Prolegomena to an Ontology of Shape

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Abstract. Influenced by the four-category ontology of Aristotle, many modern ontologies treat shapes as accidental particulars which (a) are specifically dependent on the substantial particulars which act as their bearers, and (b) instantiate accidental universals which are exemplified by those bearers. It is also common to distinguish between, on the one hand, these physical shapes which form part of the empirical world and, on the other, ideal geometrical shapes which belong to the abstract realm of mathematics. Shapes of the former kind are often said to approximate, but never to exactly instantiate, shapes of the latter kind. Following a suggestion of Frege, ideal mathematical shapes can be given precise definitions as equivalence classes under the relation of geometrical similarity. One might, analogously, attempt to define physical shape universals as equivalence classes under a relation of physical similarity, but this fails because physical similarity is not an equivalence relation. In this talk I will examine the implications of this for the ontology of shape and in particular for the relationship between mathematical shapes and the shapes we attribute to physical objects.

Keywords. shape ontology; mathematical vs physical shape; intrinsic vs embedded shape

1. Introduction

What are shapes, and how are shapes related to things which are not shapes? Are there indeed such things as shapes at all, entities of some sort that have a place in an inventory of the world's contents? Or can we explain talk about shapes in terms of an ontology in which shapes do not feature as entities of any kind?

There seem to be two distinct kinds of shapes: *physical shapes*, which we encounter in the physical world as the shapes of entities that exist in space, and *mathematical shapes*, which we encounter in geometry, the shapes of abstract mathematically-defined constructions. In both cases it seems evident that shapes are *ontologically dependent* on the objects whose shapes they are (their *bearers*), but the relationship between the two kinds of shape is not necessarily straightforward. I shall defer till later a discussion of mathematical shapes, and for the moment concentrate on physical shapes, the shapes of physical objects.

Granted, then, that physical shapes are always shapes *of* things, what kinds of things *have* shapes in this sense? A brief catalogue might be as follows:

1. *Material objects*, including chunks of matter (e.g., a pebble), organisms (e.g., a penguin), and assemblies (e.g., a bicycle).
2. *Non-material physical objects* such as holes, faces, and edges.
3. *Aggregates* such as a flock of birds or a cluster of buildings.

The boundary between material objects and aggregates is not sharp, since even a chunk of matter is, at submicroscopic resolution, an aggregate of atoms. With aggregates it is often not easy to determine an exact boundary, or therefore an exact shape [5], and to the extent that objects may be similarly indeterminate, it may likewise be impossible to assign exact shapes to them. I shall take this issue up later in the discussion of shape approximation. We should not assume uncritically that all material objects have shapes; Stroll [13] suggests that not all material objects have surfaces, and possession of a surface seems to be strongly associated with possession of a shape, even if not a necessary condition for it.

Phrases of the form "the shape of X" and "X has such-and-such a shape" attest to the intimate relation between an object and its shape, characterised as an ontological dependence of the latter on the former. Other key elements of the object–shape relationship, to be accounted for in an ontology of shape, include two objects having *the same shape*, and an object *changing shape*, expressed using the sentence forms:

(1) x and y have the same shape at time t
(2) x changed shape between times t_1 and t_2

In what follows, we will pay careful attention to these notions.

2. The dependency of shape upon objects

Shape-words in language typically come both as nouns and adjectives: in English we have, alongside nouns such as "circle", "triangle", "sphere", and "cylinder", the respective adjectives "circular", "triangular", "spherical", and "cylindrical". Cases in which the nominal and the adjectival functions are borne by the same word—e.g., "square" and "oblong"—are the exception rather than the rule. We also freely form compound adjectives such as "pear-shaped" and "heart-shaped", and in some cases the noun forming the first part of the compound refers, not to the physical object which it normally designates, but to some mathematical shape conventionally abstracted from it—e.g., the "heart" symbol ♡ only very approximately resembles the complex three-dimensional shape of an anatomical heart.

Shape adjectives point to the notion of shape as a *property* of objects, whereas shape nouns point to shapes as entities in their own right. Which of these two pictures enjoys logical or ontological primacy over the other, and how are the two pictures related?

Ontological parsimony suggests that shape-as-property should take priority over shape-as-entity. Looking around us, we see physical objects, each with its own shape, but to suggest that we see the shapes *as well as* the objects smacks of ontological over-abundance. It is more natural, when in a parsimonious mood, to say that each object is shaped in such-and-such a way, where this notion is expressed using a shape adjective. Thus we can say that the table is square, rather than that it stands in some relation to a shape entity which is *a square*.

A logical analysis of this view will invoke *shape predicates*, leading to predications of the form $Square(x)$ or $Circular(y)$—or rather, allowing for the fact that objects can change shape, $Square(x, t)$, etc. A major disadvantage lurks behind the attractive simplicity of this scheme: if we want to generalise over shapes, we have to quantify over predicates, and this requires the use of second-order logic, with all the difficulties that that brings in its wake. Thus to express the sentence-forms (1) and (2) we would have to write something like

(1a) $\forall\Phi(ShapeProperty(\Phi) \rightarrow (\Phi(x, t) \leftrightarrow \Phi(y, t)))$,
(2a) $\exists\Phi_1\exists\Phi_2(ShapeProperty(\Phi_1) \wedge ShapeProperty(\Phi_2) \wedge \Phi_1(x, t_1) \wedge \Phi_2(x, t_2) \wedge \neg\Phi_1(x, t_2) \wedge \neg\Phi_2(x, t_1))$.

A standard way of reducing such second-order predications to first-order form is by *reifying* the properties expressed by the predicates which are being quantified over [4]. By this means we introduce terms designating shape entities, and introduce a first-order $HasShape$ predicate to relate objects to the shapes that they have: thus instead of $Square(x)$, say, we would write $HasShape(x, square)$. In effect, this is to accord priority to shape-nouns over shape-adjectives. Our sentences (1) and (2) can now be expressed in first-order form as

(1b) $\forall s(HasShape(x, s, t) \leftrightarrow HasShape(y, s, t))$,
(2b) $\exists s_1\exists s_2(HasShape(x, s_1, t_1) \wedge HasShape(x, s_2, t_2) \wedge \neg HasShape(x, s_2, t_1) \wedge \neg HasShape(x, s_1, t_2))$

On this view, it is natural to regard shapes—that is, the entities designated by the s variables in (1b) and (2b)—as *generically dependent* entities. They are dependent, since a shape only exists insofar as it has bearers, and this dependence is generic because a shape is not dependent on the existence of a unique bearer but can be multiply realised in different bearers having the same shape.

Modern information systems ontologies such as BFO [7] and DOLCE [9] do not take this line; instead, they treat an object's shape as *specifically dependent* on that object, meaning that the shape belongs uniquely to that object and cannot be shared with any other. In DOLCE, shapes, along with such things as colours, volumes, weights, and densities, are classified as *qualities*. The identity of an object's shape is tied to the identity of the object itself: the shape comes into existence when the object comes into existence, and endures for as long as the object does. This does not mean that an object cannot change shape, though; what happens, according to DOLCE, is not that the object assumes a different shape, but that the object's shape assumes a different *value*. The values that may be assumed by a quality are entities of another kind, called *qualia*, which collectively constitute a domain known as a *quality space*—in the case of shape, we could speak of *shape qualia* in *shape space*. These quality spaces are similar to the *conceptual spaces* of Gärdenfors [6].

On this picture, variability of shape shows up as a time-dependency, not of the shape on its bearer, but of the value of the shape on the shape. Writing $shape(o)$ to refer to the shape which uniquely inheres in the object o, we have the rule

$$shape(x) = shape(y) \rightarrow x = y,$$

and our formulae now come out as:

(1c) $value(shape(x), t) = value(shape(y), t)$
(2c) $value(shape(x), t_1) \neq value(shape(x), t_2)$

The shape-as-quality view embraced by DOLCE has a solid pedigree in the Aristotelian four-category ontology that is encapsulated in the ontological square [8,12], which presents a cross-classification of the entities of an ontology along the dimensions of universal *vs* particular (distinguishing types from their instances) and substance *vs* accident (distinguishing independent from dependent entities). Thus *the roundness of this ball* is an accidental particular inhering in (and thus dependent on) the substantial particular *this ball*; and these two particular entities are instances of the accidental universal *roundness* and the substantial universal *ball* respectively. The ball itself is said to *exemplify* roundness.

The roundness of this ball is not quite the same as the shape of the ball conceived as a quality in DOLCE. The former is a *trope*, i.e., a specific instance of a property inhering in an object. In DOLCE terms, the "property" in question is not just a quality but a quality's having a particular value. Thus a trope could be regarded as a quality/value pair. If the quality changes value (e.g., an object changes shape), then the previous trope is superseded by a new one. When the value of some quality changes continuously, there is a continuous succession of different tropes.

3. The primacy of "same shape" over "shape"

The reified analyses discussed above are predicated on the assumption that there are such things as shapes, whether universals or particulars, with a bona fide existence that must be accounted for by according them a place within our ontology. This can, however, be questioned. Consider again the two main ways in which we describe the shape of an object:

1. Using a descriptive adjective such as "square", "round", or a combination of adjectives such as "long and thin";
2. By means of a comparison with some other object whose shape is assumed known, e.g., "heart-shaped", "hourglass-shaped".

In neither of these cases is there an explicit reference to shapes per se: we can understand "square" as a descriptive adjective without having to postulate any entity that is a square shape distinct from the square object we are talking about; and in saying that something is heart-shaped we are not saying that its shape is a heart, but rather that it is similarly-shaped to a heart. We do, of course, use the expression "it has the same shape as a heart", which seems to suppose the existence of the shape as something distinct from the heart, but it may be argued that this too is a misunderstanding, the locution "has the same shape as" being more correctly paraphrased as "is shaped the same as".

This way of arguing has venerable roots. It presents shape as one of a group of concepts X for which the notion of X itself is logically dependent on a prior notion which, once we have the concept X at our disposal, it is natural to express using the words "has the same X as". This latter notion is an equivalence relation which can be defined without any reference to the concept X itself. This idea is due to Frege [2], who noted that the concepts of *number*, *direction*, and *shape* can all be derived in this way.

In the case of (cardinal) number, the relevant relation is defined as follows:

- Set X has the same number as set Y if and only if there is a bijection (i.e., an exhaustive one-to-one correspondence) between the elements of X and the elements of Y.

Notice that bijections can be defined without reference to number; but on the other hand, according to Frege's argument, number cannot be defined without having the prior notion of "same number" to establish an identity criterion for the new concept. Thus "same number" is shown to be logically prior to "number". Instead of "has the same number as", we can use the term "equipollent".[1] We now define number in terms of equipollence as follows: The number of set X (i.e., the number of its elements, its *cardinality*) is the set of all sets equipollent to X.[2] This set is what we would now recognise as an equivalence class under the equipollence relation.[3]

Similarly, "direction" is logically dependent on the relation "has the same direction as"—which we routinely express as "is parallel to"—, and "shape" is logically dependent on "has the same shape as", i.e., is geometrically similar to. In particular, the shape of an object can be defined as the equivalence class comprising all objects which have the same shape as it. This definition works well so long as (a) a domain of "objects" is established for the relation to be defined on, and (b) within this domain "same shape" can be defined as an equivalence relation. In the next section I consider as candidate domains, first, geometrical constructions, and second, physical objects; I then go on to consider what it means to say that a physical object has the same shape as a geometrical construction.

4. Definition of the "same shape" relation

At the end of the previous section I glossed the relation "has the same shape as" as "is geometrically similar to". The latter relation, however, is first and foremost defined as a relation on geometrical objects—which, for the moment, we may understand, in standard mathematical fashion, as subsets of $\mathbb{R}^n$, for some $n \in \mathbb{Z}^+$. We therefore need to ask in what way this relation can be applied to the very different domain of physical objects: very different because physical space is not a set of real-number triples, the usefulness of $\mathbb{R}^3$ as a model for physical space

[1] Frege's term was *gleichzahlig* i.e., "equal-numbered".

[2] Frege did not himself formulate this in terms of sets: he spoke of the number which belongs to the concept F (*die Anzahl, welche dem Begriffe* F *zukommt*) and equated this to the extension of the concept "equipollent to the concept F" (*der Umfang des Begriffes "gleichzahlig dem Begriffe* F*"*).

[3] This is not unproblematic: What set is this relation defined on? Frege supposed this could be *the set of all sets*, but as Russell pointed out to him, this notion leads inexorably to devastating paradoxes. An adequate discussion of this point would take us well out of scope of this paper.

being rather that it affords constructions which capture at least some parts of the abstract essence of phenomena in physical space that we wish to model.

4.1. Similarity of geometrical objects

Considering first the notion of geometrical similarity as it applies to objects in geometrical space, the key notion is that of *distance*, which serves as a measure of the separation between two points. Writing $\Delta(p,q)$ for the distance between points $p, q \in \mathbb{R}^n$, defined by the usual Pythagorean rule, we have:

> **Definition of geometric similarity between figures in Euclidean space.**[4] Two subsets X and Y of $\mathbb{R}^n$ are geometrically similar if and only if there is a bijection ϕ from the points of X to the points of Y such that, for some constant $\kappa \in \mathbb{R}^+$, the following relation holds:
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> $$\forall x, x' \in X.\, \Delta(\phi(x), \phi(x')) = \kappa\Delta(x, x').$$

In other words, distances between points in X are multiplied by a constant factor κ when the points are mapped by ϕ into their images in Y. This is straightforward and familiar. It is of particular importance to note that the relation thereby defined is an equivalence relation, and it is this that enables the Fregean move by which the shape of a figure can be identified with the equivalence class of all figures having the same shape as it.

4.2. Similarity of physical objects

When we turn from $\mathbb{R}^n$ to the physical world, things are less straightforward. Whereas distance in $\mathbb{R}^n$ can be defined mathematically, in physical space the notion of distance is inextricably tied up with that of *measurement*, and the key fact about measurement here is that all measurement has finite precision. This means that whereas in $\mathbb{R}^n$, since distances can be arbitrary non-negative real numbers, the space of possible distances is simply $\mathbb{R}^+ \cup \{0\}$, the space of measured distances in physical space cannot take this form. To see this, note that we cannot meaningfully ask whether the length of a rod in metres is rational or irrational.

Given that in physical space we can only characterise distances in terms of measurement, and that measurement always has a finite precision, corresponding to the resolving power of the measuring instrument, it follows that geometrical similarity for physical objects can only be defined relative to a specified level of resolution. Consider two objects whose shapes we wish to compare, say P and Q, where Q is at least as big as P. Suppose the volume of P is v and that the resolving power of our measuring instrument is such that the smallest distance we can distinguish is h (I shall describe this as "resolution h"). Then in principle, within the physical space occupied by P we can distinguish, say, $n \approx v/h^3$ points,

[4] Here I am only dealing with Euclidean space—complications arise when we turn to non-Euclidean spaces. For example, on the surface of the sphere, figures cannot be similar without also being congruent. This is because, in this space, the sum of the interior angles of a triangle exceed 2π by an amount that is proportional to the area of the triangle, and hence no figure can be expanded or contracted without changing shape. Thus only in Euclidean space is shape completely independent of size.

and to each pair x, y of these points we can assign a distance $\Delta_h(x, y)$ that is some multiple kh of the minimum discernible distance.[5] Let $S_h(P)$ be this set of n discernible points in P; we may think of them, if we wish, as "blobs" of diameter h, though this is not really how they seem to us as observers.

To say that objects P and Q have the same shape is to say that the points we can discern in P at resolution h can be mapped onto some set of points we can discern in Q at that resolution, such that, first, the distances between pairs of the latter set of points are not discernibly different, at resolution h, from some constant multiple of the distances between the pairs of points from the former set to which they correspond under the mapping; and second, *every* point in Q discernible at resolution h is "sufficiently near" one of the points corresponding to a discernible point in P. In other words:

Definition of "same shape" for physical objects. Physical objects P and Q (where Q is at least as big as P) have the same shape, at resolution h, if, for some constant $\kappa \geq 1$, the set $S_h(P)$ of points discernible in P at resolution h can be mapped into the set $S_h(Q)$ of such points of Q by means of an injective mapping ϕ, such that the following relations hold:

1. $\forall x, y \in S_h(P).\, |\Delta_h(\phi(x), \phi(y)) - \kappa\Delta_h(x, y)| \leq \frac{1}{2}h$
2. $\forall x \in S_h(Q).\, \exists y \in S_h(P).\, \Delta_h(x, \phi(y)) \leq \frac{1}{2}\kappa h$

This is perhaps as near as we can come to the notion of geometric similarity in a physical setting, in which the idea of "exact distance" gains no purchase.

An immediate consequence of this is that a pair of objects which come out as having the same shape at one level of resolution may have different shapes at a finer level of resolution. For physical objects, the concept of "same shape" is inescapably tied to the level of resolution at which the objects are examined; and since, according to the Fregean argument, the concept of "shape" is logically dependent on the concept of "same shape", it follows that the notion of shape is also tied to levels of resolution. This is, of course, a familiar idea in Computer Science, where the notion of resolution, which we handled in a very crude manner here, has been considerably refined, e.g., in the technique of *multiscale representation* in which by convolving an original image with Gaussian kernels of different variances we obtain a series of images at different resolution levels (see [1, Ch.7]).

Unfortunately, the Fregean construction cannot be achieved in this instance, because the "same shape" relation on physical objects, as defined above, is not an equivalence relation. It is perfectly possible to have three objects A, B, and C, such that, at some resolution h, A has the same shape as B and B has the same shape as C, but A does not have the same shape as C. This is essentially because the "same shape" relation, as here defined, is not capturing a notion of *identical* shape so much as a notion of *indiscernible* (at resolution h) shape; and it is a familiar fact that unlike identity relations, indiscernibility relations are not transitive. The crucial implication of this, for us, is that there is no coherent

[5] This is somewhat oversimplified since in practice the resolution of our observations will not harmonise with the levels of resolution available in the system of units used for recording them—e.g., given resolving power (for lengths) of 0.03mm, recording to 1 decimal place is too coarse and recording to two decimal places is too fine. For an attempt to deal with this issue in an ontological framework, see [10].

notion of "exact shape" for a physical object, only that of objects being more or less approximately the same shape as other objects.

4.3. Similarity between physical and geometrical objects

It is often said that the shapes of physical objects can also approximate to the shapes of ideal geometrical objects (compare [11]). Thus there are many approximate spheres in the physical world, but no geometrically exact spheres, the exact sphere being an inhabitant of mathematical, not physical space. This is true so far as it goes, but we can explain what is being said here more carefully using resolution-based shape comparison. An explanation is needed since on the face of it there is something paradoxical about comparing something physical with an abstract mathematical construction: the two seem to belong to such entirely different realms that any notion of comparison ought to be out of the question.

If we are to compare the shapes of, say, a sheet P of A4 paper and a certain rectangle R defined in $\mathbb{R}^2$, then we need some way of matching up points in the former with points in the latter. There are, at least on the orthodox view, uncountably many points in R, but there does not seem to be any meaningful sense in which we can attribute uncountably many points to P. There is already something dubious about the notion of attributing *infinitely* many points to P, since as noted earlier, P can only be observed at all at some finite level of resolution, and at any such level only finitely many points can be distinguished within it. One might, of course, entertain the notion that, if there is no limit to how fine the resolution level can be, then there is no limit to how many points we can discern in P, so that the number of points in P, if not actually infinite, is at least *potentially* infinite. But it is far from clear that, in the physical world, resolution *could* be made indefinitely fine. For example, in the state of our current understanding of physics, the Planck length (approximately 1.6×10^{-35} m) is believed to provide a lower bound for the resolution of any *possible* technique of measurement.

But there is another problem we need to face, which is that while the physical piece of paper P does have an actual (albeit indeterminate) size, the mathematical rectangle R does not. How wide is the rectangle whose corners are at the points $(0, 0)$, $(0, 1)$, $(\sqrt{2}, 0)$, $(\sqrt{2}, 1)$? That's easy: it's $\sqrt{2}$! But $\sqrt{2}$ is a number, not a length. Well, then, it's $\sqrt{2}$ units. But what is a unit? How does a unit compare with a millimetre or an inch? It is a meaningless question: Objects in the mathematical world do not have actual sizes that can be compared directly to those of physical objects.

In fact this is not as serious a problem as it may seem at first sight, since in assessing geometric similarity we are only interested in relative length, not absolute length. The scale factor κ can take care of differences in absolute length, so long as the relative lengths remain unchanged. Thus in the ideal geometric rectangle, considered as a set of points in $\mathbb{R}^2$, we can follow the usual practice of recording the width as $\sqrt{2}$ units and the height as 1 unit, even though "unit" does not designate any actual physical length, since what matters for our purposes is only the ratio between the width and the height. And clearly all geometric rectangles with sides in the ratio $1 : \sqrt{2}$ are geometrically similar to one another.

What, then, does it mean to say that a piece of paper, considered at resolution h, has the shape of a rectangle with sides in a given ratio? We cannot use the definition of strict geometric similarity for geometric figures here, nor can we use the definition of "same shape at resolution h" for physical objects. Instead, we modify the criterion by combining elements from the two definitions as follows:

> **Definition of a physical object having the "same shape" as a geometrical object.** At resolution h, a physical object P has the same shape as a geometrical object Q if there is an injective mapping ϕ from the set of points $S_h(P)$ discernible in P at resolution h into the set of points in Q such that, for some constant $\kappa > 0$:
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> 1. $\forall x, y \in S_h(P).\, \Delta(\phi(x), \phi(y)) = \kappa \Delta_h(x, y)$
> 2. $\forall x \in Q.\, \exists y \in S_h(P).\, \Delta(x, \phi(y)) \leq \frac{1}{2}\kappa h$.

The point here is that the objects in the pure geometric world, being given to us by thought rather than by observation, can be specified with infinite precision: in particular, the distances between points can be arbitrary real numbers and are not constrained to being multiples of some minimal discernible distance.

5. Intrinsic vs Embedded Shape

Up to this point in our discussion, there has been an implicit assumption that the shape of an *object* can be identified with the shape of the portion of space, or *region*, which is occupied by that object. This fits in with a general presumption that the spatial properties of objects are nothing other than the properties of the portions of space they occupy. For many purposes this is not an unreasonable presumption, and has the advantage that all such properties can then be handled purely within a theory of space itself, without our having to worry about other physical properties of the objects.

In the case of shape, this does not always accord with our everyday ways of thinking. A rectangular sheet of paper, for example, is still, essentially, a rectangular piece of paper if it is folded along a diagonal, or screwed into a ball. A single long strand of wool retains this character whether it is coiled into in a skein or knitted into a scarf. A tall, thin person is still a tall thin person whatever posture he adopts; and more generally, the shape of a human body might be understood to be "something which is invariant across all the various postures that the body is capable of assuming" [3, p.201]. In such cases, we cannot identify the shape of the object with the shape of the space it occupies, since the shape, understood in this more general sense, may remain unchanged even as the object occupies a succession of differently-shaped spatial regions.

Let us distinguish between, on the one hand, the *intrinsic shape* of an object—which is the more general, abstract notion of shape described in the previous paragraph—and, on the other, its *embedded shape*, which is the shape of the region of space that it currently occupies. The idea is that the embedded shape of an object may change while its intrinsic shape remains constant. If we consider a square sheet of paper, and all the myriad origami figures which it can be folded into without tearing, we can say that across this range of figures the paper retains its intrinsic shape (i.e., square) but assumes different embedded shapes.

Like embedded shape, the notion of intrinsic shape is logically dependent on a prior notion of *same shape*: same embedded shape or same intrinsic shape respectively. Both of these notions can be defined in terms of the distances between points in the respective objects. The difference is that whereas with "same embedded shape" the distances are measured with reference to the space within which the objects are embedded, with "same intrinsic shape" they must be measured within the object itself. Given a physical object P, I define the *P-intrinsic distance* between two points x and y in P, written $\Delta_P(x, y)$, as the length of the shortest path between x and y which lies wholly within P. I contrast this with the *embedded distance* $\Delta(x, y)$ used previously. Note that, for points within a convex object, the intrinsic and embedded distances are the same.

We can now give a rough definition of "same intrinsic shape" as the existence of a bijective mapping ϕ between the points of P and the points of Q such that, for any pair of points x and x' in P we have $\Delta_Q(\phi(x), \phi(x')) = \kappa\Delta_P(x, x')$, for some constant $\kappa \in \mathbb{R}^+$. This is only a rough definition because, since we are here dealing with physical objects, we have to take into account the resolution of the distance measurements, just as we did in the embedded case. When a person adopts different bodily postures, although their intrinsic shape remains approximately the same, we will always find, if measuring sufficiently precisely, that there are differences resulting from muscular contractions and extensions which distort the shapes of individual parts of the body. Similarly, when a piece of paper is folded, at the fold there will be minute tears or stretches which disrupt the exact relationships between the intrinsic distances. But by measuring at a sufficiently coarse resolution these small-scale disruptions will disappear from view, so that intrinsic shape, *at that resolution level*, remains constant.

For a more exact definition, then, we need to introduce the notation $\Delta_{P,h}(x, y)$ to mean the P-intrinsic distance between x and y at resolution h. We then have:

> **Definition of "same intrinsic shape" for physical objects.** Physical objects P and Q (where Q is at least as big as P) have the same intrinsic shape, at resolution h, if, for some constant $\kappa \geq 1$, the set $S_h(P)$ of points discernible in P at resolution h can be mapped into the set $S_h(Q)$ of such points of Q by means of an injection ϕ, such that the following relations hold:
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> 1. $\forall x, y \in S_h(P).\, |\Delta_{Q,h}(\phi(x), \phi(y)) - \kappa\Delta_{P,h}(x, y)| \leq \frac{1}{2}h$
> 2. $\forall x \in S_h(Q).\, \exists y \in S_h(P).\, \Delta_{Q,h}(x, \phi(y)) \leq \frac{1}{2}\kappa h$

What this definition does not tell us is how widely applicable the notion of intrinsic shape is. The only examples I have given so far concern sheets of paper, strands of wool, and human bodies, but for many objects the notion of an underlying shape which is retained even as the object occupies differently-shaped regions of space does not seem to apply. We can arrive at a rough characterisation of the kinds of object for which the notion of intrinsic shape can do useful work by considering what kinds of transformations can change the embedded shape of an object while retaining its intrinsic shape.

Rigid motions—rotations, translations, and reflections—preserve *both* intrinsic and embedded shape. For rigid bodies, therefore, intrinsic shape does not convey any information beyond embedded shape. The same applies to magnification; we can see this, at least approximately, in the case of a spherical rubber ball that is being inflated; as it gets bigger, all the inter-point distances, whether measured across the embedding space or within the material of the ball itself, expand at the same rate, thus preserving both embedded and intrinsic shape.

All these transformations preserve the topology of the object, and it is certainly true that transformations that alter the topology, such as tearing or fracturing, will also alter the intrinsic shape—thus two sheets of A4 paper lying flat on the table, one of which is intact and the other has a razor-thin slit in the middle (making it topologically a torus), exhibit the same embedded shape but different intrinsic shapes. However, topological equivalence is a much coarser relation than having the same intrinsic shape: in general stretching and compression disrupt intrinsic shape while preserving topology. As noted above, transformations such as folding or "reposturing" always involve some such disruptive transformations, but because these are small compared with the objects in which they occur, intrinsic shape can be preserved even under reasonably fine resolution.

Can we characterise exactly those types of object which typically undergo transformations of a kind that result in changes of embedded shape while preserving intrinsic shape at an appropriate level of resolution? Because notions such as "typically" and "appropriate" are inherently inexact, we will never find such an exact characterisation; but it would be interesting at any rate to find a more exact characterisation than we have at present.

6. Conclusion

As with number and direction, the ontological status of shape is problematic because of its dependent character: shapes do not exist "in their own right", but only as qualities of objects. As Frege observed, for the shapes of geometrical figures characterised mathematically as subsets of $\mathbb{R}^n$, the relation of geometrical similarity provides a robust criterion of identity which, by establishing the notion of "same shape" as an equivalence relation, can support the notion of shapes as entities that could be included within an ontology.

By contrast, the "same shape" relation for physical objects, since it can only be defined relative to some finite resolution level, fails to be an equivalence relation, and therefore cannot provide a criterion of identity for a notion of physical shape. This casts doubt on the ontological integrity of the notion of shape, and we are left with the intransitive "same shape at resolution h" relation as the primary vehicle for shape-attribution to physical objects. This is reflected in the fact that, in practice, when we ascribe a shape to an object or object part it is always by comparison with something else —either another object or a geometrical figure—and relative to some (often implicit) level of resolution.

As a final observation, it should be noted that since "same shape" relations are founded on the comparison of distances amongst the points within the objects to be compared, it follows that a notion of shape should be, in principle, available in any domain where some notion of "distance" is applicable.

The notion of intrinsic, as opposed to embedded shape arises as a result of reinterpreting what is meant by distance: instead of distances measured across the space in which the objects under consideration are embedded, we measure distance along paths which are constrained to lie within the objects themselves. In this case "distance" is still essentially spatial, but by extending this term to measures of separation along non-spatial dimensions we obtain metaphorical extensions to the notion of shape.

One example is in the temporal domain, where we often speak of the *profile* of a process, meaning by this its temporal "shape"—typically rendered visible in the form of the spatial shape of some graphical representation of the process in which time is the independent variable and the dependent variables measure one or more qualities whose values are modified by the process. If we consider the process itself rather than its graphical representation, we are faced with the problem that the time dimension is not commensurable with the dimensions along which the other variables are measured: thus, for example, given a scale model of a railway train, where the non-temporal dimensions are spatial, there is no determinate answer to the question how fast the model should be run to preserve the spatio-temporal "shape" of the process in order to secure maximum verisimilitude.

An interesting extension of this is the idea of the "shape" of a musical phrase. In music, there are two dimensions that can provide analogues of distance, namely time and pitch.[6] Since these have different measurement scales, distances along these dimensions cannot be compared with each other, and this means that we can reasonably regard, e.g., augmentations or compressions along either the time axis or the pitch axis as in some sense shape-preserving. An examination of music from many different styles will furnish numerous examples of composers exploiting the expressive potential of such phenomena.

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[6] Leaving aside loudness here as a third candidate.