E-02	9.07333417E-03	26599.40	$\pm$ 7760.31	664.94	$\pm$12.26	19948.12
3	27	SVD	$f_0(t)$	2.25509707E-01	2.25509707E-01	1201.00	$\pm$ 0.00	643.40	$\pm$0.00	643.40
3	27	SVD	$f_1(t)$	3.86003391E-01	3.86003391E-01	9356.00	$\pm$ 0.00	646.80	$\pm$0.00	646.80
3	27	SVD	$f_2(t)$	2.72177103E-01	2.72177103E-01	30000.00	$\pm$ 0.00	669.26	$\pm$0.00	669.26
3	27	SVD	$f_3(t)$	4.20028794E-01	4.20028794E-01	18924.00	$\pm$ 0.00	657.58	$\pm$0.00	657.58
3	27	SVD	$f_4(t)$	2.25548877E-01	2.25548877E-01	1104.00	$\pm$ 0.00	640.39	$\pm$0.00	640.39
3	27	SVD	$f_5(t)$	4.80237162E-01	4.80237162E-01	2129.00	$\pm$ 0.00	639.77	$\pm$0.00	639.77
3	27	SVD	$f_6(t)$	1.58749439E-01	1.58749439E-01	5506.00	$\pm$ 0.00	642.76	$\pm$0.00	642.76
3	28	rand	$f_0(t)$	1.77170628E-01	1.06930945E-01	1135.47	$\pm$ 463.78	633.03	$\pm$12.33	18990.88
3	28	rand	$f_1(t)$	9.86473482E-02	8.36074689E-02	18763.33	$\pm$ 7563.59	657.85	$\pm$13.37	19735.50
3	28	rand	$f_2(t)$	6.85838245E-02	5.91540765E-02	27978.17	$\pm$ 6161.30	669.49	$\pm$11.95	20084.79
3	28	rand	$f_3(t)$	1.98890218E-02	1.51115010E-02	27506.83	$\pm$ 5928.97	670.87	$\pm$13.31	20126.02
3	28	rand	$f_4(t)$	1.44669231E-01	1.00339227E-01	2359.57	$\pm$ 1852.36	628.37	$\pm$4.74	18851.19
3	28	rand	$f_5(t)$	5.53432514E-02	4.62761481E-02	6943.50	$\pm$ 7889.49	640.98	$\pm$13.38	19229.53

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	28	rand	$f_6(t)$	1.50787647E-02	1.02800811E-02	28647.27	± 4776.47	673.67	±14.43	20209.99
3	28	SVD	$f_0(t)$	1.70637793E-01	1.70637793E-01	1756.00	± 0.00	647.24	±0.00	647.24
3	28	SVD	$f_1(t)$	4.32329163E-01	4.32329163E-01	19917.00	± 0.00	676.55	±0.00	676.55
3	28	SVD	$f_2(t)$	3.23911652E-01	3.23911652E-01	30000.00	± 0.00	681.70	±0.00	681.70
3	28	SVD	$f_3(t)$	4.35850703E-01	4.35850703E-01	30000.00	± 0.00	683.69	±0.00	683.69
3	28	SVD	$f_4(t)$	1.70698945E-01	1.70698945E-01	4639.00	± 0.00	638.62	±0.00	638.62
3	28	SVD	$f_5(t)$	5.16860238E-01	5.16860238E-01	2176.00	± 0.00	632.35	±0.00	632.35
3	28	SVD	$f_6(t)$	2.08800203E-01	2.08800203E-01	18684.00	± 0.00	658.44	±0.00	658.44
3	29	rand	$f_0(t)$	1.68701911E-01	1.19331332E-01	1285.57	± 443.89	629.96	±5.61	18898.69
3	29	rand	$f_1(t)$	1.08119692E-01	9.34337705E-02	14037.27	± 6530.59	649.33	±10.96	19479.97
3	29	rand	$f_2(t)$	9.78914199E-02	6.89473033E-02	28105.93	± 5781.61	665.32	±9.84	19959.57
3	29	rand	$f_3(t)$	2.01000722E-02	1.60433422E-02	27930.33	± 5822.35	662.62	±9.23	19878.57
3	29	rand	$f_4(t)$	1.75274935E-01	1.16729052E-01	2437.97	± 1914.65	632.28	±9.44	18968.33
3	29	rand	$f_5(t)$	6.05045299E-02	5.18149556E-02	7780.00	± 7906.66	632.56	±11.59	18976.86
3	29	rand	$f_6(t)$	1.94892321E-02	1.18275412E-02	28814.07	± 4556.96	662.62	±10.89	19878.61
3	29	SVD	$f_0(t)$	3.73042779E-01	3.73042779E-01	1045.00	± 0.00	645.20	±0.00	645.20
3	29	SVD	$f_1(t)$	4.70615317E-01	4.70615317E-01	11765.00	± 0.00	657.82	±0.00	657.82
3	29	SVD	$f_2(t)$	3.73084081E-01	3.73084081E-01	30000.00	± 0.00	697.15	±0.00	697.15

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
3	29	SVD	$f_3(t)$	4.92713723E-01	4.92713723E-01	30000.00	$\pm$ 0.00	681.26	$\pm$0.00	681.26
3	29	SVD	$f_4(t)$	3.73058908E-01	3.73058908E-01	1025.00	$\pm$ 0.00	642.48	$\pm$0.00	642.48
3	29	SVD	$f_5(t)$	5.59330694E-01	5.59330694E-01	5341.00	$\pm$ 0.00	640.13	$\pm$0.00	640.13
3	29	SVD	$f_6(t)$	2.80044259E-01	2.80044259E-01	6928.00	$\pm$ 0.00	663.88	$\pm$0.00	663.88
3	30	rand	$f_0(t)$	1.86603916E-01	1.30436656E-01	1182.37	$\pm$ 383.11	629.12	$\pm$10.30	18873.56
3	30	rand	$f_1(t)$	1.19453891E-01	9.96095521E-02	16689.13	$\pm$ 6850.73	661.00	$\pm$13.21	19830.05
3	30	rand	$f_2(t)$	1.01635251E-01	7.54044977E-02	28178.50	$\pm$ 4894.38	658.02	$\pm$7.95	19740.52
3	30	rand	$f_3(t)$	2.41443169E-02	1.85018381E-02	27421.43	$\pm$ 6281.45	664.19	$\pm$9.42	19925.69
3	30	rand	$f_4(t)$	1.98743462E-01	1.35851258E-01	1783.17	$\pm$ 1383.15	632.58	$\pm$5.34	18977.43
3	30	rand	$f_5(t)$	6.09938059E-02	5.59539930E-02	8536.93	$\pm$ 7713.88	640.84	$\pm$11.23	19225.08
3	30	rand	$f_6(t)$	1.86219982E-02	1.36967035E-02	28794.13	$\pm$ 4300.32	668.88	$\pm$8.38	20066.36
3	30	SVD	$f_0(t)$	3.41975021E-01	3.41975021E-01	620.00	$\pm$ 0.00	644.18	$\pm$0.00	644.18
3	30	SVD	$f_1(t)$	5.36706673E-01	5.36706673E-01	11886.00	$\pm$ 0.00	665.21	$\pm$0.00	665.21
3	30	SVD	$f_2(t)$	4.40978977E-01	4.40978977E-01	30000.00	$\pm$ 0.00	683.82	$\pm$0.00	683.82
3	30	SVD	$f_3(t)$	5.36112701E-01	5.36112701E-01	30000.00	$\pm$ 0.00	675.46	$\pm$0.00	675.46
3	30	SVD	$f_4(t)$	3.41975021E-01	3.41975021E-01	620.00	$\pm$ 0.00	637.17	$\pm$0.00	637.17
3	30	SVD	$f_5(t)$	6.03398089E-01	6.03398089E-01	5963.00	$\pm$ 0.00	642.81	$\pm$0.00	642.81
3	30	SVD	$f_6(t)$	3.93999851E-01	3.93999851E-01	17526.00	$\pm$ 0.00	679.16	$\pm$0.00	679.16

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	4	SVD	$f_0(t)$	2.07985420E-17	2.07985420E-17	30000.00	± 0.00	4538.17	±0.00	4538.17
4	4	SVD	$f_1(t)$	1.86072032E-17	1.86072032E-17	5525.00	± 0.00	4459.64	±0.00	4459.64
4	4	SVD	$f_2(t)$	3.31018120E-17	3.31018120E-17	30000.00	± 0.00	4555.93	±0.00	4555.93
4	4	SVD	$f_3(t)$	4.15293750E-20	4.15293750E-20	2453.00	± 0.00	4507.25	±0.00	4507.25
4	4	SVD	$f_4(t)$	8.54455662E-19	8.54455662E-19	30000.00	± 0.00	4671.94	±0.00	4671.94
4	4	SVD	$f_5(t)$	3.27863011E-18	3.27863011E-18	709.00	± 0.00	4556.38	±0.00	4556.38
4	4	SVD	$f_6(t)$	3.25254373E-17	3.25254373E-17	30000.00	± 0.00	4610.65	±0.00	4610.65
4	5	SVD	$f_0(t)$	7.03629703E-13	7.03629703E-13	30000.00	± 0.00	4611.03	±0.00	4611.03
4	5	SVD	$f_1(t)$	2.21257507E-11	2.21257507E-11	11352.00	± 0.00	4524.61	±0.00	4524.61
4	5	SVD	$f_2(t)$	6.48307045E-12	6.48307045E-12	30000.00	± 0.00	4496.05	±0.00	4496.05
4	5	SVD	$f_3(t)$	3.80097082E-11	3.80097082E-11	186.00	± 0.00	4549.83	±0.00	4549.83
4	5	SVD	$f_4(t)$	1.74674065E-12	1.74674065E-12	30000.00	± 0.00	4544.37	±0.00	4544.37
4	5	SVD	$f_5(t)$	2.16393001E-09	2.16393001E-09	124.00	± 0.00	4555.74	±0.00	4555.74
4	5	SVD	$f_6(t)$	5.22805457E-12	5.22805457E-12	30000.00	± 0.00	4573.21	±0.00	4573.21
4	6	SVD	$f_0(t)$	4.85262264E-10	4.85262264E-10	30000.00	± 0.00	4550.69	±0.00	4550.69
4	6	SVD	$f_1(t)$	3.46293751E-08	3.46293751E-08	9565.00	± 0.00	4512.04	±0.00	4512.04
4	6	SVD	$f_2(t)$	5.47769673E-10	5.47769673E-10	30000.00	± 0.00	4562.08	±0.00	4562.08
4	6	SVD	$f_3(t)$	1.95394146E-09	1.95394146E-09	5874.00	± 0.00	4515.09	±0.00	4515.09

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	6	SVD	$f_4(t)$	5.37534320E-10	5.37534320E-10	30000.00	$\pm$ 0.00	4506.49	$\pm$0.00	4506.49
4	6	SVD	$f_5(t)$	4.65060447E-07	4.65060447E-07	164.00	$\pm$ 0.00	4540.85	$\pm$0.00	4540.85
4	6	SVD	$f_6(t)$	3.57589985E-10	3.57589985E-10	30000.00	$\pm$ 0.00	4546.10	$\pm$0.00	4546.10
4	7	SVD	$f_0(t)$	9.25483361E-09	9.25483361E-09	30000.00	$\pm$ 0.00	4493.36	$\pm$0.00	4493.36
4	7	SVD	$f_1(t)$	5.22950524E-07	5.22950524E-07	10240.00	$\pm$ 0.00	4522.50	$\pm$0.00	4522.50
4	7	SVD	$f_2(t)$	9.17150419E-09	9.17150419E-09	30000.00	$\pm$ 0.00	4613.62	$\pm$0.00	4613.62
4	7	SVD	$f_3(t)$	3.43075926E-07	3.43075926E-07	4903.00	$\pm$ 0.00	4569.56	$\pm$0.00	4569.56
4	7	SVD	$f_4(t)$	1.81258911E-08	1.81258911E-08	30000.00	$\pm$ 0.00	4552.57	$\pm$0.00	4552.57
4	7	SVD	$f_5(t)$	1.24451233E-05	1.24451233E-05	159.00	$\pm$ 0.00	4499.76	$\pm$0.00	4499.76
4	7	SVD	$f_6(t)$	6.10939197E-09	6.10939197E-09	30000.00	$\pm$ 0.00	4610.81	$\pm$0.00	4610.81
4	8	SVD	$f_0(t)$	3.15189207E-08	3.15189207E-08	30000.00	$\pm$ 0.00	4611.76	$\pm$0.00	4611.76
4	8	SVD	$f_1(t)$	5.84936051E-06	5.84936051E-06	9697.00	$\pm$ 0.00	4519.10	$\pm$0.00	4519.10
4	8	SVD	$f_2(t)$	3.31079175E-07	3.31079175E-07	30000.00	$\pm$ 0.00	4600.99	$\pm$0.00	4600.99
4	8	SVD	$f_3(t)$	7.02532901E-06	7.02532901E-06	10221.00	$\pm$ 0.00	4489.52	$\pm$0.00	4489.52
4	8	SVD	$f_4(t)$	2.49941164E-08	2.49941164E-08	30000.00	$\pm$ 0.00	4624.03	$\pm$0.00	4624.03
4	8	SVD	$f_5(t)$	5.59615795E-05	5.59615795E-05	351.00	$\pm$ 0.00	4476.47	$\pm$0.00	4476.47
4	8	SVD	$f_6(t)$	4.79116098E-08	4.79116098E-08	30000.00	$\pm$ 0.00	4604.87	$\pm$0.00	4604.87
4	9	SVD	$f_0(t)$	1.34113304E-07	1.34113304E-07	30000.00	$\pm$ 0.00	4546.96	$\pm$0.00	4546.96

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	9	SVD	$f_1(t)$	2.29111145E-05	2.29111145E-05	29657.00	± 0.00	4614.41	±0.00	4614.41
4	9	SVD	$f_2(t)$	5.32150534E-07	5.32150534E-07	30000.00	± 0.00	4691.60	±0.00	4691.60
4	9	SVD	$f_3(t)$	2.11920804E-05	2.11920804E-05	24577.00	± 0.00	4618.39	±0.00	4618.39
4	9	SVD	$f_4(t)$	1.74030255E-07	1.74030255E-07	30000.00	± 0.00	4510.62	±0.00	4510.62
4	9	SVD	$f_5(t)$	2.69252658E-04	2.69252658E-04	783.00	± 0.00	4533.96	±0.00	4533.96
4	9	SVD	$f_6(t)$	2.82615462E-07	2.82615462E-07	30000.00	± 0.00	4582.16	±0.00	4582.16
4	10	SVD	$f_0(t)$	7.74520553E-08	7.74520553E-08	30000.00	± 0.00	4523.47	±0.00	4523.47
4	10	SVD	$f_1(t)$	7.94313877E-05	7.94313877E-05	16129.00	± 0.00	4547.49	±0.00	4547.49
4	10	SVD	$f_2(t)$	2.44484683E-06	2.44484683E-06	30000.00	± 0.00	4552.35	±0.00	4552.35
4	10	SVD	$f_3(t)$	1.02890061E-04	1.02890061E-04	30000.00	± 0.00	4563.06	±0.00	4563.06
4	10	SVD	$f_4(t)$	1.66450867E-07	1.66450867E-07	30000.00	± 0.00	4534.98	±0.00	4534.98
4	10	SVD	$f_5(t)$	7.63885558E-04	7.63885558E-04	698.00	± 0.00	4552.00	±0.00	4552.00
4	10	SVD	$f_6(t)$	2.04568436E-07	2.04568436E-07	30000.00	± 0.00	4598.83	±0.00	4598.83
4	11	SVD	$f_0(t)$	3.69865236E-06	3.69865236E-06	636.00	± 0.00	4534.11	±0.00	4534.11
4	11	SVD	$f_1(t)$	4.14070036E-04	4.14070036E-04	27371.00	± 0.00	4620.26	±0.00	4620.26
4	11	SVD	$f_2(t)$	1.22652130E-05	1.22652130E-05	30000.00	± 0.00	4548.65	±0.00	4548.65
4	11	SVD	$f_3(t)$	5.90287999E-04	5.90287999E-04	30000.00	± 0.00	4549.62	±0.00	4549.62
4	11	SVD	$f_4(t)$	3.69865236E-06	3.69865236E-06	636.00	± 0.00	4511.00	±0.00	4511.00

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	11	SVD	$f_5(t)$	1.64948697E-03	1.64948697E-03	1183.00	$\pm$ 0.00	4544.41	$\pm$0.00	4544.41
4	11	SVD	$f_6(t)$	1.43934846E-06	1.43934846E-06	30000.00	$\pm$ 0.00	4550.27	$\pm$0.00	4550.27
4	12	SVD	$f_0(t)$	1.23855413E-06	1.23855413E-06	30000.00	$\pm$ 0.00	4654.00	$\pm$0.00	4654.00
4	12	SVD	$f_1(t)$	6.96883750E-04	6.96883750E-04	6185.00	$\pm$ 0.00	4477.62	$\pm$0.00	4477.62
4	12	SVD	$f_2(t)$	6.08051715E-05	6.08051715E-05	30000.00	$\pm$ 0.00	4559.85	$\pm$0.00	4559.85
4	12	SVD	$f_3(t)$	1.25600307E-03	1.25600307E-03	30000.00	$\pm$ 0.00	4495.03	$\pm$0.00	4495.03
4	12	SVD	$f_4(t)$	8.51771004E-07	8.51771004E-07	30000.00	$\pm$ 0.00	4555.78	$\pm$0.00	4555.78
4	12	SVD	$f_5(t)$	3.65069801E-03	3.65069801E-03	2848.00	$\pm$ 0.00	4551.78	$\pm$0.00	4551.78
4	12	SVD	$f_6(t)$	7.46459759E-06	7.46459759E-06	30000.00	$\pm$ 0.00	4584.89	$\pm$0.00	4584.89
4	13	SVD	$f_0(t)$	2.48178378E-06	2.48178378E-06	1691.00	$\pm$ 0.00	4574.26	$\pm$0.00	4574.26
4	13	SVD	$f_1(t)$	1.60895347E-03	1.60895347E-03	19939.00	$\pm$ 0.00	4476.62	$\pm$0.00	4476.62
4	13	SVD	$f_2(t)$	2.32991493E-04	2.32991493E-04	30000.00	$\pm$ 0.00	4618.20	$\pm$0.00	4618.20
4	13	SVD	$f_3(t)$	2.00239399E-03	2.00239399E-03	14417.00	$\pm$ 0.00	4502.19	$\pm$0.00	4502.19
4	13	SVD	$f_4(t)$	2.53985698E-06	2.53985698E-06	1630.00	$\pm$ 0.00	4517.39	$\pm$0.00	4517.39
4	13	SVD	$f_5(t)$	6.40954253E-03	6.40954253E-03	3859.00	$\pm$ 0.00	4581.88	$\pm$0.00	4581.88
4	13	SVD	$f_6(t)$	6.44236119E-06	6.44236119E-06	30000.00	$\pm$ 0.00	4635.80	$\pm$0.00	4635.80
4	14	SVD	$f_0(t)$	5.61765562E-06	5.61765562E-06	30000.00	$\pm$ 0.00	4675.05	$\pm$0.00	4675.05
4	14	SVD	$f_1(t)$	3.31437541E-03	3.31437541E-03	11941.00	$\pm$ 0.00	4500.81	$\pm$0.00	4500.81

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	14	SVD	$f_2(t)$	5.23333894E-04	5.23333894E-04	30000.00	± 0.00	4582.13	±0.00	4582.13
4	14	SVD	$f_3(t)$	5.75512582E-03	5.75512582E-03	17045.00	± 0.00	4557.91	±0.00	4557.91
4	14	SVD	$f_4(t)$	5.53902086E-06	5.53902086E-06	30000.00	± 0.00	4535.35	±0.00	4535.35
4	14	SVD	$f_5(t)$	1.15863219E-02	1.15863219E-02	6182.00	± 0.00	4643.26	±0.00	4643.26
4	14	SVD	$f_6(t)$	3.72575593E-05	3.72575593E-05	30000.00	± 0.00	4616.63	±0.00	4616.63
4	15	SVD	$f_0(t)$	3.64717721E-05	3.64717721E-05	1127.00	± 0.00	4538.06	±0.00	4538.06
4	15	SVD	$f_1(t)$	3.66938796E-03	3.66938796E-03	11796.00	± 0.00	4506.83	±0.00	4506.83
4	15	SVD	$f_2(t)$	4.75633814E-04	4.75633814E-04	30000.00	± 0.00	4650.46	±0.00	4650.46
4	15	SVD	$f_3(t)$	9.55744973E-03	9.55744973E-03	19647.00	± 0.00	4587.71	±0.00	4587.71
4	15	SVD	$f_4(t)$	3.64814760E-05	3.64814760E-05	1075.00	± 0.00	4544.88	±0.00	4544.88
4	15	SVD	$f_5(t)$	1.98806721E-02	1.98806721E-02	3264.00	± 0.00	4659.52	±0.00	4659.52
4	15	SVD	$f_6(t)$	9.33500803E-05	9.33500803E-05	30000.00	± 0.00	4576.35	±0.00	4576.35
4	16	SVD	$f_0(t)$	9.48456723E-05	9.48456723E-05	1361.00	± 0.00	4517.50	±0.00	4517.50
4	16	SVD	$f_1(t)$	1.28488873E-02	1.28488873E-02	15959.00	± 0.00	4531.04	±0.00	4531.04
4	16	SVD	$f_2(t)$	2.81478135E-03	2.81478135E-03	30000.00	± 0.00	4582.59	±0.00	4582.59
4	16	SVD	$f_3(t)$	1.30088337E-02	1.30088337E-02	30000.00	± 0.00	4629.78	±0.00	4629.78
4	16	SVD	$f_4(t)$	9.48253423E-05	9.48253423E-05	1958.00	± 0.00	4576.79	±0.00	4576.79
4	16	SVD	$f_5(t)$	2.62172350E-02	2.62172350E-02	2159.00	± 0.00	4537.71	±0.00	4537.71

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	16	SVD	$f_6(t)$	2.39075982E-04	2.39075982E-04	30000.00	± 0.00	4655.79	±0.00	4655.79
4	17	SVD	$f_0(t)$	9.01116542E-05	9.01116542E-05	2843.00	± 0.00	4544.45	±0.00	4544.45
4	17	SVD	$f_1(t)$	1.49681246E-02	1.49681246E-02	12505.00	± 0.00	4707.30	±0.00	4707.30
4	17	SVD	$f_2(t)$	3.96234944E-03	3.96234944E-03	30000.00	± 0.00	4630.36	±0.00	4630.36
4	17	SVD	$f_3(t)$	1.64796355E-02	1.64796355E-02	30000.00	± 0.00	4576.12	±0.00	4576.12
4	17	SVD	$f_4(t)$	8.92872561E-05	8.92872561E-05	2745.00	± 0.00	4606.46	±0.00	4606.46
4	17	SVD	$f_5(t)$	3.83946926E-02	3.83946926E-02	4807.00	± 0.00	4700.93	±0.00	4700.93
4	17	SVD	$f_6(t)$	5.99493820E-04	5.99493820E-04	30000.00	± 0.00	4659.16	±0.00	4659.16
4	18	SVD	$f_0(t)$	5.66945914E-05	5.66945914E-05	3196.00	± 0.00	4557.92	±0.00	4557.92
4	18	SVD	$f_1(t)$	2.49808097E-02	2.49808097E-02	10511.00	± 0.00	4636.44	±0.00	4636.44
4	18	SVD	$f_2(t)$	7.39291810E-03	7.39291810E-03	30000.00	± 0.00	4693.89	±0.00	4693.89
4	18	SVD	$f_3(t)$	2.87894983E-02	2.87894983E-02	30000.00	± 0.00	4580.47	±0.00	4580.47
4	18	SVD	$f_4(t)$	5.69429554E-05	5.69429554E-05	7820.00	± 0.00	4590.07	±0.00	4590.07
4	18	SVD	$f_5(t)$	5.64946760E-02	5.64946760E-02	5132.00	± 0.00	4561.18	±0.00	4561.18
4	18	SVD	$f_6(t)$	4.44896287E-04	4.44896287E-04	30000.00	± 0.00	4626.24	±0.00	4626.24
4	19	SVD	$f_0(t)$	1.94006257E-03	1.94006257E-03	1570.00	± 0.00	4613.54	±0.00	4613.54
4	19	SVD	$f_1(t)$	3.39064703E-02	3.39064703E-02	12191.00	± 0.00	4558.85	±0.00	4558.85
4	19	SVD	$f_2(t)$	9.51106585E-03	9.51106585E-03	30000.00	± 0.00	4618.26	±0.00	4618.26

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	19	SVD	$f_3(t)$	3.81885257E-02	3.81885257E-02	30000.00	$\pm$ 0.00	4650.55	$\pm$0.00	4650.55
4	19	SVD	$f_4(t)$	1.94767447E-03	1.94767447E-03	2088.00	$\pm$ 0.00	4559.57	$\pm$0.00	4559.57
4	19	SVD	$f_5(t)$	7.51245493E-02	7.51245493E-02	1541.00	$\pm$ 0.00	4637.52	$\pm$0.00	4637.52
4	19	SVD	$f_6(t)$	6.56710225E-04	6.56710225E-04	30000.00	$\pm$ 0.00	4693.45	$\pm$0.00	4693.45
4	20	SVD	$f_0(t)$	4.75412500E-04	4.75412500E-04	2308.00	$\pm$ 0.00	4604.66	$\pm$0.00	4604.66
4	20	SVD	$f_1(t)$	4.28376749E-02	4.28376749E-02	7123.00	$\pm$ 0.00	4531.86	$\pm$0.00	4531.86
4	20	SVD	$f_2(t)$	1.19645718E-02	1.19645718E-02	30000.00	$\pm$ 0.00	4579.05	$\pm$0.00	4579.05
4	20	SVD	$f_3(t)$	4.96934428E-02	4.96934428E-02	30000.00	$\pm$ 0.00	4610.20	$\pm$0.00	4610.20
4	20	SVD	$f_4(t)$	4.72919805E-04	4.72919805E-04	5560.00	$\pm$ 0.00	4598.56	$\pm$0.00	4598.56
4	20	SVD	$f_5(t)$	9.25217686E-02	9.25217686E-02	6453.00	$\pm$ 0.00	4578.10	$\pm$0.00	4578.10
4	20	SVD	$f_6(t)$	1.01344904E-03	1.01344904E-03	30000.00	$\pm$ 0.00	4616.62	$\pm$0.00	4616.62
4	21	SVD	$f_0(t)$	1.16345678E-02	1.16345678E-02	855.00	$\pm$ 0.00	4575.24	$\pm$0.00	4575.24
4	21	SVD	$f_1(t)$	7.35218574E-02	7.35218574E-02	21123.00	$\pm$ 0.00	4573.54	$\pm$0.00	4573.54
4	21	SVD	$f_2(t)$	2.20049271E-02	2.20049271E-02	30000.00	$\pm$ 0.00	4611.56	$\pm$0.00	4611.56
4	21	SVD	$f_3(t)$	7.38738315E-02	7.38738315E-02	30000.00	$\pm$ 0.00	4746.80	$\pm$0.00	4746.80
4	21	SVD	$f_4(t)$	1.16345678E-02	1.16345678E-02	855.00	$\pm$ 0.00	4542.18	$\pm$0.00	4542.18
4	21	SVD	$f_5(t)$	1.26899743E-01	1.26899743E-01	3332.00	$\pm$ 0.00	4573.90	$\pm$0.00	4573.90
4	21	SVD	$f_6(t)$	5.76239260E-03	5.76239260E-03	13889.00	$\pm$ 0.00	4647.37	$\pm$0.00	4647.37

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	22	SVD	$f_0(t)$	5.23727071E-03	5.23727071E-03	1674.00	$\pm$ 0.00	4569.28	$\pm$0.00	4569.28
4	22	SVD	$f_1(t)$	8.46841029E-02	8.46841029E-02	25419.00	$\pm$ 0.00	4669.25	$\pm$0.00	4669.25
4	22	SVD	$f_2(t)$	3.77882529E-02	3.77882529E-02	30000.00	$\pm$ 0.00	4654.78	$\pm$0.00	4654.78
4	22	SVD	$f_3(t)$	1.03352109E-01	1.03352109E-01	6955.00	$\pm$ 0.00	4632.19	$\pm$0.00	4632.19
4	22	SVD	$f_4(t)$	5.25391633E-03	5.25391633E-03	2051.00	$\pm$ 0.00	4625.91	$\pm$0.00	4625.91
4	22	SVD	$f_5(t)$	1.51855292E-01	1.51855292E-01	10829.00	$\pm$ 0.00	4574.73	$\pm$0.00	4574.73
4	22	SVD	$f_6(t)$	9.86556332E-03	9.86556332E-03	30000.00	$\pm$ 0.00	4684.86	$\pm$0.00	4684.86
4	23	SVD	$f_0(t)$	1.08263959E-02	1.08263959E-02	2843.00	$\pm$ 0.00	4584.38	$\pm$0.00	4584.38
4	23	SVD	$f_1(t)$	1.14749529E-01	1.14749529E-01	19997.00	$\pm$ 0.00	4593.62	$\pm$0.00	4593.62
4	23	SVD	$f_2(t)$	4.75722627E-02	4.75722627E-02	30000.00	$\pm$ 0.00	4629.75	$\pm$0.00	4629.75
4	23	SVD	$f_3(t)$	1.23424443E-01	1.23424443E-01	30000.00	$\pm$ 0.00	4646.20	$\pm$0.00	4646.20
4	23	SVD	$f_4(t)$	1.13026845E-02	1.13026845E-02	5612.00	$\pm$ 0.00	4624.48	$\pm$0.00	4624.48
4	23	SVD	$f_5(t)$	1.83611770E-01	1.83611770E-01	5023.00	$\pm$ 0.00	4575.52	$\pm$0.00	4575.52
4	23	SVD	$f_6(t)$	1.20400128E-02	1.20400128E-02	30000.00	$\pm$ 0.00	4730.23	$\pm$0.00	4730.23
4	24	SVD	$f_0(t)$	2.48145462E-02	2.48145462E-02	2248.00	$\pm$ 0.00	4637.94	$\pm$0.00	4637.94
4	24	SVD	$f_1(t)$	1.42885573E-01	1.42885573E-01	6498.00	$\pm$ 0.00	4611.54	$\pm$0.00	4611.54
4	24	SVD	$f_2(t)$	7.40726269E-02	7.40726269E-02	30000.00	$\pm$ 0.00	4645.73	$\pm$0.00	4645.73
4	24	SVD	$f_3(t)$	1.53928819E-01	1.53928819E-01	28535.00	$\pm$ 0.00	4715.46	$\pm$0.00	4715.46

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	24	SVD	$f_4(t)$	2.48280595E-02	2.48280595E-02	5216.00	$\pm$ 0.00	4702.49	$\pm$0.00	4702.49
4	24	SVD	$f_5(t)$	2.24604457E-01	2.24604457E-01	8353.00	$\pm$ 0.00	4646.78	$\pm$0.00	4646.78
4	24	SVD	$f_6(t)$	2.83588547E-02	2.83588547E-02	30000.00	$\pm$ 0.00	4673.28	$\pm$0.00	4673.28
4	25	SVD	$f_0(t)$	7.18893521E-02	7.18893521E-02	1117.00	$\pm$ 0.00	4598.87	$\pm$0.00	4598.87
4	25	SVD	$f_1(t)$	1.47403775E-01	1.47403775E-01	25427.00	$\pm$ 0.00	4688.54	$\pm$0.00	4688.54
4	25	SVD	$f_2(t)$	8.26693690E-02	8.26693690E-02	19288.00	$\pm$ 0.00	4706.95	$\pm$0.00	4706.95
4	25	SVD	$f_3(t)$	1.77294849E-01	1.77294849E-01	30000.00	$\pm$ 0.00	4694.50	$\pm$0.00	4694.50
4	25	SVD	$f_4(t)$	7.18886203E-02	7.18886203E-02	1015.00	$\pm$ 0.00	4641.59	$\pm$0.00	4641.59
4	25	SVD	$f_5(t)$	2.66575744E-01	2.66575744E-01	4228.00	$\pm$ 0.00	4670.39	$\pm$0.00	4670.39
4	25	SVD	$f_6(t)$	2.69409277E-02	2.69409277E-02	30000.00	$\pm$ 0.00	4714.28	$\pm$0.00	4714.28
4	26	SVD	$f_0(t)$	1.11898399E-01	1.11898399E-01	1234.00	$\pm$ 0.00	4641.46	$\pm$0.00	4641.46
4	26	SVD	$f_1(t)$	2.31077271E-01	2.31077271E-01	21195.00	$\pm$ 0.00	4659.36	$\pm$0.00	4659.36
4	26	SVD	$f_2(t)$	1.33177818E-01	1.33177818E-01	30000.00	$\pm$ 0.00	4763.12	$\pm$0.00	4763.12
4	26	SVD	$f_3(t)$	2.27674626E-01	2.27674626E-01	30000.00	$\pm$ 0.00	4742.39	$\pm$0.00	4742.39
4	26	SVD	$f_4(t)$	1.11913696E-01	1.11913696E-01	1036.00	$\pm$ 0.00	4700.36	$\pm$0.00	4700.36
4	26	SVD	$f_5(t)$	3.16603415E-01	3.16603415E-01	8595.00	$\pm$ 0.00	4646.90	$\pm$0.00	4646.90
4	26	SVD	$f_6(t)$	8.17326166E-02	8.17326166E-02	11599.00	$\pm$ 0.00	4618.65	$\pm$0.00	4618.65
4	27	SVD	$f_0(t)$	1.25985907E-01	1.25985907E-01	1092.00	$\pm$ 0.00	4618.05	$\pm$0.00	4618.05

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	27	SVD	$f_1(t)$	2.59528902E-01	2.59528902E-01	7386.00	$\pm$ 0.00	4645.74	$\pm$0.00	4645.74
4	27	SVD	$f_2(t)$	1.40912986E-01	1.40912986E-01	30000.00	$\pm$ 0.00	4666.08	$\pm$0.00	4666.08
4	27	SVD	$f_3(t)$	2.76994350E-01	2.76994350E-01	30000.00	$\pm$ 0.00	4683.86	$\pm$0.00	4683.86
4	27	SVD	$f_4(t)$	1.25989659E-01	1.25989659E-01	1014.00	$\pm$ 0.00	4651.98	$\pm$0.00	4651.98
4	27	SVD	$f_5(t)$	3.65289172E-01	3.65289172E-01	4075.00	$\pm$ 0.00	4665.26	$\pm$0.00	4665.26
4	27	SVD	$f_6(t)$	9.18485470E-02	9.18485470E-02	30000.00	$\pm$ 0.00	4689.39	$\pm$0.00	4689.39
4	28	SVD	$f_0(t)$	1.33816658E-01	1.33816658E-01	1690.00	$\pm$ 0.00	4665.64	$\pm$0.00	4665.64
4	28	SVD	$f_1(t)$	2.99015747E-01	2.99015747E-01	24253.00	$\pm$ 0.00	4729.32	$\pm$0.00	4729.32
4	28	SVD	$f_2(t)$	1.90537616E-01	1.90537616E-01	30000.00	$\pm$ 0.00	4677.93	$\pm$0.00	4677.93
4	28	SVD	$f_3(t)$	3.36123684E-01	3.36123684E-01	30000.00	$\pm$ 0.00	4677.73	$\pm$0.00	4677.73
4	28	SVD	$f_4(t)$	1.33912659E-01	1.33912659E-01	4726.00	$\pm$ 0.00	4669.14	$\pm$0.00	4669.14
4	28	SVD	$f_5(t)$	4.07863623E-01	4.07863623E-01	2245.00	$\pm$ 0.00	4687.29	$\pm$0.00	4687.29
4	28	SVD	$f_6(t)$	1.22253762E-01	1.22253762E-01	30000.00	$\pm$ 0.00	4655.17	$\pm$0.00	4655.17
4	29	SVD	$f_0(t)$	1.72624914E-01	1.72624914E-01	2098.00	$\pm$ 0.00	4600.38	$\pm$0.00	4600.38
4	29	SVD	$f_1(t)$	3.47627542E-01	3.47627542E-01	7344.00	$\pm$ 0.00	4609.28	$\pm$0.00	4609.28
4	29	SVD	$f_2(t)$	2.64672796E-01	2.64672796E-01	30000.00	$\pm$ 0.00	4658.37	$\pm$0.00	4658.37
4	29	SVD	$f_3(t)$	3.83024449E-01	3.83024449E-01	30000.00	$\pm$ 0.00	4690.54	$\pm$0.00	4690.54
4	29	SVD	$f_4(t)$	1.72627205E-01	1.72627205E-01	5069.00	$\pm$ 0.00	4609.89	$\pm$0.00	4609.89

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
4	29	SVD	$f_5(t)$	4.68799114E-01	4.68799114E-01	4608.00	± 0.00	4645.93	±0.00	4645.93
4	29	SVD	$f_6(t)$	1.94683939E-01	1.94683939E-01	3637.00	± 0.00	4711.02	±0.00	4711.02
4	30	SVD	$f_0(t)$	2.06761663E-01	2.06761663E-01	965.00	± 0.00	4599.24	±0.00	4599.24
4	30	SVD	$f_1(t)$	4.00844202E-01	4.00844202E-01	13842.00	± 0.00	4671.43	±0.00	4671.43
4	30	SVD	$f_2(t)$	3.13414190E-01	3.13414190E-01	30000.00	± 0.00	4627.29	±0.00	4627.29
4	30	SVD	$f_3(t)$	4.53500074E-01	4.53500074E-01	30000.00	± 0.00	4691.55	±0.00	4691.55
4	30	SVD	$f_4(t)$	2.06761663E-01	2.06761663E-01	965.00	± 0.00	4581.33	±0.00	4581.33
4	30	SVD	$f_5(t)$	5.26526296E-01	5.26526296E-01	5561.00	± 0.00	4637.10	±0.00	4637.10
4	30	SVD	$f_6(t)$	1.20797315E-01	1.20797315E-01	30000.00	± 0.00	4661.40	±0.00	4661.40
5	5	SVD	$f_0(t)$	2.91045696E-19	2.91045696E-19	30000.00	± 0.00	26933.87	±0.00	26933.87
5	5	SVD	$f_1(t)$	6.62631539E-20	6.62631539E-20	4382.00	± 0.00	26534.56	±0.00	26534.56
5	5	SVD	$f_2(t)$	5.36758881E-20	5.36758881E-20	30000.00	± 0.00	26451.93	±0.00	26451.93
5	5	SVD	$f_3(t)$	3.06446748E-21	3.06446748E-21	675.00	± 0.00	26750.65	±0.00	26750.65
5	5	SVD	$f_4(t)$	6.09810777E-21	6.09810777E-21	30000.00	± 0.00	26484.75	±0.00	26484.75
5	5	SVD	$f_5(t)$	1.35178567E-20	1.35178567E-20	10146.00	± 0.00	26346.45	±0.00	26346.45
5	5	SVD	$f_6(t)$	7.75041845E-20	7.75041845E-20	30000.00	± 0.00	26139.23	±0.00	26139.23
5	6	SVD	$f_0(t)$	1.91922617E-14	1.91922617E-14	30000.00	± 0.00	26830.58	±0.00	26830.58
5	6	SVD	$f_1(t)$	3.06493394E-12	3.06493394E-12	5787.00	± 0.00	26429.13	±0.00	26429.13

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
5	6	SVD	$f_2(t)$	2.15848160E-14	2.15848160E-14	30000.00	± 0.00	26685.21	±0.00	26685.21
5	6	SVD	$f_3(t)$	1.60001996E-13	1.60001996E-13	121.00	± 0.00	26702.82	±0.00	26702.82
5	6	SVD	$f_4(t)$	1.19649392E-13	1.19649392E-13	30000.00	± 0.00	26595.69	±0.00	26595.69
5	6	SVD	$f_5(t)$	3.25863483E-12	3.25863483E-12	12901.00	± 0.00	26385.76	±0.00	26385.76
5	6	SVD	$f_6(t)$	3.95213479E-14	3.95213479E-14	30000.00	± 0.00	26562.73	±0.00	26562.73
5	7	SVD	$f_0(t)$	1.02342537E-11	1.02342537E-11	30000.00	± 0.00	26701.17	±0.00	26701.17
5	7	SVD	$f_1(t)$	9.31696711E-10	9.31696711E-10	10458.00	± 0.00	26684.35	±0.00	26684.35
5	7	SVD	$f_2(t)$	5.44251074E-12	5.44251074E-12	30000.00	± 0.00	26845.07	±0.00	26845.07
5	7	SVD	$f_3(t)$	3.03044175E-09	3.03044175E-09	102.00	± 0.00	26645.04	±0.00	26645.04
5	7	SVD	$f_4(t)$	4.85044158E-12	4.85044158E-12	30000.00	± 0.00	26694.36	±0.00	26694.36
5	7	SVD	$f_5(t)$	9.98417084E-08	9.98417084E-08	156.00	± 0.00	26235.99	±0.00	26235.99
5	7	SVD	$f_6(t)$	1.26216517E-11	1.26216517E-11	30000.00	± 0.00	25540.58	±0.00	25540.58
5	8	SVD	$f_0(t)$	6.36344375E-10	6.36344375E-10	30000.00	± 0.00	26681.30	±0.00	26681.30
5	8	SVD	$f_1(t)$	1.16634318E-07	1.16634318E-07	9208.00	± 0.00	26692.17	±0.00	26692.17
5	8	SVD	$f_2(t)$	4.16883463E-09	4.16883463E-09	30000.00	± 0.00	26590.89	±0.00	26590.89
5	8	SVD	$f_3(t)$	1.30091284E-07	1.30091284E-07	106.00	± 0.00	26708.16	±0.00	26708.16
5	8	SVD	$f_4(t)$	3.70682047E-10	3.70682047E-10	30000.00	± 0.00	26818.53	±0.00	26818.53
5	8	SVD	$f_5(t)$	2.04546615E-06	2.04546615E-06	209.00	± 0.00	26490.46	±0.00	26490.46

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
5	8	SVD	$f_6(t)$	1.99881734E-10	1.99881734E-10	30000.00	± 0.00	26466.80	±0.00	26466.80
5	9	SVD	$f_0(t)$	1.41200621E-09	1.41200621E-09	30000.00	± 0.00	26761.13	±0.00	26761.13
5	9	SVD	$f_1(t)$	7.27333952E-07	7.27333952E-07	15925.00	± 0.00	26754.83	±0.00	26754.83
5	9	SVD	$f_2(t)$	5.87216293E-08	5.87216293E-08	30000.00	± 0.00	26768.10	±0.00	26768.10
5	9	SVD	$f_3(t)$	5.44844723E-07	5.44844723E-07	5614.00	± 0.00	26690.04	±0.00	26690.04
5	9	SVD	$f_4(t)$	1.01921351E-09	1.01921351E-09	30000.00	± 0.00	26932.66	±0.00	26932.66
5	9	SVD	$f_5(t)$	1.18229167E-05	1.18229167E-05	167.00	± 0.00	26251.34	±0.00	26251.34
5	9	SVD	$f_6(t)$	6.70378318E-10	6.70378318E-10	30000.00	± 0.00	26366.80	±0.00	26366.80
5	10	SVD	$f_0(t)$	5.92763919E-09	5.92763919E-09	30000.00	± 0.00	26997.41	±0.00	26997.41
5	10	SVD	$f_1(t)$	6.10518670E-06	6.10518670E-06	22413.00	± 0.00	26490.25	±0.00	26490.25
5	10	SVD	$f_2(t)$	6.16955375E-08	6.16955375E-08	30000.00	± 0.00	26686.04	±0.00	26686.04
5	10	SVD	$f_3(t)$	9.29464357E-06	9.29464357E-06	4112.00	± 0.00	26795.21	±0.00	26795.21
5	10	SVD	$f_4(t)$	4.54500148E-09	4.54500148E-09	30000.00	± 0.00	27013.69	±0.00	27013.69
5	10	SVD	$f_5(t)$	7.11579788E-05	7.11579788E-05	864.00	± 0.00	25298.76	±0.00	25298.76
5	10	SVD	$f_6(t)$	4.00286364E-09	4.00286364E-09	30000.00	± 0.00	25597.50	±0.00	25597.50
5	11	SVD	$f_0(t)$	6.99495312E-08	6.99495312E-08	30000.00	± 0.00	26857.13	±0.00	26857.13
5	11	SVD	$f_1(t)$	1.12488898E-05	1.12488898E-05	7410.00	± 0.00	26993.14	±0.00	26993.14
5	11	SVD	$f_2(t)$	2.50294820E-07	2.50294820E-07	30000.00	± 0.00	26771.17	±0.00	26771.17

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
5	11	SVD	$f_3(t)$	2.76091022E-05	2.76091022E-05	30000.00	$\pm$ 0.00	26741.41	$\pm$0.00	26741.41
5	11	SVD	$f_4(t)$	7.06102449E-08	7.06102449E-08	30000.00	$\pm$ 0.00	27362.91	$\pm$0.00	27362.91
5	11	SVD	$f_5(t)$	1.74321743E-04	1.74321743E-04	453.00	$\pm$ 0.00	25528.47	$\pm$0.00	25528.47
5	11	SVD	$f_6(t)$	1.07170275E-07	1.07170275E-07	30000.00	$\pm$ 0.00	25756.34	$\pm$0.00	25756.34
5	12	SVD	$f_0(t)$	4.37988693E-08	4.37988693E-08	30000.00	$\pm$ 0.00	26701.59	$\pm$0.00	26701.59
5	12	SVD	$f_1(t)$	6.74559820E-05	6.74559820E-05	9877.00	$\pm$ 0.00	26417.48	$\pm$0.00	26417.48
5	12	SVD	$f_2(t)$	3.28615102E-06	3.28615102E-06	30000.00	$\pm$ 0.00	26678.72	$\pm$0.00	26678.72
5	12	SVD	$f_3(t)$	5.37862642E-05	5.37862642E-05	11225.00	$\pm$ 0.00	26819.20	$\pm$0.00	26819.20
5	12	SVD	$f_4(t)$	3.90526098E-08	3.90526098E-08	30000.00	$\pm$ 0.00	26510.31	$\pm$0.00	26510.31
5	12	SVD	$f_5(t)$	6.60155708E-04	6.60155708E-04	1128.00	$\pm$ 0.00	25497.31	$\pm$0.00	25497.31
5	12	SVD	$f_6(t)$	5.89212087E-08	5.89212087E-08	30000.00	$\pm$ 0.00	25675.55	$\pm$0.00	25675.55
5	13	SVD	$f_0(t)$	1.87429005E-07	1.87429005E-07	30000.00	$\pm$ 0.00	26989.20	$\pm$0.00	26989.20
5	13	SVD	$f_1(t)$	1.95856250E-04	1.95856250E-04	11635.00	$\pm$ 0.00	26773.28	$\pm$0.00	26773.28
5	13	SVD	$f_2(t)$	1.52925681E-05	1.52925681E-05	30000.00	$\pm$ 0.00	26723.87	$\pm$0.00	26723.87
5	13	SVD	$f_3(t)$	2.97438719E-04	2.97438719E-04	18924.00	$\pm$ 0.00	26746.89	$\pm$0.00	26746.89
5	13	SVD	$f_4(t)$	1.89721373E-07	1.89721373E-07	30000.00	$\pm$ 0.00	26720.71	$\pm$0.00	26720.71
5	13	SVD	$f_5(t)$	1.58447966E-03	1.58447966E-03	2650.00	$\pm$ 0.00	25706.87	$\pm$0.00	25706.87
5	13	SVD	$f_6(t)$	1.75645192E-07	1.75645192E-07	30000.00	$\pm$ 0.00	25510.80	$\pm$0.00	25510.80

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
5	14	SVD	$f_0(t)$	4.73246844E-07	4.73246844E-07	30000.00	± 0.00	27176.75	±0.00	27176.75
5	14	SVD	$f_1(t)$	3.39039613E-04	3.39039613E-04	30000.00	± 0.00	26606.43	±0.00	26606.43
5	14	SVD	$f_2(t)$	1.42484573E-05	1.42484573E-05	30000.00	± 0.00	27164.68	±0.00	27164.68
5	14	SVD	$f_3(t)$	1.04451703E-03	1.04451703E-03	28614.00	± 0.00	26698.89	±0.00	26698.89
5	14	SVD	$f_4(t)$	5.34432334E-07	5.34432334E-07	30000.00	± 0.00	26824.18	±0.00	26824.18
5	14	SVD	$f_5(t)$	3.16275575E-03	3.16275575E-03	1297.00	± 0.00	25433.00	±0.00	25433.00
5	14	SVD	$f_6(t)$	2.04844807E-07	2.04844807E-07	30000.00	± 0.00	25351.14	±0.00	25351.14
5	15	SVD	$f_0(t)$	2.57552256E-07	2.57552256E-07	30000.00	± 0.00	26777.31	±0.00	26777.31
5	15	SVD	$f_1(t)$	9.45755188E-04	9.45755188E-04	5155.00	± 0.00	26985.22	±0.00	26985.22
5	15	SVD	$f_2(t)$	7.02969295E-05	7.02969295E-05	30000.00	± 0.00	27010.92	±0.00	27010.92
5	15	SVD	$f_3(t)$	1.72535068E-03	1.72535068E-03	30000.00	± 0.00	27040.49	±0.00	27040.49
5	15	SVD	$f_4(t)$	2.65553259E-07	2.65553259E-07	30000.00	± 0.00	27179.99	±0.00	27179.99
5	15	SVD	$f_5(t)$	5.72936155E-03	5.72936155E-03	816.00	± 0.00	25597.79	±0.00	25597.79
5	15	SVD	$f_6(t)$	3.43331420E-06	3.43331420E-06	30000.00	± 0.00	25357.15	±0.00	25357.15
5	16	SVD	$f_0(t)$	1.24594823E-06	1.24594823E-06	30000.00	± 0.00	26912.42	±0.00	26912.42
5	16	SVD	$f_1(t)$	1.88351298E-03	1.88351298E-03	17663.00	± 0.00	26848.42	±0.00	26848.42
5	16	SVD	$f_2(t)$	1.51913485E-04	1.51913485E-04	30000.00	± 0.00	26776.71	±0.00	26776.71
5	16	SVD	$f_3(t)$	2.77473043E-03	2.77473043E-03	30000.00	± 0.00	26593.44	±0.00	26593.44

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
5	16	SVD	$f_4(t)$	1.16805339E-06	1.16805339E-06	30000.00	± 0.00	26870.65	±0.00	26870.65
5	16	SVD	$f_5(t)$	8.11208079E-03	8.11208079E-03	2746.00	± 0.00	25401.76	±0.00	25401.76
5	16	SVD	$f_6(t)$	1.47459747E-06	1.47459747E-06	30000.00	± 0.00	25442.94	±0.00	25442.94
5	17	SVD	$f_0(t)$	1.75190669E-06	1.75190669E-06	30000.00	± 0.00	26970.60	±0.00	26970.60
5	17	SVD	$f_1(t)$	3.12945478E-03	3.12945478E-03	14703.00	± 0.00	26994.52	±0.00	26994.52
5	17	SVD	$f_2(t)$	3.29313879E-04	3.29313879E-04	30000.00	± 0.00	26679.58	±0.00	26679.58
5	17	SVD	$f_3(t)$	5.42341876E-03	5.42341876E-03	30000.00	± 0.00	27002.04	±0.00	27002.04
5	17	SVD	$f_4(t)$	1.81444648E-06	1.81444648E-06	30000.00	± 0.00	26857.34	±0.00	26857.34
5	17	SVD	$f_5(t)$	1.61869933E-02	1.61869933E-02	4313.00	± 0.00	25701.03	±0.00	25701.03
5	17	SVD	$f_6(t)$	1.05270662E-05	1.05270662E-05	30000.00	± 0.00	25602.42	±0.00	25602.42
5	18	SVD	$f_0(t)$	5.87317412E-06	5.87317412E-06	1830.00	± 0.00	26752.04	±0.00	26752.04
5	18	SVD	$f_1(t)$	4.92366496E-03	4.92366496E-03	14009.00	± 0.00	27007.06	±0.00	27007.06
5	18	SVD	$f_2(t)$	8.02727143E-04	8.02727143E-04	30000.00	± 0.00	26942.83	±0.00	26942.83
5	18	SVD	$f_3(t)$	1.15573679E-02	1.15573679E-02	30000.00	± 0.00	26913.45	±0.00	26913.45
5	18	SVD	$f_4(t)$	5.88462397E-06	5.88462397E-06	5068.00	± 0.00	26707.54	±0.00	26707.54
5	18	SVD	$f_5(t)$	2.31983857E-02	2.31983857E-02	4848.00	± 0.00	25565.95	±0.00	25565.95
5	18	SVD	$f_6(t)$	2.02727236E-04	2.02727236E-04	30000.00	± 0.00	25496.83	±0.00	25496.83
5	19	SVD	$f_0(t)$	9.85564948E-06	9.85564948E-06	2168.00	± 0.00	27226.32	±0.00	27226.32

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
5	19	SVD	$f_1(t)$	1.11055836E-02	1.11055836E-02	23665.00	$\pm$ 0.00	27035.97	$\pm$0.00	27035.97
5	19	SVD	$f_2(t)$	1.70298380E-03	1.70298380E-03	30000.00	$\pm$ 0.00	27001.82	$\pm$0.00	27001.82
5	19	SVD	$f_3(t)$	1.21490146E-02	1.21490146E-02	30000.00	$\pm$ 0.00	27197.95	$\pm$0.00	27197.95
5	19	SVD	$f_4(t)$	9.87681014E-06	9.87681014E-06	5144.00	$\pm$ 0.00	27137.74	$\pm$0.00	27137.74
5	19	SVD	$f_5(t)$	3.09807572E-02	3.09807572E-02	2559.00	$\pm$ 0.00	25378.32	$\pm$0.00	25378.32
5	19	SVD	$f_6(t)$	1.23103531E-04	1.23103531E-04	30000.00	$\pm$ 0.00	25497.59	$\pm$0.00	25497.59
5	20	SVD	$f_0(t)$	1.42636099E-04	1.42636099E-04	2571.00	$\pm$ 0.00	27129.79	$\pm$0.00	27129.79
5	20	SVD	$f_1(t)$	1.58891105E-02	1.58891105E-02	20721.00	$\pm$ 0.00	27132.32	$\pm$0.00	27132.32
5	20	SVD	$f_2(t)$	3.12800207E-03	3.12800207E-03	30000.00	$\pm$ 0.00	27108.15	$\pm$0.00	27108.15
5	20	SVD	$f_3(t)$	2.45619254E-02	2.45619254E-02	30000.00	$\pm$ 0.00	27337.88	$\pm$0.00	27337.88
5	20	SVD	$f_4(t)$	1.45722471E-04	1.45722471E-04	6090.00	$\pm$ 0.00	27328.58	$\pm$0.00	27328.58
5	20	SVD	$f_5(t)$	4.58931448E-02	4.58931448E-02	4276.00	$\pm$ 0.00	25483.62	$\pm$0.00	25483.62
5	20	SVD	$f_6(t)$	4.48532367E-04	4.48532367E-04	30000.00	$\pm$ 0.00	25413.59	$\pm$0.00	25413.59
5	21	SVD	$f_0(t)$	1.94684108E-03	1.94684108E-03	1567.00	$\pm$ 0.00	27085.15	$\pm$0.00	27085.15
5	21	SVD	$f_1(t)$	2.43713711E-02	2.43713711E-02	9735.00	$\pm$ 0.00	26889.28	$\pm$0.00	26889.28
5	21	SVD	$f_2(t)$	8.82150381E-03	8.82150381E-03	30000.00	$\pm$ 0.00	27126.61	$\pm$0.00	27126.61
5	21	SVD	$f_3(t)$	2.86469418E-02	2.86469418E-02	30000.00	$\pm$ 0.00	27299.56	$\pm$0.00	27299.56
5	21	SVD	$f_4(t)$	1.95265524E-03	1.95265524E-03	1889.00	$\pm$ 0.00	27186.42	$\pm$0.00	27186.42

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
5	21	SVD	$f_5(t)$	6.73586275E-02	6.73586275E-02	4383.00	$\pm$ 0.00	25136.13	$\pm$0.00	25136.13
5	21	SVD	$f_6(t)$	8.38637187E-04	8.38637187E-04	30000.00	$\pm$ 0.00	25738.65	$\pm$0.00	25738.65
5	22	SVD	$f_0(t)$	4.86959425E-05	4.86959425E-05	2053.00	$\pm$ 0.00	27475.31	$\pm$0.00	27475.31
5	22	SVD	$f_1(t)$	3.76258617E-02	3.76258617E-02	26612.00	$\pm$ 0.00	27095.41	$\pm$0.00	27095.41
5	22	SVD	$f_2(t)$	9.48228983E-03	9.48228983E-03	30000.00	$\pm$ 0.00	27379.99	$\pm$0.00	27379.99
5	22	SVD	$f_3(t)$	3.96228109E-02	3.96228109E-02	30000.00	$\pm$ 0.00	27267.23	$\pm$0.00	27267.23
5	22	SVD	$f_4(t)$	4.92068988E-05	4.92068988E-05	5094.00	$\pm$ 0.00	27516.50	$\pm$0.00	27516.50
5	22	SVD	$f_5(t)$	8.89077299E-02	8.89077299E-02	4651.00	$\pm$ 0.00	25570.83	$\pm$0.00	25570.83
5	22	SVD	$f_6(t)$	3.90889068E-03	3.90889068E-03	30000.00	$\pm$ 0.00	25167.20	$\pm$0.00	25167.20
5	23	SVD	$f_0(t)$	7.83978393E-03	7.83978393E-03	1126.00	$\pm$ 0.00	27463.98	$\pm$0.00	27463.98
5	23	SVD	$f_1(t)$	5.64376610E-02	5.64376610E-02	30000.00	$\pm$ 0.00	27133.25	$\pm$0.00	27133.25
5	23	SVD	$f_2(t)$	1.80436702E-02	1.80436702E-02	30000.00	$\pm$ 0.00	27497.44	$\pm$0.00	27497.44
5	23	SVD	$f_3(t)$	7.89729242E-02	7.89729242E-02	30000.00	$\pm$ 0.00	27540.17	$\pm$0.00	27540.17
5	23	SVD	$f_4(t)$	7.84272435E-03	7.84272435E-03	1080.00	$\pm$ 0.00	27612.68	$\pm$0.00	27612.68
5	23	SVD	$f_5(t)$	1.08641280E-01	1.08641280E-01	4191.00	$\pm$ 0.00	25435.51	$\pm$0.00	25435.51
5	23	SVD	$f_6(t)$	8.85277377E-04	8.85277377E-04	30000.00	$\pm$ 0.00	25504.78	$\pm$0.00	25504.78
5	24	SVD	$f_0(t)$	8.87929152E-03	8.87929152E-03	1376.00	$\pm$ 0.00	27159.60	$\pm$0.00	27159.60
5	24	SVD	$f_1(t)$	4.82459925E-02	4.82459925E-02	29385.00	$\pm$ 0.00	27402.67	$\pm$0.00	27402.67

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
5	24	SVD	$f_2(t)$	1.24225888E-02	1.24225888E-02	30000.00	± 0.00	27463.62	±0.00	27463.62
5	24	SVD	$f_3(t)$	9.15858227E-02	9.15858227E-02	30000.00	± 0.00	27521.99	±0.00	27521.99
5	24	SVD	$f_4(t)$	8.88906891E-03	8.88906891E-03	2158.00	± 0.00	27411.92	±0.00	27411.92
5	24	SVD	$f_5(t)$	1.44795590E-01	1.44795590E-01	11665.00	± 0.00	25709.36	±0.00	25709.36
5	24	SVD	$f_6(t)$	1.34689671E-02	1.34689671E-02	30000.00	± 0.00	25814.69	±0.00	25814.69
5	25	SVD	$f_0(t)$	5.47544733E-03	5.47544733E-03	2809.00	± 0.00	27233.59	±0.00	27233.59
5	25	SVD	$f_1(t)$	1.05557259E-01	1.05557259E-01	12868.00	± 0.00	27143.40	±0.00	27143.40
5	25	SVD	$f_2(t)$	4.09547385E-02	4.09547385E-02	26602.00	± 0.00	27265.99	±0.00	27265.99
5	25	SVD	$f_3(t)$	1.22173738E-01	1.22173738E-01	30000.00	± 0.00	27290.67	±0.00	27290.67
5	25	SVD	$f_4(t)$	5.72700693E-03	5.72700693E-03	5864.00	± 0.00	27472.74	±0.00	27472.74
5	25	SVD	$f_5(t)$	1.90917520E-01	1.90917520E-01	333.00	± 0.00	25518.05	±0.00	25518.05
5	25	SVD	$f_6(t)$	4.57015490E-03	4.57015490E-03	30000.00	± 0.00	25588.16	±0.00	25588.16
5	26	SVD	$f_0(t)$	7.01848128E-02	7.01848128E-02	510.00	± 0.00	27392.39	±0.00	27392.39
5	26	SVD	$f_1(t)$	1.27484000E-01	1.27484000E-01	21391.00	± 0.00	27481.70	±0.00	27481.70
5	26	SVD	$f_2(t)$	5.51603334E-02	5.51603334E-02	30000.00	± 0.00	27281.52	±0.00	27281.52
5	26	SVD	$f_3(t)$	1.47734040E-01	1.47734040E-01	30000.00	± 0.00	27434.40	±0.00	27434.40
5	26	SVD	$f_4(t)$	7.01848128E-02	7.01848128E-02	510.00	± 0.00	27335.17	±0.00	27335.17
5	26	SVD	$f_5(t)$	2.28437236E-01	2.28437236E-01	2705.00	± 0.00	25831.35	±0.00	25831.35

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
5	26	SVD	$f_6(t)$	9.33534215E-03	9.33534215E-03	30000.00	$\pm$ 0.00	25895.12	$\pm$0.00	25895.12
5	27	SVD	$f_0(t)$	4.93028237E-02	4.93028237E-02	1081.00	$\pm$ 0.00	27438.62	$\pm$0.00	27438.62
5	27	SVD	$f_1(t)$	1.89805173E-01	1.89805173E-01	28768.00	$\pm$ 0.00	27431.90	$\pm$0.00	27431.90
5	27	SVD	$f_2(t)$	9.32977713E-02	9.32977713E-02	30000.00	$\pm$ 0.00	27351.63	$\pm$0.00	27351.63
5	27	SVD	$f_3(t)$	2.07678682E-01	2.07678682E-01	30000.00	$\pm$ 0.00	27341.50	$\pm$0.00	27341.50
5	27	SVD	$f_4(t)$	4.93107819E-02	4.93107819E-02	1011.00	$\pm$ 0.00	27424.83	$\pm$0.00	27424.83
5	27	SVD	$f_5(t)$	2.84131048E-01	2.84131048E-01	5042.00	$\pm$ 0.00	25650.71	$\pm$0.00	25650.71
5	27	SVD	$f_6(t)$	2.32577917E-02	2.32577917E-02	30000.00	$\pm$ 0.00	25934.71	$\pm$0.00	25934.71
5	28	SVD	$f_0(t)$	4.59551797E-02	4.59551797E-02	1306.00	$\pm$ 0.00	27308.20	$\pm$0.00	27308.20
5	28	SVD	$f_1(t)$	1.91779056E-01	1.91779056E-01	20491.00	$\pm$ 0.00	27491.66	$\pm$0.00	27491.66
5	28	SVD	$f_2(t)$	1.07312891E-01	1.07312891E-01	30000.00	$\pm$ 0.00	27437.08	$\pm$0.00	27437.08
5	28	SVD	$f_3(t)$	2.23227043E-01	2.23227043E-01	30000.00	$\pm$ 0.00	27381.83	$\pm$0.00	27381.83
5	28	SVD	$f_4(t)$	4.61065037E-02	4.61065037E-02	1034.00	$\pm$ 0.00	27200.40	$\pm$0.00	27200.40
5	28	SVD	$f_5(t)$	3.20218936E-01	3.20218936E-01	13850.00	$\pm$ 0.00	25478.57	$\pm$0.00	25478.57
5	28	SVD	$f_6(t)$	2.52787216E-02	2.52787216E-02	30000.00	$\pm$ 0.00	25676.98	$\pm$0.00	25676.98
5	29	SVD	$f_0(t)$	5.13781816E-02	5.13781816E-02	1224.00	$\pm$ 0.00	24789.85	$\pm$0.00	24789.85
5	29	SVD	$f_1(t)$	2.25301370E-01	2.25301370E-01	10817.00	$\pm$ 0.00	27333.93	$\pm$0.00	27333.93
5	29	SVD	$f_2(t)$	1.03802242E-01	1.03802242E-01	30000.00	$\pm$ 0.00	24990.34	$\pm$0.00	24990.34

k	m	**Init**	$f(t)$	**Best Min Vol**	**Avg Min Vol**	**Avg Iterations**		**Avg Run Time (s)**		**Total Time**
5	29	SVD	$f_3(t)$	2.72455858E-01	2.72455858E-01	30000.00	± 0.00	24871.60	±0.00	24871.60
5	29	SVD	$f_4(t)$	5.13967951E-02	5.13967951E-02	1487.00	± 0.00	24815.12	±0.00	24815.12
5	29	SVD	$f_5(t)$	3.75903001E-01	3.75903001E-01	7968.00	± 0.00	25918.08	±0.00	25918.08
5	29	SVD	$f_6(t)$	4.88680111E-02	4.88680111E-02	30000.00	± 0.00	25913.12	±0.00	25913.12
5	30	SVD	$f_0(t)$	9.87278732E-02	9.87278732E-02	1362.00	± 0.00	11678.43	±0.00	11678.43
5	30	SVD	$f_1(t)$	2.99445237E-01	2.99445237E-01	8855.00	± 0.00	24705.05	±0.00	24705.05
5	30	SVD	$f_2(t)$	1.54093869E-01	1.54093869E-01	30000.00	± 0.00	24832.26	±0.00	24832.26
5	30	SVD	$f_3(t)$	3.30522765E-01	3.30522765E-01	30000.00	± 0.00	24734.31	±0.00	24734.31
5	30	SVD	$f_4(t)$	9.87634072E-02	9.87634072E-02	1285.00	± 0.00	11605.66	±0.00	11605.66
5	30	SVD	$f_5(t)$	4.36825737E-01	4.36825737E-01	9802.00	± 0.00	25652.11	±0.00	25652.11
5	30	SVD	$f_6(t)$	6.54093192E-02	6.54093192E-02	30000.00	± 0.00	25763.66	±0.00	25763.66

Table B.2: Complete results of the algorithm comparison tests performed in Chapter 8. Here, m is the embedding dimension, k is the simplex dimension, and time is given in seconds.

Alg	k	m	**Init**	$f(t)$	**Max Avg Vol**	**Mean Avg Vol**	**Max Min Vol**	**Avg Min Vol**	**Avg Iterations**	**Time**
mSAP	1	2	rand		0.33356244	0.28368057	1.3735E-02	8.9331E-03	7186±2399	337.13
mSAP	1	2	SVD		0.29788170	0.29788170	8.4255E-03	8.4255E-03	9402±0	3.85
PCA	1	2			0.40344139	0.40344139	4.4405E-03	4.4405E-03		
RGP	1	2			0.00405466	0.00238679	5.0850E-05	8.4292E-06		
FLAG	1	2			0.41668110	0.41668110	4.4433E-03	4.4433E-03	0±0	1.36
SAP	1	2	rand		0.29954811	0.25411705	7.0788E-02	6.1399E-02	30000±0	192.36
SAP	1	2	SVD		0.29927939	0.29927939	7.8234E-02	7.8234E-02	30000 ±0	6.70
VOL	1	2	rand	$f_0(t)$	0.18420361	0.13255355	2.6226E-03	4.9609E-04	30000±0	8076.49
VOL	1	2	rand	$f_1(t)$	0.29931197	0.19606030	6.3412E-03	1.4600E-03	20463±7422	5686.21
VOL	1	2	rand	$f_2(t)$	0.29381125	0.18129524	2.7246E-03	7.6278E-04	29732±1441	8274.00
VOL	1	2	rand	$f_3(t)$	0.20103848	0.13316924	4.1813E-03	1.0335E-03	28257±4117	7438.76
VOL	1	2	rand	$f_4(t)$	0.21294017	0.13382224	2.5371E-03	4.8608E-04	30000±0	8420.04
VOL	1	2	rand	$f_5(t)$	0.21898978	0.15521846	3.0635E-02	1.4582E-02	4713±2415	4815.25
VOL	1	2	rand	$f_6(t)$	0.21890944	0.16044856	3.2569E-03	7.1419E-04	29698±1625	18811.49
VOL	1	2	SVD	$f_0(t)$	0.10220347	0.10220347	3.6250E-04	3.6250E-04	30000±0	359.32
VOL	1	2	SVD	$f_1(t)$	0.17941851	0.17941851	5.9164E-04	5.9164E-04	1033±0	158.09

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	1	2	SVD	$f_2(t)$	0.19365210	0.19365210	8.6226E-04	8.6226E-04	30000±0	370.41
VOL	1	2	SVD	$f_3(t)$	0.08207915	0.08207915	9.8569E-05	9.8569E-05	30000±0	404.30
VOL	1	2	SVD	$f_4(t)$	0.10246746	0.10246746	2.0775E-04	2.0775E-04	30000±0	416.02
VOL	1	2	SVD	$f_5(t)$	0.09252555	0.09252555	7.5506E-03	7.5506E-03	2562±0	12.06
VOL	1	2	SVD	$f_6(t)$	0.10584590	0.10584590	3.2225E-04	3.2225E-04	30000±0	72.82
mSAP	1	3	rand		0.44153682	0.41676832	2.9992E-02	2.2203E-02	5602±2060	433.49
mSAP	1	3	SVD		0.43273620	0.43273620	2.3364E-02	2.3364E-02	3090±0	2.31
PCA	1	3			0.51726711	0.51726711	6.2013E-03	6.2013E-03		
RGP	1	3			0.00717762	0.00397338	3.3349E-04	1.0034E-04		
FLAG	1	3			0.51933292	0.51933292	9.8255E-03	9.8255E-03	0±0	1.24
SAP	1	3	rand		0.40103841	0.35040521	1.7803E-01	1.5862E-01	29725±1480	222.60
SAP	1	3	SVD		0.37289982	0.37289982	1.7149E-01	1.7149E-01	30000±0	8.29
VOL	1	3	rand	$f_0(t)$	0.28284786	0.19798357	1.4558E-02	3.5622E-03	30000±0	7570.27
VOL	1	3	rand	$f_1(t)$	0.35587402	0.24552727	2.2712E-02	8.1397E-03	19659±7830	4935.94
VOL	1	3	rand	$f_2(t)$	0.37969232	0.23974724	1.5104E-02	6.2652E-03	29249±4042	8167.47
VOL	1	3	rand	$f_3(t)$	0.21354752	0.14622890	1.6898E-02	5.8395E-03	29530±2528	8021.42
VOL	1	3	rand	$f_4(t)$	0.33889104	0.21627508	1.6317E-02	2.9448E-03	30000±0	7445.89
VOL	1	3	rand	$f_5(t)$	0.25896821	0.18643136	6.3620E-02	3.6705E-02	4599±2082	4672.26

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	1	3	rand	$f_6(t)$	0.33904115	0.18972445	1.3120E-02	3.4084E-03	30000±0	18940.38
VOL	1	3	SVD	$f_0(t)$	0.18560545	0.18560545	9.9073E-03	9.9073E-03	30000±0	391.32
VOL	1	3	SVD	$f_1(t)$	0.35437305	0.35437305	9.4698E-03	9.4698E-03	7201±0	91.81
VOL	1	3	SVD	$f_2(t)$	0.35178111	0.35178111	7.9640E-03	7.9640E-03	30000±0	376.11
VOL	1	3	SVD	$f_3(t)$	0.26380230	0.26380230	1.5989E-02	1.5989E-02	30000 ±0	398.83
VOL	1	3	SVD	$f_4(t)$	0.18187704	0.18187704	8.9821E-03	8.9821E-03	30000±0	388.04
VOL	1	3	SVD	$f_5(t)$	0.22586481	0.22586481	5.1053E-02	5.1053E-02	7713±0	9.67
VOL	1	3	SVD	$f_6(t)$	0.15643573	0.15643573	2.3132E-03	2.3132E-03	30000 ±0	72.81
mSAP	1	5	rand		0.56061239	0.54509352	6.9182E-02	4.7340E-02	3135±1360	517.74
mSAP	1	5	SVD		0.53784223	0.53784223	3.8007E-02	3.8007E-02	2742±0	1.89
PCA	1	5			0.65492833	0.65492833	2.1631E-02	2.1631E-02		
RGP	1	5			0.00958895	0.00638331	1.3005E-03	6.6346E-04		
FLAG	1	5			0.65860812	0.65860812	2.4997E-02	2.4997E-02	0±0	1.37
SAP	1	5	rand		0.52089849	0.49205324	3.3762E-01		29555±2396	188.02
SAP	1	5	SVD		0.51235064	0.51235064	3.3395E-01	3.3395E-01	30000±0	6.05
VOL	1	5	rand	$f_0(t)$	0.46590033	0.31214607	4.8805E-02	1.2457E-02	1247±933	401.36
VOL	1	5	rand	$f_1(t)$	0.40984949	0.33263807	9.8295E-02	4.9679E-02	17640±8467	4299.30
VOL	1	5	rand	$f_2(t)$	0.43715047	0.35033557	7.2690E-02	3.3954E-02	29372 ±3376	7819.75

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	1	5	rand	$f_3(t)$	0.21427914	0.16103046	3.0216E-02	1.9571E-02	28285 ±4692	8236.06
VOL	1	5	rand	$f_4(t)$	0.43852602	0.31637737	3.7500E-02	1.2813E-02	2513 ±2229	940.49
VOL	1	5	rand	$f_5(t)$	0.27924048	0.21057757	9.0739E-02	6.1852E-02	7063±4592	6783.43
VOL	1	5	rand	$f_6(t)$	0.37214536	0.26268346	3.3571E-02	1.4692E-02	29230±3008	18640.79
VOL	1	5	SVD	$f_0(t)$	0.29466950	0.29466950	9.0344E-03	9.0344E-03	1020±0	11.33
VOL	1	5	SVD	$f_1(t)$	0.50484251	0.50484251	9.5927E-02	9.5927E-02	10047±0	118.19
VOL	1	5	SVD	$f_2(t)$	0.44166160	0.44166160	5.1838E-02	5.1838E-02	30000±0	402.12
VOL	1	5	SVD	$f_3(t)$	0.52117765	0.52117765	1.0434E-01	1.0434E-01	15774 ±0	245.79
VOL	1	5	SVD	$f_4(t)$	0.29467539	0.29467539	9.0327E-03	9.0327E-03	1041±0	14.56
VOL	1	5	SVD	$f_5(t)$	0.49827449	0.49827449	1.7532E-01	1.7532E-01	8402±0	10.59
VOL	1	5	SVD	$f_6(t)$	0.26887725	0.26887725	2.2723E-02	2.2723E-02	30000±0	47.38
mSAP	1	10	rand		0.56377097	0.54531930	7.7216E-02	5.3813E-02	1776±671	305.41
mSAP	1	10	SVD		0.54564115	0.54564115	4.1684E-02	4.1684E-02	1317±0	4.66
PCA	1	10			0.84296975	0.84296975	4.7247E-02	4.7247E-02		
RGP	1	10			0.01511400	0.01213332	4.7384E-03	2.6507E-03		
FLAG	1	10			0.84841222	0.84841222	6.7818E-02	6.7818E-02	0±0	1.81
SAP	1	10	rand		0.70966033	0.69206601	5.9833E-01	5.7911E-01	27564±6236	215.04
SAP	1	10	SVD		0.77338030	0.77338030	6.4427E-01	6.4427E-01	30000±0	8.02

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	1	10	rand	$f_0(t)$	0.58024179	0.52161537	2.6044E-01	1.4886E-01	1001±303	457.89
VOL	1	10	rand	$f_1(t)$	0.44869246	0.39307613	1.9444E-01	1.3368E-01	17005±8167	4104.23
VOL	1	10	rand	$f_2(t)$	0.49730564	0.40766669	1.7224E-01	1.0580E-01	29694±1643	7378.41
VOL	1	10	rand	$f_3(t)$	0.22918019	0.18365109	5.2202E-02	3.6868E-02	28009±4718	8200.99
VOL	1	10	rand	$f_4(t)$	0.59094814	0.49546540	1.9738E-01	1.3718E-01	1397±1437	453.14
VOL	1	10	rand	$f_5(t)$	0.30307108	0.23010010	1.1218E-01	8.6143E-02	5920±3160	5897.33
VOL	1	10	rand	$f_6(t)$	0.42853436	0.32727425	6.1123E-02	3.4801E-02	28302±5374	18231.25
VOL	1	10	SVD	$f_0(t)$	0.71586710	0.71586710	2.8870E-01	2.8870E-01	695±0	11.02
VOL	1	10	SVD	$f_1(t)$	0.76300389	0.76300389	4.5853E-01	4.5853E-01	8638±0	170.50
VOL	1	10	SVD	$f_2(t)$	0.73151955	0.73151955	3.8328E-01	3.8328E-01	30000±0	405.71
VOL	1	10	SVD	$f_3(t)$	0.76188785	0.76188785	4.0456E-01	4.0456E-01	30000±0	443.64
VOL	1	10	SVD	$f_4(t)$	0.71586710	0.71586710	2.8870E-01	2.8870E-01	695±0	12.03
VOL	1	10	SVD	$f_5(t)$	0.76870303	0.76870303	5.0693E-01	5.0693E-01	6401±0	20.95
VOL	1	10	SVD	$f_6(t)$	0.64573227	0.64573227	2.8716E-01	2.8716E-01	8559±0	18.97
mSAP	1	20	rand		0.55248886	0.54399134	7.5164E-02	5.7622E-02	1758±872	411.84
mSAP	1	20	SVD		0.54266437	0.54266437	5.5219E-02	5.5219E-02	1489±0	3.62
PCA	1	20			1.00000000	1.00000000	1.0000E+00	1.0000E+00		
RGP	1	20			0.03215551	0.02578761	1.6571E-02	1.0812E-02		

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
FLAG	1	20			1.00000000	1.00000000	1.0000E+00	1.0000E+00	0±0	1.76
SAP	1	20	rand		0.76908315	0.75984765	6.9947E-01	6.8516E-01	9904±514	203.87
SAP	1	20	SVD		1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.02
VOL	1	20	rand	$f_0(t)$	0.65062825	0.59536181	3.4459E-01	2.8305E-01	1088±382	368.15
VOL	1	20	rand	$f_1(t)$	0.49483863	0.43219551	2.3702E-01	1.9133E-01	15501±7839	4905.78
VOL	1	20	rand	$f_2(t)$	0.55283322	0.44817461	2.1625E-01	1.6547E-01	29249±3227	6435.72
VOL	1	20	rand	$f_3(t)$	0.24211295	0.20113026	6.7219E-02	5.4997E-02	29923±412	8348.04
VOL	1	20	rand	$f_4(t)$	0.68667817	0.61005431	3.9157E-01	3.0433E-01	1241±879	548.63
VOL	1	20	rand	$f_5(t)$	0.31432008	0.25858418	1.4138E-01	1.1405E-01	6323±4045	6297.88
VOL	1	20	rand	$f_6(t)$	0.42329203	0.35131314	9.2993E-02	6.2429E-02	28941±3898	18737.95
VOL	1	20	SVD	$f_0(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.13
VOL	1	20	SVD	$f_1(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.16
VOL	1	20	SVD	$f_2(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.15
VOL	1	20	SVD	$f_3(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.10
VOL	1	20	SVD	$f_4(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.18
VOL	1	20	SVD	$f_5(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.07
VOL	1	20	SVD	$f_6(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.06
mSAP	1	30	rand		0.55150863	0.54379841	8.9492E-02	6.8133E-02	1613±454	504.97

Alg	k	m	**Init**	$f(t)$	**Max Avg Vol**	**Mean Avg Vol**	**Max Min Vol**	**Avg Min Vol**	**Avg Iterations**	**Time**
mSAP	1	30	SVD		0.54019245	0.54019245	5.6942E-02	5.6942E-02	110±0	7.47
PCA	1	30			n/a	n/a	n/a	n/a		
RGP	1	30			0.04321046	0.03708930	2.2922E-02	1.7507E-02		
FLAG	1	30			1.00000000	1.00000000	1.0000E+00	1.0000E+00	0±0	1.20
SAP	1	30	rand		0.78742735	0.78313455	7.3070E-01	7.1992E-01	9846±825	273.49
SAP	1	30	SVD		1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.02
VOL	1	30	rand	$f_0(t)$	0.70562084	0.64996081	4.4839E-01	3.5950E-01	964±368	443.01
VOL	1	30	rand	$f_1(t)$	0.51810384	0.45823593	2.5917E-01	2.3329E-01	14959±6720	4167.59
VOL	1	30	rand	$f_2(t)$	0.55229893	0.49187855	2.5088E-01	2.0550E-01	28583±4700	8059.71
VOL	1	30	rand	$f_3(t)$	0.26978795	0.20392085	8.4519E-02	7.2055E-02	29351±3118	8199.73
VOL	1	30	rand	$f_4(t)$	0.70460710	0.64165759	4.2175E-01	3.5751E-01	115±928	410.67
VOL	1	30	rand	$f_5(t)$	0.32955211	0.27832811	1.5445E-01	1.3512E-01	8501±5551	7916.51
VOL	1	30	rand	$f_6(t)$	0.47616759	0.35872986	1.1077E-01	8.2381E-02	28930 ±4233	18902.79
VOL	1	30	SVD	$f_0(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.15
VOL	1	30	SVD	$f_1(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.16
VOL	1	30	SVD	$f_2(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.12
VOL	1	30	SVD	$f_3(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.24
VOL	1	30	SVD	$f_4(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.22

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	1	30	SVD	$f_5(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.13
VOL	1	30	SVD	$f_6(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.05
mSAP	2	2	rand		0.03609811	0.02441602	1.2442E-06	8.9748E-08	7618±2339	850.95
mSAP	2	2	SVD		0.02534022	0.02534022	3.1350E-07	3.1350E-07	13392±0	8.57
PCA	2	2			0.08043724	0.08043724	5.2138E-08	5.2138E-08		
RGP	2	2			0.00000936	0.00000338	1.3822E-10	6.2884E-12		
FLAG	2	2			0.08898850	0.08898850	3.0390E-09	3.0390E-09	0±0	240.72
SAP	2	2	rand		0.04422619	0.03380785	9.4365E-07	1.3282E-07	30000±0	170.52
SAP	2	2	SVD		0.04608539	0.04608539	3.4964E-10	3.4964E-10	30000±0	6.39
VOL	2	2	rand	$f_0(t)$	0.05202484	0.02440148	8.9158E-08	1.1543E-08	26002±10190	26030.81
VOL	2	2	rand	$f_1(t)$	0.04717850	0.02839715	9.0718E-08	1.6152E-08	17832±9162	19070.86
VOL	2	2	rand	$f_2(t)$	0.06999585	0.02762356	1.0636E-07	1.4424E-08	26003±10190	26063.64
VOL	2	2	rand	$f_3(t)$	0.05507915	0.02509275	3.5591E-07	3.4948E-08	15064±11056	14993.79
VOL	2	2	rand	$f_4(t)$	0.05582046	0.02408338	2.0761E-07	2.7486E-08	24005±11989	24312.99
VOL	2	2	rand	$f_5(t)$	0.06540693	0.03634915	2.5501E-05	8.9384E-06	565±324	704.06
VOL	2	2	rand	$f_6(t)$	0.05180828	0.03086374	2.2465E-07	3.5724E-08	30000±0	19502.92
VOL	2	2	SVD	$f_0(t)$	0.01545648	0.01545648	4.5521E-09	4.5521E-09	30000±0	125.59
VOL	2	2	SVD	$f_1(t)$	0.03726954	0.03726954	3.0951E-09	3.0951E-09	30000±0	502.57

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	2	2	SVD	$f_2(t)$	0.03531113	0.03531113	1.0119E-10	1.0119E-10	30000±0	473.97
VOL	2	2	SVD	$f_3(t)$	0.03171653	0.03171653	5.3866E-11	5.3866E-11	7385±0	176.75
VOL	2	2	SVD	$f_4(t)$	0.01530622	0.01530622	9.6340E-09	9.6340E-09	30000±0	498.49
VOL	2	2	SVD	$f_5(t)$	0.03617139	0.03617139	1.3717E-05	1.3717E-05	702±0	6.05
VOL	2	2	SVD	$f_6(t)$	0.01957450	0.01957450	9.1049E-09	9.1049E-09	30000±0	97.88
mSAP	2	3	rand		0.11307008	0.08499379	4.9047E-04	8.7113E-05	6650±2534	952.90
mSAP	2	3	SVD		0.07072883	0.07072883	1.7623E-05	1.7623E-05	2868±0	7.55
PCA	2	3			0.18156580	0.18156580	1.5165E-07	1.5165E-07		
RGP	2	3			0.00002259	0.00001035	3.2982E-08	7.8542E-09		
FLAG	2	3			0.18355852	0.18355852	4.4023E-06	4.4023E-06	0±0	158.99
SAP	2	3	rand		0.10036686	0.08583020	1.7078E-03	3.5333E-04	30000±0	176.91
SAP	2	3	SVD		0.09235997	0.09235997	6.9998E-04	6.9998E-04	30000±0	6.31
VOL	2	3	rand	$f_0(t)$	0.08522972	0.04455249	1.6184E-04	1.7149E-05	30000±0	27314.72
VOL	2	3	rand	$f_1(t)$	0.11337980	0.06808805	2.2213E-04	4.0610E-05	18015±6867	19666.59
VOL	2	3	rand	$f_2(t)$	0.09929215	0.05826452	1.4417E-04	2.9225E-05	29699±1620	27337.09
VOL	2	3	rand	$f_3(t)$	0.05987615	0.03956918	1.2028E-04	3.0043E-05	22422±8551	23305.48
VOL	2	3	rand	$f_4(t)$	0.08665538	0.04508418	6.5178E-05	7.7453E-06	29370±3391	26857.33
VOL	2	3	rand	$f_5(t)$	0.08843401	0.06257069	2.2748E-03	1.0705E-03	1879±1023	2115.46

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	2	3	rand	$f_6(t)$	0.07873357	0.04145687	7.3023E-05	1.2200E-05	30000±0	19337.44
VOL	2	3	SVD	$f_0(t)$	0.06639623	0.06639623	3.5824E-06	3.5824E-06	30000±0	85.61
VOL	2	3	SVD	$f_1(t)$	0.04732862	0.04732862	5.6097E-05	5.6097E-05	5396±0	20.55
VOL	2	3	SVD	$f_2(t)$	0.04783137	0.04783137	1.9085E-05	1.9085E-05	30000±0	83.17
VOL	2	3	SVD	$f_3(t)$	0.04441829	0.04441829	2.9250E-07	2.9250E-07	13777±0	229.72
VOL	2	3	SVD	$f_4(t)$	0.06607701	0.06607701	5.8646E-07	5.8646E-07	30000±0	429.38
VOL	2	3	SVD	$f_5(t)$	0.06305436	0.06305436	5.1980E-04	5.1980E-04	1604±0	7.64
VOL	2	3	SVD	$f_6(t)$	0.02350864	0.02350864	8.9110E-06	8.9110E-06	30000±0	99.32
mSAP	2	5	rand		0.23449975	0.22197711	5.6133E-03	3.1365E-03	2150±1202	991.33
mSAP	2	5	SVD		0.22664151	0.22664151	2.3877E-03	2.3877E-03	1343±0	34.73
PCA	2	5			0.35115397	0.35115397	9.4937E-04	9.4937E-04		
RGP	2	5			0.00011856	0.00003363	5.2388E-06	6.2692E-07		
FLAG	2	5			0.36023261	0.36023261	1.2498E-03	1.2498E-03	0±0	248.97
SAP	2	5	rand		0.21934371	0.20390085	7.0042E-02	3.5437E-02	30000±0	193.97
SAP	2	5	SVD		0.22392481	0.22392481	2.8835E-02	2.8835E-02	30000±0	6.74
VOL	2	5	rand	$f_0(t)$	0.19807313	0.12558943	1.4781E-03	3.8424E-04	30000±0	27493.26
VOL	2	5	rand	$f_1(t)$	0.22025976	0.16329658	9.4806E-03	4.8000E-03	1931±6103	20278.37
VOL	2	5	rand	$f_2(t)$	0.18679285	0.14014412	4.7883E-03	1.6449E-03	29752±1334	27435.01

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	2	5	rand	$f_3(t)$	0.07725105	0.05153140	3.9180E-03	1.2641E-03	24704±7747	25195.85
VOL	2	5	rand	$f_4(t)$	0.19203005	0.12250999	1.0661E-03	3.6203E-04	29755±1317	27280.92
VOL	2	5	rand	$f_5(t)$	0.14339579	0.09966804	2.3597E-02	1.4194E-02	4225±2824	4533.11
VOL	2	5	rand	$f_6(t)$	0.18369749	0.11305611	1.7592E-03	4.3625E-04	30000±0	19339.71
VOL	2	5	SVD	$f_0(t)$	0.12435383	0.12435383	2.5478E-04	2.5478E-04	30000±0	104.78
VOL	2	5	SVD	$f_1(t)$	0.17857749	0.17857749	9.3345E-03	9.3345E-03	22069±0	76.71
VOL	2	5	SVD	$f_2(t)$	0.14547258	0.14547258	2.6189E-03	2.6189E-03	30000±0	92.07
VOL	2	5	SVD	$f_3(t)$	0.17440971	0.17440971	1.0109E-02	1.0109E-02	5359±0	15.12
VOL	2	5	SVD	$f_4(t)$	0.12576652	0.12576652	2.6846E-04	2.6846E-04	30000 ±0	77.20
VOL	2	5	SVD	$f_5(t)$	0.18584725	0.18584725	3.0943E-02	3.0943E-02	5253±0	16.57
VOL	2	5	SVD	$f_6(t)$	0.05592069	0.05592069	1.9657E-04	1.9657E-04	30000±0	89.40
mSAP	2	10	rand		0.21682391	0.21062727	8.2079E-03	6.3653E-03	1820±617	838.86
mSAP	2	10	SVD		0.21098312	0.21098312	8.5530E-03	8.5530E-03	1368±0	43.88
PCA	2	10			0.66394816	0.66394816	2.1258E-02	2.1258E-02		
RGP	2	10			0.00021467	0.00014304	2.4805E-05	1.3731E-05		
FLAG	2	10			0.67877679	0.67877679	4.3477E-02	4.3477E-02	0±0	107.19
SAP	2	10	rand		0.47277171	0.44917216	3.0243E-01	2.6487E-01	30000±0	241.96
SAP	2	10	SVD		0.56908532	0.56908532	3.5763E-01	3.5763E-01	30000±0	8.42

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	2	10	rand	$f_0(t)$	0.49165183	0.36349841	7.1140E-02	2.4629E-02	1395±865	1432.23
VOL	2	10	rand	$f_1(t)$	0.39697723	0.30145176	9.4693E-02	6.0722E-02	17137±8276	18913.92
VOL	2	10	rand	$f_2(t)$	0.41592625	0.31550110	4.9803E-02	2.9897E-02	29435 ±3038	27282.59
VOL	2	10	rand	$f_3(t)$	0.09095925	0.06966536	9.8148E-03	5.5681E-03	27694±5252	26206.55
VOL	2	10	rand	$f_4(t)$	0.49956683	0.38063413	5.9249E-02	2.3756E-02	2041±1769	1480.50
VOL	2	10	rand	$f_5(t)$	0.16257037	0.13420986	4.9627E-02	3.9016E-02	6348±5595	6475.00
VOL	2	10	rand	$f_6(t)$	0.31167636	0.22269446	1.7013E-02	4.9827E-03	29804±760	19350.81
VOL	2	10	SVD	$f_0(t)$	0.50442858	0.50442858	6.8127E-02	6.8127E-02	700±0	4.71
VOL	2	10	SVD	$f_1(t)$	0.54704648	0.54704648	1.4443E-01	1.4443E-01	11838±0	46.22
VOL	2	10	SVD	$f_2(t)$	0.49961138	0.49961138	6.8482E-02	6.8482E-02	30000±0	93.99
VOL	2	10	SVD	$f_3(t)$	0.49662242	0.49662242	1.7823E-01	1.7823E-01	30000±0	90.92
VOL	2	10	SVD	$f_4(t)$	0.50442858	0.50442858	6.8127E-02	6.8127E-02	700±0	4.94
VOL	2	10	SVD	$f_5(t)$	0.51817626	0.51817626	3.0003E-01	3.0003E-01	10291±0	31.77
VOL	2	10	SVD	$f_6(t)$	0.35058239	0.35058239	1.5732E-02	1.5732E-02	30000±0	100.74
mSAP	2	20	rand		0.21351196	0.20281327	8.9375E-03	6.6957E-03	1325±442	802.23
mSAP	2	20	SVD		0.20695550	0.20695550	6.6365E-03	6.6365E-03	1636±0	14.84
PCA	2	20			1.00000000	1.00000000	1.0000E+00	1.0000E+00		
RGP	2	20			0.00079057	0.00062307	2.3777E-04	1.5128E-04		

Alg	k	m	**Init**	$f(t)$	**Max Avg Vol**	**Mean Avg Vol**	**Max Min Vol**	**Avg Min Vol**	**Avg Iterations**	**Time**
FLAG	2	20			1.00000000	1.00000000	1.0000E+00	1.0000E+00	0±0	236.09
SAP	2	20	rand		0.61266108	0.59012481	4.8628E-01	4.4806E-01	28100±5916	337.83
SAP	2	20	SVD		1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.05
VOL	2	20	rand	$f_0(t)$	0.64388909	0.58276641	2.8121E-01	2.1069E-01	1214 ±851	1004.97
VOL	2	20	rand	$f_1(t)$	0.43852842	0.38307336	1.5941E-01	1.2598E-01	16552±6260	18808.96
VOL	2	20	rand	$f_2(t)$	0.47769085	0.40866208	1.1701E-01	8.7879E-02	28559±4814	26611.97
VOL	2	20	rand	$f_3(t)$	0.10990566	0.08451044	1.5040E-02	1.1059E-02	30000±0	27357.75
VOL	2	20	rand	$f_4(t)$	0.66523070	0.59049524	2.9914E-01	2.1306E-01	2188±4489	1785.33
VOL	2	20	rand	$f_5(t)$	0.20660165	0.16812558	7.5225E-02	6.4050E-02	8906±8613	8361.85
VOL	2	20	rand	$f_6(t)$	0.34832148	0.23060754	2.2040E-02	1.1229E-02	28482±5147	19091.81
VOL	2	20	SVD	$f_0(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.13
VOL	2	20	SVD	$f_1(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.14
VOL	2	20	SVD	$f_2(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.21
VOL	2	20	SVD	$f_3(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.16
VOL	2	20	SVD	$f_4(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.16
VOL	2	20	SVD	$f_5(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.29
VOL	2	20	SVD	$f_6(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.22
mSAP	2	30	rand		0.21701495	0.20653388	1.0006E-02	7.3543E-03	1708±649	966.06

Alg	k	m	**Init**	$f(t)$	**Max Avg Vol**	**Mean Avg Vol**	**Max Min Vol**	**Avg Min Vol**	**Avg Iterations**	**Time**
mSAP	2	30	SVD		0.20677763	0.20677763	5.5615E-03	5.5615E-03	1478±0	77.62
PCA	2	30			n/a	n/a	n/a	n/a		
RGP	2	30			0.00205806	0.00141660	6.4271E-04	4.8447E-04		
FLAG	2	30			1.00000000	1.00000000	1.0000E+00	1.0000E+00	0±0	236.26
SAP	2	30	rand		0.61809280	0.59835777	4.9982E-01	4.7305E-01	9867±507	281.64
SAP	2	30	SVD		1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.05
VOL	2	30	rand	$f_0(t)$	0.73091055	0.64806440	3.8687E-01	3.1093E-01	1162±966	1311.71
VOL	2	30	rand	$f_1(t)$	0.49586003	0.41993297	1.9801E-01	1.6586E-01	19092±7561	20284.62
VOL	2	30	rand	$f_2(t)$	0.51026894	0.44356085	1.8554E-01	1.3072E-01	29996±21	27686.05
VOL	2	30	rand	$f_3(t)$	0.13197272	0.09706789	2.0799E-02	1.6203E-02	27239 ±6486	26147.07
VOL	2	30	rand	$f_4(t)$	0.72170078	0.65847976	3.9915E-01	3.0136E-01	2006±2430	2019.52
VOL	2	30	rand	$f_5(t)$	0.21999109	0.18975659	8.8589E-02	7.7889E-02	7671±5340	7595.14
VOL	2	30	rand	$f_6(t)$	0.35235513	0.24584531	3.6220E-02	1.9479E-02	27308±6184	18775.56
VOL	2	30	SVD	$f_0(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.15
VOL	2	30	SVD	$f_1(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.20
VOL	2	30	SVD	$f_2(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.12
VOL	2	30	SVD	$f_3(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.12
VOL	2	30	SVD	$f_4(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.13

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	2	30	SVD	$f_5(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.26
VOL	2	30	SVD	$f_6(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.27
mSAP	3	3	rand		0.01431983	0.01005532	6.6770E-09	7.1078E-10	5560±1633	1984.49
mSAP	3	3	SVD		0.00558318	0.00558318	1.5051E-13	1.5051E-13	3160±0	207.94
PCA	3	3			0.03191966	0.03191966	5.8267E-11	5.8267E-11		
RGP	3	3			3.6587E-08	1.0997E-08	3.4890E-15	4.5325E-16		
FLAG	3	3			0.03257789	0.03257789	4.0157E-10	4.0157E-10	0±0	3440.64
SAP	3	3	rand		0.01372314	0.01077817	8.2805E-08	5.4096E-09	30000±0	185.53
SAP	3	3	SVD		0.01114062	0.01114062	1.3791E-09	1.3791E-09	30000±0	6.31
VOL	3	3	rand	$f_0(t)$	0.02231949	0.00808329	9.2797E-10	1.0020E-10	8019±13254	7578.15
VOL	3	3	rand	$f_1(t)$	0.02110373	0.01392607	7.0359E-09	6.2972E-10	5441±8724	4997.59
VOL	3	3	rand	$f_2(t)$	0.02186956	0.01374255	1.7811E-09	2.1526E-10	9033±13726	9215.50
VOL	3	3	rand	$f_3(t)$	0.02319336	0.00839794	4.6376E-10	7.3485E-11	1809±5046	2018.24
VOL	3	3	rand	$f_4(t)$	0.02222129	0.00805314	1.0470E-09	1.1999E-10	10018±14129	10848.40
VOL	3	3	rand	$f_5(t)$	0.02015750	0.01347977	5.1292E-07	1.4773E-07	233±105	753.30
VOL	3	3	rand	$f_6(t)$	0.01508847	0.00870769	1.9009E-09	1.6005E-10	30000±0	19748.95
VOL	3	3	SVD	$f_0(t)$	0.00481294	0.00481294	1.1777E-12	1.1777E-12	30000±0	106.49
VOL	3	3	SVD	$f_1(t)$	0.02030716	0.02030716	2.4501E-11	2.4501E-11	37±0	5.44

Alg	k	m	**Init**	$f(t)$	**Max Avg Vol**	**Mean Avg Vol**	**Max Min Vol**	**Avg Min Vol**	**Avg Iterations**	**Time**
VOL	3	3	SVD	$f_2(t)$	0.02030716	0.02030716	2.4501E-11	2.4501E-11	37±0	5.55
VOL	3	3	SVD	$f_3(t)$	0.00860048	0.00860048	1.4035E-11	1.4035E-11	20±0	3.02
VOL	3	3	SVD	$f_4(t)$	0.00524168	0.00524168	6.3419E-12	6.3419E-12	30000±0	519.88
VOL	3	3	SVD	$f_5(t)$	0.01519736	0.01519736	1.2192E-07	1.2192E-07	186±0	14.86
VOL	3	3	SVD	$f_6(t)$	0.00711418	0.00711418	9.9718E-18	9.9718E-18	30000±0	121.06
mSAP	3	5	rand		0.07422597	0.06722053	1.4473E-04	3.9897E-05	2369±1177	2430.44
mSAP	3	5	SVD		0.06202679	0.06202679	5.9316E-06	5.9316E-06	1531±0	227.07
PCA	3	5			0.14552441	0.14552441	3.2188E-05	3.2188E-05		
RGP	3	5			0.00000037	0.00000013	7.1408E-10	1.6035E-10		
FLAG	3	5			0.15417498	0.15417498	5.3296E-05	5.3296E-05	0±0	4105.49
SAP	3	5	rand		0.07515475	0.06430080	1.5758E-03	5.3777E-04	29310±3713	191.59
SAP	3	5	SVD		0.07784837	0.07784837	1.8313E-04	1.8313E-04	30000±0	6.61
VOL	3	5	rand	$f_0(t)$	0.06483215	0.03925551	6.6037E-05	1.4704E-05	30000±0	27578.56
VOL	3	5	rand	$f_1(t)$	0.08036078	0.05730921	5.2403E-04	9.7828E-05	19642±6544	21576.90
VOL	3	5	rand	$f_2(t)$	0.07595380	0.04607912	3.4028E-04	4.0867E-05	30000±0	27553.88
VOL	3	5	rand	$f_3(t)$	0.02775429	0.01858816	1.0496E-04	2.8038E-05	17628±8456	19407.17
VOL	3	5	rand	$f_4(t)$	0.07187891	0.04260052	6.2092E-05	1.1533E-05	29244±4066	27650.57
VOL	3	5	rand	$f_5(t)$	0.05836189	0.04390957	2.1067E-03	1.1787E-03	1591±841	2201.74

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	3	5	rand	$f_6(t)$	0.06606423	0.03888715	2.0888E-04	1.9351E-05	28112±6302	19085.94
VOL	3	5	SVD	$f_0(t)$	0.02270320	0.02270320	4.7016E-08	4.7016E-08	30000±0	106.33
VOL	3	5	SVD	$f_1(t)$	0.05563482	0.05563482	1.3292E-04	1.3292E-04	24420±0	93.54
VOL	3	5	SVD	$f_2(t)$	0.04586225	0.04586225	5.1027E-05	5.1027E-05	30000±0	120.99
VOL	3	5	SVD	$f_3(t)$	0.05591712	0.05591712	6.3176E-05	6.3176E-05	28434±0	97.53
VOL	3	5	SVD	$f_4(t)$	0.02303204	0.02303204	2.8990E-08	2.8990E-08	30000±0	109.87
VOL	3	5	SVD	$f_5(t)$	0.06301351	0.06301351	2.7125E-03	2.7125E-03	504±0	15.53
VOL	3	5	SVD	$f_6(t)$	0.02982061	0.02982061	6.9682E-06	6.9682E-06	30000±0	100.39
mSAP	3	10	rand		0.07132097	0.06803753	8.6290E-04	5.7614E-04	1588±502	2209
mSAP	3	10	SVD		0.07314491	0.07314491	4.2321E-04	4.2321E-04	1358±0	170
PCA	3	10			0.48480614	0.48480614	2.3422E-03	2.3422E-03		
RGP	3	10			0.00000219	0.00000135	1.4349E-07	5.9599E-08		
FLAG	3	10			0.50661738	0.50661738	1.2787E-02	1.2787E-02	0±0	2842.63
SAP	3	10	rand		0.29523946	0.27093954	9.1503E-02	5.8651E-02	30000±0	245.19
SAP	3	10	SVD		0.39351163	0.39351163	9.6124E-02	9.6124E-02	30000±0	8.73
VOL	3	10	rand	$f_0(t)$	0.32022278	0.24386460	5.1239E-03	1.1921E-03	14624±14388	16484.07
VOL	3	10	rand	$f_1(t)$	0.28974504	0.24226782	4.3507E-02	2.3409E-02	21194±6552	22607.28
VOL	3	10	rand	$f_2(t)$	0.31886440	0.24248048	1.9816E-02	7.7587E-03	30000±0	27582.45

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	3	10	rand	$f_3(t)$	0.05151009	0.03557878	2.2958E-03	1.1799E-03	27904±4629	27126.50
VOL	3	10	rand	$f_4(t)$	0.33852639	0.24864473	5.1583E-03	1.2045E-03	16961±13702	19310.77
VOL	3	10	rand	$f_5(t)$	0.12014594	0.09780144	2.3878E-02	1.9763E-02	4591±5093	5320.83
VOL	3	10	rand	$f_6(t)$	0.23630109	0.18226280	1.8267E-03	6.2050E-04	30000±0	19634.54
VOL	3	10	SVD	$f_0(t)$	0.30197345	0.30197345	5.4139E-03	5.4139E-03	733±0	18.22
VOL	3	10	SVD	$f_1(t)$	0.37636521	0.37636521	3.8276E-02	3.8276E-02	30000±0	99.02
VOL	3	10	SVD	$f_2(t)$	0.32991916	0.32991916	1.3592E-02	1.3592E-02	30000±0	131.47
VOL	3	10	SVD	$f_3(t)$	0.32001429	0.32001429	7.1081E-02	7.1081E-02	25513±0	94.66
VOL	3	10	SVD	$f_4(t)$	0.30197345	0.30197345	5.4139E-03	5.4139E-03	733±0	15.82
VOL	3	10	SVD	$f_5(t)$	0.35515693	0.35515693	1.3753E-01	1.3753E-01	8381±0	38.44
VOL	3	10	SVD	$f_6(t)$	0.26383860	0.26383860	3.1820E-03	3.1820E-03	30000±0	114.91
mSAP	3	20	rand		0.07077651	0.06660894	1.0582E-03	8.2474E-04	1494±458	2012.72
mSAP	3	20	SVD		0.06623353	0.06623353	1.0516E-03	1.0516E-03	1290±0	166.28
PCA	3	20			1.00000000	1.00000000	1.0000E+00	1.0000E+00		
RGP	3	20			0.00003356	0.00001387	6.7007E-06	2.3440E-06		
FLAG	3	20			1.00000000	1.00000000	1.0000E+00	1.0000E+00	0±0	1691.34
SAP	3	20	rand		0.41060843	0.39020918	2.1034E-01	1.7750E-01	9909±488	195.11
SAP	3	20	SVD		1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.23

Alg	k	m	Init	$f(t)$	**Max Avg Vol**	**Mean Avg Vol**	**Max Min Vol**	**Avg Min Vol**	**Avg Iterations**	**Time**
VOL	3	20	rand	$f_0(t)$	0.80591383	0.66829001	3.9785E-01	2.2473E-01	1591±1057	1876.96
VOL	3	20	rand	$f_1(t)$	0.44934205	0.38499169	1.4469E-01	9.7866E-02	17174±6537	19621.37
VOL	3	20	rand	$f_2(t)$	0.52419558	0.41130951	1.2673E-01	6.5453E-02	28313±4864	27062.66
VOL	3	20	rand	$f_3(t)$	0.06642811	0.04931045	6.1560E-03	3.5461E-03	28553±5634	27313.98
VOL	3	20	rand	$f_4(t)$	0.71486704	0.65104743	3.2266E-01	2.2980E-01	1786±1631	2159.18
VOL	3	20	rand	$f_5(t)$	0.15423297	0.13820047	5.5167E-02	4.4008E-02	8848±8200	8652.80
VOL	3	20	rand	$f_6(t)$	0.36093804	0.22692118	1.8615E-02	4.6365E-03	29103±4157	19487.52
VOL	3	20	SVD	$f_0(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.66
VOL	3	20	SVD	$f_1(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.59
VOL	3	20	SVD	$f_2(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.54
VOL	3	20	SVD	$f_3(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.64
VOL	3	20	SVD	$f_4(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.55
VOL	3	20	SVD	$f_5(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.84
VOL	3	20	SVD	$f_6(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.62
mSAP	3	30	rand		0.07343305	0.06775903	1.4176E-03	1.0171E-03	1698±635	2132.40
mSAP	3	30	SVD		0.06792662	0.06792662	9.4428E-04	9.4428E-04	2259±0	358.38
PCA	3	30			n/a	n/a	n/a	n/a		
RGP	3	30			0.00008370	0.00005226	2.2014E-05	1.2714E-05		

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
FLAG	3	30			1.00000000	1.00000000	1.0000E+00	1.0000E+00	0±0	3584.80
SAP	3	30	rand		0.06792662	0.06792662	9.4428E-04	9.4428E-04	2259±0	358.38
SAP	3	30	SVD		1.00000000	1.00000000	1.0000E+00	1.0000E+00	0±0	3584.80
VOL	3	30	rand	$f_0(t)$	0.81877747	0.74408222	4.9377E-01	3.8834E-01	1272±961	1685.86
VOL	3	30	rand	$f_1(t)$	0.47342283	0.43874683	1.9013E-01	1.4434E-01	17877±7918	20102.79
VOL	3	30	rand	$f_2(t)$	0.55621245	0.47874711	1.7501E-01	1.1949E-01	29954±245	27798.26
VOL	3	30	rand	$f_3(t)$	0.07486918	0.05907375	8.8430E-03	5.5112E-03	29491±2717	27635.12
VOL	3	30	rand	$f_4(t)$	0.81194281	0.73266025	4.6669E-01	3.8961E-01	1698±2119	2134.61
VOL	3	30	rand	$f_5(t)$	0.18375859	0.16127398	6.5559E-02	5.9713E-02	7836±6749	8057.43
VOL	3	30	rand	$f_6(t)$	0.34165921	0.21139602	1.9272E-02	7.2081E-03	26792±6617	18917.11
VOL	3	30	SVD	$f_0(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.53
VOL	3	30	SVD	$f_1(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.40
VOL	3	30	SVD	$f_2(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.52
VOL	3	30	SVD	$f_3(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.59
VOL	3	30	SVD	$f_4(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.48
VOL	3	30	SVD	$f_5(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.76
VOL	3	30	SVD	$f_6(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	0.80
mSAP	5	5	rand		0.00107947	0.00085289	1.6658E-12	9.0933E-14	1988±1234	48481.94

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
mSAP	5	5	SVD		0.00118930	0.00118930	3.1775E-14	3.1775E-14	3290±0	4400.44
PCA	5	5			0.00647336	0.00647336	2.5545E-13	2.5545E-13		
RGP	5	5			1.8581E-12	3.9689E-13	4.9075E-21	2.9020E-22		
FLAG	5	5			0.00752256	0.00752256	1.4485E-12	1.4485E-12	0±0	193456.39
SAP	5	5	rand		0.00237457	0.00166499	1.8026E-11	2.3214E-12	29982±94	218.87
SAP	5	5	SVD		0.00256078	0.00256078	2.4030E-12	2.4030E-12	30000±0	7.31
VOL	5	5	rand	$f_0(t)$	0.00446829	0.00200792	2.7121E-11	4.3871E-12	23016±12658	26620.24
VOL	5	5	rand	$f_1(t)$	0.00550401	0.00271813	3.3327E-11	7.1564E-12	7815±6883	11479.71
VOL	5	5	rand	$f_2(t)$	0.00525082	0.00241630	2.5098E-11	5.9697E-12	19932±14053	24570.89
VOL	5	5	rand	$f_3(t)$	0.00295147	0.00220932	2.7238E-12	5.7367E-13	10481±8698	15903.95
VOL	5	5	rand	$f_4(t)$	0.00482377	0.00251514	3.2305E-11	7.7737E-12	17027±14835	21386.33
VOL	5	5	rand	$f_5(t)$	0.00389510	0.00252525	2.8443E-11	5.8263E-12	1215±5345	5195.15
VOL	5	5	rand	$f_6(t)$	0.00247732	0.00160817	1.5607E-12	2.0181E-13	29961±205	21432.08
VOL	5	5	SVD	$f_0(t)$	0.00391384	0.00391384	1.2930E-11	1.2930E-11	60±0	65.45
VOL	5	5	SVD	$f_1(t)$	0.00199140	0.00199140	1.0009E-13	1.0009E-13	6886±0	426.58
VOL	5	5	SVD	$f_2(t)$	0.00159925	0.00159925	4.0350E-14	4.0350E-14	30000±0	676.86
VOL	5	5	SVD	$f_3(t)$	0.00243739	0.00243739	1.5768E-12	1.5768E-12	20566±0	566.80
VOL	5	5	SVD	$f_4(t)$	0.00391384	0.00391384	1.2930E-11	1.2930E-11	60±0	244.15

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	5	5	SVD	$f_5(t)$	0.00182231	0.00182231	7.3670E-13	7.3670E-13	185±0	117.77
VOL	5	5	SVD	$f_6(t)$	0.00092418	0.00092418	1.3488E-15	1.3488E-15	30000±0	205.80
mSAP	5	10	rand		0.00243399	0.00211273	1.1224E-06	4.3361E-07	1673±676	48383.25
mSAP	5	10	SVD		0.00221461	0.00221461	2.7225E-07	2.7225E-07	2268±0	4399.45
PCA	5	10			0.19759362	0.19759362	7.3756E-05	7.3756E-05		
RGP	5	10			3.0831E-10	9.8411E-11	4.1377E-12	8.5514E-13		
FLAG	5	10			0.21891543	0.21891543	2.5530E-03	2.5530E-03	0±0	162076.79
SAP	5	10	rand		0.08795755	0.07360555	4.9101E-03	2.6542E-03	29723±1487	266.69
SAP	5	10	SVD		0.14853920	0.14853920	7.5851E-03	7.5851E-03	30000±0	9.40
VOL	5	10	rand	$f_0(t)$	0.11915928	0.07482243	1.1893E-04	3.0047E-05	25104±10946	6196.72
VOL	5	10	rand	$f_1(t)$	0.12912685	0.10651241	5.0940E-03	1.7196E-03	20228±7284	24539.91
VOL	5	10	rand	$f_2(t)$	0.11173659	0.09153336	1.2703E-03	2.4630E-04	29967±172	28780.18
VOL	5	10	rand	$f_3(t)$	0.01695961	0.01240395	2.5795E-04	1.0549E-04	23013±8616	26785.05
VOL	5	10	rand	$f_4(t)$	0.10625562	0.06940750	2.7148E-04	3.3105E-05	24877±10944	27294.33
VOL	5	10	rand	$f_5(t)$	0.06137446	0.04684635	7.2023E-03	4.7932E-03	4338±7125	8137.17
VOL	5	10	rand	$f_6(t)$	0.11211197	0.06711901	6.8132E-04	6.7797E-05	27522±7789	20449.71
VOL	5	10	SVD	$f_0(t)$	0.07771362	0.07771362	4.1707E-05	4.1707E-05	30000±0	215.41
VOL	5	10	SVD	$f_1(t)$	0.15783010	0.15783010	1.5475E-03	1.5475E-03	10247±0	146.73

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	5	10	SVD	$f_2(t)$	0.14566402	0.14566402	1.2637E-04	1.2637E-04	30000±0	244.14
VOL	5	10	SVD	$f_3(t)$	0.07715838	0.07715838	2.4478E-03	2.4478E-03	30000±0	227.03
VOL	5	10	SVD	$f_4(t)$	0.07797844	0.07797844	4.8169E-05	4.8169E-05	30000±0	202.36
VOL	5	10	SVD	$f_5(t)$	0.12542250	0.12542250	1.5656E-02	1.5656E-02	3500±0	126.38
VOL	5	10	SVD	$f_6(t)$	0.04107370	0.04107370	5.6185E-05	5.6185E-05	30000±0	200.86
mSAP	5	20	rand		0.00278250	0.00244210	5.3622E-06	2.4091E-06	1476±567	46649.77
mSAP	5	20	SVD		0.00244398	0.00244398	2.0884E-06	2.0884E-06	2466±0	3810.09
PCA	5	20			1.00000000	1.00000000	1.0000E+00	1.0000E+00		
RGP	5	20			1.8954E-08	6.3307E-09	2.2096E-09	5.0420E-10		
FLAG	5	20			1.00000000	1.00000000	1.0000E+00	1.0000E+00	0±0	175778.67
SAP	5	20	rand		0.22788223	0.19636664	7.0432E-02	5.1894E-02	9933±358	231.52
SAP	5	20	SVD		1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	1.70
VOL	5	20	rand	$f_0(t)$	0.89342579	0.81975384	6.6390E-01	5.2610E-01	708±350	3467.53
VOL	5	20	rand	$f_1(t)$	0.51594064	0.44359672	1.6656E-01	1.0509E-01	15419±7378	5835.57
VOL	5	20	rand	$f_2(t)$	0.57585980	0.48247262	1.2946E-01	6.5921E-02	30000±0	7911.51
VOL	5	20	rand	$f_3(t)$	0.04417544	0.02948000	2.8948E-03	1.1637E-03	29848±817	7569.32
VOL	5	20	rand	$f_4(t)$	0.90186875	0.83549595	6.7247E-01	5.1877E-01	915±881	3526.48
VOL	5	20	rand	$f_5(t)$	0.15533110	0.12111325	5.2195E-02	4.0757E-02	8826±9229	11225.33

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	5	20	rand	$f_6(t)$	0.50514774	0.25995313	4.9165E-02	5.0615E-03	28870±3446	21068.58
VOL	5	20	SVD	$f_0(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.56
VOL	5	20	SVD	$f_1(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.47
VOL	5	20	SVD	$f_2(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.45
VOL	5	20	SVD	$f_3(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	4.00
VOL	5	20	SVD	$f_4(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.82
VOL	5	20	SVD	$f_5(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.99
VOL	5	20	SVD	$f_6(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.68
mSAP	5	30	rand		0.00316300	0.00266805	1.1096E-05	5.4341E-06	1580±615	52790.07
mSAP	5	30	SVD		0.00270251	0.00270251	5.0132E-06	5.0132E-06	1458±0	4295.51
PCA	5	30			n/a	n/a	n/a	n/a		
RGP	5	30			1.0727E-07	5.7109E-08	2.4660E-08	9.5915E-09		
FLAG	5	30			1.00000000	1.00000000	1.0000E+00	1.0000E+00	0±0	181180.57
SAP	5	30	rand		0.28205396	0.25179004	1.3503E-01	9.8833E-02	28232±4809	499.41
SAP	5	30	SVD		1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	1.26
VOL	5	30	rand	$f_0(t)$	0.94925006	0.91119235	8.2534E-01	7.7190E-01	728±647	3543.37
VOL	5	30	rand	$f_1(t)$	0.61797064	0.53174429	2.7964E-01	1.7844E-01	19553±7949	6718.11
VOL	5	30	rand	$f_2(t)$	0.69847776	0.57834797	2.4708E-01	1.5494E-01	28628±3866	7915.55

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	5	30	rand	$f_3(t)$	0.04775830	0.03874951	5.6601E-03	2.6083E-03	28471±4800	7954.08
VOL	5	30	rand	$f_4(t)$	0.95074347	0.90989219	8.1836E-01	7.6900E-01	752±519	3812.80
VOL	5	30	rand	$f_5(t)$	0.17678376	0.15726890	7.4517E-02	6.4251E-02	8802±7922	11391.88
VOL	5	30	rand	$f_6(t)$	0.46418394	0.27232835	2.5610E-02	6.3476E-03	28670±5109	21244.07
VOL	5	30	SVD	$f_0(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	4.26
VOL	5	30	SVD	$f_1(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.62
VOL	5	30	SVD	$f_2(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.76
VOL	5	30	SVD	$f_3(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.95
VOL	5	30	SVD	$f_4(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.70
VOL	5	30	SVD	$f_5(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	4.02
VOL	5	30	SVD	$f_6(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	3.87
mSAP	10	10	rand		n/a	n/a	n/a	n/a		
mSAP	10	10	SVD		1.0319E-10	1.0319E-10	3.9261E-29	3.9261E-29	3613±0	26942.01
PCA	10	10			0.00071233	0.00071233	2.4498E-17	2.4498E-17		
RGP	10	10			1.1794E-22	3.0742E-23	5.8434E-33	5.2827E-34		
FLAG	10	10			n/a	n/a	n/a			
SAP	10	10	rand		0.00012700	0.00006428	9.0249E-15	1.5121E-15	29258±3995	387.16
SAP	10	10	SVD		0.00049839	0.00049839	4.6931E-15	4.6931E-15	30000±0	14.00

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
VOL	10	10	rand	$f_0(t)$	0.00041099	0.00023749	7.4869E-15	1.1651E-15	30000±0	19649.47
VOL	10	10	rand	$f_1(t)$	0.00042726	0.00033411	5.5220E-14	3.9929E-15	8265±2301	16076.61
VOL	10	10	rand	$f_2(t)$	0.00035923	0.00025293	1.3642E-14	9.4854E-16	28514±4316	20190.76
VOL	10	10	rand	$f_3(t)$	0.00047237	0.00028389	3.1948E-14	1.8533E-15	2489±2563	15378.95
VOL	10	10	rand	$f_4(t)$	0.00038765	0.00023102	6.1286E-14	3.5582E-15	30000±0	20270.20
VOL	10	10	rand	$f_5(t)$	0.00053320	0.00040877	1.0506E-13	1.4271E-14	4576±8982	19722.99
VOL	10	10	rand	$f_6(t)$	0.00032311	0.00022844	2.1726E-14	1.5681E-15	29953±248	29327.24
VOL	10	10	SVD	$f_0(t)$	0.00020866	0.00020866	5.4480E-15	5.4480E-15	30000±0	633.65
VOL	10	10	SVD	$f_1(t)$	0.00030966	0.00030966	1.9456E-15	1.9456E-15	5744±0	1062.42
VOL	10	10	SVD	$f_2(t)$	0.00025438	0.00025438	9.5891E-17	9.5891E-17	24349±0	1240.88
VOL	10	10	SVD	$f_3(t)$	0.00022034	0.00022034	5.5460E-16	5.5460E-16	4659±0	568.73
VOL	10	10	SVD	$f_4(t)$	0.00020670	0.00020670	8.8557E-16	8.8557E-16	30000±0	610.85
VOL	10	10	SVD	$f_5(t)$	0.00038238	0.00038238	2.3826E-14	2.3826E-14	30000±0	593.39
VOL	10	10	SVD	$f_6(t)$	0.00020010	0.00020010	4.0850E-16	4.0850E-16	30000±0	609.89
mSAP	10	20	rand		n/a	n/a	n/a	n/a		
mSAP	10	20	SVD		5.6233E-09	5.6233E-09	3.4491E-16	3.4491E-16	3894±0	27767.30
PCA	10	20			1.00000000	1.00000000	1.00000000	1.00000000		
RGP	10	20			1.8559E-17	5.7439E-18	4.9807E-19	1.4516E-19		

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
FLAG	10	20			n/a	n/a	n/a			
SAP	10	20	rand		0.00924065	0.00651884	6.6831E-04	3.6044E-04	9874±677	341.07
SAP	10	20	SVD		1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	7.25
VOL	10	20	rand	$f_0(t)$	1.00000000	0.99938287	1.0000E+00	9.9857E-01	364±129	15535.22
VOL	10	20	rand	$f_1(t)$	0.93627517	0.87282515	8.8668E-01	7.3743E-01	17859±8545	18584.00
VOL	10	20	rand	$f_2(t)$	0.99264549	0.93386006	9.8381E-01	8.5129E-01	24568±6732	19528.83
VOL	10	20	rand	$f_3(t)$	0.04227855	0.03515666	7.5443E-03	3.6449E-03	30000±0	20810.76
VOL	10	20	rand	$f_4(t)$	0.99982913	0.99933400	9.9959E-01	9.9844E-01	365±129	15439.49
VOL	10	20	rand	$f_5(t)$	0.27381551	0.22272605	1.9136E-01	1.3393E-01	22399±10696	27914.67
VOL	10	20	rand	$f_6(t)$	0.88761396	0.36437105	7.3231E-01	1.0204E-01	18365±9938	25700.41
VOL	10	20	SVD	$f_0(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	16.02
VOL	10	20	SVD	$f_1(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	57.65
VOL	10	20	SVD	$f_2(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	16.28
VOL	10	20	SVD	$f_3(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	15.50
VOL	10	20	SVD	$f_4(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	41.80
VOL	10	20	SVD	$f_5(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	16.45
VOL	10	20	SVD	$f_6(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	16.84
mSAP	10	30	rand		n/a	n/a	n/a	n/a		

Alg	k	m	Init	$f(t)$	Max Avg Vol	Mean Avg Vol	Max Min Vol	Avg Min Vol	Avg Iterations	Time
mSAP	10	30	SVD		1.8410E-08	1.8410E-08	3.4295E-14	3.4295E-14	1659±0	26402.20
PCA	10	30			n/a	n/a	n/a	n/a		
RGP	10	30			1.9621E-15	1.0090E-15	1.7908E-16	8.5243E-17		
FLAG	10	30			n/a	n/a	n/a			
SAP	10	30	rand		0.02873905	0.02432665	5.8491E-03	3.7620E-03	30000±0	653.47
SAP	10	30	SVD		1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	5.78
VOL	10	30	rand	$f_0(t)$	0.99978007	0.99949901	9.9942E-01	9.9886E-01	362±121	16294.02
VOL	10	30	rand	$f_1(t)$	0.97263400	0.93594521	9.5305E-01	8.6412E-01	18535±11104	18793.47
VOL	10	30	rand	$f_2(t)$	0.99370443	0.96357165	9.8568E-01	9.1714E-01	20819±7423	19262.52
VOL	10	30	rand	$f_3(t)$	0.07289881	0.05872938	1.8565E-02	1.0283E-02	30000±0	21048.45
VOL	10	30	rand	$f_4(t)$	0.99988443	0.99948463	9.9972E-01	9.9883E-01	371±137	16376.03
VOL	10	30	rand	$f_5(t)$	0.42351800	0.34358931	3.4546E-01	2.5400E-01	29090±2851	30139.23
VOL	10	30	rand	$f_6(t)$	0.94096787	0.37070740	8.5270E-01	1.2534E-01	17700±10492	27268.85
VOL	10	30	SVD	$f_0(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	16.58
VOL	10	30	SVD	$f_1(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	15.55
VOL	10	30	SVD	$f_2(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	15.99
VOL	10	30	SVD	$f_3(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	16.85
VOL	10	30	SVD	$f_4(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	16.46

Alg	k	m	**Init**	$f(t)$	**Max Avg Vol**	**Mean Avg Vol**	**Max Min Vol**	**Avg Min Vol**	**Avg Iterations**	**Time**
VOL	10	30	SVD	$f_5(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	17.29
VOL	10	30	SVD	$f_6(t)$	1.00000000	1.00000000	1.0000E+00	1.0000E+00	1±0	16.36