SPACES OF ANALYTIC FUNCTIONS AND THEIR APPLICATIONS

A Dissertation

by

MISHKO MITKOVSKI

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2010

Major Subject: Mathematics

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ABSTRACT

Spaces of Analytic Functions and Their Applications. (August 2010)

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In this dissertation we consider several problems in classical complex analysis and operator theory.

In the first part we study basis properties of a system of complex exponentials with a given frequency sequence. We show that most of these basis properties can be characterized in terms of the invertibility properties of certain Toeplitz operators. We use this reformulation to give a metric description of the radius of l^2-dependence. Using similar methods we solve the classical Beurling gap problem in the case of separated real sequences.

In the second part we consider the classical Polýa-Levinson problem asking for a description of all real sequences with the property that every zero type entire function which is bounded on such a sequence must be a constant function. We first give a description in terms of injectivity of certain Toeplitz operators and then use this to give a metric description of all such sequences.

In the last part we study the spectral changes of a partial isometry under unitary perturbations. We show that all the spectra can be described in terms of the characteristic function of the partial isometry that is being perturbed. Our main tool in the proofs is a Herglotz-type representation for generalized spectral measures. We furthermore use this representation to give a new proof of the classical Naimark's dilation theorem and to generalize Aleksandrov's disintegration theorem.

To my parents

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TABLE OF CONTENTS

Page

ABSTRACT iii

DEDICATION iv

ACKNOWLEDGMENTS v

TABLE OF CONTENTS vi

CHAPTER

I INTRODUCTION AND BACKGROUND 1

II PRELIMINARIES 4

2.1. Function theory 4
2.1.1. Harmonic functions 4
2.1.2. Hardy spaces $H^p(\mathbb{C}_+)$ 6
2.1.3. Nevanlinna class $N(\mathbb{C}_+)$ 6
2.1.4. Smirnov class $N^+(\mathbb{C}_+)$ 9
2.1.5. Herglotz functions 10
2.1.6. Paley-Wiener theorem 11
2.1.7. Model spaces $\mathcal{K}_\Theta$ 12
2.1.8. de Branges spaces $\mathcal{B}_E$ 14
2.2. Dilation theory 16
2.2.1. Discrete case 16
2.2.2. Continuous case 18

III BASIS PROPERTIES OF COMPLEX EXPONENTIALS AND RELATED TOPICS 21

3.1. Introduction 21
3.2. Beurling-Malliavin results in terms of Toeplitz operators . 25
3.3. Criterion for l^2-independence 26
3.4. Radius of dependence 30
3.5. Criterion for Riesz sequences 33

CHAPTER Page

3.6. Criterion for Parseval frames 36
3.7. Basis properties of systems of eigenfunctions of one-dimensional Schrödinger operators 37
3.8. Beurling gap problem 39

IV PÓLYA-LEVINSON PROBLEM 44

4.1. Introduction 44
4.2. Characterization of Pólya sequences 46
4.3. Cartwright theorem 53

V HERGLOTZ REPRESENTATION OF GENERALIZED SPECTRAL MEASURES AND SOME APPLICATIONS 55

5.1. Representation theorem 56
5.2. Application to Naimark's dilation theorem 61
5.3. Application to rank-one unitary perturbations of partial isometries 63
5.4. Application to higher rank unitary perturbations of partial isometries 65
5.5. Generalization of the Aleksandrov disintegration theorem 65

VI SUMMARY AND FUTURE DIRECTIONS 68

6.1. Summary 68
6.2. Future directions 69
6.2.1. Schauder basis for complex exponentials 69
6.2.2. An old problem for entire functions 69
6.2.3. Unitary perturbations of a contraction 70
6.2.4. De Branges spaces of entire functions 70

REFERENCES 72

VITA 77

CHAPTER I

INTRODUCTION AND BACKGROUND

One of the most fundamental facts of classical harmonic analysis says that the sequence $(e^{int})_{n\in\mathbb{Z}}$ (with appropriate normalization) forms an orthonormal basis in $L^2(0,2\pi)$. It is natural to examine what happens if one replaces the integer frequencies with some other sequence of real numbers. Clearly, the very useful orthogonality property will be lost. However, in many cases the sequence will still possess some basis property. Thus, the following natural problem arises: For a given frequency sequence $\Lambda := (\lambda_n)_{n\in\mathbb{Z}}$, describe the basis properties of the corresponding sequence of complex exponentials $(e^{i\lambda_n t})$. Questions of this type have a very long history, with origins in the works of Paley and Wiener [35], and Levinson [25]. In the early years, the most interesting was the question whether $(e^{i\lambda_n t})$ forms a Riesz basis in $L^2(0,2\pi)$. However, later many other types of basis properties were also considered.

Some of these questions turned out to be very easy to answer. For example, $(e^{i\lambda_n t})$ is a Bessel sequence in $L^2(0,2\pi)$ if and only if its sequence of frequencies $(\lambda_n)_{n\in\mathbb{Z}}$ can be represented as a finite union of separated ones [46] (we say that $(\mu_n)_{n\in\mathbb{Z}}$ is separated if there exists $\delta > 0$ such that $|\mu_n - \mu_m| \geq \delta > 0,\ (n \neq m)$). However, many other questions turned out to be much more difficult. For example, the Riesz basis problem remained open until Pavlov's impact in the late seventies [37] (see also [17]), despite the fact that many famous mathematicians worked on it since the early thirties. Finally, for some basis properties the question is still open. Moreover, for most of them no reasonably condensed answer is expected.

Because of these difficulties, the following natural subproblem has been consid-

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This dissertation follows the style of Advances in Mathematics.

ered. For a fixed basis property and a given frequency sequence $\Lambda := (\lambda_n)_{n\in\mathbb{Z}}$ find the upper bound of all $c > 0$ for which $(e^{i\lambda_n t})$ has (or doesn't have) this basis property in $L^2(0,c)$. This upper bound is usually called the radius for the corresponding basis property. For example, we have a radius of completeness, frame radius, etc. By now, most of these problems have been completely resolved. It is interesting, however, that the natural l^2-independence problem remained open. One of the main results of this dissertation is the solution to this problem in the case when the frequency sequence is real and separated. We also give a criterion for a sequence of complex exponential to be a Riesz basis and Parseval frame. Our characterization is in terms of the invertibility properties of a certain Toeplitz operator (whose symbol depends on an inner function that is associated to the frequency sequence Λ.) Let us mention the important paper of Hruschev, Nikolskii, and Pavlov [17], where the idea to use Toeplitz operators in the study of basis properties of complex exponentials was introduced. These results are presented in chapter III.

Another classical problem that we consider in this dissertation is the so called Polya-Levinson problem. The Polya-Levinson problem asks for a description of separated real sequences Λ having the property that any entire function of zero exponential type which is bounded on Λ must be constant. This problem was first explicitly posed by Polya in [40] although some partial cases were considered much earlier by Valiron [45]. Using involved classical techniques Levinson [25] obtained some very strong results regarding the problem. Except for de Branges' improvement in [11] these results remained strongest to date. In chapter IV we give a complete solution to the Polya-Levinson problem. Again the inner function that is associated to the sequence Λ plays a major role.

The final chapter of the dissertation has a somewhat different flavor, being much more operator theoretic. However, even in this chapter a major role is played by a certain inner function which is operator valued in this case. We consider the old question concerning the spectral changes which occur when a partial isometry in a Hilbert space is subjected to unitary perturbations. These questions were first systematically treated by Donoghue [13] for the case of a symmetric operator, and by Clark [6] for the partial isometry case. Both of these authors considered only the case when the corresponding deficiency indices are both equal to one. Even this simplest case attracted great deal of interest among mathematicians in the past (see [39] and references therein). In this dissertation we treat the general possibly infinite-dimensional case. The main novelty in our approach is the new Herglotz-type representation for generalized spectral measures. This representation allows us to translate many properties from the very well studied rank-one case in the general infinite-dimensional case.

CHAPTER II

PRELIMINARIES

Here we summarize some well known facts that will be used in the forthcoming chapters.

2.1. Function theory

References for this section are [14, 19, 33, 12].

2.1.1. Harmonic functions

Each function $f \in L^p(\mathbb{R})$, $1 < p < \infty$ can be extended in a natural way to a harmonic function on the upper half-plane $\mathbb{C}_+$. The natural harmonic extension is defined by the Poisson formula:

$$f(x+iy) := \frac{1}{\pi}\int \frac{yf(t)}{(x-t)^2+y^2}dt.$$

It is easy to check that each such function, besides being obviously harmonic, must satisfy:

$$\sup_{y>0}\int |f(x+iy)|^p dx < \infty. \tag{2.1}$$

Conversely, any harmonic function $f : \mathbb{C}_+ \to \mathbb{C}$ satisfying (2.1) can be obtained from some (boundary) function in $L^p(\mathbb{R})$ using the Poisson formula above. The space of all such functions forms a Banach space with respect to the norm

$$\|f\|_p := (\sup_{y>0}\int |f(x+iy)|^p dx)^{\frac{1}{p}} = (\lim_{y\to 0}\int |f(x+iy)|^p dx)^{\frac{1}{p}}$$

and it will be denoted by $h^p(\mathbb{C}_+)$. Given $f(z) \in h^p(\mathbb{C}_+)$ there are several ways to recover the boundary function. One way is as an L^p limit of the family of level functions $f_y(x) := f(x+iy)$ as $y \to 0$. As a consequence, the following important relation follows:

$$\|f\|_p = (\int |f(x)|^p dx)^{\frac{1}{p}}.$$

This shows that the correspondence between $h^p(\mathbb{C}_+)$ and $L^p(\mathbb{R})$ is isometric. Another, perhaps more natural way, to recover $f(z) \in h^p(\mathbb{C}_+)$ from its boundary function is by taking vertical limits, i.e., by

$$f(x) := \lim_{y \to 0} f(x+iy).$$

This limit exists for a.e. $x \in \mathbb{R}$ and completely recovers the boundary function f. All this results continue to hold in the case $p = \infty$, except for the fact that one cannot recover the boundary function by taking L^∞ limits of f_y. In the case when $p = 1$ the situation is a little bit different. Namely, now the collection of all harmonic functions $f : \mathbb{C}_+ \to \mathbb{C}$ satisfying (2.1) is in 1-1 correspondence with the collection of all finite (complex) measures on $\mathbb{R}$. In one direction the correspondence is given again by the Poisson formula. However, in the opposite direction taking the vertical limits (or L^1-limit of f_y) recovers only the absolutely continuous part of the measure.

Each nonnegative harmonic function $f(z)$ on $\mathbb{C}_+$ can be represented by the formula

$$f(x+iy) := ky + \frac{1}{\pi} \int \frac{y}{(x-t)^2 + y^2} \mu(dt), \tag{2.2}$$

where μ is some positive measure on the real line satisfying the Poisson integrability condition $\int \mu(dt)/(1+t^2) < \infty$ and $k \geq 0$ is some nonnegative number (which should be thought of as a point mass at infinity). The correspondence is one to one since obviously each such pair determines a nonnegative harmonic function on $\mathbb{C}_+$ by the formula above. The measure μ can be recovered as a weak* limit of the family of level functions $f_y(x) := f(x+iy)$ as $y \to 0$, whereas one way to recover the point mass k is the formula $k = \lim_{y \to \infty} f(iy)/y$.

2.1.2. Hardy spaces $H^p(\mathbb{C}_+)$

An important linear subspace of $h^p(\mathbb{C}_+)$ (and consequently of $L^p(\mathbb{R})$) is obtained by taking only the holomorphic functions $f : \mathbb{C}_+ \to \mathbb{C}$ for which (2.1) holds. This subspace is closed and is therefore a Banach space itself. It is called Hardy space and is denoted by $H^p(\mathbb{C}_+)$. The corresponding space of boundary functions is denoted by $H^p(\mathbb{R})$ (and is also called Hardy space). In this case even for $p = 1$ there is a perfect isometric 1-1 correspondence between $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{R})$. Furthermore, besides the Poisson formula, now, there is another way to recover $f(z) \in H^p(\mathbb{C}_+)$ from its boundary values on $\mathbb{R}$. It is given by the Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \int \frac{f(t)}{t-z} dt.$$

The right hand side in the last formula makes sense for any function $f \in L^p(\mathbb{R})$, $p < \infty$. Moreover, the Cauchy operator $C : L^p(\mathbb{R}) \to H^p(\mathbb{R})$ defined by

$$Cf(z) := \frac{1}{2\pi i} \int \frac{f(t)}{t-z} dt$$

is bounded and onto for $1 < p < \infty$. In the case when $p = 2$ it defines an orthogonal projection of $L^2(\mathbb{R})$ onto $H^2(\mathbb{R})$. If $p = 1$ the Cauchy operator fails to be bounded. However, it still defines an analytic function in the upper half-plane which belongs to the Smirnov class (see below).

2.1.3. Nevanlinna class $N(\mathbb{C}_+)$

A simple fact that for any function $f \in H^p(\mathbb{R})$, $\log^+ |f(z)|$ has a positive harmonic majorant $|f(z)|^p$ implies by itself many nice properties for functions in $H^p(\mathbb{R})$. This is why the class of holomorphic functions for which $\log^+ |f(z)|$ has a harmonic majorant (or equivalently $\log |f(z)|$ has a positive harmonic majorant) deserved to be studied on its own. Such a class is called Nevanlinna class and is denoted by $N(\mathbb{C}_+)$. As in the case of Hardy spaces, functions in $N(\mathbb{C}_+)$ have vertical limits a.e. on the real line which determine them uniquely. This unique determination is strong in the sense that the boundary values of two different functions in $N(\mathbb{C}_+)$ cannot coincide even on a set of positive measure. Moreover, the nonzero boundary functions possess the following integrability property

$$f(t) \in L^1(\frac{dt}{1+t^2}). \tag{2.3}$$

Conversely, each nonnegative function on the real line satisfying (2.3) is a boundary modulus of some function in $N(\mathbb{C}_+)$.

Another nice property of this class is the fact that there is a complete description for the zero sets of its functions. Namely, if $\{z_k\}$ is a sequence of zeros of a function $f \in N(\mathbb{C}_+)$ then

$$\sum \frac{\Im z_k}{1+|z_k|^2} < \infty. \tag{2.4}$$

Conversely, for any such sequence $\{z_k\}$ there exists a (non unique) function in $N(\mathbb{C}_+)$ with zeroes exactly $\{z_k\}$. One such function is given by the product

$$b(z) := \prod \frac{1-\frac{z}{z_k}}{1-\frac{z}{\bar{z}_k}}.$$

and is known as the Blaschke product. It is easy to see that such a function is bounded by 1 in the upper half-plane and is therefore in $N(\mathbb{C}_+)$. Being bounded, every Blaschke product has vertical limits a.e. on $\mathbb{R}$. Furthermore, it is not hard to see that $|B(t)| = 1$ a.e. on $\mathbb{R}$. As a byproduct one also obtains that

$$\log|f(t)| \in L^1(\frac{dt}{1+t^2}).$$

Besides having easy to describe zero sets, functions in $N(\mathbb{C}_+)$ also have a very nice and extremely important factorization property. Namely, if $f(z) \in N(\mathbb{C}_+)$ then

$$f(z) = Ce^{iaz} B(z) \frac{s_1(z)}{s_2(z)} h(z), \tag{2.5}$$

where C is some unimodular constant, $a \in \mathbb{R}$, $B(z)$ is the Blaschke product containing all the zeros of $f(z)$,

$$s_k(z) := \exp(-\frac{1}{\pi i} \int \frac{tz+1}{t-z} \frac{\mu_{k,s}(dt)}{1+t^2})$$

($\mu_{k,s}$ is the singular part of μ_k) and

$$h(z) := \exp(\frac{1}{\pi i} \int \frac{tz+1}{t-z} \frac{\log |f(t)|}{1+t^2} dt).$$

The factorization above is unique up to a unimodular constant and it is called an inner-outer factorization. The inner-outer terminology is due to Beurling. Namely, a holomorphic function in $\mathbb{C}_+$ which is bounded by 1 on $\mathbb{C}_+$ and is unimodular a.e. on $\mathbb{R}$ is called an inner function. On the other hand, a holomorphic function $h(z)$ in $\mathbb{C}_+$ satisfying

$$\log |h(z)| = \frac{1}{\pi} \int \frac{y \log |h(t)|}{(x-t)^2 + y^2} dt$$

or equivalently

$$h(z) = \exp(\frac{1}{\pi i} \int \frac{tz+1}{t-z} \frac{\log |f(t)|}{1+t^2} dt)$$

is called an outer function. Every function $s(z)$ of the form

$$s(z) := \exp(-\frac{1}{\pi i} \int \frac{tz+1}{t-z} \frac{\nu(dt)}{1+t^2}),$$

where ν is a singular measure, is an inner function. This particular kind of inner functions are known as singular inner functions. Therefore, in the inner-outer factorization above, each of the factors e^{iaz}, $b(z)$, $s_k(z)$ is either an inner function or an outer function. Using the inner-outer factorization one easily obtains the following alternative description of the Nevanlinna class. A holomorphic function on $\mathbb{C}_+$ belongs to the Nevanlinna class, $N(\mathbb{C}_+)$, if and only if it can be represented as a quotient of two bounded holomorphic functions in $\mathbb{C}_+$.

Functions in $N(\mathbb{C}_+)$ are sometimes called functions of bounded type. Below we explain the reason for this. A holomorphic function $f(z)$ in $\mathbb{C}_+$ is said to have a type T if the infimum of all $M \in \mathbb{R}$ for which

$$|f(z)| \leq Ce^{M|z|}$$

is equal to T. A theorem of Hayman (see [24]) can be used to show that any function in $N(\mathbb{C}_+)$ has a finite type (which is the reason for the name functions of bounded type). Furthermore, using the same theorem one can also show that for a.e. $\theta \in (0, 2\pi)$

$$\lim_{r\to 0} \frac{\log |f(re^{i\theta})|}{r} = T.$$

One way to compute the type is the formula

$$T = \limsup_{y\to 0} \frac{\log |f(iy)|}{y}.$$

T is exactly the opposite of the number $a \in \mathbb{R}$ in the inner-outer factorization (2.5) of $f(z)$.

2.1.4. Smirnov class $N^+(\mathbb{C}_+)$

The Smirnov class $N^+(\mathbb{C}_+)$ is obtained by taking only those functions $f(z) \in N(\mathbb{C}_+)$ for which in the inner-outer factorization (2.5) of $f(z)$, $a \geq 0$, i.e., the type is non positive, and the singular inner function in the denominator $s_2(z) \equiv 1$. Each function in the Smirnov class can be therefore factored in a unique way as $f(z) = g(z)h(z)$, where $g(z)$ is inner and $h(z)$ is an outer function. It trivially follows from (2.5) that for each function $f(z) \in N^+(\mathbb{C}_+)$ the following inequality holds

$$\log|f(z)| \leq \frac{1}{\pi} \int \frac{y \log|f(t)|}{(x-t)^2 + y^2} dt, \tag{2.6}$$

for all $z = x + iy \in C_+$. Conversely, each holomorphic function satisfying this inequality everywhere in $\mathbb{C}_+$ must be in the Smirnov class. This characterization of $N^+(\mathbb{C}_+)$ yields that $H^p(\mathbb{R}) \subset N^+(\mathbb{C}_+)$ for all $1 \leq p \leq \infty$. Moreover, every function in $N^+(\mathbb{C}_+)$ whose boundary function belongs in $L^p(\mathbb{R})$ must be in $H^p(\mathbb{R})$, i.e.,

$$H^p(\mathbb{R}) = L^p(\mathbb{R}) \cap N^+(\mathbb{C}_+).$$

The last fact follows by applying Jensen's inequality to (2.6).

2.1.5. Herglotz functions

Holomorphic functions with nonnegative imaginary part are called Herglotz functions. Each such function $m(z)$ can be represented by the formula

$$m(z) = a + bz + \frac{1}{\pi} \int \frac{tz+1}{t-z} \frac{\mu(dt)}{1+t^2},$$

where $a \in \mathbb{R}$, $b \geq 0$ (point mass at infinity) and μ is a positive Poisson integrable measure. The constant b is determined by the formula $b = \lim_{y\to\infty} \Im m(iy)/y$.

In the case when $y\Im m(iy)$ is bounded and $\lim_{y\to\infty} m(iy) = 0$ (and only in this case) the Herglotz function $m(z)$ has the following representation

$$m(z) = \frac{1}{\pi} \int \frac{\mu(dt)}{t-z},$$

with μ being with finite total mass.

There is a close relationship between Herglotz functions and functions $\Theta(z) \in$

$H^\infty(\mathbb{C}_+)$ that are bounded by 1 everywhere on $\mathbb{C}_+$. Indeed, the Cayley transform

$$m(z) = i\frac{1+\Theta(z)}{1-\Theta(z)},$$

gives a one-to-one correspondence between these two classes. In this case, $\Theta(z)$ is inner if and only if the measure μ in the representation of $m(z)$ is singular. This also gives a one-to-one correspondence between inner functions and singular Poisson integrable measures on the real line taken with the (not necessarily nonzero) point mass at infinity. Since this correspondence plays a major role in this dissertation it is worth giving it explicitly:

$$\Re\frac{1+\Theta(z)}{1-\Theta(z)} = by + \frac{1}{\pi}\int \frac{yd\mu(t)}{(x-t)^2+y^2}. \tag{2.7}$$

There is also a one-to-one correspondence between Herglotz functions $m(z)$ and holomorphic functions $\phi(z)$ on $\mathbb{C}_+$ which are nonzero everywhere on C_+ and satisfy $|\phi(z)| \le 1$. The correspondence is given by the formula

$$\phi(z) = e^{im(z)}.$$

In this case, $\phi(z)$ is a singular inner function if and only if the measure μ associated to $m(z)$ is nonzero singular measure (μ is a zero measure if and only if $\phi(z) = e^{ibz}$). Therefore, the class of all inner functions $\Theta(z)$ and the class of nowhere zero inner functions $\phi(z)$ are also in a one-to-one correspondence given by

$$\phi(z) = e^{i\frac{1+\Theta(z)}{1-\Theta(z)}}.$$

2.1.6. Paley-Wiener theorem

One of the most fundamental results for $H^2(\mathbb{C}_+)$ is the Paley-Wiener Theorem. It says that $H^2(\mathbb{C}_+)$ is just the Fourier transform of $L^2[0,\infty)$. More precisely, if $f \in L^2[0,\infty)$

then its Fourier transform

$$F(f)(z) := \int_0^\infty e^{itz} f(t)dt$$

belongs in $H^2(\mathbb{C}_+)$, and conversely every function in $H^2(\mathbb{C}_+)$ can be represented in this way. Furthermore, the Fourier transform $F : L^2[0,\infty) \to H^2(\mathbb{C}_+)$ (multiplied by 2π) defines an isometry.

The subspace of $H^2(\mathbb{R})$ that one gets after applying the Fourier transform to $L^2[0,a]$ is equal to $\mathcal{K}_{S^a} := H^2(\mathbb{C}_+) \ominus S^a H^2(\mathbb{C}_+)$. This is a reproducing kernel space with reproducing kernels given by

$$k_{S^a}(z,w) := \frac{1}{2\pi i} \frac{1 - S^a(z)\overline{S^a(w)}}{\bar{w} - z}.$$

Each reproducing kernel $k_{S^a}(z,w)$ is obtained as an image of $e^{i\bar{w}t} \in L^2[0,a]$ under the Fourier transformation F. Therefore, the orthonormal basis $(e^{2\pi nit/a})_{n\in\mathbb{Z}}$ for $L^2[0,a]$ is mapped onto an orthonormal basis $(k_{S^a}(z, 2\pi n/a))_{n\in\mathbb{Z}}$ for $\mathcal{K}_{S^a}$.

Applying Fourier transform to the space $L^2[-a,a]$ yields a familiar space of entire functions known as the Paley-Wiener space $\mathcal{PW}_a$. Each such space consists of entire functions of exponential type no greater than a which are square integrable on the real line. Recall that the exponential type of an entire functions is computed by the formula

$$\tau := \limsup_{|z|\to\infty} \frac{\log|F(z)|}{|z|}.$$

Conversely, each Paley-Wiener space is obtained in this way.

2.1.7. Model spaces $\mathcal{K}_\Theta$

Each inner function $\Theta(z)$ defines a model space

$$\mathcal{K}_\Theta := H^2(\mathbb{C}_+) \ominus \Theta H^2(\mathbb{C}_+).$$

This is a reproducing kernel space with reproducing kernels given by

$$k_\Theta(z,w) := \frac{1}{2\pi i}\frac{1-\Theta(z)\overline{\Theta(w)}}{\bar{w}-z}.$$

Let μ be the singular measure corresponding to $\Theta(z)$ with b being the point mass at infinity (see (2.7)). In the case when $b = 0$ (and this happens precisely when $\lim_{y\to\infty}(1-|\Theta(iy)|^2)/|1-\Theta(iy)|^2 = 0$) there is an isometric isomorphism between $\mathcal{K}_\Theta$ and $L^2(\mu)$. In one direction this isomorphism is given by the formula

$$f(z) = \frac{1-\Theta(z)}{2\pi i}\int \frac{f(t)}{t-z}d\mu(t). \tag{2.8}$$

As usual, in the other direction one just takes the vertical limits. The fact that the vertical limits exist μ a.e. is a rather deep result [38].

Every function $f(z) \in \mathcal{K}_\Theta$ can be extended analytically exactly at those points at which $\Theta(z)$ can be extended. An inner function $\Theta(z) = Ce^{iaz}b(z)s(z)$ can be extended analytically through a point on the real line if and only if this point is not a density point of its zeros nor lies in the support of the singular measure that determines its singular factor. Therefore, $\Theta(z)$ can be extended to a meromorphic function on the whole complex plane if an only if $s(z) \equiv 1$ and the zeros of $b(z)$ are bounded away from the real axis. In this case, every function $f(z) \in K_\Theta$ also has a meromorphic extension in $\mathbb{C}$ given by the formula (2.8). The measure μ corresponding to $\Theta(z)$ is then discrete with atoms at the points of $\Lambda = \{t|\Theta(t) = 1\}$ given by $\mu(\{x\}) = 2\pi/|\Theta'(x)|$. Furthermore, the system $(k_\Theta(z,\lambda_n)/\|k_\Theta(z,\lambda_n)\|)_{\lambda_n\in\Lambda}$ is an orthonormal basis for $\mathcal{K}_\Theta$ and (2.8) represents just an expansion with respect to this basis.

Each meromorphic inner function $\Theta(z)$ can be written as $\Theta(t) = e^{i\phi(t)}$ on $\mathbb{R}$, where $\phi(t)$ is a real analytic and strictly increasing function. The function $\phi(t) = \arg\Theta(t)$ is the continuous argument of $\Theta(z)$ and is called the phase function of $\Theta(z)$.

A subset of $\mathbb{R}$ is called discrete if it has no finite density points. For every discrete set $\Lambda \subset \mathbb{R}$, there exists a (far from unique) meromorphic inner function $\Theta(z)$ such that $\{t|\Theta(t) = 1\} = \Lambda$. In the case of a separated sequence Λ, we will be especially interested in the inner function that corresponds to the counting measure on Λ (measure with point masses equal to 1 at each point of Λ). In this case, we will say that $\Theta(z)$ *is associated to* Λ. An important property of this inner function is the fact that its phase $\arg \Theta(t)$ has a bounded derivative (see for instance Lemma 16 in [12]).

2.1.8. de Branges spaces $\mathcal{B}_E$

An important class of entire functions consists of those functions $F(z)$ of exponential type $\leq a$ that satisfy $\log|F(t)| \in L^1(dt/(1+t^2))$. This class is mostly known as the Cartwright class and is denoted by $\mathcal{C}_a$. A classical theorem of Krein gives a connection between the Smirnov-Nevanlinna class $N^+(\mathbb{C}_+)$ and the Cartwright class $\mathcal{C}_a$. An entire function $F(z)$ belongs to the Cartwright class $\mathcal{C}_a$ if and only if

$$\frac{F(z)}{S^{-a}(z)} \in N^+(\mathbb{C}_+), \qquad \frac{F^\#(z)}{S^{-a}(z)} \in N^+(\mathbb{C}_+),$$

where $F^\#(z) = \overline{F(\bar{z})}$.

As an immediate consequence one obtains a connection between the Hardy space $H^2(\mathbb{C}_+)$ and the Paley-Wiener space PW_a. Namely, an entire function $F(z)$ belongs to the Paley-Wiener class PW_a if and only if

$$\frac{F(z)}{S^{-a}(z)} \in H^2(\mathbb{C}_+), \qquad \frac{F^\#(z)}{S^{-a}(z)} \in H^2(\mathbb{C}_+).$$

The definition of the de Branges spaces of entire functions may be viewed as a generalization of the above definition of the Payley-Wiener spaces with $S^{-a}(z)$ replaced by a more general entire function. Consider an entire function $E(z)$ satisfying the inequality

$$|E(z)| > |E(\bar{z})|, \qquad z \in \mathbb{C}_+.$$

Such functions are usually called de Branges functions. The de Branges space $\mathcal{B}_E$ associated with $E(z)$ is defined to be the space of entire functions $F(z)$ satisfying

$$\frac{F(z)}{E(z)} \in H^2(\mathbb{C}_+), \qquad \frac{F^{\#}(z)}{E(z)} \in H^2(\mathbb{C}_+).$$

It is a Hilbert space equipped with the norm $\|F\|_E := \|F/E\|_{L^2(\mathbb{R})}$. If $E(z)$ is of exponential type then all the functions in the de Branges space $\mathcal{B}_E$ will be of exponential type not greater then the type of $E(z)$. A de Branges space is called short (or regular) if together with every function $F(z)$ it contains $(F(z) - F(a))/(z - a)$ for any $a \in \mathbb{C}$. If a de Branges space $\mathcal{B}_E$ is short then its de Branges function $E(z)$ must be in Cartwright class.

Every de Branges function $E(z)$ gives rise to a meromorphic inner function $\Theta(z) = E^{\#}(z)/E(z)$. We say that an inner function $\Theta(z)$ in $\mathbb{C}_+$ is a meromorphic inner function if it allows a meromorphic extension to the whole complex plane. The meromorphic extension to the lower half-plane $\mathbb{C}_-$ is given by:

$$\Theta(z) = \frac{1}{\Theta^{\#}(z)}.$$

Conversely, every meromorphic inner function $\Theta(z)$ can be represented in the form $\Theta(z) = E^{\#}(z)/E(z)$, for some de Branges function $E(z)$ (see, for instance, Lemma 2.1 in [16]). Such a function is unique up to a factor of an entire function that is real on $\mathbb{R}$ and has only real zeros. There is an important relationship between the model subspaces $\mathcal{K}_\Theta$ and the de Branges spaces $\mathcal{B}_E$ of entire functions. If $E(z)$ is a de Branges function and $\Theta(z) = E^{\#}(z)/E(z)$ is the corresponding meromorphic inner function, then the multiplication operator $f \mapsto Ef$ is an isometric isomorphism

$K_\Theta \to \mathcal{B}_E$.

Let μ be a positive measure on $\mathbb{R}$ satisfying $\int d\mu(t)/(1+t^2) < \infty$. Then there exists a nest of short de Branges spaces $\mathcal{B}_{E_a}$ contained isometrically in $L^2(\mu)$. By a nest we mean that $\mathcal{B}_{E_a}$ is isometrically contained in $\mathcal{B}_{E_b}$ whenever $a \leq b$. Two possibilities exist. Either the union $\cup_{a\geq 0}\mathcal{B}_{E_a}$ is dense in $L^2(\mu)$ (so called limit point case) or there exists a finite number $a \geq 0$ such that $\mathcal{B}_{E_a} = \mathcal{B}_{E_b}$ for all $b > a$ (so called limit circle case). Moreover, the exponential type $\tau(a)$ of $\mathcal{B}_{E_a}$ is a nondecreasing function of a. Therefore, if there exists a space $\mathcal{B}_E$ in the nest with $E(z)$ of positive exponential type, then there also exists a space $\mathcal{B}_E$ in the nest that is contained properly in $L^2(\mu)$.

General treatment of de Branges' theory is given in [12]. However, the proofs are more accessible in de Branges' earlier work [7, 8, 9, 10].

2.2. Dilation theory

References for this section are [22, 23, 32, 44].

2.2.1. Discrete case

A bounded operator $T \in \mathcal{B}(\mathcal{H})$ is said to be a contraction if $\|T\| \leq 1$. The self adjoint operators $D_T := (I - T^*T)^{1/2}$ and $D_{T^*} := (I - TT^*)^{1/2}$ are called defect operators of T and the spaces $\mathcal{D}_T := \overline{D_T\mathcal{H}}$, $\mathcal{D}_{T^*} := \overline{D_{T^*}\mathcal{H}}$ are called defect spaces for T. The defect indices are defined by $\partial_T := \dim\mathcal{D}_T$ and $\partial_{T^*} := \dim\mathcal{D}_{T^*}$. These indices measure, in a certain sense, how much a contraction differs from a unitary operator.

Every contraction T can be dilated to a unitary operator U acting on a larger space $\mathcal{H}'$ such that $T^n = P_{\mathcal{H}}U^n$ for $n \geq 0$. In the case when T^n and T^{*n} converge strongly to zero as $n \to \infty$ this larger space $\mathcal{H}'$ can be represented as $\mathcal{H}' =$

$\sum_{n\in\mathbb{Z}} U^n(\mathcal{D}_T)$ and $\mathcal{H}' = \sum_{n\in\mathbb{Z}} U^n(\mathcal{D}_{T^*})$ (both sums are orthogonal). Therefore $\mathcal{H}'$ will be isomorphic to both $L^2(\mathbb{T}, \mathcal{D}_T)$ and $L^2(\mathbb{T}, \mathcal{D}_{T^*})$. (Here $L^2(\mathbb{T}, \mathcal{E})$ represents the space of all $\mathcal{E}$ valued square summable functions on $\mathbb{T}$.) Denote by $L : L^2(\mathbb{T}, \mathcal{D}_T) \to \mathcal{H}'$ and $L_* : L^2(\mathbb{T}, \mathcal{D}_{T^*}) \to \mathcal{H}'$ these isomorphisms. It can be shown that $L_*^* L : L^2(\mathbb{T}, \mathcal{D}_T) \to L^2(\mathbb{T}, \mathcal{D}_{T^*})$ maps the vector valued Hardy space $H^2(\mathbb{T}, \mathcal{D}_T)$ into the vector valued Hardy space $H^2(\mathbb{T}, \mathcal{D}_{T^*})$. Therefore, there exists an operator valued inner function $\Theta_T(z)$ (for each $z \in \mathbb{D}$, $\Theta_T(z)$ is a contraction mapping $\mathcal{D}_T$ into $\mathcal{D}_{T^*}$ and $\Theta_T(\xi)$ is unitary for a.e $\xi \in \mathbb{T}$) such that

$$L_*^* L f(\xi) = \Theta_T(\xi) f(\xi),$$

for every $f \in L^2(\mathbb{T}, \mathcal{D}_T)$. This inner function $\Theta_T(z)$ is called a characteristic function for T. It can be explicitly computed by the formula

$$\Theta_T(z) := -T + z D_{T^*}(I - zT^*)^{-1} D_T | \mathcal{D}_T.$$

Now, the space $\mathcal{H}$ can be represented as

$$\mathcal{H} = H^2(\mathbb{T}, \mathcal{D}_{T^*}) \ominus \Theta_T H^2(\mathbb{T}, \mathcal{D}_T),$$

and the contraction T can be represented as a shift (multiplication by z) compressed on $H^2(\mathbb{T}, \mathcal{D}_{T^*}) \ominus \Theta_T H^2(\mathbb{T}, \mathcal{D}_T)$.

A similar theory exists even in the case when T^n and T^{*n} do not converge strongly to zero as $n \to \infty$. In this case $\mathcal{H}'$ is not necessarily isomorphic to $L^2(\mathbb{T}, \mathcal{D}_T)$ and $L^2(\mathbb{T}, \mathcal{D}_{T^*})$, however if T is completely non-unitary (c.n.u) these spaces embed into $\mathcal{H}'$ and their embeddings span $\mathcal{H}'$ (A contraction T is said to be completely nonunitary (c.n.u.) if it is not unitary on any of its invariant subspaces.) Therefore, in defining

$\Theta_T(z)$ one should take

$$L_*^* PL : L^2(\mathbb{T}, \mathcal{D}_T) \to L^2(\mathbb{T}, \mathcal{D}_{T^*}),$$

where P is the projection from the embedding of $L^2(\mathbb{T}, \mathcal{D}_T)$ onto the embedding of $L^2(\mathbb{T}, \mathcal{D}_{T^*})$. Now, the characteristic function will only be contraction valued holomorphic function, i.e, in general it will not be inner. The form of the model space will also be more complicated. However, the formula for $\Theta_T(z)$ remains the same.

As seen above, every contraction T determines a contraction valued holomorphic function $\Theta_T(z)$. Every such function $\Theta_T(z)$ is pure in the sense that $\|\Theta(0)h\| < \|h\|$ for $h \in \mathcal{D}_T$, $h \neq 0$. Conversely, for any given pure inner function $\Theta : \mathbb{D} \to \mathcal{B}(\mathcal{L}, \mathcal{L}^*)$ there exists a contraction T whose characteristic function coincides with $\Theta(z)$.

2.2.2. Continuous case

One-parameter family $\{T(t) | t \geq 0\}$ of contractions on $\mathcal{H}$ satisfying $T(0) = I$ and $T(t_1 + t_2) = T(t_1)T(t_2)$ for $t_1, t_2 > 0$ is called a semigroup of contractions. The semigroup is strongly continuous if $\lim_{t\to 0} T(t)h = h$ for all $h \in \mathcal{H}$. Every such subgroup possesses an infinitesimal generator B defined by

$$Bh := -i \lim_{t\to 0} \frac{T(t)h - h}{t}.$$

This is a densely defined (possibly unbounded) closed operator which uniquely determines the semigroup. There is an interesting relation connecting the semigroup to the resolvent of its generator. It is given by

$$(\lambda I - B)^{-1} = \int_0^\infty T(t) e^{i\lambda t} dt,$$

for all $\lambda \in \mathbb{C}_+$. A closed densely defined operator B generates a strongly continuous group of contractions if and only if

$$\|(iyI - B)^{-1}\| \leq \frac{1}{y},$$

for all $y > 0$. This condition can be rewritten as $\Im \langle Bh|h \rangle \geq 0$ for all h lying in the domain of B. Such possibly unbounded operators B are called dissipative. The Cayley transform $T = (B - iI)(B + iI)^{-1}$ gives a one-to-one correspondence between dissipative operators B and contractions T. This contractions are called cogenerators for the corresponding semigroups. It is not hard to check that $T(t)$ forms a group of unitary operators if and only if its generator B is self-adjoint (or eqivalently its cogenerator T is unitary).

Every continuous semigroup of contractions $T(t)$ can be dilated to a group of unitary operators $U(t)$ acting on a larger space $\mathcal{H}'$ such that $T(t) = P_{\mathcal{H}} U(t)$ for $t \geq 0$. In the case when $T(t)$ and $T(t)^*$ converge strongly to zero as $t \to \infty$ this larger space $\mathcal{H}'$ will be isomorphic to both $L^2(\mathbb{R}, \mathcal{D}_T)$ and $L^2(\mathbb{R}, \mathcal{D}_{T^*})$. (Here $L^2(\mathbb{R}, \mathcal{E})$ represents the space of all $\mathcal{E}$ valued square summable functions on $\mathbb{R}$.) Denote by $L : L^2(\mathbb{R}, \mathcal{D}_T) \to \mathcal{H}'$ and $L_* : L^2(\mathbb{R}, \mathcal{D}_{T^*}) \to \mathcal{H}'$ these isomorphisms. It can be shown that $L_*^* L : L^2(\mathbb{T}, \mathcal{D}_T) \to L^2(\mathbb{T}, \mathcal{D}_{T^*})$ maps the vector valued Hardy space $H^2(\mathbb{R}, \mathcal{D}_T)$ into the vector valued Hardy space $H^2(\mathbb{R}, \mathcal{D}_{T^*})$. Therefore, there exists an operator valued inner function $\Phi_T(z)$ (for each $z \in \mathbb{C}_+$, $\Phi_T(z)$ is a contraction mapping $\mathcal{D}_T$ into $\mathcal{D}_{T^*}$ and $\Phi_T(x)$ is unitary for a.e $x \in \mathbb{R}$) such that

$$L_*^* L f(x) = \Phi_T(x) f(x),$$

for every $f \in L^2(\mathbb{R}, \mathcal{D}_T)$. This inner function $\Phi_T(z)$ is called a scattering operator for the semigroup $T(t)$. If T is the cogenerator of $T(t)$ then the characteristic function

$\Theta_T(z)$ of T is related to the scattering operator $\Phi_T(z)$ of $T(t)$ by

$$\Phi_T(z) = \Theta_T(\frac{z+i}{z-i}).$$

The space $\mathcal{H}$ can be represented as

$$\mathcal{H} = H^2(\mathbb{R}, \mathcal{D}_{T^*}) \ominus \Phi_T H^2(\mathbb{R}, \mathcal{D}_T),$$

and the semigroup of contractions $T(t)$ can be represented as a translation (multiplication by e^{itx}) compressed on $H^2(\mathbb{R}, \mathcal{D}_{T^*}) \ominus \Phi_T H^2(\mathbb{R}, \mathcal{D}_T)$. This is the reason why spaces of this form are called model spaces. Notice that in the case when the defect indices are both equal to one these spaces are exactly the model spaces $\mathcal{K}_\Theta$ that we considered above.

CHAPTER III

BASIS PROPERTIES OF COMPLEX EXPONENTIALS AND RELATED TOPICS

In this chapter we solve the problem of characterizing those separated sequences of real frequencies $(\lambda_n)_{n\in\mathbb{Z}}$ for which the corresponding system of complex exponentials $(e^{i\lambda_n t})$ is l^2 -independent, Riesz sequence and Parseval frame in $L^2(0,c)$. Furthermore, we show that the l^2 -dependence radius is equal to the interior Beurling-Malliavin density of $(\lambda_n)_{n\in\mathbb{Z}}$.

3.1. Introduction

First we recall the definition of the most fundamental basis properties.

Definition 3.1.1. A sequence of vectors (f_j) in a Hilbert space $\mathcal{H}$ is said to be

- a Bessel sequence for $\mathcal{H}$ if $\sum |\langle f|f_j\rangle|^2 \preceq \|f\|^2$, for all $f \in \mathcal{H}$ or equivalently if $\|\sum a_j f_j\| \preceq \|(a_j)\|_{l^2}$, for all $(a_j) \in l^2$.
- a complete sequence for $\mathcal{H}$ if $|\langle f|f_j\rangle| = 0$ for all j implies that $f = 0$
- an ω-independent sequence for $\mathcal{H}$ if $\|\sum a_j f_j\| = 0$ for $(a_j) \in l^2$ implies that $a_j = 0$.
- a frame for $\mathcal{H}$ if $\sum |\langle f|f_j\rangle|^2 \simeq \|f\|^2$, for all $f \in \mathcal{H}$
- a Parseval frame for $\mathcal{H}$ if $\sum |\langle f|f_j\rangle|^2 = \|f\|^2$, for all $f \in \mathcal{H}$
- a Riesz sequence for $\mathcal{H}$ if $\|\sum a_j f_j\| \simeq \|(a_j)\|_{l^2}$, for all $(a_j) \in l^2$
- a Riesz basis for $\mathcal{H}$ if it is an l^2-independent frame (or equivalently if it is a complete Riesz sequence).

The definition can be also given in the following condensed form (which gives a more unified view).

A sequence of vectors (f_j) in a Hilbert space $\mathcal{H}$ is called a Bessel sequence for $\mathcal{H}$ if for some/any orthonormal basis (e_j) the operator $T : \mathcal{H} \to \mathcal{H}$ defined on (e_j) by $Te_j := f_j$ can be extended to a bounded operator on $\mathcal{H}$. Now, a Bessel sequence (f_j) is

- a complete sequence for $\mathcal{H}$ if and only if T has a dense range.
- an l^2-independent sequence for $\mathcal{H}$ if and only if T is injective.
- a frame for $\mathcal{H}$ if and only if T is surjective.
- a Parseval frame for $\mathcal{H}$ if and only if T is a coisometry (T^* is an isometry).
- a Riesz sequence for $\mathcal{H}$ if and only if T is injective with a dense range.
- a Riesz basis for $\mathcal{H}$ if and only if T is invertible.

We will first consider vector systems of complex exponentials $(e^{i\lambda_n t})$ in $L^2(0, c)$. As mentioned in the previous chapter, one of the central problems in the classical harmonic analysis can be stated as follows: For a certain fixed basis property and a given frequency sequence $\Lambda := (\lambda_n)_{n\in\mathbb{Z}}$ find the upper bound of all $c > 0$ for which $(e^{i\lambda_n t})$ has (or doesn't have) this basis property in $L^2(0, c)$. This upper bound is usually called radius for the corresponding basis property. For example, the radius of completeness is the supremum of all $c > 0$ for which $(e^{i\lambda_n t})$ is complete in $L^2(0, c)$, whereas the radius of l^2-dependence is the supremum of all $c > 0$ for which $(e^{i\lambda_n t})$ is l^2-dependent in $L^2(0, c)$. By now, most of these problems have been resolved completely. It is interesting, however, that the natural l^2- independence problem remained open. The main result of this chapter is the solution of this problem in the case when the frequency sequence is real and separated. Before stating our result we give some necessary definitions. We also summarize some of the previous work in order to illustrate how our result fits in the complete picture.

As mentioned above, we will be only interested in separated real sequences. Such a sequence $\Lambda := (\lambda_n)_{n\in\mathbb{Z}}$ is said to be uniformly regular with density $d > 0$ if there exists some $C > 0$ such that

$$|n_\Lambda(x) - n_{\frac{1}{d}\mathbb{Z}}(x)| < C.$$

Here, for a sequence Γ, $n_\Gamma(x)$ denotes the usual counting function defined by

$$n_\Gamma(t) = \begin{cases} \#(\Gamma \cap [0,t]) \text{ if } t > 0, \\ -\#(\Gamma \cap [t,0]) \text{ if } t < 0, \\ 0 \text{ if } t = 0. \end{cases}$$

and

$$\frac{1}{d}\mathbb{Z} = \{\frac{n}{d} | n \in \mathbb{Z}\}.$$

Now, the interior uniform density (which is also known as interior Beurling density) $D_B^-(\Lambda)$ of Λ can be defined as

$$D_B^-(\Lambda) := \sup\{d \mid \exists \text{ strongly uniformly regular subsequence } \ \Lambda' \subset \Lambda \text{ with density } d\}.$$

Similarly, the exterior uniform density is defined as

$$D_B^+(\Lambda) := \inf\{d \mid \exists \text{ strongly uniformly regular supsequence } \ \Lambda' \supset \Lambda \text{ with density } d\}.$$

These densities are used to describe the frame radius and the Riesz sequence radius. Note that by a frame (Riesz sequence) radius we mean the upper bound of all $c > 0$ for which our sequence is a frame (not a Riesz sequence) in $L^2(0, c)$.

Theorem 3.1.2 ([18, 21, 41])**.** *Let Λ be a separated sequence of real numbers.*

(i) The frame radius of Λ is equal to $2\pi D_B^-(\Lambda)$.

(ii) The Riesz sequence radius of Λ is equal to $2\pi D_B^+(\Lambda)$.

The description of the completeness and the l^2-dependence radius depends on different kind of densities. We say that Λ is regular with density d if

$$\int \frac{|n_\Lambda(x) - n_{\frac{1}{d}\mathbb{Z}}(x)|}{1+x^2} < \infty.$$

The interior BM (Beurling-Malliavin) density $D_{BM}^-(\Lambda)$ of Λ is defined as

$$D_{BM}^-(\Lambda) := \sup\{a \mid \exists \text{ regular subsequence } \ \Lambda' \subset \Lambda \text{ with density } D\}.$$

(If no such sequence exists $D_{BM}^-(\Lambda) := 0$.) Similarly, the exterior BM density is defined as

$$D_{BM}^+(\Lambda) := \inf\{D \mid \exists \text{ regular supsequence } \ \Lambda' \supset \Lambda \text{ with density } D\}.$$

(If no such sequence exists $D_{BM}^+(\Lambda) := \infty$.)

Now we can state the famous Beurling-Malliavin description of the completeness radius. More about this description can be found in [20].

Theorem 3.1.3 ([4])**.** *Let Λ be a sequence of real numbers. The radius of completeness of Λ is equal to $2\pi D_{BM}^+(\Lambda)$.*

It is natural to expect that the only density which was not used so far will play a role in the description of the l^2-dependence radius. This is indeed the case. Recall that the radius of l^2-dependence of $(e^{i\lambda_n t})$ is the supremum of all $c > 0$ such that $(e^{i\lambda_n t})$ is not l^2-independent in $L^2(0, c)$.

Theorem 3.1.4. *Let* $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ *be a separated sequence of real numbers. The radius of dependence* $R(\Lambda)$ *of the sequence of complex exponentials* $(e^{i\lambda_n t})$ *is equal to* $2\pi D^-_{BM}(\Lambda)$.

Notice that every uniformly regular sequence must be regular. Therefore

$$D^-_B(\Lambda) \leq D^-_{BM}(\Lambda) \leq D^+_{BM}(\Lambda) \leq D^+_B(\Lambda).$$

As an easy consequence we obtain the following

Corollary 3.1.5. *Let* $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ *be a sequence of real numbers. If* $(e^{i\lambda_n t})$ *is a Riesz basis in* $L^2(0, c)$ *then*

$$c = D^-_B(\Lambda) = D^-_{BM}(\Lambda) = D^+_{BM}(\Lambda) = D^+_B(\Lambda).$$

3.2. Beurling-Malliavin results in terms of Toeplitz operators

A Toeplitz operator T_U with a symbol $U \in L^\infty(\mathbb{R})$ is the map $T_U : H^2 \to H^2$ defined by

$$T_U F := P_+(UF),$$

where P_+ is the orthogonal projection in $L^2(\mathbb{R})$ onto the Hardy space $H^2 = H^2(\mathbb{C}_+)$. Along with the usual H^2-kernels, one defines Toeplitz kernels in the Smirnov class $N^+(\mathbb{C}_+)$,

$$\ker^+ T_U = \{f(z) \in N^+(\mathbb{C}_+) \cap L^1_{loc}(\mathbb{R}) : \bar{U}(t)\bar{f}(t) \in N^+(\mathbb{C}_+)\}.$$

The following theorem is a version of the Beurling-Malliavin multiplier theorem [3] in the language of Toeplitz operators.

Theorem 3.2.1 ([27, Section 4.2])**.** *Suppose that* $\Theta(z)$ *is a meromorphic inner function with the derivative of* $\arg\Theta(t)$ *bounded on* $\mathbb{R}$. *Then for any meromorphic inner* *function $J(z)$, we have*

$$\ker^+ T_{\bar{\Theta}J} \neq 0 \quad \Rightarrow \quad \forall \epsilon > 0, \quad \ker T_{\bar{S}^\epsilon \bar{\Theta} J} \neq 0.$$

To state the next result we will need the following definitions.

A sequence of disjoint intervals (I_n) on the real line is called short (in the sense of Beurling and Malliavin) if

$$\sum_n \frac{|I_n|^2}{1 + dist^2(I_n, 0)} < \infty,$$

and it is called long otherwise. Here $|I_n|$ denotes the length of the interval I_n.

Let $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\gamma(\mp\infty) = \pm\infty$. i.e.

$$\lim_{x\to-\infty} \gamma(x) = +\infty, \qquad \lim_{x\to+\infty} \gamma(x) = -\infty.$$

The family $\mathcal{BM}(\gamma)$ is defined as the collection of the connected components of the open set

$$\left\{ x \in \mathbb{R} : \ \gamma(x) \neq \max_{t\in[x,+\infty)} \gamma(t) \right\}.$$

We say that γ is almost decreasing if $\gamma(\mp\infty) = \pm\infty$ and the family of intervals $\mathcal{BM}(\gamma)$ is short.

Theorem 3.2.2 ([27, Section 4.3])**.** *Suppose* $\gamma'(t) > -\text{const}.$

(i) If γ is not almost decreasing, then for every $\epsilon > 0$, $\ker^+ T_{S^\epsilon e^{i\gamma}} = 0.$

(ii) If γ is almost decreasing, then for every $\epsilon > 0$, $\ker^+ T_{\bar{S}^\epsilon e^{i\gamma}} \neq 0.$

3.3. Criterion for l^2-independence

Let Λ be a separated sequence of real numbers. In this case the corresponding sequence of complex exponentials $(e^{i\lambda_n t})$ is a Bessel sequence and therefore we have that $\sum_n a_n e^{i\lambda_n t} = 0$ converges in $L^2(0, c)$ for each $(a_n) \in l_2$.

We will use the following simple lemma.

Lemma 3.3.1. *Let* $\{\lambda_n\}_{n\in\mathbb{Z}}$ *be a separated sequence of real numbers and let* $(a_n) \in l_2$*. Then the series*

$$\sum_n \frac{a_n}{\lambda_n - z}$$

converges locally uniformly for all $z \in \mathbb{C}_+$ *and*

$$\lim_{y\to\infty} \sum_n \frac{a_n}{\lambda_n - iy} = 0.$$

Proof. Notice that

$$\sum_n \frac{1}{|\lambda_n - z|^2} = \sum_n \frac{1}{\pi y} \frac{y}{(x-\lambda_n)^2 + y^2} \le \frac{C}{y},$$

where the constant C depends only on the separation constant of $\{\lambda_n\}_{n\in\mathbb{Z}}$. The rest follows from the Cauchy-Schwartz inequality. □

Recall that for a given separated sequence Λ we say that the inner function $\Theta(z)$ is *associated* to Λ if

$$\Re\frac{1+\Theta(z)}{1-\Theta(z)} = \frac{1}{\pi}\int \frac{y\,d\mu(t)}{(x-t)^2+y^2},$$

where μ denotes the counting measure on Λ.

Theorem 3.3.2. *Let* $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ *be a separated sequence of real numbers and let* $\Theta(z)$ *be the meromorphic inner function associated to* Λ*. A sequence of complex exponentials* $(e^{i\lambda_n t})$ *is* l^2*-independent in* $L^2(0,c)$ *if and only if the Toeplitz operator* $T_{\bar{\Theta}S^c}$ *is injective.*

Remark 1. In one direction we also have the following slightly more general statement. If a sequence of complex exponentials $(e^{i\lambda_n t})$ is l^2-independent in $L^2(0,c)$ then the Toeplitz operator $T_{\bar{\Theta}S^c}$ is injective for every meromorphic inner function $\Theta(z)$ such that $\{t|\Theta(t) = 1\} = \{\lambda_n\}_{n\in\mathbb{Z}}$ and $1 - |\Theta(iy)| \neq O(1/y)$ as $y \to \infty$.

Proof. Assume that $(e^{i\lambda_n t})$ is not l^2-independent in $L^2(0,c)$. Then there exists a non zero sequence $(a_n) \in l^2$ such that $\sum_n a_n e^{i\lambda_n t} = 0$ in $L^2(0,c)$. By applying the Fourier transform we have that

$$\sum_n a_n k_{S^c}(z, \lambda_n) = 0$$

in the space K_{S^c}. Therefore

$$\sum_n \frac{a_n}{\lambda_n - z} = S^c(z) \sum_n \frac{a_n S^{-c}(\lambda_n)}{\lambda_n - z},$$

for all $z \in \mathbb{C}_+$.

Define

$$g(z) := \sum_n a_n k_\Theta(z, \lambda_n) \in K_\Theta.$$

Then $g(z) = S^c(z) f(z)$ for

$$f(z) = \frac{1 - \Theta(z)}{2\pi i} \sum_n a_n \frac{S^{-c}(\lambda_n)}{\lambda_n - z} = \sum_n a_n S^{-c}(\lambda_n) k_\Theta(z, \lambda_n) \in K_\Theta.$$

Therefore, $f(z) \in \ker T_{\bar{\Theta} S^c}$, i. e., $T_{\bar{\Theta} S^c}$ is not injective.

Conversely, assume that $T_{\bar{\Theta} S^c}$ is not injective. Let $f(z)$ be some non-zero element from $\ker T_{\bar{\Theta} S^c}$. Then $g(z) = S^c(z) f(z) \in K_\Theta$ and by the Clark formula

$$S^c(z) f(z) = \frac{1 - \Theta(z)}{2\pi i} \int \frac{g(t)}{t - z} d\sigma(t),$$

where $d\sigma$ is the Clark measure corresponding to $\Theta(z)$ (it is actually the counting measure for Λ). Define

$$h(z) := \frac{1}{2\pi i} \int \frac{1 - e^{icz} e^{-ict}}{t - z} g(t) d\sigma(t).$$

Clearly, $h(z) \in K_{S^c}$ and

$$h(z) = S^c(z) \left(\frac{2\pi i f(z)}{1 - \Theta(z)} - \int \frac{e^{-ict}}{t - z} g(t) d\sigma(t) \right).$$

Since the function in parenthesis is obviously in the Smirnov class $N^+(\mathbb{C}_+)$ it must be also in $H^2(\mathbb{C}_+)$ since $h(t)S^{-c}(t) \in L^2(\mathbb{R})$. Therefore $h(z) \in K_{S^c} \cap S^c H^2$ and hence $h(z) \equiv 0$. So we have

$$0 \equiv h(z) = \frac{1}{2\pi i} \int \frac{1 - S^c(z)S^{-c}(t)}{t - iy} g(t) d\sigma(t).$$

Now, by setting $a_n := g(\lambda_n)$ we obtain an non-zero l^2 sequence such that

$$\sum_n a_n k_{S^c}(z, \lambda_n) = 0,$$

for all $z \in \mathbb{C}_+$. Since this is equivalent to $\sum_n a_n e^{i\lambda_n t} = 0$ in $L^2(0, c)$ we are done. □

It was shown in [27] that a sequence of complex exponentials $(e^{i\lambda_n t})$ is complete in $L^2(0, c)$ if and only if the Toeplitz operator $T_{\Theta \bar{S}^c}$ is injective for every meromorphic inner function $\Theta(z)$ such that $\{t | \Theta(t) = 1\} = \{\lambda_n\}_{n \in \mathbb{Z}}$. Using the well known Coburn lemma, saying that a Toeplitz operator cannot have a nontrivial kernel and cokernel in the same time, we obtain the following

Corollary 3.3.3. *Let $(\lambda_n)_{n \in \mathbb{Z}}$ be a separated sequence of real numbers. If the sequence of complex exponentials $(e^{i\lambda_n t})$ is l_2-dependent in $L^2(0, c)$ then it must be complete in $L^2(0, c)$.*

As seen above, the problem of describing the basis properties for a sequence of complex exponentials $(e^{i\lambda_n t})$ in $L^2(0, c)$ is equivalent to the corresponding problem for the sequence of reproducing kernels $(k_{S^c}(z, \lambda_n))$ in K_{S^c}. Therefore, it is natural to consider the following more general problem. Given a meromorphic inner function $\Psi(z)$ and a real separated sequence $\{\lambda_n\}_{n \in \mathbb{Z}}$, describe the basis properties of the system of reproducing kernels $(k_\Psi(z, \lambda_n))$ in K_Ψ. Furthermore, one can also consider the analogous problem for a system of reproducing kernels in a de Branges space. In view of the isometric relation $B_E = EK_\Psi$ the last two problems are equivalent under the assumption $0 < c < |E(\lambda_n)| < C < \infty$.

In general, the separation condition on a sequence $(\lambda_n)_{n\in\mathbb{Z}}$ does not imply that the corresponding sequence of reproducing kernels $(k_\Psi(z,\lambda_n))$ is a Bessel sequence in K_Ψ. Therefore, in the following theorem we must add this condition as an assumption.

Theorem 3.3.4. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a separated sequence of real numbers and let $\Theta(z)$ be the meromorphic inner function associated to Λ. Furthermore, assume that the corresponding sequence of reproducing kernels $(k_\Psi(z,\lambda_n))$ is a Bessel sequence in K_Ψ. Then a sequence of reproducing kernels $(k_\Psi(z,\lambda_n))$ is l_2-independent in K_Ψ if and only if the Toeplitz operator $T_{\bar{\Theta}\Psi}$ is injective.*

The proof is the same as the one of Theorem 3.3.2. One just needs to replace $S^c(z)$ with $\Psi(z)$.

We also have an analog of Corollary 3.3.3.

Corollary 3.3.5. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a separated sequence of real numbers. Assume that the corresponding sequence of reproducing kernels $(k_\Psi(z,\lambda_n))$ is a Bessel sequence in K_Ψ. If the sequence of reproducing kernels $(k_\Psi(z,\lambda_n))$ is l_2-dependent in K_Ψ then it must be complete in K_Ψ.*

3.4. Radius of dependence

Next we prove the Theorem 3.1.4 from the introduction.

Proof. Let $R(\Lambda)$ be the radius of dependence for Λ and let $\Theta(z)$ be the the meromorphic inner function associate to Λ. By Theorem 3.3.2

$$R(\Lambda) = \sup\{c \geq 0 | \ker T_{\bar{\Theta}S^c} \neq 0\}. \tag{3.1}$$

Applying the Beurling-Malliavin multiplier Theorem 3.2.1, we obtain

$$R(\Lambda) = \sup\{c \geq 0 | \ker^+ T_{\bar{\Theta}S^c} \neq 0\}. \tag{3.2}$$

Now, Theorem 3.2.2 gives

$$R(\Lambda) = \sup\{c \geq 0 | \gamma_c(t) := ct - \phi(t) \text{is almost decreasing}\}, \tag{3.3}$$

where $\phi(t)$ is phase function of $\Theta(z)$.

Let $c < R(\Lambda)$. Then γ_c is almost decreasing and the collection of open intervals $\mathcal{BM}(\gamma_c)$ is short. $\mathcal{BM}(\gamma_c)$ gives a decomposition of the real line in two parts. The intervals in $\mathcal{BM}(\gamma_c)$ represent the part in which the densities of Λ and $(2\pi/c)\mathbb{Z}$ are "comparable", whereas the complement of $\mathcal{BM}(\gamma_c)$ represents the part in which the density of Λ is much larger then the density of $(2\pi/c)\mathbb{Z}$.

Let Σ be a subsequence of Λ obtained by throwing away points from Λ which lie in the complement of $\mathcal{BM}(\gamma_c)$ so that the densities of Σ and $(2\pi/c)\mathbb{Z}$ in this part become essentially the same. In this case we will have

$$|n_\Sigma(t) - \frac{c}{2\pi}t| \asymp \gamma_c^*(t) - \gamma_c(t),$$

where

$$\gamma_c^*(t) := \sup\{\gamma_c(x) | x \in [t, \infty]\}.$$

Therefore,

$$\int \frac{|n_\Sigma(t) - n_{\frac{2\pi}{c}\mathbb{Z}}(t)|}{1+t^2} \asymp \sum_{I \in \mathcal{BM}(\gamma_c)} \int_I \frac{|\gamma_c^*(t) - \gamma_c(t)|}{1+t^2} \lesssim \sum_{I \in \mathcal{BM}(\gamma_c)} \frac{1}{dist(0,I)^2} \int_0^{|I|} tdt$$

$$\asymp \sum_{I \in \mathcal{BM}(\gamma_c)} \frac{|I|^2}{1 + dist(0,I)^2} < \infty.$$

(In the second inequality we used the fact that $\gamma_c'(t)$ is bounded from below.) Thus,

$R(\Lambda) \le 2\pi D^-_{BM}(\Lambda)$.

Let now $c < 2\pi D^-_{BM}(\Lambda)$. There exists a subsequence Σ of Λ which is c-regular, i.e.,

$$\int \frac{|n_\Sigma(t) - n_{\frac{1}{c}\mathbb{Z}}(t)|}{1+t^2} < \infty.$$

Let $\Theta(z)$ be the meromorphic inner function associated to Σ and denote by $\phi(t)$ the phase function of $\Theta(z)$. Since $\phi'(t)$ is bounded on $\mathbb{R}$ we have that $|\phi(t) - n_{\frac{2\pi}{c}\mathbb{Z}}(t)|$ is bounded and hence

$$\int \frac{|\phi(t) - \frac{c}{2\pi}t|}{1+t^2} < \infty. \tag{3.4}$$

Let $0 \le d < c$ be arbitrary. Our goal will be to show that

$$\gamma_d(t) := \frac{d}{2\pi}t - \phi(t)$$

is almost decreasing. This will imply that

$$2\pi D^-_{BM}(\Lambda) \le R(\Sigma) \le R(\Lambda)$$

and we will be done.

First, it is well known (and elementary to check) that 3.4 implies

$$\lim_{t\to\pm\infty} \frac{\phi(t)}{t} = \frac{2\pi}{c}.$$

Therefore, $\gamma_d(\pm\infty) = \mp\infty$.

So, it remains to show that $\mathcal{BM}(\gamma_d)$ is short. We split the intervals in $\mathcal{BM}(\gamma_d)$ in two groups as follows. Let $I = (a,b) \in \mathcal{BM}(\gamma_d)$. Consider the intersection between the line $y = (d-c)/2\pi x$ and the horizontal line extending the restriction of $y = \gamma_d^*(x)$ on I (γ_d^* is defined as above). If this point is to the right of the midpoint of I (which we denote by $m := (a+b)/2$) then we put the interval I in the first group. We put all the other intervals from $\mathcal{BM}(\gamma_d)$ in the second group.

Let I be some interval from the first group. We can underestimate the area between $y = (d-c)/2\pi x$ and $y = c/2\pi x - \phi(x)$ by the triangle formed from $y = (d-c)/2\pi x$ the vertical line $y = a$ and the horizontal line extending the restriction of $y = \gamma_d^*(x)$ on I. Therefore we obtain

$$\int_I \frac{|\phi(t) - \frac{c}{2\pi}t|}{1+t^2} \geq \frac{c-d}{8}\frac{|I|^2}{1+b^2}.$$

This together with 3.4 shows that the collection of intervals in the first group is short.

Similar argument shows that the collection of intervals in the second group is short and hence $\mathcal{BM}(\gamma_d)$ is short. We are done.

□

Remark 2. Notice that since $R(\Lambda) = \sup\{c \geq 0 | \ker T_{\bar{\Theta}S^c} \neq 0\}$ we have also proved that

$$2\pi D^-_{BM}(\Lambda) = \sup\{c \geq 0 | \ker T_{\bar{\Theta}S^c} \neq 0\}.$$

We will use this fact in this and the next chapter.

3.5. Criterion for Riesz sequences

Next, we characterize those sequences of complex exponentials which form a Riesz sequence in $L^2(0,c)$.

Theorem 3.5.1. *Let $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ be a separated sequence of real numbers and let $\Theta(z)$ be the meromorphic inner function associated to Λ. A sequence of complex exponentials $(e^{i\lambda_n t})$ is a Riesz sequence in $L^2(0,c)$ if and only if the Toeplitz operator $T_{\bar{\Theta}S^c}$ is injective with closed range (i.e., it is bounded from below).*

Remark 3. Again, in one direction we also have the following slightly more general statement. If a sequence of complex exponentials $(e^{i\lambda_n t})$ is l_2-independent in $L^2(0,c)$

then the Toeplitz operator $T_{\bar{\Theta}S^c}$ is injective with closed range for every meromorphic inner function $\Theta(z)$ such that $\{t|\Theta(t) = 1\} = \{\lambda_n\}_{n\in\mathbb{Z}}$ and $1 - |\Theta(iy)| \neq O(1/y)$ as $y \to \infty$.

Proof. Notice first that since we work only with separated real sequences the Bessel condition is always fulfilled. Therefore in our setting $(e^{i\lambda_n t})$ is a Riesz sequence in $L^2(0, c)$ if and only if

$$C(\sum_n |a_n|^2)^{1/2} \le \|\sum_n a_n e^{i\lambda_n t}\|,$$

for some $C > 0$ and all $(a_n) \in l_2$.

Assume that $(e^{i\lambda_n t})$ is not a Riesz sequence in $L^2(0, c)$. Then there exists a sequence in k of elements $(a_n^k) \in l_2$ with all elements of norm 1 ($\sum_n |a_n^k|^2 = 1$) such that $\sum_n a_n^k e^{i\lambda_n t} \to 0$ in $L^2(0, c)$ as $k \to \infty$. By applying the Fourier transform we have that

$$\sum_n a_n^k k_{S^c}(z, \lambda_n) \to 0$$

in the space K_{S^c} as $k \to \infty$. Therefore

$$h_k(z) := \sum_n \frac{a_n^k}{\lambda_n - z} - S^c(z) \sum_n \frac{a_n^k S^{-c}(\lambda_n)}{\lambda_n - z} \to 0,$$

in the space K_{S^c} as $k \to \infty$.

Define

$$g_k(z) := \sum_n a_n^k k_\Theta(z, \lambda_n) \in K_\Theta.$$

Then

$$g_k(z) = \frac{1 - \Theta(z)}{2\pi i} h_k(z) - S^c(z) f_k(z)$$

for

$$f_k(z) = \frac{1 - \Theta(z)}{2\pi i} \sum_n a_n^k \frac{S^{-c}(\lambda_n)}{\lambda_n - z} = \sum_n a_n^k S^{-c}(\lambda_n) k_\Theta(z, \lambda_n) \in K_\Theta.$$

Notice that $\|f_k\| = 1$ since $\sum_n |a_n^k|^2 = 1$. Therefore,

$$T_{\bar{\Theta}S^c} f_k(t) = P_+ \bar{\Theta}(t) g_k(t) - P_+ \frac{\bar{\Theta}(t) - 1}{2\pi i} h_k(t).$$

The first term in the difference above is always zero since $g_k(z) \in K_\Theta$ and the second term goes to zero as $k \to \infty$. Therefore $T_{\bar{\Theta}S^c}$ is not bounded below.

Conversely, assume that $T_{\bar{\Theta}S^c}$ is not bounded below. Then there exists a sequence $(f_k(z))$ of elements in $H^2(\mathbb{C}_+)$ all with norm one such that $T_{\bar{\Theta}S^c} f_k \to 0$ as $k \to \infty$. Let $h_k := T_{\bar{\Theta}S^c} f_k \in H^2$ and let $g_k(z) := \bar{\Theta} S^c f_k - h_k \in \overline{H^2}$. Notice that since $h_k \to 0$ and $\|f_k\| = 1$ we have that the norms of g_k are bounded from below.

Since $\Theta(z) g_k(z) \in K_\Theta$ we have by the Clark formula

$$\Theta(z) g_k(z) = \frac{1 - \Theta(z)}{2\pi i} \sum_n \frac{g_n^k}{\lambda_n - z},$$

where $g_n^k := \Theta(\lambda_n) g_k(\lambda_n) = g_k(\lambda_n)$. By the remark above the l_2 norms of the sequence (g_n^k) are bounded from below. Now we have the following equality

$$\Theta(z) h_k(z) = \frac{1}{2\pi i} \sum_n \frac{1 - S^c(z) S^{-c}(\lambda_n)}{\lambda_n - z} g_n^k - S^c(z)(f_k(z) - \frac{1}{2\pi i} \sum_n \frac{S^{-c}(\lambda_n) g_n^k}{\lambda_n - z}).$$

In the difference above the first term is in K_{S^c} and the second one is in $S^c H^2$. Thus they are orthogonal for all k. Since the difference tends to 0 we have that both terms must tend to 0. Therefore,

$$\frac{1}{2\pi i} \sum_n \frac{1 - S^c(z) S^{-c}(\lambda_n)}{\lambda_n - z} g_n^k \to 0$$

and hence $\sum_n g_n^k e^{i\lambda_n t} \to 0$ in $L^2(0, c)$ as $k \to \infty$. Since (g_n^k) is bounded from below in l_2 it follows that $(e^{i\lambda_n t})$ cannot be a Riesz sequence in $L^2(0, c)$. □

As mentioned above, it was shown in [27] that a sequence of complex exponentials $(e^{i\lambda_n t})$ is complete in $L^2(0, c)$ if and only if the Toeplitz operator $T_{\Theta \bar{S}^c}$ is injective for every meromorphic inner function $\Theta(z)$ such that $\{t|\Theta(t)=1\}=\{\lambda_n\}_{n\in\mathbb{Z}}$. Combining this result with the theorem above gives the following:

Corollary 3.5.2. *Let $\Lambda := (\lambda_n)_{n\in\mathbb{Z}}$ be a separated sequence of real numbers and let $\Theta(z)$ be the meromorphic inner function associated to Λ. The sequence of complex exponentials $(e^{i\lambda_n t})$ is a Riesz basis in $L^2(0,c)$ if and only if the Toeplitz operator $T_{\bar{\Theta}S^c}$ is invertible.*

3.6. Criterion for Parseval frames

Below we describe Parseval frames of complex exponentials. The description is very similar to the description of frames given in [34].

Theorem 3.6.1. *Let $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ be a separated sequence of real numbers and let $\Theta(z)$ be the meromorphic inner function associated to Λ. A sequence of complex exponentials $(e^{i\lambda_n t})$ is a Parseval frame in $L^2(0,c)$ if and only if $\Theta(z)$ is divisible by $S^c(z)$. Consequently, the largest c for which $(e^{i\lambda_n t})$ is a Parseval frame in $L^2(0,c)$ is equal to the mean type of $\Theta(z)$.*

Proof. Assume that $(e^{i\lambda_n t})$ is a Parseval frame in $L^2(0,c)$ and equivalently $(k_{S^c}(z,\lambda_n))$ is a Parseval frame in K_{S^c}. Then for every $f(z)\in K_{S^c}$ we have

$$\|f\|^2 = \int |f(t)|^2 d\mu(t),$$

where μ is the counting measure for Λ. By Aleksandrov's theorem [1] there exists a function $h(z)$ holomorphic in the upper half-plane with $|h(z)|\le 1$ on $\mathbb{C}_+$ and

$$\frac{y}{\pi}\sum_n \frac{1}{(\lambda_n - x)^2 + y^2} = \Re\frac{1+h(z)S^c(z)}{1-h(z)S^c(z)}.$$

But this says that $h(z)S^c(z)$ is the inner function associated to Λ. Therefore $h(z)$ is also inner function and hence $S^c(z)$ divides $\Theta(z)$.

The opposite is similar and easier.

□

Remark 4. It seems that the description of the mean type of $\Theta(z)$ in terms of Λ is a difficult problem.

We also have the following more general statement with the same proof

Theorem 3.6.2. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a separated sequence of real numbers and let $\Theta(z)$ be the meromorphic inner function associated to Λ. A sequence of reproducing kernels $(k_\Psi(z,\lambda_n))$ is a Parseval frame in K_Ψ if and only if $\Theta(z)$ is divisible by $\Psi(z)$.*

3.7. Basis properties of systems of eigenfunctions of one-dimensional Schrödinger operators

Consider the differential equation

$$-u''(t) + q(t)u(t) = zu(t) \tag{3.5}$$

on some finite interval $[0,c]$ (As usual we assume that the potential $q \in L^1(0,c)$). Fix some self-adjoint boundary condition at 0 of the form

$$\cos\alpha u(t) + \sin\alpha u'(t) = 0. \tag{3.6}$$

Denote by $u(t,z)$ ($z \in \mathbb{C}$) the non-trivial solution of 3.5 satisfying the boundary condition at 0.

Each such equation gives rise to a short de Branges space with a de Branges function

$$E(z) := \sqrt{\pi}(u'(c,z) - iu(c,z)).$$

The corresponding generalized Fourier transform $\mathcal{F} : L^2(0,c) \to B_E$ defined by

$$\mathcal{F}(f)(z) := \int_0^c f(t)u(t,z)dt$$

is a unitary transformation which maps $u(t,\lambda)$ to a reproducing kernel $K_E(z,\lambda)$.

The next proposition shows that for a given separated real sequence $(\lambda_n)_{n\in\mathbb{N}}$ the basis properties of the system $(u(t,\lambda_n))$ in $L^2(0,c)$ are identical to the basis properties of the system $(cos\sqrt{\lambda_n}t)$ in $L^2(0,c)$.

Proposition 3.7.1. *Let $\Lambda := (\lambda_n)_{n\in\mathbb{N}}$ be a separated sequence of positive real numbers. Let $u(t,\lambda)$ be the non-trivial solution of 3.5 satisfying some non-Dirichlet boundary condition at 0 ($\alpha \neq 0$ in 3.6). Then the system $(u(t,\lambda_n))$ is l^2- independent in $L^2(0,c)$ if and only if the system $(cos\sqrt{\lambda_n}t)$ is l^2- independent in $L^2(0,c)$*

Proof. Consider the equation 3.5 with a zero potential and Neumann boundary condition at 0 ($\alpha = \pi/2$ in 3.6). Notice that its non-trivial solution is given by $(cos\sqrt{z}t)$ and its de Branges function is given by $E_N(z) := \cos c\sqrt{z} + i\sqrt{z}\sin c\sqrt{z}$. Therefore, all we need to show is that the systems of reproducing kernels $(K_{E_N}(z,\lambda_n))$ and $(K_E(z,\lambda_n))$ have the same basis properties. Here, $E(z)$ is the de Branges function of the equation with solutions $u(t,\lambda)$.

Standard asymptotic formulae for $u(t,\lambda)$ imply that $0 < c < |E(t)|/|E_N(t)| < C < \infty$ on $\mathbb{R}$ (see [27]). Therefore, B_E and B_{E_N} are equal as sets with equivalent norms. Now, showing that $(K_{E_N}(z,\lambda_n))$ is l^2- independent (complete, a frame, a Riesz sequence, a Riesz basis) if and only if the system $(K_E(z,\lambda_n))$ is l^2- independent (complete, a frame, a Riesz sequence, a Riesz basis) is easy. We will prove the l^2- independence only since the rest are similar. Indeed, $\sum_n a_n K_E(z,\lambda_n) = 0$ for some $(a_n) \in l^2$ is equivalent to $\sum_n a_n F(\lambda_n) = 0$ for every $F(z) \in B_E$. The last equality is equivalent to $\sum_n b_n F_1(\lambda_n) = 0$, where $b_n := E(\lambda_n)a_n/E_N(\lambda_n)$ and $F_1(z) :=$

$E_N(z)F(z)/E(z)$. The last two equations are unitary transformations. The first one is unitary in l_2 whereas the second one is unitary between B_E and B_{E_N}. Thus, we obtain that l_2- dependence of $(K_E(z,\lambda_n))$ is equivalent to l_2- dependence of $(K_{E_N}(z,\lambda_n))$. We are done. □

3.8. Beurling gap problem

The Beurling gap problem that we consider here may be formulated as follows. For a given real separated sequence $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ find the supremum of all $a > 0$ for which there exists a nonzero l_1 sequence (a_n) such that $\sum_n a_n e^{i\lambda_n t} = 0$ for all $t \in (-a, a)$. Notice that if $(a_n) \in l_1$ then $\sum_n a_n e^{i\lambda_n t}$ defines a continuous function in t. Denote this supremum by $R_1(\Lambda)$.

Theorem 3.8.1. *With the notation above the following equality holds*

$$R_1(\Lambda) = \pi D^-_{BM}(\Lambda).$$

Notice that since $l^1 \subset l^2$ from Theorem 3.1.4 we immediately have $R_1(\Lambda) \leq R(\Lambda)/2 = \pi D^-_{BM}(\Lambda)$. The proof of the equality now follows from 2 combined with the following two lemmas.

Lemma 3.8.2. *Let $(\lambda_n)_{n\in\mathbb{Z}}$ be a separated sequence of real numbers and let $\Theta(z)$ be the meromorphic inner function associated to Λ. If $\ker T_{\bar{\Theta}S^{2a}} \neq 0$ for some $a > 0$, then for any $\epsilon > 0$ there exists $(h_n) \in l^1$ such that*

$$\lim_{y\to\pm\infty} e^{xy} \sum_n \frac{h_n}{\lambda_n - iy} = 0$$

for every $x \in (-a+\epsilon, a-\epsilon)$.

Proof. Let

$$b(z) := \frac{z-i}{z+i}.$$

Since $\ker T_{\bar{\Theta}S^{2a}} \neq 0$ we have that $\ker T_{\bar{\Theta}S^{2a-2\epsilon}b} \neq 0$ for any $\epsilon > 0$. Hence there exists a non-zero

$$f \in \ker T_{\bar{\Theta}S^{2a-2\epsilon}b} \neq 0 \subset H^2(\mathbb{C}_+).$$

Then $S^{2a-2\epsilon}(z)b(z)f(z) \in K_\Theta$. Define $h(z) := S^{a-\epsilon}(z)b(z)f(z)/(z-i)$. Clearly $h(z)$ belongs to K_Θ, and therefore

$$h(z) = \frac{1-\Theta(z)}{2\pi i}\int \frac{h(t)}{t-z}d\sigma(t)$$

where σ is the counting measure for Λ. In particular, for $x < a - \epsilon$,

$$\lim_{y\to\infty} e^{xy}\int \frac{h(t)}{t-iy}d\sigma(t) = 0$$

because $f(iy) \to 0$, since $f \in H^2(\mathbb{C}_+)$, and because the outer function $1 - \Theta(iy)$ cannot go to zero exponentially fast.

Denote $g = \bar{\Theta}h \in \bar{H}^2 = H^2(\mathbb{C}_-)$. Then $h = \Theta g$ in the lower half-plane. Note also that $g = S^{-a+\epsilon}k$ where

$$k(z) = \bar{\Theta}(z)S^{2a-2\epsilon}(z)b(z)f(z)/(z-i) \in \bar{H}^2 = H^2(\mathbb{C}_-).$$

Hence for $x > -a + \epsilon$,

$$\lim_{y\to-\infty} e^{xy}\int \frac{h(t)}{t-iy}d\sigma(t) = \lim_{y\to-\infty} 2\pi i\frac{k(iy)\Theta(iy)}{e^{(a-\epsilon+x)y}(1-\Theta(iy))} = 0.$$

The last equality follows from the facts that $k(z) \in H^2(\mathbb{C}_-)$ and that

$$\frac{1-\Theta(iy)}{\Theta(iy)} = \overline{\Theta}(-iy) - 1.$$

It is left to notice that $h(z) = l(z)/(z-i)$ where both $l = S^{2a-2\epsilon}bf$ and $(z-i)^{-1}$ belong to $L^2(\sigma)$. Thus, $h \in L^1(\sigma)$, i.e., $(h_n) = (h(\lambda_n)) \in l^1$. □

The following lemma is a well known fact that we include here for completeness.

Lemma 3.8.3. *Let μ be a measure with finite total variation. Then the Fourier transform of μ vanishes on $[-a,a]$ if and only if*

$$\lim_{y\to\pm\infty} e^{xy}\int \frac{d\mu(t)}{t-iy}=0,$$

for every $x\in[-a,a]$.

Proof. Suppose that $\int e^{ixt}d\mu(t)=0$ for all $x\in[-a,a]$. Then

$$e^{-ixz}\int_{-\infty}^{+\infty}\frac{e^{ixt}-e^{ixz}}{i(t-z)}=\int_{-\infty}^{+\infty}\int_0^x e^{iu(t-z)}dud\mu(t)=$$

$$=\int_0^x\int_{-\infty}^{+\infty}e^{iut}d\mu(t)e^{-iuz}du=0,$$

for every $x\in[-a,a]$ and $z\in\mathbb{C}$. Therefore,

$$\int\frac{e^{ixt}-e^{ixz}}{t-z}d\mu(t)=0$$

for every $x\in[-a,a]$. Obviously,

$$\lim_{y\to\pm\infty}\int\frac{e^{ixt}}{t-iy}d\mu(t)=0 \tag{3.7}$$

and therefore

$$\lim_{y\to\pm\infty}e^{xy}\int\frac{d\mu(t)}{t-iy}=0$$

for every $x\in[-a,a]$.

Conversely, for $x\in[-a,a]$, define

$$H(z):=\int\frac{e^{ixt}-e^{ixz}}{t-z}d\mu(t).$$

Then $H(z)$ is an entire function of Cartwright class. To show that $H(z)$ is identically zero it suffices to check that $\lim_{y\to\pm\infty}H(iy)=0$. Recall that for $x\in[-a,a]$,

$$\lim_{y\to\pm\infty}\int\frac{e^{ix(iy)}}{t-iy}d\mu(t)=0.$$

Together with (3.7) this implies $H(z) \equiv 0$.

Thus

$$\int e^{ixt} d\mu(t) = \lim_{y\to\infty} -iy \int \frac{e^{ixt} d\mu(t)}{t-iy} = \lim_{y\to\infty} -iye^{-xy} \int \frac{d\mu(t)}{t-iy} = 0$$

for all $-a \le x \le a$. □

An extreme point method of de Branges [11] can be used to give another proof of the inequality $R(\Lambda) \le \pi D^{-}_{BM}(\Lambda)$. This proof does not depend on Toeplitz kernels techniques. Below, we give a sketch of it. We will follow de Branges's argument from [11]. The only difference is that we work with l_1-sequences instead of finite measures.

Assume that there exists a non-zero sequence $(a_n) \in l^1$ such that $\sum_n a_n e^{i\lambda_n t} = 0$ for all $t \in (-a, a)$. By applying the Fourier transform we obtain a relation for the corresponding sequence of reproducing kernels in the Paley-Wiener space $\mathcal{PW}_c$ and therefore

$$\sum_n a_n F(\lambda_n) = 0, \tag{3.8}$$

for all $F(z) \in \mathcal{PW}_c$. The fact that $F(z) \in \mathcal{PW}_c$ implies $F^{\#}(z) \in \mathcal{PW}_c$ assures existence of a real sequence $(a_n) \in l^1$ satisfying 3.8.

Consider the set of all real sequences $(a_n) \in l^1$ with l^1 norm at most 1 and satisfying 3.8. It is a nonempty weak* compact and convex set. Therefore by the Krein-Millman Theorem it must have a nonzero extreme point. Fix one such extreme point $(a_n) \in l^1$. Let $\Lambda' \subset \Lambda$ be the subsequence consisting of those λ_n for which $a_n \neq 0$. The goal is to construct an entire function $H(z)$ of Cartwright class vanishing at Λ' such that

$$\limsup_{y\to\infty} \frac{\log |H(iy)|}{y} = \frac{c_1}{2} = \limsup_{y\to-\infty} \frac{\log |H(iy)|}{-y}, \tag{3.9}$$

for some $c_1 \geq c$. The natural candidate for such an entire function is

$$H(z) := \frac{1}{\sum_n \frac{a_n}{\lambda_n - z}}.$$

Of course, such a function $H(z)$ will not be entire for all $(a_n) \in l^1$ satisfying 3.8. However, it turns out that it will be entire for our extremal choice $(a_n) \in l^1$.

First, it is easy to show that $\sum_n |a_n| = 1$. Denote by $\mathcal{B}$ the closure of $\mathcal{PW}_c$ with respect to the norm

$$\|F\| := \sum_n |F(\lambda_n)||a_n|.$$

It can be shown that $\mathcal{B}$ consists of entire function of Cartwright class $\mathcal{C}_c$. Moreover, the fact that (a_n) is an extreme point implies that for any sequence (b_n) satisfying $\sum_n |a_n b_n| < \infty$ and $\sum_n a_n b_n = 0$ there exists a function $G(z) \in \mathcal{B}$ such that $G(\lambda_n) = b_n$. Without loss of generality assume that $\lambda_0, \lambda_1 \in \Lambda'$. There exists an entire function $G(z) \in \mathcal{B}$ such that $G(\lambda_0) = 1/a_0(\lambda_0 - \lambda_1)$, $G(\lambda_1) = 1/a_1(\lambda_1 - \lambda_0)$, and $G(\lambda_n) = 0$ for all other elements in Λ'. Finally, define $H(z) := G(z)(z - \lambda_1)(z - \lambda_0)$. It can be shown that

$$H(z) := \frac{1}{\sum_n \frac{a_n}{\lambda_n - z}},$$

and $H(z)$ satisfies 3.9. This implies that Λ is a regular sequence in the sense of [4]. Finally, using the well known description of the interior BM-density in terms of regular subsequences we obtain the desired inequality $R(\Lambda) \leq \pi D^{-}_{BM}(\Lambda)$.

Remark 5. A little more work shows that

$$\frac{S^c(z)}{H(z)} = \sum_n \frac{a_n e^{ic\lambda_n}}{\lambda_n - z}.$$

CHAPTER IV

PÓLYA-LEVINSON PROBLEM

A separated sequence Λ on the real line is called a Pólya sequence if any entire function of zero exponential type bounded on Λ must be a constant function. In this chapter we solve the problem of Pólya and Levinson that asks for a description of Pólya sequences.

4.1. Introduction

Recall that an entire function $F(z)$ is said to have exponential type zero if

$$\limsup_{|z|\to\infty} \frac{\log|F(z)|}{|z|} = 0.$$

We call a separated real sequence $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ (a sequence satisfying $|\lambda_n - \lambda_m| \geq \delta > 0, \ (n \neq m)$) a *Pólya sequence* if any entire function of exponential type zero that is bounded on Λ is constant.

Historically, first results on Pólya sequences were obtained in the work of Valiron [45], where it was proved that the set of integers $\mathbb{Z}$ is a Pólya sequence. Later this result was popularized by Pólya, who posted it as a problem in [40]. Subsequently many different proofs and generalizations were given (see for example section 21.2 of [24] or chapter 10 of [5] and references therein).

In his 1940' book [25] Levinson showed that if $|\lambda_n - n| \leq p(n)$, where $p(t)$ satisfies

$$\int \frac{p(t)}{1+t^2} \log|\frac{t}{p(t)}| dt < \infty$$

and some smoothness conditions, then $\Lambda = \{\lambda_n\}$ is a Pólya sequence. In the same time for each such $p(t)$ satisfying $\int p(t)dt/(1+t^2) = \infty$ he was able to construct a sequence $\Lambda = \{\lambda_n\}$ that is not Pólya sequence. As it often happens in problems from this area, the construction took a considerable effort (see [25], pp. 153-185). Closing the gap between Levinson's sufficient condition and the counterexample remained an open problem for almost 25 years until de Branges [11] essentially solved it by showing that Λ is a Pólya sequence if $\int p(t)dt/(1+t^2) < \infty$ (but assuming extra regularity conditions on the sequence).

The results of [25] and [11] remain strongest to date. However, none of them gives a complete answer, since there are Pólya sequences for which $\int p(t)dt/(1+t^2) = \infty$. For example, as will be clear from our results below, the sequence

$$\lambda_n := n + n/\log(|n|+2),\ n \in \mathbb{Z}$$

is a Pólya sequence.

In the opposite direction, [11] contains the following necessary condition. A sequence of disjoint intervals I_n on the real line is called long (in the sense of Beurling and Malliavin) if

$$\sum_n \frac{|I_n|^2}{1+dist(I_n,0)^2} = \infty,$$

and it is called short otherwise. De Branges [11] proved that if the complement of a closed set $X \subset \mathbb{R}$ is long then there exists a non constant zero type entire function that is bounded on X. In particular, if the complement of a sequence Λ is long then Λ is not a Pólya sequence. The sequence $\lambda_n := n^2$ shows that this condition is not sufficient. Indeed, $\lambda_n = n^2$ is the zero set of the zero type function $F(z) := \cos\sqrt{2\pi z}\cos\sqrt{-2\pi z}$ and thus is not a Pólya sequence. On the other hand, the real complement of this sequence is short.

Most of the results from this chapter can be found in [29].

4.2. Characterization of Pólya sequences

As was mentioned in the introduction, a sequence of real numbers is called separated if $|\lambda_n - \lambda_m| \geq \delta > 0, \ (n \neq m)$. It is natural to introduce a separation condition in the Pólya-Levinson problem because of the following obvious reasons. If one takes a zero set of a zero-type entire function and adds a large number of points close enough to each zero, the entire function will still be bounded on the new sequence. At the same time, this way one can obtain non-Pólya sequences of arbitrarily large density, in any reasonable definition of density. Hence, if one hopes to obtain a description of Pólya sequences based on densities or similar terms, it is necessary to include a separation condition, as it was done in the classical results cited above.

The following theorem gives a complete characterization of Pólya sequences.

Theorem 4.2.1. *Let $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ be a separated sequence of real numbers and let $\Theta(z)$ be the meromorphic inner function associated to Λ. The following are equivalent:*

(i) $\Lambda = \{\lambda_n\}$ *is a Pólya sequence.*

(ii) The interior Beurling-Malliavin density of Λ, $D^-_{BM}(\Lambda)$, is positive.

(iii) $\ker T_{\bar{\Theta}S^{2c}} \neq 0$*, for some $c > 0$.*

We will use the following lemma.

Lemma 4.2.2. *Let $\Lambda = (\lambda_n)_{n\in\mathbb{Z}}$ be a separated sequence of real numbers and let $\Theta(z)$ be the meromorphic inner function associated to Λ. Denote by σ the counting measure on Λ (i.e. the Clark measure corresponding to $\Theta(z)$). If $\ker T_{\bar{\Theta}S^{2c}} \neq 0$ for some $c > 0$ then there exists $h \in L^2(\sigma)$*

$$\lim_{y\to\pm\infty} e^{xy} \int \frac{h(t)}{t - iy} d\sigma(t) = 0,$$

for all $x \in (-c, c)$.

Proof. Let $f(z)$ be some nonzero element of $\ker T_{\bar{\Theta}S^{2c}} \subset H^2(\mathbb{C}_+)$. Then $S^{2c}(z)f(z) \in K_\Theta$. Define $h(z) := S^c f(z)$. Clearly, $h(z)$ belongs to K_Θ and therefore

$$h(z) = \frac{1-\Theta(z)}{2\pi i}\int \frac{h(t)}{t-z}d\sigma(t).$$

In particular, for $x < c$

$$\lim_{y\to\infty} e^{xy}\int \frac{h(t)}{t-iy}d\sigma(t) = 0$$

because $f(iy) \to 0$, since $f \in H^2(\mathbb{C}_+)$, and because the outer function $1-\Theta(iy)$ cannot go to zero exponentially fast.

Let $g(t) := \bar{\Theta}(t)h(t) \in \overline{H^2(\mathbb{R})} = H^2(\mathbb{C}_-)$. Then the meromorphic continuation of $h(z)$ to the lower half plane $\mathbb{C}_-$ is given by $h(z) = \Theta(z)g(z)$. Note that $g(z) = S^{-c}(z)k(z)$ where

$$k(t) = \bar{\Theta}(t)S^{2c}(t)f(t) \in \overline{H^2(\mathbb{R})} = H^2(\mathbb{C}_-).$$

Therefore, for $x > -a$,

$$\lim_{y\to-\infty} e^{xy}\int \frac{h(t)}{t-iy}d\sigma(t) = \lim_{y\to-\infty} 2\pi i\frac{k(iy)\Theta(iy)}{e^{(c+x)y}(1-\Theta(iy))} = 0.$$

The last equality follows from the facts that $k(z) \in H^2(\mathbb{C}_-)$ and that

$$\frac{1-\Theta(iy)}{\Theta(iy)} = \overline{\Theta}(-iy) - 1.$$

□

Proof. By Remark 2 in the previous chapter it follows that *(ii)* and *(iii)* are equivalent.

(i) ⇒ *(iii)* Assume *(iii)* is not true, i.e., for the meromorphic inner function $\Theta(z)$ associated with Λ, $\ker T_{\bar{\Theta}S^{2c}} = 0$ for every $c > 0$. In this case we will construct a nonconstant zero type entire function which is bounded on Λ, which will mean that

Λ is not a Pólya set. Let σ be the counting measure on Λ. Then since Λ is separated $\int d\sigma(t)/(1+t^2) < \infty$. Therefore, there exists a short de Branges space B_E contained isometrically and properly in $L^2(\mu)$. First, let us show that B_E cannot contain a function of positive exponential type $c > 0$.

Suppose that $F(z) \in B_E$ has a positive exponential type. We can assume that $F(iy)$ grows exponentially in y as $y \to \infty$. Since $B_E \neq L^2(\sigma)$, there exists $g(t) \in L^2(\sigma)$ with $\bar{g}(t) \perp B_E$. Then

$$0 = \int \frac{F(t) - F(w)}{t - w} g(t) d\sigma(t) = \int \frac{F(t)}{t - w} g(t) d\mu(t) - F(w) \int \frac{1}{t - w} g(t) d\sigma(t),$$

for any $w \in \mathbb{C}$ and therefore

$$F(w) = \frac{\int \frac{F(t)}{t-w} g(t) d\sigma(t)}{\int \frac{1}{t-w} g(t) d\sigma(t)}.$$

Since $F(w)$ grows exponentially along $i\mathbb{R}_+$, the integral in the denominator must decay exponentially in w along $i\mathbb{R}_+$. Thus the function

$$G(z) := \frac{1 - \Theta(z)}{2\pi i} \int \frac{1}{t - z} g(t) d\sigma(t)$$

can be represented as $G(z) = S^c(z)h(z)$ for some nonzero $h(z) \in H^2(\mathbb{C}_+)$ and $c > 0$, and belongs to K_Θ, where $\Theta(z)$ is the inner function associated to Λ. Hence $h(z) \in \ker T_{\bar{\Theta} S^{2c}}$ and we have a contradiction.

Therefore any $F(z) \in B_E$ has zero type. It is left to notice that

$$|F(\lambda_n)| \leq \sqrt{\sum_m |F(\lambda_m)|^2} = \|F\|_{L^2(\mu)} < \infty,$$

which means that $F(z)$ is bounded on Λ.

(iii) $\Rightarrow$ *(i)* Let $F(z)$ be a zero type entire function bounded on Λ by some

constant $M > 0$. For any integer $n \in \mathbb{N}$, $F^n(z)$ is also a zero type function. Let σ be the counting measure on Λ. If $\ker T_{\bar{\Theta}S^{2c}} \neq 0$ for some $c > 0$ then by the Lemma 4.2.2 there exists $h(t) \in L^2(\sigma)$ such that

$$\lim_{y \to \pm\infty} e^{xy} \int \frac{h(t)}{t - iy} d\sigma(t) = 0, \tag{4.1}$$

for all $x \in (-c, c)$.

Define

$$H(z) := \int \frac{F^n(t) - F^n(z)}{t - z} h(t) d\sigma(t)$$

for all $z \in \mathbb{C}$. It is clear that $H(z)$ is an entire function of zero type. To show that $H(z) \equiv 0$ it is enough to check that $H(iy) \to 0$ as $y \to \pm\infty$.

Using 4.1 and Lemma 3.3.1 we obtain

$$\lim_{y \to \pm\infty} H(iy) = \lim_{y \to \pm\infty} \left[\int \frac{F^n(t)}{t - iy} h(t) d\sigma(t) - F^n(iy) \int \frac{h(t)}{t - iy} d\sigma(t) \right] = 0.$$

Therefore,

$$\int \frac{F^n(t) - F^n(z)}{t - z} h(t) d\sigma(t) \equiv 0.$$

Now,

$$|F(z) \left(\int \frac{h(t) d\sigma(t)}{t - z} \right)^{1/n} | \leq M \left(\|h\|_2 \sum_n \frac{1}{|\lambda_n - z|^2} \right)^{1/n} \leq M \left(\frac{C \|h\|_2}{|\Im z|} \right)^{1/n},$$

for every non real z. Since this is true for all $n \in \mathbb{N}$, we have that $|F(z)| \leq M$ for all non real $z \in \mathbb{C}$ for which $\int d\mu(t)/(t - z) \neq 0$. Since σ is a non-zero measure, by continuity, $F(z)$ is bounded on the whole plane and hence it is identically 0.

□

As an immediate consequence we obtain that the sequence of integers $\mathbb{Z}$ is a Pólya sequence, as known from Valiron's original statement. This follows from (iii)

and also from (iv) by taking $\Theta(z) = S^{2\pi}(z)$. Another consequence is that a separated real sequence with classical density zero cannot be a Pólya sequence. However, there are sequences with positive classical density which are not Pólya. As was mentioned in the introduction, the first example of such a sequence was given by Levinson [25]. New examples in both directions can now be constructed using the following description.

Corollary 4.2.3. *Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}}$ be a separated sequence of real numbers. Then Λ is a Pólya sequence if and only if for every long sequence of intervals $\{I_n\}$ the sequence $\#(\Lambda \cap I_n)/|I_n|$ is not a null sequence, i.e., $\#(\Lambda \cap I_n)/|I_n| \nrightarrow 0$.*

Proof. Suppose that there exists a long sequence of intervals $\{I_n\}$ such that $\#(\Lambda \cap I_n)/|I_n| \to 0$. Let $\Theta(z)$ be any meromorphic inner function with $\{\Theta = 1\} = \Lambda$ whose argument $\phi(t) := \arg\Theta(t)$ has a bounded derivative. Also, let $b > 0$ and $\epsilon = b/4\pi > 0$. Without loss of generality, $|I_n| \to \infty$. Then there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, $|I_n| > 2\pi/b$ and $\#(\Lambda \cap I_n)/|I_n| < \epsilon$. Let $k_n := \#(\Lambda \cap I_n) \in \mathbb{N}$ and let $I_n = (a_n, b_n)$. Denote by λ_1^n the largest element of Λ smaller than a_n and by $\lambda_{k_n+2}^n$ the smallest element larger than b_n. Denote the first half of I_n by J_n. For every $t \in J_n$ we have

$$2b(\lambda_{k_n+2}^n - t) > 2b\frac{|I_n|}{2} > b\frac{k_n}{\epsilon} \geq b\frac{k_n+1}{4\epsilon} = \pi(k_n+1) > \phi(\lambda_{k_n+2}^n) - \phi(t)$$

if $k_n > 0$, and similarly

$$2b(\lambda_{k_n+2}^n - t) > 2b\frac{|I_n|}{2} > 2\pi > \phi(\lambda_{k_n+2}^n) - \phi(t)$$

if $k_n = 0$. Therefore, for $\gamma_b(t) := 2bt - \phi(t)$, we obtain $\gamma_b(\lambda_{k_n+2}^n) > \gamma_b(t)$, for all $t \in J_n$, and hence $J_n \in \mathcal{BM}(\gamma_b)$. Thus, $\mathcal{BM}(\gamma_b)$ is long, and consequently $\gamma_b(t)$ is not almost decreasing. Since $\gamma_b'(t) > -const.$, Theorem 3.2.2 applies and we get $\ker T_{S^\epsilon e^{i\gamma_b}} \subseteq \ker T^+_{S^\epsilon e^{i\gamma_b}} = 0$ for every $\epsilon > 0$. Now, for any $a > 0$, by choosing $b = \epsilon = a/3 > 0$, we obtain $\ker T_{\bar{\Theta}S^{2a}} = 0$. Finally, by the remark after Theorem 4.2.1, Λ is not a Pólya set.

Conversely, assume that Λ is not a Pólya set. Choose a meromorphic inner function $\Theta(z)$ with $\{\Theta = 1\} = \Lambda$ which has a continuous argument $\phi(t)$ with bounded derivative. We will show that for every $b > 0$, $\gamma_b(t) := 2bt - \phi(t)$ is not almost decreasing. Indeed, let $b > 0$ be such that $\gamma_b(t)$ is almost decreasing. Then, by Theorem II, $\ker T^+_{S^{2b-\epsilon}e^{-i\phi}} = \ker T^+_{\bar{S}^{\epsilon}e^{i\gamma_b}} \neq 0$ for every $\epsilon > 0$. Now, using Theorem 3.2.2, we obtain $N[\bar{\Theta}S^{2a}] \neq 0$, for some $a > 0$. This is in contradiction with Theorem 4.2.1. So, $\gamma_b(t)$ is not almost decreasing for all $b > 0$. This implies that for every $b > 0$ either $\gamma_b(\pm\infty) \neq \mp\infty$ or $\mathcal{BM}(\gamma_b)$ is long.

First, assume that there exists a decreasing null sequence (b_n) such that

$$\gamma_{b_n}(\pm\infty) \neq \mp\infty.$$

By passing to a subsequence we can assume without loss of generality that $\gamma_{b_n}(\infty) \neq -\infty$ (the case $\gamma_{b_n}(-\infty) \neq \infty$ is analogous). There exists $\epsilon^1 > 0$ and a sequence $t_n^1 \to \infty$ such that $\gamma_{b_1}(t_n^1) > \epsilon^1$ for all $n \in \mathbb{N}$. Therefore, for $n_1 \in \mathbb{N}$ large enough we have that

$$\frac{\#(\Lambda \cap (0, t_{n_1}^1))}{t_{n_1}^1} < b_1$$

Now, since $\gamma_{b_2}(\infty) - \gamma_{b_2}(t_{n_1}^1) \neq -\infty$ there exists $\epsilon^2 > 0$ and a sequence $t_n^2 \to \infty$ such that $\gamma_{b_2}(t_n^2) - \gamma_{b_2}(t_{n_1}^1) > \epsilon^2$. for all $n \in \mathbb{N}$. Hence, for $n_2 \in \mathbb{N}$ large enough we have that $t_{n_2}^2 > 2t_{n_1}^1$ and

$$\frac{\#(\Lambda \cap (t_{n_1}^1, t_{n_2}^2))}{t_{n_2}^2} < b_2.$$

Continuing the same procedure we obtain a sequence $(t_{n_k}^k)$ such that $t_{n_{k+1}}^{k+1} - t_{n_k}^k > t_{n_k}^k$ and

$$\frac{\#(\Lambda \cap (t_{n_{k-1}}^{k-1}, t_{n_k}^k))}{t_{n_k}^k} < b_k,$$

for all $k \in \mathbb{N}$. Finally, by taking $I_k := (t^k_{n_k}, t^{k+1}_{n_{k+1}})$ we obtain a long sequence of intervals (I_k) such that

$$\frac{\#(\Lambda \cap I_k)}{|I_k|} \to 0.$$

If there is no decreasing null sequence (b_n) such that $\gamma_{b_n}(\pm\infty) \neq \mp\infty$ then there exists $b_0 > 0$ such that $\gamma_b(\pm\infty) = \mp\infty$ for all $b < b_0$. For $r \in \mathbb{N}$, since $\gamma_{b_0/r}(\pm\infty) = \mp\infty$, we have that $\mathcal{BM}(\gamma_{b_0/r}) = \cup_n I^r_n$ is long. Let $I^r_n = (a^r_n, b^r_n)$ and let $k^r_n := \#(\Lambda \cap I^r_n)$. Then, since

$$\gamma_{\frac{b_0}{r}}(a^r_n) = \gamma_{\frac{b_0}{r}}(b^r_n),$$

we have

$$2\pi(k^r_n - 1) \geq \phi(b^r_n) - \phi(a^r_n) = \frac{b_0}{r}(b^r_n - a^r_n) = \frac{b_0}{r}|I^r_n|$$

and hence

$$\frac{\#(\Lambda \cap I^r_n)}{|I^r_n|} \leq \frac{1}{2\pi k} + \frac{b_0}{|I^r_n|}.$$

So, for each $r \in \mathbb{N}$, there is a large enough n_r such that for $n \geq n_r$,

$$\frac{\#(\Lambda \cap I^r_n)}{|I^r_n|} \leq \frac{1}{\pi k}.$$

Choose $l_1 < m_1 \in \mathbb{N}$ such that

$$\sum_{n=l_1+1}^{m_1} \frac{|I^{(1)}_n|^2}{d(I^{(1)}_n)^2} > 1$$

and define $I_n = I^{(1)}_{l_1+n}$ for $1 \leq n \leq m_1 - l_1$. Next, choose $l_2 < m_2 \in \mathbb{N}$ such that

$$\sum_{n=l_2+1}^{m_2} \frac{|I^{(2)}_n|^2}{d(I^{(2)}_n)^2} > 1,$$

and $I^{(1)}_{m_1}$ is to the left of $I^{(2)}_{l_2+1}$. Define $I_{m_1-l_1+n} = I^{(2)}_{l_2+n}$ for $1 \leq n \leq m_2 - l_2$. Continue the same procedure. It is clear that the sequence $\{I_n\}$ defined in this way is long and that $\#(\Lambda \cap I_n)/|I_n| \to 0$.

□

4.3. Cartwright theorem

It is natural to ask weather there exist Pólya type sequences for entire functions of exponential type bigger than 0. The most famous result in this direction is the classical theorem of Cartwright.

Theorem 4.3.1. *If $F(z)$ is an entire function of exponential type $< \pi$ which is bounded on the integers Z (i.e. there exists $M > 0$ such that $|F(n)| < M$ for all $n \in \mathbb{Z}$) then $F(z)$ must be bounded on the whole real line.*

It is interesting to try to extend this classical result to more general real sequences then $\mathbb{Z}$. Below we give some results in this direction that can be obtained using the techniques from the proof of Theorem 4.2.1.

Let Λ be a separated sequence of real numbers and let $\Theta(z)$ be the meromorphic inner function associated to Λ. Denote by σ the counting measure on Λ. Let $F(z)$ be an entire function of exponential type $c < \pi D^-_{BM}(\Lambda)$ such that $|F(\lambda_n)| < M$.

By the Lemma 4.2.2 for every $h(z) \in \ker T_{\bar{\Theta}S^c}$ we have that

$$\lim_{y \to \pm\infty} e^{xy} \int \frac{h(t)}{t - iy} d\sigma(t) = 0,$$

for all $x \in (-c, c)$.

Define

$$H(z) := \int \frac{F(t) - F(z)}{t - z} h(t) d\sigma(t)$$

for all $z \in \mathbb{C}$. It is clear that $H(z)$ is an entire function of Carthwright class. Reasoning as in the proof of Theorem 4.2.1 one shows that $H(z) \equiv 0$. Therefore we obtain the following identity

$$F(z) = \frac{\int \frac{F(t)h(t)}{t-z} d\sigma(t)}{\int \frac{h(t)}{t-z} d\sigma(t)}. \tag{4.2}$$

Now, since $h(z) \in \ker T_{\bar{\Theta}S^c} \subset K_\Theta$ we can rewrite the last identity as

$$h(z)F(z) = \frac{1-\Theta(z)}{2\pi i} \int \frac{F(t)h(t)}{t-z} d\sigma(t).$$

Therefore, for every $x \in \mathbb{R}$ and $\|h\|_2 = 1$ we obtain

$$|h(x)F(x)| \leq M \|k_\Theta(\cdot, x)\|_2 = M\phi'(x).$$

Since $\Theta(z)$ is associated to a separated sequence Λ, $\phi'(t)$ must be bounded on $\mathbb{R}$. Thus, $|h(x)F(x)| < C$ for all $x \in \mathbb{R}$ and $h \in \ker T_{\bar{\Theta}S^c}$ with norm 1. Therefore, to obtain boundedness of $F(z)$ on the real line we must show that for all $x \in \mathbb{R}$,

$$\sup\{|h(x)| \mid h \in \ker T_{\bar{\Theta}S^c}, \|h\|_2 = 1\} > B > 0.$$

At this point we do not know if this is possible.

In the case when $\Lambda = \mathbb{Z}$ we have $\Theta(z) = S^{2\pi}(z)$. Therefore for $c < \pi D^-_{BM}(\mathbb{Z}) = \pi$ and $x \in \mathbb{R}$ we can take $h(z) = k_{\S^{2\pi - c}}(z, x)$. Since $h(x) = k_{\S^{2\pi-c}}(x, x) = 2\pi - c > 0$ we obtain that $F(z)$ is bounded on the real line which gives another proof of Cartwright's theorem.

CHAPTER V

HERGLOTZ REPRESENTATION OF GENERALIZED SPECTRAL MEASURES AND SOME APPLICATIONS

Let $\mathcal{B}$ be a family of all Borel sets on the unit circle $\mathbb{T}$. By a generalized spectral measure on $\mathbb{T}$ we mean a function $B : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ whose values are positive bounded self-adjoint operators on $\mathcal{H}$ such that $B(\emptyset) = 0$, $B(\mathbb{T}) = I$ and for every sequence $\Delta_1, \Delta_2, ...$ of mutually disjoint Borel sets, we have

$$B(\Delta_1 \cup \Delta_2 \cup ...) = \sum_{i=1}^{\infty} B(\Delta_i)$$

in the strong operator topology. If we require all the values to be orthogonal projections then we have an (ordinary) spectral measure. A classical theorem of Naimark [30] says that any generalized spectral measure can be represented as a projection of an ordinary spectral measure. This theorem is considered by many as the beginning of Dilation Theory. Since then many different proofs and generalization have appeared (e.g. [31, 44, 43, 36]). We propose yet another approach, which involves in a natural way characteristic functions and spectral representations of unitary operators; it also relates Naimark's theorem for the first time to the subject of rank-one perturbations of a given operator. The latter is another classical subject with a rich literature behind (see [39] and the references therein). The key idea in our approach is to obtain a representation for a generalized spectral measure which is reminiscent to the well known one:

$$\left\langle (U + zI)(U - zI)^{-1} h_1 | h_2 \right\rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \, d \left\langle (E(\xi) h_1 | h_2 \right\rangle$$

relating a unitary operator U to its spectral measure E. We will show that if $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ and the generalized spectral measure $B : \mathcal{B} \to \mathcal{B}(\mathcal{K})$ is obtained from a spectral measure $E : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ by $B(\Delta) = P_{\mathcal{K}} E(\Delta)$ then the above mentioned representation is given by

$$\left\langle (A + \Theta_S(z))(A - \Theta_S(z))^{-1} k_1 | k_2 \right\rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \, d \left\langle (B(\xi) k_1 | k_2 \right\rangle$$

where $A : U^*(\mathcal{K}) \to \mathcal{K}$ is the restriction of $P_{\mathcal{K}} U$ on $U^*(\mathcal{K})$ and $\Theta_S(z)$ is the characteristic function corresponding to the operator $S := (I - P_{\mathcal{K}})U$. In the case when $\mathcal{K}$ is one-dimensional this relation reduces to a relation that D. Clark [6] used in his treatment of the rank-one perturbations of a restricted shift. This is the point where the connection between these seemingly unrelated concepts is made.

5.1. Representation theorem

In what follows we will be most concerned with partial isometries $S : \mathcal{H} \to \mathcal{H}$ satisfying $\dim \ker S = \dim \ker S^*$. First, we prove some simple, but useful, facts about them.

Lemma 5.1.1. *If S is a partial isometry then $\ker S = \mathcal{D}_S$ and $\ker S^* = \mathcal{D}_{S^*}$. Furthermore, the defect operators D_S and D_{S^*} are just orthogonal projections onto $\mathcal{D}_S$ and $\mathcal{D}_{S^*}$ respectively.*

Proof. It is well known that S is a partial isometry if and only if $SS^*S = S$ (see e.g. [15]) which is equivalent to $S{D_S}^2 = O$. Since the elements of the form ${D_S}^2 h$ span $\mathcal{D}_S$ we have that $\mathcal{D}_S \subset \ker S$. For the other direction just notice that $h \in \ker S$ implies $h = D_S h \in \mathcal{D}_S$. The other equality is proved in the same way having in mind that if S is a partial isometry then so is S^*. Next, to show that D_S is an orthogonal projection it is enough to prove that D_S^2 is. If $h \in \mathcal{D}_S = \ker S$ it is clear that $D_S^2 h = h$.

If $h \perp \mathcal{D}_S = \ker S$ then for all $k \in \mathcal{H}$

$$\left\langle D_S^2 h | k \right\rangle = \left\langle h | D_S^2 k \right\rangle = 0.$$

Therefore, $D_S^2 h = 0$ and hence D_S is an orthogonal projection onto $\mathcal{D}_S$. The argument for D_{S^*} is the same. □

Lemma 5.1.2. *A contraction S is a partial isometry if and only if $\Theta_S(0) = O$.*

Proof. If S is a partial isometry then by the previous lemma S restricted on $\mathcal{D}_S$ is zero and hence $\Theta_S(0) = -S|\mathcal{D}_S = O$. In the other direction we need to prove that $S|\mathcal{D}_S = O$ implies that S is a partial isometry. But this is clear since in this case $SD_S^2 = O$ which is equivalent to $SS^*S = S$. □

Lemma 5.1.3. *Every contraction $S : \mathcal{H} \to \mathcal{H}$ satisfying $\dim \mathcal{D}_S = \dim \mathcal{D}_{S^*}$ (i.e. the defect indices are the same) can be represented as $S = (I - P)U$, where $P : \mathcal{H} \to \mathcal{H}$ is an orthogonal projection onto $\mathcal{D}_{S^*}$ and $U : \mathcal{H} \to \mathcal{H}$ is some unitary operator. The representation is unique up to a unitary operator $A : \mathcal{D}_S \to \mathcal{D}_{S^*}$. Conversely, any operator S representable in this way must be a partial isometry satisfying $\dim \mathcal{D}_S = \dim \mathcal{D}_{S^*}$.*

Proof. Let $S^* = S_1 Q$ be the polar decomposition of S^*, where $Q = (SS^*)^{\frac{1}{2}}$ and $S_1 : (\mathcal{R}\dashv\backslash T)^- \to (\mathcal{R}\dashv\backslash T^*)^-$ is unitary. S being a partial isometry implies that $Q = (I - D_{S^*}^2)^{\frac{1}{2}}$ is an orthogonal projection onto $\mathcal{D}_S$. Since $\dim \ker S = \dim \mathcal{D}_S = \dim \mathcal{D}_{S^*} = \dim \ker S^*$, S_1 can be extended to a unitary operator on the whole $\mathcal{H}$. The extension will of course be not unique. Now, we obtain the desired representation by setting $P := I - Q$ and $U := S_1^*$. The converse is plain. □

Let $S : \mathcal{H} \to \mathcal{H}$ be a partial isometry. By the previous lemma S can be represented as $S = (I - P)U$. The unitary operator U represents a unitary perturbation of

S. It depends on the choice of the unitary operator $A : \mathcal{D}_S \to \mathcal{D}_{S^*}$. When necessary we will stress this dependence by writing U_A. Notice that both $\Theta_S(z)$ and A act between same spaces.

Fix $h \in \mathcal{H}$. Then the function

$$\big\langle (U + zI)(U - zI)^{-1} h | h \big\rangle$$

is holomorphic in $z \in \mathbb{D}$ with positive real part. Thus, there exists a unique measure μ_h on the unit circle $\mathbb{T}$ such that

$$\big\langle (U + zI)(U - zI)^{-1} h | h \big\rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \, d\mu_h(\xi).$$

Using polarization, for any $h_1, h_2 \in \mathcal{H}$ there exists a measure μ_{h_1,h_2} such that

$$\big\langle (U + zI)(U - zI)^{-1} h_1 | h_2 \big\rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \, d\mu_{h_1,h_2}(\xi).$$

Let Δ be any Borel set on the unit circle $\mathbb{T}$. Then $\mu_{h_1,h_2}(\Delta)$ is a skew-symmetric function of h_1 and h_2, linear in h_1, and bounded by $\|h_1\|\|h_2\|$. Therefore, it can be represented as

$$\mu_{h_1,h_2}(\Delta) = \langle E(\Delta) h_1 | h_2 \rangle ,$$

for some positive bounded operator $E(\Delta)$. It is well known that $E : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ is an ordinary spectral measure.

For fixed $k \in \mathcal{D}_{S^*}$, again $\langle (A + \Theta_S(z))(A - \Theta_S(z))^{-1} k | k \rangle$ is a holomorphic function in $z \in \mathbb{D}$ with positive real part and consequently there exists a unique measure σ_k satisfying

$$\big\langle (A + \Theta_S(z))(A - \Theta_S(z))^{-1} k | k \big\rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \, d\sigma_k(\xi).$$

By polarization again, for any $k_1, k_2 \in \mathcal{D}_{S^*}$ there exists a measure σ_{k_1,k_2} such that:

$$\left\langle (A + \Theta_S(z))(A - \Theta_S(z))^{-1} k_1 | k_2 \right\rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \, d\sigma_{k_1,k_2}(\xi).$$

For any Borel set Δ on the unit circle let $B(\Delta)$ be the positive self-adjoint operator satisfying

$$\sigma_{k_1,k_2}(\Delta) = \langle B(\Delta) k_1 | k_2 \rangle \,,$$

for all $k_1, k_2 \in \mathcal{D}_{S^*}$. It can be shown that $B : \mathcal{B} \to \mathcal{B}(\mathcal{D}_{S^*})$ is a generalized spectral measure.

Theorem 5.1.4. *For any $k \in \mathcal{D}_{S^*}$ the following equality holds:*

$$\left\langle (U + zI)(U - zI)^{-1} k | k \right\rangle = \left\langle (A + \Theta_S(z))(A - \Theta_S(z))^{-1} k | k \right\rangle .$$

Proof. The left- and right-hand side of the equality we are aiming to prove are equal to $2 \langle (I - \Theta_S(z) A^*)^{-1} k | k \rangle - \|k\|^2$ and $2 \langle (I - zU^*)^{-1} k | k \rangle - \|k\|^2$, respectively. Hence, it is equivalent to show that:

$$\left\langle (I - \Theta_S(z) A^*)^{-1} k | k \right\rangle = \left\langle (I - zU^*)^{-1} k | k \right\rangle .$$

To prove this equality, we first notice that

$$\begin{aligned}
\left\langle (I-\Theta_S(z)A^*)^{-1}k|k\right\rangle &= \left\langle (I-zP(I-zS^*)^{-1}A^*)^{-1}k|k\right\rangle && (5.1)\\
&= \left\langle (AA^*-zP(I-zS^*)^{-1}A^*)^{-1}k|k\right\rangle && (5.2)\\
&= \left\langle A(A-zP(I-zS^*)^{-1})^{-1}k|k\right\rangle && (5.3)\\
&= \left\langle A(A(I-zS^*)(I-zS^*)^{-1}-zP(I-zS^*)^{-1})^{-1}k|k\right\rangle && (5.4)\\
&= \left\langle A(I-zS^*)(A(I-zS^*)-zP)^{-1}k|k\right\rangle && (5.5)\\
&= \left\langle A(I-zS^*)(A(I-zS^*)-zAA^*P)^{-1}k|k\right\rangle && (5.6)\\
&= \left\langle A(I-zS^*)(I-zS^*-zA^*P)^{-1}A^*k|k\right\rangle. && (5.7)
\end{aligned}$$

Now, let $T := PU$. It is easy to check that $A^*P = T^*$ and $T^*k = A^*k$, for all $k \in \mathcal{D}_{S^*}$. Therefore, we obtain

$$\begin{aligned}
&\left\langle (I-\Theta_S(z)A^*)^{-1}k|k\right\rangle = \left\langle (I-zS^*)(I-zS^*-zT^*)^{-1}A^*k|A^*k\right\rangle\\
&= \left\langle (I-zS^*-zT^*)^{-1}T^*k|T^*k\right\rangle - \left\langle z(I-zS^*-zT^*)^{-1}T^*k|ST^*k\right\rangle\\
&= \left\langle (I-zS^*-zT^*)^{-1}T^*k|T^*k\right\rangle.
\end{aligned}$$

The last equality is a consequence of $ST^*k = 0$. We finally have

$$\begin{aligned}
\left\langle (I-\Theta_S(z)A^*)^{-1}k|k\right\rangle &= \left\langle (I-z(T+S)^*)^{-1}T^*k|T^*k\right\rangle\\
&= \left\langle (I-zU^*)^{-1}U^*k|U^*k\right\rangle\\
&= \left\langle U^*(I-zU^*)^{-1}k|U^*k\right\rangle\\
&= \left\langle (I-zU^*)^{-1}k|k\right\rangle.
\end{aligned}$$

□

Remark 6. If $\mathcal{D}_{S^*} = \mathcal{H}$ then $S = O$ on $\mathcal{H}$ and hence $\Theta_S(z) = zI$. Also, clearly $A = U$. Thus, we can view $(A+\Theta_S(z))(A-\Theta_S(z))^{-1}$ as a substitution for $(U+zI)(U-zI)^{-1}$ in the case when $\mathcal{D}_{S^*}$ is a true subspace.

Corollary 5.1.5. *For any Borel set $\Delta \subset \mathbb{T}$, $B(\Delta) = PE(\Delta)$.*

Proof. To show the claim it is equivalent to show

$$\langle PE(\Delta)k_1|k_2\rangle = \langle B(\Delta)k_1|k_2\rangle$$

for all $k_1, k_2 \in \mathcal{D}_{S^*}$. Since P is an orthogonal projection, this is equivalent to:

$$\langle E(\Delta)k_1|k_2\rangle = \langle B(\Delta)k_1|k_2\rangle .$$

Therefore, using polarization it suffices to prove

$$\langle E(\Delta)k|k\rangle = \langle B(\Delta)k|k\rangle$$

for all $k \in \mathcal{D}_{S^*}$. But, this easily follows from Theorem 5.1.4.

□

5.2. Application to Naimark's dilation theorem

Next we give a new proof of Naimark's dilation theorem.

Theorem 5.2.1 (Naimark [30])**.** *Let $B : \mathcal{B} \to \mathcal{B}(\mathcal{K})$ be a generalized spectral measure. Then there exist $\mathcal{H} \supset \mathcal{K}$ and an ordinary spectral family $E : \mathcal{B} \to \mathcal{B}(\mathcal{H})$ such that for any Borel set $\Delta \subset \mathbb{T}$, $B(\Delta) = P_{\mathcal{K}}E(\Delta)$.*

Proof. Let $k_1, k_2 \in \mathcal{K}$. Define $\sigma_{k_1,k_2}(\Delta) = \langle B(\Delta)k_1|k_2\rangle$ for any Borel set $\Delta \subset \mathbb{T}$. Clearly, σ_{k_1,k_2} is a Borel measure on $\mathbb{T}$. For any $z \in \mathbb{D}$ define $F(z) : \mathcal{K} \to \mathcal{K}$ such that

$$\langle F(z)k_1|k_2\rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\sigma_{k_1,k_2}(\xi)$$

for every $k_1, k_2 \in \mathcal{K}$. It is easy to see that $F(z)$ is a linear operator with a positive real part and $F(0) = I$. It is also analytic as an operator-valued function in $z \in \mathbb{D}$. The inverse $(I + F(z))^{-1}$ is defined on a dense subset of $\mathcal{K}$ since $F(z)$ has a positive real part. Set $\Theta(z) = (F(z) - I)(F(z) + I)^{-1}$ on that dense set. By continuity, $\Theta(z)$ can be extended on the whole $\mathcal{K}$ with $\|\Theta(z)\| \leq 1$. Moreover, $\Theta(z)$ is an analytic contraction-valued function with $\Theta(0) = O$ (and hence is pure). Thus, there exists a completely non-unitary contraction $S : \mathcal{H}_1 \to \mathcal{H}_1$ whose characteristic function $\Theta_S(z)$ coincides with $\Theta(z)$. More precisely, there exist unitary operators $\omega : \mathcal{D}_S \to \mathcal{K}$ and $\omega_* : \mathcal{D}_{S^*} \to \mathcal{K}$ such that $\Theta_S(z) = {\omega_*}^{-1}\Theta(z)\omega$. Define $A = {\omega_*}^{-1}\omega : \mathcal{D}_S \to \mathcal{D}_{S^*}$ and set $T : \mathcal{H}_1 \to \mathcal{H}_1$ to be

$$Th = \begin{cases} Ah, & h \in \mathcal{D}_S \\ 0, & h \in \ker D_S. \end{cases}$$

Finally, define $U_1 = T + S : \mathcal{H}_1 \to \mathcal{H}_1$. Clearly, U_1 is unitary. Let $E_1(\Delta)$ be the spectral measure corresponding to U_1 and let $B_1(\Delta)$ be the generalized spectral measure such that

$$\left\langle (A + \Theta_S(z))(A - \Theta_S(z))^{-1} s_1 | s_2 \right\rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\left\langle B_1(\xi) s_1 | s_2 \right\rangle$$

for every $s_1, s_2 \in \mathcal{D}_{S^*}$. By the previous Corrolary, for any Borel set Δ we have

$$B_1(\Delta) = P_{\mathcal{D}_{S^*}} E_1(\Delta).$$

Now, since

$$\left\langle (A + \Theta_S(z))(A - \Theta_S(z))^{-1} s_1 | s_2 \right\rangle = \left\langle (I + \Theta(z))(I - \Theta(z))^{-1} \omega_* s_1 | \omega_* s_2 \right\rangle$$

and $F(z) = (I + \Theta(z))(I - \Theta(z))^{-1}$, it follows that $B(\Delta) = \omega_* B_1(\Delta) \omega_*^{-1}$.

Define $\mathcal{H} = \mathcal{K} \oplus \ker D_{S^*}$ and $W_* = \omega_* \oplus I : \mathcal{H}_1 \to \mathcal{H}$. Clearly, W_* is unitary.

Then $B(\Delta) = P_{\mathcal{K}} E(\Delta)$ where $P_{\mathcal{K}} = W_* P_{\mathcal{D}_{S^*}} W_*^{-1}$ and $E(\Delta) = W_* E_1(\Delta) W_*^{-1}$. One readily sees that $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{H}$ defined this way is the orthogonal projection onto $\mathcal{K}$ and $E(\Delta)$ is a spectral measure on $\mathcal{H}$.

□

5.3. Application to rank-one unitary perturbations of partial isometries

Let $S : \mathcal{H} \to \mathcal{H}$ be a c.n.u. partial isometry with both defect indices equal to 1, i.e., $\partial_S = \partial_{S^*} = 1$. Fix two unit vectors $k \in \mathcal{D}_{S^*}$ and $\tilde{k} \in \mathcal{D}_S$. For any complex number α of modulus 1, define the linear operator U_α by

$$U_\alpha h = \begin{cases} Sh, & \text{if } h \perp \mathcal{D}_S \\ \alpha k, & \text{if } h = \tilde{k}. \end{cases}$$

Clearly, U_α are unitary operators; these are the only unitary rank-one perturbations of S.

Lemma 5.3.1. *With the notation above the following equality holds*

$$\overline{span}\{U_\alpha^n k : n \in \mathbb{Z}\} = \mathcal{H}$$

Proof. Consider $\mathcal{K}_\alpha := \overline{\text{span}}\{U_\alpha^n k : n \in \mathbb{Z}\}$. One can show by induction that for any $h \in \mathcal{K}_\alpha^\perp$ and $\beta \in \mathbb{T}$ we have that $U_\beta^n h = S^n h$ and $U_\beta^{*n} h = S^{*n} h$. Therefore, $\mathcal{K}_\beta^\perp \subset \mathcal{K}_\alpha^\perp$ and, by symmetry, $\mathcal{K}_\beta^\perp = \mathcal{K}_\alpha^\perp$. This space $\mathcal{K}_\alpha^\perp$ is reducing for S and S is unitary there. Since S is c.n.u., we have that $\mathcal{K}_\alpha = \mathcal{H}$ for all $\alpha \in \mathbb{T}$. □

Next we will describe the spectrum of U_α. Let $\Theta_S(z)$ be the characteristic function of S. For each $z \in \mathbb{D}$, $\Theta_S(z)$ is an operator between two one-dimensional spaces $\mathcal{D}_S$ and $\mathcal{D}_{S^*}$. Denote by $\theta(z)$ the scalar valued analytic function such that $\Theta_S(z)\tilde{k} = \theta(z)k$ for each $z \in \mathbb{C}$. Clearly, $|\theta(z)| \leq 1$. Define also $A_\alpha : \mathcal{D}_S \to \mathcal{D}_{S^*}$ to be the operator sending $\tilde{k}$ to αk. Let σ_α be the measure on $\mathbb{T}$ for which

$$\left\langle (A_\alpha + \Theta_S(z))(A_\alpha - \Theta_S(z))^{-1} k | k \right\rangle = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\sigma_\alpha(\xi).$$

This is equivalent to

$$\frac{\alpha + \theta(z)}{\alpha - \theta(z)} = \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d\sigma_\alpha(\xi),$$

i.e., σ_α is a Clark measure for $\theta(z)$. It is also immediate that the corresponding generalized spectral measure $B_\alpha(\Delta)$ is simply given by $B_\alpha(\Delta)k = \sigma_\alpha(\Delta)k$. If $E_\alpha(\Delta)$ is the spectral measure corresponding to U_α then it follows from Corollary 5.1.5 that

$$\langle E_\alpha(\Delta)k|k\rangle = \sigma_\alpha(\Delta).$$

Since $\overline{\operatorname{span}}\{E_\alpha(\Delta)k : \Delta \text{ Borel subset of } \mathbb{T}\} = \mathcal{H}$ the last equality proves the following:

Corollary 5.3.2. *The spectrum of U_α coincides with the support of σ_α. Thus, it consists of the union of those points in $\mathbb{T}$ at which $\theta(z)$ cannot be analytically continued and those $\zeta \in \mathbb{T}$ at which $\theta(z)$ is analytically continuable with $\theta(\zeta) = \alpha$. The set of eigenvalues of U_α coincides with the set of all the atoms of σ_α.*

Remark 7. There are several well-known conditions describing the atoms of a Clark measure σ_α. An important one (goes back to M. Riesz) is the following: σ_α has a point mass at ζ if and only if $\theta(z)$ has the nontangential limit α at ζ and for all (or one) β in $\mathbb{T}$ different from α,

$$\int_{\mathbb{T}} \frac{d\sigma_\beta(\zeta)}{|\xi - \zeta|^2} < \infty.$$

Remark 8. In [6], a similar description (although with different methods) of the spectra is obtained for unitary rank-one perturbations of a restricted shift. Notice that not every partial isometry can be represented as a restricted shift on the space $H^2(\mathbb{D}) \ominus \Theta H^2(\mathbb{D})$ of scalar-valued functions. Thus, the proposition above is not implied by the results in [6].

5.4. Application to higher rank unitary perturbations of partial isometries

We can also consider the general case when S is c.n.u. with equal (possibly infinite) defect indices. For a unitary operator $A : \mathcal{D}_S \to \mathcal{D}_{S^*}$, similarly as in [2], we can define a unitary perturbation U_A of S by

$$U_A h = \begin{cases} Sh, & \text{if } h \perp \mathcal{D}_S \\ Ah, & \text{if } h \in \mathcal{D}_S. \end{cases}$$

Notice that these are just the unitary perturbations for which $S = (I - P)U_A$ with P orthogonal projection onto $\mathcal{D}_{S^*}$ as in Lemma 5.1.3.

As in the case of the rank-one perturbations, one can show that S c.n.u. implies $\overline{\text{span}}\{U_A^n k : k \in \mathcal{D}_{S^*}, n \in \mathbb{Z}\} = \mathcal{H}$. To describe the spectrum of U_A notice that by Theorem 5.1.4 we have

$$\begin{aligned} \left\langle (A + \Theta_S(z))(A - \Theta_S(z))^{-1} k | k \right\rangle &= \left\langle (A_A + \Theta_S(z))(A_A - \Theta_S(z))^{-1} k | k \right\rangle \\ &= \int_{\mathbb{T}} \frac{\xi + z}{\xi - z}\, d \left\langle E_A(\xi) k | k \right\rangle . \end{aligned}$$

Corollary 5.3.2 has the following analogue in this general case.

Corollary 5.4.1. *The spectrum of U_A consists of the union of those points in $\mathbb{T}$ at which $\Theta_S(z)$ cannot be analytically continued and those $\zeta \in \mathbb{T}$ at which $\Theta_S(z)$ is analytically continuable with $\Theta_S(\zeta) - A$ not invertible.*

5.5. Generalization of the Aleksandrov disintegration theorem

In this section we generalize the well known Aleksandrov disintegration theorem to the case of finite rank unitary perturbation.

Let $S : \mathcal{H} \to \mathcal{H}$ be a c.n.u. partial isometry with equal finite defect indices $\partial_S = \partial_{S^*} < \infty$. Furthermore, assume that S^n and S^{*n} converge strongly to zero as

$n \to \infty$. This implies that $\Theta_S(z)$ is inner.

Now, fix a unitary operator $K : \mathcal{D}_S \to \mathcal{D}_{S^*}$. For a unitary operator $A : \mathcal{D}_{S^*} \to \mathcal{D}_{S^*}$, define a unitary perturbation U_A of S by

$$U_A h = \begin{cases} Sh, & \text{if } h \perp \mathcal{D}_{S^*} \\ AKh, & \text{if } h \in \mathcal{D}_{S^*}. \end{cases}$$

Let M be the left-invariant Haar measure on the group of unitaries $\mathbb{U}(\mathcal{D}_{S^*})$. Notice that such measure exists since $\dim \mathcal{D}_{S^*} < \infty$ which implies that the group $\mathbb{U}(\mathcal{D}_{S^*})$ is compact. For a continuous function $f : \mathbb{U}(\mathcal{D}_{S^*}) \to \mathcal{L}(\mathcal{H})$ define $\int_{\mathbb{U}(\mathcal{D}_{S^*})} f(A) dM(A)$ to be the bounded operator on $\mathcal{H}$ for which

$$\langle \left(\int_{\mathbb{U}(\mathcal{D}_{S^*})} f(A)\, dM(A) \right) h_1 | h_2 \rangle = \int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle f(A) h_1 | h_2 \rangle \, dM(A)$$

for all $h_1, h_2 \in \mathcal{H}$.

Theorem 5.5.1. *Let $f : \mathbb{T} \to \mathbb{C}$ be a continuous function. Then*

$$\int_{\mathbb{U}(\mathcal{D}_{S^*})} f(U_A)\, dM(A) k = \left(\int_{\mathbb{T}} f(\xi)\, dm(\xi) \right) k,$$

for all $k \in \mathcal{D}_{S^*}$.

Proof. By density, it is enough to prove the statement for $f(\xi) = P_z(\xi) := \frac{1-|z|^2}{|\xi - z|^2}$. Therefore, by polarization it is enough to show that

$$\int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle P_z(U_A) k | k \rangle \, dM(A) = \langle \left(\int_{\mathbb{T}} P_z(\xi)\, dm(\xi) \right) P_{\mathcal{D}_{S^*}} k | k \rangle = \langle P_{\mathcal{D}_{S^*}} k | k \rangle = 1,$$

for all norm one $k \in \mathcal{D}_{S^*}$.

$$\begin{aligned}
\int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle P_z(U_A)k|k\rangle \, dM(A) &= \int_{\mathbb{U}(\mathcal{D}_{S^*})} \int_{\mathbb{T}} P_z(\xi) \, d\langle E_A(\xi)k|k\rangle \, dM(A) \\
&= \Re \int_{\mathbb{U}(\mathcal{D}_{S^*})} \int_{\mathbb{T}} \frac{\xi + z}{\xi - z} \, d\langle E_A(\xi)k|k\rangle \, dM(A) \\
&= \Re \int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle (AK + \Theta_S(z))(AK - \Theta_S(z))^{-1}k|k\rangle \, dM(A) \\
&= \Re \int_{\mathbb{U}(\mathcal{D}_{S^*})} 2\sum_{n=0}^{\infty} \langle (\Theta_S(z)(AK)^*)^n k|k\rangle \, dM(A) - 1 \\
&= 2\Re \sum_{n=0}^{\infty} \int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle (\Theta_S(z)(AK)^*)^n k|k\rangle \, dM(A) - 1
\end{aligned}$$

Define $\phi_n(z) := \sum_{i=1}^{n} \int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle (\Theta_S(z)(AK)^*)^n k_1|k_2\rangle \, dM(A)$. This is a bounded holomorphic function on the unit disc. Since $\Theta_S(z)$ is inner, it is unitary for almost every $\xi \in \mathbb{T}$. For each such ξ the radial limit

$$\lim_{z\to\xi} \phi_n(z) = \sum_{n=0}^{k} \int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle (\Theta_S(\xi)(AK)^*)^n k|k\rangle \, dM(A) = 0.$$

Therefore, $\phi_n(z) = 0$ for every $z \in \mathbb{D}$ and $n \geq 1$. So, we have

$$\begin{aligned}
&2\Re \sum_{n=0}^{\infty} \int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle (\Theta_S(z)(AK)^*)^n k|k\rangle \, dM(A) - 1 \\
&= 2\Re \int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle k|k\rangle \, dM(A) + 2\Re \sum_{n=1}^{\infty} \int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle (\Theta_S(z)(AK)^*)^n k|k\rangle \, dM(A) - 1 = 1.
\end{aligned}$$

Thus,

$$\int_{\mathbb{U}(\mathcal{D}_{S^*})} \langle P_z(U_A)k|k\rangle \, dM(A) = 1$$

and we are done.

□

CHAPTER VI

SUMMARY AND FUTURE DIRECTIONS

6.1. Summary

In this dissertation we considered three problems in classical complex analysis and operator theory.

In the third chapter we considered the following classical problem: Describe in terms of $\{\lambda_n\}$ the basis properties of $\{\epsilon^{i\lambda_n t}\}$ in $L^2[0,a]$. We gave a characterization of the l_2-independent, Riesz basic sequences and Parseval frames of complex exponentials in terms of invertibility properties of certain Toeplitz operators. Moreover, using this characterization we completely described the l^2-dependence radius showing that it is equal to the interior Beurling-Malliavin density of the frequency sequence $\{\lambda_n\}$.

In the fourth chapter we gave a solution to the classical Polya-Levinson problem. More precisely, we gave a description of all separated real sequences on which a nonconstant zero type entire function cannot be bounded. Again, an important intermediate step was to translate the problem in the language of Toeplitz operators.

In the fifth chapter we made the first steps in trying to understand the general (not necessarily rank-one) unitary perturbations of a given partial isometry. The key thing in our approach was the Herglotz-type representation relating a partial isometry to a related generalized spectral measure. This is an analog to the well known formula relating a unitary operator to its spectral measure. We used this representation formula to describe the spectra of all unitary perturbations of a given partial isometry in terms of the characteristic function. Furthermore, we applied this formula to derive a generalization of the Aleksandrov's disintegration theorem and to give a new proof of the classical Naimark's dilation theorem.

6.2. Future directions

6.2.1. Schauder basis for complex exponentials

One of the few basis properties of non-harmonic complex exponentials for which the complete description has not been obtained yet is the property of being a Schauder basis.

Problem 1. Describe all sequences $\{\lambda_n\}$ for which the sequence of complex exponentials represents a Schauder basis for $L^2[0, 2\pi]$.

The key step will be to reformulate the question in terms of certain properties of appropriate Toeplitz operators.

It is clear that for a sequence of vectors in a Hilbert space the property of being a Riesz basis is stronger then just being a Schauder basis. It is interesting, however, that in classical function spaces it is in general very difficult to construct a Schauder basis which is not a Riesz basis. For the sequence of complex exponentials this is still open.

Problem 2. Is there a Schauder basis of complex exponentials which is not a Riesz basis?

Solution of Problem 1 would give a solution to Problem 2 since the complete description of complex exponentials which represent a Riesz basis is known.

6.2.2. An old problem for entire functions

A well known classical result in function theory (Cartwright Theorem) says that every entire function of exponential type less than π which is bounded on the sequence of integers must be bounded on the whole real line. The natural question of extending this result to an arbitrary sequence of real numbers has received a considerable amount of attention in the past. However, to the best of my knowledge, a complete solution is still not known.

Problem 3. For the given separated sequence of real numbers $\{\lambda_n\}$ find the optimal restriction on the type $\tau > 0$ such that every entire function of exponential type less than τ which is bounded on $\{\lambda_n\}$ is bounded on the whole real line.

6.2.3. Unitary perturbations of a contraction

The Herglotz-type representation formula that we obtained holds only for partial isometries. This raises a natural question if a similar formula hold true for general contractions.

Problem 4. Can the Herglotz-type representation formula be extended to the case of general contractions with equal defect indices?

Another direction (and perhaps a more important one) is to find an analog of the Simon-Wolff criterion in this more general situation of possibly infinite-dimensional perturbations. In other words, to find a criterion under which "most" of the unitary perturbations have a complete set of eigenvectors. Such a result might have nice implications to the spectral theory of multi-dimensional Schrodinger operators.

Problem 5. Is there an analog of the Simon-Wolff criterion and other one-dimensional results in the case of infinite-dimensional perturbations?

6.2.4. De Branges spaces of entire functions

These spaces can be viewed as an analog of the Paley-Wiener space obtained by replacing the differentiation operator with more general second-order differential operators (so called canonical systems). Besides the Paley-Wiener space, important examples of de Branges spaces are given by the spaces generated by classical orthogonal polynomials. Another important class is generated by the one-dimensional Schrodinger operators.

It was noticed by Barry Simon [42] that there are many similarities between the theory of orthogonal polynomials and the theory of one-dimensional Schrodinger operators. He was able to translate many of the results from one area to the other and vice versa. I strongly believe that the simple reason behind these similarities is the fact that all of them come from the results that hold generally in a larger class of de Banges spaces. For certain results, the extension to this larger class is very desirable because it may reveal properties that are not visible in the classical, more restricted setting. The following is, in my opinion, an example of a problem of this kind.

Many important results about the distribution of eigenvalues of random matrices are usually proved by reducing them to certain technical limits involving orthogonal polynomials. An important example of such a limit is:

$$\lim_{n\to\infty} \frac{K_n(\xi + \frac{a}{\tilde{K}_n(\xi,\xi)}, \xi + \frac{b}{\tilde{K}_n(\xi,\xi)})}{K_n(\xi,\xi)} = \frac{\sin \pi(b-a)}{\pi(b-a)},$$

where $K_n(\xi,\xi)$ is the reproducing kernel for the space generated by certain orthogonal polynomials of degree not larger than n and $\tilde{K}_n(\xi,\xi)$ is their normalized version. It was observed by Lubinsky [26] that by replacing ξ with a sequence ξ_n the nice limiting behavior (now on the subsequence level) is still preserved. However, in this more general setting there is a possibility for the limit function to be a reproducing kernel of a class of de Branges spaces known as weighted Paley-Wiener spaces. Inspired probably by Simon's program an analog of the above limit was very recently obtained in [28] for the case of Schrodinger operators. All this strongly suggests the existence of a more general version holding for a larger class of de Branges spaces. Exploring this might give a better understanding of the above mentioned results of Lubinsky.

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