RESEARCH ON COMBINATORIAL STATISTICS:
CROSSINGS AND NESTINGS IN DISCRETE STRUCTURES

A Dissertation

by

SVETLANA POZNANOVIKJ

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2010

Major Subject: Mathematics

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Approved by:

Chair of Committee,	Catherine H. Yan
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ABSTRACT

Research on Combinatorial Statistics:

Crossings and Nestings in Discrete Structures. (August 2010)

Svetlana Poznanovikj, B.S., Ss. Cyril and Methodius University

Chair of Advisory Committee: Dr. Catherine H. Yan

We study the distribution of combinatorial statistics that exhibit a structure of crossings and nesting in various discrete structures, in particular, in set partitions, matchings, and fillings of moon polyominoes with entries 0 and 1.

Let π and λ be two set partitions with the same number of blocks. Assume π is a partition of $[n]$. For any integers $l, m \geq 0$, let $\mathcal{T}(\pi, l)$ be the set of partitions of $[n+l]$ whose restrictions to the last n elements are isomorphic to π, and $\mathcal{T}(\pi, l, m)$ the subset of $\mathcal{T}(\pi, l)$ consisting of those partitions with exactly m blocks. Similarly define $\mathcal{T}(\lambda, l)$ and $\mathcal{T}(\lambda, l, m)$. We prove that if the statistic cr (ne), the number of crossings (nestings) of two edges, coincides on the sets $\mathcal{T}(\pi, l)$ and $\mathcal{T}(\lambda, l)$ for $l = 0, 1$, then it coincides on $\mathcal{T}(\pi, l, m)$ and $\mathcal{T}(\lambda, l, m)$ for all $l, m \geq 0$. These results extend the ones obtained by Klazar on the distribution of crossings and nestings for matchings.

Moreover, we give a bijection between partially directed paths in the symmetric wedge $y = \pm x$ and matchings, which sends north steps to nestings. This gives a bijective proof of a result of E. J. Janse van Rensburg, T. Prellberg, and A. Rechnitzer that was first discovered through the corresponding generating functions: the number of partially directed paths starting at the origin confined to the symmetric wedge $y = \pm x$ with k north steps is equal to the number of matchings on $[2n]$ with k nestings.

Furthermore, we propose a major index statistic on 01-fillings of moon polyominoes which, when specialized to certain shapes, reduces to the major index for permutations and set partitions. We consider the set $\mathbf{F}(\mathcal{M}, \mathbf{s}; A)$ of all 01-fillings of a moon polyomino $\mathcal{M}$ with given column sum $\mathbf{s}$ whose empty rows are A, and prove that this major index has the same distribution as the number of north-east chains, which are the natural extension of inversions (resp. crossings) for permutations (resp. set partitions). Hence our result generalizes the classical equidistribution results for the permutation statistics inv and maj. Two proofs are presented. The first is an algebraic one using generating functions, and the second is a bijection on 01-fillings of moon polyominoes in the spirit of Foata's second fundamental transformation on words and permutations.

To Mom and Dad

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CHAPTER I

INTRODUCTION

In this chapter we will introduce the notion of crossings in several combinatorial structures and summarize the results that have been obtained about them by various authors. The objects of main interest for us are set partitions, matchings as an interesting subclass, and fillings of moon polyominoes. Since they can be viewed as a generalization of permutations and words, the first section is dedicated to them. Only results that have motivated the work for this dissertation are presented here and this is by no means an exhaustive list of results in the area.

1.1. Mahonian statistics on words and permutations

Given a multiset S of n positive integers, a word on S is a sequence $w = w_1 w_2 \dots w_n$ that reorders the elements in S. When $S = [n] := \{1, \dots, n\}$ the word is a permutation. A pair (w_i, w_j) is called an *inversion* of w if $i < j$ and $w_i > w_j$. One well-known statistic on words and permutations is $\mathrm{inv}(w)$, defined as the number of inversions of w. The *descent set* and *descent statistic* of a word w are defined as

$$\mathrm{Des}(w) = \{i : 1 \leq i \leq n-1, w_i > w_{i+1}\}, \qquad \mathrm{des}(w) = \#\mathrm{Des}(w).$$

In [35] MacMahon defined the *major index* statistic for a word w as

$$\mathrm{maj}(w) = \sum_{i \in \mathrm{Des}(w)} i,$$

and showed the remarkable result that its distribution over all words on S is equal to that of the inversion number over the same set. Precisely, for the set W_S of all words

This dissertation follows the style of SIAM Journal on Discrete Mathematics.

on $S = \{1^{m_1}, \ldots, k^{m_k}\}$,

$$\sum_{w \in W_S} q^{\mathrm{maj}(w)} = \sum_{w \in W_S} q^{\mathrm{inv}(w)} = \binom{m_1 + \cdots + m_k}{m_1, \ldots, m_k}_q. \tag{1.1}$$

The first proof of (1.1) given by MacMahon relied on combinatorial analysis. This raised the question of constructing a canonical bijection $\Phi : W_S \to W_S$ such that $\mathrm{maj}(w) = \mathrm{inv}(\Phi(w))$. Foata answered the question by constructing an elegant map Φ [20], which is referred to as the *second fundamental transformation* [21]. Here we review Foata's map $\Phi : W_S \to W_S$.

Let $w = w_1 w_2 \cdots w_n$ be a word on $\mathbb{N}$ and let a be an integer. If $w_n \leq a$, the a-factorization of w is $w = v_1 b_1 \cdots v_p b_p$, where each b_i is a letter less than or equal to a, and each v_i is a word (possibly empty), all of whose letters are greater than a. Similarly, if $w_n > a$, the a-factorization of w is $w = v_1 b_1 \cdots v_p b_p$, where each b_i is a letter greater than a, and each v_i is a word (possibly empty), all of whose letters are less than or equal to a. In both cases one defines

$$\gamma_a(w) = b_1 v_1 \cdots b_p v_p.$$

With the above notation, let $a = w_n$ and let $w' = w_1 \cdots w_{n-1}$. The second fundamental transformation Φ is defined recursively by $\Phi(w) = w$ if w has length 1, and

$$\Phi(w) = \gamma_a(\Phi(w'))a,$$

if w has length $n > 1$. The map Φ has the property that it preserves the last letter of the word, and $\mathrm{inv}(\Phi(w)) = \mathrm{maj}(w)$.

A statistic on words/permutations is called *Mahonian* if it has the same generating function as the inversion number. So, the major index is one example of a Mahonian statistic. Since the beginning of MacMahon's systematic study of permutation statistics, numeorus other Mahonian statistics have been discovered and the relations and interpolations between them have been studied (e.g. *den* [14], *mak* [22], *mad* [12], [5, 16, 23, 24, 25, 29]). Babson and Steingrimsson in [1] gave a unified view to a lot of Mahonian permutation statistics by showing that they can all be written as linear combinations of what the authors call generalized permutation patterns.

1.2. Matchings and set partitions

A (set) partition of $[n] = \{1, 2, \ldots, n\}$ is a collection of disjoint nonempty subsets of $[n]$, called blocks, whose union is $[n]$. A matching of $[2n]$ is a partition of $[2n]$ in n two-element blocks, which we also call *edges*. Let Π_n and $\mathcal{M}_n$ denote the sets of partitions of $[n]$ and matchings of $[2n]$, respectively. If a partition π has k blocks, we write $|\pi| = k$. A partition π is often represented as a graph on the vertex set $[n]$, drawn on a horizontal line in the increasing order from left to right, whose edge set consists of arcs connecting the elements of each block in numerical order. We write an arc e as a pair (i, j) with $i < j$. For example, the graph of the partition $1, 3, 6, 2, 4, 5$ has three arcs $(1, 3), (3, 6)$, and $(2, 4)$. The type of a partition is defined to be $type(\pi) = (S, T)$ where S and T are the sets of minimal and maximal elements in the blocks of π, respectively.

For a partition π of $[n]$, we say that the arcs $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ form a k-crossing if $i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$, and they form a k-nesting if $i_1 < i_2 < \cdots < i_k < j_k < \cdots < j_2 < j_1$. When the number k is omitted it is assumed to be 2. So, the terms crossings and nestings mean 2-crossings and 2-nestings, respectively. The *crossing number* of π is the maximal k such that π has a k-crossing. The *nesting number* is defined analogously. By $cr(\pi)$ (resp. $ne(\pi)$), we denote the number of crossings (resp. nestings) of π. In order to avoid possible confusion we remark that some authors that study the crossing and nesting numbers use cr and ne to denote these numbers, respectively. The distribution of the statistics cr and ne on matchings has been studied in a number of articles. Here we summarize the main results.

Theorem 1 ([15, 31]). *Let $S, T \subset [n]$. Then*

$$|\{\pi \in \Pi_n : type(\pi) = (S,T), cr(\pi) = k, ne(\pi) = l\}|$$
$$= |\{\pi \in \Pi_n : type(\pi) = (S,T), ne(\pi) = k, cr(\pi) = l\}|.$$

This implies symmetric distribution of crossings and nestings over set partitions, as wells as over the subclass of matchings.

Theorem 2 ([42, 38]). *The generating function for cr over the set $\mathcal{M}_n$ is given by*

$$\sum_{M \in \mathcal{M}_n} q^{cr(M)} = \frac{1}{(1-q)^n} \sum_{k=-n}^{n} (-1)^k q^{k(k-1)/2} \begin{bmatrix} 2n \\ n+k \end{bmatrix}.$$

Theorem 3. *Let X_n be the random variable equal to the value of cr taken over the set $\mathcal{M}_n$ endowed with the uniform probability distribution.*

(a) ([38, 19, 18]) The mean and the variance of the distribution of X_n are

$$\mu_n := E(X_n) = \frac{n(n-1)}{6} \qquad \sigma_n^2 := Var(X_n) = \frac{n(n-1)(n+3)}{45},$$

(b) ([18]) The distribution of X_n is Gaussian in the asymptotic limit, i.e., for all x one has

$$\lim_{n\to\infty} Pr\left(\frac{X_n - \mu_n}{\sigma_n} \le x\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy.$$

Analogues of Thorems 2 and 3 for set partitions are not known. In section 2.4 we give a continued fraction expansion of $\sum_{\pi \in \Pi_n} q^{cr(\pi)} p^{ne(\pi)}$. Unfortunately, k-crossings and k-nestings for $k \ge 3$ seem to be a lot more complicated than the basic $k = 2$ case. Namely, the equidistribution result does not hold for these values of k.

On the other hand, there are some very elegant results concerning avoidance of large crossings and nestings. Partitions that do not have a k-crossing are called *k-noncrossing*. Similarly, one defines *k-nonnesting partitions*. 2-noncrossing objects are commonly said to be simply noncrossing.

Theorem 4 ([6])**.** *The number of partitions of $[n]$ of type (S,T) that are k-noncrossing and l-nonnesting is equal to the number of partitions of the same type that are l-noncrossing and k-nonnesting.*

The number of noncrossing partitions of $[n]$ as well as the number of noncrossing matchings on $[2n]$ is equal to the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$. Noncrossing partitions arise in the context of algebraic combinatorics, geometric combinatorics, in relation to topological problems, questions in probability theory and mathematical biology [41]. The set of 3-noncrossing matchings is in one-to-one correspondence with the set of pairs of noncrossing Dyck paths. For general k, the number of k-noncrossing matchings of [2n] is equal to the number of closed lattice walks of length $2n$ in the set

$$V_k = \{(a_1, a_2, \ldots, a_{k-1}) : a_1 \geq a_2 \geq \cdots \geq a_{k-1} \geq 0, a_i \in \mathbb{Z}\}$$

from the origin to itself with unit steps in any coordinate direction or its negative [6]. Similarly, the number of k-noncrossing partitions is equal to the number of vacillating walks in the same region starting and ending at the origin. Another result is the following.

Theorem 5 ([4])**.** *The number $C_3(n)$ of 3-noncrossing partitions is given by $C_3(0) = C_3(1) = 1$ and, for $n \geq 0$*

$$9n(n+3)C_3(n) - 2(5n^2 + 32n + 42)C_3(n+1) + (n+7)(n+6)C_3(n+2) = 0.$$

Equivalently, the associated generating function $\mathcal{C}_n(t) = \sum_{n\geq 0} C(n)t^n$ *satisfies*

$$t^2(1-9t)(1-t)\frac{d^2}{dt^2}\mathcal{C}(t) + 2t(5-27t+18t^2)\frac{d}{dt}\mathcal{C}(t) + 10(2-3t)\mathcal{C}(t) = 20.$$

Finally, as n *tends to infinity,*

$$C_3(n) \sim \frac{3^9 \cdot 5}{2^5}\frac{\sqrt{3}}{\pi}\frac{9^n}{n^7}.$$

The authors in [7] defined major index *pmaj* for set partitions which is an analogue of the classical major index for permutations. We present their definition next. Given $\pi \in \Pi_n$, first label the arcs of π by $1, 2, \ldots, k$ from right to left in order of their left-hand endpoints. That is, if the arcs are $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ with $i_1 > i_2 > \cdots > i_k$, then (i_r, j_r) has label r, for $1 \leq r \leq k$. Next we associate a sequence $\sigma(r)$ to each right-hand endpoint r. Assume that the right-hand endpoints are $r_1 > r_2 > \cdots > r_k$. The sequence $\sigma(r_i)$ is defined backward recursively: let $\sigma(r_1) = a$ if r_1 is the right-hand endpoint of the arc with label a. In general, after defining $\sigma(r_i)$, assume that the left-hand endpoints of the arcs labeled $a_1, \ldots, a_t$ are lying between r_{i+1} and r_i, including r_{i+1}. Then $\sigma(r_{i+1})$ is obtained from $\sigma(r_i)$ by deleting entries $a_1, \ldots, a_t$ and adding b at the very beginning, where b is the label for the arc whose right-hand endpoint is r_{i+1}. Finally, define $pmaj(\pi)$ by

$$pmaj(P) := \sum_{r_i} des(\sigma(r_i)).$$

The statistic pmaj is an analogue of maj for permutations in the sense that when permutations are embedded into set partitions in a natural way (a permutation σ of $[n]$ can be represented as the partition with arcs connecting $n+1-\sigma(i)$ and $n+i$ for $1 \leq i \leq n$), the major index of each permutation is equal to the major index of the corresponding set partition. Moreover,

$$\sum_{type(\pi)=(S,T)} q^{pmaj(\pi)} = \sum_{type(\pi)=(S,T)} q^{cr(\pi)}. \tag{1.2}$$

1.3. Fillings of moon polyominoes

A *polyomino* is a finite subset of $\mathbb{Z}^2$, where we represent every element of $\mathbb{Z}^2$ by a square cell. The polyomino is *convex* if its intersection with any column or row is connected. It is *intersection-free* if every two columns are comparable, i.e., the row-coordinates of one column form a subset of those of the other column. Equivalently, it is intersection-free if every two rows are comparable. A *moon polyomino* is a convex intersection-free polyomino. If the rows (resp. columns) of the moon polyomino are left-aligned (resp. top-aligned), we will call it a *left-aligned stack polyomino* (resp. *top-aligned stack polyomino*). A *Ferrers diagram* is a left-aligned and top-aligned stack polyomino. See Figure 1.1 for an illustration. The term 'moon polyomino' was first used by Jonsson in [27] where he used fillings of such polyominoes to study generalized triangulations.

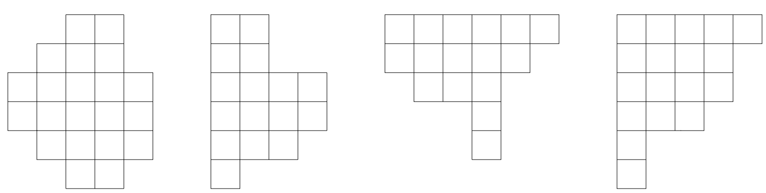


FIG. 1.1 *A moon polyomino, a left-aligned and a top-aligned stack polyomino, and a Ferrers diagram.*

Fillings of moon polyominoes with nonnegative entries have been subject of study of various authors in several articles. A *north-east chain*, or shortly *ne-chain* of length k in an arbitrary filling of a moon polyomino is a chain of k non-zero entries, such that each element is strictly to the right and strictly above the preceding element of the sequence and such that the smallest rectangle containing the chain is contained in the moon polyomino. Similarly, in a *south-east chain*, or shortly *se-chain* each element is strictly to the right and strictly below the preceding element and the smallest rectangle that contains it is contained in the moon polyomino. The length of such a chain is the number of its elements.

Fillings of Ferrers shapes with entries 0 and 1 correspond to multigraphs [13], and if the column and row sums are at most 1, then the fillings correspond to set partitions. If the shape is rectangular, the filling corresponds to a word. The correspondence is explained in section 4.2. In view of this correspondence ne-chains are a generalization of increasing subsequences in words and crossings in set partitions. Several analogues of Theorem 4 have been derived. We mention some of them, others including more general theorems are given in [2, 3, 27, 28, 34, 39].

Theorem 6. *(a) ([2, 34]) The number of* 01*-fillings of a given Ferrers shape with exactly one 1 in each row and column with lengths of the longest north-east chain k and the longest south-east chain l is equal to the number of fillings with lengths of the longest north-east chain l and the longest south-east chain k.*

(b) ([27, 28, 39]) Two moon polyominoes that differ only by a permutation of their columns (without any vertical shifts) permit the same number of 01*-fillings with a given length of the longest north-east chain and a given number of non-zero entries in each row.*

(c) ([39]) Two moon polyominoes that differ only by a permutation of their columns (without any vertical shifts) permit the same number of arbitrary fillings with a given length of the longest north-east chain and given row and column sums.

In this dissertation, we will derive results about 01-fillings of moon polyominoes with restricted row sums. That is, given a moon polyomino $\mathcal{M}$, we assign a 0 or a 1 to each cell of $\mathcal{M}$ so that there is at most one 1 in each row. The following result is an analogue of Theorem 1.

Theorem 7 ([30]). *The number of 01-fillings of a given moon polyomino with fixed column and row sums and with at most one 1 in each row that have k north-east and l south-east chains of length 2 is equal to the number of fillings with l north-east and k south-east chains of length 2.*

In the following chapters we will derive generalizations and analogues of some of the theorems mentioned above. In chapter II we study the tree of set partitions and the distribution of crossings and nestings over its subtrees. In chapter III we construct a bijection between matchings with k nestings and partially directed self-avoiding paths in the wedge $y = \pm x$ with k north steps. This serves as a bijective proof of the result that was first obtained algebraically which says that the number of matchings on $[2n]$ with k nestings is equal to the number of aforementioned paths that start at the origin, end at the point $(n, -n)$ and have k north steps. In chapter IV, we define a major index of 01-fillings of moon polyominoes with at most one 1 in each row which generalizes the major index for words, permutations, and set partitions. We find the generating function and construct a Foata-type bijection to prove a generalization of (1.1) and (1.2).

CHAPTER II

CROSSINGS AND NESTINGS OF TWO EDGES IN SET PARTITIONS

In [33] Klazar studied distributions of the numbers of crossings and nestings of two edges in matchings. All matchings form an infinite tree $\mathcal{T}$ rooted at the empty matching $\emptyset$, in which the children of a matching M are the matchings obtained from M by adding to M in all possible ways a new first edge. Given two matchings M and N on $[2n]$, Klazar decided when the number of crossings (nestings) have identical distribution on the levels of the two subtrees of $\mathcal{T}$ rooted at M and N. In this chapter we investigate the distribution of the statistics $cr(\pi)$ and $ne(\pi)$ over the partitions of $[n]$ with a prefixed restriction to the last k elements.

Denote by Π_n the set of all partitions of $[n]$, and by $\Pi_{n,k}$ the set of partitions of $[n]$ with k blocks. For $n = 0$, Π_0 contains the empty partition. Let $\Pi = \cup_{n=0}^{\infty} \Pi_n = \cup_{n=0}^{\infty} \cup_{k \leq n} \Pi_{n,k}$. We define *the tree* $\mathcal{T}(\Pi)$ *of partitions* as a rooted tree whose nodes are partitions such that:

1. The root is the empty partition;

2. The partition π of $[n+1]$ is a child of λ, a partition of [n], if and only if the restriction of π on $\{2, \ldots, n+1\}$ is order-isomorphic with λ. (If $\pi = \{B_1, \ldots, B_k\}$ is a partition of the set A and $C \subseteq A$ then the restriction of π on C is the partition of C with blocks $C \cap B_1, \ldots, C \cap B_k$, where the empty intersections are dropped.)

See Figure 2.1 for an illustration of $\mathcal{T}(\Pi)$.

Observe that if λ is a partition of $[n]$ with $|\lambda| = k$, then λ has $k+1$ children in $\mathcal{T}(\Pi)$. Let $B_1, \ldots, B_k$ be the blocks of λ ordered in increasing order with respect to their minimal elements. For a set S, let $S + 1 = \{a + 1 : a \in S\}$. We denote the

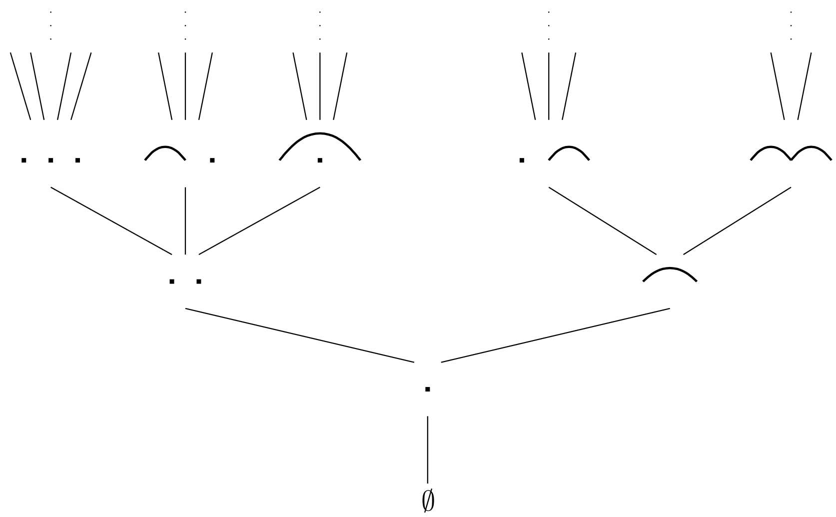


FIG. 2.1 *The tree of partitions* $\mathcal{T}(\Pi)$.

children of λ by $\lambda^0, \lambda^1, \ldots, \lambda^k$ as follows: λ^0 is a partition of $[n+1]$ with $k+1$ blocks,

$$\lambda^{(0)} = \{\{1\}, B_1 + 1, \ldots, B_k + 1\};$$

for $1 \leq i \leq k$, λ^i is a partition of $[n+1]$ with k blocks,

$$\lambda^{(i)} = \{\{1\} \cup (B_i + 1), B_1 + 1, \ldots, B_{i-1} + 1, B_{i+1} + 1, \ldots, B_k + 1\}.$$

For a partition λ, let $\mathcal{T}(\lambda)$ denote the subtree of $\mathcal{T}(\Pi)$ rooted at λ, and let $\mathcal{T}(\lambda, l)$ be the set of all partitions at the l-th level of $\mathcal{T}(\lambda)$. $\mathcal{T}(\lambda, l, m)$ is the set of all partitions on the l-th level of $\mathcal{T}(\lambda)$ with m blocks. Note that $\mathcal{T}(\lambda, l, m) \neq \emptyset$ if and only if $k \leq m \leq k + l$.

Let G be an abelian group and α, β two elements in G. Consider the statistics $s_{\alpha,\beta} : \Pi \to G$ given by $s_{\alpha,\beta}(\lambda) = cr(\lambda)\alpha + ne(\lambda)\beta$. In [33] Klazar defined a tree of matchings and showed that for two matchings M and N, if the statistic $s_{\alpha,\beta}$ coincides at the first two levels of $\mathcal{T}(M)$ and $\mathcal{T}(N)$ then it coincides at all levels, and similarly for the pair of statistics $s_{\alpha,\beta}, s_{\beta,\alpha}$. Here we prove that in the tree of partitions defined above, the same results hold. Precisely,

Theorem 8. *Let $\lambda, \pi \in \mathcal{T}(\Pi)$ be two non-empty partitions, and $s_{\alpha,\beta}(T)$ be the multiset containing $\{s_{\alpha,\beta}(t) : t \in T\}$. We have*

(a) If $s_{\alpha,\beta}(\mathcal{T}(\lambda, l)) = s_{\alpha,\beta}(\mathcal{T}(\pi, l))$ for $l = 0, 1$ then
$s_{\alpha,\beta}(\mathcal{T}(\lambda, l, m)) = s_{\alpha,\beta}(\mathcal{T}(\pi, l, m))$ for all $l, m \geq 0$.

(b) If $s_{\alpha,\beta}(\mathcal{T}(\lambda, l)) = s_{\beta,\alpha}(\mathcal{T}(\pi, l))$ for $l = 0, 1$ then
$s_{\alpha,\beta}(\mathcal{T}(\lambda, l, m)) = s_{\beta,\alpha}(\mathcal{T}(\pi, l, m))$ for all $l, m \geq 0$.

In other words, if the statistic $s_{\alpha,\beta}$ coincides on the first two levels of the trees $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$ then it coincides on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, m)$ on all levels, and similarly for the pair of statistics $s_{\alpha,\beta}, s_{\beta,\alpha}$.

Note that the conditions of Theorem 8 imply that λ and π have the same number of blocks. But they are not necessarily partitions of the same $[n]$.

There are several differences between the structure of crossing and nesting of set partitions and that of matchings:

1. The tree of partitions $\mathcal{T}(\Pi)$ and the tree of matchings are different. In $\mathcal{T}(\Pi)$, children of a partition π is obtained by adding a new vertex, instead of adding a first edge. Hence Klazar's tree of matchings is not a sub-poset of $\mathcal{T}(\Pi)$. The definition of $\mathcal{T}(\Pi)$ allows us to define the analogous operators $R_{\alpha,\beta,i}$, as in [33, §2]. Since some descendants of π are obtained by adding isolated points, we need to introduce an extra operator M, (c.f. Definition 1), and supply some new arguments to work with our structure and M.

2. The type of a matching is encoded by a Dyck path, while for set partitions,

the corresponding structure is restricted bicolored Motzkin paths (RBM), (c.f. section 3).

3. The enumeration of crossing/nesting similarity classes is different. A crossing-similarity class is determined by a value $cr(M)$ ($cr(\pi)$) and a composition $(a_1, a_2, \ldots, a_m)$ of n. For matchings $cr(M)$ can be any integer between 0 and $1 + a_2 + 2a_3 + \cdots + (m-1)a_m$. But for partitions the possible value of $cr(\pi)$ depends only on m, but not the a_i's.

   In matchings there is a bijection between the set of nesting sequences of matchings of $[2n]$ and the set of Dyck paths $\mathcal{D}(n)$. There is no analogous result between set partitions and restricted bicolored Motzkin paths.

4. For matchings every nesting-similarity class is a subset of a crossing-similarity class. This is not true for set partitions.

2.1. Proof of Theorem 8

Formally a multiset is a pair (A, m), where A is a set, called the underlying set, and $m : A \to \mathbb{N}$ is a mapping that determines the multiplicities of the elements of A. We often write multisets by repeating the elements according to their multiplicities.

For a map $f : X \to Y$ and $Z \subset X$, let $f(Z)$ denote the multiset whose underlying set is $\{f(z) : z \in Z\}$ and in which each element y appears with multiplicity equal to the cardinality of the set $\{z : z \in Z \text{ and } f(z) = y\}$. $\mathcal{S}(X)$ denotes the set of all finite multisets with elements in the set X. Any function $f : X \to S(Y)$ naturally extends to $f : \mathcal{S}(X) \to \mathcal{S}(Y)$ by $f(Z) = \bigcup_{z \in Z}\{f(z)\}$, where $\bigcup$ is union of multisets (the multiplicities of elements are added).

For each $b_i = \min B_i$ of λ define $u_i(\lambda)$ to be the number of edges (p, q) such that $p < b_i < q$ and $v_i(\lambda)$ to be the number of edges (p, q) such that $p < q < b_i$. They satisfy the obvious recursive relations

$$u_i(\lambda^0) = \begin{cases} 0 & \text{if } i = 1 \\ u_{i-1}(\lambda) & \text{if } 2 \le i \le k+1 \end{cases} \tag{2.1}$$

$$v_i(\lambda^0) = \begin{cases} 0 & \text{if } i = 1 \\ v_{i-1}(\lambda) & \text{if } 2 \le i \le k+1 \end{cases} \tag{2.2}$$

$$u_i(\lambda^j) = \begin{cases} 0 & \text{if } i = 1 \\ u_{i-1}(\lambda) + 1 & \text{if } 2 \le i \le j \\ u_i(\lambda) & \text{if } j+1 \le i \le k \end{cases} \tag{2.3}$$

$$v_i(\lambda^j) = \begin{cases} 0 & \text{if } i = 1 \\ v_{i-1}(\lambda) & \text{if } 2 \le i \le j \\ v_i(\lambda) + 1 & \text{if } j+1 \le i \le k \end{cases} \tag{2.4}$$

for $j = 1, \ldots, k$, where $k = |\lambda| \ge 1$. For the statistics $s_{\alpha,\beta} : \Pi \to G$ defined by $s_{\alpha,\beta}(\lambda) = cr(\lambda)\alpha + ne(\lambda)\beta$, we have that

$$s_{\alpha,\beta}(\lambda^0) = s_{\alpha,\beta}(\lambda^1) = s_{\alpha,\beta}(\lambda), \tag{2.5}$$

$$s_{\alpha,\beta}(\lambda^j) = s_{\alpha,\beta}(\lambda) + u_j(\lambda)\alpha + v_j(\lambda)\beta, \qquad j \ge 1. \tag{2.6}$$

For simplicity, we will write λ^{ij} for $(\lambda^i)^j$.

Lemma 1. *For* $|\lambda| \ge 1$*,*

$$s_{\alpha,\beta}(\lambda^{0j})) = \begin{cases} s_{\alpha,\beta}(\lambda) & \textit{if } j = 0, 1 \\ s_{\alpha,\beta}(\lambda^{j-1}) & \textit{if } j \ge 2, \end{cases} \tag{2.7}$$

and for $i \geq 1$,

$$s_{\alpha,\beta}(\lambda^{ij}) = \begin{cases} s_{\alpha,\beta}(\lambda^i) & \text{if } j = 0, 1 \\ s_{\alpha,\beta}(\lambda^i) + s_{\alpha,\beta}(\lambda^{j-1}) - s_{\alpha,\beta}(\lambda^1) + \alpha & \text{if } 2 \leq j \leq i \\ s_{\alpha,\beta}(\lambda^i) + s_{\alpha,\beta}(\lambda^j) - s_{\alpha,\beta}(\lambda^1) + \beta & \text{if } j \geq i+1. \end{cases} \tag{2.8}$$

Proof. We first show (2.8). The first line in (2.8) follows directly from (2.5). For the other two,

$$\begin{aligned} s_{\alpha,\beta}(\lambda^{ij}) &= s_{\alpha,\beta}(\lambda^i) + u_j(\lambda^i)\alpha + v_j(\lambda^i)\beta \\ &= \begin{cases} s_{\alpha,\beta}(\lambda^i) + u_{j-1}(\lambda)\alpha + \alpha + v_{j-1}(\lambda)\beta & \text{if } 2 \leq j \leq i \\ s_{\alpha,\beta}(\lambda^i) + u_j(\lambda)\alpha + v_j(\lambda)\beta + \beta & \text{if } j \geq i+1 \end{cases} \\ &= \begin{cases} s_{\alpha,\beta}(\lambda^i) + s_{\alpha,\beta}(\lambda^{j-1}) - s_{\alpha,\beta}(\lambda^1) + \alpha & \text{if } 2 \leq j \leq i \\ s_{\alpha,\beta}(\lambda^i) + s_{\alpha,\beta}(\lambda^j) - s_{\alpha,\beta}(\lambda^1) + \beta & \text{if } j \geq i+1. \end{cases} \end{aligned}$$

The first and third equality follow from (2.6) and the second one follows from (2.3) and (2.4). Similarly, (2.7) follows from (2.1), (2.2), (2.5), and (2.6). □

To each partition λ with k blocks, ($k \geq 1$), we associate a sequence

$$seq_{\alpha,\beta}(\lambda) := s_{\alpha,\beta}(\lambda^1)s_{\alpha,\beta}(\lambda^2)\ldots s_{\alpha,\beta}(\lambda^k)$$

The sequence $seq_{\alpha,\beta}(\lambda)$ encodes the information about the distribution of $s_{\alpha,\beta}$ on the children of λ in $\mathcal{T}(\Pi)$, in which $s_{\alpha,\beta}(\lambda^1)$ plays a special role when we analyze the change of $seq_{\alpha,\beta}(\lambda)$ below . This is due to the fact that $s_{\alpha,\beta}(\lambda^1)$ carries information about λ and two children of λ, namely, λ^0 and λ^1.

For an abelian group G, let G_l^* denote the set of finite sequences of length l over G, and $G^* = \cup_{l\geq 1} G_l^*$. If $u = x_1x_2\ldots x_k \in G^*$ and $y \in G$, then we use the convention that the sequence $(x_1 + y)(x_2 + y) \dots (x_k + y)$ is denoted by $x_1x_2 \dots x_k + y$.

Definition 1. *For $\alpha, \beta \in G$ and $i \geq 1$, define $R_{\alpha,\beta,i} : G_l^* \to G_l^*$, ($i \leq l$) by setting*

$$R_{\alpha,\beta,i}(x_1x_2 \dots x_l) = x_i(x_1 \dots x_{i-1} + (x_i - x_1 + \alpha))(x_{i+1} \dots x_l + (x_i - x_1 + \beta))$$

and $R_{\alpha,\beta} : G^* \to S(G^*)$ by setting

$$R_{\alpha,\beta}(x_1x_2 \dots x_l) = \{R_{\alpha,\beta,i}(x_1x_2 \dots x_l) : 1 \leq i \leq l\}.$$

In addition, define $M : G^* \to G^*$ by setting

$$M(x_1x_2 \dots x_l) = x_1x_1x_2 \dots x_l.$$

Lemma 1 immediately implies that

$$seq_{\alpha,\beta}(\lambda^0) = M(seq_{\alpha,\beta}(\lambda)),$$
$$seq_{\alpha,\beta}(\lambda^i) = R_{\alpha,\beta,i}(seq_{\alpha,\beta}(\lambda)), \qquad \text{for } 1 \leq i \leq |\lambda|.$$

For $l \geq 0$, let $E_{\alpha,\beta}(\lambda, l, m) = \{seq_{\alpha,\beta}(\mu) : \mu \in \mathcal{T}(\lambda, l, m)\}$, the multiset of sequences $seq_{\alpha,\beta}(\mu)$ associated to partitions $\mu \in \mathcal{T}(\lambda, l, m)$. Then for $l \geq 1$,

$$E_{\alpha,\beta}(\lambda, l, m) = R_{\alpha,\beta}(E_{\alpha,\beta}(\lambda, l-1, m)) \cup M(E_{\alpha,\beta}(\lambda, l-1, m-1)). \qquad (2.9)$$

Next, we define an auxiliary function f which reflects the change of the statistic $s_{\alpha,\beta}$ along $\mathcal{T}(\Pi)$. Then we prove two general properties of f and use these properties to prove Theorem 8. For an integer $r \geq 0$ and $\gamma \in G$, the function $f : G^* \to S(G)$ is defined by

$$f_\gamma^r(x_1x_2 \dots x_l) := \{x_{a_1} + x_{a_2} + \dots + x_{a_r} - (r-1)x_1 + \gamma : 1 < a_1 < a_2 < \dots < a_r \leq l\}$$

In particular,

$$f_0^0(x_1x_2\dots x_l) = \{x_1\},$$
$$f_0^1(x_1x_2\dots x_l) = \{x_2,\dots,x_l\}.$$

Lemma 2. *Let $X, Y \in S(G^*)$ be two multisets such that $f_\gamma^r(X) = f_\gamma^r(Y)$ for every $r \geq 0$ and $\gamma \in G$. Then*

(a) $f_\gamma^r(M(X)) = f_\gamma^r(M(Y))$,

(b) $f_\gamma^r(R_{\alpha,\beta}(X)) = f_\gamma^r(R_{\alpha,\beta}(Y))$,

(c) $f_\gamma^r(R_{\alpha,\beta}(X)) = f_\gamma^r(R_{\beta,\alpha}(Y))$,

for every $r \geq 0$ and $\gamma \in G$.

Proof. (a) The elements in $f_\gamma^r(M(X))$ have the form $y_{a_1}+y_{a_2}+\dots+y_{a_r}-(r-1)y_1+\gamma$ for some $y_1y_2\dots y_{l+1} \in M(X)$, where $y_1y_2\dots y_{l+1} = x_1x_1x_2\dots x_l$ for some $x_1x_2\dots x_l \in X$. For $r = 0$,

$$f_\gamma^0(M(X)) = \{y_1+\gamma : y_1y_2\dots y_{l+1} \in M(X)\} = \{x_1+\gamma : x_1x_2\dots x_l \in X\} = f_\gamma^0(X).$$

Hence $f_\gamma^0(X) = f_\gamma^0(Y)$ implies $f_\gamma^0(M(X)) = f_\gamma^0(M(Y))$.

For $r \geq 1$, divide the multiset $f_\gamma^r(M(X))$ into two disjoint multisets,

$$A = \{y_{a_1}+y_{a_2}+\dots+y_{a_r}-(r-1)y_1+\gamma : y_1y_2\dots y_{l+1} \in M(X),\ a_1 = 2\}$$

and

$$B = \{y_{a_1}+y_{a_2}+\dots+y_{a_r}-(r-1)y_1+\gamma : y_1y_2\dots y_{l+1} \in M(X),\ a_1 > 2\}.$$

The elements of A can be written as

$$\begin{aligned} y_{a_1} + y_{a_2} + \cdots + y_{a_r} - (r-1)y_1 + \gamma &= x_1 + y_{a_2} + \cdots + y_{a_r} - (r-1)x_1 + \gamma \\ &= y_{a_2} + \cdots + y_{a_r} - (r-2)x_1 + \gamma \\ &= x_{a_2-1} + \cdots + x_{a_r-1} - (r-2)x_1 + \gamma. \end{aligned}$$

Since $a_2 - 1 > a_1 - 1 = 1$, the multiset A is equal to $f_\gamma^{r-1}(X)$. The elements in B can be written as

$$y_{a_1} + y_{a_2} + \cdots + y_{a_r} - (r-1)y_1 + \gamma = x_{a_1-1} + x_{a_2-1} + \cdots + x_{a_r-1} - (r-1)x_1 + \gamma.$$

Since $a_1 \geq 3$, the indices on the right-hand side run through all the increasing r-tuples $1 < a_1 - 1 < a_2 - 1 < \cdots < a_r - 1 \leq l$. Therefore, B is equal to $f_\gamma^r(X)$. So,

$$f_\gamma^r(M(X)) = f_\gamma^{r-1}(X) \cup f_\gamma^r(X). \tag{2.10}$$

By assumption we have

$$f_\gamma^r(M(X)) = f_\gamma^{r-1}(X) \cup f_\gamma^r(X) = f_\gamma^{r-1}(Y) \cup f_\gamma^r(Y) = f_\gamma^r(M(Y)).$$

(c) We will prove only (c) because the proof of (b) is similar and easier. Since $f_\gamma^r(X)$ is a translation of $f_0^r(X)$ by γ, it is enough to prove the result for $\gamma = 0$ only. The elements of $f_0^r(R_{\alpha,\beta}(X))$ have the form $y_{a_1} + y_{a_2} + \cdots + y_{a_r} - (r-1)y_1$, where $y_1 y_2 \ldots y_l \in R_{\alpha,\beta}(X)$ is equal to $x_i(x_1 \ldots x_{i-1} + x_i - x_1 + \alpha)(x_{i+1} \ldots x_l + x_i - x_1 + \beta)$ for some $x_1 x_2 \ldots x_l \in X$ and $i \in [l]$.

For $0 \leq t \leq r$, let

$$\begin{aligned} C_{t,\alpha,\beta}(X) = \{ & y_{a_1} + y_{a_2} + \cdots + y_{a_r} - (r-1)y_1 : \\ & y_1 y_2 \ldots y_l \in R_{\alpha,\beta,i}(X) \text{ and } a_t \leq i < a_{t+1}, \text{ for some } i \in [l]\}. \end{aligned}$$

An element $y_{a_1} + y_{a_2} + \cdots + y_{a_r} - (r-1)y_1 \in C_{t,\alpha,\beta}(X)$ is equal to

$$\begin{aligned} x_{a_1-1} + \cdots x_{a_t-1} + t(x_i - x_1 + \alpha) + x_{a_{t+1}} + \cdots \\ + x_{a_r} + (r-t)(x_i - x_1 + \beta) - (r-1)x_i \\ = x_{a_1-1} + \cdots x_{a_t-1} + x_i + x_{a_{t+1}} + \cdots + x_{a_r} - rx_1 + t\alpha + (r-t)\beta. \end{aligned} \tag{2.11}$$

Again, we consider two cases, according to the value of a_1. By (2.11), the submultiset of $C_{t,\alpha,\beta}(X)$ for $a_1 > 2$ is equal to $f^{r+1}_{t\alpha+(r-t)\beta}(X)$, and for $a_1 = 2$ the corresponding submultiset is equal to $f^{r}_{t\alpha+(r-t)\beta}(X)$. Therefore,

$$C_{t,\alpha,\beta}(X) = f^{r+1}_{t\alpha+(r-t)\beta}(X) \cup f^{r}_{t\alpha+(r-t)\beta}(X). \tag{2.12}$$

Similarly,

$$C_{t,\beta,\alpha}(Y) = f^{r+1}_{t\beta+(r-t)\alpha}(Y) \cup f^{r}_{t\beta+(r-t)\alpha}(Y). \tag{2.13}$$

So,

$$\begin{aligned} f^r_0(R_{\alpha,\beta}(X)) &= \bigcup_{t=0}^{r} C_{t,\alpha,\beta}(X) \\ &= \bigcup_{t=0}^{r} f^{r+1}_{t\alpha+(r-t)\beta}(X) \cup \bigcup_{t=0}^{r} f^{r}_{t\alpha+(r-t)\beta}(X) \\ &= \bigcup_{t=0}^{r} f^{r+1}_{t\alpha+(r-t)\beta}(Y) \cup \bigcup_{t=0}^{r} f^{r}_{t\alpha+(r-t)\beta}(Y) \\ &= \bigcup_{t=0}^{r} f^{r+1}_{(r-t)\alpha+t\beta}(Y) \cup \bigcup_{t=0}^{r} f^{r}_{(r-t)\alpha+t\beta}(Y) \\ &= \bigcup_{t=0}^{r} C_{t,\beta,\alpha}(Y) \\ &= f^r_0(R_{\beta,\alpha}(Y)). \end{aligned}$$

The second and fifth equality follow from (2.12) and (2.13) respectively. The third equality follows from the assumption of the lemma, while the fourth equality is just a reordering of the unions. □

Lemma 3. *If $X, Y \in S(G^*)$ are one-element sets such that $f_0^0(X) = f_0^0(Y)$ and $f_0^1(X) = f_0^1(Y)$, then $f_\gamma^r(X) = f_\gamma^r(Y)$ for every $r \geq 0$ and $\gamma \in G$.*

Proof. We need to prove that if $u, v \in G^*$ are two sequences beginning with the same term and having equal numbers of occurrences of each $g \in G$, then $f_\gamma^r(u) = f_\gamma^r(v)$ for every $r \geq 0$ and $\gamma \in G$. It suffices to prove the statement for $\gamma = 0$, because $f_\gamma^r(u)$ is a translation of $f_0^r(u)$ by γ. Let $u = u_1 \dots u_l$ and $v = v_1 \dots v_l$. Since $u_1 = v_1$, it suffices to prove that the multisets $\{u_{a_1} + u_{a_2} + \cdots + u_{a_r} : 1 < a_1 < a_2 < \cdots < a_r \leq l\}$ and $\{v_{a_1} + v_{a_2} + \cdots + v_{a_r} : 1 < a_1 < a_2 < \cdots < a_r \leq l\}$ are equal. That is clear because $\{u_2, \dots, u_l\}$ and $\{v_2, \dots, v_l\}$ are equal as multisets. □

Proof of Theorem 8 . We prove (b), the proof of (a) is similar. First, we prove by induction on l that

$$f_\gamma^r(E_{\alpha,\beta}(\lambda, l, m)) = f_\gamma^r(E_{\beta,\alpha}(\pi, l, m)) \text{ for every } r \geq 0 \text{ and } \gamma \in G. \tag{2.14}$$

Before we proceed with the induction, it is useful to observe that the assumption

$$s_{\alpha,\beta}(\mathcal{T}(\lambda, l)) = s_{\beta,\alpha}(\mathcal{T}(\pi, l)) \text{ for } l = 0, 1$$

of Theorem 8 (b) is equivalent to

$$s_{\alpha,\beta}(\mathcal{T}(\lambda, l, m)) = s_{\beta,\alpha}(\mathcal{T}(\pi, l, m)) \text{ for } l = 0, 1 \text{ and } k \leq m \leq k + l, \tag{2.15}$$

where $k = |\lambda|$. One direction is clear, the other one follows from the following equations.

$$s_{\alpha,\beta}(\mathcal{T}(\lambda,1,k+1)) = s_{\alpha,\beta}(\mathcal{T}(\lambda,0,k)) = s_{\alpha,\beta}(\mathcal{T}(\lambda,0)),$$
$$s_{\alpha,\beta}(\mathcal{T}(\lambda,1,k)) = s_{\alpha,\beta}(\mathcal{T}(\lambda,1))\backslash s_{\alpha,\beta}(\mathcal{T}(\lambda,0)),$$

where $\backslash$ is the difference of multisets. For the same reason the assumption of part (a) is equivalent to

$$s_{\alpha,\beta}(\mathcal{T}(\lambda,l,m)) = s_{\alpha,\beta}(\mathcal{T}(\pi,l,m)) \text{ for } l=0,1 \text{ and } k \le m \le k+l.$$

Now we show (2.14). For $l=0$ we need to show

$$f_\gamma^r(E_{\alpha,\beta}(\lambda,0,k)) = f_\gamma^r(E_{\beta,\alpha}(\pi,0,k)).$$

By Lemma 3 we only need to check that $f_0^0(X) = f_0^0(Y)$ and $f_0^1(X) = f_0^1(Y)$ for $X = \{seq_{\alpha,\beta}(\lambda)\}$ and $Y = \{seq_{\beta,\alpha}(\pi)\}$. This follows from (2.15), because $f_0^0(X) = s_{\alpha,\beta}(\mathcal{T}(\lambda,0,k))$ and $f_0^1(X) = s_{\alpha,\beta}(\mathcal{T}(\lambda,1,k))$.

Suppose $f_\gamma^r(E_{\alpha,\beta}(\lambda,s,m)) = f_\gamma^r(E_{\beta,\alpha}(\pi,s,m))$ for all $0 \le s < l$ and all m. Then using (2.9), the induction hypothesis, and Lemma 2 we have

$$\begin{aligned} f_\gamma^r(E_{\alpha,\beta}(\lambda,l,m)) &= f_\gamma^r(R_{\alpha,\beta}(E_{\alpha,\beta}(\lambda,l-1,m))) \cup f_\gamma^r(M(E_{\alpha,\beta}(\lambda,l-1,m-1))) \\ &= f_\gamma^r(R_{\beta,\alpha}(E_{\beta,\alpha}(\pi,l-1,m))) \cup f_\gamma^r(M(E_{\beta,\alpha}(\pi,l-1,m-1))) \\ &= f_\gamma^r(E_{\beta,\alpha}(\pi,l,m)), \end{aligned}$$

and the induction is completed. This proves (2.14). Now

$$s_{\alpha,\beta}(\mathcal{T}(\lambda,l,m)) = f_0^0(E_{\alpha,\beta}(\lambda,l,m)) = f_0^0(E_{\beta,\alpha}(\pi,l,m)) = s_{\beta,\alpha}(\mathcal{T}(\pi,l,m)).$$

$\square$

2.2. Applications and examples

As a direct corollary, we obtain a result of Kasraoui and Zeng [31, Eq.(1.6)].

Corollary 1. *The joint distribution of crossings and nestings of partitions is symmetric i.e.*

$$\sum_{\pi\in\Pi_n} p^{cr(\pi)}q^{ne(\pi)} = \sum_{\pi\in\Pi_n} p^{ne(\pi)}q^{cr(\pi)}$$

Proof. Let $G = (\mathbb{Z}\oplus\mathbb{Z}, +), \alpha = (1,0)$ and $\beta = (0,1)$. The result follows from the second part of Theorem 8 for $\lambda = \pi = \{\{1\}\}$. □

For a partition λ we say that two edges form an alignment if they neither form a crossing nor a nesting. The total number of alignments in λ is denoted by $al(\lambda)$. A stronger result of Kasraoui and Zeng [31, Eq. (1.4)] can also be derived from Theorem 8.

Corollary 2.

$$\sum_{\pi\in\Pi_n} p^{cr(\pi)}q^{ne(\pi)}t^{al(\pi)} = \sum_{\pi\in\Pi_n} p^{ne(\pi)}q^{cr(\pi)}t^{al(\pi)}$$

Proof. Again we use $G = (\mathbb{Z}\oplus\mathbb{Z}, +)$, $\alpha = (1,0)$, $\beta = (0,1)$, and $\lambda = \pi = \{\{1\}\}$. Any partition $\mu \in \Pi_n$ with k blocks has $n-k$ edges. Hence $cr(\mu)+ne(\mu)+al(\mu) = \binom{n-k}{2}$. The result follows from the second part of Theorem 8. □

Corollary 3. *Let λ and π be two partitions of $[n]$ with same number of blocks k. If the statistic al is equidistributed on the first two levels of $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$, it is equidistributed on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, m)$ for all $l, m \geq 0$.*

Proof. Again we use the identity $cr(\mu) + ne(\mu) + al(\mu) = \binom{n-k}{2}$, which holds for any partition $\mu \in \Pi_n$ with k blocks. Moreover, $al(\lambda) = al(\lambda^0)$. Therefore the condition that the statistic al is equidistributed on the first two levels of $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$ implies that the statistic $cr+ne$ is equidistributed on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, m)$ for all $l = 0, 1$

and all m. In other words, if we set $G = \mathbb{Z}$ and $\alpha = \beta = 1$ then the the assumption of Theorem 8 is satisfied, and hence $cr + ne$ is equidistributed on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, m)$ for all $l, m \geq 0$. This, in return, implies that al is equidistributed on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, m)$ for all $l, m \geq 0$. □

Example 9. *Let* $\lambda = \{\{1,2,5\},\{3,4\}\}$ *and* $\pi = \{\{1,2,4\},\{3,5\}\}$*. There are as many partitions on* $[n]$ *with* m *crossings and* l *nestings which restricted to the last five points form a partition isomorphic to* λ *as there are partitions of* $[n]$ *with* l *crossings and* m *nestings which restricted to the last five points form a partition isomorphic to* π*.*

Proof. Set $G = (\mathbb{Z} \oplus \mathbb{Z}, +), \alpha = (1,0)$ and $\beta = (0,1)$, $s_{\alpha,\beta} = (cr, ne)$. The claim follows from part (b) of Theorem 8 since $s_{\alpha,\beta}(\lambda) = (0,1) = s_{\beta,\alpha}(\pi)$ and $s_{\alpha,\beta}(\mathcal{T}(\lambda,1)) = \{(0,1),(0,1),(1,2)\} = s_{\beta,\alpha}(\mathcal{T}(\pi,1))$. □

Example 10. *Consider the partitions* $\lambda = \{\{1,7\},\{2,6\},\{3,4\},\{5,8\}\}$ *and* $\pi = \{\{1,8\},\{2,4\},\{3,6\},\{5,7\}\}$*. There are as many partitions on* $[n]$ *with* m *crossings and* l *nestings which restricted to the last eight points form a partition isomorphic to* λ *as there are ones which restricted to the last eight points form a partition isomorphic to* π*.*

Proof. Again set $G = (\mathbb{Z} \oplus \mathbb{Z}, +), \alpha = (1,0)$ and $\beta = (0,1)$. Then $s_{\alpha,\beta} = (cr, ne)$. The claim follows from part (a) of Theorem 8 since

$$s_{\alpha,\beta}(\lambda) = (2,3) = s_{\alpha,\beta}(\pi)$$

and

$$s_{\alpha,\beta}(\mathcal{T}(\lambda,1)) = \{(2,3),(2,3),(3,3),(4,3),(4,4)\} = s_{\alpha,\beta}(\mathcal{T}(\pi,1)).$$

□

2.3. Number of crossing- and nesting-similarity classes

In this section we consider equivalence relations $\sim_{cr}$ and $\sim_{ne}$ on set partitions in the same way Klazar defines them on matchings [33]. We determine the number of crossing-similarity classes in $\Pi_{n,k}$. For $\sim_{ne}$, we find a recurrence relation for the number of nesting-similarity classes in $\Pi_{n,k}$, and compute the total number of such classes in Π_n.

Define an equivalence relation $\sim_{cr}$ on Π_n: $\lambda \sim_{cr} \pi$ if and only if $cr(\mathcal{T}(\lambda, l, m)) = cr(\mathcal{T}(\pi, l, m))$ for all $l, m \geq 0$. The relation $\sim_{cr}$ partitions $\Pi_{n,k}$ into equivalence classes. Theorem 8 implies that

$$\lambda \sim_{cr} \pi \text{ if and only if } cr(\lambda) = cr(\pi) \text{ and } f_0^1(seq_{1,0}(\lambda)) = f_0^1(seq_{1,0}(\pi)).$$

Define $crseq(\lambda) = seq_{1,0}(\lambda) - cr(\lambda)$. For the upcoming computations it is useful to observe that $\lambda \sim_{cr} \pi$ if and only if $cr(\lambda) = cr(\pi)$ and $f_0^1(crseq(\lambda)) = f_0^1(crseq(\pi))$, i.e., λ and π are equivalent if and only if they have the same number of crossings and their sequences $crseq(\lambda)$ and $crseq(\pi)$ are equal as multisets. Denote the multiset consisting of the elements of $crseq(\lambda)$ by $crset(\lambda)$.

Similarly, define $\lambda \sim_{ne} \pi$ if and only if $ne(\mathcal{T}(\lambda, l, m)) = ne(\mathcal{T}(\pi, l, m))$ for all $l, m \geq 0$. Again, from Theorem 8 we have that $\lambda \sim_{ne} \pi$ if and only if $ne(\lambda) = ne(\pi)$ and $f_0^1(seq_{0,1}(\lambda)) = f_0^1(seq_{0,1}(\pi))$. Since the sequence $seq_{0,1}(\lambda)$ is nondecreasing, $\lambda \sim_{ne} \pi$ if and only if $ne(\lambda) = ne(\pi)$ and $seq_{0,1}(\lambda) - ne(\lambda) = seq_{0,1}(\pi) - ne(\pi)$. With the notation at the beginning of section 2.1, $seq_{0,1}(\lambda) - ne(\lambda) = v_1 \dots v_k$. Denote this sequence by $neseq(\lambda)$.

A *Motzkin path* $M = (s_1, \dots, s_n)$ is a path from $(0,0)$ to $(n,0)$ consisting of steps $s_i \in \{(1,1), (1,0), (1,-1)\}$ which does not go below the x-axis. We say that the step s_i is of height l if its left endpoint is at the line $y = l$. A restricted bicolored Motzkin path is a Motzkin path with each horizontal step colored red or blue which does not have a blue horizontal step of height 0. We will denote the steps $(1,1)$, $(1,-1)$, red $(1,0)$, and blue $(1,0)$ by NE (northeast), SE (southeast), RE (red east), and BE (blue east) respectively. The set of all restricted bicolored Motzkin paths of length n is denoted by RBM_n. A *Charlier diagram* of length n is a pair $h = (M, \xi)$ where $M = (s_1, \ldots, s_n) \in RBM_n$ and $\xi = (\xi_1, \ldots, \xi_n)$ is a sequence of integers such that $\xi_i = 1$ if s_i is a NE or RE step, and $1 \leq \xi_i \leq l$ if s_i is a SE or BE step of height l. Γ_n will denote the set of Charlier diagrams of length n.

It is well known that partitions are in a one-to-one correspondence with Charlier diagrams. Here we use two maps described in [31], which are based on similar constructions in [17, 43]. For our purpose, we reformulate the maps $\Phi_r, \Phi_l : \Gamma_n \to \Pi_n$ as follows. Given $(M, \xi) \in \Gamma_n$, construct $\lambda \in \Pi_n$ step by step. The path $M = (s_1, \ldots, s_n)$ determines the type of λ: $i \in [n]$ is

- a minimal but not a maximal element of a block of λ (opener) if and only if s_i is a NE step;
- a maximal but not a minimal element of a block of λ (closer) if and only if s_i is a SE step;
- both a minimal and a maximal element of a bock of λ (singleton) if and only if s_i is a RE step;
- neither a minimal nor a maximal element of a block of λ (transient) if and only if s_i is a BE step.

To draw the edges in $\Phi_r((M, \xi))$, we process the closers and transients one by one from left to right. Each time we connect the vertex i that we are processing to the ξ_i-th available opener or transient to the left of i, where the openers and transients are ranked from right to left. If we rank the openers and transients from left to right, we get $\Phi_l((M, \xi))$. It can be readily checked that Φ_r and Φ_l are well defined. Moreover:

Proposition 1. *The maps $\Phi_r, \Phi_l : \Gamma_n \to \Pi_n$ are bijections.*

The proof can be found in [17, 31] and their references.

Example 11. *If (M, ξ) is the Charlier diagram in Figure 2.2, then*

$$\Phi_r((M, \xi)) = \{\{1, 7, 10\}, \{2, 4, 6, 8\}, \{3\}, \{5, 9\}, \{11, 12\}\}$$
$$\Phi_l((M, \xi)) = \{\{1, 4, 6, 7, 9\}, \{2, 10\}, \{3\}, \{5, 8\}, \{11, 12\}\}$$

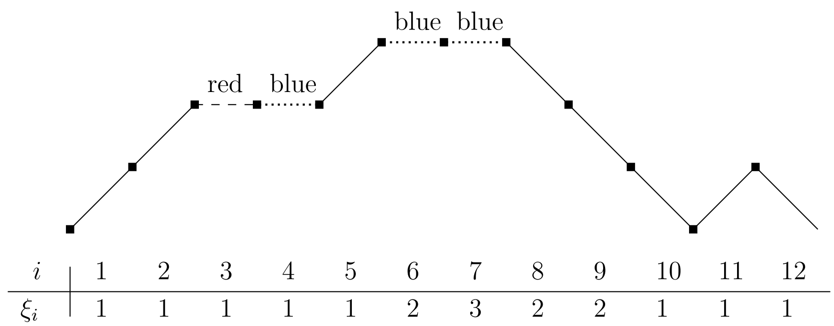


FIG. 2.2 *A Charlier diagram.*

For $M \in RBM_n$ let d_i be the number of NE and RE steps that start at height i, $(i \geq 0)$. The *profile* of M is the sequence $pr(M) = (d_0, \ldots, d_l)$, where $l = \max\{i : d_i \neq 0\}$. Note that this implies that $d_i \geq 1$ for each $i = 0, \ldots, l$, and that the path M is of height l or $l + 1$. The *semi-type* of $M = (s_1, \ldots, s_i)$ is the sequence $st(M) = (\epsilon_1, \ldots, \epsilon_n)$ where $\epsilon_i = 0$ if s_i is a NE or RE step, and $\epsilon_i = 1$ if s_i is a SE or BE step. For example, if M is the path in Figure 2.2, then $pr(M) = (2, 1, 2)$, and $st(M) = (0, 0, 0, 1, 0, 1, 1, 1, 1, 1, 0, 1)$.

Let $\lambda \in \Pi_n$ and $\Phi_r^{-1}(\lambda) = (M, \xi^r)$, $\Phi_l^{-1}(\lambda) = (M, \xi^l)$. Define $\varphi(\lambda) = M \in RBM_n$. Note that for a given λ, $\varphi(\lambda)$ can be easily constructed using the four steps above. The next lemma gives the relation between a partition and its corresponding restricted bicolored Motzkin path and Charlier diagram.

Lemma 4. *Let Φ_r, Φ_l and φ be the maps defined above and $\Phi_r^{-1}(\lambda) = (M, \xi^r)$, $\Phi_l^{-1}(\lambda) = (M, \xi^l)$.*

(a) The number of blocks of λ is equal to the total number of NE and RE steps of M.

(b) $cr(\lambda) = \sum_{i=1}^{n} (\xi_i^r - 1)$, $ne(\lambda) = \sum_{i=1}^{n} (\xi_i^l - 1)$.

(c) $pr(M) = (d_0, \ldots, d_l)$ if and only if $crset(\lambda) = \{0^{d_0}, \ldots, l^{d_l}\}$.

(d) $neseq(\lambda) = v_1 \ldots v_k$ if and only if the zeros in $st(M) = (\epsilon_1, \ldots, \epsilon_n)$ are in the positions $v_1 + 1, v_2 + 2, \ldots, v_k + k$.

Proof. (a) The result follows from the fact that the number of blocks of λ is equal to the total number of openers and singletons.

(b) Denote by E be the set of arcs of λ. For $e = (i, j) \in E$ let $c_e = |\{(p, q) \in E : i < p < j < q\}|$. Then $cr(\lambda) = \sum_{e \in E} c_e$. Similarly, if $n_e = |\{(p, q) \in E : p < i < j < q\}|$, then $ne(\lambda) = \sum_{e \in E} n_e$. From the definitions of Φ_r and Φ_l it follows that $c_{(i,j)} = \xi_j^r - 1$ and $n_{(i,j)} = \xi_j^l - 1$. Hence the claim.

(c) Using the notation at the beginning of section 2.1, we have $crseq(\lambda) = (u_1, \ldots, u_k)$, where u_i is the number of edges (p, q) such that $p < b_i < q$. Here $b_i = \min B_i$, that is, b_i is the i-th opener or singleton from left to right. But the step in M which corresponds to b_i is of height h if and only if $u_i = h$.

(d) It follows directly from the definitions of $neseq(\lambda)$ and $st(M)$. □

A composition of k is an ordered tuple $(d_0, \ldots, d_l)$ of positive integers whose sum is k.

Lemma 5. *Let* $l \geq 0, k \geq 1$, *and* $n \geq k$.

(a) If $\lambda \in \Pi_{n,k}$ *and* $crset(\lambda) = \{0^{d_0}, \ldots, l^{d_l}\}$, *then* $(d_0, \ldots, d_l)$ *is a composition of* k *into* $l+1$ *parts, where* $l \leq n-k$, *and* $0 \leq cr(\lambda) \leq (n-k-1)l - \frac{l(l-1)}{2}$.

(b) Given a composition $(d_0, \ldots, d_l)$ *of* k *into* $l+1 \leq n-k+1$ *parts and an integer* c *such that* $0 \leq c \leq (n-k-1)l - \frac{l(l-1)}{2}$, *there exists* $\lambda \in \Pi_{n,k}$ *with* $crset(\lambda) = \{0^{d_0}, \ldots, l^{d_l}\}$ *and* $cr(\lambda) = c$.

Proof. (a) It is clear that $d_0 + \cdots + d_l = k$. It follows that all the d_i's are positive from part (c) of Lemma 4. Moreover, λ has at least l openers and, therefore, at least l closers. So, $k + l \leq n$, i.e., $l + 1 \leq n - k + 1$. Let c_i (respectively t_i) be the number of SE (respectively BE) steps at level i, $1 \leq i \leq l+1$. Then $\sum_{i=1}^{l+1}(c_i + t_i) = n - k$ and $c_i \geq 1$, $1 \leq i \leq l$. Using part (b) of Lemma 4, we have

$$0 \leq cr(\lambda) \leq \sum_{i=1}^{l}(1+0)(i-1) + (n-k-l)l = (n-k-1)l - \frac{l(l-1)}{2}.$$

(b) Suppose first that $l+1 \leq n-k$. Let $M \in RBM_n$ consist of $d_0 - 1$ RE steps followed by a NE step, then $d_1 - 1$ RE steps followed by one NE step, etc., $d_l - 1$ RE steps followed by a NE step, then $n-k-l-1$ BE steps, and $l+1$ SE steps. It is not hard to see that indeed $M \in RBM_n$. The path never crosses the x-axis and all the BE steps, if any, are at hight $l+1 \geq 1$. Also $pr(M) = (d_0, \ldots, d_l)$. Consider all the sequences $\xi = (\xi_1, \ldots, \xi_n)$ such that (M, ξ) is a Charlier diagram. Then

$$\begin{aligned} \xi_i = 1, &\qquad 1 \le i \le k \\ 1 \le \xi_i \le l+1, &\qquad k+1 \le i \le n-l-1 \\ 1 \le \xi_{n-i+1} \le i, &\qquad 1 \le i \le l+1. \end{aligned} \tag{2.16}$$

Hence

$$0 \le \sum_{i=1}^{n} (\xi_i - 1) \le (n-k-l-1)l + l + \cdots + 1 = (n-k-1)l - \frac{l(l-1)}{2}. \tag{2.17}$$

In the case $l = n - k$, construct $M \in RBM_n$ similarly: $d_0 - 1$ RE steps followed by a NE step, then $d_1 - 1$ RE steps followed by one NE step, etc., d_l RE steps, followed by l SE steps. (Note that, unlike in the case $l < n - k$, the path M is of height l) All the sequences $\xi = (\xi_1, \ldots, \xi_n)$ such that (M, ξ) is a Charlier diagram satisfy the following properties:

$$\begin{aligned} \xi_i = 1, &\qquad 1 \le i \le k \\ 1 \le \xi_{n-i+1} \le i, &\qquad 1 \le i \le l. \end{aligned} \tag{2.18}$$

Hence

$$0 \le \sum_{i=1}^{n} (\xi_i - 1) \le (l-1) + \cdots + 1 = (n-k-1)l - \frac{l(l-1)}{2}. \tag{2.19}$$

Because of (2.17) (respectively (2.19)), for any integer c between 0 and $(n-k-1)l - \frac{l(l-1)}{2}$, ξ can be chosen to satisfy the conditions (2.16) (respectively (2.18)) and such that $\sum_{i=1}^{n} (\xi_i - 1) = c$. Since Φ is a bijection, there is $\lambda \in \Pi_{n,k}$ such that $\Phi(\lambda) = (M, \xi)$ and, by part (b) and (c) of Lemma 4, $cr(\lambda) = c$ and $crset(\lambda) = \{0^{d_0}, \ldots, l^{d_l}\}$. □

Theorem 12. *Let* $n \geq k \geq 1$ *and* $m = \min\{n-k, k-1\}$. *Then*

$$|\Pi_{n,k}/\sim_{cr}| = \sum_{l=0}^{m} \binom{k-1}{l}[(n-k-1)l - \frac{l(l-1)}{2} + 1]. \tag{2.20}$$

In particular, if $n \geq 2k-1$,

$$|\Pi_{n,k}/\sim_{cr}| = (n-k-1)(k-1)2^{k-2} + 2^{k-1} - (k-1)(k-2)2^{k-4}. \tag{2.21}$$

Proof. Recall that $\lambda \sim_{cr} \pi$ if and only if $cr(\lambda) = cr(\pi)$ and $crset(\lambda) = crset(\pi)$. Therefore, $|\Pi_{n,k}/\sim_{cr}| = |\{(crset(\lambda), cr(\lambda)) : \lambda \in \Pi_{n,k}\}|$. Using Lemma 5 and the fact that the number of compositions of k into $l+1$ parts, $0 \leq l \leq k-1$, is $\binom{k-1}{l}$, we derive (2.20). In particular, when $n \geq 2k-1$,

$$|\Pi_{n,k}/\sim_{cr}| = \sum_{l=0}^{k-1} \binom{k-1}{l}[(n-k-1)l - \frac{l(l-1)}{2} + 1].$$

But

$$\begin{aligned}
\sum_{l=0}^{k-1} \binom{k-1}{l} &= (1+x)^{k-1}|_{x=1} = 2^{k-1}, \\
\sum_{l=0}^{k-1} l\binom{k-1}{l} &= \left(\frac{d}{dx}(1+x)^{k-1}\right)|_{x=1} = (k-1)(1+x)^{k-2}|_{x=1} \\
&= (k-1)2^{k-2}, \\
\sum_{l=0}^{k-1} l(l-1)\binom{k-1}{l} &= \left(\frac{d^2}{dx^2}(1+x)^{k-1}\right)|_{x=1} \\
&= (k-1)(k-2)(1+x)^{k-3}|_{x=1} = (k-1)(k-2)2^{k-3},
\end{aligned}$$

and (2.21) follows. □

Theorem 12 implies that there are many more examples of different partitions λ and π for which the statistic cr has same distribution on the the levels of $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$. For example, $|\Pi_{2k,k}| > (2k-1)!! \approx \sqrt{2}\left(\frac{2k}{e}\right)^k$ while $|\Pi_{2k,k}/\sim_{cr}| \approx 3k^2 2^{k-4}$.

Next we analyze the number of nesting-similarity classes. First we derive a recurrence for the numbers $f_{n,k} = |\Pi_{n,k}/\sim_{ne}|$.

Theorem 13. *Let $n \geq k \geq 1$. Then*

$$f_{n,1} = 1, \tag{2.22}$$

$$f_{n,k} = \sum_{r=k-1}^{n-1} f_{r,k-1} + (k-1)\binom{n-2}{k}, \qquad k \geq 2. \tag{2.23}$$

Proof. Equation (2.22) is clear since $|\Pi_{n,1}| = 1$.

Recall that $\lambda \sim_{ne} \pi$ if and only if $ne(\lambda) = ne(\pi)$ and $neseq(\lambda) = neseq(\pi)$. By Lemma 4, $f_{n,k}$ is equal to the number of pairs (ϵ, c) such that there exists $\lambda \in \Pi_{n,k}$ with $ne(\lambda) = c$ and $st(\varphi(\lambda)) = \epsilon$. It is not hard to see that for a given a sequence $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{0,1\}^n$, there exists $\lambda \in \Pi_{n,k}$ such that $st(\varphi(\lambda)) = \epsilon$ if and only if ϵ has k zeros and $\epsilon_1 = 0$. Denote the set of all such sequences by $S^0_{n,k}$ and denote the set of all $\epsilon \in \{0,1\}^n$ with k zeros by $S_{n,k}$.

For a sequence $\epsilon \in S_{n,k}$ define a bicolored Motzkin path $M = M(\epsilon) = (s_1, \ldots, s_n)$ as follows. For i from n to 1 do:

- If $\epsilon_i = 0$ and s_i is not defined yet, then set s_i to be a RE step;
- If $\epsilon_i = 1$ and there is $j < i$ such that $\epsilon_j = 0$ and s_j is not defined yet, then set s_i to be a SE step and s_{j_0} to be a NE step, where $j_0 = \min\{j : \epsilon_j = 0 \text{ and } s_j \text{ is not defined yet}\}$;
- If $\epsilon_i = 1$ and there is no $j < i$ such that $\epsilon_j = 0$ and s_j has not been defined yet, set s_i to be a BE step.

Note that we build M backwards, from $(n, 0)$ to $(0, 0)$. Let h_i be the height of s_i and $ne(\epsilon) = \sum(h_i - 1)$, where the sum is over all the indices i such that $\epsilon_i = 1$. For example, if $\epsilon = (0,0,0,1,0,1,1,1,1,1,0,1)$, then

$$M(\epsilon) = (NE, NE, NE, BE, NE, BE, BE, SE, SE, SE, RE, SE).$$

The sequence of the heights of all the steps of M is $(0,1,2,3,3,4,4,4,3,2,1,1)$ and $ne(\epsilon) = (3-1)+(4-1)+(4-1)+(4-1)+(3-1)+(2-1)+(1-1) = 14$.

Although clearly $M(\epsilon)$ stays above the x-axis, it is not necessarily a restricted bicolored Motzkin path. The reason is that any 1 in ϵ before the first zero would produce a BE step on the x-axis. Hence, $M(\epsilon) \in RBM_n$ if and only if $\epsilon_1 = 0$, or equivalently, $\epsilon \in S^0_{n,k}$.

We claim that for a fixed $\epsilon \in S^0_{n,k}$, there is $\lambda \in \Pi_{n,k}$ such that $st(\varphi(\lambda)) = \epsilon$ and $ne(\lambda) = c$ if and only if $0 \le c \le ne(\epsilon)$. To show the if part, one can choose a sequence $\xi = (\xi_1, \dots, \xi_n)$ with $1 \le \xi_i \le h_i$ if $\epsilon_i = 1$, $\xi_i = 1$ if $\epsilon_i = 0$, and $\sum_{i=1}^n (\xi_i - 1) = c$. Then Lemma 4 implies that $\Phi_l((M,\xi))$ satisfies the requirements. Conversely, suppose $\lambda \in \Pi_{n,k}$ is such that $st(\varphi(\lambda)) = \epsilon$. Let $\Phi_l^{-1}(\lambda) = (M', \xi')$. Then the height $h_i^{'}$ of each BE and SE step of M' satisfies

$$h_i^{'} \le \min\{\#\text{ zeros in } (\epsilon_1, \dots, \epsilon_{i-1}), (\#\text{ ones in } (\epsilon_{i+1}, \dots, \epsilon_n)) + 1\} = h_i.$$

Now, by Lemma 4,

$$ne(\lambda) = \sum (\xi_i^{'} - 1) \le \sum (h_i^{'} - 1) \le \sum (h_i - 1) = ne(\epsilon).$$

The claim is proved. Back to the proof of Theorem 13, we have

$$f_{n,k} = \sum_{\epsilon \in S^0_{n,k}} (ne(\epsilon) + 1) = \sum_{\epsilon \in S^0_{n,k}} (\sum (h_i - 1) + 1) = \sum_{\epsilon \in S^0_{n,k}} \sum h_i - (n-k-1)\binom{n-1}{k-1}.$$

Set

$$g_{n,k} = \sum_{\epsilon \in S^0_{n,k}} \sum h_i \qquad \text{and} \qquad g^*_{n,k} = \sum_{\epsilon \in S_{n,k}} \sum h_i,$$

where the inner sums are taken over all the indices i such that $\epsilon_i = 1$. With this notation,

$$f_{n,k} = g_{n,k} - (n-k-1)\binom{n-1}{k-1}. \tag{2.24}$$

The sequences $g_{n,k}$ and $g^*_{n,k}$ satisfy the following recurrence relations:

$$g_{n,k} = g_{n-1,k-1} + g^*_{n-2,k-1} + (n-k)\binom{n-2}{k-1}, \tag{2.25}$$

$$g^*_{n,k} = \sum_{r=k}^{n} g_{r,k}. \tag{2.26}$$

To see (2.25), note that if $\epsilon_n = 0$ then $(\epsilon_1, ..., \epsilon_{n-1}) \in S^0_{n-1,k-1}$ and the path $M(\epsilon)$ is $M(\epsilon_1, ..., \epsilon_{n-1})$ with one RE step appended, and if $\epsilon_n = 1$ then $(\epsilon_2, ..., \epsilon_{n-1}) \in S_{n-2,k-1}$ and $M(\epsilon_2, ..., \epsilon_{n-1})$ is obtained from $M(\epsilon)$ by deleting the first NE and the last SE step. For (2.26), if $\epsilon_1 = \cdots = \epsilon_{r-1} = 1$ and $\epsilon_r = 0$, then $M(\epsilon_r, \ldots, \epsilon_n)$ is obtained from $M(\epsilon)$ by deleting the first $r-1$ BE steps at level 0. Substituting (2.26) into (2.25) gives

$$g_{n,k} = \sum_{r=k-1}^{n-1} g_{r,k-1} + (n-k)\binom{n-2}{k-1}. \tag{2.27}$$

Finally, by substituting $g_{n,k}$ from (2.24) into (2.27) and simplifying, we obtain (2.23). □

Corollary 4.

$$|\Pi_1/\sim_{ne}| = 1, \qquad |\Pi_2/\sim_{ne}| = 2$$

$$|\Pi_n/\sim_{ne}| = 2^{n-5}(n^2 - 5n + 22), \qquad n \geq 3$$

Proof. Denote $|\Pi_n/\sim_{ne}|$ by F_n. Using $F_n = \sum_{k=1}^{n} f_{n,k}$, (2.22), and (2.23), we get

$$\begin{aligned} F_n &= 1 + F_1 + \cdots + F_{n-1} + \sum_{k=2}^{n}(k-1)\binom{n-2}{k} \\ &= F_1 + \cdots + F_{n-1} + (n-4)2^{n-3} + 2, \qquad n \geq 2. \end{aligned}$$

Table 2.1

The two equivalence relations $\sim_{cr}$ and $\sim_{ne}$ on set partitions are not compatible.

crossing-similarity classes

$n \setminus k$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	2	1			
4	1	3	3	1		
5	1	4	7	4	1	
6	1	5	11	4	5	1

nesting-similarity classes

$n \setminus k$	1	2	3	4	5	6
1	1					
2	1	1				
3	1	2	1			
4	1	4	3	1		
5	1	7	9	4	1	
6	1	11	22	16	5	1

This yields the recurrence relation

$$F_n = 2F_{n-1} + (n-3)2^{n-4}, \qquad n \geq 3$$

with initial values $F_1 = 1$ and $F_2 = 2$, which has the solution

$$F_n = 2^{n-5}(n^2 - 5n + 22), \qquad n \geq 3.$$

□

Table 2.1 gives the number of crossing/nesting-similarity classes on $\Pi_{n,k}$ for small n and k. The two equivalence relations $\sim_{cr}$ and $\sim_{ne}$ on set partitions are not compatible. From Table 2.1 it is clear that $\sim_{cr}$ is not a refinement of $\sim_{ne}$. On the other hand, let $\pi = \{\{1,3\},\{2,4\},\{5,6\}\}$ and $\lambda = \{\{1,3,6\},\{2,4\},\{5\}\}$. It is easy to check that $\pi \sim_{ne} \lambda$, but $\pi \not\sim_{cr} \lambda$, as $cr(\pi) = 1$ and $cr(\lambda) = 2$.

2.4. Generating function for crossings and nestings

In this section we analyze the generating function

$$S_\pi(q,p,z) = \sum_{l\geq 0} \sum_{\lambda\in\mathcal{T}(\pi,l)} q^{cr(\lambda)} p^{ne(\lambda)} z^l$$

for a given partition π, and derive a continued fraction expansion for $S_\pi(q,p,z)$. For this we work with the group $G=\mathbb{Z}\oplus\mathbb{Z}$ and $\alpha=(1,0), \beta=(0,1)$. Fix a partition π with k blocks. Define $E_{\alpha,\beta}(\pi,l) = \cup_{m=k}^{k+l} E_{\alpha,\beta}(\pi,l,m)$, i.e., $E_{\alpha,\beta}(\pi,l)$ is the multiset of sequences $seq_{\alpha,\beta}(\mu)$ associated to the partitions $\mu\in\mathcal{T}(\pi,l)$. A recurrence analogous to (2.9) holds. Namely, for $l\geq 1$

$$E_{\alpha,\beta}(\lambda,l) = R_{\alpha,\beta}(E_{\alpha,\beta}(\lambda,l-1)) \cup M(E_{\alpha,\beta}(\lambda,l-1)). \tag{2.28}$$

For simplicity we write E_l instead of $E_{\alpha,\beta}(\pi,l)$ when there is no confusion. Define $b_{l,r}$ to be the generating function of the multiset $f_0^r(E_l)$, i.e.,

$$b_{l,r}(q,p) = \sum_{(x,y)\in f_0^r(E_l)} q^x p^y,$$

where $(x,y)\in f_0^r(E_l)$ contributes to the sum above according to its multiplicity in $f_0^r(E_l)$. By convention, let $b_{l,r}(q,p)=0$ if $f_0^r(E_l)=\emptyset$, or, one of l,r is negative. For simplicity we write $b_{l,r}$ for $b_{l,r}(q,p)$. Note that $b_{l,0}=\sum_{\lambda\in\mathcal{T}(\pi,l)} q^{cr(\lambda)}p^{ne(\lambda)}$ and hence

$$S_\pi(q,p,z) = \sum_{l\geq 0} b_{l,0} z^l. \tag{2.29}$$

By the formulas (2.28), (2.10), and the proof of part (c) of Lemma 2, we get

$$\begin{aligned} f_0^r(E_l) &= f_0^r(M(E_{l-1})) \cup f_0^r(R_{\alpha,\beta}(E_{l-1})) \\ &= f_0^{r-1}(E_{l-1}) \cup f_0^r(E_{l-1}) \cup \bigcup_{t=0}^{r} f_{t\alpha+(r-t)\beta}^{r+1}(E_{l-1}) \cup \bigcup_{t=0}^{r} f_{t\alpha+(r-t)\beta}^{r}(E_{l-1}), \end{aligned}$$

which leads to a recurrent relation for $b_{l,r}$:

$$b_{l,r} = b_{l-1,r-1} + b_{l-1,r} + (\sum_{t=0}^{r} q^t p^{r-t}) b_{l-1,r+1} + (\sum_{t=0}^{r} q^t p^{r-t}) b_{l-1,r}.$$

Using the standard notation $[r]_{q,p} := \frac{q^r - p^r}{q-p}$, we can write this as

Proposition 2.

$$b_{l,r} = b_{l-1,r-1} + (1 + [r+1]_{q,p}) b_{l-1,r} + [r+1]_{q,p} b_{l-1,r+1}.$$

If the sequence associated to the partition π is $x_1 x_2 \dots x_k$, with $x_i = u_i \alpha + v_i \beta,\ 1 \le i \le k$, then

$$\begin{aligned} b_{0,0} &= q^{u_1} p^{v_1} \\ b_{0,r} &= \sum_{1 < i_1 < \cdots < i_r \le k} q^{u_{i_1} + \cdots + u_{i_r} - (r-1)u_1} p^{v_{i_1} + \cdots + v_{i_r} - (r-1)v_1} \quad \text{for } r \ge 1. \end{aligned} \tag{2.30}$$

In particular, $b_{0,r} = 0$ if $r \ge k$.

Given l and s, nonnegative integers, consider the paths from $(l,0)$ to $(0,s)$ using steps $(-1,0)$, $(-1,1)$, and $(-1,-1)$ which do not go below the x-axis. Each step $(-1,0)$ $((-1,1)$, $(-1,-1)$ respectively) starting at the line $y = r$ has weight $[r+1]_{q,p}$ $(1 + [r+1]_{q,p}$, 1, respectively). The weight $w(M)$ of such a path M is defined to be the product of the weights of its steps. Let $c_{l,s} = \sum w(M)$, where the sum is over all the paths M described above. Then from Proposition 2 one has

$$b_{l,0} = \sum_{0 \le s \le k-1} c_{l,s} b_{0,s}.$$

Set $a_r = [r+1]_{q,p}$ and $c_r = [r+1]_{q,p} + 1$. By the well-known theory of continued fractions (see [17]), $c_{l,s}$ is equal to the coefficient in front of z^l in

$$K_s(z) := J^{/0/}(z) a_0 z J^{/1/}(z) a_1 z \cdots J^{/s/}(z) = \frac{1}{z^s}(Q_{s-1}(z) J(z) - P_{s-1}(z)) \tag{2.31}$$

where

$$J^{/h/}(z) = \cfrac{1}{1 - c_h z - \cfrac{a_h z^2}{1 - c_{h+1} z - \cfrac{a_{h+1} z^2}{\ddots}}}$$

and $\frac{P_k(z)}{Q_k(z)}$ is the k-th convergent of $J(z) := J^{/0/}(z)$. Hence

Theorem 14. *Let π be a set partition with k blocks whose associated sequence is $x_1x_2 \dots x_k$, where $x_i = u_i\alpha + v_i\beta$ for $1 \le i \le k$. Then*

$$S_\pi(q, p, z) = \sum_{0 \le s \le k-1} b_{0,s} K_s(z),$$

where $b_{0,s}$ is given by the formula (2.30)*, and $K_s(z)$ is given by* (2.31)*.*

In particular, when $k = 1$, i.e., π is a partition with only one block, then $b_{0,0} = 1$ and $b_{l,0} = c_{l,0} b_{0,0} = c_{l,0}$. Therefore

Corollary 5. *If $|\pi| = 1$, then*

$$S_\pi(q, p, z) = \cfrac{1}{1 - ([1]_{q,p} + 1)z - \cfrac{[1]_{q,p} z^2}{1 - ([2]_{q,p} + 1)z - \cfrac{[2]_{q,p} z^2}{\ddots}}}.$$

Corollary 5 leads to a continued fraction expansion for the generating function of crossings and nestings over Π: Just taking π to be the partition of $\{1\}$, and bearing in mind that we are counting the empty partition as well, we get

$$\sum_{n\geq 0}\sum_{\lambda\in\Pi_n} q^{cr(\lambda)}p^{ne(\lambda)}z^n = 1 + zS_{\{1\}}(q,p,z)$$

$$= 1 + \cfrac{z}{1-([1]_{q,p}+1)z-\cfrac{[1]_{q,p}z^2}{1-([2]_{q,p}+1)z-\cfrac{[2]_{q,p}z^2}{\ddots}}}. \qquad (2.32)$$

A different expansion was given in [31], as

$$\sum_{n\geq 0}\sum_{\lambda\in\Pi_n} q^{cr(\lambda)}p^{ne(\lambda)}z^n = \cfrac{1}{1-z-\cfrac{z^2}{1-([1]_{q,p}+1)z-\cfrac{[2]_{q,p}z^2}{1-([2]_{q,p}+1)z-\cfrac{[3]_{q,p}z^2}{\ddots}}}}. \qquad (2.33)$$

The fractions (2.33) and (2.32) can be transformed into each another by applying twice the following contraction formula for continued fraction, (for example, see [11]):

$$\cfrac{c_0}{1-\cfrac{c_1z}{1-\cfrac{c_2z}{\ddots}}} = c_0 + \cfrac{c_0c_1z}{1-(c_1+c_2)z-\cfrac{c_2c_3z^2}{1-(c_3+c_4)z-\cfrac{c_4c_5z^2}{\ddots}}}.$$

CHAPTER III

A BIJECTION BETWEEN PARTIALLY DIRECTED PATHS IN THE SYMMETRIC WEDGE AND MATCHINGS

E.J. Janse van Rensburg, T. Prellberg, and A. Rechnitzer in [26] studied the self-avoiding partially directed paths in the wedge $y = \pm px$ consisting of east, north and south steps. Using a variation of the kernel method they were able to derive explicitly the generating function for the case $p = 1$. The generating function revealed that the number of such paths which end at $(n, -n)$ with k north steps is the same as the number of matchings on $[2n]$ with k nestings. In this chapter we give a bijective proof of this fact.

A partially directed path in the plane is a path starting at the origin and consisting of unit east, north, and south steps. We consider all such self-avoiding paths confined to the symmetric wedge defined by the lines $y = \pm x$. Let $\mathcal{P}_n$ be the set of all such paths ending at the line $y = -x$ with n horizontal steps.

Theorem 15. *There is a bijection $\Phi : \mathcal{P}_n \to \mathcal{M}_n$ that takes the number of north steps of $P \in \mathcal{P}_n$ to the number of nestings of $\Phi(P)$.*

Below we define a bijection $\Phi : \mathcal{P}_n \to \mathcal{M}_n$ that takes the number of north steps of $P \in \mathcal{P}_n$ to the number of nestings of $\Phi(P)$. The map Φ is defined as the composition of two maps: $\Phi = \phi \circ \psi$, where $\psi : \mathcal{P}_n \to \mathcal{M}_n$ and $\phi : \mathcal{M}_n \to \mathcal{M}_n$.

3.1. Bijection ψ from $\mathcal{P}_n$ to $\mathcal{M}_n$

Every path $P \in \mathcal{P}_n$ is determined by the y-coordinates of its east steps, i.e., a sequence $a_1, \ldots, a_n$ of integers such that $-(i-1) \le a_i \le i-1$. Set $b_i = a_{n+1-i} + n + 1 - i$. Note that $1 \le b_i \le 2(n+1-i) - 1$. Define a matching M on $[2n]$ by connecting the first available vertex from the left to the b_i-th available vertex to its right, one by one for each $i = 1, \ldots, n$ in that order. Note that before the i-th step there are $2(n+1-i)$ vertices that are not connected yet, so each step is possible. We define $\psi(P) = M$. It is not hard to see that knowing M, one can reverse the steps one by one and find the b_i's, which determine a path P. So ψ is a bijection. Figure 3.1 shows a path $P \in \mathcal{P}_7$ and $\psi(P)$.

Definition 2. *Let $M \in \mathcal{M}_n$. Suppose the edges $e_1, \ldots, e_n$ of M are ordered according to their left endpoints in ascending order. Suppose $e_i = (a, b)$ and $e_{i+1} = (c, d)$. Define*

$$st_i(M) := \begin{cases} |\{v : d \leq v \leq b, v \text{ is a vertex of } e_k, k > i\}|, & \text{if } e_i \text{ and } e_{i+1} \text{ are nested} \\ 0, & \text{otherwise} \end{cases}$$

and

$$st(M) = \sum_{i=1}^{n-1} st_i(M).$$

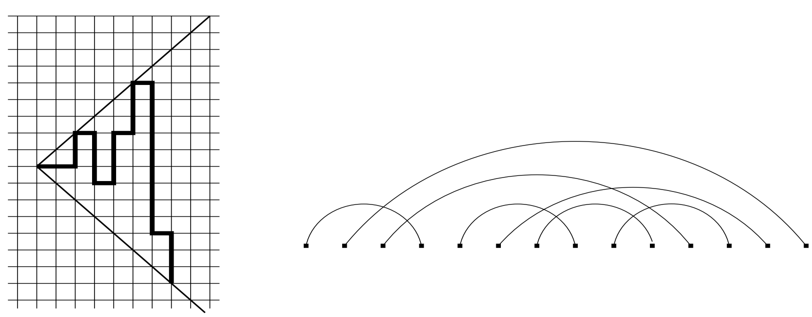


FIG. 3.1 *Path $P \in \mathcal{P}_7$ and the corresponding matching $\psi(P)$.*

Lemma 6. *The number of north steps of P is equal to $st(\psi(P))$.*

Proof. Let $M = \psi(P)$. The number of north steps of P is

$$\sum_{a_{i+1} > a_i} (a_{i+1} - a_i) = \sum_{b_{n-i} \geq b_{n-i+1}+2} (b_{n-i} - b_{n-i+1} - 1) \tag{3.1}$$

So, it suffices to show that

$$st_i(M) = \begin{cases} b_i - b_{i+1} - 1, & \text{if } b_i \geq b_{i+1} + 2 \\ 0, & \text{otherwise} \end{cases} \tag{3.2}$$

After the i-th edge e_i is drawn in the construction of M, there are $b_i - 1$ unconnected vertices below it. In the case $b_i \geq b_{i+1} + 2$, we have $b_i - 1 \geq b_{i+1} + 1$ which implies e_{i+1} is nested below e_i and $st_i(M) = b_i - b_{i+1} - 1$. In the other case, when $b_i < b_{i+1} + 2$, we have $b_i - 1 < b_{i+1} + 1$ and hence the edge e_{i+1} and e_i are crossed (if $b_i > 1$) or aligned (if $b_i = 1$). In either case, $st_i(M) = 0$. □

3.2. Bijection ϕ from $\mathcal{M}_n$ to $\mathcal{M}_n$

We describe ϕ by a series of transformations on the diagrams of the matchings. This map preserves the first edge. For $M \in \mathcal{M}_n$, $N = \phi(M)$ is constructed inductively as follows. If $n = 1$ set $\phi(M) = M$. If $n > 1$, let M_1 be the matching obtained from M by deleting its first edge $e_1 = (1, r)$ and let $N_1 = \phi(M_1)$. Let N_2 be the matching obtained by adding back the edge e_1 in the same position as it was in M. Denote by e_2 the second edge of N_2 (which was also the second edge of M). There are three cases:

case 1: e_1 and e_2 were aligned

In this case set $N = \phi(M) = N_2$.

case 2: e_1 and e_2 were crossed

Let $f_2 = e_2 = (l_2, r_2), f_3 = (l_3, r_3), \ldots, f_k = (l_k, r_k)$ be the edges in N_2 crossing e_1 ordered by their left endpoints $2 = l_2 < l_3 < \cdots < l_k$. Rearrange them in the following way: connect r_2 to l_3, r_3 to $l_4, \ldots, r_{k-1}$ to l_k. Finally, insert one additional vertex right before r and connect it to r_k. Delete the vertex l_2

and renumber the remaining vertices (see Figure 3.2). Note that the position of the first edge in the matching N obtained this way is the same as in M. Set $\phi(M) = N$.

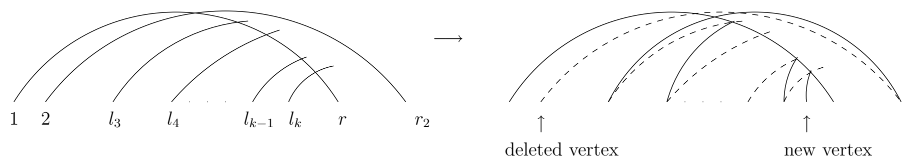


FIG. 3.2 *Definition of ϕ when e_1 and e_2 are crossed. Dashed lines are used to represent edges whose left endpoints have been changed.*

case 3: e_1 and e_2 were nested

In N_2, let $f_1 = (l_1, r_1), \ldots, f_p = (l_p, r_p)$ be the edges crossing both $e_1 = (1, r)$ and $e_2 = (2, q)$, and let $f_{p+1} = (l_{p+1}, r_{p+1}), \ldots, f_{p+s} = (l_{p+s}, r_{p+s})$ be the edges crossing e_1 but not e_2, such that $l_1 < \cdots < l_p < q < l_{p+1} < \cdots < l_{p+s}$. For easier notation let $\{l_1 < \cdots < l_p < q < l_{p+1} < \cdots < l_{p+s}\} = \{v_1 < \cdots < v_p < v_{p+1} < v_{p+2} < \cdots < v_{p+s+1}\}$. Add one vertex right before r and connect it to v_{s+1}. "Rearrange" the edges $f_1, \ldots, f_{p+s}$ so that $r_1, \ldots, r_{p+s}$ are connected to $v_1, \ldots, v_s, v_{s+2}, \ldots, v_{p+s+1}$ in that order. Finally, delete the vertex 2 and renumber the remaining vertices. See Figure 3.3 for an illustration when $p = 3$ and $s = 2$. Call the matching obtained this way N. The first edge of N is the same as in M. Set $\phi(M) = N$.

Example 16. *Figure 3.4 shows step-by-step construction of $\phi(M)$ for the matching M from Figure 3.1. So, for the path P given in Figure 3.1, the corresponding matching is $\Phi(P) = \{(1,4), (2,14), (3,12), (5,8), (6,9), (7,11), (10,13)\}$.*

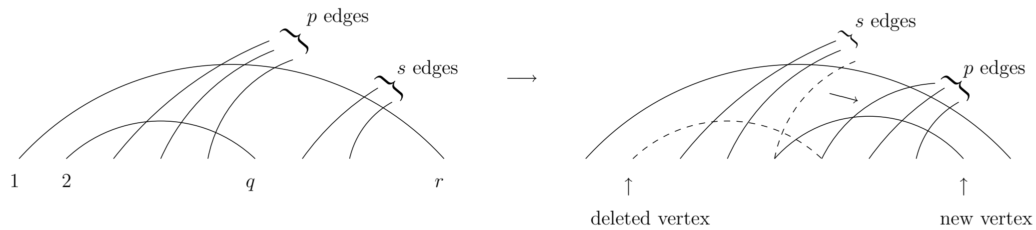


FIG. 3.3 *Example of case 3 for* $p = 3$ *and* $s = 2$.

Theorem 17. *The map* ϕ *is a bijection and* $ne(\phi(M)) = st(M)$.

Proof. To show that ϕ is bijective, we explain how to define the inverse map. Note that the matching resulting from case 1 above has the property that its first edge is (1,2). In the matching resulting from case 2 (case 3 respectively), the vertex preceding the right endpoint of the first edge e_1 is a left endpoint (right endpoint respectively) of an edge different than e_1. Since all the steps in the definition of ϕ are invertible, we simply perform the inverse steps of the corresponding case.

It is left to prove $ne(\phi(M)) = st(M)$. For shortness, for any matching M, let $ne(e, M)$ denote the number of edges in M below the edge e. Let M, M_1, N_1, N_2, and N be the same as in the definition of ϕ. By inductive hypothesis, $ne(N_1) = st(M_1) = st(M) - st_1(M)$. So we just need to prove

$$ne(N) = ne(N_1) + st_1(M) \tag{3.3}$$

It is clear that

$$ne(N_2) = ne(N_1) + ne(e_1, N_2) \tag{3.4}$$

In the first case of the definition of ϕ, (3.3) clearly follows since $st_1(M) = 0$ and we do not add nestings to N_1 by adding back e_1.

In the second case, $st_1(M) = 0$, so we need to show that $ne(N) = ne(N_1)$. To this

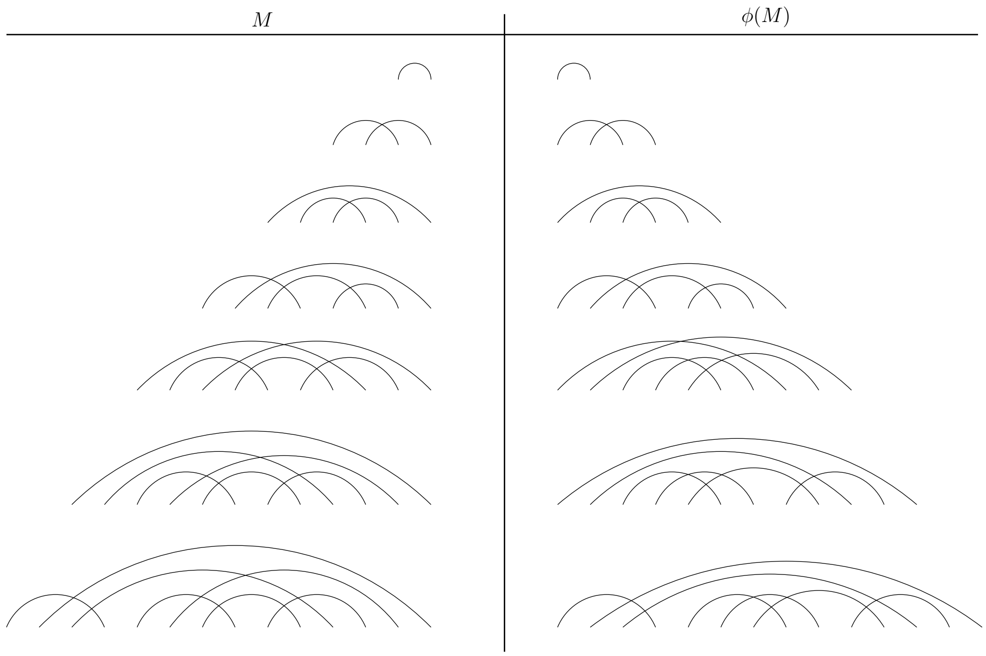


Fig. 3.4 *Example of construction of* $\phi(M)$.

end, if e is an edge in N_2 different from $f_2, \ldots, f_k$ (notation from the definition of ϕ), let $r(e)$ be the edge in N that corresponds to e in the obvious way, and let $r(f_i)$ be the edge with right endpoint r_i, for $i = 2, \ldots, k$. It is clear that $ne(e, N_2) = ne(r(e), N)$ for any edge $e \notin \{e_1, f_2, \ldots, f_k\}$. Note that the left endpoint of $r(f_i)$ in N is $l_{i+1} - 1$ because the vertex 2 from N_2 was deleted (see Figure 3.2). So, for $2 \leq i < k$

$$ne(f_i, N_2) - ne(r(f_i), N) =$$

$$= |\{\text{edges in } N \text{ below } e_1 \text{ with left endpoint between } l_i - 1 \text{ and } l_{i+1} - 1\}| \tag{3.5}$$

$$ne(f_k, N_2) - ne(r(f_k), N) =$$

$$= |\{\text{edges in } N \text{ below } e_1 \text{ with left endpoint between } l_k - 1 \text{ and } r\}| \tag{3.6}$$

By subtracting the following equalities

$$ne(N_2) = \sum_{i=2}^{k} ne(f_i, N_2) + \sum_{e \notin \{f_2,\ldots,f_k\}} ne(e, N_2) \tag{3.7}$$

$$ne(N) = \sum_{i=2}^{k} ne(r(f_i), N) + \sum_{e \notin \{f_2,\ldots,f_k\}} ne(r(e), N) \tag{3.8}$$

and using (3.5) and (3.6) we get

$$ne(N_2) - ne(N) = ne(e_1, N) = ne(e_1, N_2) \tag{3.9}$$

This together with (3.4) gives $ne(N) = ne(N_1)$.

In the third case, similarly, denote by $r(f_i)$ the edge in N that ends with vertex r_i, $i = 1, \ldots, p+s$, by $r(e_2)$ the edge that ends with the vertex $r-1$, and for every other edge e in N_2, denote by $r(e)$ the edge in N that corresponds to e in the natural way. In N_2, define a to be the number of edges below e_1 and crossing $e_2 = (2, q)$ and b to be the number of those edges below e_1 with a left endpoint right of q. In what follows, v_i are the vertices defined in case 3 of the definition of ϕ. Then

$$st_1(M) = 1 + a + 2b + s \tag{3.10}$$

$$ne(r(e_2), N) = |\{\text{edges in } N_2 \text{ below } e_1 \text{ with left endpoint between } v_{s+1} \text{ and } r\}| \tag{3.11}$$

$$ne(N_2) = ne(N_1) + ne(e_2, N_2) + 1 + a + b \tag{3.12}$$

$$ne(N_2) = ne(e_1, N_2) + ne(e_2, N_2) + \sum_{i=1}^{p+s} ne(f_i, N_2) + \sum_{e \notin \{e_1,e_2,f_1,\ldots,f_{p+s}\}} ne(e, N_2) \tag{3.13}$$

$$ne(N) = ne(e_1, N) + ne(r(e_2), N) + \sum_{i=1}^{p+s} ne(r(f_i), N) + \sum_{e \notin \{e_1,e_2,f_1,\ldots,f_{p+s}\}} ne(r(e), N_2) \tag{3.14}$$

To complete the proof, we need to distinguish two cases: $s \geq p$ and $p > s$. When $s \geq p$, close inspection of the "rearrangement" of the edges reveals:

$$ne(r(f_i), N) - ne(f_i, N_2) = \\ = \begin{cases} 1, 1 \leq i \leq & p \\ 1 + \left|\{\text{edges in } N_2 \text{ below } e_1 \text{ with left vertex between } v_i \text{ and } v_{i+1}\}\right|, p < i \leq & s \\ 0, s < i \leq p+ & s \end{cases} \tag{3.15}$$

while when $p > s$, similar equalities hold:

$$ne(r(f_i)) - ne(f_i) = \\ = \begin{cases} 1, 1 \leq i \leq & s \\ -\left|\{\text{edges in } N_2 \text{ below } e_1 \text{ with left vertex between } v_i \text{ and } v_{i+1}\}\right|, s < i \leq & p \\ 0, p < i \leq p+ & s \end{cases} \tag{3.16}$$

Now, we add the equations (3.12) and (3.14) and subtract (3.13) from them. Using (3.10), (3.11), and (3.15), i.e., (3.16), we get (3.3). □

3.3. Some properties of Φ

First we need few definitions. We say that $\{l, l{+}1, \ldots, k\}$ is a component of a matching $M \in \mathcal{M}_n$ if the restrictions of M on each of the sets $\{1, \ldots, l-1\}$, $\{l, l+1, \ldots, k\}$, and $\{k+1, \ldots, n\}$ are matchings themselves. A matching is called irreducible if it has only one component. In terms of diagrams, a matching is irreducible if it cannot be split by vertical bars into disjoint matchings.

A component of a path $P \in \mathcal{P}_n$ is a subsequence of consecutive steps beginning at $(l, -l)$ and ending at $(k, -k)$ such that both parts of P between $(l, -l)$ and $(k, -k)$, and between $(k, -k)$ and $(n, -n)$ when translated by the appropriate vector to the origin represent paths in $\mathcal{P}_{k-l}$ and $\mathcal{P}_{n-k}$ respectively. A component which does not have nontrivial subcomponents is called irreducible.

Proposition 3. *For $P \in \mathcal{P}_n$ the following are true:*

(a) P has k south steps on the line $x = n$ if and only if in $\Phi(P)$, 1 is connected to $k+1$.

(b) The irreducible components of P read backwards are in one-to-one correspondence with the irreducible components of $\Phi(P)$ from left to right.

Proof. (a) From the definition of ψ, it is clear that P has k south steps on the line $x = n$ if and only in $\psi(P)$, 1 is connected to $k+1$. Thus, the claim follows from the fact that ϕ preserves the first edge.

(b) This statement is clearly true if we replace Φ by ψ. Hence, it suffices to observe that if the irreducible components of $\psi(P)$ are $C_1, \ldots, C_k$, then $\phi(C_1), \ldots, \phi(C_k)$ are the irreducible components of $\Phi(P)$.

□

Proposition 4. *If P is a path with no north steps (Dyck path) then $M = \Phi(P)$ is the unique matching with no nestings such that i is a left endpoint in M exactly when the $(2n+1-i)$-th step of P is a south step.*

In other words, the set of left and right endpoints of M is determined by P traced backwards.

Proof. It follows from the definition of ψ that the statement is true for $\psi(P)$. Moreover, since $\psi(P)$ has no nestings, ϕ leaves $\psi(P)$ unchanged. □

CHAPTER IV

MAJOR INDEX FOR 01-FILLINGS OF MOON POLYOMINOES

In this chapter we extend the major index to certain 01-fillings of moon polyominoes which include words and set partitions. In section 4.1 we introduce the necessary notations, and describe the definition of the major index for 01-fillings of moon polyominoes. We explain in section 4.2 how the classical definition of major index on words and set partitions can be obtained by considering moon polyominoes of special shapes. In section 4.3 we show that the maj statistic is equally distributed as ne, the number of north-east chains of length 2, by computing the corresponding generating functions. In the fillings of special shapes, ne corresponds to the number of inversions for words and crossings for set partitions. Therefore, our main result, Theorem 21, leads to a generalization of (1.1) and the analogous result for set partitions. In section 4.4, we present a bijective proof for the equidistribution of maj and ne, which consists of three maps. The first is from the fillings of left-aligned stack polyominoes to itself that sends maj to ne, constructed in the spirit of Foata's second fundamental transformation. The other two are maps that transform a moon polyomino to a left-aligned stack polyomino with the same set of columns, while preserving the statistics maj and ne, respectively. Composing these three maps yields the desired bijection.

4.1. Definition of major index for fillings of moon polyominoes

We are concerned with 01-fillings of moon polyominoes with restricted row sums. That is, given a moon polyomino $\mathcal{M}$, we assign a 0 or a 1 to each cell of $\mathcal{M}$ so that there is at most one 1 in each row. In this chapter we will simply use the term *filling* to denote such 01-filling. Also we will use the term north-east chain, or shortly NE chain, to denote a north-east chain of length 2. The number of NE chains of M will be denoted ne(M). We say a cell is empty if it is assigned a 0, and it is a 1-cell otherwise. Given a moon polyomino $\mathcal{M}$ with m columns, we label them $c_1, \ldots, c_m$ from left to right . Let $\mathbf{s} = (s_1, \ldots, s_m)$ be an m-tuple of nonnegative integers and A be a subset of rows of $\mathcal{M}$. We denote by $\mathbf{F}(\mathcal{M}, \mathbf{s}; A)$ the set of fillings M of $\mathcal{M}$ such that the empty rows of M are exactly those in A and the column c_i has exactly s_i many 1's, $1 \leq i \leq m$. The filling M in Figure 4.1 is in $\mathbf{F}(\mathcal{M}, \mathbf{s}; A)$ for $A = \{3\}$ and $\mathbf{s} = (1, 1, 2, 1, 1, 0)$.

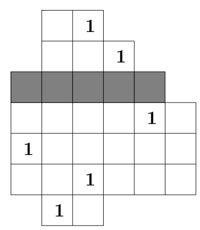


FIG. 4.1 *Filling M of a moon polyomino $\mathcal{M}$ with* ne$(M) = 5$.

The set $\mathbf{F}(\mathcal{M}, \mathbf{s}; A)$ was first studied by Kasraoui [30], who showed that the numbers of north-east and south-east chains are equally distributed. The generating functions for these statistics are also presented in [30, Theorem 2.2].

To define the major index for fillings of moon polyominoes, we first state it for rectangular shapes, which is essentially the classical definition of major index for words, (c.f. section 4.2.1).

Let R be a filling of a rectangle whose n *nonempty* rows $r_1, \ldots, r_n$ are numbered from top to bottom. We define the *descent statistic* for R as

$$\text{des}(R) = |\{i \mid \text{ the 1-cells in rows } r_i \text{ and } r_{i+1} \text{ form an NE chain}\}|.$$

That is, descents of a rectangular filling are NE chains in consecutive nonempty rows.

For each nonempty row r_i of R, let $R(r_i)$ denote the rectangle that contains the row r_i and all rows in R above the row r_i.

Definition 3. *The major index for the rectangular filling R is defined to be*

$$\mathrm{maj}(R) = \sum_{i=1}^{n} \mathrm{des}(R(r_i)). \tag{4.1}$$

Clearly, the empty rows of R do not play any role in $\mathrm{maj}(R)$. That is, if we delete them, the major index of the resulting filling remains the same.

Definition 4. *Let M be a filling of a moon polyomino $\mathcal{M}$. Let $\mathcal{R}_1, \ldots, \mathcal{R}_r$ be the list of all the maximal rectangles contained in $\mathcal{M}$ ordered increasingly by height, and denote by R_i the filling M restricted on the rectangle $\mathcal{R}_i$. Then the major index of M is defined to be*

$$\mathrm{maj}(M) = \sum_{i=1}^{r} \mathrm{maj}(R_i) - \sum_{i=1}^{r-1} \mathrm{maj}(R_i \cap R_{i+1}), \tag{4.2}$$

where $\mathrm{maj}(R_i)$ *and* $\mathrm{maj}(R_i \cap R_{i+1})$ *are defined by* (4.1).

In particular, $\mathrm{maj}(M) = \mathrm{maj}(R_1)$ when M is a rectangular shape. It is also clear that $\mathrm{maj}(M)$ is always nonnegative since $\mathrm{maj}(R_{i+1}) \geq \mathrm{maj}(R_i \cap R_{i+1})$, $i = 1, \ldots, r-1$.

Example 18. *Consider the filling from Figure 4.1. It has four maximal rectangles and* $\mathrm{maj}(M) = (2+2+1+1) - (2+0+0) = 4$. *See Figure 4.2.*

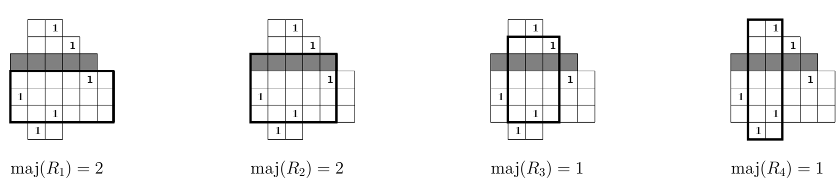


FIG. 4.2 *Calculation of* $\mathrm{maj}(M)$ *for a moon polyomino using Definition 4.*

The major index can be equivalently defined in a slightly more complicated way. However, this way is useful in proofs, especially when one uses induction on the number of columns of the filling, (c.f. Theorem 4.1). We state this equivalent definition next. First we need some notation. Let $Left(\mathcal{M})$ be the set of columns of $\mathcal{M}$ which are to left of the columns of maximal length and let $Right(\mathcal{M})$ consist of the remaining columns of $\mathcal{M}$.

We order the columns of $\mathcal{M}$, $c_1, \ldots, c_m$, by a total order $\prec$ as follows: $c_i \prec c_j$ if and only if

- $|c_i| < |c_j|$ or
- $|c_i| = |c_j|$, $c_i \in Left(\mathcal{M})$ and $c_j \in Right(\mathcal{M})$, or
- $|c_i| = |c_j|$, $c_i, c_j \in Left(\mathcal{M})$ and c_i is to the left of c_j, or
- $|c_i| = |c_j|$, $c_i, c_j \in Right(\mathcal{M})$ and c_i is to the right of c_j,

where $|c|$ denotes the length of the column c. A similar ordering of rows was used in [30].

For every column $c_i \in Left(\mathcal{M})$ define the rectangle $\mathcal{M}(c_i)$ to be the largest rectangle that contains c_i as the leftmost column. For $c_i \in Right(\mathcal{M})$, the rectangle $\mathcal{M}(c_i)$ is taken to be the largest rectangle that contains c_i as the rightmost column and does not contain any columns from $Left(\mathcal{M})$ of same length as c_i.

Definition 4'. Let $c_{i_1} \prec c_{i_2} \prec \cdots \prec c_{i_m}$ be the ordering of the columns of $\mathcal{M}$ and let M_j be the restriction of M on the rectangle $\mathcal{M}(c_{i_j})$, $1 \leq j \leq m$. Then

$$\mathrm{maj}(M) = \sum_{j=1}^{m} \mathrm{maj}(M_j) - \sum_{j=1}^{m-1} \mathrm{maj}(M_j \cap M_{j+1}), \tag{4.3}$$

where $\mathrm{maj}(M_j)$ and $\mathrm{maj}(M_j \cap M_{j+1})$ are defined by (4.1).

Example 19. *Consider the filling M from Figure 4.1. The order $\prec$ on the columns of $\mathcal{M}$ is $c_6 \prec c_1 \prec c_5 \prec c_4 \prec c_3 \prec c_2$. So, $\mathcal{M}_1 = \mathcal{M}(c_6)$, $\mathcal{M}_2 = \mathcal{M}(c_1)$, $\mathcal{M}_3 = \mathcal{M}(c_5)$, $\mathcal{M}_4 = \mathcal{M}(c_4)$, $\mathcal{M}_5 = \mathcal{M}(c_3)$, and $\mathcal{M}_6 = \mathcal{M}(c_2)$, as illustrated in Figure 4.3. By Definition 4', $\mathrm{maj}(M) = (2+2+1+1+1+0) - (2+1+0+0+0) = 4$.*

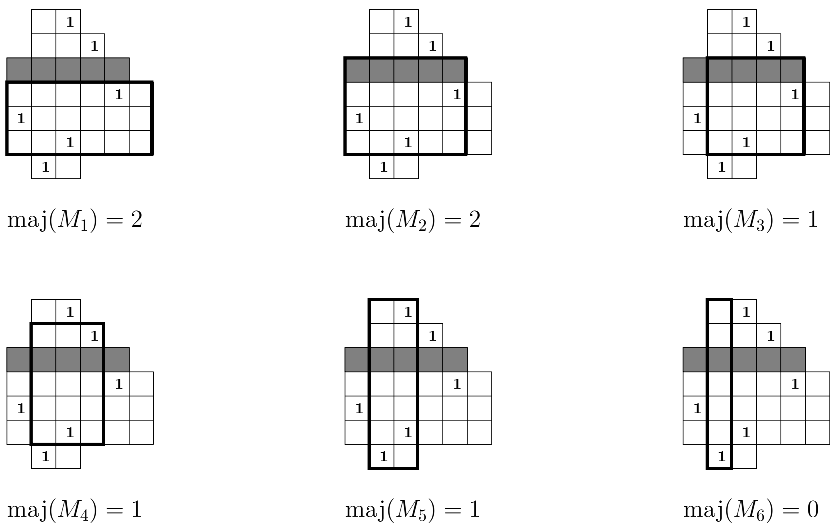


FIG. 4.3 *Calculation of* $\mathrm{maj}(M)$ *for a moon polyomino using Definition 4'.*

Proposition 5. *Definitions 4 and 4' are equivalent.*

Proof. Some of the rectangular fillings M_j in Definition 4' may contain M_{j+1}, in which case $\mathrm{maj}(M_{j+1}) - \mathrm{maj}(M_j \cap M_{j+1}) = 0$ and formula (4.3) can be simplified. More precisely, there are uniquely determined indices $1 = j_1 < j_2 < \cdots < j_r \leq m$ such that $M_1 = M_{j_1} \supseteq M_2 \supseteq \cdots \supseteq M_{j_2-1} \not\supseteq M_{j_2} \supseteq \cdots \supseteq M_{j_r-1} \not\supseteq M_{j_r} \supseteq \cdots \supseteq M_m$. That is, if the rectangles M_i are ordered by containment, then $M_{j_1}, M_{j_2}, \ldots, M_{j_r}$ are the maximal elements. Following the notation in Definition 4, the filling M_{j_k} is the filling R_k of the k-th maximal rectangle contained in $\mathcal{M}$. Then, after cancellation of some terms, the right-hand side of (4.3) becomes the right-hand side of (4.2). For instance, in the previous example, the rectangles M_3 and M_6 are not maximal and therefore $\mathrm{maj}(M_3) - \mathrm{maj}(M_2 \cap M_3) = 0$ and $\mathrm{maj}(M_6) - \mathrm{maj}(M_5 \cap M_6) = 0$. □

4.2. maj(M) for special shapes $\mathcal{M}$

It is well-known that permutations and set-partitions can be represented as fillings of Ferrers diagrams (e.g. [34]). Here we will describe such presentations to show how to get the classical major index for words and set partitions from Definition 4.

4.2.1. When $\mathcal{M}$ is a rectangle: words and permutations

For any rectangle, label the rows from top to bottom, and columns from left to right. Fillings of rectangles are in bijection with words. More precisely, let $w = w_1 \dots w_n$ be a word with letters in the set $[m]$. The word w can be represented as a filling M of an $n \times m$ rectangle $\mathcal{M}$ in which the cell in row $n - i + 1$ and column $m - j + 1$ is assigned the integer 1 if $w_i = j$, and is empty otherwise. Conversely, each filling of the rectangle $\mathcal{M}$ corresponds to a word w. It is easy to check that $\mathrm{ne}(M) = \mathrm{inv}(w)$, $\mathrm{des}(M) = \mathrm{des}(w)$, and $\mathrm{maj}(M) = \mathrm{maj}(w)$. The latter follows from the fact that $\mathrm{maj}(w) = \sum_{i=1}^{n} \mathrm{des}(w_i w_{i+1} \dots w_n)$.

4.2.2. When $\mathcal{M}$ is a Ferrers diagram: matchings and set partitions

As explained in [13], general fillings of Ferrers diagrams correspond to multigraphs. Here we briefly describe the correspondence when restricted to 01-fillings with row sum at most 1. The Ferrers diagram is bounded by a vertical line from the left, a horizontal line from above, and a path consisting of east and north steps. Following this path starting from the bottom left end, we label the steps of the path by $1, 2, \dots, n$. This gives a labeling of the columns and rows of the diagram. Draw n vertices on a horizontal line and draw an edge connecting vertices i and j if and only if there is a 1 in the cell of the column labeled i and the row labeled j (see Figure 4.4). The resulting graph has no loops or multiple edges and every vertex is a right endpoint of at most one edge. The NE chains correspond to crossings of edges in the graph. If the column sums are at most one, the graph is a matching. If, moreover, in the labeling of the steps we allow to have a vertical step followed by a horizontal one with a same label, then the corresponding graph will have vertices which are both right and left endpoints of edges. Such graphs represent set partitions. In either case, the major index of the filling is equal to the major index of the corresponding matching or set partition as defined in [7]. Restricting to the triangular Ferrers diagram, the 01-fillings considered in this paper also include *linked partitions*, a combinatorial structure that was studied in [10, 30].

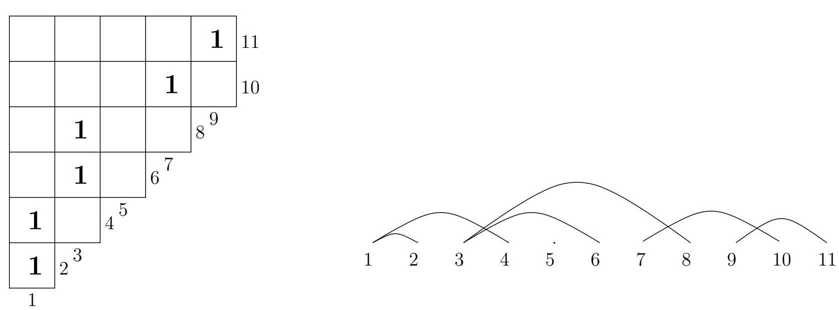


FIG. 4.4 *A filling of a Ferrers diagram and the corresponding graph.*

4.2.3. When $\mathcal{M}$ is a top-aligned stack polyomino

In the case when $\mathcal{M}$ is a top-aligned stack polyomino, $\mathrm{maj}(M)$ has a simpler form in terms of the des statistic, just as in words and permutations. Explicitly, if the nonempty rows of M are $r_1, \ldots, r_n$, denote by $M(r_i)$ the filling M restricted to the largest rectangle contained in $\mathcal{M}$ whose bottom row is r_i.

Proposition 6. *Let M be a filling of a top-aligned stack polyomino and $M(r_i)$ be as defined above. Then*

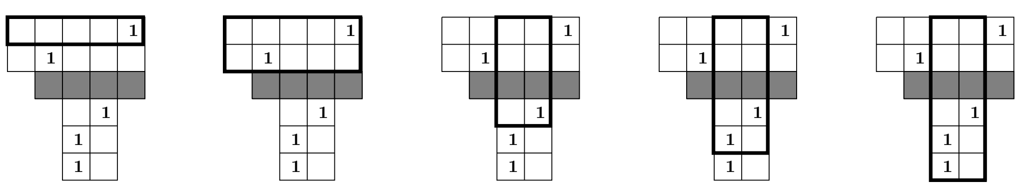
$$\operatorname{maj}(M) = \sum_{i=1}^{n} \operatorname{des}(M(r_i)). \tag{4.4}$$

Proof. Let $r_{k+1}, \ldots, r_n$ be all the nonempty rows of M that are of same length as the last row r_n. Denote by $M' = M \backslash \{r_{k+1}, \ldots, r_n\}$ and by R the filling of the largest rectangle of $\mathcal{M}$ containing r_n. We proceed by induction on the number n of rows of M. The claim is trivial when M has only one row or is a rectangle. Otherwise,

$$\begin{aligned} \operatorname{maj}(M) &= \operatorname{maj}(M') + \operatorname{maj}(R) - \operatorname{maj}(M' \cap R) && \text{(Definition 4)} \\ &= \sum_{i=1}^{k} \operatorname{des}(M(r_i)) + \sum_{i=k+1}^{n} \operatorname{des}(M(r_i)), && \text{(Ind. hyp. and Definition 3)} \end{aligned}$$

which completes the proof. □

Example 20. *Consider the filling M of a top-aligned stack polyomino in Figure 4.5 with $A = \{3\}$ and $\boldsymbol{s} = (0, 1, 2, 1, 1)$. M has five nonempty rows and, by Proposition 6,* $\operatorname{maj}(M) = \sum_{i=1}^{5} \operatorname{des}(M(r_i)) = 0 + 1 + 0 + 1 + 1 = 3.$



$\operatorname{des}(M(r_1)) = 0$ $\operatorname{des}(M(r_2)) = 1$ $\operatorname{des}(M(r_3)) = 0$ $\operatorname{des}(M(r_4)) = 1$ $\operatorname{des}(M(r_5)) = 1$

FIG. 4.5 *Calculation of* maj(M) *using Proposition 6.*

Proposition 6 will be used in the proof of Lemma 9.

4.3. The generating function for the major index

In this section we state and prove the main theorem which gives the generating function for maj over $\mathbf{F}(\mathcal{M}, \mathbf{s}; A)$, the set of all fillings of the moon polyomino $\mathcal{M}$ with column sums given by the integer sequence $\mathbf{s}$ and empty rows given by A . Let $|c_1|, \ldots, |c_m|$ be the column lengths of M and let a_i the number of rows in A that intersect the column c_i. Suppose that, under the ordering defined in section 4.1, $c_{i_1} \prec c_{i_2} \prec \cdots \prec c_{i_m}$. Then for $j = 1, \ldots, m$, define

$$h_{i_j} = |c_{i_j}| - a_{i_j} - (s_{i_1} + s_{i_2} + \cdots + s_{i_{j-1}}) \tag{4.5}$$

The numbers h_i have the following meaning: if one fills in the columns of M from smallest to largest according to the order $\prec$, then h_{i_j} is the number of available cells in the j-th column to be filled.

Theorem 21. *For a moon polyomino $\mathcal{M}$ with m columns,*

$$\sum_{M \in \boldsymbol{F}(\mathcal{M}, \boldsymbol{s}; A)} q^{\mathrm{maj}(M)} = \prod_{i=1}^{m} \begin{bmatrix} h_i \\ s_i \end{bmatrix}_q \tag{4.6}$$

where h_i is defined by (4.5).

We postpone the proof of Theorem 21 until the end of this section.

Example 22. *Suppose $\mathcal{M}$ is the first moon polyomino in Figure 1.1, $A = \{5\}$, and $\boldsymbol{s} = (1, 0, 2, 1, 1)$. The $\prec$ order on the columns of $\mathcal{M}$ is: $c_1 \prec c_5 \prec c_2 \prec c_4 \prec c_3$. The fifth row intersects all columns except c_1, so, $a_1 = 0$ and $a_2 = a_3 = a_4 = a_5 = 1$.*

Therefore,

$$h_1 = h_{i_1} = |c_1| - a_1 = 2 - 0 = 2$$

$$h_5 = h_{i_2} = |c_5| - a_5 - s_1 = 3 - 1 - 1 = 1$$

$$h_2 = h_{i_3} = |c_2| - a_2 - (s_1 + s_5) = 4 - 1 - (1 + 1) = 1$$

$$h_4 = h_{i_4} = |c_4| - a_4 - (s_1 + s_5 + s_2) = 6 - 1 - (1 + 1 + 0) = 3$$

$$h_3 = h_{i_5} = |c_3| - a_3 - (s_1 + s_5 + s_2 + s_4) = 6 - 1 - (1 + 1 + 0 + 1) = 2$$

By Theorem 21, $\sum_{M \in \boldsymbol{F}(\mathcal{M}, \boldsymbol{s}; A)} q^{\mathrm{maj}(M)} = \prod_{i=1}^{m} \begin{bmatrix} h_i \\ s_i \end{bmatrix}_q = \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \begin{bmatrix} 1 \\ 0 \end{bmatrix}_q \begin{bmatrix} 2 \\ 2 \end{bmatrix}_q \begin{bmatrix} 3 \\ 1 \end{bmatrix}_q \begin{bmatrix} 1 \\ 1 \end{bmatrix}_q = (1+q)(1+q+q^2) = 1 + 2q + 2q^2 + q^3$. *The fillings in* $\boldsymbol{F}(\mathcal{M}, \boldsymbol{s}; A)$ *are listed in Figure 4.6.*

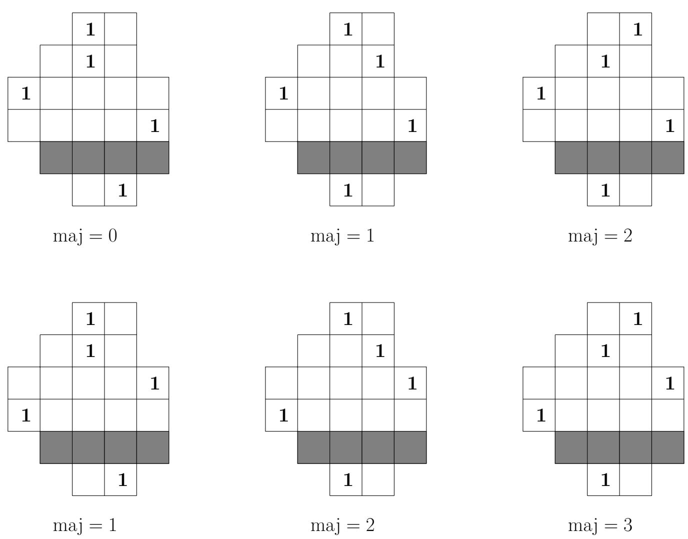


FIG. 4.6 *Illustration of Theorem 21.*

Corollary 6. *Let* σ *be a permutation of* $[m]$ *and let* $\mathcal{M}$ *be a moon polyomino with columns* $c_1, \ldots, c_m$*. Suppose the shape* $\mathcal{N}$ *with columns* $c'_i = c_{\sigma(i)}$ *is also a moon* *polyomino and* $\boldsymbol{s}' = (s'_1, \ldots, s'_m)$ *with* $s'_i = s_{\sigma(i)}$*. Then*

$$\sum_{M \in \boldsymbol{F}(\mathcal{M}, \boldsymbol{s}; A)} q^{\mathrm{maj}(M)} = \sum_{N \in \boldsymbol{F}(\mathcal{N}, \boldsymbol{s}'; A)} q^{\mathrm{maj}(N)}. \tag{4.7}$$

That is, the generating function $\sum_{M \in \boldsymbol{F}(\mathcal{M}, \boldsymbol{s}; A)} q^{\mathrm{maj}(M)}$ *does not depend on the order of the columns of* $\mathcal{M}$*.*

Proof. The formula (4.6) seems to depend on the order $\prec$ of the columns of the moon polyomino. However, note that $c_i \prec c_j$ if $|c_i| < |c_j|$; and columns of same length are consecutive in the order $\prec$. Hence it suffices to compare the terms in the right-hand side of (4.6) that come from columns of fixed length. If $c_{i_{j+1}} \prec \cdots \prec c_{i_{j+k}}$ are all the columns of $\mathcal{M}$ of fixed length, then

$$h_{i_{j+r}} = h_{i_{j+1}} - (s_{i_{j+1}} + \cdots + s_{i_{j+r-1}}),\ 2 \le r \le k. \tag{4.8}$$

Thus the contribution of these columns to the right-hand side of (4.6) is

$$\prod_{r=1}^{k} \begin{bmatrix} h_{i_{j+r}} \\ s_{i_{j+r}} \end{bmatrix}_q = \begin{bmatrix} h_{i_{j+1}} \\ s_{i_{j+1}}, \ldots, s_{i_{j+k}}, h_{i_{j+1}} - (s_{i_{j+1}} + \cdots + s_{i_{j+k}}) \end{bmatrix}_q. \tag{4.9}$$

The last q-multinomial coefficient is invariant under permutations of $s_{i_{j+1}}, \ldots, s_{i_{j+k}}$, and the claim follows.

□

The right-hand side of formula (4.6) also appeared in the generating function of NE chains, as shown in [30]:

Theorem 23 (Kasraoui)**.**

$$\sum_{M \in \boldsymbol{F}(\mathcal{M}, \boldsymbol{s}; A)} q^{\mathrm{ne}(M)} = \prod_{i=1}^{m} \begin{bmatrix} h_i \\ s_i \end{bmatrix}_q.$$

Combining Theorems 21 and 23, we have

Theorem 24.

$$\sum_{M\in \boldsymbol{F}(\mathcal{M},\boldsymbol{s};A)} q^{\mathrm{maj}(M)} = \sum_{M\in \boldsymbol{F}(\mathcal{M},\boldsymbol{s};A)} q^{\mathrm{ne}(M)}.$$

That is, the maj statistic has the same distribution as the ne statistic over the set $\mathbf{F}(\mathcal{M},\mathbf{s},A)$. Since $\mathrm{ne}(M)$ is the natural analogue of inv for words and permutations, Theorem 24 generalizes MacMahon's equidistribution result. Unfortunately, as noted in [7], the refinement of MacMahon's theorem which asserts that the joint distribution of maj and inv over permutations is symmetric does not hold even for matchings.

We shall prove a lemma about the major index for words and then use it to prove Theorem 21.

Lemma 7. *Let $w = w_1 \dots w_k$ be a word such that $w_i < n$ for all i. Consider the set $S(w)$ of all the words w' that can be obtained by inserting m many n's between the letters of w. Then the difference* $\mathrm{maj}(w') - \mathrm{maj}(w)$ *ranges over the multiset* $\{i_1 + i_2 + \cdots + i_m \mid 0 \le i_1 \le i_2 \le \cdots \le i_m \le k\}$. *It follows that*

$$\sum_{w'\in S(w)} q^{\mathrm{maj}(w')} = q^{\mathrm{maj}(w)} \sum_{0\le i_1\le i_2\le\cdots\le i_m\le k} q^{i_1+i_2+\cdots+i_m} = q^{\mathrm{maj}(w)} \begin{bmatrix} k+m \\ m \end{bmatrix}_q . \qquad (4.10)$$

The same statement holds if $w = w_1 \dots w_k$ is a word such that $w_i > 1$ for all i and $S(w)$ is the set of all the words w' obtained by inserting m many 1's between the letters of w.

Proof. We give an elementary proof for the case of inserting n's. The case of inserting 1's is dealt with similarly. For a word $w = w_1 \dots w_k$ we define the *descent sequence* $\mathrm{desseq}(w) = a_1 \dots a_{k+1}$ by letting $a_i = \mathrm{maj}(w^{(i)}) - \mathrm{maj}(w)$, where $w^{(i)}$ is obtained by inserting one n in the i-th gap of w, i.e., $w^{(i)} = w_1 \dots w_{i-1} n w_i \dots w_k$. Note that

$$\mathrm{maj}(w^{(i)}) = \mathrm{maj}(w) + \begin{cases} 1 + \mathrm{des}(w_i \dots w_k), & \text{if } w_{i-1} > w_i,\ 2 \le i \le k \text{ or } i = 1 \\ i + \mathrm{des}(w_i \dots w_k), & \text{if } w_{i-1} \le w_i,\ 2 \le i \le k \\ 0, & \text{if } i = k+1. \end{cases}$$

Hence, if $t = \mathrm{des}(w)$ then $\mathrm{desseq}(w)$ is a shuffle of the sequences $t+1, t, \dots, 0$ and $t+2, t+3, \dots, k$ which begins with $t+1$ and ends with 0. Moreover, every such sequence is $\mathrm{desseq}(w)$ for some word w. By definition, the multiset $\{\mathrm{maj}(w') - \mathrm{maj}(w) \mid w' \in S(w)\}$ depends only on $\mathrm{desseq}(w)$ and not on the letters of w.

First we consider the case when $\mathrm{desseq}(w) = k(k-1)\dots 0$, i.e., $\mathrm{Des}(w) = [k-1]$. Inserting b n's between w_{i-1} and w_i increases the major index by $b(k-i+1)$. Hence, if one inserts b_i many n's between w_{i-1} and w_i for $1 \le i \le k+1$, the major index is increased by

$$\sum_{i=1}^{k+1} b_i(k-i+1) = \underbrace{k + \cdots + k}_{b_1} + \underbrace{(k-1) + \cdots + (k-1)}_{b_2} + \cdots + \underbrace{0 + \cdots + 0}_{b_{k+1}}.$$

Letting $(b_1, \dots, b_{k+1})$ range over all $(k+1)$-tuples with $\sum_{i=1}^{k+1} b_i = m$, the increased amount ranges over the multiset $\{i_1 + i_2 + \cdots + i_m \mid 0 \le i_1 \le i_2 \le \cdots \le i_m \le k\}$.

To deal with the other possible descent sequences, we note that any descent sequence $\mathrm{desseq}(w)$ can be obtained by applying a series of adjacent transpositions to $k(k-1)\dots 0$, where we only transpose adjacent elements i, j which correspond to ascent and descent positions, respectively. Suppose

$$\mathrm{desseq}(w) = a_1 \dots a_{i-1} a_{i+1} a_i a_{i+2} \dots a_{k+1}$$

is obtained by transposing the i-th and $(i+1)$-st element in

$$\mathrm{desseq}(v) = a_1 \dots a_{i-1} a_i a_{i+1} a_{i+2} \dots a_{k+1},$$

where a_i and a_{i+1} correspond to an ascent and a descent in v, respectively. Consequently, $w_{i-1} > w_i \leq w_{i+1}$ and $v_{i-1} \leq v_i > v_{i+1}$, while $w_j \leq w_{j+1}$ if and only if $v_j \leq v_{j+1}$ for $j \neq i-1, i$. It suffices to show that the multisets $\{\text{maj}(w') - \text{maj}(w) \mid w' \in S(w)\}$ and $\{\text{maj}(v') - \text{maj}(v) \mid v' \in S(v)\}$ are equal. Suppose $w' \in S(w)$ is obtained by inserting b_j many n's in the j-th gap of w, $1 \leq j \leq k+1$. Let $v' \in S(v)$ be the word obtained by inserting c_j n's in the j-th gap of v where c_j are defined as follows. If $b_{i+1} = 0$ then

$$c_j = \begin{cases} 0, & \text{if } j = i \\ b_i, & \text{if } j = i+1 \\ b_j, & \text{if } j \neq i, i+1. \end{cases} \tag{4.11}$$

If $b_{i+1} > 0$ then

$$c_j = \begin{cases} b_i + 1, & \text{if } j = i \\ b_{i+1} - 1, & \text{if } j = i+1 \\ b_j, & \text{if } j \neq i, i+1. \end{cases} \tag{4.12}$$

This defines a map between sequences (b_j) with $b_{i+1} = 0$ (resp. $b_{i+1} > 0$), and (c_j) with $c_i = 0$ (resp. $c_i > 0$), which is a bijection. The n's inserted in the j-th gap of w for $j \neq i, i+1$ contribute to $\text{maj}(w') - \text{maj}(w)$ the same amount as do the n's inserted in the same gaps in v to $\text{maj}(v') - \text{maj}(v)$. The major index contributed to $\text{maj}(w')$ by the segment

$$w_{i-1} \underbrace{n \ldots n}_{b_i} w_i \underbrace{n \ldots n}_{b_{i+1}}$$

is the sum of $i-1$ (which equals the contribution of $w_{i-1}w_i$ to $\text{maj}(w)$) and

$$\begin{cases} (b_1 + \cdots + b_i) + (b_1 + \cdots + b_{i+1} + i), & \text{if } b_{i+1} > 0 \\ (b_1 + \cdots + b_i), & \text{if } b_{i+1} = 0. \end{cases}$$

Similarly, the major index contributed to $\mathrm{maj}(v')$ by the segment

$$v_{i-1}\underbrace{n\dots n}_{c_i}v_i\underbrace{n\dots n}_{c_{i+1}}$$

is the sum of i (which equals the contribution of $v_{i-1}v_i$ to $\mathrm{maj}(v)$) and

$$\begin{cases} (c_1+\cdots+c_i+i-1)+(c_1+\cdots+c_{i+1}), & \text{if } c_i>0 \\ (c_1+\cdots+c_{i+1}), & \text{if } c_i=0. \end{cases}$$

Now, one readily checks that $\mathrm{maj}(w')-\mathrm{maj}(w)=\mathrm{maj}(v')-\mathrm{maj}(v)$. This completes the proof.

□

Proof of Theorem 21. Let c be the smallest column of $\mathcal{M}$ in the order $\prec$ and $\mathcal{M}'=\mathcal{M}\backslash c$. Note that c is the leftmost or the rightmost column of $\mathcal{M}$. Let M be a filling of $\mathcal{M}$ with s nonempty cells in c and let M' be its restriction on $\mathcal{M}'$. From Definition 4', we derive that

$$\mathrm{maj}(M)=\mathrm{maj}(M')+\mathrm{maj}(R_c)-\mathrm{maj}(R_c\cap M'),$$

where R_c is the filling of the rectangle $\mathcal{M}(c)$ determined by the column c. R_c is obtained by adding a column with s many 1-cells to the rectangular filling $R_c\cap M'$. Using the bijection between fillings of rectangles and words described in section 4.2.1, one sees that if c is the leftmost (resp. rightmost) column in M, this corresponds to inserting s maximal (resp. minimal) elements in the word determined by the filling $R_c\cap M'$. When the column c varies over all possible $\binom{h}{s}$ fillings, by Lemma 4.10, the value $\mathrm{maj}(R_c)-\mathrm{maj}(R_c\cap M')$ varies over the multiset $\{i_1+i_2+\cdots+i_s \mid 0\le i_1\le i_2\le\cdots\le i_s\le(h-s)\}$ with generating function $\sum_{0\le i_1\le\cdots\le i_s\le(h-s)} q^{i_1+\cdots+i_s}=\begin{bmatrix} h \\ s \end{bmatrix}_q$, where h is the value h_1 (resp. h_m) if c is the leftmost (resp. the rightmost) column of M, as defined in (4.5). Therefore,

$$\sum_{M\in\mathbf{F}(\mathcal{M},\mathbf{s};A)} q^{\mathrm{maj}(M)} = \begin{bmatrix} h \\ s \end{bmatrix}_q \sum_{M'\in\mathbf{F}(\mathcal{M}',\mathbf{s}';A)} q^{\mathrm{maj}(M')}$$

where $\mathbf{s}'$ is obtained from $\mathbf{s}$ by removing the first (resp. last) component if c is the leftmost (resp. rightmost) column of M. Equation (4.6) now follows by induction on the number of columns of $\mathcal{M}$. □

4.4. The Foata-type bijection for moon polyominoes

The objective of this section is to give a bijective proof for Theorem 24. For the set W_S of all words of a multiset S, the equidistribution of maj and inv was first proved by MacMahon by combinatorial analysis.

Foata's map Φ is constructed recursively with certain "local operations" to eliminate the difference caused by adding the last letter in the words. Inspired by this idea, we construct a Foata-type bijection $\phi : \mathbf{F}(\mathcal{M},\mathbf{s};A) \to \mathbf{F}(\mathcal{M},\mathbf{s};A)$ with the property $\mathrm{maj}(M) = \mathrm{ne}(\phi(M))$. The map ϕ can be defined directly for left-aligned stack polyominoes. But we describe it first for Ferrers diagrams in section 4.4.1, because this case contains all the essential steps and is easy to understand. This map is a revision of the Foata-type bijection presented in the preprint of [7] for set partitions, which correspond to fillings with row and column sums at most 1. Then in section 4.4.2 we extend the construction to left-aligned stack polyominoes and prove that $\mathrm{maj}(M) = \mathrm{ne}(\phi(M))$. In section 4.4.3, we construct two bijections, f and g, that transform a filling of a moon polyomino to a filling of a left-aligned stack polyomino and preserve the statistics maj and ne, respectively. Composing these maps with ϕ defined on left-aligned stack polyominoes yields a bijection on $\mathbf{F}(\mathcal{M},\mathbf{s};A)$ that sends maj to ne.

4.4.1. The bijection ϕ for Ferrers diagrams

The empty rows in the filling $M \in \mathbf{F}(\mathcal{M}, \mathbf{s}; A)$ do not play any role in the definitions of $\mathrm{maj}(M)$ or $\mathrm{ne}(M)$. Therefore, in what follows we assume that $A = \emptyset$ and describe ϕ for fillings without empty rows. For $A \neq \emptyset$, one can first delete the empty rows of the filling, apply the map ϕ, and reinsert the empty rows back.

Let $\mathcal{F}$ be a Ferrers diagram and F a filling of $\mathcal{F}$. The bijection ϕ is defined inductively on the number of rows of F. If F has only one row, then $\phi(F) = F$. Otherwise, we denote by F_1 the filling obtained by deleting the top row r of F. Let $F_1' = \phi(F_1)$ and let F_2 be the filling obtained by performing the algorithm γ_r described below. Then $F' = \phi(F)$ is obtained from F_2 by adding the top row r. So, by definition, ϕ preserves the first row of F.

The algorithm γ_r

If C is the 1-cell of r, then denote by $\mathcal{R}$ the set of all rows of F_1' that intersect the column of C. The 1-cells in $\mathcal{R}$ that are strictly to the left of C are called left and the 1-cells in $\mathcal{R}$ that are weakly to the right of C are called right. The cell C is neither left nor right.

Let CLC (critical left cell) be the topmost left 1-cell, and CRC (critical right cell) be the leftmost right 1-cell that is above CLC. If there is more than one such cell the CRC is defined to be the lowest one. Note that CLC and CRC need not exist. Denote by $\mathcal{R}_1$ the set of all rows weakly below the row of CRC that intersect the column of CRC and $\mathcal{R}_2$ is the set of all rows in $\mathcal{R}$ that do not intersect the column of CRC. If CLC does not exist then both $\mathcal{R}_1$ and $\mathcal{R}_2$ are empty. If CLC exists but CRC does not, then $\mathcal{R}_1$ is empty and $\mathcal{R}_2$ contains all the rows in $\mathcal{R}$ weakly below CLC. See Figure 4.7 for an illustration.

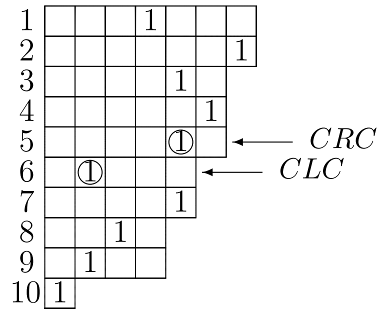


FIG. 4.7 *The CRC and CLC are in rows 5 and 6, respectively. $\mathcal{R}_1$ contains rows 5, 6, and 7, while $\mathcal{R}_2$ contains rows 8 and 9.*

Definition 5. *Let C_1 and C_2 be two 1-cells with coordinates (i_1, j_1) and (i_2, j_2), respectively. We swap the cells C_1 and C_2 by deleting the 1's from these two cells and write 1's in the cells with coordinates (i_1, j_2) and (i_2, j_1).*

For a cell C, denote by $col(C)$ the column of C, and $|col(C)|$ the length of $col(C)$.

Algorithm γ_r^1 on $\mathcal{R}_1$

Let ptr_1 and ptr_2 be two pointers.

(A) Set ptr_1 on the highest row of $\mathcal{R}_1$ and ptr_2 on the next row in $\mathcal{R}_1$ below ptr_1.

(B) If ptr_2 is null, then go to (D). Otherwise, the pointers ptr_1 and ptr_2 point at 1-cells C_1 and C_2, respectively.

(B1) If C_2 is a left cell then swap the cells C_1 and C_2 and move ptr_1 to the row of ptr_2.

(B2) If C_2 is a right cell, then

(B2.1) If $|col(C_1)| = |col(C_2)|$ then move ptr_1 to the row of ptr_2.

(B2.2) If C_1 is to the left of C_2 and $|col(C_1)| > |col(C_2)|$ then do nothing.

(B2.3) If C_1 is to the right of C_2 and $|col(C_1)| < |col(C_2)|$, then find the lowest left 1-cell L that is above C_1. Suppose that the row-column coordinates of the cells L, C_1, and C_2 are (i_1, j_1), (i_2, j_2), and (i_3, j_3), respectively. Delete the 1's from these three cells and write them in the cells with coordinates (i_1, j_3), (i_2, j_1), and (i_3, j_2). Move ptr_1 to the row of ptr_2.[†]

(C) Move ptr_2 to the next row in $\mathcal{R}_1$. Go to (B).

(D) Stop.

See Figure 4.8 for illustration of the steps.

[†]**Note:** This step is well-defined since it cannot occur before ptr_2 reaches CLC (that would contradict the definition of CRC) and, after that, ptr_1 is always below CLC. Therefore, the 1-cell L always exists. The fact that the square (i_3, j_2) belongs to $\mathcal{F}$ follows from the definition of $\mathcal{R}_1$ and Lemma 8 (c).

When the algorithm γ_r^1 stops, we continue processing the rows of $\mathcal{R}_2$ by algorithm γ_r^2.

Algorithm γ_r^2 on $\mathcal{R}_2$

(A′) Set ptr_1 on the highest row of $\mathcal{R}_2$ and ptr_2 on the next row in $\mathcal{R}_2$ below ptr_1.

(B′) (***Borrowing***) If ptr_1 points to a right 1-cell then find the lowest left 1-cell above it and swap them. Now ptr_1 points to a left 1-cell.

(C′) If ptr_2 is null, then go to (E′). Otherwise, the pointers ptr_1 and ptr_2 point at 1-cells C_1 and C_2 respectively.

(C′1) If C_2 is a right cell then swap C_1 and C_2.

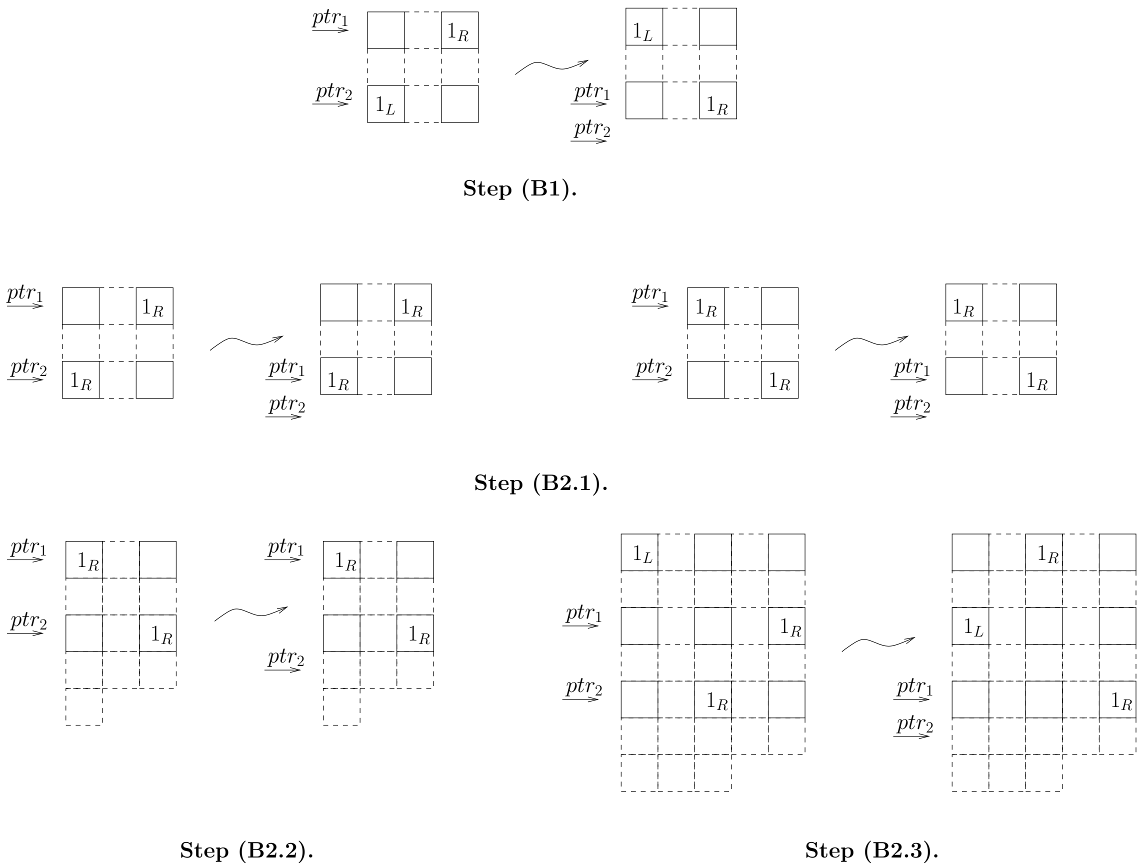


Fig. 4.8 *The algorithm* γ_r^1 *on* $\mathcal{R}_1$.

(C′2) If C_2 is a left cell then do nothing.

(D′) Move ptr_1 to the row of ptr_2 and ptr_2 to the next row in $\mathcal{R}_2$ below. Go to (C′).

(E′) Stop.

See Figure 4.9 for illustration of the steps in γ_r^2.

Let us note the following easy but useful properties of the algorithm γ_r.

1. The pointers ptr_1 and ptr_2 process the rows of $\mathcal{R}_1$ and $\mathcal{R}_2$ from top to bottom. However, while ptr_2 always moves from one row to the next one below it, ptr_1

Step (C′1). **Step (C′2).**

FIG. 4.9 *The algorithm* γ_r^2 *on* $\mathcal{R}_2$*.*

sometimes stays on the same row (cf. step (B2.2)) and sometimes "jumps" several rows below (cf. step (B1)).

2. Pointer ptr_1 always points to a right 1-cell in γ_r^1 and to a left 1-cell in γ_r^2.

Further we show some nontrivial properties which will be used to show that the map ϕ has the desired property $\mathrm{maj}(F) = \mathrm{ne}(\phi(F))$.

Lemma 8. *Suppose in* γ_r^1 ptr_1 *is pointing to* C_1 *and* ptr_2 *is pointing to* C_2*. Then*

(a) In the rows between C_1 *and* C_2 *there are no 1's weakly to the left of* C_1*.*

(b) If L *is the lowest left* 1*-cell above* C_1*, then in the rows between* L *and* C_1 *there are no 1's weakly to the left of* C_1*.*

(c) The 1*-cell* C_1 *is in a column with same length as the column of the critical right cell* CRC*. Moreover, when* γ_r^1 *stops* ptr_1 *points to the last row in* $\mathcal{R}_1$*.*

Proof. The first two parts are proved by induction on the number of steps performed in the algorithm.

(a) The only step which leaves the gap between ptr_1 and ptr_2 nonempty is (B2.2). But after this step, the number of 1's in the rows between ptr_1 and ptr_2 weakly to the left of C_1 does not change.

(b) The case when the last step is (B2.2) follows from the induction hypothesis. If the last step is (B1) or (B2.3) the claim follows from part (a). The case when the last step was (B2.1) and the cells C_1 and C_2 formed an NE chain the claim also follows from part (a) and the induction hypothesis. If in (B2.1), the cells C_1 and C_2 do not form an NE chain, in addition to the induction hypothesis, one uses the fact that there are no 1's in the rows between these two cells which are weakly to the left of C_2.

(c) The first part follows immediately from the definition of the steps. If the second part is not true, then the last step must be (B2.2). and the column of C_1 is longer than the column of C_2. Since this is the last step of γ_r^1, it follows that the column of C_1 intersects the top row of $\mathcal{R}_2$. However, the column of CRC does not intersect the top row of $\mathcal{R}_2$ by definition of $\mathcal{R}_2$. This contradicts the fact that C_1 and CRC are in columns of same length.

□

Proposition 7. *During algorithm γ_r^1 the number of NE chains decreases by the total number of left* 1*-cells in $\mathcal{R}_1$, whereas during algorithm γ_r^2 the number of NE chains increases by the total number of right* 1*-cells in $\mathcal{R}_2$.*

Proof. To prove the first part note that pointer ptr_2 points to each left 1-cell in $\mathcal{R}_1$ exactly once in the algorithm γ_r^1. When that happens, step (B1) is performed during which, by Lemma 8(a), the number of NE chains decreases by one. In the other steps, the number of NE chains remains unchanged. This is trivial for steps (B2.1) and (B2.2), whereas for step (B2.3) it follows from Lemma 8 (a) and (b).

For the second part, note that pointer ptr_2 points to each right 1-cell in $\mathcal{R}_2$ exactly once in the algorithm γ_r^2 with possible exception being the top 1-cell in $\mathcal{R}_2$ to which it never points. In those steps the number of NE chains is increased by one, while otherwise it remains the same. When the top 1-cell is right, then Borrowing occurs. It follows from parts (b) and (c) of Lemma 8 that the number of NE chains is also increased by one. □

Theorem 25. *The map* $\phi : \boldsymbol{F}(\mathcal{F}, \boldsymbol{s}; A) \to \boldsymbol{F}(\mathcal{F}, \boldsymbol{s}; A)$ *is a bijection.*

Proof. To show that the map $\phi : \mathbf{F}(\mathcal{F}, \mathbf{s}; A) \to \mathbf{F}(\mathcal{F}, \mathbf{s}; A)$ is a bijection, it suffices to describe the inverse of the algorithm γ_r. First, we need to determine what $\mathcal{R}_1$ and $\mathcal{R}_2$ were. For that we use the following properties of γ_r.

1. When γ_r^1 stops, the pointer ptr_1 is at the lowest row of $\mathcal{R}_1$ pointing at a right 1-cell C_1. The algorithm γ_r^2 does not change the cell C_1 and when the whole algorithm γ_r terminates, there is no 1-cell below C_1 that together with it forms an NE chain.

2. After performing γ_r^2, all the right 1-cells in $\mathcal{R}_2$ have at least one 1-cell below them with which they form an NE chain.

3. If $\mathcal{R}_2$ was nonempty then, after performing γ_r, the lowest row of $\mathcal{R}_2$ contains a left 1-cell.

4. If any borrowing occurred and the right cell $C_1 = (i_1, j_1)$ and the left cell $C_2 = (i_2, j_2)$ were swapped, then right after this step the cell (i_1, j_2) is a right 1-cell in $\mathcal{R}_1$ which forms an NE chain with a left 1-cell from $\mathcal{R}_2$ and this is the lowest 1-cell in $\mathcal{R}_1$ with this property.

Therefore, if $\mathcal{R}$ has no left 1-cells then $\mathcal{R}_2 = \mathcal{R}_1 = \emptyset$. Otherwise, find the lowest right 1-cell C^* in $\mathcal{R}$ such that there is no 1-cell in $\mathcal{R}$ below it that together with C^* forms an NE chain. Then, by Properties 1 and 2 all the nonempty rows in $\mathcal{R}$ below C^* are in $\mathcal{R}_2$. If there is no 1-cell C^* with that property, then $\mathcal{R}_2$=$\mathcal{R}$.

Algorithm δ_r^2: the inverse of γ_r^2

(IA′) Initially, set ptr_1 to the lowest row of $\mathcal{R}_2$ and ptr_2 to the next row in $\mathcal{R}_2$ above ptr_1.

(IB′) If ptr_2 is null then go to step (ID′). Otherwise, suppose ptr_1 and ptr_2 are pointing at 1-cells C_1 and C_2.

(IB′1) If C_2 is a left cell do nothing.

(IB′2) If C_2 is a right cell then swap the cells C_1 and C_2.

(IC′) Move ptr_1 to the row of ptr_2 and ptr_2 to the next row in $\mathcal{R}_2$ above it. Go to (IB′).

(ID′) (***Inverse borrowing***) Note that ptr_1 always points to a left 1-cell. When it reaches the highest row of $\mathcal{R}_2$ and points to a 1-cell C_1 first we need to determine whether there was any borrowing. For that purpose find the lowest right 1-cell C_2 above C_1 such that the two cells form an NE chain. Using Property 4 we conclude:

(ID′1) If there is no such a cell C_2, then there was no borrowing so do nothing.

(ID′2) If there is such a cell C_2, and there is a left 1-cell in the rows between C_1 and C_2, then there was no borrowing so do nothing.

(ID′3) If there is such a cell C_2, and there is no left 1-cell between C_1 and C_2, then swap C_1 and C_2.

(IE′) Stop.

When the algorithm δ_r^2 stops, continue by applying the algorithm δ_r^1 on $\mathcal{R}\backslash\mathcal{R}_2$.

Algorithm δ_r^1: the inverse of γ_r^1

(IA) Position ptr_1 on the lowest row in $\mathcal{R}\backslash\mathcal{R}_2$ and ptr_2 on the lowest nonempty row in $\mathcal{R}\backslash\mathcal{R}_2$ above it.

(IB) If ptr_2 is null then go to step (ID). Otherwise, suppose that ptr_1 and ptr_2 point at the 1-cells C_1 and C_2 respectively.

(IB1) If C_2 is a left cell then check whether there exists a right 1-cell R above C_2 in a column longer than the column of C_1 such that there are no left 1-cells between R and C_2.

(IB1.1) If such a cell R exists, then suppose that the row-column coordinates of C_1, C_2, and R are (i_1, j_1), (i_2, j_2), and (i_3, j_3), respectively. Delete the 1's from these three cells and write them in the cells with coordinates (i_1, j_3), (i_2, j_1), and (i_3, j_2). Move ptr_1 to the row of ptr_2.

(IB1.1) If there is no such a cell R, then swap C_1 and C_2 and move ptr_1 to the row of ptr_2.

(IB2) If C_2 is a right cell, then

(IB2.1) If $|col(C_1)| = |col(C_2)|$ then move ptr_1 to the row of ptr_2.

(IB2.2) If $|col(C_1)| \neq |col(C_2)|$ then do nothing.

(IC) Move ptr_2 to the next row in $\mathcal{R}\backslash\mathcal{R}_2$ above it. Go to (IB).

(ID) Stop.

Suppose M' is a filling obtained by applying the algorithm γ_r to the filling $M \in \mathbf{F}(\mathcal{M}, \mathbf{s}; A)$. Next we show that by applying δ_r^2 and δ_r^1 to M' one obtains the filling M.

Steps (IB′1) and (IB′2) clearly invert the steps (C′2) and (C′1), respectively. Immediately after the borrowing step in γ_r^2, by Lemma 8, (b) and (c), and Property 4, we can conclude that borrowing has been performed if and only if there is a right 1-cell in $\mathcal{R}\backslash\mathcal{R}_2$ which forms an NE chain with a left 1-cell from $\mathcal{R}_2$ and does not form an NE chain with any left cells in $\mathcal{R}_1$. It is clear that step (ID′) detects whether there was any borrowing and inverts it if it occurred. So, the first algorithm δ_r^2 is indeed the inverse of γ_r^2.

In δ_r^1, when ptr_2 points to a left 1-cell we need to invert either (B1) or (B2.3). Using Lemma 8 (c) one sees that step (IB1) detects exactly which of (B1) and (B2.3) occurred and inverts it. If ptr_2 points to a right cell, it is clear that (IB2.1) and (IB2.2) invert the steps (B2.1) and (B2.2) of γ_r^1. Moreover, if both ptr_1 and ptr_2 point to right 1-cells, and there is no left 1-cell above them, then clearly the algorithm δ_r^1 leaves them unchanged. That is, the algorithm δ_r^1 preserves all rows that are above the rows in $\mathcal{R}_1$. Therefore, we have

$$M \xrightarrow{\gamma_r} M' \xrightarrow{\delta_r^2+\delta_r^1} M.$$

This implies that $\gamma_r : \mathbf{F}(\mathcal{M}, \mathbf{s}; A) \to \mathbf{F}(\mathcal{M}, \mathbf{s}; A)$ is injective. Since $\mathbf{F}(\mathcal{M}, \mathbf{s}; A)$ is finite, it follows that γ_r is bijective.

□

An example of the bijection ϕ is presented in Figure 4.10 and we next explain how it includes Foata's second fundamental transformation. If the moon polyomino is rectangular, then the algorithm γ_r has a simpler description because some of the cases can never arise. Namely, if the cell in the second row forms an NE chain with the top cell (a descent) then $\mathcal{R}_1 = \emptyset$, and only the steps (C'1) and (C'2) are performed. Otherwise, $\mathcal{R}_2 = \emptyset$ and, since all columns are of equal height, only the steps (B1) and (B2.1) are performed. In both these cases, via the correspondence between words and rectangular fillings (c.f. section 4.2.1) the algorithms γ_r^2 and γ_r^1 for rectangles are equivalent to Foata's transformation for words [20].

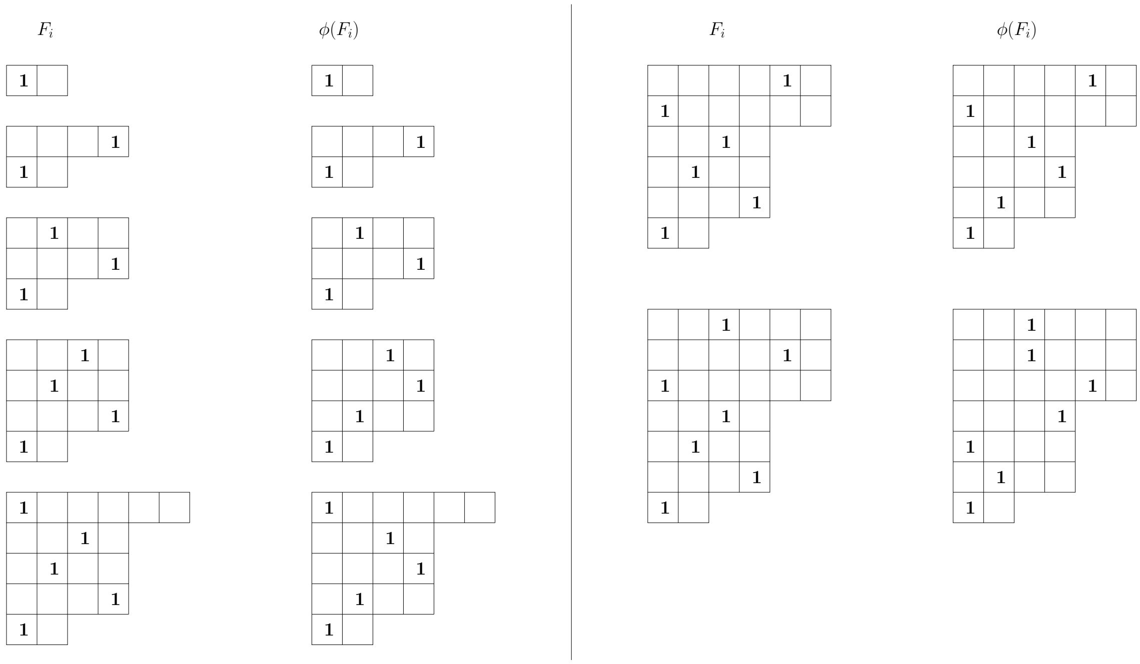


FIG. 4.10 *Example of the map ϕ applied inductively on $F = F_7$. The fillings F_i are restrictions of F on the last i rows.*

4.4.2. The bijection ϕ for left-aligned stack polyominoes

Next we extend the bijection ϕ to fillings of left-aligned stack polyominoes which sends maj to ne. Let $\mathcal{M}$ be a left-aligned stack polyomino. As in the case of Ferrers diagrams, we can assume that $A = \emptyset$ and only consider fillings without empty rows. Suppose M is a filling of $\mathcal{M}$ with top row r. The map, which is again denoted by ϕ, is defined inductively on the number of rows of M. If M has only one row, then $\phi(M) = M$. Otherwise, let $\mathcal{F}$ be the maximal Ferrers diagram in $\mathcal{M}$ that contains the top row and $\mathcal{F}_1$ be $\mathcal{F}$ without the top row. Let $F = M \cap \mathcal{F}$ and $F_1 = M \cap \mathcal{F}_1$. To obtain $\phi(M)$ we perform the following steps. (See Figure 4.11).

1. Delete the top row of M and get $M_1 = M\backslash r$.

2. Apply ϕ to M_1 and get $M_1' = \phi(M_1)$.

3. Apply the algorithm γ_r to the filling $F_1' = M_1' \cap \mathcal{F}_1$ and leave the cells in M_1' outside of F_1' unchanged. Denote the resulting filling by M_2.

4. $\phi(M)$ is obtained by adding row r back to M_2.

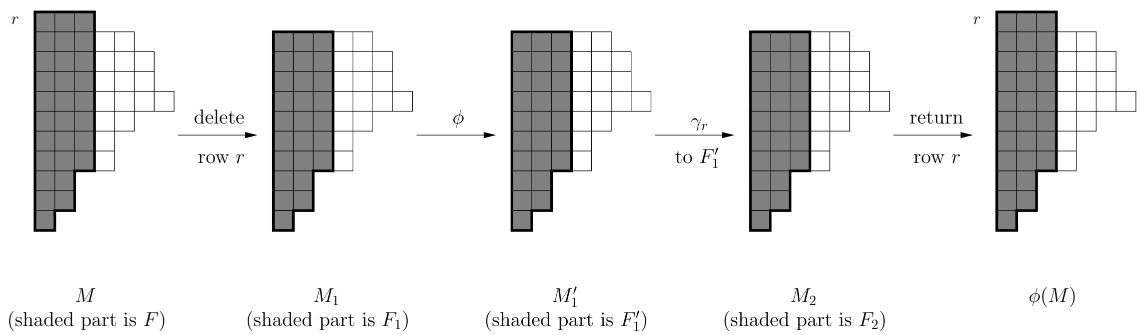


FIG. 4.11 *Illustration of ϕ for left-aligned stack polyominoes.*

It is a bijection from $\mathbf{F}(\mathcal{M}, \mathbf{s}; A)$ to itself since every step is invertible. That Step 3 is invertible follows from the proof of Theorem 25.

Lemma 9. *Using the notation introduced above,*

$$\text{maj}(M) - \text{maj}(M_1) = \text{maj}(F) - \text{maj}(F_1) = \#\mathcal{R}_2(F_1) \tag{4.13}$$

where $\mathcal{R}_2(F_1)$ is defined with respect to the top row r of F as in section 4.4.1.

Proof. Let S be the filling M restricted to the rows which are not completely contained in $\mathcal{F}$. From the definition of maj, we have

$$\text{maj}(M) = \text{maj}(F) + \text{maj}(S) - \text{maj}(F \cap S) \tag{4.14}$$

$$\text{maj}(M_1) = \text{maj}(F_1) + \text{maj}(S) - \text{maj}(F_1 \cap S) \tag{4.15}$$

Note that $F \cap S = F_1 \cap S$, hence (4.14) and (4.15) give the first equality in (4.13). The second one is readily checked using Proposition 4.4 for major index of stack polyominoes. □

The rest of this subsection is devoted to proving that the map ϕ has the desired property $\text{maj}(M) = \text{ne}(\phi(M))$. We shall need one more lemma.

Let C be a 1-cell in a filling M of a left-aligned stack polyomino. We say that C is *maximal* if there is no 1 in M which is strictly above and weakly to the left of C. Clearly, a maximal cell must be the highest 1-cell in its column. For a maximal cell C in M, we use $t(C, M)$ to denote the index of the leftmost column (the columns are numbered from left to right) to the right of C that contains a 1-cell which together with C forms an NE chain. If such a column does not exist set $t(C, M) = \infty$.

Lemma 10. *A 1-cell C in M is maximal if and only if the highest 1-cell C' in the same column in $M' = \phi(M)$ exists and is maximal. Moreover, $t(C, M) = t(C', M')$ when they are both maximal.*

Proof. We proceed by induction on the number of rows of M and keep the notation as in Figure 4.11. The case when M has one row is trivial.

Suppose that the top row r of M has a 1-cell C^* in the column i^*. The cell C^* is maximal in both M and M' and $t(C^*, M) = \infty = t(C^*, M')$. Note that a 1-cell $C \neq C^*$ in M (resp. M') is maximal if and only if C is in column $i < i^*$ and is maximal in M_1 (resp. M_2).

Denote by C_1, C_1', and C_2 the highest 1-cells in column $i < i^*$ in the fillings M_1, $M_1' = \phi(M_1)$, and M_2, respectively. By the induction hypothesis, C_1 is maximal in M_1 if and only if C_1' is maximal in M_1'. M_2 is obtained from M_1' by applying the algorithm γ_r with respect to the row r to F_1'. C_1' is a left cell in this algorithm, and since γ_r does not change the relative position of the left 1-cells, C_2 is maximal in M_2.

It only remains to prove that $t(C, M) = t(C', M')$. Suppose that $t(C_1, M_1) = t(C_1', M_1') = a$ (they are equal by the inductive hypothesis).

If $a < i^*$ then $t(C, M) = a$. In M_1' it corresponds to a 1-cell D' in a column left of C^* such that C_1', D' form an NE chain. Since the algorithm γ_r preserves the column sums and the relative positions of left 1-cells, we have $t(C_2, M_2) = a$, and hence $t(C', M') = a$.

If $a \geq i^*$ then $t(C, M) = i^*$. The fact that $t(C_1', M_1') = a \geq i^*$ means that there is no left 1-cell in M_1' above and to the right of C_1' and, since γ_r does not change the relative positions of the left 1-cells, there will be no left 1-cell above and to the right of C_2 in M_2. So, in this case, we conclude $t(C', M') = i^* = t(C, M)$. □

Theorem 26. *If M is a filling of a left-aligned stack polyomino $\mathcal{M}$, then* $\mathrm{maj}(M) = \mathrm{ne}(\phi(M))$.

Proof. Again we proceed by induction on the number of rows of M. The case when M has one row is trivial. We use the notation as in Figure 4.11 and let $F_2 = M_2 \cap \mathcal{F}_1$.

By Proposition 7, the algorithm γ_r on M_1' decreases ne by one for each left 1-cell in $\mathcal{R}_1(M_1')$ and increases it by one for each right 1-cell in $\mathcal{R}_2(M_1')$. Therefore,

$$\begin{aligned}\mathrm{ne}(\phi(M)) &= \mathrm{ne}(M_2) + \#\{\text{left 1-cells in } \mathcal{R}(F_2)\} \\ &= \mathrm{ne}(M_1') - \#\{\text{left 1-cells in } \mathcal{R}_1(F_1')\} + \#\{\text{right 1-cells in } \mathcal{R}_2(F_1')\} \\ &\quad + \#\{\text{left 1-cells in } \mathcal{R}(F_2)\}\end{aligned}$$

Since γ_r preserves the column sums,

$$\#\{\text{left 1-cells in } \mathcal{R}(F_2)\} = \#\{\text{left 1-cells in } \mathcal{R}(F_1')\}$$

hence

$$\mathrm{ne}(\phi(M)) = \mathrm{ne}(M_1') + \#\mathcal{R}_2(F_1').$$

One the other hand, from Lemma 9 and the induction hypothesis, one gets

$$\mathrm{maj}(M) = \mathrm{maj}(M_1) + \#\mathcal{R}_2(F_1) = \mathrm{ne}(M_1') + \#\mathcal{R}_2(F_1).$$

So, it suffices to show that

$$\#\mathcal{R}_2(F_1) = \#\mathcal{R}_2(F_1'). \tag{4.16}$$

To show equation (4.16), note that for a filling of a Ferrers diagram F the number of rows in $\mathcal{R}_2(F)$ is determined by the column indices of CLC (critical left cell) and CRC (critical right cell). The CLC is the topmost left 1-cell which, if exists, is the topmost *maximal* cell besides the 1-cell in the first row. By Lemma 10, CLC in $F_1 \subseteq M_1$ is in the same column as CLC' in $F_1' \subseteq M_1'$, or neither of them exists. The latter case is trivial, as $\mathcal{R}_2(F)$ is empty. In the first case, let $|r|$ denote the length of row r. Then CRC, the critical right cell of F_1, is in the column $t(CLC, M_1)$ if $t(CLC, M_1) \leq |r|$ and does not exist otherwise. Similarly, the critical right cell of F_1', CRC', is in the column $t(CLC', M_1')$ if $t(CLC, M_1') \leq |r|$, and does not exist otherwise. By Lemma 10, $t(CLC, M_1) = t(CLC', M_1')$, and this implies (4.16). □

4.4.3. The case of a general moon polyomino

Now we consider the case when $\mathcal{M}$ is a general moon polyomino. We label the rows of $\mathcal{M}$ by $r_1, \ldots, r_n$ from top to bottom. Let $\mathcal{N}$ be the unique left-aligned stack polyomino whose sequence of row lengths is equal to $|r_1|, \ldots, |r_n|$ from top to bottom. In other words, $\mathcal{N}$ is the left-aligned polyomino obtained by rearranging the columns of $\mathcal{M}$ by length in weakly decreasing order from left to right. For the definitions that follow, we use an idea of Rubey [39] to describe an algorithm that rearranges the columns of $\mathcal{M}$ to obtain $\mathcal{N}$ (Figure 4.12).

Algorithm α for rearranging $\mathcal{M}$:

1. Set $\mathcal{M}' = \mathcal{M}$.

2. If $\mathcal{M}'$ is left aligned go to (4).

3. If $\mathcal{M}'$ is not left-aligned consider the largest rectangle $\mathcal{R}$ completely contained in $\mathcal{M}'$ that contains c_1, the leftmost column of $\mathcal{M}'$. Update $\mathcal{M}'$ by letting $\mathcal{M}'$ be the polyomino obtained by moving the leftmost column of $\mathcal{R}$ to the right end. Go to (2).

4. Set $\mathcal{N} = \mathcal{M}'$.

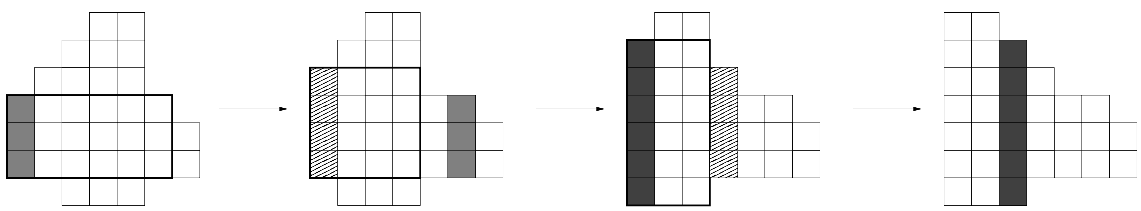


FIG. 4.12 *The algorithm α.*

In this section we give two bijections $f, g : \mathbf{F}(\mathcal{M}, \mathbf{s}; A) \to \mathbf{F}(\mathcal{N}, \mathbf{s}'; A)$ which preserve the major index and the number of NE chains, respectively. The sequence $\mathbf{s}'$ is obtained by rearranging the sequence $\mathbf{s}$ in the same way $\mathcal{N}$ is obtained by rearranging the columns of $\mathcal{M}$.

4.4.3.1. Bijection $f : \mathbf{F}(\mathcal{M}, \mathbf{s}; A) \to \mathbf{F}(\mathcal{N}, \mathbf{s}'; A)$ such that $\mathrm{maj}(M) = \mathrm{maj}(f(M))$

Let R be a filling of a rectangle $\mathcal{R}$ with column sums $s_1, s_2, \ldots, s_m$. First, we describe a transformation τ which gives a filling of $\mathcal{R}$ with column sums $s_2, s_3, \ldots, s_m, s_1$ and preserves the major index. Recall that a descent in R is a pair of 1's in consecutive nonempty rows that form an NE chain. Define an *ascent* to be a pair of 1's in two consecutive nonempty rows that does not form an NE chain. Suppose the first column of R has k many 1-cells: $C_1, \ldots, C_k$ from top to bottom.

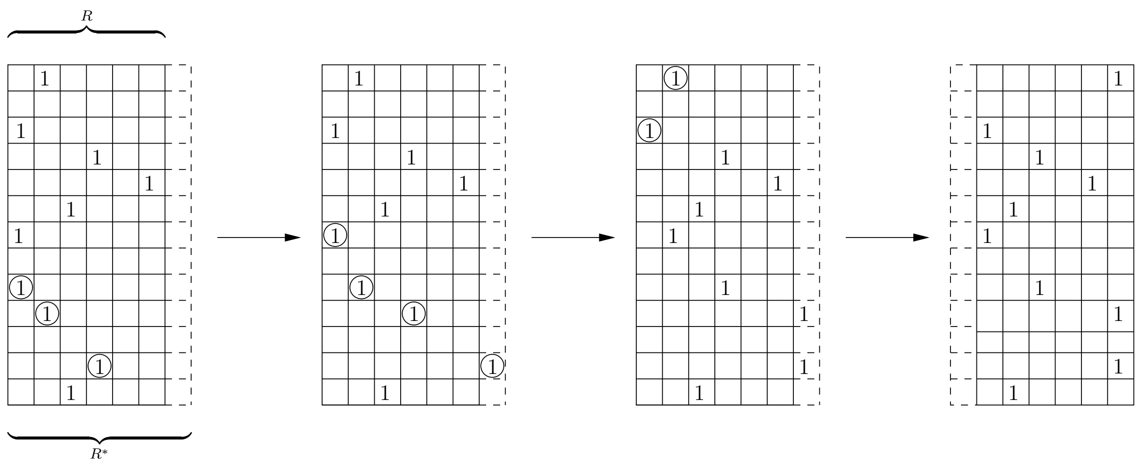


FIG. 4.13 *Illustration of the transformation τ for rectangles.*

Transformation τ on rectangles:

1. Let R^* be the rectangle obtained by adding one empty column to R from right. Process the 1-cells $C_1, \ldots, C_k$ from bottom to top (see Figure 4.13):

2. For $r = k, k-1, \ldots, 1$ do the following:

   Let D_a be the lowest 1-cell above C_r in R^* and D_b be the highest 1-cell below C_r in R^*.

   (a) If D_a does not exist or $D_a = C_{r-1}$, and D_b does not exist, just move C_r horizontally to the last column of R^*;

   (b) If (i) D_a does not exist or $D_a = C_{r-1}$, but D_b exists, or (ii) both D_a and D_b exist, and D_a is strictly to the right of D_b:

Let $D_1 = C_r, D_2 = D_b, D_3, \ldots, D_p$ be the maximal chain of consecutive ascents in R^*. Move D_i horizontally to the column of D_{i+1}, for $i = 1, \ldots, p-1$, and move D_p horizontally to the last column of R^*. (If D_p is in the last column of R^*, it remains there.)

(c) If $D_a \neq C_{r-1}$, and (D_a is weakly to the left of D_b or D_b does not exist):

Let $D_1 = C_r, D_2 = D_a, D_3, \ldots, D_p$ be the maximal chain of consecutive descents in R^*. Move D_i horizontally to the column of D_{i+1}, for $i = 1, \ldots, p-1$, and move D_p horizontally to the last column of R^*.

3. Delete the first column of R^*.

Note that each iteration in Step 2 decreases the number of 1's in the first column of R^* by one and increases the number of 1's in the last column by one, while the other column sums remain unchanged.

Lemma 11. *The transformation τ is invertible. Moreover, it preserves the descents and hence* $\mathrm{maj}(R) = \mathrm{maj}(\tau(R))$.

Proof. The map τ is invertible since τ^{-1} can be obtained by taking $\rho \circ \tau \circ \rho$, where ρ is the rotation of rectangles by 180°. Moreover, each iteration of Step 2 of τ preserves the positions of descents, hence the second part of the claim holds. □

Now we are ready to define the map f. Suppose $M \in \mathbf{F}(\mathcal{M}, \mathbf{s}; A)$. We perform the algorithm α on M to transform the shape $\mathcal{M}$ to $\mathcal{N}$. While we are in Step 3, instead of just moving the first column of $\mathcal{R}$ to the right end, we perform the algorithm τ on the filling R of $\mathcal{R}$. $f(M)$ is defined to be the resulting filling of $\mathcal{N}$.

Proposition 8. *The map $f : \boldsymbol{F}(\mathcal{M}, \boldsymbol{s}; A) \to \boldsymbol{F}(\mathcal{N}, \boldsymbol{s}'; A)$ is a bijection and* $\mathrm{maj}(M) = \mathrm{maj}(f(M))$.

Proof. The first part follows from Lemma 11 and the fact that α is invertible. The filling $f(M)$ is obtained after several iterations of Step 3 in the algorithm α combined with application of τ. For the second part of the claim, it suffices to show that maj is preserved after one such iteration. Let M' be the filling of the shape $\mathcal{M}'$ obtained from M after one step of f in which the rectangle R is transformed into $\tau(R)$. We shall use Definition 4 to compare $\mathrm{maj}(M)$ and $\mathrm{maj}(M')$. Let $R_1, \ldots, R_k$ (resp. $R'_1, \ldots, R'_k$) be the maximal rectangles of M (resp. M'), ordered by height. In particular, R_i and R'_i are of same size. Suppose $R = R_s$, then $R'_s = \tau(R)$. We partition the set of rectangles into three subsets according to $i < s$, $i = s$, and $i > s$, and compare the contribution of R_i's in each subset.

1. For $1 \leq i \leq s-1$, the rectangles R_i and $R_i \cap R_{i+1}$ are of smaller height than R and contain several consecutive rows of R. Using Lemma 11 we know that the descents in $R'_i \cap \tau(R)$ (resp. $R'_i \cap R'_{i+1} \cap \tau(R)$) $1 \leq i \leq s-1$, are in the same rows as in $R_i \cap R$ (resp. $R_i \cap R_{i+1} \cap R$). And descents formed by one cell in R and one cell on the right of R are preserved since τ preserves the set of rows that contain a non-zero entry. Consequently,

$$\mathrm{maj}(R_i) = \mathrm{maj}(R'_i),\ \mathrm{maj}(R_i \cap R_{i+1}) = \mathrm{maj}(R'_i \cap R'_{i+1}),\ 1 \leq i \leq s-1. \quad (4.17)$$

2. For $i = s$, by Lemma 11, we have

$$\mathrm{maj}(R_s) = \mathrm{maj}(R'_s). \quad (4.18)$$

3. For $i > s$, the height of the rectangle R_i or $R_{i-1} \cap R_i$ is greater than or equal to that of R. Each of these rectangles contains several consecutive columns of R excluding the first one. The filling of each subrectangle of R consisting of the columns $c_j, \ldots, c_{j+m}$ $(j > 1)$ is equal, up to a rearrangement of the

empty rows, to the filling of the subrectangle of $\tau(R)$ consisting of the columns $c_{j-1}, \ldots, c_{j+m-1}$. To see this, note that the shifting of the 1-cells $D_1, \cdots, D_p$ horizontally in the algorithm τ can be viewed as moving the 1-cell D_i vertically to the row of D_{i-1} and moving D_1 to the last column of R^* in the row of D_p. Thus, the descent positions in the fillings $R'_i \cap \tau(R)$ (resp. $R'_{i-1} \cap R'_i \cap \tau(R)$), $i > s$ are the same as in $R_i \cap R$ (resp. $R_{i-1} \cap R_i \cap R$). Finally, one checks that the descents formed by one cell in R and one cell outside R are also preserved. Therefore,

$$\text{maj}(R_i) = \text{maj}(R'_i),\ \text{maj}(R_{i-1} \cap R_i) = \text{maj}(R'_{i-1} \cap R'_i),\ i > s. \qquad (4.19)$$

Combining (4.17), (4.18), and (4.19) gives $\text{maj}(M) = \text{maj}(M')$. □

4.4.3.2. Bijection $g : \mathbf{F}(\mathcal{M}, \mathbf{s}; A) \to \mathbf{F}(\mathcal{N}, \mathbf{s}'; A)$ such that $\text{ne}(M) = \text{ne}(g(M))$

Suppose $M \in \mathbf{F}(\mathcal{M}, \mathbf{s}; A)$. To obtain $g(M)$, we perform the algorithm α to transform the shape $\mathcal{M}$ to $\mathcal{N}$ and change the filling when we move columns in Step 3 so that the number of 1's in each row and column is preserved.

Let R be the rectangular filling in Step 3 of α that contains the column c_1 of the current filling. Suppose c_1 contains k many 1-cells $C_1, \ldots, C_k$ from top to bottom. Shade the empty rows of R and the cells in R to the right of $C_1, \ldots, C_k$. Let l_i denote the number of empty white cells in R above C_i. If $\mathcal{R}'$ is the rectangle obtained by moving the column c_1 from first to last place, fill it to obtain a filling R' as follows.

1. The rows that were empty remain empty. Shade these rows.

2. Write k many 1's in the last column from bottom to top so that there are l_i white empty cells below the i-th 1, $1 \leq i \leq k$.

3. Shade the cells to the left of the nonempty cells in the last column.

4. Fill in the rest of the rectangle by writing 1's in the unshaded rows of $\mathcal{R}'$ so that the unshaded part of R to the right of the first column is the same as the unshaded part of R' to the left of the last column.

See Figure 4.14 for an example.

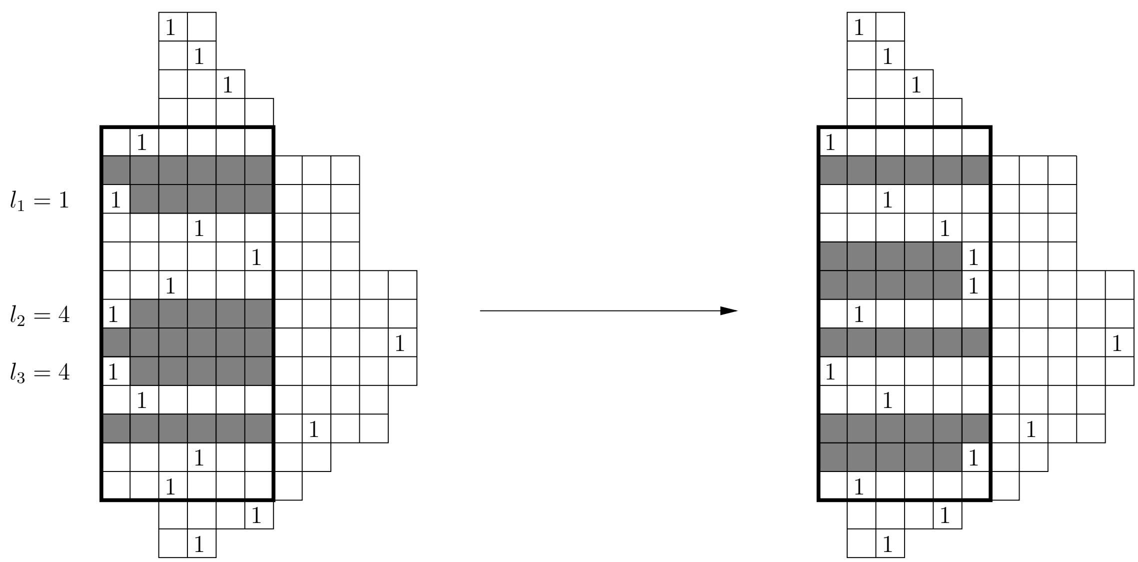


FIG. 4.14 *One step of the map g.*

Proposition 9. *The map $g : \boldsymbol{F}(\mathcal{M}, \boldsymbol{s}; A) \to \boldsymbol{F}(\mathcal{N}, \boldsymbol{s}'; A)$ is a bijection and* $\mathrm{ne}(M) = \mathrm{ne}(g(M))$.

Proof. Clearly, every step of g is invertible. To see that ne is preserved, it suffices to show that it is preserved in each step of g, when one column is moved from left to right. First note that $\mathrm{ne}(R) = \mathrm{ne}(R')$. This follows from the fact that the 1's in the first column of R form $l_1, \ldots, l_k$ NE chains from top to bottom, while the 1's in the last column of R' form $l_1, \ldots, l_k$ NE chains from bottom to top and the remaining parts of R and R' are essentially equal. The numbers of NE chains in M and M'

respectively made of at least one 1 which is outside of R and R' respectively are equal because each step of g preserves the row and column sums. □

Theorem 27. *Let ϕ denote the Foata-type bijection described in section 4.4.2. For a general moon polyomino M the map $\psi = g^{-1} \circ \phi \circ f : \boldsymbol{F}(\mathcal{M}, \boldsymbol{s}; A) \rightarrow \boldsymbol{F}(\mathcal{M}, \boldsymbol{s}; A)$ is a bijection with the property*

$$\mathrm{maj}(M) = \mathrm{ne}(\psi(M)) \tag{4.20}$$

Proof. The maps f and g permute the columns of M together with the corresponding number of 1's, while ϕ preserves the column sums. Moreover, all three of them preserve the empty rows. Therefore, ψ is indeed a map from $\mathbf{F}(\mathcal{M}, \mathbf{s}; A)$ to itself. Finally, (4.20) follows from Proposition 8, Proposition 9, and Theorem 26. □

CHAPTER V

SUMMARY AND FUTURE DIRECTIONS

We have obtained several results about the distribution of crossings and nestings in set partitions, matchings, and 01-fillings of moon polyominoes.

In chapter II we defined the rooted tree of set partitions $\mathcal{T}(\Pi)$ and studied the distribution of the statistic $s_{\alpha,\beta}(\lambda) = cr(\lambda)\alpha + ne(\lambda)\beta$ along the subtrees of $\mathcal{T}(\Pi)$. Our main result is that if the statistic $s_{\alpha,\beta}$ coincides on the first two levels of the trees $\mathcal{T}(\lambda)$ and $\mathcal{T}(\pi)$ then it coincides on $\mathcal{T}(\lambda, l, m)$ and $\mathcal{T}(\pi, l, k)$ on all levels, and similarly for the pair of statistics $s_{\alpha,\beta}$, $s_{\beta,\alpha}$.

We further considered equivalence relations $\sim_{cr}$ and $\sim_{ne}$ on set partitions in the same way Klazar defined them on matchings [33]. We determined the number of crossing-similarity classes in $\Pi_{n,k}$. For $\sim_{ne}$, we found a recurrence relation for the number of nesting-similarity classes in $\Pi_{n,k}$, and computed the total number of such classes in Π_n.

Finally, we analyzed the generating function

$$S_\pi(q, p, z) = \sum_{l \geq 0} \sum_{\lambda \in \mathcal{T}(\pi, l)} q^{cr(\lambda)} p^{ne(\lambda)} z^l$$

for the pair (cr, ne) over a subtree $\mathcal{T}(\pi)$ rooted at a partition π with k blocks and found that it can be written as a sum of k continued fractions. In the special case when π is the partition of $\{1\}$, this yields a continued fraction expansion of the generating function for (cr, ne) over all partitions.

In chapter III we defined a bijection Φ between self-avoiding directed paths in the symmetric wedge $y = \pm x$ that begin at the origin and end at the line $y = -x$. The map Φ is a composition of two maps. The first map ψ maps the paths to matchings and sends the number of north steps to a new statistic on matchings, which we denote by st. The second map ϕ maps the set of matchings onto itself, preserves the first edge of the matching and sends st to the number of nestings. The composition $\Phi = \phi \circ \psi$ sends paths that end at $(n, -n)$ to matchings on $[2n]$ and has the property that the number of north steps in the path is equal to the number of nestings in the corresponding matching. The construction of Φ provides a completely combinatorial proof of this fact that was first discovered algebraically as a side result in the study of self-avoiding directed paths in the symmetric wedge $y = \pm x$ [26] .

In chapter IV we introduced a major index for fillings of moon polyominoes with entries 0 and 1 such that there is at most one 1 in each row. This major index reduces to the classical major index for words and permutations when the polyomino is a rectangle and to the major index for set partitions introduced in [7] in the case of a Ferrers shape. Moreover, our major index has the property that it is equally distributed with the number of north-east chains (ne) in the filling, which can be viewed as a generalization of inversions in permutations, and crossings in set partitions. We gave two proofs of this fact. In the first one we computed the generating function of maj and compared it to the generating function of ne. In the second one we constructed a bijection in the spirit of Foata's second fundamental transformation which sends maj to ne.

Chapter II is a joint work with Catherine Yan and a paper based on these results is published in the SIAM Journal of Discrete Mathematics [37]. The contents of chapter III are accepted for publication in the Annals of Combinatorics [36]. The contents of chapter IV are a joint work with William Chen, Catherine Yan, and Arthur Yang, and have been accepted for publication in the Journal of Combinatorial Theory Series A [8].

Below we list some questions related to the work in this dissertation which we hope to address in the future.

While there are many results concerning regular crossings and nestings, not much is known for general k-crossings/k-nestings. Since the problem of finding a recurrence or an algebraic equation for 3-crossings is hard even for permutations, one cannot hope to find the generating function for 3-crossings/3-nestings over matchings. It is known, however, that these two generating functions are not equal. The question that remains is whether they have the same limiting distribution and what this distribution is. As mentioned in chapter II, it is known that the limiting distribution for $k = 2$ is normal.

Another avenue for research is to study subclasses of matchings that appear in the literature in connection to other areas of mathematics. For example, certain families of matchings defined by specifying conditions on the crossing structure provide models for RNA pseudoknot structures. It would be interesting to explore the combinatorial properties of these subclasses.

A lot of results about matchings have been based on studying the subclasses with fixed type, i.e., fixed minimal and maximal elements in the blocks. The authors in [32] have started investigating noncrossing pairings on bit strings in which zeros and ones on a line are paired by arcs as in the picture for matchings. Note that while there is still a restriction in the pairings of the vertices, this is not equivalent to fixing the type. Finding a closed formula for the number of noncrossing pairings for the most general case seems hard, but the case $1^{m_1}0^{m_1}\dots 1^{m_k}0^{m_k}$, which is extremal in a certain sense, gives rise to interesting combinatorics. In particular, if $m_1 \leq m_2 \leq \cdots \leq m_k$ the noncrossing pairings correspond to certain lattice paths in $\mathbb{Z}^2$. As far as we know, a simple bijection which shows this fact is not known. It would be interesting to continue this line of work and understand the combinatorial structure of the parings in the more general case.

Adopting the point of view that crossings are a generalization of the inversion number of permutations, one can define as Mahonian the statistics on matchings/set partitions which have the same generating function as crossings do. The major index for matchings which was defined in [7] is a natural analogue of the major index for permutations and was shown to be Mahonian. However, the nice property that maj and inv are symmetrically distributed, which holds for permutations, ceases to hold in this more general setting. Thus, it is desirable to find a definition of a major index for matchings which extends the one for permutations and still has the nice symmetry property with cr.

Another direction connected to this is the study of Eulerian statistics and Eulerian-Mahonian pairs on matchings. A permutation statistic *stat* is said to be Eulerian if it is equally distributed with the number of descents. For example, one Eulerian statistic is the number of excedances $exc(\sigma) = \#\{i : \sigma(i) > i\}$. The first pair of equidistributed Euler-Mahonian permutation bistatistics to be discovered was that of (des, inv) and $(des, imaj)$ where $imaj$ is the major index of the inverse permutation. Later, other proper pairs were found, for example (des, maj) and (exc, den). However, there is no proper useful definition of an Eulerian statistic for matchings. It would be interesting to investigate if there are meaningful extensions of the well-known Eulerian permutation statistics to matchings and which distribution properties they have.

Recently, some research has been devoted to the study of involutions and fixed point free involutions as subclasses of permutations. Several results for permutations have analogues for these subclasses. Involutions correspond to symmetric fillings of rectangles. This motivates the exploration of the properties of symmetric moon polyominoes with symmetric fillings. One interesting question is what are the analogues of the mixed statistics, which are defined in [9] as certain combinations of crossings and nestings, for symmetric fillings.

There exist natural analogues of set partitions in type B (C) and D, which correspond to intersections of the reflection hyperplanes of the Coxeter groups of type

B and D, respectively, while the classical set partitions arise in type A. Noncrossing partitions for the classical reflection groups of type B (C) and D were introduced by Victor Reiner and were shown to have similar enumerative and structural properties as those of the noncrossing partitions, which are associated to the reflection groups of type A. A uniform definition of nonnesting partitions was given by Alexander Postnikov for all irreducible root systems Φ associated to Weyl groups. For $\Phi = A_{n-1}$ they are naturally in bijection with nonnesting partitions of $[n]$. The crossing and nesting structure in partitions of these types has not been completely understood. In [40] the authors give definitions for crossings and nestings for partitions of types B and C and show that they are equally distributed. The definitions they give for type C are natural but the type B case is more complicated and one needs more insight in what crossings and nestings should be and what they correspond to. Furthermore, there is no natural definition of crossings or nestings in type D at all. Moon polyominoes with certain symmetries defined on them might provide the right approach.

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