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ON PARAMETRIC AND NONPARAMETRIC METHODS FOR DEPENDENT DATA

A Dissertation

by

SOUTIR BANDYOPADHYAY

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2010

Major Subject: Statistics

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Approved by:

Chair of Committee,	Soumendra N. Lahiri
Committee Members,	Raymond J. Carroll
	Michael Sherman
	Gerald R. North
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ABSTRACT

On Parametric and Nonparametric Methods for Dependent Data. (August 2010)

Soutir Bandyopadhyay, B.Sc., St. Xavier's College;

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Chair of Advisory Committee: Dr. Soumendra N. Lahiri

In recent years, there has been a surge of research interest in the analysis of time series and spatial data. While on one hand more and more sophisticated models are being developed, on the other hand the resulting theory and estimation process has become more and more involved. This dissertation addresses the development of statistical inference procedures for data exhibiting dependencies of varied form and structure.

In the first work, we consider estimation of the mean squared prediction error (MSPE) of the best linear predictor of (possibly) nonlinear functions of finitely many future observations in a stationary time series. We develop a resampling methodology for estimating the MSPE when the unknown parameters in the best linear predictor are estimated. Further, we propose a bias corrected MSPE estimator based on the bootstrap and establish its second order accuracy. Finite sample properties of the method are investigated through a simulation study.

The next work considers nonparametric inference on spatial data. In this work the asymptotic distribution of the Discrete Fourier Transformation (DFT) of spatial data under pure and mixed increasing domain spatial asymptotic structures are studied under both deterministic and stochastic spatial sampling designs. The deterministic design is specified by a scaled version of the integer lattice in $\mathbb{R}^d$ while the data-sites under the stochastic spatial design are generated by a sequence of independent random vectors, with a possibly nonuniform density. A detailed account of the asymptotic joint distribution of the DFTs of the spatial data is given which, among other things, highlights the effects of the geometry of the sampling region and the spatial sampling density on the limit distribution. Further, it is shown that in both deterministic and stochastic design cases, for "asymptotically distant" frequencies, the DFTs are asymptotically independent, but this property may be destroyed if the frequencies are "asymptotically close". Some important implications of the main results are also given.

To Sucharita Bandyopadhyay and Tushar Bandyopadhyay

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CHAPTER I

INTRODUCTION

We consider analysis of dependent data in which interest focuses on developing different inference procedures for the population-level quantities. In the first work a parametric approach has been adopted for the estimation of the mean squared prediction error of the best linear predictor of (possibly) nonlinear functions of finitely many future observations in a stationary time series.

A random field or a spatial process is a generalization of a stochastic process such that the underlying parameter need no longer be a simple real, but can instead be a multidimensional vector space or even a manifold. Due to advances in science and technology, high-throughput spatial data from neuroscience and environmental (among other) applications are being generated at a rapid pace. This has created an urgent need for methodology and tools for analyzing regularly as well as irregularly spaced spatial data. In biological applications, advances in microscope automation are yielding an abundance of data on spatial patterns of neuronal activation. Using digital imaging modalities, it is now possible to collect replicated patterns over large areas of the brain. Another important application in epidemiology is the geographical disease surveillance using scan statistics. Spatial scan statistics are widely used for count data to detect geographical disease clusters of high or low incidence, mortality or prevalence and to evaluate their statistical significance. In the environmental and climate sciences, measurements are often recorded at different locations on many variables. Formulating correct statistical models that make sense of all the many com-

This dissertation follows the style of the *Journal of the Royal Statistical Society*.

plex relationships and multivariate dependencies that are in the data, investigating the properties of these models and developing inferential procedures have provided major challenges for statisticians, probabilists and others working in this field. In the next work we concentrate on developing a nonparametric inference method for regularly as well as irregularly spaced spatial data.

In recent years, there has been a surge of research interest in the analysis of spatial data using the frequency domain approach. At a heuristic level, the popularity of the frequency domain approach lies in the fact that for equi-spaced time series data, the discrete Fourier transform (DFT) of the observations are asymptotically independent. As a result, it allows one to avoid accounting for the dependence in the data explicitly. However, validity of the asymptotic independence of the DFTs for spatial data remains largely unexplored. In contrast to the time series case where observations are usually taken at a regular interval of time and asymptotics is driven by the unidirectional flow of time, for random processes observed over space, several different types of spatial sampling designs and spatial asymptotic structures are relevant for practical applications. For example, image data are equi-spaced in the plane, but locations of the drilling-sites for mineral ores in a mine are usually irregularly spaced. Thus, the type of asymptotics that are appropriate in these applications are inherently different. We investigated in detail the asymptotic properties of the DFT for equi-spaced as well as irregularly spaced spatial data under different types of spatial asymptotic structures.

CHAPTER II

RESAMPLING-BASED BIAS-CORRECTED TIME SERIES PREDICTION

II.1. Introduction

Let $\{X_t\}_{t=-\infty}^{\infty}$ be a second order stationary time series with auto-covariance function $\gamma(\cdot)$ and spectral density $f(\cdot)$. Suppose that a finite stretch, $X_1, \ldots, X_n$ of the series is observed. In many applications, it is important to predict an unobserved future value X_{n+k} in the time series or more generally, a suitable functional of a set of future values $X_{n+1}, \ldots, X_{n+k}$:

$$\Psi = \psi(X_{n+1}, \ldots, X_{n+k})$$

where $\psi : \mathbb{R}^k \to \mathbb{R}$ is a known function and $k \in \mathbb{N}$. Here and in the following, $\mathbb{N}$ and $\mathbb{Z}$ respectively denote the set of all positive integers and the set of all integers. A popular predictor of Ψ is given by the best linear predictor (BLP) $\tilde{\Psi}_n = \alpha_1 X_1 + \ldots + \alpha_n X_n$, where

$$(\lambda_1, \ldots, \lambda_n) = \text{argmin}_{a_1,\ldots,a_n} E\Big(\Psi - [a_1 X_1 + \ldots + a_n X_n]\Big)^2. \qquad \text{(II.1.1)}$$

The co-efficients $\lambda_1, \ldots, \lambda_n$ can be found by standard optimization arguments from calculus; See (II.2.1), Section II.2 below for an explicit expression for $\lambda_1, \ldots, \lambda_n$. Typically, $\lambda_1, \ldots, \lambda_n$ in $\tilde{\Psi}_n$ depend on the auto-covariance function $\gamma(\cdot)$ and, for a nonlinear $\psi(\cdot)$, on other population parameters of the $\{X_t\}$-process and hence, are typically unknown in practice. Here we restrict attention to parametric time series models and highlight the dependence of the BLP on the underlying parameters by writing

$$\tilde{\Psi}_n = \tilde{\Psi}_n(\theta)$$

where $\theta \in \mathbb{R}^p$ ($p \in \mathbb{N}$) is the vector of unknown parameters of the $\{X_t\}$-process. Since $\tilde{\Psi}(\theta)$ depends on unknown θ, it is not usable in practice. A common approach is to plug-in an estimator $\hat{\theta}_n$ of the unknown parameter θ in $\tilde{\Psi}_n(\theta)$, yielding the estimated best linear predictor (EBLP):

$$\hat{\Psi}_n = \tilde{\Psi}_n(\hat{\theta}_n). \tag{II.1.2}$$

An important problem in time series analysis is to accurately estimate the mean squared prediction error (MSPE) of the EBLP:

$$M(\theta) \equiv E\Big(\hat{\Psi}_n - \Psi_n(\theta)\Big)^2. \tag{II.1.3}$$

Like the BLP $\tilde{\Psi}_n(\theta)$, the MSPE also depends on the unknown parameter vector θ. Note that the function $M(\theta) \equiv M_n(\theta)$ can be represented as

$$\begin{aligned} M(\theta) &= E\Big(\tilde{\Psi}_n(\theta) - \Psi\Big)^2 + 2E\Big[\{\hat{\Psi}_n - \tilde{\Psi}_n(\theta)\}\{\tilde{\Psi}_n(\theta) - \Psi\}\Big] + E\Big(\hat{\Psi}_n - \tilde{\Psi}_n(\theta)\Big)^2 \\ &\equiv M_1(\theta) + M_2(\theta) + M_3(\theta), \quad \text{say.} \end{aligned} \tag{II.1.4}$$

The first term $M_1(\theta) \equiv M_{1n}(\theta)$ is the MSPE of the ideal predictor $\tilde{\Psi}_n(\theta)$, the third term $M_3(\theta) \equiv M_{3n}(\theta)$ is the estimation error due to the substitution of $\hat{\theta}_n$ in place of θ in $\tilde{\Psi}_n(\cdot)$, and the second one is a cross-product term. Thus, the MSPE of the EBLP depends on the MSPE of the ideal predictor as well as on the particular estimator $\hat{\theta}_n$ used for estimating the unknown parameter vector θ. Except for some very specific cases, analytic expressions for the functions $M_i(\theta)$, $i = 1, 2, 3$ (particularly, $M_2(\theta)$ and $M_3(\theta)$) are not available in the literature, making the estimation of the MSPE $M(\theta)$ difficult by the traditional plug-in approach. In this chapter, we propose a bootstrap based method to derive an estimator of the MSPE $M(\theta)$. The key advantage of the bootstrap methodology is that it produces an estimator of the MSPE of the EBLP *for any given* estimator $\hat{\theta}_n$ of θ, without requiring any analytical computation of the functions $M_2(\theta)$ and $M_3(\theta)$ which critically depend on the choice of $\hat{\theta}_n$. We show that under fairly mild regularity conditions on the $\{X_t\}$-process and on the estimators $\hat{\theta}_n$, the bootstrap MSPE estimator is consistent.

Next we consider higher order accuracy of the resulting bootstrap estimator. Typically, of the three terms $M_i(\theta)$, $i = 1, 2, 3$, the first one is $O(1)$, while the second and the third terms are typically $O(n^{-1})$, as the sample size n goes to infinity. As a result, usual consistency of the "ordinary" bootstrap MSPE estimator of $M(\theta)$ is not adequate in many applications where the sample size only moderately large and the effects of the $O(n^{-1})$ terms can not be ignored. Indeed, it can be shown that the bootstrap MSPE estimator has a bias of the order $O(n^{-1})$, which is of the same order as the orders of the terms $M_i(\theta)$, $i = 2, 3$. Thus, the "ordinary" bootstrap MSPE estimator masks the contributions coming from parameter estimation in $\tilde{\Psi}_n(\theta)$ to the overall MSPE of $\hat{\Psi}_n$. What is needed is an estimator of the MSPE of $\hat{\Psi}_n$ that has a bias of order $o(n^{-1})$ and still retains the standard order of convergence; Following Prasad and Rao (1990), we call such estimators of the MSPE $M(\theta)$ *second order correct*. A common way to construct a second order correct MSPE estimator is to use the explicit bias correction to a plug-in estimator of $M(\theta)$. However, this is impractical and undesirable in our situation mainly because of two reasons, namely, (i) explicit analytical expressions for $M_i(\theta)$, $i = 1, 2, 3$ are very rarely available in the literature (only in some simple toy models) as these are very difficult to derive in reasonable generality, and (ii) the explicit bias correction leads to a negative estimator of the MSPE with a positive probability. An important contribution of this work is to develop a new method for constructing a second order correct MSPE estimator that is non negative with probability one. The key idea is to "tilt" the estimator $\hat{\theta}_n$ suitably so that it balances out the bias of the "ordinary" bootstrap MSPE estimator to the order $O(n^{-1})$. The tilting factor used here is based on certain iterations of the bootstrap step and on a simple formula to combine them. As a result, the computation of the proposed second order correct MSPE estimator is very much feasible with today's computing power, and the methodology works any choice of the estimator $\hat{\theta}_n$ satisfying the mild regularity conditions of the main result. Most importantly, the proposed method does not require any analytical derivation on the part of the user.

The rest of the chapter is organized as follows. We conclude this section with a brief literature review. In Section II.2, we describe the "ordinary" bootstrap estimator of the MSPE and prove its consistency. The tilted version of the MSPE estimator and its theoretical properties are stated in Section II.3. In Section II.4, we develop some bootstrap based approximations for different functions appearing in the tilted MSPE estimator, for which exact analytical expressions are either unavailable or intractable. In Section II.5, we report the results from a simulation study on finite sample properties of the proposed tilted MSPE estimator. Proofs of the main results are presented in Section II.6.

The literature on time series prediction is huge and is well documented in the case where the target variable $\Psi = X_{n+k}$ for some k; See Brockwell and Davis (1991), Priestley (1981). For standard stationary time series models, like the autoregressive (AR) processes and autoregressive and moving average (ARMA) processes, explicit expressions for $M_1(\theta)$ is known (cf. Brockwell and Davis (1991)), although expressions for $M_2(\theta)$ and $M_3(\theta)$ are not common. The masking-effect of the naive plug-in approach on MSPE estimation was pointed out by Prasad and Rao (1990), who also introduced the concept of second order bias corrected MSPE estimators, in the context of small area estimation. For a detailed account of the literature on issues and solutions in the small area estimation problem until 2003, see Rao (2003). In the time series context, Ansley and Kohn (1986) and Quenneville and Singh (2000) proposed different MSPE estimators based on analytical considerations for the state-space model. More recently, Pfeffermann and Tiller (2005) proposed a bootstrap based method for MSPE estimation of the best linear unbiased predictor (BLUP), also for the state-space model under a Gaussian assumption. The second order correct MSPE estimation methodology presented here is different from the earlier work on the problem in the time series literature; It is based on the approach developed by Lahiri and Maiti (2003) and Lahiri et al. (2007) in the context of small area estimation.

II.2. Bootstrap estimation of the MSPE

II.2.1. Preliminaries

In this section, we formalize the basic framework for bootstrap estimation of the MSPE. As in Section II.1, let $\{X_t\}_{t=-\infty}^{\infty}$ be a second order stationary time series with an absolutely summable auto-covariance function $\gamma(\cdot)$ and spectral density $f(\cdot)$, and for a known function $\psi : I\!R^k \rightarrow I\!R$ let

$$\Psi = \psi(X_{n+1}, \ldots, X_{n+k})$$

is to be predicted using the observations $X_1, \ldots, X_n$. For the ease of exposition and as it is customary in the time series literature (see Chapter 5, Brockwell and Davis (1991)), for the rest of this chapter, we shall suppose that the variables X_t's and Ψ have mean zero. Thus, the focus of the work is on the prediction of the random part; The deterministic mean part, if any, can be estimated by any of the standard methods, such as (quasi-)maximum likelihood, method of moments, etc., which in turn, can be used for mean correction. Under the zero-mean assumption, it is easy to derive an explicit expression for the BLP $\tilde{\Psi}_n$ using standard arguments. Thus, by differentiating the expression on the right side of (II.1.1), it is easy to show that the vector $\boldsymbol{\lambda}_n \equiv (\lambda_1, \ldots, \lambda_n)'$ of co-efficients in $\tilde{\Psi}_n$ are given by

$$\boldsymbol{\lambda}_n \equiv \boldsymbol{\lambda}_n(\theta) = \Gamma_n^{-1}\boldsymbol{\gamma}_n, \tag{II.2.1}$$

where $\Gamma_n \equiv \Gamma_n(\theta)$ is the $n \times n$ matrix with (i,j)th element $\text{cov}(X_i, X_j)$, $1 \leq i, j \leq n$, and where $\boldsymbol{\gamma}_n \equiv \boldsymbol{\gamma}_n(\theta) = (\text{cov}(\Psi, X_1), \ldots, \text{cov}(\Psi, X_n))'$. Here and in the following, we drop θ from population quantities, except when it is important to highlight the dependence on θ and similarly, often drop n from subscript, for simplicity of exposition. Note that by the Pythagorus theorem, the MSPE of the ideal predictor $\tilde{\Psi}_n$ is given by

$$M_{1n}(\theta) = \text{var}(\Psi) - \boldsymbol{\gamma}_n'\Gamma_n^{-1}\boldsymbol{\gamma}_n.$$

However, exact expressions for the second and the third terms in (II.1.4) are not easy to write down and both of these terms depend on the particular estimator $\hat{\theta}_n$ is used. In the next section, we describe a resampling method for estimating all three components of the MSPE of the EBLP $\hat{\Psi}_n$.

II.2.2. Ordinary bootstrap estimator of the MSPE

Let $\tilde{\theta}_n$ be an estimator of θ based on $X_1, \ldots, X_n$. We shall use $\tilde{\theta}_n$ to produce the bootstrap estimator of the MSPE $M(\theta)$. In principle, one may take $\tilde{\theta}_n = \hat{\theta}_n$, but a different choice of $\hat{\theta}_n$ may be more appropriate in a specific application. The main steps in the ordinary bootstrap estimation procedure are as follows:

A. Generate a bootstrap sample $X_1^*, \ldots, X_n^*$ under the $\theta = \tilde{\theta}_n$. Let θ_n^* denote the bootstrap version of $\hat{\theta}_n$ obtained by replacing $X_1, \ldots, X_n$ by $X_1^*, \ldots, X_n^*$.

B. Compute $\hat{\Psi}_n^* = \tilde{\Psi}_n(\theta_n^*)$ by replacing θ in $\boldsymbol{\lambda}(\theta)$ (see (II.2.1)) by θ_n^*.

C. The bootstrap estimator of $M(\theta)$ is given by

$$\widehat{\text{mspe}}_n^{\text{OR}} = E_*\left(\hat{\Psi}_n^* - \Psi_n^*\right)^2, \tag{II.2.2}$$

where $\Psi_n^* = \psi(X_1^*, \ldots, X_n^*)$ is the bootstrap version of the predictand Ψ and where E_* denotes conditional expectation given $X_1, \ldots, X_n$.

In practice, evaluation of the conditional expectation is done by the Monte-Carlo method. For this, steps A-C are repeated a large number (say, B) of times and the resulting bootstrap replicates are combined. Specifically, for each $b = 1, \ldots, B$, one generates the bth resample $X_1^{*b}, \ldots, X_n^{*b}$ under the $\theta = \tilde{\theta}_n$ (independently of the other replicates) and then computes θ_n^{*b}, $\hat{\Psi}_n^{*b} = \tilde{\Psi}_n(\theta_n^{*b})$ and $\Psi_n^{*b} = \psi(X_1^{*b}, \ldots, X_n^{*b})$ based on $X_1^{*b}, \ldots, X_n^{*b}$ as in steps A-C. The Monte-Carlo approximation to $\widehat{\text{mspe}}_n^{\text{OR}}$ is given by

$$\widehat{\text{mspe}}_n^{\text{OR:MC}} = B^{-1} \sum_{b=1}^{B} \left(\hat{\Psi}_n^{*b} - \Psi_n^{*b}\right)^2. \tag{II.2.3}$$

The following result shows that the ordinary bootstrap estimator of the MSPE is consistent under mild conditions on the underlying time series $\{X_t\}_{t=-\infty}^{\infty}$ and on the estimator sequences $\{\hat{\theta}_n\}_{n\geq 1}$ and $\{\tilde{\theta}_n\}_{n\geq 1}$.

Theorem II.2.1. *Let θ_0 denote the true value of the parameter θ and let $\Theta_0 = \{\theta \in \Theta : \|\theta - \theta_0\| \leq \delta_0\}$ for some $\delta_0 \in (0, \infty)$. Suppose that $\tilde{\theta}_n - \theta_0 = o_p(1)$ as $n \to \infty$ and that the following conditions hold:*

(A.1) There exists $\delta \in (0, 1]$ such that

(i) $\sup\{E_\theta \Psi^2 : \theta \in \Theta_0\} < \delta^{-1}$, and

(ii) $\delta < f_\theta(\omega) \leq \delta^{-1}$ for all $\omega \in (-\pi, \pi)$ and $\theta \in \Theta_0$.

(A.2) (i) For each $j \leq 0$, $g_j(\theta)$ is continuous at $\theta = \theta_0$ and $|g_j(\theta)| \leq a_j$ for all $\theta \in \Theta_0$, where $\sum_{j=-\infty}^{0} a_j < \infty$.

(ii) $f_\theta(\cdot)$ is continuous at $\theta = \theta_0$ in the $\|\cdot\|_\infty$-norm.

(A.3) $\sup\{M_{3n}(\theta) : \theta \in \Theta_0\} \to 0$ *as* $n \to \infty$.

Then

$$\widehat{mspe}_n^{\mathrm{OR}} - M_n(\theta_0) \to_p 0 \quad as \quad n \to \infty. \tag{II.2.4}$$

Conditions (A.1)-(A.3) are local uniformity conditions on various second order population quantities (moments) related to the time series $\{X_i\}_{i=-\infty}^{\infty}$, and essentially requires continuity of the parametric model at $\theta = \theta_0$. Condition (A.1)(i) requires that the second moment of the predictand Ψ be bounded in a neighborhood of the true parameter value θ_0, which would hold if $E_{\theta_0}\Psi^2 < \infty$ and $E_\theta \Psi^2$, as a function of θ, is continuous at $\theta = \theta_0$. Condition (A.1)(ii) is a crucial condition that is used all through the chapter. It is used to obtain some bounds on the spectral norm of the matrix $\Gamma_n(\theta)$ and its inverse. This condition is satisfied when $\{X_i\}_{i=-\infty}^{\infty}$ is an ARMA(p,q)-process where all roots of the corresponding characteristic polynomial lie outside the unit circle (see Brockwell and Davis (1991)). Next consider (A.2). Continuity of $g_j(\cdot)$ at $\theta = \theta_0$ is tied down to the continuity of the model parametrization at $\theta = \theta_0$. The local uniform summability of $g_j(\theta)$'s can be replaced by requiring finiteness and continuity of the absolute sum $\sum_{j\le 0} |g_j(\theta)|$ on Θ_0, in which case a_j corresponds to $|g_j(\theta_1)|$ for all $j \le 0$, for a common $\theta_1 \in \Theta_0$. Alternatively, it is guaranteed if some standard mixing and moment conditions hold. More specifically, suppose that the process $\{X_i\}_{i=-\infty}^{\infty}$ is strongly mixing with the mixing co-efficient

$$\alpha(n;\theta) \equiv \sup\{|P_\theta(A \cap B) - P_\theta(A)P_\theta(B)| : A \in \mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{-\infty}, j \in \mathbb{Z}\}, \tag{II.2.5}$$

where $\mathcal{F}_a^b = \sigma\langle X_t : t \in [a,b] \cap \mathbb{Z}\rangle$, $-\infty \le a \le b \le \infty$. Let $\alpha_0(n) = \sup_{\theta\in\Theta_0} \alpha(n;\theta)$, $n \ge 1$. If $E|\Psi|^{2+\delta} < \infty$ and $\sum_{n=1}^{\infty} \alpha_0(n)^{\frac{\delta}{2+\delta}} < \infty$, then (A.2)(i) holds. Condition (A.2)(ii) requires a form of continuity of the parametric model at $\theta = \theta_0$, and is satisfied in many examples, including the class of ARMA (p,q)-models mentioned above. It can be further ascertained if the function $E_\theta X_0 X_j$ is continuous at $\theta = \theta_0$ for each $j \geq 0$, and for some $\delta > 0$, $E|X_1|^{2+\delta} < \infty$ and $\sum_{n=1}^{\infty} \alpha_0(n)^{\frac{\delta}{2+\delta}} < \infty$. Finally, consider (A.3). As pointed out before, the function $M_{3n}(\theta)$ quantifies the effect of replacing the unknown true value of the parameter by the estimator $\hat{\theta}_n$ in the (ideal) BLP, and hence, it critically depends on the properties of the estimator sequence $\{\hat{\theta}_n\}_{n\geq 1}$. Typically, for a sequence of consistent estimators $\{\hat{\theta}_n\}_{n\geq 1}$, $M_{3n}(\theta_0) \to 0$. Condition (A.3) requires the convergence to be uniform in a neighborhood of θ_0. We impose the condition directly on $M_{3n}(\cdot)$ to keep the statement of Theorem II.2.1 simple, which only claims consistency of the ordinary bootstrap estimator of the MSPE. A set of sufficient conditions for (A.3) is given in Section II.3, where a more precise bound (namely, $O(n^{-1})$) on the order of $M_{3n}(\cdot)$ is obtained.

II.2.3. Limitations of the ordinary bootstrap estimator

It is easy to see that the ordinary bootstrap estimator of the true MSPE $M_n(\theta_0)$ is equivalent to the plug-in estimator $M_n(\tilde{\theta}_n)$, and has the added advantage that it does not require an explicit expression for the three components $M_{in}(\theta_0)$, $i = 1, 2, 3$ (see (II.1.4)). However, as explained earlier, of the three terms in (II.1.4), only the leading term $M_{1n}(\theta_0) = O(1)$ while the terms $M_{in}(\theta_0)$, $i = 2, 3$ are typically of the order $O(n^{-1})$. Therefore, Theorem II.2.1 asserts consistency of bootstrap estimator $\widehat{\text{mspe}}_n^{\text{OR}}$ for $M_{1n}(\theta_0)$, the MSPE of the ideal predictor $\tilde{\Psi}_n$, only and fails to capture the effects of estimating the unknown θ_0 by $\hat{\theta}_n$, leading to the terms $M_{in}(\theta_0)$, $i = 2, 3$ in the overall MSPE $M_n(\theta_0)$ of the EBLP $\hat{\Psi}_n$. For a better approximation, effects of the terms $M_{in}(\theta_0)$, $i = 2, 3$ must be taken into account. In the next section, we describe an implicit bias-correction method based on the bootstrap that achieves this goal.

II.3. Second order accurate estimation of the MSPE

II.3.1. The tilting method

We first describe the tilting method in our MSPE estimation problem. The basic idea behind the tilting method is to replace the original estimator $\hat{\theta}_n$ with a suitably *tilted* (or *perturbed*) estimator of θ that annihilates the bias contribution of $\hat{\theta}_n$ to $M_{1n}(\cdot)$, up to the second order accuracy. Suppose that

$$\sum_{j=1}^{p} |M_{1n}^{(j)}(\theta_0)| > \epsilon_0, \tag{II.3.1}$$

for some $\epsilon_0 > 0$, where for a smooth function $f : I\!\!R^p \to I\!\!R$, $f^{(i)}$, $f^{(i,j)}$ and $f^{(i,j,k)}$ denote the first, the second and the third order partial derivatives with respect to the i-th co-ordinate, the (i,j)-th co-ordinates, and the (i,j,k)-th co-ordinates, respectively, $i,j,k = 1,\cdots,p$. Condition (II.3.1) says that $M_{1n}^{(i)}(\theta_0) \neq 0$ for some i. For notational simplicity, we suppose that $M_{1n}^{(1)}(\theta_0) \neq 0$. Next let $\beta_n \equiv \beta_n(\theta)$ and $\Sigma_n = \Sigma_n(\theta)$ respectively denote the the bias and variance of $\hat{\theta}_n$. We shall also suppose that some consistent estimators $\hat{\beta}_n$ and $\hat{\Sigma}_n$ of β_n and Σ_n, respectively, are available. For example, under mild conditions on $\hat{\theta}_n$ and $\{X_t\}$, such estimators can be generated using the bootstrap method (see Lahiri (2003a)). Then the *preliminary tilted estimator* of $\boldsymbol{\theta}$ is defined as $\hat{\theta}_n + \mathbf{r}_n$, where $\mathbf{r}_n$ is given by,

$$\mathbf{r}_n = -\left[\sum_{i=1}^{p} M_{1n}^{(i)}(\hat{\theta}_n)\hat{\beta}_{n,i} + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p} M_{1n}^{(i,j)}(\hat{\theta}_n)\hat{\Sigma}_n(i,j)\right]\left\{M_{1n}^{(1)}(\hat{\theta}_n)\right\}^{-1}\mathbf{e}_1, \tag{II.3.2}$$

where, $\hat{\beta}_{n,i}$ and $\hat{\Sigma}_n(i,j)$ denote the ith component of $\hat{\beta}_n$ and (i,j)th component of $\hat{\Sigma}_n$, respectively, and where the vector $\mathbf{e}_\ell \in I\!\!R^p$ has one in the ℓth position and zeros elsewhere, $1 \leq \ell \leq p$. Thus, the preliminary tilted estimator is obtained from the initial estimator $\hat{\theta}_n$ by adding a correction factor to the first component of $\hat{\theta}_n$ only. Note that if, instead of $M_{1n}^{(1)}(\cdot)$, a different partial derivative $M_{1n}^{(i)}(\cdot)$ were nonzero, then we would define the preliminary tilted estimator by replacing the factor $\left\{M_{1n}^{(1)}(\hat{\theta}_n)\right\}^{-1}\mathbf{e}_1$ in (II.3.2) with $\left\{M_{1n}^{(i)}(\hat{\theta}_n)\right\}^{-1}\mathbf{e}_i$.

To make the MSPE estimator well-defined and to ensure its consistency, we need to modify the preliminary tilted estimator $\hat{\theta}_n+\mathbf{r}_n$. The modifications are needed either if $\hat{\theta}_n+\mathbf{r}_n$ falls outside Θ, in which case $M_n(\hat{\theta}_n+\mathbf{r}_n)$ is not well defined, or if $M_{1i}^{(1)}(\hat{\theta}_n)$ becomes too small, in which case, it scales up the variability of the correction factor $\mathbf{r}_n$. Under appropriate regularity conditions, the probability of getting a preliminary estimator $\hat{\theta}_n+\mathbf{r}_n$ outside Θ or that of getting a value of $M_{1n}^{(1)}(\hat{\theta}_n)$ below the threshold $(1+\log n)^{-2}$ tends to zero rapidly as $n \to \infty$. As a consequence, the perturbed estimator $\check{\theta}_n$ coincides with the preliminary perturbed estimator $\hat{\theta}_n+\mathbf{r}_n$ with high probability.

The *tilted estimator of the MSPE* is now defined as

$$\widehat{\text{MSPE}}_n = M_n(\check{\theta}_n) \tag{II.3.3}$$

where $\check{\theta}_n$ is the *tilted estimator* of θ, defined by

$$\check{\theta}_n = \begin{cases} \hat{\theta}_n+\mathbf{r}_n & \text{if } \hat{\theta}_n+\mathbf{r}_n \in \Theta \text{ and } |M_{1n}^{(1)}(\hat{\theta}_n)|^{-1} \le (1+\log n)^2 \\ \hat{\theta}_n & \text{otherwise.} \end{cases} \tag{II.3.4}$$

Although an explicit expression for the function $M_n(\cdot)$ is typically unknown, it is not difficult to see that the tilted estimator $\widehat{\text{MSPE}}_n$ is equivalently given by (II.2.2) with $\tilde{\theta}_n = \check{\theta}_n$; The latter can be computed using the algorithm given in Section II.2.2.

In the next section, we state the regularity conditions and show that the tilted estimator of the MSPE in (II.3.3) achieves second order bias accuracy.

II.3.2. Theoretical properties

As before, let θ_0 denote the true value of the parameter θ and let $\Theta_0 = \{\theta \in \Theta : \|\theta - \theta_0\| \le \delta_0\}$ denote a open neighborhood of θ_0. Let P_θ and E_θ denote the probability and expectation under θ. For notational simplicity, we set $P_{\theta_0} = P$ and $E_{\theta_0} = E$. For $j \in \mathbb{Z}$, define

$$g_j(\theta) = E_\theta[\psi(X_1, \ldots, X_k)X_j], \quad \theta \in \Theta.$$

Note that $g_j(\theta)$ is the covariance between X_{n+j} and $\Psi = \psi(X_{n+1}, \ldots, X_{n+k})$ under θ, which decreases to zero as $j \to -\infty$ under suitable weak dependence and moment conditions on $\{X_i\}_{i=-\infty}^{\infty}$. Let $\Delta\boldsymbol{\lambda}_n(\theta)$ be the $p \times n$ matrix, with ith column given by the $p \times 1$ vector of partial derivatives of the ith component of $\boldsymbol{\lambda}_n(\theta) = \gamma_n(\theta)\Gamma_n(\theta)^{-1}$. Define

$$\begin{aligned} \mu_{2n}(\theta) &\equiv nE_\theta\Big([\hat{\theta}_n - \theta]'\Delta\boldsymbol{\lambda}_n(\theta)\mathbf{X}_n\{\boldsymbol{\lambda}_n(\theta)\mathbf{X}_n - \Psi\}\Big) \\ \mu_{3n}(\theta) &\equiv E_\theta\Big(n^{1/2}[\hat{\theta}_n - \theta]'\Delta\boldsymbol{\lambda}_n(\theta)\mathbf{X}_n\Big)^2, \end{aligned}$$

which respectively give approximations to the functions $M_{2n}(\theta)$ and $M_{3n}(\theta)$, upto an error of order $o(n^{-1})$. We shall use the following regularity conditions to prove the results.

(C.1) Suppose that there exists a $\delta \in (0, \infty)$ such that

$$\liminf_{n\to\infty} M_{1n}^{(1)}(\theta_0) \ge \delta.$$

(C.2) Suppose that there exists $\kappa, c_0 \in (0, \infty)$ such that for all $\theta \in \Theta$:

(i) $E_\theta\Psi^2 < c_0$,

(ii) $E_\theta|X_1|^{4+\kappa} < c_0$,

(iii) $\limsup_{n\to\infty} E_\theta\Big\{\sqrt{n}\|\hat{\theta}_n - \theta\|\Big\}^8 < c_0$.

(C.3) Suppose that $g_j(\theta)$ and f_θ are twice differentiable on Θ, and that there exist a constant $c_1 \in (0,\infty)$ and a sequence $\{a_n\}_{n\ge 1} \subset (0,\infty)$ with $\sum_{n-1}^{\infty} a_j < \infty$ such that for all $k,l \in \{1,\dots,p\}$,

(i) $\max\{g_j(\theta), |g_j^{(k)}(\theta)|, |g_j^{(k,l)}(\theta)|\} < a_j$ for all $\theta \in \Theta$,

(ii) $\max\{\|f_\theta\|_\infty, \|f_\theta^{-1}\|_\infty, \|f_\theta^{(k)}\|_\infty, \|f_\theta^{(k,l)}\|_\infty\} < c_1$ for all $|a| \le 2$ and for all $\theta \in \Theta$, and

(iii) $\|f_\theta^{(k,l)} - f_{\theta_0}^{(k,l)}\| \le c_1 \|\theta - \theta_0\|^\delta$, for all $\theta \in \Theta_0$ for some $\delta > 0$.

(C.4) Suppose that there exists a $c_2 \in (0,\infty)$ such that $\sup\{|\mu_{kn}(\theta)| : \theta \in \Theta\} < c_2$ for all $n \ge c_2$, and $\mu_{kn}(\cdot)$ is equi-continuous at $\theta = \theta_0$, $\ k = 2,3$.

(C.5) Suppose that $\beta_n(\theta) = n^{-1}\beta_0(\theta) + o(n^{-1})$ and $\Sigma_n(\theta) = n^{-1}\Sigma_0 + o(n^{-1})$ uniformly in $\theta \in \Theta$ and $\Delta_0 \equiv \sup\{\|\beta_0\| + \|\Sigma_0(\theta)\| : \theta \in \Theta\} < \infty$.

Condition (C.1) is a specialized version of (II.3.1) for the given formula for the correction factor $\mathbf{r}_n$, which says that the function $M_{1n}(\cdot)$ has a non-zero derivative along one of the directions $i \in \{1,\dots,p\}$ at the true value θ_0, and is typically satisfied in most applications. See the discussion following (II.3.1) for implications and alternative versions of this. Condition (C.2)(i) is needed to make $M_{1n}(\cdot)$ well-defined while Conditions (C.2)(ii) and (iii) are used to establish exact orders of the functions $M_{kn}(\cdot)$ for $k = 2,3$ (see Lemma II.6.2 below). Condition (C.3) is a smoothness condition on the spectral density of the process $\{X_t\}$ and on the cross-covariances $g_j(\theta) = \text{cov}_\theta(\Psi, X_j)$, which would hold if the underlying model-parametrization is suitably smooth. The same comment applies to Condition (C.4), which requires boundedness and equi-continuity of the approximating functions μ_{kn}, $k = 2,3$. Finally, Condition (C.5) is a condition on the bias and the variance of the estimator sequence $\{\hat{\theta}_n\}_{n\ge 1}$. We have decided to state Conditions (C.4) and (C.5) in terms of the original sequence $\{\hat{\theta}_n\}_{n\geq 1}$ to allow for generality. For a specific choice of $\hat{\theta}_n$, these conditions have to be checked directly. To indicate the type of arguments one would need to verify (suitable variants) of these conditions, consider the class of estimator sequences $\{\hat{\theta}_n\}_{n\geq 1}$ that admit a representation of the form:

$$\hat{\theta}_n - \theta = \frac{\beta_0(\theta)}{n} + n^{-1}\sum_{i=1}^{n} \xi_i + R_n \tag{II.3.5}$$

for some function $\beta_0(\cdot) : \Theta \to I\!R^p$, zero mean random vectors $\xi_i \in \sigma\langle X_i\rangle$, $i \geq 1$ and a remainder term R_n. Suppose that there exist constants $\delta, c_3 \in (0, \infty)$ and a sequence $\{d_n\}_{n\geq 1}$ satisfying $d_n = o(n^{-1/2})$ such that $\|\beta(\theta)\| < c_3$, $E_\theta\|\xi_i\|^8 < c_3$, $E_\theta\|d_n^{-1}R_n\|^8 < c_3$ and $\sum_{n=1}^{\infty} n^3\alpha(n;\theta)^{\frac{\delta}{8+\delta}} < c_3$ for all $\theta \in \Theta$. Then, it is easy to check that Conditions (C.2)(iii) and (C.5) hold. Under (II.3.5), it can be shown that a variant of Condition (C.4) holds where the factor $n^{1/2}(\hat{\theta}_n - \theta)$ in the functions μ_{kn} are replaced by the leading to terms from (II.3.5). For example, for $k = 3$, it can be shown that under (C.3),

$$\sup\Big\{\Big|\mu_{3n}(\theta) - \tilde{\mu}_{3n}(\theta) : \theta \in \Theta\Big\} = o(1). \tag{II.3.6}$$

where

$$\tilde{\mu}_{3n}(\theta) \equiv E_\theta\Big(\Big[n^{-1/2}\sum_{i=1}^{n}\xi_i\Big]'\Delta\boldsymbol{\lambda}_n(\theta)\mathbf{X}_n\Big)^2.$$

As a result, one can use $\tilde{\mu}_{3n}(\theta)$ in place of $\mu_{3n}(\theta)$ as an approximation to $M_{3n}(\theta)$ to establish Theorem II.3.1 (retracing the steps given in Section II.6). Note that the equi-continuity of $\tilde{\mu}_{3n}(\theta)$ at $\theta = \theta_0$ can now be proved under a continuity condition on the individual lag-covariance functions $E_\theta[X_1, \xi_1][X_{k+1}, \xi_{k+1}]'$, $k \geq 0$ (as functions of θ) as in Condition (A.2) and the discussion following the statement of Theorem II.2.1. We give a proof of (II.3.6) in Section II.6. A similar treatment is possible also for the term $\mu_{2n}(\theta)$. Hence, it follows that for an estimator sequence $\{\hat{\theta}_n\}_{n\geq 1}$ satisfying

(II.3.5), Conditions (C.2) - (C.5) hold under mild moment conditions on the variables X_t's and ξ_t's and under mild weak dependence conditions on the underlying process. With this, we are now ready to state the main result of this section.

Theorem II.3.1. *Suppose that Conditions (C.1) - (C.5) hold. Then*

$$E\Big(\widehat{\text{MSPE}}_n - M_n(\theta_0)\Big) = o(n^{-1}) \quad \text{(II.3.7)}$$

$$var\big(\widehat{\text{MSPE}}_n - M_n(\theta_0)\Big) = O(n^{-1}). \quad \text{(II.3.8)}$$

Theorem II.3.1 shows that under suitable regularity conditions, the tilted MSPE estimator attains second order bias accuracy. Further, the variance of the tilted estimator continues to be of the same order as the untiled (naive) MSPE estimator, and is guaranteed to be non-negative. Thus, the tilted MSPE estimator may be preferred over the ordinary MSPE estimator that fails to capture the effects of parameter estimation in the EBLP on the overall MSPE. In the next section, we describe some important issues related to the implementation of the titling method in practice.

II.4. Practical implementation based on the bootstrap

Note that the tilting method described above involves computing the functions $M_{in}(\cdot)$, $i = 1, 2, 3$, and its first and second order partial derivatives, for which explicit expressions are not always available. In this section, we develop bootstrap based approximations to these quantities, so that the tilted MSPE estimator can be used in practice, without any analytical derivations. To that end, first we define a bootstrap-based approximation to the function $M_{1n}(\cdot)$ at a given value $\theta = \theta_1$ (which may depend on the data). The steps are similar to those used for generating the Monte-carlo approximation to $\widehat{\text{mspe}}_n^{\text{OR}}$ in (II.2.3). Specifically, for $b = 1, \cdots, B$,

A. Generate bootstrap samples $(X_1^{*b}, \cdots, X_{n+k}^{*b})$ under θ_1,

B. Compute $\tilde{\Psi}_n^{*b}$ and Ψ^{*b} by replacing $X_1, \ldots, X_n$ with $X_1^{*b}, \cdots, X_{n+k}^{*b}$. The Monte-carlo approximation to $M_{1n}(\theta_1)$ is given by

$$M_{1n}^*(\theta_1) = B^{-1} \sum_{b=1}^{B} (\tilde{\Psi}_n^{*b} - \Psi^{*b})^2. \quad \text{(II.4.1)}$$

Next we construct estimates of the partial derivatives of the function $M_{1n}(\cdot)$ for computing the correction factor $\mathbf{r}_n$. To motivate the construction, first consider a smooth function $g : I\!R \to I\!R$. Then, for any $x \in I\!R$, using Taylor series expansion,

$$g(x + \epsilon) - g(x - \epsilon) = 2\epsilon g^{'}(x) + o(\epsilon),$$

as $\epsilon \to 0$, where $g^{'}(x)$ denotes the derivative of $g(x)$ at x. Hence we can use the scaled difference $(2\epsilon)^{-1}\{g(x + \epsilon) - g(x - \epsilon)\}$ as an approximation to $g^{'}(x)$ for small values of $\epsilon > 0$. Relying on this fact, we can now define suitable bootstrap approximations to the first order partial derivatives of $M_{1n}(\cdot)$ at $\hat{\theta}_n$. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers converging to zero. Then, with M_{1n}^* as in (II.4.1), we define the bootstrap approximation to the first order partial derivatives as,

$$M_{1n}^{*(j)}(\hat{\theta}_n) = (2a_n)^{-1} \left[M_{1n}^*(\hat{\theta}_n + a_n \mathbf{e}_j) - M_{1n}^*(\hat{\theta}_n - a_n \mathbf{e}_j) \right], \quad \text{(II.4.2)}$$

$j = 1, \cdots, p$. Similarly, we can define the bootstrap approximations to the second order partial derivatives as:

$$\begin{aligned} M_{1n}^{*(j,j)}(\hat{\theta}_n) &= a_n^{-2} \left[M_{1n}^*(\hat{\theta}_n + a_n \mathbf{e}_j) + M_{1n}^*(\hat{\theta}_n - a_n \mathbf{e}_j) - 2M_{1n}^*(\hat{\theta}_n) \right], \quad 1 \leq j \leq p, \\ M_{1n}^{*(i,j)}(\hat{\theta}_n) &= 2a_n^{-2} \left[\left\{ M_{1n}^*(\hat{\theta}_n + a_n \mathbf{e}_{i,j}) + M_{1n}^*(\hat{\theta}_n - a_n \mathbf{e}_{i,j}) - 2M_{1n}^*(\hat{\theta}_n) \right\} \right. \\ &\qquad \left. - a_n^2 \left\{ M_{1n}^{*(i,i)}(\hat{\theta}_n) + M_{1n}^{*(j,j)}(\hat{\theta}_n) \right\} \right], \quad 1 \leq i \neq j \leq p. \quad \text{(II.4.3)} \end{aligned}$$

where, $\mathbf{e}_{i,j} = \mathbf{e}_i + \mathbf{e}_j$.

Then, we have the following result on the accuracy of the bootstrap estimates of the partial derivatives:

Proposition II.4.1. *Suppose Conditions (C.2) and (C.3) hold. Then,*

$$E_*\left|M_{1n}^{*(j)}(\hat{\theta}_n) - M_{1n}^{(j)}(\hat{\theta}_n)\right|^2 = O(B^{-1}a_n^{-2} + a_n^2) \quad \textit{almost surely}$$

for all $1 \le j \le p$ *and*

$$E_*\left|M_{1n}^{*(i,j)}(\hat{\theta}_n) - M_{1n}^{(i,j)}(\hat{\theta}_n)\right|^2 = O(B^{-1}a_n^{-4} + a_n^2) \quad \textit{almost surely}$$

for all $1 \le i, j \le p$.

Thus, by choosing a_n small and then choosing the number of bootstrap replicates B suitably large, we can generate accurate approximations to the first and second order partial derivatives of the function $M_{1n}(\cdot)$. Analytical derivations of the partial derivatives, therefore, can be completely bypassed by using the bootstrap (and hence, necessary computing resources).

Next, we define the bootstrap estimators of the bias and variance of $\hat{\theta}_n$ by,

$$\begin{aligned} \beta_n^* &= \frac{1}{B}\sum_{b=1}^{B} \hat{\theta}_n^{*b} - \hat{\theta}_n, \\ \Sigma_n^* &= \left\{\frac{1}{B}\sum_{b=1}^{B} \hat{\theta}_n^{*b}(\hat{\theta}_n^{*b})'\right\} - \left(\frac{1}{B}\sum_{b=1}^{B} \hat{\theta}_n^{*b}\right)\left(\frac{1}{B}\sum_{b=1}^{B} \hat{\theta}_n^{*b}\right)' \end{aligned} \qquad \text{(II.4.4)}$$

respectively, where $\hat{\theta}_n^{*b}$ denote the bth bootstrap replicate of $\hat{\theta}_n$, obtained by replacing $X_1, \ldots, X_n$ with $X_1^{*b}, \cdots, X_n^{*b}$ and $\{(X_1^{*b}, \cdots, X_n^{*b}) : b = 1, \ldots, B\}$ are independent bootstrap replicates under $\theta = \hat{\theta}_n$.

Proposition II.4.2. *Suppose Conditions (C.2), (C.3) and (C.5) hold. Then,*

$$E_*\left\|\beta_n^* - \beta_n\right\|^2 = O(B^{-1}n^{-2}) + o(n^{-2}) \quad \textit{almost surely}$$

and

$$E_*\left\|\Sigma_n^* - \Sigma_n\right\|^2 = O(B^{-1}n^{-2}) + o(n^{-2}) \quad \textit{almost surely.}$$

Combining (II.4.2), (II.4.3) and (II.4.4) , we now define the bootstrap based correction factor as

$$\mathbf{r}_n^* = -\left[\sum_{i=1}^{p} M_{1n}^{*(i)}(\hat{\theta}_n)\beta_{n,i}^* + \frac{1}{2}\sum_{i=1}^{p}\sum_{j=1}^{p}\{M_{1n}^{*(i,j)}(\hat{\theta}_n)\}\{\Sigma_n^*(i,j)\}\right] \times \frac{\mathbf{e}_1}{\left\{M_{1n}^{*(1)}(\hat{\theta}_n)\right\}}, \tag{II.4.5}$$

where $\beta_{n,i}^*$ and $\Sigma_n^*(i,j)$ respectively denote the ith component of β_n^* and the (i,j)th element of Σ_n^*, $1 \leq i,j \leq p$. The bootstrap-based bias-corrected MSPE estimate is given by

$$\widehat{\text{MSPE}}_n^* = \widehat{\text{mspe}}_n^{\text{OR:MC}}(\check{\theta}_n^*) \tag{II.4.6}$$

where $\widehat{\text{mspe}}_n^{\text{OR:MC}}(\check{\theta}_n^*)$ is defined by (II.2.3) with $\tilde{\theta}_n = \check{\theta}_n^*$, and $\check{\theta}_n^*$ is defined by replacing $\mathbf{r}_n$ and $M_{1n}^{(1)}(\hat{\theta}_n)$ in (II.3.4) by $\mathbf{r}_n^*$ and $M_{1n}^{*(1)}(\hat{\theta}_n)$, respectively.

In view of Theorem II.3.1 and Propositions 4.1 and 4.2, $\widehat{\text{MSPE}}_n^*$ gives an accurate approximation to the bias-corrected estimator of the MSPE that can be evaluated without any analytical work, provided Conditions (C.1)-(C.5) hold. However, finite sample performance of the MSPE estimator depends on the choice of different factors, such as a_n, B, etc. In the next section, we explore these issues further through a simulation study.

II.5. Simulation study

For the simulation study, we consider one-step-ahead best linear prediction, i.e., we take the predictand Ψ to be X_{n+1}. We shall consider the the following time series models.

Model 1: In the linear autoregressive model of order 2, AR(2),

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \epsilon_t, \tag{II.5.1}$$

for $t \in \mathbb{Z}$, where $\{\epsilon_t\}$ are independent and identically distributed $N(0, \sigma^2)$. Values of ϕ_1, ϕ_2 are chosen to be $(0.2, 0.5)$ and the value of σ^2 is 4.

Model 2: In the linear autoregressive moving-average model of order (1,1), ARMA (1,1),

$$X_t = \phi_1 X_{t-1} + \epsilon_t + \psi_1 \epsilon_{t-1}, \tag{II.5.2}$$

for $t \in \mathbb{Z}$, where $\{\epsilon_t\}$ are independent and identically distributed $N(0, \sigma^2)$. Here we take the parameter values to be $\phi_1 = 0.2, \psi_1 = 0.5$ and we take $\sigma^2 = 4$.

In this simulation study, we will perturb the estimator in the direction of σ^2. In implementing the method, we use $B = 1000$ bootstrap samples to estimate the bias, variances and for all other approximations. All simulation results are based on $N = 500$ replications. The simulations are done for $n = 50, 120$ and 500. Table 1 reports the empirical measures of bias and root mean squared error (RMSE) for both bias-corrected and not bias-corrected estimators $\widehat{\text{MSPE}}_n^*$ and $\widehat{\text{mspe}}_n^{\text{OR:MC}}$ of MSPE for three different values of n. The bias and mean squared error (MSE) are estimated empirically by taking the average over the replicates of the bias and MSE for each dataset. From Table 1 we can see that the bias correction method gives us significantly better results for different values of n under the models (II.5.1) and (II.5.2). However, it is worth mentioning that due to the bias correction, the RMSE's of the bias-corrected estimators are seemed to be slightly higher than the not bias-corrected

Table 1. Bias and root mean squared error (RMSE) for the estimators (with and without bootstrap based bias correction) of the mean squared prediction errors for models in (II.5.1)-(II.5.2) for sample size(n) = 50, 120 and 500, number of replications(N) = 500 and number of bootstrap samples(N_0) = 1000.

		Not bias-corrected		Bias-corrected	
n	Model	Mean	RMSE	Mean	RMSE
50	1	0.966	0.976	0.547	1.216
	2	0.534	0.940	0.488	0.964
120	1	0.317	0.564	0.103	0.568
	2	0.169	0.547	0.135	0.604
500	1	0.043	0.297	0.003	0.353
	2	0.098	0.311	0.049	0.377

estimators. This is expected, as the randomness in the various approximation steps in the construction of the bias-corrected estimators adds to its total variability. Boxplots for RMSE's of the two estimators of MSPE over $N = 500$ simulations under different models are presented in Figure 1. The boxplots also support the conclusions obtained from Table 1. In these boxplots we can see that due to the bias correction the RMSE's of the tilted estimators seem to be higher than the unperturbed estimators.

II.6. Proofs

For a $l \times l$ matrix A, let $\|A\| = \sup\{\|Ax\| : x \in \mathbb{R}^l, \|x\| = 1\}$ denote the spectral norm, where $1 \leq l \leq \infty$ and where $\| \cdot \|$ denotes the ℓ^2 norm on $\mathbb{R}^2$. Let $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. For $a = (a_1, \ldots, a_p)' \in \mathbb{Z}_+^p$, let $|a| = |a_1| + \ldots + |a_p|$, $a! = \prod_{i=1}^p a_i!$ and $D^a = D_1^{a_1} \ldots D_p^{a_p}$, where D_j denotes the partial derivative w.r.t. the jth co-ordinate, $1 \leq j \leq p$. Let $C, C(\cdot)$ denote generic constants with values in $(0, \infty)$ that depend on their arguments, if any, but not on n. Unless otherwise specified, limits in order symbols are taken by letting $n \to \infty$.

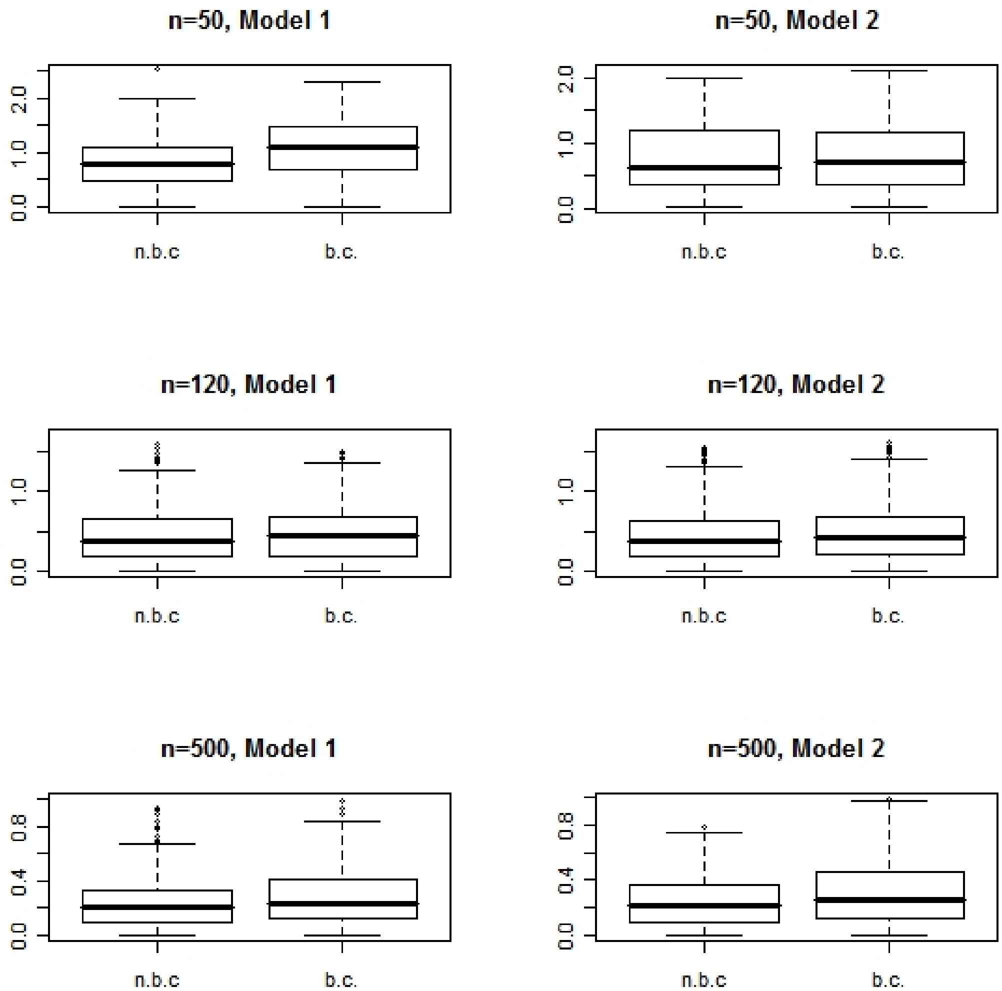

Fig. 1 Boxplots of RMSE values of the estimators (with bias-correction (b.c.) and without bias correction (n.b.c.)) of the mean squared prediction errors for n=50, 120 and 500 under the three models as in (II.5.1)-(II.5.2)

II.6.1. Auxiliary lemmas

Lemma II.6.1. *Suppose that $g_j(\theta)$ and f_θ are twice differentiable on Θ, and that there exists a constant $C_0 > 0$ such that*

(i) $\sum_{j\leq 0} |D^a g_j(\theta)|^2 < C_0$ *for all* $\theta \in \Theta$,

(ii) $\|f_\theta\|_\infty + \|f_\theta^{-1}\|_\infty + \|D^a f_\theta\|_\infty < C_0$ *for all* $|a| \leq 2$ *and for all* $\theta \in \Theta$, *and*

(iii) $\|D^a f_\theta - D^a f_{\theta_0}\| \leq C_0 \|\theta - \theta_0\|^\delta$, *for all* $\theta \in \Theta_0$ *for some* $\delta > 0$.

Then, there exists a constant $C_1 \in (0, \infty)$ such that $|M_{1n}^{(j)}(\theta)| + |M_{1n}^{(i,j)}(\theta)| < C_1$ for all $\theta \in \Theta$ and for all $n \geq C_1$, where $1 \leq i, j \leq p$. Further, $\sum_{1\leq i,j\leq p} |M_{1n}^{(i,j)}(\theta_0) - M_{1n}^{(i,j)}(\theta)| \leq C_1 \|\theta_0 - \theta\|^\delta$ for all $\theta \in \Theta_0$ and for all $n \geq C_1$.

Proof It is easy to check that for $\theta_1, \theta_2 \in \Theta$,

$$\begin{aligned}
& \gamma_n(\theta_1)'\Gamma_n(\theta_1)^{-1}\gamma_n(\theta_1) - \gamma_n(\theta_2)'\Gamma_n(\theta_2)^{-1}\gamma_n(\theta_2) \\
= & \Big(\gamma_n(\theta_1) - \gamma_n(\theta_2)\Big)'\Gamma_n(\theta_1)^{-1}\gamma_n(\theta_1) + \gamma_n(\theta_2)\Gamma_n(\theta_1)^{-1}\Big(\gamma_n(\theta_1) - \gamma_n(\theta_2)\Big) \\
& +\gamma_n(\theta_2)'\Gamma_n(\theta_2)^{-1}\Big[\Gamma_n(\theta_2) - \Gamma_n(\theta_1)\Big]\Gamma_n(\theta_1)^{-1}\gamma_n(\theta_2),
\end{aligned}$$

which, in view of conditions (i), (ii) and (II.6.4), readily implies that

$$\begin{aligned}
D_j M_{1n}(\theta) &= [D_j\gamma_n(\theta)]'\Gamma_n(\theta)^{-1}\gamma_n(\theta) + \gamma_n(\theta)'\Gamma_n(\theta)^{-1}[D_j\gamma_n(\theta)] \\
&\quad -\gamma_n(\theta)'\Gamma_n(\theta)^{-1}[D_j\Gamma_n(\theta)]\Gamma_n(\theta)^{-1}\gamma_n(\theta).
\end{aligned}$$

Next using similar arguments for the second derivation (which is now given by nine terms), one can complete the proof of the lemma. We omit the details.

Lemma II.6.2. *Suppose that there exists $\kappa, C \in (0, \infty)$ such that*

(i) $E_\theta |X_1|^{4+\kappa} < C$,

(ii) $\limsup_{n\to\infty} E_\theta \|\hat{\theta}_n - \theta\|^8 < C$

(iii) $\limsup_{n\to\infty} \|\gamma_n(\theta)\| < C$, *and*

(iv) $\|f_\theta^{-1}\|_\infty < C$

for all $\theta \in \Theta$. *Then,* $\sup_{\theta\in\Theta}\Big[\Big|M_{3n}(\theta) - n^{-1}\mu_{3n}(\theta)\Big| + \Big|M_{2n}(\theta) - n^{-1}\mu_{2n}(\theta)\Big|\Big] = o(n^{-1})$.

Proof Note that on the set $A_n \equiv \{\|\theta - \hat{\theta}_n\| \le \epsilon\}$,

$$\begin{aligned} & [\boldsymbol{\lambda}_n(\hat{\theta}_n) - \boldsymbol{\lambda}(\theta)]\mathbf{X}_n \\ = \; & [\hat{\theta}_n - \theta]'[\Delta\boldsymbol{\lambda}_n(\theta)]\mathbf{X}_n + \sum_{|a|=2}[\hat{\theta}_n - \theta]^a D^a \boldsymbol{\lambda}(\theta_1)\mathbf{X}_n/a! \end{aligned}$$

where θ_1 is a point in A_n. Hence,

$$\begin{aligned} & \Big|M_{3n}(\theta) - E_\theta\Big([\hat{\theta}_n - \theta]'\Delta\boldsymbol{\lambda}_n(\theta)\mathbf{X}_n\Big)^2\Big| \\ \le \; & C(p)\sup\{|D^a\boldsymbol{\lambda}(t)|\,|\,\|t-\theta\| \le \epsilon, |a| = 2\}\dot{E}_\theta\|\hat{\theta}_n - \theta\|^4\|\mathbf{X}_n\|^2 \mathbb{1}(A_n) \\ & + E_\theta\Big([|\boldsymbol{\lambda}_n(\hat{\theta}_n)\mathbf{X}_n| + |\boldsymbol{\lambda}_n(\theta)\mathbf{X}_n|]\cdot \mathbb{1}(\|\hat{\theta}_n - \theta\| \ge \delta)\Big)^2 \\ \equiv \; & I_1 + I_2 + I_3, \quad \text{say.} \end{aligned}$$

First consider I_2. Let $\mathbf{X}_{n,i} = (X_{1,i}, \ldots, X_{n,i})'$, $i = 1, 2$ where $X_{j,1} = X_j \mathbb{1}(|X_j| \le c_n)$ and $X_{j,2} = X_j - X_{j,1}$, $1 \le j \le n$, where $c_n = n^{1/2-\kappa/16}$. By (iii) and (iv), there exists $c_0 \in (0,\infty)$ such that $\|\boldsymbol{\lambda}_n(\theta)\|^2 = \gamma_n(\theta)'\Gamma(\theta)^{-2}\gamma_n(\theta) < c_0^2$ for all $\theta \in \Theta$, for n large. Hence, we have, for any $\epsilon > 0$,

$$\begin{aligned} & E_\theta\Big([\boldsymbol{\lambda}_n(\hat{\theta}_n)\mathbf{X}_n]\cdot \mathbb{1}(\|\hat{\theta}_n - \theta\| \ge \epsilon)\Big)^2 \\ \le \; & 2E_\theta\Big([\boldsymbol{\lambda}_n(\hat{\theta}_n)'\mathbf{X}_{n,1}]\cdot \mathbb{1}(A_n^c)\Big)^2 + 2E_\theta\Big(\boldsymbol{\lambda}_n(\hat{\theta}_n)'\mathbf{X}_{n,2}\Big)^2 \\ \le \; & 2c_0^2\Big[\Big|E_\theta\Big(\sum_{i=1}^n [X_{i,1}^2 - E_\theta X_{1,1}^2]\mathbb{1}(A_n^c)\Big)\Big| + nE_\theta X_{1,1}^2 P_\theta(A_n^c)\Big] + 2c_0^2 E_\theta\|\mathbf{X}_{n,2}\|^2 \\ \le \; & Cc_0^2\Big[\Big(n\sum_{i=1}^\infty |\mathrm{cov}(X_{1,1}^2, X_{i,1}^2)|\Big)^{1/2}\Big(P_\theta(A_n^c)\Big)^{1/2} + nE_\theta X_1^2 P_\theta(A_n^c) \\ & + E_\theta X_1^2 \mathbb{1}(|X_1| > c_n)\Big] \end{aligned}$$

$$\leq \; Cc_0^2 n^{-1-\epsilon}$$

for some $\epsilon = \epsilon(\kappa) > 0$. By similar arguments, $I_3 \leq Cc_0^2 n^{-1-\epsilon}$. Also,

$$\begin{aligned}
I_1 &\leq CE_\theta \|\hat{\theta}_n - \theta\|^4 \|bfX_n\|^2 1\!\!1(A_n) \\
&\leq E_\theta\Big(\|\hat{\theta}_n - \theta\|^4 \Big[\sum_{i=1}^n (X_{i,1}^2 - E_\theta X_{i,1}^2) + nE_\theta X_{1,1}^2 + \sum_{i=1}^n (X_{2,1}^2 \Big] 1\!\!1(A_n) \Big) \\
&\leq CE_\theta \Big(\|\hat{\theta}_n - \theta\|^8 1\!\!1(A_n) \Big)^{1/2} \Big(n \sum_{i=1}^\infty |\mathrm{cov}(X_{1,1}^2, X_{i,1}^2)| \Big)^{1/2} \\
&\quad + CnE_\theta X_1^2 E_\theta \|\hat{\theta}_n - \theta\|^4 1\!\!1(A_n) + Cn(E_\theta X_{2,1}^4)^{1/2} (E_\theta \|\hat{\theta}_n - \theta\|^8 1\!\!1(A_n))^{1/2} \\
&\leq Cc_0^2 n^{-1-\epsilon}
\end{aligned}$$

for some $\epsilon = \epsilon(\kappa) > 0$.

The proof of the second relation follows by repeating the same arguments, and therefore, it is omitted.

Lemma II.6.3. *For $j \geq 1$ and $1 \leq k \leq 4$, let ξ_{kj} be a $\sigma\langle X_j \rangle$-measurable zero-mean random variable such that for some $\delta, c_1 \in (0, \infty)$, $E_\theta |\xi_{kj}|^{4+\delta} < c_1$ for all j, k and $\sum_{n=1}^\infty n^3 \alpha(n;\theta)^{\frac{\delta}{4+\delta}} < c_1$ for all $\theta \in \Theta$. Let $\{e_{kjn} : 1 \leq j \leq n\}_{n \geq 1} \subset I\!R$ be such that $\sum_{j=1}^n e_{kjn}^2 = O(1)$ for $1 \leq k \leq 4$. Then there exists a constant C_1 (depending on c_1, but not on θ) such that*

$$\limsup_{n\to\infty} \Big\{ \Big| E_\theta \Big[\Big(\sum_{i=1}^n \xi_{1i} \Big) \prod_{k=2}^3 \Big(\sum_{i=1}^n e_{kjn} \xi_{ki} \Big) \Big] \Big| + E_\theta \Big[\prod_{k=1}^4 \Big(\sum_{i=1}^n e_{kjn} \xi_{ki} \Big) \Big] \Big\} < C_1$$

for all $\theta \in \Theta$.

Proof We shall give a proof of the bound on the second term only; the proof of the bound on the first term is similar (and somewhat simpler). Clearly, for any $1 \leq k, l \leq 4$,

$$E_\theta \Big(\sum_{i=1}^n e_{kin} \xi_{ki} \Big) \Big(\sum_{j=1}^n e_{ljn} \xi_{lj} \Big)$$

$$
\begin{aligned}
&\le \sum_{|m|=0}^{n-1} \sum_{\{(i,j):\, i-j=m, 1\le i,j\le n\}} e_{kin} e_{ljn} E_\theta \xi_{ki}\xi_{lj} \\
&\le \sum_{|m|=0}^{n-1} \sum_{\{(i,j):\, i-j=m, 1\le i,j\le n\}} |e_{kin} e_{ljn}| \Big(E_\theta |\xi_{ki}|^{2+\delta}\Big)^{\frac{1}{2+\delta}} \Big(E_\theta |\xi_{lj}|^{2+\delta}\Big)^{\frac{1}{2+\delta}} \alpha(|m|;\theta)^{\frac{\delta}{2+\delta}} \\
&\le \sum_{|m|=0}^{n-1} \Big[\sum_{i=1}^n e_{kin}^2\Big]^{1/2} \Big[\sum_{i=1}^n e_{lin}^2\Big]^{1/2} C(c_1,\delta)\alpha(|m|;\theta)^{\frac{\delta}{2+\delta}} \\
&\le C(c_1,\delta) \sum_{m=0}^{n-1} \alpha(m;\theta)^{\frac{\delta}{2+\delta}}.
\end{aligned}
$$

Let $\mathcal{K}_4(V_1, V_2, V_3, V_4)$ denote the fourth order (mixed) cumulant of a set of random variables V_1, V_2, V_3, V_4 under θ, defined by

$$
\mathcal{K}_4(V_1, V_2, V_3, V_4;\theta) = \frac{\partial}{\partial v_1}\frac{\partial}{\partial v_2}\frac{\partial}{\partial v_3}\frac{\partial}{\partial v_4} E_\theta \exp(\sqrt{-1}[v_1V_1+v_2V_2+v_3V_3+v_4V_4])\Big|_{v_1=\ldots=v_4=0}.
$$

Then, by using multi-linearity of $\mathcal{K}_4(\cdot)$, it follows that

$$
\begin{aligned}
& E_\theta\Big[\prod_{k=1}^4 \Big(\sum_{i=1}^n e_{kjn}\xi_{ki}\Big)\Big] \\
\le\ & \Big|\mathcal{K}_4(\sum_{i=1}^n e_{1jn}\xi_{1i}, \ldots, \sum_{i=1}^n e_{4jn}\xi_{4i})\Big| \\
& \quad + \sum_{I\subset\{1,2,3,4\}, |I|=2} \Big|\mathcal{K}_2(\sum_{i=1}^n e_{kjn}\xi_{ki},\ k\in I)(\sum_{i=1}^n e_{kjn}\xi_{ki},\ k\in I^c)\Big|.
\end{aligned}
$$

Note that

$$
\begin{aligned}
& |\mathcal{K}_2(\sum_{i=1}^n e_{kjn}\xi_{1i},\ k\in I)| \\
\le\ & \prod_{k\in I}\Big[\mathrm{var}\Big(\sum_{i=1}^n e_{kin}\xi_{ki}\Big)\Big]^{1/2} \\
\le\ & \prod_{k\in I}\Big[\sum_{i=1}^n e_{kin}^2 E_\theta(\xi_{ki})^2 + 2\sum_{j=1}^{n-1}\Big(\sum_{i=1}^{n-j} e_{kin}^2\Big)^{1/2}\Big(\sum_{i=j+1}^{n} e_{kin}^2\Big)^{1/2} |\mathrm{cov}(\xi_{ki}, \xi_{k(i+j)})|\Big]^{1/2} \\
=\ & O(1) \quad \text{uniformly in} \quad \theta\in\Theta.
\end{aligned}
$$

Next writing $\check{e}_{in} = \max\{|e_{kin}| : k = 1,2,3,4\}$, $1 \le i \le n$, and writing $\sum_b$ for the sum over all $i_1, \ldots, i_4 \in \{1, \ldots, n\}$ with maximal gap b, $0 \le b \le n-1$, we have,

$$
\begin{aligned}
&\Big|\mathcal{K}_4(\sum_{i=1}^{n} e_{1jn}\xi_{1i}, \ldots, \sum_{i=1}^{n} e_{4jn}\xi_{4i})\Big| \\
\le\ & C\sum_{b=0}^{n-1}\sum_{b}\prod_{k=1}^{4}|e_{ki_kn}||\mathcal{K}_4(\xi_{1i_1}, \ldots, \xi_{4i_4})| \\
\le\ & C\sum_{b=0}^{n-1}\sum_{b}\prod_{k=1}^{4}|e_{ki_kn}|c_1^{\frac{4}{4+\delta}}\alpha(b;\theta)^{\frac{\delta}{4+\delta}} \\
\le\ & C(c_1,\delta)\sum_{b=0}^{n-1}\Big[\sum_{i=1}^{n}\check{e}_{in}\Big\{\sum_{|i_k-i|\le b, k=1,2,3}\prod_{k=1}^{3}\check{e}_{i_kn}\Big\}\Big]\alpha(b;\theta)^{\frac{\delta}{4+\delta}} \\
\le\ & C(c_1,\delta)\sum_{b=0}^{n-1}\Big[\sum_{i=1}^{n}\check{e}_{in}\prod_{k=1}^{3}b^{1/2}\Big(\sum_{|i_k-i|\le b}\check{e}_{i_kn}^2\Big)^{1/2}\Big]\alpha(b;\theta)^{\frac{\delta}{4+\delta}} \\
\le\ & C(c_1,\delta)\sum_{b=0}^{n-1}b^{3/2}\Big[\Big\{\sum_{i=1}^{n}\check{e}_{in}^2\Big\}^{1/2}\Big\{\sum_{i=1}^{n}\prod_{k=1}^{3}\Big(\sum_{|i_k-i|\le b}\check{e}_{i_kn}^2\Big)\Big\}^{1/2}\Big]\alpha(b;\theta)^{\frac{\delta}{4+\delta}} \\
\le\ & C(c_1,\delta)\sum_{b=0}^{n-1}b^{3/2}\Big[\Big\{\sum_{i=1}^{n}\check{e}_{in}^2\Big\}^{1/2}\Big\{\Big(b\sum_{i=1}^{n}\check{e}_{in}\Big)\Big(b\max\{\check{e}_{in}^2 : 1 \le i \le n\}\Big)^2\Big\}\Big]\alpha(b;\theta)^{\frac{\delta}{4+\delta}} \\
\le\ & C(c_1,\delta)\Big[\sum_{b=0}^{n-1}b^3\alpha(b;\theta)^{\frac{\delta}{4+\delta}}\Big]\times\Big[\sum_{n=1}^{\infty}n^3\alpha(n;\theta)^{\frac{\delta}{4+\delta}}\Big]\times\Big[\max\{\check{e}_{in}^2 : 1 \le i \le n\}\Big)^2\}\Big] \\
=\ & O(1) \quad \text{uniformly in} \quad \theta \in \Theta.
\end{aligned}
$$

II.6.2. Proofs of the main results

Proof of Theorem II.2.1 It is enough to show that,

$$|M_{1n}(\tilde{\theta}_n) - M_{1n}(\theta_0)| + \sum_{i=2}^{3}\Big(|M_{in}(\tilde{\theta}_n)| + |M_{in}(\theta_0)|\Big) = o_p(1). \qquad \text{(II.6.1)}$$

Note that by (C.2) and the condition $\tilde{\theta}_n \xrightarrow{p} \theta_0$, $|M_{1n}(\tilde{\theta}_n) - M_{1n}(\theta_0)| = o_p(1)$ if,

$$\gamma_n(\tilde{\theta}_n)'\Gamma_n^{-1}(\tilde{\theta}_n)\gamma_n(\tilde{\theta}_n) - \gamma(\theta_0)'\Gamma_n^{-1}(\theta_0)\gamma_n(\theta_0) = o_p(1). \qquad \text{(II.6.2)}$$

It is easy to check that the absolute value of the right side of (II.6.1) is bounded above by

$$\begin{aligned}
&|\gamma_n(\tilde{\theta}_n)'(\Gamma_n^{-1}(\tilde{\theta}_n) - \Gamma_n^{-1}(\theta_0))\gamma_n(\tilde{\theta}_n)| \\
&+2\|\gamma_n(\tilde{\theta}_n) - \gamma_n(\theta_0)\|\|\Gamma_n^{-1}(\theta_0)\| \left(\|\gamma_n(\tilde{\theta}_n)\| + \|\gamma_n(\theta_0)\|\right) \\
&\equiv I_{1n} + I_{2n}, \text{ say.} \qquad \text{(II.6.3)}
\end{aligned}$$

By using the standard isometric isomorphism between $\ell^2(\mathbb{Z})$ and $L^2(0, 2\pi)$ through the Fourier-Plancherel transform (see Bhatia (2003), Rudin (1987)), we have,

$$\begin{aligned}
\|\Gamma_n^{-1}(\theta)\| &\leq C\|f_\theta^{-1}\|_\infty \\
\|\Gamma_n(\theta_1) - \Gamma_n(\theta_2)\| &\leq C\|f_{\theta_1} - f_{\theta_2}\|_\infty, \text{ for all } \theta_1, \theta_1 \in \Theta, n \geq 1. \qquad \text{(II.6.4)}
\end{aligned}$$

By (II.6.4) and conditions (C.2) and (C.3),

$$\begin{aligned}
I_{1n} &= |\gamma_n(\tilde{\theta}_n)'\Gamma_n^{-1}(\theta_0)(\Gamma_n(\tilde{\theta}_n) - \Gamma_n(\theta_0))\Gamma_n^{-1}(\tilde{\theta}_n)\gamma_n(\tilde{\theta}_n)| \\
&\leq \|\gamma_n(\tilde{\theta}_n)\|^2\|\Gamma_n^{-1}(\theta_0)\|\|\Gamma_n(\tilde{\theta}_n)\|\|\Gamma_n(\tilde{\theta}_n) - \Gamma_n(\theta_0)\| \\
&= o_p(1).
\end{aligned}$$

By similar arguments, on the set $\{\|\tilde{\theta}_n - \theta_0\| < \epsilon\}$, $(0 < \epsilon < \delta)$,

$$\begin{aligned}
I_{2n}^2 &\leq C\|\gamma_n(\tilde{\theta}_n) - \gamma_n(\theta_0)\|^2 \\
&= C\left[\sum_{j=0}^{M-1} |g_j(\tilde{\theta}_n) - g_j(\theta_0)|^2 + \sum_{j=M}^{n-1} |g_j(\tilde{\theta}_n) - g_j(\theta_0)|^2\right] \\
&\leq C\left[\sum_{j=0}^{M-1} \sup_{\|x\|\leq\epsilon} |g_j(\theta_0 + x) - g_j(\theta_0)| + \sum_{j=M}^{\infty} \sup_{\theta\in\Theta_0} |g_j(\theta)|\right]
\end{aligned}$$

Given any $\eta > 0$, there exist $M \geq 2$, such that, $\sum_{j=M}^{\infty} \sup_{\theta \in \Theta_0} |g_j(\theta)| < \frac{\eta}{[3C]}$. Next, given $M \geq 1$ and $\eta > 0$, there exists $\epsilon \in (0, \delta)$ such that

$$\sup_{\|x\| \leq \epsilon} |g_j(\theta_0 + x) - g_j(\theta_0)| < \frac{\eta}{3MC}, \text{ for all } j = 0, \ldots, M.$$

Hence,

$$\begin{aligned} P(I_{2n}^2 > \eta) &\leq P(\|\tilde{\theta}_n - \theta_0\| > \epsilon) + P(I_{2n}^2 > \eta, \|\tilde{\theta}_n - \theta_0\| < \delta) \\ &\leq P(\|\tilde{\theta}_n - \theta_0\| > \epsilon) + 0 \quad \text{for large} \quad n \\ &= o(1). \end{aligned}$$

By similar arguments,

$$\begin{aligned} P(M_{3n}(\tilde{\theta}_n) > \epsilon) &\leq P(\sup\{M_{3n}(\theta) : \theta \in \Theta_0\} > \epsilon, \tilde{\theta}_n \in \Theta_0 + P(\tilde{\theta}_n \notin \Theta_0) \\ &= o(1). \end{aligned}$$

Since $M_{2n}(\theta) \leq 2\Big[M_{1n}(\theta)M_{3n}(\theta)\Big]^{1/2}$ for all θ, the theorem is proved.

Proof of (II.3.6) Note that by Lemma II.6.3, $\sup\{|\tilde{\mu}_{3n}(\theta)| : \theta \in \Theta\} = O(1)$. Hence, noting that $\sup\{(E_\theta \|R_n\|^8)^{1/8} : \theta \in \Theta\} = O(d_n) = o(n^{-1/2})$, it is enough to show that

$$\sup\bigg\{\Big|\tilde{\mu}_{3n}(\theta) - E_\theta\Big(\Big[n^{-1/2}\beta_0(\theta) + n^{-1/2}\sum_{i=1}^{n}\xi_i\Big]'\Delta\boldsymbol{\lambda}_n(\theta)\mathbf{X}_n\Big)^2\Big| : \theta \in \Theta\bigg\} = o(1). \quad \text{(II.6.5)}$$

Now expanding the second term and applying the first part of Lemma II.6.3, one can conclude that the left side of (II.6.5) is in fact $O(n^{-1})$. This completes the proof of (II.3.6).

Proof of Theorem II.3.1 By (C.1), there exists $C \in (0, \infty)$ such that $\sup\{|M_{1n}^{(1)}(\theta)|^{-1} : \theta \in \Theta_0, j, l = 1, \cdots, p; n \geq 1\} < C$. Let

$$\hat{D}_n = \sum_{j=1}^{p} M_{1n}^{(j)}(\hat{\theta}_n)\hat{\beta}_{n,j} + \sum_{j=1}^{p}\sum_{l=1}^{p} w(j,l) M_{1i}^{(j,l)}(\hat{\theta}_n)\hat{\Sigma}_n(j,l)$$

$$\tilde{D}_n = \sum_{j=1}^{p} M_{1i}^{(j)}(\theta_0)\hat{\beta}_{n,j} + \sum_{j=1}^{p}\sum_{l=1}^{p} w(j,l) M_{1i}^{(j,l)}(\theta_0)\hat{\Sigma}_n(j,l),$$

$n \geq 1$, where $w(j,l) = 1/2$ for $j \neq l$ and $w(j,l) = 1$ for $j = l$. Then by Taylor's expansion, it follows that there exists a constant $C \in (0,\infty)$ such that on the set $\{\hat{\theta} \in \Theta_0\}$,

$$\hat{D}_n = \tilde{D}_n + R_{1n}, \quad \text{and} \quad \mathbf{r}_n = -\frac{\tilde{D}_n}{M_{1n}^{(1)}(\theta_0)}\boldsymbol{e}_1 + R_{2n}\boldsymbol{e}_1 \tag{II.6.6}$$

where $|R_{1n}| \leq C\Big\{\|\hat{\beta}_n\|.\|\hat{\theta}_n - \theta_0\| + \|\hat{\theta}_n - \theta_0\|^{\gamma}\|\hat{\Sigma}_n\|\Big\}$ and $|R_{2n}| \leq C\Big\{|\hat{D}_n|.\|\hat{\theta}_n - \theta_0\| + |R_{1n}|\Big\}$.

Let $A_{1n} \equiv \{\hat{\theta}_n \in \Theta_0\} \cap \{\hat{\theta}_n + \mathbf{r}_n \in \Theta\}$. Using similar arguments, on the set A_{1n}, for all $u \in [0,1]$, we have

$$\begin{aligned} & \sum_{j=1}^{p}\sum_{l=1}^{p} w(j,l) M_{1n}^{(jl)}(\hat{\theta}_n + u\mathbf{r}_n)\Big([\hat{\theta}_n + \mathbf{r}_n] - \theta_0\Big)^{\boldsymbol{e}_j + \boldsymbol{e}_l} \\ = & \sum_{j=1}^{p}\sum_{l=1}^{p} w(j,l) M_{1n}^{(jl)}(\theta_0)\Big(\hat{\theta}_n - \theta_0\Big)^{\boldsymbol{e}_j + \boldsymbol{e}_l} + R_{3n}(u) \end{aligned}$$

where $\sup_{u\in[0,1]} |R_{3n}(u)| \leq C\Big[\|(\hat{\theta}_n + \mathbf{r}_n) - \theta_0\|^{2+\gamma} + \|\ (\hat{\theta}_n + \mathbf{r}_n) - \theta_0\| \cdot \|\hat{\theta}_n - \theta_0\|$ $+\|\mathbf{r}_n\|^2]$ for some $C \in (0,\infty)$.

Next define the set $A_{2n} = A_{1n} \cap \{\hat{\theta}_n + \mathbf{r}_n \in \Theta_0\}$. Then, on $A_{2n} = \{\hat{\theta}_n, \hat{\theta}_n + \mathbf{r}_n \in \Theta_0\}$, by Taylor's expansion, there exists a point $\theta_n^{\dagger}$ on the line joining $\hat{\theta}_n + \mathbf{r}_n$ and θ_0 such that

$$\begin{aligned} & M_{1i}(\hat{\theta}_n + \mathbf{r}_n) - M_{1i}(\theta_0) \\ = & \sum_{j=1}^{p} M_{1n}^{(j)}(\theta_0)\Big\{[\hat{\theta}_n + \mathbf{r}_n] - \theta_0\Big\}^{\boldsymbol{e}_j} + \sum_{j=1}^{p}\sum_{l=1}^{p} w(j,l) M_{1i}^{(jl)}(\theta_n^{\dagger})\Big\{[\hat{\theta}_n + \mathbf{r}_n] - \theta_0\Big\}^{\boldsymbol{e}_j + \boldsymbol{e}_l} \\ = & \sum_{j=1}^{p} M_{1n}^{(j)}(\theta_0)\Big(\hat{\theta}_n - \theta_0\Big)^{\boldsymbol{e}_j} + M_{1n}^{(1)}(\theta_0)\left(-\frac{\tilde{D}_n}{M_{1n}^{(1)}(\theta_0)} + R_{2n}\right) \\ & + \sum_{j=1}^{p}\sum_{l=1}^{p} w(j,l) M_{1n}^{(jl)}(\theta_0)\Big\{\hat{\theta}_n - \theta_0\Big\}^{\boldsymbol{e}_j + \boldsymbol{e}_l} + R_{3n}^{\dagger} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{p} M_{1n}^{(j)}(\theta_0)\left\{\left(\hat{\theta}_n-\theta_0\right)^{\boldsymbol{e}_j}-\hat{\beta}_{n,j}\right\} \\
&\quad +\sum_{j=1}^{p}\sum_{l=1}^{p} w(j,l)M_{1i}^{(j,l)}(\theta_0)\left\{\left(\hat{\theta}_n-\theta_0\right)^{\boldsymbol{e}_j+\boldsymbol{e}_l}-\hat{\Sigma}_n(j,l)\right\}+M_{1n}^{(1)}(\theta_0)R_{2n}+R_{3n}^{\dagger} \\
&\equiv Q_{1n}+M_{1n}^{(1)}(\theta_0)R_{2n}+R_{3n}^{\dagger},\ \text{say}
\end{aligned} \tag{II.6.7}
$$

where $R_{3n}^{\dagger}=R_{3n}(u)$ with the u corresponding to $\theta_n^{\dagger}$.

Hence, on the set $A_{3n}\equiv\{|M_{1n}^{(1)}(\hat{\theta}_n)|^{-1}\le(1+\log n)^2\}$,

$$
\begin{aligned}
& M_{1n}(\check{\theta}_n)-M_{1n}(\theta_0) \\
&= [M_{1n}(\hat{\theta}_n+\mathbf{r}_n)-M_{1n}(\theta_0)]\mathbb{1}\Big(\Big\{\hat{\theta}_n+\mathbf{r}_n\in\Theta\Big\}\cap A_{3n}\Big) \\
&\quad +[M_{1n}(\hat{\theta}_n)-M_{1n}(\theta_0)]\mathbb{1}\Big(\Big\{\hat{\theta}_n+\mathbf{r}_n\notin\Theta\Big\}\cup A_{3n}^c\Big) \\
&= [M_{1n}(\hat{\theta}_n+\mathbf{r}_n)-M_{1n}(\theta_0)]\Big\{\mathbb{1}(A_{2n})+\mathbb{1}(\hat{\theta}_n+\mathbf{r}_n\in\Theta)-\mathbb{1}(A_{2n})\Big\}\mathbb{1}(A_{3n}) \\
&\quad +[M_{1n}(\hat{\theta}_n)-M_{1n}(\theta_0)]\mathbb{1}\Big(\Big\{\hat{\theta}_n+\mathbf{r}_n\notin\Theta\Big\}\cup A_{3n}^c\Big) \\
&\equiv \Big[Q_{1n}+M_{1n}^{(1)}(\theta_0)R_{2n}+R_{3n}^{\dagger}\Big]\mathbb{1}(A_{2n}\cap A_{3n})+R_{4n},\ \text{say} \\
&\equiv Q_{1i}+R_{5n},\text{say},
\end{aligned} \tag{II.6.8}
$$

where $|R_{5n}|\le|R_{4n}|+|R_{2n}+R_{3n}^{\dagger}|\mathbb{1}(A_{2n})+|Q_{1n}|\mathbb{1}(A_{2n}^c\cap A_{3n}^c)$ and

$$
\begin{aligned}
|R_{4n}| &\le \Big|M_{1n}(\hat{\theta}_n+\mathbf{r}_n)-M(\theta_0)\Big|\cdot\Big|\mathbb{1}(\hat{\theta}_n+\mathbf{r}_n\in\Theta)-\mathbb{1}(A_{2n})\Big|\mathbb{1}(A_{3n}) \\
&\quad +\Big|M_{1n}(\hat{\theta}_n-M_{1n}(\theta_0)\Big|\mathbb{1}(\{\hat{\theta}_n+\mathbf{r}_n\notin\Theta_0\}\cup A_{3n}^c) \\
&\equiv R_{41n},\ \text{say.}
\end{aligned}
$$

Note that by definition,

$$
\begin{aligned}
& \Big|\mathbb{1}(\hat{\theta}_n+\mathbf{r}_n\in\Theta)-\mathbb{1}(A_{2n})\Big| \\
&\le \mathbb{1}(\hat{\theta}_n+\mathbf{r}_n\in\Theta)\mathbb{1}(A_{2n}^c)+\mathbb{1}(\hat{\theta}_n+\mathbf{r}_n\notin\Theta)\mathbb{1}(A_{2n}) \\
&\le \{\mathbb{1}(\hat{\theta}_n\notin\Theta_0)+\mathbb{1}(\hat{\theta}_n+\mathbf{r}_n\in\Theta\setminus\Theta_0)\}+\mathbb{1}(\emptyset).
\end{aligned}
$$

Hence, with $A_{4n}^c \equiv \{\hat{\theta}_n + \mathbf{r}_n \notin \Theta_0\} \cap A_{3n}$,

$$\begin{aligned}
R_{41n} &\leq \Big| M_{1n}(\hat{\theta}_n + \mathbf{r}_n) - M_{1n}(\hat{\theta}_n) \Big| \mathbb{1}(A_{3n}) \{ \mathbb{1}(\hat{\theta}_n \notin \Theta_0) + \mathbb{1}(\hat{\theta}_n + \mathbf{r}_n \in \Theta \setminus \Theta_0)\} \\
&\quad +2 \Big| M_{1n}(\hat{\theta}_n) - M_{1n}(\theta_0) \Big| \Big\{ \mathbb{1}(\hat{\theta}_n \notin \Theta_0) + \mathbb{1}\Big(\{\hat{\theta}_n + \mathbf{r}_n \notin \Theta_0\} \cap A_{3n} \Big) \\
&\qquad + \mathbb{1}(A_{3n}^c) \Big\} \\
&\leq C \|\mathbf{r}_n\| \mathbb{1}(A_{3n}) \Big\{ \mathbb{1}(\hat{\theta}_n \notin \Theta_0) + \mathbb{1}(\hat{\theta}_n + \mathbf{r}_n \in \Theta \setminus \Theta_0) \Big\} \\
&\quad + C \|\hat{\theta}_n - \theta_0\| \Big\{ \mathbb{1}(\hat{\theta}_n \notin \Theta_0) + \mathbb{1}(A_{4n}^c) + \mathbb{1}(A_{3n}^c) \Big\} \\
&\leq C \cdot (\log n)^2 \{ \|\hat{\beta}_n\| + \|\hat{\Sigma}_n\| \} \{ \mathbb{1}(\hat{\theta}_n \notin \Theta_0) + \mathbb{1}(A_{4n}^c) \} \\
&\quad + C \cdot \|\hat{\theta}_n - \theta_0\| \Big\{ \mathbb{1}(\hat{\theta}_n \notin \Theta_0) + \mathbb{1}(A_{4n}^c) + \mathbb{1}(A_{3n}^c) \Big\}. \qquad \text{(II.6.9)}
\end{aligned}$$

By condition, there exist $C \in (0, \infty)$ and $\epsilon_1 \in (0, \frac{\epsilon_0}{2})$ such that

$$\begin{aligned}
A_{4n}^c &\subset \{ \|\hat{\theta}_n - \theta_0\| > \frac{\epsilon_0}{2} \} \cup \{ \|\mathbf{r}_n\| > \frac{\epsilon_0}{2} \} \\
&\subset \{ \|\hat{\theta}_n - \theta_0\| > \epsilon_1 \} \cup \{ (\log n)^2 (\|\hat{\beta}_n\| + \|\hat{\Sigma}_n\|) > C \} \qquad \text{(II.6.10)}
\end{aligned}$$

and $A_{3n}^c \subset \{ \|\hat{\theta}_n - \theta_0\| > \epsilon_1 \}$ for all $n \geq 1$. Hence, it follows that

$$\begin{aligned}
R_{41n} &\leq C \cdot (\log n)^2 \{ \|\hat{\beta}_n\| + \|\hat{\Sigma}_n\| \} \Big[\mathbb{1}(\|\hat{\theta}_n - \theta_0\| > \epsilon_1) \\
&\quad + \mathbb{1}\Big([\log n]^2 (\|\hat{\beta}_n\| + \|\hat{\Sigma}_n\|) > C \Big) \Big] + C \cdot \|\hat{\theta}_n - \theta_0\| \Big[\mathbb{1}(\|\hat{\theta}_n - \theta_0\| > \epsilon_1) \\
&\quad + \mathbb{1}\Big([\log n]^2 (\|\hat{\beta}_n\| + \|\hat{\Sigma}_n\|) > C \Big) \Big] \qquad \text{(II.6.11)}
\end{aligned}$$

for all $n \geq 1$. Let $W_n = (n\|\hat{\beta}_n\| + n\|\hat{\Sigma}_n\|)$. Note that by uniform integrability of $\{(\sqrt{n}\|\hat{\theta} - \theta\|)^2\}_{m \geq 1}$ and the fact that $E \mid W_n \mid^{1+\eta} = O(1)$,

$$\begin{aligned}
& E(R_{41n}) \\
&\leq C n^{-1} (\log n)^2 \Big[\Big(E \mid W_n \mid^{1+\eta} \Big)^{\frac{1}{1+\eta}} \Big(P(\|\hat{\theta}_n - \theta_0\| > \epsilon_1 \Big)^{\frac{\eta}{1+\eta}} \\
&\quad + E \mid |W_n \mid^{1+\eta} \{ n^{-1} (\log n)^2 \}^{\eta} \Big] \\
&\quad + C \Big[\epsilon_1^{-1} E \|\hat{\theta}_n - \theta_0\|^2 \mathbb{1}(\|\hat{\theta}_n - \theta_0\| > \epsilon_1)
\end{aligned}$$

$$+\Big(E\|\hat{\theta}_n-\theta_0\|^2\Big)^{1/2}\Big\{P(n^{-1}(\log n)^2|W_n|>C)\Big\}^{\frac{1}{2}}\Big]$$
$$= \quad o(n^{-1}) \quad \text{as} \quad m\to\infty. \tag{II.6.12}$$

This proves the first part of Theorem II.3.1.

Next we consider the bound on the variance of the tilted MSPE estimator. Since $\sup\{|M_{kn}(\theta)|^2:\theta\in\Theta\}=O(n^{-2})$ for $k=2,3$, by Cauchy-Schwarz inequality, it is enough to show that

$$\mathrm{Var}\Big(M_{1n}(\check{\theta}_n)\Big)=O(n^{-1}). \tag{II.6.13}$$

By Taylor's expansion,

$$M_{1n}(\check{\theta}_n)=M_{1n}(\theta_0)+\sum_{j=1}^{p}M_{1n}^{(j)}(\theta_0)[\check{\theta}_n-\theta_0]^{\boldsymbol{e}_j}+R_{6n}$$

where $|R_{6n}|\le C(p)\Delta\|\check{\theta}_n-\theta_0\|^2$ and $\Delta_r=\limsup_{n\to\infty}\sup\{|M_{1n}^{\alpha}(\theta)|:\theta\in\Theta,|\alpha|=r\},r=1,2$. Also, let $A_{5n}=\{\hat{\theta}_n+\mathbf{r}_n\in\Theta,|M_{1n}^{(1)}(\hat{\theta}_n)|^{-1}\le(1+\log n)^2\}$. Thus, it follows that

$$\begin{aligned}
ER_{6n}^2 &\le C(p,\Delta_2)E\|\check{\theta}_n-\theta_0\|^4\\
&= C(p,\Delta_2)\Big[E\|\hat{\theta}_n+\mathbf{r}_n-\theta_0\|^4\mathbb{1}(A_{5n})+E\|\hat{\theta}_n-\theta_0\|^4\mathbb{1}(A_{5n}^c)\Big]\\
&\le C(p,\Delta_2)2^3\Big[E\|\hat{\theta}_n-\theta_0\|^4+E\|\mathbf{r}_n\|^4\mathbb{1}(A_{5n})\Big]\\
&\le C(p,\Delta_0,\Delta_1,\Delta_2)\Big[E\|\hat{\theta}_n-\theta_0\|^4+(1=\log n)^8n^{-4}\Big]\\
&= O(n^{-2}).
\end{aligned} \tag{II.6.14}$$

By similar arguments and Cauchy-Schwarz inequality,

$$\begin{aligned}
&E\Big[\check{\theta}_n-\theta_0\Big]^{\boldsymbol{e}_i+\boldsymbol{e}_j}\\
=\;& E\Big[\hat{\theta}_n-\theta_0\Big]^{\boldsymbol{e}_i+\boldsymbol{e}_j}+O\Big(E\|\mathbf{r}_n\|^2\mathbb{1}(A_{5n})+\Big\{E\|\hat{\theta}_n-\theta_0\|^2\Big\}^{1/2}\Big\{E\|\mathbf{r}_n\|^2\mathbb{1}(A_{5n})\Big\}^{1/2}\\
=\;& O(n^{-1})+O(n^{-3/2}[\log n]^2].
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
&\mathrm{Var}\Big(\sum_{j=1}^{p} M_{1n}^{(j)}(\theta_0)[\check{\theta}_n-\theta_0]^{\boldsymbol{e}_j}\Big)\\
&= \sum_{i=1}^{p}\sum_{j=1}^{p} M_{1n}^{(i)}(\theta_0)M_{1n}^{(j)}(\theta_0)\mathrm{cov}\Big([\check{\theta}_n-\theta_0]^{\boldsymbol{e}_i},[\check{\theta}_n-\theta_0]^{\boldsymbol{e}_j}\Big)\\
&= \sum_{i=1}^{p}\sum_{j=1}^{p} M_{1n}^{(i)}(\theta_0)M_{1n}^{(j)}(\theta_0)\mathrm{cov}\Big([\check{\theta}_n-\theta_0]^{\boldsymbol{e}_i},[\check{\theta}_n-\theta_0]^{\boldsymbol{e}_j}\Big)\\
&= O(n^{-1}). \qquad \text{(II.6.15)}
\end{aligned}$$

Hence, by (II.6.14), (II.6.15), and Cauchy-Schwarz inequality, (II.6.13) follows. This completes the proof of Theorem II.3.1.

Proof of Proposition II.4.1 For $b=1,\ldots,B$, let $\Upsilon_{1j}^{*b}=\tilde{\Psi}_n^{*b}(\hat{\theta}_n+a_n\boldsymbol{e}_j)-\Psi_n^{*b}(\hat{\theta}_n+a_n\boldsymbol{e}_j)$ and let Υ_{2j}^{*b} be defined by replacing $\hat{\theta}_n+a_n\boldsymbol{e}_j$ by $\hat{\theta}_n-a_n\boldsymbol{e}_j$ in Υ_{1j}^{*b}, $1\le j\le p$. Then, by Taylor's expansion

$$\begin{aligned}
&\Big|E_*M_{1n}^{*(j)}(\hat{\theta}_n)-M_{1n}(\hat{\theta}_n)\Big|\\
&= \Big|(2a_n)^{-1}\Big[M_{1n}(\hat{\theta}_n+a_n\boldsymbol{e}_j)-M_{1n}(\hat{\theta}_n-a_n\boldsymbol{e}_j)\Big]-M_{1n}(\hat{\theta}_n)\Big|\\
&\le Ca_n\sup\{M_{1n}(\theta):\theta\in\Theta\}.
\end{aligned}$$

Next, by (conditional) independence of $\{\Upsilon_{kj}^{*b}:b=1,\ldots,B\}$, $k=1,2$,

$$\mathrm{var}_*([2Ba_n]^{-1}\sum_{b=1}^{B}[\Upsilon_{1j}^{*b}-\Upsilon_{2j}^{*b}])=O(a_n^{-2}B^{-1}),\quad k=1,2.$$

This proves the first part of Proposition II.4.1. The proof of the second part is similar and hence, is omitted.

Proof of Proposition II.4.2 Similar to the proof of Proposition II.4.1 and hence is omitted.

CHAPTER III

ASYMPTOTIC PROPERTIES OF DISCRETE FOURIER TRANSFORMS FOR SPATIAL DATA

III.1. Introduction

In recent years, there has been a surge of research interest in the analysis of spatial data using the frequency domain approach; see for example, Hall and Patil (1994), Im et al. (2007), Fuentes (2002, 2005, 2007), and the references therein. At a heuristic level, the popularity of the frequency domain approach lies in the fact that for *equi-spaced* time series data, the discrete Fourier transform (DFT) of the observations are asymptotically independent (see Kawata (1966,1969), Fuller (1976) and Brockwell and Davis (1991), Lahiri (2003c)). As a result, it allows one to avoid accounting for the dependence in the data explicitly. However, validity of the asymptotic independence of the DFTs for spatial data remains largely unexplored. In contrast to the time series case where observations are usually taken at a regular interval of time and asymptotics is driven by the unidirectional flow of time, for random processes observed over space, several different types of spatial sampling designs and spatial asymptotic structures are relevant for practical applications. For example, image data are equi-spaced in the plane, but locations of the drilling-sites for mineral ores in a mine are usually irregularly spaced. Thus, the type of asymptotics that are appropriate in these applications are inherently different. In this chapter, we investigate in detail the asymptotic properties of the DFT for equi-spaced as well as irregularly spaced spatial data under different types of spatial asymptotic structures.

For spatial data, there are two basic types of spatial asymptotic structures (see

Cressie (1993)): (i) *pure increasing domain* (PID) and (ii) *infill*. When the neighboring data-sites remain separated by a minimum positive distance (in the limit) and the sampling region becomes unbounded with the sample size, one gets the PID asymptotic structure. This is the most common framework used for studying the large sample properties in the spatial case and may be considered as the spatial analogue of the asymptotic structure used in the time-series case. In contrast, when the sampling region remains bounded and the data-sites fill in the sampling region increasingly densely, one gets the *infill* asymptotic structure. This kind of asymptotic framework is mainly used in Mining and other Geostatistical applications. In some situations, a combination of these two frameworks, called the *mixed increasing domain* (MID) asymptotic structure is used (see Hall and Patil (1994)). Under MID asymptotics, the sampling region becomes unbounded and at the same time, the distances between the neighboring sampling sites tend to zero, as the sample size increases.

In this work, we study the asymptotic joint distribution of a finite collection of DFTs of spatial data under the PID and MID asymptotic structures. It has been noted that the large sample behaviors of many standard inference procedures under the infill asymptotics are noticeably different from what can be obtained under the PID or MID asymptotic frameworks; See, for example, Cressie (1993), Lahiri (1996), Loh (2005), Stein (1999), Ying (1993) and the references therein. Indeed, unlike the PID and MID cases, the asymptotic distributions of the DFTs under infill asymptotics are typically non-normal and the DFTs are typically asymptotically dependent for the general class of underlying spatial processes considered here. As a result, we do not consider the case of pure infill asymptotics here and concentrate only on the PID and MID asymptotic structures for regularly (gridded) and irregularly spaced spatial data.

To describe the main results of the chapter, let $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ be a zero mean stationary random field which is observed at finitely many locations $\mathbf{S}_N = \{\mathbf{s}_i : i = 1, \ldots, N\}$ in the sampling region $\mathcal{D} \subset I\!R^d$. We shall assume that in the equi-spaced case, the data-sites $\{\mathbf{s}_i : i = 1, \ldots, N\}$ lie on a scaled version of the integer grid (call it $\mathcal{Z}^d$), while in the irregularly spaced spatial data case, the data-sites are generated by a stochastic sampling scheme. The Discrete Fourier Transform (DFT) of $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_N)\}$ is given by,

$$d_N(\boldsymbol{\omega}) \;=\; N^{-1/2} \sum_{j=1}^{N} Z(\mathbf{s}_j) \exp\left(\iota \boldsymbol{\omega}' \mathbf{s}_j\right), \quad \boldsymbol{\omega} \in I\!R^d, \tag{III.1.1}$$

where $\iota = \sqrt{-1}$ and B' denote the transpose of a matrix B. For $\boldsymbol{\omega} \in I\!R^d$, also define

$$\begin{aligned} C_N(\boldsymbol{\omega}) &= N^{-1/2} \sum_{j=1}^{N} \cos(\boldsymbol{\omega}' \mathbf{s}_j) Z(\mathbf{s}_j), \\ S_N(\boldsymbol{\omega}) &= N^{-1/2} \sum_{j=1}^{N} \sin(\boldsymbol{\omega}' \mathbf{s}_j) Z(\mathbf{s}_j), \end{aligned} \tag{III.1.2}$$

the cosine and the sine transforms of the data. Then, $d_N(\boldsymbol{\omega}) = C_N(\boldsymbol{\omega}) + \iota S_N(\boldsymbol{\omega})$. In the deterministic case, the main findings of our work are:

(i) As in the time series case, under suitable regularity conditions, the asymptotic joint distributions of finite collections of the sine and cosine transforms are multivariate Gaussian.

(ii) DFTs at unequal nonzero limiting frequencies are asymptotically independent.

(iii) For sampling regions of a *general shape* and for DFTs at ordinates converging to a *common* limiting frequency, asymptotic independence holds *if and only if* the ordinates are *asymptotically distant*. In the PID case, we say that $\{\boldsymbol{\omega}_{jn}\}$ and $\{\boldsymbol{\omega}_{kn}\}$ are asymptotically distant if $N^{1/d}\|\boldsymbol{\omega}_{jn} - \boldsymbol{\omega}_{kn}\| \to \infty$ as $N \to \infty$.

(iv) For two discrete Fourier frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ converging to the zero frequency, the corresponding sine and cosine transforms may exhibit different behavior depending on whether the frequency sequences approach zero at the same rate (*asymptotically symmetrically close case*) or at a different rate (*asymptotically close case*). See Section III.3.1.3 for details.

(v) For sampling sites located on the d-dimensional integer grid, DFTs at all discrete Fourier frequencies are asymptotically independent when the sampling region is cubic. However, this is false for a sampling region of a general shape (including spheres, hyper-rectangles, etc.).

(vi) For sampling sites on a scaled version of $\mathbb{Z}^d$ and a *rectangular sampling region*, asymptotic independence holds, provided the grid-increment in each direction is inversely proportional to the sides of the sampling region.

Thus, although in the deterministic case the sampling sites are located on a regular grid, it turns out that the geometry of the sampling region plays an important role in determining the asymptotic independence of the DFTs. The main tool used in the regular-grid case is a discrete version of the Riemann-Lebesgue Lemma (see Section III.6) that may be of some independent interest. For more details on the properties of the DFTs based on regularly spaced spatial data, see Section III.3.

Next we consider the case of DFTs based on irregularly spaced data-locations, specified by a stochastic design. The main findings in this case are:

(i) As in the deterministic case, under suitable regularity conditions, the asymptotic joint distributions of finite collections of the sine and cosine transforms are multivariate Gaussian. However, the asymptotic covariance critically depends on the spatial sampling density and the spatial asymptotic structure (PID *vs* MID); We give a complete description of their effects on the resulting limit distributions.

(ii) DFTs at unequal nonzero limiting frequencies are asymptotically independent, but for a general sampling density, DFTs at ordinates converging to a *common* limiting frequency, asymptotic independence holds *if and only if* the ordinates are *asymptotically distant*. Thus, although the data-sites are irregularly spaced, the asymptotic behavior of the DFTs remains similar to that for regularly spaced spatial data. This is rather surprising and contrary to the folklore about lack of independence of DFTs for irregularly spaced time series data.

(iii) For a *hyper-rectangular sampling region* and a *uniform* sampling density, asymptotic independence of DFTs holds even for asymptotically close frequency sequences. See Section III.4 for more details.

There are several important implications of the main results on asymptotic independence of the DFTs in the context of statistical inference for spatial data in the frequency domain, particularly under PID in the stochastic design case. For example, the usual formulation of the frequency domain bootstrap (FDB) (see Franke and Hardle (1992)), which makes use of the asymptotic independence of the full set of DFTs, may not work for spatial data when the sampling region is non-rectangular. Similarly, the popular nonparametric estimator of the covariance function of Hall and Patil (1994) for irregularly spaced spatial data will have a nontrivial bias under PID asymptotic structure and hence, will be inconsistent. See Section III.5 for further discussion and details.

The rest of the chapter is organized as follows. In Section III.2, we introduce the theoretical framework for studying the asymptotic distributions of the DFTs for equi-spaced and irregularly spaced spatial data. In Section III.3, we present the main results for the equi-spaced case under the PID and MID asymptotic structures, while in Section III.4, we do the same for the stochastic design case. In Section III.5, we discuss various implications of the main results in the context of frequency domain statistical inference for spatial data. Proofs of the two cases require qualitatively different arguments and are presented in Sections III.6 and III.7, respectively.

III.2. Theoretical framework

We will follow the spatial asymptotic framework of Lahiri (2003b). Denote the variable driving the asymptotics by n. In Section III.2.1, we give a formulation for the sampling region which is common to both deterministic and stochastic design cases. The descriptions of the two spatial designs for the regularly- and irregularly-spaced data-sites are next given in Sections III.2.2 and III.2.3, respectively. Regularity conditions on the random field $\{Z(\cdot)\}$ are given in Section III.2.4.

III.2.1. Sampling region

Let $\mathcal{D}_0$ be the prototype set for the sampling region $\mathcal{D} \equiv \mathcal{D}_n$, satisfying $\tilde{\mathcal{D}}_0 \subset \mathcal{D}_0 \subset \text{closure}(\tilde{\mathcal{D}}_0)$ for some open connected subset $\tilde{\mathcal{D}}_0$ of $(-1/2, 1/2]^d$ containing the origin. The sampling region $\{\mathcal{D}_n : n \geq 1\}$ is obtained by multiplying the prototype set $\mathcal{D}_0$ by λ_n, where $\{\lambda_n\}_{n\geq 1} \subset [1, \infty)$ is a sequence of real numbers such that $\lambda_n \uparrow \infty$ as $n \to \infty$, *i.e.*,

$$\mathcal{D}_n = \lambda_n \mathcal{D}_0.$$

It may be noted that under this formulation of the sampling region, we can consider sampling regions of a variety of shapes, such as polygonal, ellipsoidal, and star-shaped regions that can be non-convex. Also to avoid pathological cases, we suppose that for any sequence of real numbers $\{b_n\}_{n\geq 1}$ such that $b_n \to 0+$ as $n \to \infty$, the number of cubes of the form $b_n(\mathbf{j} + [0,1)^d)$, $\mathbf{j} \in \mathbb{Z}^d$ that intersects both $\mathcal{D}_0$ and $\mathcal{D}_0^c$ is of the order $O([b_n]^{-(d-1)})$ as $n \to \infty$. This boundary condition holds for most regions of practical interest.

III.2.2. Sampling design for regularly-spaced data-sites

To describe the deterministic design case, we define first a $d \times d$ diagonal matrix Δ with finite positive diagonal elements $\delta_k, k = 1, \cdots, d$ and let $\mathcal{Z}^d = \{\Delta \mathbf{i} : \mathbf{i} \in \mathbb{Z}^d\}$. Thus, the lattice $\mathcal{Z}^d$ has an increment δ_k in the kth direction, $k = 1, \cdots, d$. For the PID, we suppose that the random process $Z(\mathbf{s})$ is observed at the sampling sites $\{\mathbf{s}_1, \cdots, \mathbf{s}_{N_n}\}$ defined by

$$\{\mathbf{s}_1, \cdots, \mathbf{s}_{N_n}\} = \{\mathbf{s} \in \mathcal{Z}^d : \mathbf{s} \in \mathcal{D}_n\} = \mathcal{D}_n \cap \mathcal{Z}^d.$$

Note that under the PID, the sampling sites are separated by a minimum distance $\delta_0 \equiv \min\{\delta_k : k = 1, \ldots, d\}$ for all n, the sampling region $\mathcal{D}_n$ grows to $\mathbb{R}^d$ as $n \to \infty$ and the sample size N_n satisfies the relation

$$N_n \sim \text{ vol.}[\Delta^{-1}\mathcal{D}_0]\lambda_n^d, \tag{III.2.1}$$

where vol.$[A]$ denotes the volume (*i.e.,* the Lebesgue measure) of a set A in $\mathbb{R}^d$ and for two positive sequences $\{s_n\}$ and $\{t_n\}$ we write, $s_n \sim t_n$ if $\lim_{n\to\infty} s_n/t_n = 1$.

Next we describe the MID structure in the fixed design case. Let $\{\eta_n\}_{n\geq 1}$ be a sequence of non-increasing positive real numbers such that $\eta_n \downarrow 0$ as $n \to \infty$. We suppose that the random process $Z(\mathbf{s})$ is observed at the sampling sites $\{\mathbf{s}_1, \cdots, \mathbf{s}_{N_n}\}$, defined by

$$\{\mathbf{s}_1, \cdots, \mathbf{s}_{N_n}\} = \{\mathbf{s} \in \eta_n\mathcal{Z}^d : \mathbf{s} \in \mathcal{D}_n\} = \mathcal{D}_n \cap \eta_n\mathcal{Z}^d.$$

Thus, the data-sites $\{\mathbf{s}_1, \cdots, \mathbf{s}_{N_n}\}$ are given by the points on the scaled lattice $\eta_n\mathcal{Z}^d$ that lie in the sampling region $\mathcal{D}_n$. In MID, the lattice $\eta_n\mathcal{Z}^d$ becomes finer as $n \to \infty$ and thus, fills any given region of $\mathbb{R}^d$ (and hence, of $\mathcal{D}_n$) with an increasing density.

Note that in MID case,

$$N_n \sim \text{ vol.}[\Delta^{-1}\mathcal{D}_0]\lambda_n^d \eta^{-d}, \tag{III.2.2}$$

implying that under the MID structure, the sample size N_n is of a *larger order* of magnitude than the volume of $\mathcal{D}_n$, given by vol.$[\Delta^{-1}\mathcal{D}_0]\lambda_n^d$.

III.2.3. Stochastic sampling design - irregularly spaced case

Let $f(\mathbf{x})$ be a continuous probability density function on $\mathcal{D}_0$ such that the support of $f(\cdot)$ is the closure of $\mathcal{D}_0$. Let $\{\mathbf{X}_k\}_{k\geq 1}$ be a sequence of independent and identically distributed (iid) random vectors with probability density $f(\mathbf{x})$. In the stochastic design case, for simplicity of notation, we will denote the sample size by n (Note that in the fixed design case, sample size equals the size of $\mathcal{D}_n \cap \mathcal{Z}^d$ which need not be equal to n for a given prototype set $\mathcal{D}_0$ and for a given sequence $\{\lambda_n\}$, leading to the notation N_n. But, due to the absence of a regular grid structure, this problem does not appear in stochastic design case and we may simply use n to denote the sample size). We suppose that in the stochastic design case, the sampling sites $\mathbf{s}_i$'s are obtained by the following relation

$$\mathbf{s}_i \;\equiv\; \mathbf{s}_{in} \;=\; \lambda_n \mathbf{x}_i, \quad 1 \leq i \leq n.$$

This formulation improves upon the standard approach to modeling the irregularly spaced sampling sites $\{\mathbf{s}_1, \cdots, \mathbf{s}_n\}$ using the homogeneous Poisson point process. For such process, the expected number of points in a region is proportional to the volume of the region and given the total number of points in any region, the points are independent and form a random sample from the uniform distribution over the region. However, our formulation allows the number of sampling sites to grow at a different rate than the volume of the sampling region and also allows the sampling sites to have a *non-uniform density* over the sampling region.

In the stochastic design case, the concepts of the PID and the MID structures are determined by the relative growth rates of the sample size n and the volume of the sampling region $\mathcal{D}_n$ (see Cressie (1993), Lahiri (2003b)). When $n/\lambda_n^d \to c_*$ for some finite positive constant c_*, it is regarded as the PID asymptotic structure (see (III.2.1)) under the stochastic design. On the other hand, if $n/\lambda_n^d \to \infty$ as $n \to \infty$, it corresponds to the MID case (see (III.2.2)).

III.2.4. Regularity conditions on the random field

Let $I\!N = \{1, 2, \ldots\}$ denote the set of all positive integers. For $p \in I\!N$, let I_p denote the identity matrix of order p. In addition to the standard ℓ^2-distance $\|\cdot\|$, let $\|\cdot\|_1$ denote the ℓ^1 distance on $I\!R^d$. We assume that $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ is a weakly dependent random field with an integrable autocovariance function $\rho(\mathbf{s}) = \mathrm{cov}(Z(\mathbf{s}), Z(\mathbf{0}))$. Then, the $Z(\cdot)$-process has a spectral density ψ on $I\!R^d$ satisfying

$$\rho(\mathbf{s}) = \int \exp(\iota \mathbf{s}'\boldsymbol{\omega})\psi(\boldsymbol{\omega})d\boldsymbol{\omega}, \quad \mathbf{s} \in I\!R^d. \tag{III.2.3}$$

We also suppose that the random field $Z(\cdot)$ satisfies a spatial version of the strong mixing condition, which is defined as follows: For $E_1, E_2 \subset I\!R^d$, let

$$\alpha_1(E_1, E_2) = \sup_{A_i \in \sigma_Z(E_i), i=1,2} |P(A_1 \cap A_2) - P(A_1)P(A_2)|,$$

where $\sigma_Z(E)$ denotes the σ-algebra generated by the random variables $\{Z(\mathbf{s}) : \mathbf{s} \in E\}$. Let $\delta(E_1, E_2) = \inf\{\|\mathbf{x} - \mathbf{s}\|_1 : \mathbf{x} \in E_1, \mathbf{s} \in E_2\}$. For $a > 0, b > 0$, the mixing coefficient of the random field $\{Z(\cdot)\}$ is defined as

$$\alpha(a; b) = \sup\{\alpha_1(E_1, E_2) : E_i \in \mathbb{C}_b,\ i = 1, 2,\ \delta(E_1, E_2) \geq a\},$$

where $\mathbb{C}_b$ is the collection of d-dimensional sets with volume b or less. Note that in the definition above, the sets E_1, E_2 are of finite volumes. For $d \geq 2$, this is important (see Bradley, 1989, 1993); unbounded E_1's and E_2's in the definition of the strong mixing coefficient makes the random field ρ-mixing (which is a *smaller* class). For simplicity of exposition, we shall further assume that

$$\alpha(a,b) \leq \gamma_1(a)\gamma_2(b), \quad a,b \in (0,\infty), \tag{III.2.4}$$

where, without loss of generality (w.l.g.), $\gamma_1(\cdot)$ is a left continuous, non increasing function satisfying $\lim_{m\to\infty} \gamma_1(m) = 0$ and $\gamma_2(\cdot)$ is a right continuous, non decreasing function that is bounded for $d = 1$ (but it may be unbounded for $d > 1$) (see Lahiri (2003b)). We shall use the following regularity conditions to prove the results.

(A.1) There exists a $\tau \in (0,\infty)$ such that $E|Z(\mathbf{s})|^{2+\tau} < \infty$ and $\int_1^\infty a^{d-1}\gamma_1(a)^{\tau/(2+\tau)}da < \infty$ for some $\tau > 0$.

(A.2) For $d \geq 2$, $\gamma_2(b) = o\left(b^\kappa\right)$ as $b \to \infty$, where, with τ is as in (A.1), $\kappa = 2/3(\tau - 1)$ for $\tau \in (2,\infty)$ and $\kappa = 2/3$ for $\tau \in (0,2]$.

III.3. Results in the regularly-spaced case

III.3.1. Results under PID

III.3.1.1. Definition of the DFTs

For the PID asymptotic structure in the equi-spaced case, the discrete Fourier transform (DFT) of $\{Z(\mathbf{s}_1),\ldots,Z(\mathbf{s}_{N_n})\}$ is given by,

$$\begin{aligned} d_n^P(\boldsymbol{\omega}) &\equiv N_n^{-1/2}\sum_{i=1}^{N_n} Z(\mathbf{s}_i)\exp\left(\iota\boldsymbol{\omega}'\mathbf{s}_i\right) \\ &= N_n^{-1/2}\sum_{\mathbf{j}\in J_n} Z(\Delta\mathbf{j})\exp\left(\iota\boldsymbol{\omega}'\Delta\mathbf{j}\right), \end{aligned} \tag{III.3.1}$$

where $J_n = \{\mathbf{j} \in \mathbb{Z}^d : \Delta\mathbf{j} \in \mathcal{D}_n\}$ and where recall that $\iota = \sqrt{-1}$ and B' denote the transpose of a matrix B. Similarly, for $\boldsymbol{\omega} \in I\!R^d$, we define

$$\begin{aligned} C_n^P(\boldsymbol{\omega}) &= N_n^{-1/2} \sum_{\mathbf{j} \in J_n} Z(\Delta\mathbf{j}) \cos\left(\boldsymbol{\omega}' \Delta\mathbf{j}\right), \\ S_n^P(\boldsymbol{\omega}) &= N_n^{-1/2} \sum_{\mathbf{j} \in J_n} Z(\Delta\mathbf{j}) \sin\left(\boldsymbol{\omega}' \Delta\mathbf{j}\right), \end{aligned} \tag{III.3.2}$$

the cosine and the sine transforms of the data. Then, $d_n^P(\boldsymbol{\omega}) = C_n^P(\boldsymbol{\omega}) + \iota S_n^P(\boldsymbol{\omega})$. Under the fixed design case with the PID asymptotic structure, the process $Z(\cdot)$ is observed at regularly spaced locations on the grid $\mathcal{Z}^d$. In such a case, the spectrum of the observations is concentrated within the frequency band

$$\Pi_\Delta \equiv \Delta^{-1}(-\pi, \pi]^d = (-\pi\delta_1^{-1}, \pi\delta_1^{-1}] \times \ldots \times (-\pi\delta_d^{-1}, \pi\delta_d^{-1}].$$

The whole frequency space $I\!R^d$ is partitioned into (hyper-)rectangles of volume $(2\pi)^d$ $\prod_{i=1}^d \delta_i^{-1}$, and the spectrum at a given point in the "principal band" Π_Δ is obtained by superimposing the spectra at congruent points from the partition $\{(i_1\pi\delta_1^{-1} \pm \pi\delta_1^{-1}] \times \ldots \times (i_1\pi\delta_d^{-1} \pm \pi\delta_d^{-1}] : (i_1, \ldots, i_d)' \in \mathbb{Z}^d\}$. Thus, it is easy to check that the spectral density ψ_Δ (say) of the $Z(\cdot)$-process on the lattice $\mathcal{Z}^d$ can be expressed in terms of the spectral density ψ of the continuous stationary process $Z(\cdot)$ as

$$\psi_\Delta(\boldsymbol{\omega}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} \psi\left(\boldsymbol{\omega} + 2\pi\Delta^{-1}\mathbf{k}\right), \ \boldsymbol{\omega} \in \Pi_\Delta. \tag{III.3.3}$$

It turns out that the asymptotic distribution of the DFTs in the fixed design PID case depends on the spatial dependence structure of the random field $Z(\cdot)$ only through ψ_Δ.

III.3.1.2. Asymptotic distribution at nonzero frequencies

Next we investigate the asymptotic joint distribution of $(d_n^P(\boldsymbol{\omega}_{1n}), \cdots, d_n^P(\boldsymbol{\omega}_{rn}))$ for a finite collection of frequencies $\boldsymbol{\omega}_{1n}, \ldots, \boldsymbol{\omega}_{rn}$, $1 \leq r < \infty$. In analogy to the equi-spaced observations in the time series case, here we shall suppose that the $\boldsymbol{\omega}_{jn}$'s are of the form

$$\boldsymbol{\omega}_{jn} = 2\pi\lambda_n^{-1}\Delta^{-1}\mathbf{k}_{jn},\ \mathbf{k}_{jn} \in \mathbb{Z}^d, \quad \text{and} \quad \boldsymbol{\omega}_{jn} \to \boldsymbol{\omega}_j \in \Pi_\Delta \quad \text{as} \quad n \to \infty. \tag{III.3.4}$$

The first result concerns the asymptotic joint distribution of the cosine and sine transforms at $\boldsymbol{\omega}_{1n}, \ldots, \boldsymbol{\omega}_{rn}$ in the case where $\pm\boldsymbol{\omega}_j$'s are distinct and nonzero elements of Π_Δ^0, where $\Pi_\Delta^0 = (-\pi\delta_1^{-1}, \pi\delta_1^{-1}) \times \ldots \times (-\pi\delta_d^{-1}, \pi\delta_d^{-1})$, is the interior of Π_Δ.

Theorem III.3.1. *Suppose that, $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ is a zero mean stationary random field satisfying assumptions (A.1) and (A.2). Also suppose that for $j = 1, \ldots, r$, $r \in I\!N$, $\{\boldsymbol{\omega}_{jn}\}$ are sequences of the form (III.3.4) such that $\boldsymbol{\omega}_j \in \Pi_\Delta^0 \setminus \{\mathbf{0}\}$ and $\boldsymbol{\omega}_j \pm \boldsymbol{\omega}_k \in \Pi_\Delta^0 \setminus \{\mathbf{0}\}$ for all $1 \leq j \neq k \leq r$. Then,*

$$\begin{aligned} &[C_n^P(\boldsymbol{\omega}_{1n}), S_n^P(\boldsymbol{\omega}_{1n}), \cdots, C_n^P(\boldsymbol{\omega}_{rn}), S_n^P(\boldsymbol{\omega}_{rn})]' \\ &\xrightarrow{d} N\left[\mathbf{0}, \begin{pmatrix} A_1 I_2 & \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \cdots & A_r I_2 \end{pmatrix}\right], \end{aligned}$$

where, with $c_0 = (1/2)(2\pi)^d/[\prod_{i=1}^d \delta_i]$, $A_j = c_0\psi_\Delta(\boldsymbol{\omega}_j)$ for $j = 1, \cdots, r$ and where $\mathbf{0}_{2\times 2}$ is the 2×2 matrix of zeros.

Theorem III.3.1 implies that for each single sequence $\{\boldsymbol{\omega}_{jn}\}$ converging to a non-zero frequency $\boldsymbol{\omega}_j \in \Pi_\Delta^0$, the corresponding sine and cosine transforms are asymptotically independent. Further, since the co-variance matrix of the limiting Gaussian distribution is diagonal, any collection of *disjoint* subsets of the $2r$ cosine and sine transforms are also asymptotically independent. In particular, under the conditions of the theorem, the DFTs $(d_n^P(\boldsymbol{\omega}_{1n}), \cdots, d_n^P(\boldsymbol{\omega}_{rn}))$ are asymptotically independent and their asymptotic distribution depends on the dependence structure of the spatial process $\{Z(\cdot)\}$ only through the folded spectral density $\psi_\Delta(\cdot)$.

Next we consider the case when the limit frequencies are not necessarily distinct. In this case, the joint asymptotic normality continues to hold under the regularity conditions of Theorem III.3.1 on the random field $\{Z(\cdot)\}$. However, the asymptotic independence of the DFTs may no longer hold. To state the main results in a transparent manner, we shall restrict our attention to the case $r = 2$ with a common nonzero limit frequency, although the conclusions do generalize to the case $r > 2$ in an obvious manner. Accordingly, consider the asymptotic joint distribution of $(d_n^P(\boldsymbol{\omega}_{1n}), d_n^P(\boldsymbol{\omega}_{2n}))'$, with $\boldsymbol{\omega}_{jn} \to \boldsymbol{\omega}_j$ for $j = 1, 2$, where $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega} \neq \mathbf{0}$. Let $\boldsymbol{\omega}_{1n}^{(p)}$ and $\boldsymbol{\omega}_{2n}^{(p)}$ denote the p-th ordinate of the respective vectors $\boldsymbol{\omega}_{1n}$ and $\boldsymbol{\omega}_{2n}$, $p = 1, \cdots, d$. The limit behavior of the DFTs can be different depending on the closeness of the sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$, as specified below:

Definition (i) We say that $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are *asymptotically distant* if

$$|\lambda_n(\omega_{1n}^{(p)} - \omega_{2n}^{(p)})| \to \infty \quad \text{as} \quad n \to \infty \quad \text{for at least one} \quad p = 1, \ldots, d. \tag{III.3.5}$$

(ii) We say that $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are *asymptotically close* if

$$\left.\begin{array}{l} \lambda_n(\omega_{1n}^{(p)} - \omega_{2n}^{(p)}) \to 2\pi\Delta^{-1}\ell_p \quad \text{as} \quad n \to \infty \quad \text{for all} \;\; p = 1, \cdots, d \\ \quad \text{with} \quad \boldsymbol{\ell} = (\ell_1, \ldots, \ell_d)' \in \mathbb{Z}^d \quad \text{and} \quad \sum_{p=1}^d |\ell_p| \neq 0. \end{array}\right\} \tag{III.3.6}$$

Then we have the following result.

Theorem III.3.2. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ is a zero mean stationary random field satisfying Assumptions (A.1) and (A.2). Also, suppose that $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are two sequences satisfying (III.3.4) with $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega}$ where $\boldsymbol{\omega} \neq \mathbf{0}$ and $2\boldsymbol{\omega} \in \Pi_\Delta^0$.*

(a) *(Asymptotically distant frequencies): Suppose that* $\{\boldsymbol{\omega}_{1n}\}$ *and* $\{\boldsymbol{\omega}_{2n}\}$ *satisfy (III.3.5). Then,*

$$[C_n^P(\boldsymbol{\omega}_{1n}), S_n^P(\boldsymbol{\omega}_{1n}), C_n^P(\boldsymbol{\omega}_{2n}), S_n^P(\boldsymbol{\omega}_{2n})]' \xrightarrow{d} N\left[\mathbf{0}, \Sigma\right], \tag{III.3.7}$$

where $\Sigma = c_0\psi_\Delta(\boldsymbol{\omega})I_4$, *with* $c_0 = 2^{-1}[(2\pi)^d / \prod_{i=1}^d \delta_i]$.

(b) *(Asymptotically close frequencies): Suppose that* $\{\boldsymbol{\omega}_{1n}\}$ *and* $\{\boldsymbol{\omega}_{2n}\}$ *satisfy (III.3.6). Then, (III.3.7) holds with*

$$\Sigma = \begin{bmatrix} A_1 & 0 & A_2 & A_3 \\ & A_1 & -A_3 & A_2 \\ & & A_1 & 0 \\ & & & A_1 \end{bmatrix},$$

where $A_1 = c_0\psi_\Delta(\boldsymbol{\omega})$, $A_2 = c_0\psi_\Delta(\boldsymbol{\omega})\phi_1(2\pi\boldsymbol{\ell})$, *and* $A_3 = c_0\psi_\Delta(\boldsymbol{\omega})\phi_2(2\pi\boldsymbol{\ell})$. *Here,* $\phi_1(\cdot)$ *and* $\phi_2(\cdot)$ *respectively denote the real and the imaginary parts of the characteristic function of the uniform distribution on* $\Delta^{-1}\mathcal{D}_0$.

Theorem III.3.2 shows that for any two *asymptotically distant* sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ of frequencies, all four sine and cosine transforms are asymptotically independent and hence, the corresponding DFT's are asymptotically independent. However, for *asymptotically close* frequencies in the neighborhood of a nonzero frequency $\boldsymbol{\omega}$, this may no longer be true. Theorem III.3.2 reveals some interesting behavior of the sine and the cosine transforms in this case. From the form of the asymptotic covariance matrix, it is clear that the sine and the cosine transforms *along a given* sequence of ordinates $\boldsymbol{\omega}_{jn}$ (with a fixed $j \in \{1, 2\}$) are asymptotically independent, but any combination of sine and cosine transforms corresponding to *different* ordinates (say, one at $\boldsymbol{\omega}_{jn}$ and the other at $\boldsymbol{\omega}_{kn}$ for $j \neq k$) may have a *non-zero correlation* in the limit. Note that if $\Delta^{-1}\mathcal{D}_0$ is symmetric around zero (in the sense $\mathbf{x} \in \Delta^{-1}\mathcal{D}_0$ implies $-\mathbf{x} \in \Delta^{-1}\mathcal{D}_0$), then the function $\phi_2(\cdot)$ is identically zero and the cross-correlation between the sine and cosine transforms vanish ($A_3 = 0$). However, for sampling regions of a *general* shape, the correlation between the two cosine transforms need not vanish, and therefore, the DFTs along $\boldsymbol{\omega}_{1n}$ and $\boldsymbol{\omega}_{2n}$ are typically not asymptotically independent.

Next consider the important *special case, where* $\Delta^{-1}\mathcal{D}_0$ *is an integer multiple of* $(-1/2, 1/2]^d$. In this case, both $A_2 = 0$ and $A_3 = 0$, and therefore, the cross-correlation between the sine and the cosine transforms vanish in the limit. As a result, the corresponding DFTs are asymptotically independent. Some instances of this special case are:

(i) $\delta_i = 1$ for all $i = 1, \ldots, d$, and $\mathcal{D}_0 = (-1/2, 1/2]^d$.

(ii) The prototype set $\mathcal{D}_0 = (-a_1, a_1] \times \ldots \times (-a_d, a_d]$ for some $a_1, \ldots, a_d \in (0, 1/2)$ and $\delta_i = a_i^{-1}$ for all $i = 1, \ldots, d$.

Under (i), the spatial sampling sites lie on the integer grid $\mathbb{Z}^d$ and the sampling region is a (hyper-)cube in $\mathbb{R}^d$. Here, the DFTs are asymptotically independent even when the frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are asymptotically close. The behavior of the DFTs in this case is analogous to that for time series data observed over equispaced time points. However, it turns out that the asymptotic independence of the DFTs at asymptotically close frequencies can also hold for non-cubic regions and for non-equispaced spatial data-sites. Instance (ii) above corresponds to a hyper-rectangular sampling region. Here, the same conclusions as in Instance (i) are possible, if for each $i = 1, \ldots, d$, we set the grid increment δ_i in the ith direction as the inverse of the length of the rectangular prototype set in the that direction.

To summarize the main implications of Theorem III.3.2, DFTs at asymptotically distant frequencies are asymptotically independent, and the asymptotic independence of DFTs may also hold for asymptotically close frequencies in certain *special* cases.

However, for a non-rectangular sampling region, asymptotic independence of DFTs at asymptotically close ordinates typically fails in the spatial case, even for regularly spaced sampling sites. Thus, the shape of the sampling region and the sampling grid plays an important role in determining the behavior of the DFTs in the spatial case, which sets it apart from the familiar weakly dependent time series set up.

Remark There is a dual to Theorem III.3.2, where $\boldsymbol{\omega}_1 = -\boldsymbol{\omega}_2 = \boldsymbol{\omega}$ and $2\boldsymbol{\omega} \in \Pi_\Delta^0 \setminus \{\mathbf{0}\}$. In this case, conclusions similar to part (a) of Theorem III.3.2 hold if $\|\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n})\| \to \infty$. For $\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n}) \to 2\pi\Delta^{-1}\boldsymbol{\ell}$ for some $\boldsymbol{\ell} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, the asymptotic normality of the cosine and sine transforms as in (III.3.7) continues to hold under (A.1) and (A.2), but with the following limiting covariance matrix:

$$\Sigma = \begin{bmatrix} A_1 & 0 & A_2 & A_3 \\ & A_1 & A_3 & -A_2 \\ & & A_1 & 0 \\ & & & A_1 \end{bmatrix}, \tag{III.3.8}$$

where A_1, A_2, A_3 are as in part (b) of Theorem III.3.2. In particular, for the special cases (i) and (ii) of cubic and rectangular sampling regions considered above, the asymptotic independence of the DFTs holds even for the asymptotically close frequencies, but not necessarily for sampling regions of a general shape.

III.3.1.3. Asymptotic distribution for the zero limiting frequency

Next we consider the case where $\boldsymbol{\omega}_{jn}$'s converge to the zero frequency. Here, some extra care must be taken while studying the asymptotic behavior of the DFTs due to the special role played by the zero frequency in the definitions of the sine and cosine transforms. For the zero frequency limit, we shall suppose that the discrete Fourier ordinates $\boldsymbol{\omega}_n = 2\pi\lambda_n^{-1}\Delta^{-1}\mathbf{k}_n$ satisfy the following regularity condition:

$$\boldsymbol{\omega}_n = 2\pi\lambda_n^{-1}\Delta^{-1}\mathbf{k}_n,\ \mathbf{k}_n \in \mathbb{Z}^d \setminus \{\mathbf{0}\}, \quad \text{and} \quad \boldsymbol{\omega}_n \to \mathbf{0} \quad \text{as} \quad n \to \infty. \qquad \text{(III.3.9)}$$

We rule out the choice $\mathbf{k}_n = \mathbf{0}$ in (III.3.9) in order to avoid some nonstandard asymptotic behavior of the DFTs. To appreciate why, suppose that $\mathbf{k}_n = \mathbf{0}$ along a subsequence and $\mathbf{k}_n \neq \mathbf{0}$ along a different subsequence, but $\boldsymbol{\omega}_n \to \mathbf{0}$. Then the sine transform is (identically) equal to zero along the first subsequence, but it has a non-degenerate limit distribution along the other subsequence, thereby destroying the convergence of the full sequence.

As before, for clarity of exposition, we restrict attention to the asymptotic distribution of the DFTs along two sequences of frequencies $\{\boldsymbol{\omega}_{jn}\}$, $j = 1, 2$ satisfying (III.3.9). The case of an arbitrary finite number of such frequency sequences can be handled in a straightforward manner, with added notational complexity. In comparison to a non-zero limit frequency, here we need to consider three situations arising from the relative orders of magnitude of the sequences $\{\boldsymbol{\omega}_{jn}\}$, $j = 1, 2$:

(i) $\|\lambda_n(\boldsymbol{\omega}_{1n} \pm \boldsymbol{\omega}_{2n})\| \to \infty$

(ii) Exactly one of the sequences $\{\|\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n})\|\}$ and $\{\|\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n})\|\}$ tends to infinity and the other has a finite limit.

(iii) Both sequences $\{\|\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n})\|\}$ and $\{\|\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n})\|\}$ have finite limits.

The first two cases are the analogs of the "asymptotically distant" and "asymptotically close" cases considered in the last section. However, the situation covered in the third part can occur *only* for the zero limit frequency case, as both $\boldsymbol{\omega}_{1n}$ and $-\boldsymbol{\omega}_{1n}$ can be close to $\boldsymbol{\omega}_{2n}$ *simultaneously*. We will refer to this as the "asymptotically symmetrically close" case. The asymptotic behaviors of the cosine- and sine-transforms of spatial data under PID in these three cases are given by the following theorem.

Theorem III.3.3. *Suppose that* $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ *is a zero mean stationary random field satisfying Assumptions (A.1) and (A.2). Also, suppose that* $\{\boldsymbol{\omega}_{1n}\}$ *and* $\{\boldsymbol{\omega}_{2n}\}$ *are two sequences satisfying (III.3.9).*

(a) If $\|\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n})\| \to \infty$ *and* $\|\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n})\| \to \infty$*, then*

$$[C_n^P(\boldsymbol{\omega}_{1n}), S_n^P(\boldsymbol{\omega}_{1n}), C_n^P(\boldsymbol{\omega}_{2n}), S_n^P(\boldsymbol{\omega}_{2n})]' \xrightarrow{d} N\left[\mathbf{0}, \Sigma^{(a)}\right], \qquad \text{(III.3.10)}$$

where $\Sigma^{(a)} = c_0\psi_\Delta(\mathbf{0})I_4$.

(b) Suppose that $\|\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n})\| \to \infty$ *but* $\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n}) \to \mathbf{z}_{12} \in \mathbb{R}^d$*. Then, (III.3.10) holds with* $\Sigma^{(a)}$ *replaced by* $\Sigma^{(b)}$*, where*

$$\Sigma^{(b)} = \begin{bmatrix} A_1^{[0]} & 0 & A_2^{[0]} & A_3^{[0]} \\ & A_1^{[0]} & -A_3^{[0]} & A_2^{[0]} \\ & & A_1^{[0]} & 0 \\ & & & A_1^{[0]} \end{bmatrix},$$

with $A_1^{[0]} = c_0\psi_\Delta(\mathbf{0})$, $A_2^{[0]} = c_0\psi_\Delta(\mathbf{0})\phi_1(\Delta\mathbf{z}_{12})$, *and* $A_3^{[0]} = c_0\psi_\Delta(\mathbf{0})\phi_2(\Delta\mathbf{z}_{12})$.

(c) Suppose that $\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n}) \to \mathbf{y}_{12} \in \mathbb{R}^d$ *and* $\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n}) \to \mathbf{z}_{12} \in \mathbb{R}^d$*. Then, (III.3.10) holds with* $\Sigma^{(a)}$ *replaced by* $\Sigma^{(c)}$*, where*

$$\Sigma^{(c)} = \sigma_\infty^2 \begin{bmatrix} [1+\tilde{\phi}_1(2\mathbf{y}_1)] & \tilde{\phi}_2(2\mathbf{y}_1) & [\tilde{\phi}_1(\mathbf{y}_{12}) + \tilde{\phi}_1(\mathbf{z}_{12})] & [\tilde{\phi}_1(\mathbf{y}_{12}) + \tilde{\phi}_1(\mathbf{z}_{12})] \\ & [1-\tilde{\phi}_1(2\mathbf{y}_1)] & [\tilde{\phi}_2(\mathbf{y}_{12}) - \tilde{\phi}_2(\mathbf{z}_{12})] & [\tilde{\phi}_1(\mathbf{z}_{12}) - \tilde{\phi}_2(\mathbf{y}_{12})] \\ & & [1+\tilde{\phi}_1(2\mathbf{y}_2)] & \tilde{\phi}_2(2\mathbf{y}_2) \\ & & & [1-\tilde{\phi}_1(2\mathbf{y}_2)] \end{bmatrix}$$

with $\sigma_\infty^2 = c_0\psi_\Delta(\mathbf{0})$, $\mathbf{y}_1 = (\mathbf{y}_{12} + \mathbf{z}_{12})/2$, $\mathbf{y}_2 = (\mathbf{y}_{12} - \mathbf{z}_{12})/2$, *and* $\tilde{\phi}_j(\boldsymbol{\omega}) = \phi_j(\Delta\boldsymbol{\omega})$, $\boldsymbol{\omega} \in \mathbb{R}^d$, $j = 1, 2$.

Thus, from Theorem III.3.3, it follows that the sine and the cosine transforms at *non-zero ordinates converging to the zero frequency* have very similar asymptotic behavior as in the case of a non-zero limit frequency for the "asymptotically distant" and "asymptotically close" parts (see parts (a) and (b) of Theorems III.3.2 and III.3.3). In particular, asymptotic independence of the DFTs continues to hold for "asymptotically distant" discrete Fourier ordinates converging to zero. For "asymptotically close" ordinates converging to zero, DFTs are typically asymptotically dependent; For such sequences of ordinates, asymptotic independence of the DFTs holds in the special case where $\Delta^{-1}\mathcal{D}_0$ is an integer multiple of the d-cube $(-1/2, 1/2]^d$, as noted in the discussion of Theorem III.3.2 above. Finally, for the "asymptotically symmetrically close" ordinates, it is clear that *every* possible pairs of sine and cosine transforms may have nontrivial asymptotic correlations for sampling regions of a general shape and hence, the DFTs typically are not asymptotically independent.

Next consider the special case where $\Delta^{-1}\mathcal{D}_0$ is d-cubic. Note that, in this case, $\tilde{\phi}_k(\mathbf{y}_j) = \phi_k(2\pi\boldsymbol{\ell}_j)$ for some $\boldsymbol{\ell}_j \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, for all $j, k \in \{1, 2\}$ and thus, all off-diagonal terms in $\Sigma^{(c)}$ vanish. In this special case, asymptotic independence of the DFTs hold even for 'asymptotically symmetrically close" ordinates converging to the zero frequency, as in the time series case. But, for sampling regions of a general shape, the DFTs are typically dependent in the limit in the 'asymptotically close' and 'asymptotically symmetrically close' cases.

Remark As in the last section, there is a dual to part (b) of Theorem III.3.3. For $\|\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n})\| \to \infty$ but $\lambda_n(\boldsymbol{\omega}_{1n} + \boldsymbol{\omega}_{2n}) \to \mathbf{z}_{12} \in I\!R^d$, asymptotic normality of the cosine- and sine-transforms holds where the limiting covariance matrix is given by (III.3.8) with A_i's replaced by $A_i^{[0]}$'s, $i = 1, 2, 3$.

Remark For completeness, consider the case where $\boldsymbol{\omega}_n = \mathbf{0}$ for all $n \geq 1$. In the case, $S_n(\boldsymbol{\omega}_n) = 0$ for all $n \geq 1$, while $C_n(\boldsymbol{\omega}_n) = N_n^{-1/2} \sum_{i=1}^{N_n} Z(\mathbf{s}_i)$ and $C_n(\boldsymbol{\omega}_n) \to^d N(0, 2c_0\psi_\Delta(\mathbf{0}))$ solely under Assumption (A.1) and (A.2) (see Theorem 4.3, Lahiri (2003b))

III.3.1.4. Results for mean-corrected DFTs

In many applications, the random field $\{Z(\mathbf{s})\}$ has a mean $\mu = EZ(\mathbf{0})$ that is unknown. In such situations, the DFT defined in Section III.3.1.1 is often replaced by its mean corrected version:

$$\tilde{d}_n^P(\boldsymbol{\omega}) \equiv N_n^{-1/2} \sum_{\mathbf{j} \in J_n} [Z(\Delta\mathbf{j}) - \bar{Z}_n] \exp\left(\iota \boldsymbol{\omega}' \Delta\mathbf{j}\right) \qquad \text{(III.3.11)}$$

where, as before, $J_n = \{\mathbf{j} \in \mathbb{Z}^d : \Delta\mathbf{j} \in \mathcal{D}_n\}$ and $\bar{Z}_n = N_n^{-1} \sum_{\mathbf{j} \in J_n} Z(\Delta\mathbf{j})$ is the sample mean. Similarly, we define

$$\begin{aligned} \tilde{C}_n^P(\boldsymbol{\omega}) &= N_n^{-1/2} \sum_{\mathbf{j} \in J_n} [Z(\Delta\mathbf{j}) - \bar{Z}_n] \cos\left(\boldsymbol{\omega}' \Delta\mathbf{j}\right), \\ \tilde{S}_n^P(\boldsymbol{\omega}) &= N_n^{-1/2} \sum_{\mathbf{j} \in J_n} [Z(\Delta\mathbf{j}) - \bar{Z}_n] \sin\left(\boldsymbol{\omega}' \Delta\mathbf{j}\right), \end{aligned} \qquad \text{(III.3.12)}$$

the mean corrected versions of the cosine and the sine transforms of the data. Then, $\tilde{d}_n^P(\boldsymbol{\omega}) = \tilde{C}_n^P(\boldsymbol{\omega}) + \iota \tilde{S}_n^P(\boldsymbol{\omega})$. The following result gives the asymptotic behavior of the DFTs in different cases treated in Theorems III.3.1-III.3.3.

Theorem III.3.4. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ is a zero mean stationary random field satisfying Assumptions (A.1) and (A.2). Also, suppose that $\{\boldsymbol{\omega}_{1n}\}, \ldots, \{\boldsymbol{\omega}_{rn}\}$ are sequences satisfying (III.3.4).*

(a) If the limiting frequencies $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_r$ satisfy the conditions of Theorem III.3.1, then

$$[\tilde{C}_n^P(\boldsymbol{\omega}_{1n}), \tilde{S}_n^P(\boldsymbol{\omega}_{1n}), \ldots, \tilde{C}_n^P(\boldsymbol{\omega}_{rn}), \tilde{S}_n^P(\boldsymbol{\omega}_{rn})]'$$

has the same limit distribution as that of the mean uncorrected version $[C_n^P(\boldsymbol{\omega}_{1n})$, $S_n^P(\boldsymbol{\omega}_{1n}), \ldots, C_n^P(\boldsymbol{\omega}_{rn}), S_n^P(\boldsymbol{\omega}_{rn})]$, given by Theorem III.3.1.

(b) Suppose that $r = 2$ and the sequences $\{\boldsymbol{\omega}_{1n}\}, \{\boldsymbol{\omega}_{2n}\}$ satisfy the conditions of one of the two parts of Theorem III.3.2. Then, $[\tilde{C}_n^P(\boldsymbol{\omega}_{1n}), \tilde{S}_n^P(\boldsymbol{\omega}_{1n}), \tilde{C}_n^P(\boldsymbol{\omega}_{2n}),$

$\tilde{S}_n^P(\boldsymbol{\omega}_{2n})]' \rightarrow^d N[\mathbf{0}, \Sigma]$, with Σ as in the respective part of Theorem III.3.2.

(c) Suppose that $r = 2$ and the sequences $\{\boldsymbol{\omega}_{1n}\}, \{\boldsymbol{\omega}_{2n}\}$ satisfy the conditions of any one of the three parts Theorem III.3.3. Then,

$$[\tilde{C}_n^P(\boldsymbol{\omega}_{1n}), \tilde{S}_n^P(\boldsymbol{\omega}_{1n}), \tilde{C}_n^P(\boldsymbol{\omega}_{2n}), \tilde{S}_n^P(\boldsymbol{\omega}_{2n})]' \rightarrow^d N[\mathbf{0}, \tilde{\Sigma}],$$

where $\tilde{\Sigma} = \Sigma^i - \Sigma_0$ for the ith part, $i = (a), (b), (c)$ and

$$\Sigma_0 = \sigma_\infty^2 \begin{bmatrix} \tilde{\phi}_1^2(\mathbf{y}_1) & 2\tilde{\phi}_1(\mathbf{y}_1)\tilde{\phi}_2(\mathbf{y}_1) & 2\tilde{\phi}_1(\mathbf{y}_1)\tilde{\phi}_1(\mathbf{y}_2) & 2\tilde{\phi}_1(\mathbf{y}_1)\tilde{\phi}_2(\mathbf{y}_2) \\ & \tilde{\phi}_2^2(\mathbf{y}_1) & 2\tilde{\phi}_1(\mathbf{y}_2)\tilde{\phi}_2(\mathbf{y}_1) & 2\tilde{\phi}_2(\mathbf{y}_1)\tilde{\phi}_2(\mathbf{y}_2) \\ & & \tilde{\phi}_1^2(\mathbf{y}_2) & 2\tilde{\phi}_1(\mathbf{y}_2)\tilde{\phi}_2(\mathbf{y}_2) \\ & & & \tilde{\phi}_2^2(\mathbf{y}_2) \end{bmatrix}.$$

Thus, the asymptotic distributions of the sine and cosine transforms remain unchanged in all cases where the discrete Fourier frequencies converge to a nonzero limit. However, for frequency sequences converging to the zero frequency, the asymptotic covariance is different.

III.3.2. Results under the MID case

For the MID case, we define the discrete Fourier transform (DFT) of $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_{N_n})\}$ by

$$d_n^M(\boldsymbol{\omega}) = N_n^{-1/2} \sum_{\mathbf{j} \in \mathbb{Z}^d : \Delta \mathbf{j} \eta_n \in \mathcal{D}_n} Z(\Delta \mathbf{j} \eta_n) \exp\left\{\iota \boldsymbol{\omega}' \Delta \mathbf{j} \eta_n\right\} \tag{III.3.13}$$

and we can define the corresponding cosine and sine transforms $C_n^M(\boldsymbol{\omega})$ and $S_n^M(\boldsymbol{\omega})$ in a similar way. Although for each fixed n, the observations in the deterministic MID case lie on a grid, the asymptotic distribution of the DFT depends on the dependence structure of the random field $\{Z(\cdot)\}$ through the *full* spectral density function $\psi(\boldsymbol{\omega})$; knowledge of the folded spectral density ψ_Δ is no longer adequate as in the PID case.

This is mainly due to the fact that the asymptotic variances of the relevant transforms in the MID case are given by certain integrals of the auto-covariance function over $\mathbb{R}^d$ as compared to infinite sums in the PID case. Further, we also *drop* the restriction on the limiting frequencies to lie in the set Π_Δ, as it is now possible to infer about the full spectral density $\psi(\cdot)$ by considering the DFT at any given $\boldsymbol{\omega} \in \mathbb{R}^d$.

For the MID case, we will make use of the following additional assumption:

(A.3) $\lambda_n \eta_n \to \infty$, *i.e.,* the sites fill in any given region at a slower rate than the inflating factor λ_n.

The following result gives the asymptotic joint distribution of the sine and cosine transforms in the case of distinct *nonzero* limits.

Theorem III.3.5. *Suppose that* $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ *is a zero mean stationary random field satisfying Assumptions (A.1) and (A.2) and that*

(A.3) $\lambda_n \eta_n \to \infty$.

Also suppose that for $r \in \mathbb{N}$, $\{\boldsymbol{\omega}_{1n}\}, \ldots, \{\boldsymbol{\omega}_{rn}\}$ *are frequency sequences of the form* $\boldsymbol{\omega}_{jn} = 2\pi \mathbf{k}_{jn}/\lambda_n$ *for* $\mathbf{k}_{jn} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ *such that* $\mathbf{k}_{jn} \to \boldsymbol{\omega}_j \in \mathbb{R}^d \setminus \{\mathbf{0}\}$ *and* $\boldsymbol{\omega}_j \pm \boldsymbol{\omega}_k \neq \mathbf{0}$ *for all* $1 \leq j \neq k \leq r$. *Then,*

$$\eta_n^{d/2}[C_n^M(\boldsymbol{\omega}_{1n}), S_n^M(\boldsymbol{\omega}_{1n}), \cdots, C_n^M(\boldsymbol{\omega}_{rn}), S_n^M(\boldsymbol{\omega}_{rn})]'$$

$$\xrightarrow{d} N\left[\mathbf{0}, \begin{pmatrix} B_1 I_{2\times 2} & \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \cdots & B_r I_{2\times 2} \end{pmatrix}\right],$$

where, $B_\ell = (1/2)(\prod_{i=1}^n \delta_i)^{-1}(2\pi)^d \psi(\boldsymbol{\omega}_\ell)$ *for* $\ell = 1, \cdots, r$.

Theorem III.3.5 implies that for $\boldsymbol{\omega}_{1n}, \cdots, \boldsymbol{\omega}_{rn}$ converging to *different non-zero* limits, the corresponding DFTs are also asymptotically independent and the asymptotic variances depend on the spectral density function of the process at the limiting frequencies $\boldsymbol{\omega}_1, \ldots, \boldsymbol{\omega}_r$. Note that the conditions on the frequency sequences $\{\boldsymbol{\omega}_{jn}\}$ in Theorem III.3.5 are weaker than those required in the PID case (see Theorem III.3.1). Also note that the DFTs defined in (III.3.13) do not have a nondegenerate limit *unless* they are rescaled by the damping factor $\eta_n^{d/2}$. As the spacing of the grid goes to zero, observations at neighboring locations tend to have very strong correlations, and as a result, the variances of the sine and cosine transforms $S_n^M(\cdot)$ and $C_n^M(\cdot)$ grow at a rate faster than the sample size. As a result, the natural scaling by the inverse-square root of the sample size is not adequate under the MID, and the additional multiplicative factor $\eta_n^{d/2}$ is needed to make the sine and cosine transforms converge to a nondegenerate normal limit.

Remark Conclusions on the asymptotic independence of the DFTs do not change in the other scenarios covered by Theorems III.3.2-III.3.4. Specifically, with the additional $\eta_n^{d/2}$ multiplicative factor, the sine- and the cosine- transforms in the MID case continue to have the same limits as their PID counterparts in the set ups of Theorems III.3.2-III.3.4, where the folded spectral density ψ_Δ in the limit is replaced by ψ in every occurrence. We do not restate the theorems for each of these cases to save space.

III.4. Results under the stochastic design

III.4.1. Definition of the DFT and some preliminaries

In the stochastic design case, the observations are given by $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_n)\}$, under both the PID and the MID asymptotic structures. Thus, we define the (scaled) DFT of the sample $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_n)\}$ under the stochastic design as

$$\check{d}_n(\boldsymbol{\omega}) \quad = \quad \lambda_n^{d/2} n^{-1} \sum_{j=1}^{n} Z(\mathbf{s}_j) \exp\left(\iota \boldsymbol{\omega}' \mathbf{s}_j\right), \quad \boldsymbol{\omega} \in \mathbb{R}^d \qquad \text{(III.4.1)}$$

for both PID and MID cases. Note that under both asymptotic structures, we use a common scaling $\lambda_n^{d/2}$, which is asymptotically equivalent to the square root of the sample size under the PID, but grows at a slower rate (than $n^{1/2}$) in the MID case. That this is the correct scaling sequence for a non-degenerate limit in both cases will be clear in the next section where we state the main results. In analogy to (III.4.1), also define the (scaled) cosine and the sine transforms of $\{Z(\mathbf{s}_1), \ldots, Z(\mathbf{s}_n)\}$ as

$$\begin{aligned} \check{C}_n(\boldsymbol{\omega}) &= \lambda_n^{d/2} n^{-1} \sum_{j=1}^{n} \cos(\boldsymbol{\omega}' \mathbf{s}_j) Z(\mathbf{s}_j), \\ \check{S}_n(\boldsymbol{\omega}) &= \lambda_n^{d/2} n^{-1} \sum_{j=1}^{n} \sin(\boldsymbol{\omega}' \mathbf{s}_j) Z(\mathbf{s}_j), \end{aligned} \tag{III.4.2}$$

$\boldsymbol{\omega} \in \mathbb{R}^d$. Then, $\check{d}_n(\boldsymbol{\omega}) = \check{C}_n(\boldsymbol{\omega}) + \iota \check{S}_n(\boldsymbol{\omega})$.

Note that under the stochastic design, the sampling sites are generated by a realization of the sequence $\{\mathbf{X}_n\}$. As a consequence, the distributions of the DFTs discussed in this section actually refer to their *conditional distribution* given $\{\mathbf{X}_n\}$ and the CLTs under the stochastic design assert *weak convergence of these conditional distributions to respective normal limits for almost all realizations of the sequence* $\{\mathbf{X}_n\}$ *under* $P_{\mathbf{X}}$, $P_{\mathbf{X}}$ denotes the joint distribution of the $\mathbf{X}_i$'s. Also, for brevity, *we shall use the convention that* $(\infty)^{-1} a = 0$ *for all* $a \in \mathbb{R}$. Thus, in the statements of the theorems below, the condition $n/\lambda_n^d \to c_* \in (0, \infty]$, will cover both the cases, $c_* \in (0, \infty)$ for the PID asymptotic structure and $c_* = \infty$ for the MID, in a unified way, and for an expression of the form $c_*^{-1} a$ where $a \in \mathbb{R}$, we would interpret $c_*^{-1} a$ as zero in the MID (i.e., $c_* = \infty$) case. Finally, set $K = \int f^2(\mathbf{x}) d\mathbf{x}$ and $I_\psi = \int_{\mathbb{R}^d} \psi(\boldsymbol{\omega}) d\boldsymbol{\omega}$.

With this, we are now ready to state the main results for the stochastic design case in the sections below.

III.4.2. Asymptotic distribution at nonzero frequencies

In this section, we investigate the asymptotic joint distribution of the sine and cosine transforms at a finite collection of frequencies $\boldsymbol{\omega}_{1n}, \cdots, \boldsymbol{\omega}_{rn}, 1 \leq r < \infty$, where

$$\boldsymbol{\omega}_{jn} \to \boldsymbol{\omega}_j \in I\!R^d \text{ as } n \to \infty. \tag{III.4.3}$$

Since under the stochastic design the data-sites are randomly distributed, we do not require the sequences $\{\boldsymbol{\omega}_{jn}\}$'s to satisfy (III.3.4). The first result concerns the asymptotic joint distribution at $\boldsymbol{\omega}_{1n}, \cdots, \boldsymbol{\omega}_{rn}$ in the case where $\pm\boldsymbol{\omega}_j$'s are distinct and nonzero.

Theorem III.4.1. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ is a zero mean stationary random field satisfying (A.1), (A.2) and that it satisfies Assumption (A.4):*

(A.4) $\lambda_n \gg n^\epsilon$ *for some* $\epsilon > 0$ *and* $\lim_{n\to\infty} n/\lambda_n = c_* \in (0, \infty]$.

Also suppose that for $j = 1, \cdots, r$, $r \in I\!N$, $\{\boldsymbol{\omega}_{jn}\}$ are sequences satisfying $\boldsymbol{\omega}_{jn} \to \boldsymbol{\omega}_j \in I\!R^d \setminus \{\mathbf{0}\}$ and $\boldsymbol{\omega}_j \pm \boldsymbol{\omega}_k \neq \mathbf{0}$ for all $1 \leq j \neq k \leq r$. Then,

$$\begin{aligned}&[\check{C}_n(\boldsymbol{\omega}_{1n}), \check{S}_n(\boldsymbol{\omega}_{1n}), \cdots, \check{C}_n(\boldsymbol{\omega}_{rn}), \check{S}_n(\boldsymbol{\omega}_{rn})]' \\ &\xrightarrow{d} N\left[\mathbf{0}, \begin{pmatrix} \check{A}_1 I_2 & \mathbf{0}_{2\times 2} & \cdots & \mathbf{0}_{2\times 2} \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{0}_{2\times 2} & \mathbf{0}_{2\times 2} & \cdots & \check{A}_r I_2 \end{pmatrix}\right], \quad a.s. \quad (P_{\mathbf{X}}),\end{aligned}$$

where $2\check{A}_j = c_*^{-1} I_\psi + K \cdot (2\pi)^d \psi(\boldsymbol{\omega}_j)$ and where, recall that, $K = \int f^2(\mathbf{x}) d\mathbf{x}$, $I_\psi = \int_{I\!R^d} \psi(\boldsymbol{\omega}) d\boldsymbol{\omega}$, and $\mathbf{0}_{2\times 2}$ is the 2×2 matrix of zeros.

As in the fixed design case, Theorem III.4.1 implies that any collection of disjoint subsets of the $2r$ cosine and sine transforms are also asymptotically independent. However, the asymptotic distribution of the DFTs $(\check{d}_n(\boldsymbol{\omega}_{1n}), \cdots, \check{d}_n(\boldsymbol{\omega}_{rn}))$ under the stochastic design depends on three factors, namely, (i) on the dependence structure of the spatial process $\{Z(\cdot)\}$, through the spectral density $\psi(\cdot)$, (ii) on the design density $f(\cdot)$, through the constant K, and (iii) on the spatial asymptotic framework (PID *vs.* MID). Note that the asymptotic variance has a somewhat simpler form under the MID (i.e., $c_* = \infty$) case where the first term in $\check{A}_j$ drops out.

Next we consider the case where the limit frequencies are not necessarily distinct. As before, to state the main results in a transparent manner, we shall restrict our attention to the case $r = 2$; The conclusions can be generalized to the case $r > 2$ in an obvious manner. For $\mathbf{z} \in I\!\!R^d$, define

$$\begin{aligned} \int \cos(\mathbf{z}'\mathbf{x}) f(\mathbf{x}) d\mathbf{x} &= K_1(\mathbf{z}) \quad , \quad \int \sin(\mathbf{z}'\mathbf{x}) f(\mathbf{x}) d\mathbf{x} = K_2(\mathbf{z}), \\ \int \cos(\mathbf{z}'\mathbf{x}) f^2(\mathbf{x}) d\mathbf{x} &= K_3(\mathbf{z}) \quad , \quad \int \sin(\mathbf{z}'\mathbf{x}) f^2(\mathbf{x}) d\mathbf{x} = K_4(\mathbf{z}). \end{aligned}$$

Then we have the following results.

Theorem III.4.2. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in I\!\!R^d\}$ is a zero mean stationary random field satisfying Assumptions (A.1), (A.2)and (A.4). Also, suppose that $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are two sequences satisfying (III.4.3) with $\boldsymbol{\omega}_1 = \boldsymbol{\omega}_2 = \boldsymbol{\omega} \in I\!\!R^d \setminus \{\mathbf{0}\}$.*

(a) *(Asymptotically distant frequencies): Suppose that $\|\lambda_n(\boldsymbol{\omega}_{1n} - \boldsymbol{\omega}_{2n})\| \to \infty$. Then,*

$$[\check{C}_n(\boldsymbol{\omega}_{1n}), \check{S}_n(\boldsymbol{\omega}_{1n}), \check{C}_n(\boldsymbol{\omega}_{2n}), \check{S}_n(\boldsymbol{\omega}_{2n})]' \xrightarrow{d} N[\mathbf{0}, \Sigma] \quad a.s. \quad (P_{\mathbf{X}}). \qquad \text{(III.4.4)}$$

where $\Sigma = \check{A} I_4$, with $2\check{A} = c_*^{-1} I_\psi + K \cdot (2\pi)^d \psi(\boldsymbol{\omega})$.

(b) *(Asymptotically close frequencies): Suppose that*

$$\left. \begin{array}{l} \lambda_n(\omega_{1n}^{(p)} - \omega_{2n}^{(p)}) \to z_p \quad as \quad n \to \infty \quad for\ all \quad p = 1, \cdots, d \\ \quad with \quad \mathbf{z} = (z_1, \ldots, z_d)' \in I\!\!R^d \setminus \{\mathbf{0}\}. \end{array} \right\} \qquad \text{(III.4.5)}$$

Then, (III.4.4) holds with

$$\Sigma = \begin{bmatrix} \check{A}_1 & 0 & \check{A}_2 & \check{A}_3 \\ & \check{A}_1 & -\check{A}_3 & \check{A}_2 \\ & & \check{A}_1 & 0 \\ & & & \check{A}_1 \end{bmatrix},$$

where $2\check{A}_1 = c_*^{-1} I_\psi + K \cdot (2\pi)^d \psi(\boldsymbol{\omega})$, $2\check{A}_2 = c_*^{-1} I_\psi K_1(\mathbf{z}) + K_3(\mathbf{z})(2\pi)^d \psi(\boldsymbol{\omega})$, $2\check{A}_3 = c_*^{-1} I_\psi K_2(\mathbf{z}) + K_4(\mathbf{z})(2\pi)^d \psi(\boldsymbol{\omega})$.

Theorem III.4.2 gives the asymptotic distribution of the DFTs when both frequency sequences converge to a common *non-zero* frequency. Note that the limiting covariance matrix for for the asymptotically close frequencies depends on the spatial sampling density $f(\cdot)$ through all four functionals $K_1(\cdot)$-$K_4(\cdot)$, and the constant K. This typically makes the corresponding DFTs asymptotically dependent. In contrast, DFTs along *asymptotically distant* frequency sequences are asymptotically independent.

Next, consider the special case, where the sampling region $\mathcal{D}_0$ is of the form

$$\mathcal{D}_0 = (-a_1, b_1) \times \ldots \times (-a_d, b_d), \tag{III.4.6}$$

for some $0 < a_j, b_j \leq 1/2$, $j = 1, \ldots, d$ and the sampling density $f(\cdot)$ is *uniform* over $\mathcal{D}_0$. In this case, $K_i(\mathbf{z}) = 0$ *for all* $i = 1, \ldots, 4$, whenever $\mathbf{z}$ is of the form

$$\mathbf{z} = \big((2\pi\ell_1)(a_1 + b_1)^{-1}, \ldots, (2\pi\ell_d)(a_d + b_d)^{-1}\big)', \tag{III.4.7}$$

for some $\ell_1, \ldots, \ell_d \in \mathbb{Z}$ with $\sum_{j=1}^d |\ell_j| \neq 0$. As a result, asymptotic independence of the DFTs holds even for asymptotically close frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$, provided $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ satisfy (III.4.5) for some $\mathbf{z}$ of the form (III.4.7). The asymptotic independence property of the DFTs can be guaranteed *for all* distinct asymptotically close sequences if one restricts attention to DFTs based on frequency sequences of the form

$$\boldsymbol{\omega}_n = \big((2\pi k_{1n})(a_1+b_1)^{-1}, \ldots, (2\pi k_{dn})(a_d+b_d)^{-1}\big)', \tag{III.4.8}$$

with $k_{1n}, \ldots, k_{dn} \in \mathbb{Z}$, $\sum_{j=1}^{d} |k_{jn}| \neq 0$.

Remark It is worth noting that for a rectangular sampling region (with $\mathcal{D}_0$ as in (III.4.6)), a similar conclusion on asymptotic independence of the DFTs holds for a *more general* class of sampling densities that can be expressed as a convolution of a general probability distribution with a suitable uniform density. Specifically, the class of such sampling densities is given by $\mathcal{F} = \Big\{ f : f$ has support $\mathcal{D}_0$, and $f(\cdot) = \int g_{\boldsymbol{\ell}}(\cdot - \mathbf{y}) dG_{\boldsymbol{\ell}}(\mathbf{y})$ for some probability distribution $G_{\boldsymbol{\ell}}$ and for some $\boldsymbol{\ell} \in \mathcal{N} \Big\}$ where $g_{\boldsymbol{\ell}}$ is the density of the uniform distribution on $(-a_1/\ell_1, b_1/\ell_1) \times \ldots \times (-a_d/\ell_d, b_d\ell_d)$ and $\mathcal{N} = \{(\ell_1, \ldots, \ell_d)' \in \mathbb{Z}^d : \ell_j \geq 2$ for all $j = 1, \ldots, d\}$.

Remark As in the deterministic case, there is a dual to Theorem III.4.2, where $\boldsymbol{\omega}_1 = -\boldsymbol{\omega}_2 = \boldsymbol{\omega} \neq \mathbf{0}$. The formulation of the dual parallels that in the deterministic case with the A_j's in (III.3.8) replaced by $\check{A}_j$'s from Theorem III.4.2.

III.4.3. Asymptotic distribution for the zero limiting frequency

In this section we consider the case where the fourier frequencies converge to the zero frequency. The asymptotic behavior of the cosine and sine transforms of spatial data under the PID and the MID asymptotic structures in the stochastic design case are given by the following theorem.

Theorem III.4.3. *Suppose that $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ is a zero mean stationary random field satisfying Assumptions (A.1), (A.2) and (A.4). Also, suppose that $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are two sequences in $\mathbb{R}^d \setminus \{\mathbf{0}\}$, converging to $\mathbf{0}$.*

(a) *If* $\|\lambda_n(\boldsymbol{\omega}_{1n}+\boldsymbol{\omega}_{2n})\| \to \infty$ *and* $\|\lambda_n(\boldsymbol{\omega}_{1n}-\boldsymbol{\omega}_{2n})\| \to \infty$*, then*

$$[\check{C}_n(\boldsymbol{\omega}_{1n}), \check{S}_n(\boldsymbol{\omega}_{1n}), \check{C}_n(\boldsymbol{\omega}_{2n}), \check{S}_n(\boldsymbol{\omega}_{2n})]' \xrightarrow{d} N\left[\mathbf{0}, \Sigma^{(a)}\right], \quad a.s. \quad (P_{\mathbf{X}}) \qquad \text{(III.4.9)}$$

where $\Sigma^{(a)} = \check{A}^{[0]} I_4$*, with* $2\check{A}^{[0]} = c_*^{-1} I_\psi + K \cdot (2\pi)^d \psi(\mathbf{0})$.

(b) *Suppose that* $\|\lambda_n(\boldsymbol{\omega}_{1n}+\boldsymbol{\omega}_{2n})\| \to \infty$ *but* $\lambda_n(\boldsymbol{\omega}_{1n}-\boldsymbol{\omega}_{2n}) \to \mathbf{z}_{12} \in I\!R^d$*. Then, (III.4.9) holds with* $\Sigma^{(a)}$ *replaced by* $\Sigma^{(b)}$*, where*

$$\Sigma^{(b)} = \begin{bmatrix} \check{A}_1^{[0]} & 0 & \check{A}_2^{[0]} & \check{A}_3^{[0]} \\ & \check{A}_1^{[0]} & -\check{A}_3^{[0]} & \check{A}_2^{[0]} \\ & & \check{A}_1^{[0]} & 0 \\ & & & \check{A}_1^{[0]} \end{bmatrix},$$

with $2\check{A}_1^{[0]} = c_*^{-1} I_\psi + K \cdot (2\pi)^d \psi(\mathbf{0})$, $2\check{A}_2^{[0]} = c_*^{-1} I_\psi K_1(\mathbf{z}_{12}) + K_3(\mathbf{z}_{12})(2\pi)^d \psi(\mathbf{0})$, $2\check{A}_3^{[0]} = c_*^{-1} I_\psi K_2(\mathbf{z}_{12}) + K_4(\mathbf{z}_{12})(2\pi)^d \psi(\mathbf{0})]$.

(c) *Suppose that* $\lambda_n(\boldsymbol{\omega}_{1n}+\boldsymbol{\omega}_{2n}) \to \mathbf{y}_{12} \in I\!R^d$ *and* $\lambda_n(\boldsymbol{\omega}_{1n}-\boldsymbol{\omega}_{2n}) \to \mathbf{z}_{12} \in I\!R^d$*. Then, (III.4.9) holds with* $\Sigma^{(a)}$ *replaced by* $\Sigma^{(c)}$*, where the elements of* $\Sigma^{(c)}$ *are given by*

$$\begin{aligned}
\Sigma_{11}^{(c)} &= (c_*^{-1} I_\psi/2)\{K_1(2\mathbf{y}_1)+1\} + (2\pi)^d(\psi(\mathbf{0})/2)\{K_3(2\mathbf{y}_1)+K\}, \\
\Sigma_{12}^{(c)} &= c_*^{-1} I_\psi K_2(2\mathbf{y}_1) + (2\pi)^d \psi(\mathbf{0}) K_4(2\mathbf{y}_1), \\
\Sigma_{13}^{(c)} &= c_*^{-1} I_\psi \{K_1(\mathbf{y}_{12}) + K_1(\mathbf{z}_{12})\} + (2\pi)^d \psi(\mathbf{0})\{K_3(\mathbf{y}_{12}) + K_3(\mathbf{z}_{12})\}, \\
\Sigma_{14}^{(c)} &= c_*^{-1} I_\psi \{K_2(\mathbf{y}_{12}) - K_2(\mathbf{z}_{12})\} + (2\pi)^d \psi(\mathbf{0})\{K_4(\mathbf{y}_{12}) - K_4(\mathbf{z}_{12})\}, \\
\Sigma_{22}^{(c)} &= (c_*^{-1} I_\psi/2)\{1-K_1(2\mathbf{y}_1)\} + (2\pi)^d(\psi(\mathbf{0})/2)\{K-K_3(2\mathbf{y}_1)\}, \\
\Sigma_{23}^{(c)} &= c_*^{-1} I_\psi \{K_2(\mathbf{y}_{12}) + K_2(\mathbf{z}_{12})\} + (2\pi)^d \psi(\mathbf{0})\{K_4(\mathbf{y}_{12}) + K_4(\mathbf{z}_{12})\}, \\
\Sigma_{24}^{(c)} &= c_*^{-1} I_\psi \{K_1(\mathbf{z}_{12}) - K_1(\mathbf{y}_{12})\} + (2\pi)^d \psi(\mathbf{0})\{K_3(\mathbf{z}_{12}) - K_3(\mathbf{y}_{12})\}, \\
\Sigma_{33}^{(c)} &= (c_*^{-1} I_\psi/2)\{K_1(2\mathbf{y}_2)+1\} + (2\pi)^d(\psi(\mathbf{0})/2)\{K_3(2\mathbf{y}_2)+K)\},
\end{aligned}$$

$$\Sigma_{34}^{(c)} = c_*^{-1} I_\psi K_2(2\mathbf{y}_2) + (2\pi)^d \psi(\mathbf{0}) K_4(2\mathbf{y}_2),$$
$$\Sigma_{44}^{(c)} = (c_*^{-1} I_\psi/2)\{1 - K_1(2\mathbf{y}_2)\} + (2\pi)^d (\psi(\mathbf{0})/2)\{K - K_3(2\mathbf{y}_2)\},$$

where $\mathbf{y}_1 = (\mathbf{y}_{12} + \mathbf{z}_{12})/2$ *and* $\mathbf{y}_2 = (\mathbf{y}_{12} - \mathbf{z}_{12})/2$.

As in Theorem III.3.3, Theorem III.4.3 shows that asymptotic independence of DFTs continues to hold for “asymptotically distant” Fourier frequency sequences converging to zero. For “asymptotically close” frequencies converging to zero, DFTs are typically asymptotically dependent. In the special case of the rectangular sampling region with a sampling design $f \in \mathcal{F}$, asymptotic independence of every pair of distinct sine and cosine transforms continues to hold, provided the frequency sequences are of the form (III.4.8).

Remark As in the last section, there is a dual to part (b) of Theorem III.4.3, which is straight-forward to formulate. Also, if $\boldsymbol{\omega}_n = \mathbf{0}$ for all $n \geq 1$, $\check{S}_n(\boldsymbol{\omega}_n) = 0$ for all $n \geq 1$, while $\check{C}_n(\boldsymbol{\omega}_n) = n^{-1/2} \sum_{i=1}^n Z(\mathbf{s}_i) \to^d N(0, c_*^{-1} I_\psi + K(2\pi)^d \psi(\mathbf{0}))$, $a.s.$ $(P_{\mathbf{X}})$. (see Lahiri (2003b)).

III.4.3.1. Results for mean-corrected DFTs

For stochastic design, the mean corrected DFT is defined as follows:

$$\tilde{d}_n(\boldsymbol{\omega}) \equiv \lambda_n^{d/2} n^{-1} \sum_{j=1}^n [Z(\mathbf{s}_j) - \bar{Z}_n] \exp\left(\iota \boldsymbol{\omega}' \mathbf{s}_j\right), \quad \boldsymbol{\omega} \in I\!R^d \qquad \text{(III.4.10)}$$

where, $\bar{Z}_n = n^{-1} \sum_{j=1}^n Z(\mathbf{s}_j)$ is the sample mean. Similarly, we define,

$$\tilde{C}_n(\boldsymbol{\omega}) = \lambda_n^{d/2} n^{-1} \sum_{j=1}^n [Z(\mathbf{s}_j) - \bar{Z}_n] \cos\left(\boldsymbol{\omega}' \mathbf{s}_j\right),$$
$$\tilde{S}_n(\boldsymbol{\omega}) = \lambda_n^{d/2} n^{-1} \sum_{j=1}^n [Z(\mathbf{s}_j) - \bar{Z}_n] \sin\left(\boldsymbol{\omega}' \mathbf{s}_j\right), \qquad \text{(III.4.11)}$$

the mean corrected version of the cosine and the sine transforms of the data. The following result gives the asymptotic behavior of the DFTs in different cases treated in Theorems III.4.1-III.4.3.

Theorem III.4.4. *Suppose that* $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ *is a zero mean stationary random field satisfying Assumptions (A.1), (A.2) and (A.4). Also, suppose that* $\{\boldsymbol{\omega}_{1n}\}, \ldots,$ $\{\boldsymbol{\omega}_{rn}\}$ *are sequences satisfying (III.4.3).*

(a) If $\{\boldsymbol{\omega}_{1n}\}, \ldots, \{\boldsymbol{\omega}_{rn}\}$ *satisfy the conditions of Theorem III.4.1, then* $[\tilde{C}_n(\boldsymbol{\omega}_{1n}), \tilde{S}_n(\boldsymbol{\omega}_{1n}), \ldots, \tilde{C}_n(\boldsymbol{\omega}_{rn}), \tilde{S}_n(\boldsymbol{\omega}_{rn})]$ *has the same limit distribution as that of the mean uncorrected version given by Theorem III.4.1.*

(b) Suppose that the sequences $\{\boldsymbol{\omega}_{1n}\}$ *and* $\{\boldsymbol{\omega}_{2n}\}$ *satisfy the conditions of one of the two parts of Theorem III.4.2. Then* $[\tilde{C}_n(\boldsymbol{\omega}_{1n}), \tilde{S}_n(\boldsymbol{\omega}_{1n}), \tilde{C}_n(\boldsymbol{\omega}_{2n}), \tilde{S}_n(\boldsymbol{\omega}_{2n})]$ *has the same limit distribution as its mean uncorrected version given by the respective part of Theorem III.4.2.*

(c) Suppose that the sequences $\{\boldsymbol{\omega}_{1n}\}$ *and* $\{\boldsymbol{\omega}_{2n}\}$ *satisfy the conditions of any one of the three parts Theorem III.4.3. Then,*

$$[\tilde{C}_n(\boldsymbol{\omega}_{1n}), \tilde{S}_n(\boldsymbol{\omega}_{1n}), \tilde{C}_n(\boldsymbol{\omega}_{2n}), \tilde{S}_n(\boldsymbol{\omega}_{2n})] \to^d N[\mathbf{0}, \tilde{\Sigma}],\ a.s.\,(P_{\mathbf{X}})$$

where $\tilde{\Sigma} = \Sigma^i - \Sigma_0$ *for the ith part,* $i = (a), (b), (c)$ *and*

$$\Sigma_0 = \sigma_\infty^2 \begin{bmatrix} K_1^2(\mathbf{y}_1) & 2K_1(\mathbf{y}_1)K_2(\mathbf{y}_1) & 2K_1(\mathbf{y}_1)K_1(\mathbf{y}_2) & 2K_1(\mathbf{y}_1)K_2(\mathbf{y}_2) \\ & K_2^2(\mathbf{y}_1) & 2K_1(\mathbf{y}_2)K_2(\mathbf{y}_1) & 2K_2(\mathbf{y}_1)K_2(\mathbf{y}_2) \\ & & K_1^2(\mathbf{y}_2) & 2K_1(\mathbf{y}_2)K_2(\mathbf{y}_2) \\ & & & K_2^2(\mathbf{y}_2) \end{bmatrix},$$

with $2\sigma_\infty^2 = c_*^{-1} I_\psi + K \cdot (2\pi)^d \psi(\mathbf{0})$.

Thus, as in the fixed design case, the asymptotic distributions of the sine and cosine transforms with mean correction remain unchanged in all cases where the

Fourier frequencies converge to a nonzero limit. However, for frequency sequences converging to the zero frequency, the asymptotic covariance can be different; the correction factor Σ_0 in the stochastic design case depends on the spatial sampling density $f(\cdot)$.

III.5. Some implications of the main results

The results on asymptotic joint distribution of the DFTs have some important implications for various frequency domain statistical inference methodologies. For example, the formulation of the frequency domain bootstrap (FDB) (see Franke and Härdle (1992)) critically depends on the asymptotic independence of the *full* set of DFTs. The results of Sections III.3 and III.4 show that for a sampling region of a general shape and/or for a general sampling density, the DFTs at asymptotically close ordinates are asymptotically correlated. As a result, formulation of the spatial version of the FDB must take into account the geometry of the sampling region under both the designs (i.e., fixed and stochastic) and, in addition, the non-uniformity of the sampling density in the stochastic design case.

Next consider the problem of non-parametric estimation of the spectral density and the auto-covariance function of the spatial process $\{Z(\cdot)\}$. In the regularly spaced data-sites case, the analogs of the standard time series formulas and bounds on the covariance between the DFTs (which is $O(n^{-1})$ where n is the sample size; See Priestley (1981)) no longer holds for a sampling region of a general shape, as the asymptotic correlation between neighboring DFTs do not vanish. As a result, consistency of the standard spectral density estimators based on kernel-smooth of the sample periodograms need not hold. The situation gets worse in the case of irregularly spaced data-sites, as in this case, not only the geometry of the sampling region plays a crucial role, but the sampling density $f(\cdot)$ has a nontrivial effect on the asymptotic distribution. For consistent estimation of the spectral density in the stochastic design case, one must also explicitly estimate various functionals of $f(\cdot)$ (e.g., the constant K, and the functions $K_1(\cdot), \ldots, K_4(\cdot)$) appearing in the asymptotic covariance matrix of the DFTs (see Theorems III.4.2 and III.4.3). Further, between the two asymptotic structures under the stochastic design case, the problem of estimating the spectral density and the auto-covariance function of the $Z(\cdot)$-process is *trickier* in the PID case. This is because under PID, $n/\lambda_n \to c_* \in (0, \infty)$ and the term $c_*^{-1} I_\psi$ in the asymptotic covariance matrices must be explicitly estimated. This observation has important implications for the popular nonparametric estimator of the auto-covariance function given by Hall and Patil (1994). Indeed, consistency of Hall and Patil (1994)'s estimator under the stochastic design is proved only under the MID asymptotic structure. Because of the presence of the extra term $c_*^{-1} I_\psi$, its consistency is no longer guaranteed under the PID case.

III.6. Proofs of the results from Section III.3

III.6.1. Preliminaries

Let $\mathcal{U} = [0, 1)^d$ denote the unit cube in $I\!R^d$. Let the autocovariance function of the stationary random field $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ be given by $\rho(\mathbf{s}) = cov(Z(\mathbf{s}), Z(\mathbf{0})), \mathbf{s} \in I\!R^d$. Then,

$$\begin{aligned} \rho(\mathbf{s}) &= \int_{I\!R^d} \exp(\iota \mathbf{s}' \boldsymbol{\omega}) \psi(\boldsymbol{\omega}) d\boldsymbol{\omega}, \ \mathbf{s} \in I\!R^d, \text{ and} \\ \rho(\Delta \mathbf{i}) &= \int_{\Pi_\Delta} \exp(\iota [\Delta \mathbf{i}]' \boldsymbol{\omega}) \psi_\Delta(\boldsymbol{\omega}) d\boldsymbol{\omega}, \ \mathbf{i} \in \mathbb{Z}^d. \end{aligned}$$

Let $\nu_\Delta(\cdot)$ denote the uniform distribution on $\Delta^{-1}\mathcal{D}_0$. Recall that the characteristic function of $\nu_\Delta(\cdot)$ is given by,

$$\int \exp(\iota \mathbf{t}' \mathbf{x}) d\nu_\Delta(d\mathbf{x}) = \phi_1(\mathbf{t}) + \iota \phi_2(\mathbf{t}), \ \mathbf{t} \in I\!R^d.$$

Let $\mathbf{e}_1, \ldots, \mathbf{e}_d$ denote the standard basis of $I\!R^d$. Thus, $\mathbf{e}_1 = (1, 0, \ldots, 0)'$, $\mathbf{e}_2 = (0, 1, 0, \ldots, 0)'$, etc. Let $C, C(\cdot)$ denote generic positive constants that depend on their arguments (if any), but not on n. Also, unless otherwise specified, limits in the order symbols are taken by letting $n \to \infty$.

The first lemma is a CLT for weighted sums of the form $\sum_{i=1}^{n} w_n(\mathbf{s}_i) Z(\mathbf{s}_i)$ under the deterministic spatial asymptotic framework as discussed earlier, where $w_n(\cdot)$ is a non-random weight function.

Lemma III.6.1. *Let $\{Z(\mathbf{s}) : \mathbf{s} \in I\!R^d\}$ be a zero mean stationary random field satisfying (A.1) and (A.2) and let $w_n(\cdot) : \mathcal{D}_n \to I\!R$ be a non-random bounded weight function. Suppose that there exists a function $Q : I\!R^d \to I\!R$ such that for any $\mathbf{h}_n \to \mathbf{h}$ in $I\!R^d$,*

$$\lim_{n\to\infty} c_n^{-2} \sum_{i:\mathbf{s}_i, \mathbf{s}_i+\mathbf{h}_n \in \mathcal{D}_n} w_n(\mathbf{s}_i) w_n(\mathbf{s}_i + \mathbf{h}_n) = Q(\mathbf{h}). \qquad \text{(III.6.1)}$$

where $c_n^2 = \sum_{i=1}^{N_n} w_n(\mathbf{s}_i)^2$. Let $\{\eta_n\}_{n\geq 1} \subset (0, \infty)$ be a nonrandom sequence of real numbers such that $\eta_n \equiv 1$ for all $n \geq 1$ in the PID case and $\eta_n \downarrow 0$ as $n \to \infty$ in the MID case. Suppose that

$$\max\{w_n^2(\mathbf{s}) : \mathbf{s} \in \mathcal{D}_n\} \lambda_n^d \eta_n^{-d} c_n^{-2} = O(1). \qquad \text{(III.6.2)}$$

Then, with $N_n \equiv |\{\mathbf{j} : \mathbf{j} \in \mathbb{Z}^d, \eta_n \mathbf{j}/\lambda_n \in \Delta^{-1}\mathcal{D}_0\}|$,

$$\eta_n^{d/2} c_n^{-1} \sum_{i=1}^{N_n} w_n(\mathbf{s}_i) Z(\mathbf{s}_i) \xrightarrow{d} N(0, \sigma_\infty^2) \qquad \text{(III.6.3)}$$

where $\sigma_\infty^2 = \sum_{\mathbf{i}\in\mathbb{Z}^d} \rho(\Delta\mathbf{i}) Q(\Delta\mathbf{i})$ for the PID case and $\sigma_\infty^2 = (\prod_{i=1}^{d} \delta_i)^{-1} \int_{I\!R^d} \rho(\mathbf{x}) Q(\mathbf{x}) d\mathbf{x}$ for the MID case.

Proof To prove the lemma, we will verify the conditions of Theorem 4.3 of Lahiri (2003b) for the PID case and Theorem 4.2 of Lahiri (2003a) for the MID case. First note that under Assumption (A.1), since $\gamma_1(\cdot)$ is monotone (and left-continuous), there exists a $t_0 \in (0, \infty)$ such that

$$\gamma_1(t) \le t^{-d(2+\tau)/\tau} \quad \text{for all} \quad t \ge t_0.$$

Write $\gamma_1^0(t) \equiv t^{-d(2+\tau)/\tau}$ for all $t > 0$. Since the weight function $w_n(\cdot)$ is bounded, by Remark 4.1 and Proposition 4.1 of Lahiri (2003b), it is enough to verify that

$$\gamma_2(t) = o\Big([f_1^{-1}(t)]^d / [t\gamma_1^0(t) f_1^{-1}(t)]\Big) \quad \text{as} \quad t \to \infty, \tag{III.6.4}$$

where $f_1(t) = t^d \int_1^t y^{2d-1}\gamma_1^0(y)dy$, $t \in [1, \infty)$.

Write $\tau_0 = d(2+\tau)/\tau$. Then, $\tau_0 > 2d$ for $0 < \tau < 2$, $\tau_0 = 2d$ for $\tau = 2$, and $d < \tau_0 < 2d$ for $\tau \in (2, \infty)$. For $\tau \in (2, \infty)$, (III.6.4) follows from relation (4.4) of Lahiri (2003b) (with τ replaced by τ_0 and $a = 0$ therein). Next, check that for $0 < \tau < 2$,

$$f_1(t) = Ct^d(1 + o(1)) \quad \text{as} \quad t \to \infty.$$

and for $\tau = 2$,

$$f_1(t) = [t^d \log t](1 + o(1)) \quad \text{as} \quad t \to \infty.$$

Consider $\tau = 2$ first. In this case, $f_1^{-1}(t) = d^{1/d}[t/\log t]^{1/d}(1 + o(1))$ as $t \to \infty$ and hence,

$$[f_1^{-1}(t)]^d / [t\gamma_1^0(t) f_1^{-1}(t)] = t^2[\log t]^{-3}(1 + o(1)) \quad \text{as} \quad t \to \infty.$$

Hence, (III.6.4) holds. One can similarly establish (III.6.4) for the case $0 < \tau < 2$ using the growth rate of $f_1(\cdot)$ above, as in (4.5) of Lahiri (2003b). The lemma now follows from Theorems 4.2 and 4.3 of Lahiri (2003b).

Lemma III.6.2. *Suppose that $\mathcal{D}_0$ be a Borel subset of $(-1/2, 1/2]^d$ such that the d-dimensional Lebesgue measure of its boundary $\partial\mathcal{D}_0$ is zero. Let $\{\eta_n\}_{n\geq 1}$ and N_n be as in Lemma III.6.1. Also let $J_n = \{\mathbf{j} : \mathbf{j} \in \mathbb{Z}^d, \eta_n \mathbf{j}/\lambda_n \in \Delta^{-1}\mathcal{D}_0\}$ and $N_n = |J_n|$. Then*

(a) For any $K \in (0, \infty)$,

$$\sup_{\|\mathbf{z}\|\leq K} \left| \eta_n^d \lambda_n^{-d} \sum_{\mathbf{j}\in J_n} \exp(\iota 2\pi \mathbf{z}'\mathbf{j}\eta_n/\lambda_n) - \int_{\Delta^{-1}\mathcal{D}_0} \exp(\iota 2\pi \mathbf{z}'\mathbf{x}) d\mathbf{x} \right| \to 0 \quad as \quad n \to \infty.$$

(b) (A discrete version of the Reimann-Lebesgue lemma in d-dimension): Let $\{\mathbf{z}_n\}_{n\geq 1} \subset \mathbb{R}^d$ be a sequence satisfying $\|\mathbf{z}_n\|^{-1} = o(1)$ and $\limsup_{n\to\infty} |\mathbf{e}_i'\mathbf{z}_n\eta_n| /\lambda_n < 1/2$ for all $i = 1, \ldots, d$. Then,

$$\left| N_n^{-1} \sum_{\mathbf{j}\in J_n} \exp(\iota 2\pi \mathbf{z}_n'\mathbf{j}/\lambda_n) \right| \to 0 \quad as \quad n \to \infty.$$

Proof For simplicity, first consider the PID case. Then, $\eta_n \equiv 1$. Let $J_{1n} = \{\mathbf{j} : (\mathbf{j} + \mathcal{U})\lambda_n^{-1} \subset \Delta^{-1}\mathcal{D}_0\}$ and let $J_{2n} = J_n \setminus J_{1n}$. Also for $k = 1, 2$, write $\sum_k$ for summation over $\mathbf{j} \in J_{kn}$. Part (a) is a uniform version of the Riemann sum approximation to integrals and can be proved easily. Here we give an outline of the proof for completeness. For any $\mathbf{z} \in \mathbb{R}^d$ with $||\mathbf{z}|| \leq K$,

$$\begin{aligned}
&\left| \lambda_n^{-d} \sum_{\mathbf{j}\in J_n} \exp(\iota 2\pi \mathbf{z}'\mathbf{j}/\lambda_n) - \int_{\Delta^{-1}\mathcal{D}_0} \exp(\iota 2\pi \mathbf{z}'\mathbf{x}) d\mathbf{x} \right| \\
&\leq \sum\nolimits_1 \left| \int_{(\mathbf{j}+\mathcal{U})/\lambda_n} \left[\exp(\iota 2\pi \mathbf{z}'\mathbf{j}/\lambda_n) - \exp(\iota 2\pi \mathbf{z}'\mathbf{x}) \right] d\mathbf{x} \right| \\
&\quad + \sum\nolimits_2 \left| \int_{[(\mathbf{j}+\mathcal{U})/\lambda_n]\cap\Delta^{-1}\mathcal{D}_0} \exp(\iota 2\pi \mathbf{z}'\mathbf{x}) d\mathbf{x} \right| + \lambda_n^{-d}|J_{2n}| \\
&\leq \sum\nolimits_1 \int_{(\mathbf{j}+\mathcal{U})/\lambda_n} (2\pi ||\mathbf{z}|| \sqrt{d}/\lambda_n) d\mathbf{x} + 2\lambda_n^{-d}|J_{2n}| \\
&\leq K(2\pi\sqrt{d})|J_{1n}|\lambda_n^{-d-1} + 2\lambda_n^{-d}|J_{2n}|.
\end{aligned}$$

Part (a) follows from this.

Nest consider part (b). Let $\phi(\cdot)$ denote the characteristic function of the uniform distribution on the unit cube $\mathcal{U}$. Then, it is easy to check that for any $\epsilon \in (0, 1/2)$,

$$\inf \{|\phi(\mathbf{t})| : \mathbf{t} \in [-\pi + \epsilon, \pi - \epsilon]^d\} \in (0, 1]. \tag{III.6.5}$$

Also,

$$\begin{aligned} &\lambda_n^{-d} \sum_{\mathbf{j} \in J_{1n}} \exp(\iota 2\pi \mathbf{z}_n^{'} \mathbf{j}/\lambda_n) \phi(2\pi \mathbf{z}_n/\lambda_n) \\ &= \lambda_n^{-d} \sum_{\mathbf{j} \in J_{1n}} \int_{\mathcal{U}} \exp(\iota 2\pi \mathbf{z}_n^{'}[\mathbf{j} + \mathbf{x}]/\lambda_n) d\mathbf{x} \\ &= \sum_{\mathbf{j} \in J_{1n}} \int_{(\mathbf{j}+\mathcal{U})/\lambda_n} \exp(\iota 2\pi \mathbf{z}_n^{'} \mathbf{x}) d\mathbf{x}. \end{aligned} \tag{III.6.6}$$

Hence by (III.6.5) and (III.6.6) and the Riemann Lebesgue lemma, for any $\{\mathbf{z}_n\}_{n \geq 1}$ satisfying the conditions of part (b), we have,

$$\begin{aligned} &\left| \lambda_n^{-d} \sum_{\mathbf{j} \in J_n} \exp(\iota 2\pi \mathbf{z}_n^{'} \mathbf{j}/\lambda_n) \right| \\ &\leq \lambda_n^{-d} \left| \sum_{\mathbf{j} \in J_{1n}} \exp(\iota 2\pi \mathbf{z}_n^{'} \mathbf{j}/\lambda_n) \right| + |J_{2n}| \lambda_n^{-d} \\ &\leq \left| \phi(2\pi \mathbf{z}_n \lambda_n^{-1}) \right|^{-1} \left\{ \left| \int_{\Delta^{-1}\mathcal{D}_0} \exp(\iota 2\pi \mathbf{z}_n^{'} \mathbf{x}) d\mathbf{x} \right| + |J_{2n}| \lambda_n^{-d} \right\} + |J_{2n}| \lambda_n^{-d} \\ &= o(1). \end{aligned}$$

In the MID case, both parts of Lemma III.6.2 can be proved by retracing the above steps with λ_n replaced by $\eta_n^{-1} \lambda_n$ in every occurrence. We omit the routine details.

Remark Note that the conditions imposed on the boundary of $\mathcal{D}_0$ in Section III.2.1 implies that the d-dimensional Lebesgue measure of its boundary $\partial \mathcal{D}_0$ is zero.

Remark For the MID case, analogs of parts (a) and (b) hold, provided in each appearance, λ_n is replaced by $\lambda_n \eta_n^{-1}$. Thus, J_n should be replaced by $J_n^{[1]} \equiv \{\mathbf{j} \in \mathbb{Z}^d : \eta \mathbf{j}/\lambda_n \in \Delta^{-1}\mathcal{D}_0\}$ and λ_n^{-d} in part (a) is replaced by $\lambda_n^{-d} \eta_n^d$. In addition, for part (b),

N_n is now the sample size under MID.

III.6.2. Proof of Theorem III.3.1

Fix $a_1, \cdots, a_r, b_1, \cdots, b_r \in I\!R$ with $\sum_{i=1}^r (a_i^2 + b_i^2) \neq 0$. Then, we may write,

$$\begin{aligned}
&\sum_{p=1}^{r} [a_p C_n^P(\boldsymbol{\omega}_{pn}) + b_p S_n^P(\boldsymbol{\omega}_{pn})] \\
&= \sum_{\mathbf{j} \in J_n} Z(\Delta\mathbf{j}) \left[N_n^{-1/2} \sum_{p=1}^{r} \{a_p \text{cos}(\boldsymbol{\omega}_{pn}^{'} \Delta\mathbf{j}) + b_p \text{sin}(\boldsymbol{\omega}_{pn}^{'} \Delta\mathbf{j})\} \right] \\
&= \sum_{\mathbf{j} \in J_n} Z(\Delta\mathbf{j}) w_n(\Delta\mathbf{j}), \text{ (say).} \qquad \text{(III.6.7)}
\end{aligned}$$

Hence, to prove Theorem III.3.1, it is enough to establish the asymptotic distribution of the weighted sum in (III.6.7). Note that, by (III.3.4), $\boldsymbol{\omega}_{jn}^{'} \Delta\mathbf{j} = 2\pi\lambda_n^{-1}\mathbf{k}_{jn}^{'}\mathbf{j}$ and hence,

$$\begin{aligned}
&\sum_{\mathbf{j} \in J_n} w_n^2(\Delta\mathbf{j}) \\
&= (2N_n)^{-1} \sum_{p=1}^{r} \sum_{q=1}^{r} \Bigg[a_p a_q \sum_{\mathbf{j} \in J_n} \left\{ \cos(2\pi(\mathbf{k}_{pn} + \mathbf{k}_{qn})^{'}\mathbf{j}/\lambda_n) + \cos(2\pi(\mathbf{k}_{pn} - \mathbf{k}_{qn})^{'}\mathbf{j}/\lambda_n) \right\} \\
&\quad + a_p b_q \sum_{\mathbf{j} \in J_n} \left\{ \sin(2\pi(\mathbf{k}_{pn} + \mathbf{k}_{qn})^{'}\mathbf{j}/\lambda_n) - \sin(2\pi(\mathbf{k}_{pn} - \mathbf{k}_{qn})^{'}\mathbf{j}/\lambda_n) \right\} \\
&\quad + a_q b_p \sum_{\mathbf{j} \in J_n} \left\{ \sin(2\pi(\mathbf{k}_{pn} + \mathbf{k}_{qn})^{'}\mathbf{j}/\lambda_n) + \sin(2\pi(\mathbf{k}_{pn} - \mathbf{k}_{qn})^{'}\mathbf{j}/\lambda_n) \right\} \\
&\quad + b_p b_q \sum_{\mathbf{j} \in J_n} \left\{ \cos(2\pi(\mathbf{k}_{pn} - \mathbf{k}_{qn})^{'}\mathbf{j}/\lambda_n) - \cos(2\pi(\mathbf{k}_{pn} + \mathbf{k}_{qn})^{'}\mathbf{j}/\lambda_n) \right\} \Bigg]. \qquad \text{(III.6.8)}
\end{aligned}$$

Since under the conditions of Theorem III.3.1, $2\pi(\mathbf{k}_{pn} \pm \mathbf{k}_{qn})/\lambda_n$ converges to a point in $\Pi_\Delta^0 \setminus \{\mathbf{0}\}$ for all $1 \leq p \neq q \leq r$, by Lemma III.6.2, part (b), it follows that

$$\sum_{\mathbf{j} \in J_n} w_n^2(\Delta\mathbf{j}) \to (1/2) \sum_{p=1}^{r} (a_p^2 + b_p^2).$$

Also, for any $\mathbf{h}_n \to \mathbf{h} \in I\!R^d$, by Lemma III.6.2, part (a) and arguments similar to (III.6.8),

$$\lim_{n\to\infty} \sum_{i:\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}_n\in\mathcal{D}_n} w_n(\mathbf{s}_i)w_n(\mathbf{s}_i+\mathbf{h}_n) = (1/2)\sum_{p=1}^{r}(a_p^2+b_p^2)\cos(\boldsymbol{\omega}_p'\mathbf{h}). \qquad \text{(III.6.9)}$$

Hence, by (III.6.8) and (III.6.9), condition (III.6.1) of Lemma III.6.1 holds with

$$Q(\mathbf{h}) = [\sum_{p=1}^{r}(a_p^2+b_p^2)\cos(\boldsymbol{\omega}_p'\mathbf{h})]\Big/\sum_{p=1}^{r}(a_p^2+b_p^2).$$

Further, by (III.6.8) and the boundedness of $\cos(\cdot)$ and $\sin(\cdot)$, condition (III.6.2) of Lemma III.6.1 holds. Next note that by the inversion formula,

$$\psi_\Delta(\boldsymbol{\omega}) = [vol(\Pi_\Delta)]^{-1}\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho(\Delta\mathbf{j})\exp(-2\boldsymbol{\omega}'\Delta\mathbf{j}),\ \boldsymbol{\omega}\in\Pi_\Delta. \qquad \text{(III.6.10)}$$

Hence, by Lemma III.6.1, it follows that,

$$\begin{aligned}&\sum_{p=1}^{r}[a_pC_n^P(\boldsymbol{\omega}_{pn})+b_pS_n^P(\boldsymbol{\omega}_{pn})]\\ &\quad\to^d N\left(0,\sum_{\mathbf{j}\in\mathbb{Z}^d}\rho(\Delta\mathbf{j})Q(\Delta\mathbf{j})[(1/2)\sum_{p=1}^{r}(a_p^2+b_p^2)]\right),\\ &\quad= N\left(0,(1/2)\sum_{p=1}^{r}(a_p^2+b_p^2)\psi_\Delta(\boldsymbol{\omega}_p)\,vol(\Pi_\Delta)\right).\end{aligned}$$

This completes the proof of Theorem III.3.1.

III.6.3. Proof of Theorem III.3.2

Fix $a_1, a_2, b_1, b_2 \in I\!R$ with $\sum_{i=1}^{2}(a_i^2+b_i^2)\neq 0$. Then, as in (III.6.7) we have,

$$\sum_{p=1}^{2}[a_pC_n^P(\boldsymbol{\omega}_{pn})+b_pS_n^P(\boldsymbol{\omega}_{pn})] = \sum_{\mathbf{j}\in J_n}Z(\Delta\mathbf{j})w_n(\Delta\mathbf{j}), \qquad \text{(III.6.11)}$$

where, $w_n(\mathbf{s}) = N_n^{-1/2}\sum_{p=1}^{2}\{a_p\cos(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}) + b_p\sin(\boldsymbol{\omega}_{pn}^{'}\mathbf{s})\}$. Note that for any $\mathbf{h}_n \to \mathbf{h} \in I\!R^d$,

$$\begin{aligned}
&\sum_{i:\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}_n\in\mathcal{D}_n} w_n(\mathbf{s}_i)w_n(\mathbf{s}_i+\mathbf{h}_n)\\
&= (2N_n)^{-1}\sum_{p=1}^{2}\sum_{q=1}^{2}\Bigg[a_pa_q\sum_{\mathbf{j}\in J_n}\Big\{\cos(2\pi(\mathbf{k}_{pn}+\mathbf{k}_{qn})^{'}\mathbf{j}\lambda_n^{-1}+\boldsymbol{\omega}_{qn}^{'}\mathbf{h}_n)+\\
&\quad\cos(2\pi(\mathbf{k}_{pn}-\mathbf{k}_{qn})^{'}\mathbf{j}\lambda_n^{-1}-\boldsymbol{\omega}_{qn}^{'}\mathbf{h}_n)\Big\}\\
&+a_pb_q\sum_{\mathbf{j}\in J_n}\Big\{\sin(2\pi(\mathbf{k}_{pn}+\mathbf{k}_{qn})^{'}\mathbf{j}\lambda_n^{-1}+\boldsymbol{\omega}_{qn}^{'}\mathbf{h}_n)+\sin(2\pi(\mathbf{k}_{qn}-\mathbf{k}_{pn})^{'}\mathbf{j}\lambda_n^{-1}+\boldsymbol{\omega}_{qn}^{'}\mathbf{h}_n)\Big\}\\
&+a_qb_p\sum_{\mathbf{j}\in J_n}\Big\{\sin(2\pi(\mathbf{k}_{pn}+\mathbf{k}_{qn})^{'}\mathbf{j}\lambda_n^{-1}+\boldsymbol{\omega}_{qn}^{'}\mathbf{h}_n)-\sin(2\pi(\mathbf{k}_{qn}-\mathbf{k}_{pn})^{'}\mathbf{j}\lambda_n^{-1}+\boldsymbol{\omega}_{qn}^{'}\mathbf{h}_n)\Big\}\\
&+b_pb_q\sum_{\mathbf{j}\in J_n}\Big\{\cos(2\pi(\mathbf{k}_{qn}-\mathbf{k}_{pn})^{'}\mathbf{j}\lambda_n^{-1}+\boldsymbol{\omega}_{qn}^{'}\mathbf{h}_n)\\
&\quad-\cos(2\pi(\mathbf{k}_{pn}+\mathbf{k}_{qn})^{'}\mathbf{j}\lambda_n^{-1}+\boldsymbol{\omega}_{qn}^{'}\mathbf{h}_n)\Big\}\Bigg] \qquad \text{(III.6.12)}
\end{aligned}$$

Note that, under the conditions of the theorem, for every pair $p, q \in \{1, 2\}$, $\{2\pi[\mathbf{k}_{pn} + \mathbf{k}_{qn}]\}$ converges to a non-zero element in $(-\pi/2, \pi/2)^d$. Thus, when the frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are asymptotically distant (see (III.3.5)), by Lemma III.6.2(b),

$$\lim_{n\to\infty}\sum_{i:\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}_n\in\mathcal{D}_n} w_n(\mathbf{s}_i)w_n(\mathbf{s}_i+\mathbf{h}_n) = (1/2)\sum_{p=1}^{2}(a_p^2+b_p^2)\cos(\boldsymbol{\omega}^{'}\mathbf{h}).$$

On the other hand, when $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ satisfy (III.3.6), by Lemma III.6.2, (both parts (a) and (b)),

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i:\mathbf{s}_i,\mathbf{s}_i+\mathbf{h}_n\in\mathcal{D}_n} w_n(\mathbf{s}_i)w_n(\mathbf{s}_i+\mathbf{h}_n)\\
&= (1/2)\sum_{p=1}^{2}\sum_{q=1}^{2}\Bigg[a_pa_q\int\cos(2\pi\boldsymbol{\ell}_{pq}^{'}\mathbf{x}-\boldsymbol{\omega}^{'}\mathbf{h})d\nu_\Delta(\mathbf{x})+a_pb_q\int\sin(2\pi\boldsymbol{\ell}_{21}^{'}\mathbf{x}+\boldsymbol{\omega}^{'}\mathbf{h})\\
&d\nu_\Delta(\mathbf{x})-a_qb_p\int\sin(2\pi\boldsymbol{\ell}_{21}^{'}\mathbf{x}+\boldsymbol{\omega}^{'}\mathbf{h})d\nu_\Delta(\mathbf{x})+b_pb_q\int\cos(2\pi\boldsymbol{\ell}_{21}^{'}\mathbf{x}+\boldsymbol{\omega}^{'}\mathbf{h})d\nu_\Delta(\mathbf{x})\Bigg],
\end{aligned}$$

where, $\boldsymbol{\ell}_{12} = \boldsymbol{\ell} = -\boldsymbol{\ell}_{21}$ and $\boldsymbol{\ell}_{11} = \boldsymbol{\ell}_{22} = 0$. The proof of Theorem III.3.2 can now be completed using Lemma III.6.1, the inversion formula (III.6.10) and the arguments in the proof of Theorem III.3.1.

III.6.4. Proof of Theorem III.3.3

Fix $a_1, a_2, b_1, b_2 \in I\!R$ with $\sum_{p=1}^{2}(a_p^2 + b_p^2) \neq 0$. Then, $\sum_{p=1}^{2}[a_p C_n^P(\boldsymbol{\omega}_{pn}) + b_p S_n^P(\boldsymbol{\omega}_{pn})]$ can be expressed in the from (III.6.11) and the corresponding weight function $w_n(\cdot)$ satisfies (III.6.12). For part (c), using Lemma III.6.2 in (III.6.12), for any $\mathbf{h}_n \to \mathbf{h} \in I\!R^d$, we get

$$
\begin{aligned}
&\lim_{n\to\infty} \sum_{i:\mathbf{s}_i, \mathbf{s}_i+\mathbf{h}_n \in \mathcal{D}_n} w_n(\mathbf{s}_i) w_n(\mathbf{s}_i + \mathbf{h}_n) \\
&= 2^{-1} \sum_{p,q=1}^{2} \Bigg[a_p a_q \left\{ \int \cos(2\pi(\mathbf{k}_p + \mathbf{k}_q)'\mathbf{x}) d\nu_\Delta(\mathbf{x}) + \int \cos(2\pi(\mathbf{k}_p - \mathbf{k}_q)'\mathbf{x}) d\nu_\Delta(\mathbf{x}) \right\} \\
&+ a_p b_q \left\{ \int \sin(2\pi(\mathbf{k}_p + \mathbf{k}_q)'\mathbf{x}) d\nu_\Delta(\mathbf{x}) + \int \sin(2\pi(\mathbf{k}_q - \mathbf{k}_p)'\mathbf{x}) d\nu_\Delta(\mathbf{x}) \right\} \\
&+ a_q b_p \left\{ \int \sin(2\pi(\mathbf{k}_p + \mathbf{k}_q)'\mathbf{x}) d\nu_\Delta(\mathbf{x}) - \int \sin(2\pi(\mathbf{k}_q - \mathbf{k}_p)'\mathbf{x}) d\nu_\Delta(\mathbf{x}) \right\} \\
&+ b_p b_q \left\{ \int \cos(2\pi(\mathbf{k}_q - \mathbf{k}_p)'\mathbf{x}) d\nu_\Delta(\mathbf{x}) - \int \cos(2\pi(\mathbf{k}_p + \mathbf{k}_q)'\mathbf{x}) d\nu_\Delta(\mathbf{x}) \right\} \Bigg],
\end{aligned}
$$

where, $\mathbf{k}_i = \lim_{n\to\infty} [\lambda_n \Delta \boldsymbol{\omega}_{in}/2\pi]$, $i = 1, 2$ (which exist as $\lim_{n\to\infty} \lambda_n \boldsymbol{\omega}_{in}$ exist for both $i = 1, 2$ and are finite). The proof of part (c) now can be completed by applying the inversion formula (III.6.10) and the central limit theorem of Lemma III.6.1. Parts (a) and (b) can be proved by repeating the steps in the proofs of parts (a) and (b) of Theorem III.3.2 respectively, setting $\boldsymbol{\omega} = \mathbf{0}$. We omit the details to save space.

III.6.5. Proof of Theorem III.3.4

Fix $a_1, \cdots, a_r, b_1, \cdots, b_r \in I\!R$ with $\sum_{i=1}^r (a_i^2 + b_i^2) \neq 0$. Note that,

$$\sum_{p=1}^{r} [a_p \tilde{C}_n^P(\boldsymbol{\omega}_{pn}) + b_p \tilde{S}_n^P(\boldsymbol{\omega}_{pn})] = \sum_{p=1}^{r} [a_p C_n^P(\boldsymbol{\omega}_{pn}) + b_p S_n^P(\boldsymbol{\omega}_{pn})] - N_n^{1/2} \bar{Z}_n \beta_n$$
$$= \sum_{i=1}^{N_n} Z(\mathbf{s}_i)[w_n(\mathbf{s}_i) - N_n^{-1/2}\beta_n] \tag{III.6.13}$$

where $w_n(\mathbf{s}) = N_n^{-1/2} \sum_{p=1}^r \{a_p \cos(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}) + b_p \sin(\boldsymbol{\omega}_{pn}^{'}\mathbf{s})\}$ and $\beta_n = N_n^{-1} \sum_{i=1}^{N_n} \sum_{p=1}^r \{a_p \cos(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}_i) + b_p \sin(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}_i)\}$. Now suppose, $\boldsymbol{\omega}_{pn} \to \boldsymbol{\omega} \in \Pi_\Delta^0 \setminus \{\mathbf{0}\}$, for all $p = 1, 2, \cdots, r$. Then by Lemma III.6.2(b), $\beta_n \to 0$. On the other hand, if $\lambda_n \boldsymbol{\omega}_{pn} \to \mathbf{y}_p \in I\!R^d$ for all $p = 1, 2, \cdots, r$, then by Lemma III.6.2(b),

$$\beta_n \to \beta \equiv \sum_{p=1}^{r} \{a_p \int \cos(\mathbf{y}_p^{'}\Delta\mathbf{x}) d\nu_\Delta(\mathbf{x}) + b_p \int \sin(\mathbf{y}_p^{'}\Delta\mathbf{x}) d\nu_\Delta(\mathbf{x})\}. \tag{III.6.14}$$

For any $\mathbf{h}_n \to \mathbf{h} \in I\!R^d$, let $Q(\mathbf{h}) = \lim_{n\to\infty} \sum_{i:\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}_n \in \mathcal{D}_n} w_n(\mathbf{s}_i) w_n(\mathbf{s}_i + \mathbf{h}_n)$. From the proofs of theorems III.3.1-III.3.3, it follows that under the hypothesis of each of the parts (a)-(c) of Theorem III.3.4, $Q(\mathbf{h})$ exists. Hence, for any $\mathbf{h}_n \to \mathbf{h} \in I\!R^d$,

$$\lim_{n\to\infty} \sum_{i:\mathbf{s}_i, \mathbf{s}_i + \mathbf{h}_n \in \mathcal{D}_n} \left(w_n(\mathbf{s}_i) - N_n^{-1/2}\beta_n\right)\left(w_n(\mathbf{s}_i + \mathbf{h}_n) - N_n^{-1/2}\beta_n\right)$$
$$= \begin{cases} Q(\mathbf{h}) & : \ \boldsymbol{\omega}_{pn} \to \boldsymbol{\omega} \in \Pi_\Delta^0 \setminus \{\mathbf{0}\}, \text{ for all } p = 1, \cdots, r, \\ Q(\mathbf{h}) - \beta^2 & : \ \boldsymbol{\omega}_{pn} \to \mathbf{0}, \text{ for all } p = 1, \cdots, r. \end{cases} \tag{III.6.15}$$

Note that,

$$\beta^2 = \sum_{p=1}^{r} \sum_{q=1}^{r} \Big[a_p a_q \tilde{\phi}_1(\mathbf{y}_p)\tilde{\phi}_1(\mathbf{y}_q) + a_p b_q \tilde{\phi}_1(\mathbf{y}_p)\tilde{\phi}_2(\mathbf{y}_q)$$
$$+ a_q b_p \tilde{\phi}_1(\mathbf{y}_q)\tilde{\phi}_2(\mathbf{y}_p) + b_p b_q \tilde{\phi}_2(\mathbf{y}_p)\tilde{\phi}_2(\mathbf{y}_q)\Big] \tag{III.6.16}$$

The result now follows from (III.6.13)-(III.6.16) and Lemma III.6.1.

III.6.6. Proof of Theorem III.3.5

The proof follows by retracing the above steps and employing Lemma III.6.1 and III.6.2 for the MID case along with the inversion formula,

$$\psi(\boldsymbol{\omega}) = (2\pi)^{-d} \int \rho(\mathbf{x}) \exp(-\iota \boldsymbol{\omega}' \mathbf{x}) d\mathbf{x}, \ \boldsymbol{\omega} \in \mathbb{R}^d. \tag{III.6.17}$$

We omit the routine details.

III.7. Proofs of the results from Section III.4

III.7.1. Preliminaries

Lemma III.7.1. *Suppose that* $\{Z(\mathbf{s}) : \mathbf{s} \in \mathbb{R}^d\}$ *is a zero mean stationary random field satisfying (A.1), (A.2) and (A.4) and*

$$\max\{w_n^2(\mathbf{s}) : \mathbf{s} \in \mathcal{D}_n\} s_n^{-2} = O(1), \tag{III.7.1}$$

where $s_n^2 = E w_n^2(\lambda_n \mathbf{X}_1)$*. Also, suppose that there exists a function* $Q_1(\cdot)$ *such that for all* $\mathbf{h} \in \mathbb{R}^d$*,*

$$\left[\int w^2(\lambda_n \mathbf{x}) f(\mathbf{x}) d\mathbf{x}\right]^{-1} \int w(\lambda_n \mathbf{x} + \mathbf{h}) w(\lambda_n \mathbf{x}) f^2(\mathbf{x}) d\mathbf{x} \to Q_1(\mathbf{h}) \text{ as } n \to \infty. \tag{III.7.2}$$

(i) If $n/\lambda_n^d \to c_* \in (0, \infty)$ *as* $n \to \infty$*, then,*

$$(n s_n^2)^{-1/2} \sum_{i=1}^n w_n(\mathbf{s}_i) Z(\mathbf{s}_i) \quad \xrightarrow{d} \quad N\left(0, \rho(\mathbf{0}) + c_* \int_{\mathbb{R}^d} \rho(\mathbf{x}) Q_1(\mathbf{x}) d\mathbf{x}\right) \ a.s. \ (P_{\mathbf{X}}).$$

(ii) If $n/\lambda_n^d \to \infty$ *as* $n \to \infty$*, then,*

$$(n^2 \lambda_n^{-d} s_n^2)^{-1/2} \sum_{i=1}^n w_n(\mathbf{s}_i) Z(\mathbf{s}_i) \quad \xrightarrow{d} \quad N\left(0, \int_{\mathbb{R}^d} \rho(\mathbf{x}) Q_1(\mathbf{x}) d\mathbf{x}\right) a.s. \ (P_{\mathbf{X}}).$$

Proof : See Lahiri (2003b).

Lemma III.7.2. *(Multivariate Riemann-Lebesgue Lemma): Let $f \in L^1(\mathbb{R}^d)$. Then,*

$$\lim_{\|\mathbf{t}\|\to\infty} \int_{\mathbb{R}^d} f(\mathbf{x}) cos(\mathbf{t}'\mathbf{x}) d\mathbf{x} \quad = 0 = \quad \lim_{\|\mathbf{t}\|\to\infty} \int_{\mathbb{R}^d} f(\mathbf{x}) sin(\mathbf{t}'\mathbf{x}) d\mathbf{x}.$$

Proof Follows by approximating f by a sequence of finite linear combinations of indicator functions of rectangular sets in $\mathbb{R}^d$ (which are dense in $L^1(\mathbb{R}^d)$) as in the one-dimensional case.

III.7.2. Proof of Theorem III.4.1

Fix $a_1, \cdots, a_r, b_1, \cdots, b_r \in \mathbb{R}$ with $\sum_{p=1}^r (a_p^2 + b_p^2) \neq 0$. Let $\hat{C}_n(\boldsymbol{\omega}) = n^{-1/2} \sum_{j=1}^n \cos(\boldsymbol{\omega}'\mathbf{s}_j) Z(\mathbf{s}_j)$ and $\hat{S}_n(\boldsymbol{\omega}) = n^{-1/2} \sum_{j=1}^n \sin(\boldsymbol{\omega}'\mathbf{s}_j) Z(\mathbf{s}_j)$, $\boldsymbol{\omega} \in \mathbb{R}^d$. Note that

$$\sum_{p=1}^r [a_p \hat{C}_n(\boldsymbol{\omega}_{pn}) + b_p \hat{S}_n(\boldsymbol{\omega}_{pn})] = \sum_{j=1}^n Z(\mathbf{s}_j) w_n(\mathbf{s}_j), \qquad \text{(III.7.3)}$$

where, $w_n(\mathbf{s}_j) = n^{-1/2} \sum_{p=1}^r \{a_p \cos(\boldsymbol{\omega}_{pn}'\mathbf{s}_j) + b_p \sin(\boldsymbol{\omega}_{pn}'\mathbf{s}_j)\}$. Proceeding similarly as (III.6.8), we may write

$$\begin{aligned}
&\int w_n^2(\lambda_n \mathbf{x}) f(\mathbf{x}) d\mathbf{x} \\
&= (2n)^{-1} \sum_{p=1}^r \sum_{q=1}^r \Bigg[a_p a_q \int \Big\{ \cos(\lambda_n (\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})'\mathbf{x}) + \cos(\lambda_n (\boldsymbol{\omega}_{pn} - \boldsymbol{\omega}_{qn})'\mathbf{x}) \Big\} f(\mathbf{x}) d\mathbf{x} \\
&\quad + a_p b_q \int \Big\{ \sin(\lambda_n (\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})'\mathbf{x}) - \sin(\lambda_n (\boldsymbol{\omega}_{pn} - \boldsymbol{\omega}_{qn})'\mathbf{x}) \Big\} f(\mathbf{x}) d\mathbf{x} \\
&\quad + a_q b_p \int \Big\{ \sin(\lambda_n (\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})'\mathbf{x}) + \sin(\lambda_n (\boldsymbol{\omega}_{pn} - \boldsymbol{\omega}_{qn})'\mathbf{x}) \Big\} f(\mathbf{x}) d\mathbf{x} \\
&\quad + b_p b_q \int \Big\{ \cos(\lambda_n (\boldsymbol{\omega}_{pn} - \boldsymbol{\omega}_{qn})'\mathbf{x}) - \cos(\lambda_n (\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})'\mathbf{x}) \Big\} f(\mathbf{x}) d\mathbf{x} \Bigg]. \qquad \text{(III.7.4)}
\end{aligned}$$

Hence under the conditions of Theorem III.4.1 and Lemma III.7.2, it follows that

$$n \int w_n^2(\lambda_n \mathbf{x}) f(\mathbf{x}) d\mathbf{x} \to (1/2) \sum_{p=1}^r (a_p^2 + b_p^2).$$

Also, for any $\mathbf{h} \in I\!R^d$, by Lemma III.7.2 and arguments similar to (III.7.4),

$$\lim_{n\to\infty} n \int w_n(\lambda_n \mathbf{x} + \mathbf{h}) w_n(\lambda_n \mathbf{x}) f^2(\mathbf{x}) d\mathbf{x} = (K/2) \sum_{p=1}^{r} (a_p^2 + b_p^2) \cos(\boldsymbol{\omega}_p' \mathbf{h}). \quad \text{(III.7.5)}$$

Hence, by (III.7.4) and (III.7.5), condition (III.7.2) holds with

$$Q_1(\mathbf{h}) = [K \sum_{p=1}^{r} (a_p^2 + b_p^2) \cos(\boldsymbol{\omega}_p' \mathbf{h})] \Big/ \sum_{p=1}^{r} (a_p^2 + b_p^2).$$

Further, by (III.7.4) and the boundedness of $\cos(\cdot)$ and $\sin(\cdot)$, condition (III.7.1) of Lemma III.7.1 holds. Next by the inversion formula as in (III.6.17), it follows that

$$\begin{aligned} &\sum_{p=1}^{r} [a_p \check{C}_n(\boldsymbol{\omega}_{pn}) + b_p \check{S}_n(\boldsymbol{\omega}_{pn})] \\ &\quad \to^d N\left(\mathbf{0}, (c_*^{-1} I_\psi/2) \sum_{p=1}^{r} (a_p^2 + b_p^2) + (K/2)(2\pi)^d \sum_{p=1}^{r} \psi(\boldsymbol{\omega}_p)(a_p^2 + b_p^2)\right). \end{aligned}$$

This completes the proof of Theorem III.4.1.

III.7.3. Proof of Theorem III.4.2

Fix $a_1, a_2, b_1, b_2 \in I\!R$ with $\sum_{i=1}^{2} (a_i^2 + b_i^2) \neq 0$. Then, as in (III.7.3) we have,

$$\sum_{p=1}^{2} [a_p \hat{C}_n(\boldsymbol{\omega}_{pn}) + b_p \hat{S}_n(\boldsymbol{\omega}_{pn})] = \sum_{j=1}^{n} Z(\mathbf{s}_j) w_n(\mathbf{s}_j), \quad \text{(III.7.6)}$$

where, $w_n(\mathbf{s}) = n^{-1/2} \sum_{p=1}^{2} \{a_p \cos(\boldsymbol{\omega}_{pn}' \mathbf{s}) + b_p \sin(\boldsymbol{\omega}_{pn}' \mathbf{s})\}$. Then for any $\mathbf{h} \in I\!R^d$,

$$\begin{aligned} &\int w_n(\lambda_n \mathbf{x} + \mathbf{h}) w_n(\lambda_n \mathbf{x}) f^2(\mathbf{x}) d\mathbf{x} \\ &= (2n)^{-1} \sum_{p=1}^{2} \sum_{q=1}^{2} \Big[a_p a_q \int \Big\{ \cos(\lambda_n (\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})' \mathbf{x} + \boldsymbol{\omega}_{qn}' \mathbf{h}) \\ &\quad + \cos(\lambda_n (\boldsymbol{\omega}_{qn} - \boldsymbol{\omega}_{pn})' \mathbf{x} + \boldsymbol{\omega}_{qn}' \mathbf{h}) \Big\} f^2(\mathbf{x}) d\mathbf{x} \\ &+ a_p b_q \int \Big\{ \sin(\lambda_n (\boldsymbol{\omega}_{pn} + \boldsymbol{\omega}_{qn})' \mathbf{x} + \boldsymbol{\omega}_{qn}' \mathbf{h}) + \sin(\lambda_n (\boldsymbol{\omega}_{qn} - \boldsymbol{\omega}_{pn})' \mathbf{x} + \boldsymbol{\omega}_{qn}' \mathbf{h}) \Big\} f^2(\mathbf{x}) d\mathbf{x} \end{aligned}$$

$$+a_q b_p \int \left\{\sin(\lambda_n(\boldsymbol{\omega}_{pn}+\boldsymbol{\omega}_{qn})'\mathbf{x}+\boldsymbol{\omega}'_{qn}\mathbf{h}) - \sin(\lambda_n(\boldsymbol{\omega}_{qn}-\boldsymbol{\omega}_{pn})'\mathbf{x}+\boldsymbol{\omega}'_{qn}\mathbf{h})\right\} f^2(\mathbf{x})d\mathbf{x}$$
$$+b_p b_q \int \left\{\cos(\lambda_n(\boldsymbol{\omega}_{qn}-\boldsymbol{\omega}_{pn})'\mathbf{x}+\boldsymbol{\omega}'_{pn}\mathbf{h}) - cos(\lambda_n(\boldsymbol{\omega}_{pn}+\boldsymbol{\omega}_{qn})'\mathbf{x}+\boldsymbol{\omega}'_{qn}\mathbf{h})\right\}$$
$$f^2(\mathbf{x})d\mathbf{x}\Big].$$

When the frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ are asymptotically distant, by Lemma III.7.2,

$$\lim_{n\to\infty}\int w_n(\lambda_n\mathbf{x}+\mathbf{h})w_n(\lambda_n\mathbf{x})f^2(\mathbf{x})d\mathbf{x} = (K/2)\cos(\boldsymbol{\omega}'\mathbf{h})\sum_{p=1}^{2}(a_p^2+b_p^2).$$

On the other hand, when $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ satisfy (III.4.5), then by Lemma III.7.2 and proceeding similarly as in Theorem III.3.2, we get the required result.

III.7.4. Proof of Theorem III.4.3

Fix $a_1, a_2, b_1, b_2 \in I\!R$ with $\sum_{p=1}^2(a_p^2+b_p^2) \neq 0$. Then, $\sum_{p=1}^2[a_p\hat{C}_n(\boldsymbol{\omega}_{pn})+b_p\hat{S}_n(\boldsymbol{\omega}_{pn})]$ can be expressed in the from (III.7.6) and the corresponding weight function $w_n(\cdot)$ satisfies (III.7.7). For part (c), for any $\mathbf{h}\in I\!R^d$, we get

$$\lim_{n\to\infty} n\int w_n(\lambda_n\mathbf{x}+\mathbf{h})w_n(\lambda_n\mathbf{x})f^2(\mathbf{x})d\mathbf{x}$$
$$= (1/2)\sum_{p=1}^{2}\sum_{q=1}^{2}\Big[a_p a_q\Big\{\int\cos((\mathbf{y}_p+\mathbf{y}_q)'\mathbf{x})f^2(\mathbf{x})d(\mathbf{x}) + \int\cos((\mathbf{y}_p-\mathbf{y}_q)'\mathbf{x})f^2(\mathbf{x})$$
$$d(\mathbf{x})\} + a_p b_q\left\{\int\sin((\mathbf{y}_p+\mathbf{y}_q)'\mathbf{x})f^2(\mathbf{x})d(\mathbf{x}) + \int\sin((\mathbf{y}_q-\mathbf{y}_p)'\mathbf{x})f^2(\mathbf{x})d(\mathbf{x})\right\}$$
$$+a_q b_p\left\{\int\sin((\mathbf{y}_p+\mathbf{y}_q)'\mathbf{x})f^2(\mathbf{x})d(\mathbf{x}) - \int\sin((\mathbf{y}_q-\mathbf{y}_p)'\mathbf{x})f^2(\mathbf{x})d(\mathbf{x})\right\}$$
$$+b_p b_q\left\{\int\cos((\mathbf{y}_q-\mathbf{y}_p)'\mathbf{x})f^2(\mathbf{x})d(\mathbf{x}) - \int\cos(\mathbf{y}_p+\mathbf{y}_q)'\mathbf{x})f^2(\mathbf{x})d(\mathbf{x})\right\}\Big],$$

where, $\mathbf{y}_i = \lim_{n\to\infty}\lambda_n\boldsymbol{\omega}_{in}$, $i = 1,2$. The proof of part (c) now can be completed by applying the inversion formula (III.6.17) and the central limit theorem of Lemma III.7.1. Parts (a) and (b) can be proved by repeating the steps in the proofs of parts

(a) and (b) of Theorem III.4.2, respectively, setting $\boldsymbol{\omega} = \mathbf{0}$. We omit the details to save space.

III.7.5. Proof of Theorem III.4.4

Fix $a_1, \cdots, a_r, b_1, \cdots, b_r \in I\!R$ with $\sum_{i=1}^{r}(a_i^2 + b_i^2) \neq 0$. With an obvious definition of $\tilde{\hat{C}}_n(\boldsymbol{\omega})$ and $\tilde{\hat{S}}(\boldsymbol{\omega})$, we have

$$\sum_{p=1}^{r}[a_p\tilde{\hat{C}}_n(\boldsymbol{\omega}_{pn}) + b_p\tilde{\hat{S}}(\boldsymbol{\omega}_{pn})] = \sum_{i=1}^{n} Z(\mathbf{s}_i)[w_n(\mathbf{s}_i) - n^{-1/2}\beta_n] \qquad \text{(III.7.7)}$$

where $w_n(\mathbf{s}) = n^{-1/2}\sum_{p=1}^{r}\{a_p\cos(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}) + b_p\sin(\boldsymbol{\omega}_{pn}^{'}\mathbf{s})\}$ and $\beta_n = n^{-1}\sum_{i=1}^{n}\sum_{p=1}^{r}\{a_p \cos(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}_i) + b_p\sin(\boldsymbol{\omega}_{pn}^{'}\mathbf{s}_i)\}$. Now suppose, $\boldsymbol{\omega}_{pn} \to \boldsymbol{\omega} \neq \mathbf{0}$, for all $p = 1, 2, \cdots, r$. Then by SLLN, $\beta_n \to^{a.s.} 0$. On the other hand, if $\lambda_n\boldsymbol{\omega}_{pn} \to \mathbf{y}_p \in I\!R^d$ for all $p = 1, 2, \cdots, r$, then,

$$\beta_n \to^{a.s.} \beta_0 \equiv \sum_{p=1}^{r}\{a_p\int\cos(\mathbf{y}_p^{'}\mathbf{x})f(\mathbf{x})d(\mathbf{x}) + b_p\int\sin(\mathbf{y}_p^{'}\mathbf{x})f(\mathbf{x})d(\mathbf{x})\}. \qquad \text{(III.7.8)}$$

Now,

$$\sum_{p=1}^{r}[a_p\tilde{\hat{C}}_n(\boldsymbol{\omega}_{pn}) + b_p\tilde{\hat{S}}(\boldsymbol{\omega}_{pn})] = \sum_{i=1}^{n} Z(\mathbf{s}_i)[w_n(\mathbf{s}_i) - n^{-1/2}\beta_0] + R_n \qquad \text{(III.7.9)}$$

where, $R_n = (\beta_n - \beta_0)n^{-1/2}\sum_{i=1}^{n} Z(\mathbf{s}_i) = o_p(1),\ a.s.\ (P_{\mathbf{X}})$. Therefore, it is enough to find the asymptotic distribution for $\sum_{i=1}^{n} Z(\mathbf{s}_i)[w_n(\mathbf{s}_i) - n^{-1/2}\beta_0]$. Let $w_n^*(\mathbf{s}) = w_n(\mathbf{s}) - n^{-1/2}\beta_0$. Then,

$$\begin{aligned} & n\int w_n^{*2}(\lambda_n\mathbf{x})f(\mathbf{x})d\mathbf{x} \\ & = n\int w_n^2(\lambda_n\mathbf{x})f(\mathbf{x})d\mathbf{x} - \beta_0^2 + o_p(1), \quad \text{a.s.} \quad (P_{\mathbf{X}}). \end{aligned} \qquad \text{(III.7.10)}$$

and,

$$\begin{aligned} &n \int w_n^*(\lambda_n \mathbf{x} + \mathbf{h}) w_n^*(\lambda_n \mathbf{x}) f^2(\mathbf{x}) d\mathbf{x} \\ &= n \int w_n(\lambda_n \mathbf{x} + \mathbf{h}) w_n(\lambda_n \mathbf{x}) f^2(\mathbf{x}) d\mathbf{x} - K\beta_0^2 + o_p(1), \quad \text{a.s.} \quad (P_{\mathbf{X}}). \end{aligned} \tag{III.7.11}$$

The results now follow from (III.7.7)-(III.7.11) and Lemma III.7.1.

CHAPTER IV

CONCLUSION

In the first work, we propose a bootstrap based method to derive an estimator of the mean squared prediction error $M(\theta)$ due to the prediction of a suitable functional of a set of future values $X_{n+1}, \ldots, X_{n+k}$ based on observed time-series $X_1, \ldots, X_n$. The key advantage of the bootstrap methodology is that it produces an estimator of the MSPE of the estimated best linear predictor for any given estimator $\hat{\theta}_n$ of θ, without requiring any analytical computation of the functions $M_2(\theta)$ (a cross-product term) and $M_3(\theta)$ (the estimation error due to the substitution of $\hat{\theta}_n$ in place of θ in $\tilde{\Psi}_n(\cdot)$) which critically depend on the choice of $\hat{\theta}_n$. We show that under fairly mild regularity conditions on the $\{X_t\}$-process and on the estimators $\hat{\theta}_n$, the bootstrap MSPE estimator is consistent.

An important contribution of this work is to develop a new method for constructing a second order correct MSPE estimator that is non negative with probability one. The key idea is to "tilt" the estimator $\hat{\theta}_n$ suitably so that it balances out the bias of the "ordinary" bootstrap MSPE estimator to the order $O(n^{-1})$. The tilting factor used here is based on certain iterations of the bootstrap step and on a simple formula to combine them. As a result, the computation of the proposed second order correct MSPE estimator is very much feasible with today's computing power, and the methodology works any choice of the estimator $\hat{\theta}_n$ satisfying the mild regularity conditions of the main result. Most importantly, the proposed method does not require any analytical derivation on the part of the user.

In the next work, we study the asymptotic joint distribution of a finite collection of DFTs of regularly and irregularly spaced spatial data under the pure and mixed increasing domain asymptotic structures. The main findings of our paper under both deterministic and stochastic sampling designs are:

(i) As in the time series case, under suitable regularity conditions, the asymptotic joint distributions of finite collections of the sine and cosine transforms are multivariate Gaussian under both deterministic and stochastic designs. However, in the stochastic design case, the asymptotic covariance critically depends on the spatial sampling density and the spatial asymptotic structure (PID *vs* MID); A complete description of their effects on the resulting limit distributions is given.

(ii) DFTs at unequal nonzero limiting frequencies are asymptotically independent.

(iii) In the fixed design case, for sampling regions of a *general shape* and for DFTs at ordinates converging to a *common* limiting frequency, asymptotic independence holds *if and only if* the ordinates are *asymptotically distant*. $\{\boldsymbol{\omega}_{jn}\}$ and $\{\boldsymbol{\omega}_{kn}\}$ are called asymptotically distant if $(\text{vol.}(\mathcal{D}))^{1/d}\|\boldsymbol{\omega}_{jn} - \boldsymbol{\omega}_{kn}\| \to \infty$ as $N \to \infty$. In the stochastic design case, similar result holds for a general sampling density. Thus, although the data-sites are irregularly spaced, the asymptotic behavior of the DFTs remains similar to that for regularly spaced spatial data. This is rather surprising and contrary to the folklore about lack of independence of DFTs for irregularly spaced time series data.

(iv) For two discrete Fourier frequency sequences $\{\boldsymbol{\omega}_{1n}\}$ and $\{\boldsymbol{\omega}_{2n}\}$ converging to the zero frequency, the corresponding sine and cosine transforms may exhibit different behavior depending on whether the frequency sequences approach zero at the same rate (*asymptotically symmetrically close case*) or at a different rate (*asymptotically close case*). See Section III.3.1.3 and Section III.4.3 for details.

(v) For sampling sites located on the d-dimensional integer grid, DFTs at all discrete Fourier frequencies are asymptotically independent when the sampling region is cubic. However, this is false for a sampling region of a general shape (including spheres, hyper-rectangles, etc.). Also for sampling sites on a scaled version of $\mathbb{Z}^d$ and a *rectangular sampling region*, asymptotic independence holds, provided the grid-increment in each direction is inversely proportional to the sides of the sampling region.

(vi) Under the stochastic design, for a *hyper-rectangular sampling region* and a *uniform* sampling density, asymptotic independence of DFTs holds even for asymptotically close frequency sequences. See Section III.4 for more details.

Thus, in contrast to the time series case, the geometry of the sampling region plays an important role in determining the asymptotic independence of the DFTs of spatial data.

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