# TWO CLOSED FORMS FOR THE BERNOULLI POLYNOMIALS

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ABSTRACT. In the paper, the authors find two closed forms involving the Stirling numbers of the second kind and in terms of a determinant of combinatorial numbers for the Bernoulli polynomials and numbers.

## 1. INTRODUCTION

It is common knowledge that the Bernoulli numbers and polynomials $B_k$ and $B_k(u)$ for $k \ge 0$ satisfy $B_k(0) = B_k$ and can be generated respectively by

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} = 1 - \frac{z}{2} + \sum_{k=1}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}, \quad |z| < 2\pi$$

and

$$\frac{ze^{uz}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(u) \frac{z^k}{k!}, \quad |z| < 2\pi.$$

Because the function $\frac{x}{e^x-1} - 1 + \frac{x}{2}$ is odd in $x \in \mathbb{R}$, all of the Bernoulli numbers $B_{2k+1}$ for $k \in \mathbb{N}$ equal 0. It is clear that $B_0 = 1$ and $B_1 = -\frac{1}{2}$. The first few Bernoulli numbers $B_{2k}$ are

$$\begin{aligned} B_2 &= \frac{1}{6}, & B_4 &= -\frac{1}{30}, & B_6 &= \frac{1}{42}, & B_8 &= -\frac{1}{30}, \\ B_{10} &= \frac{5}{66}, & B_{12} &= -\frac{691}{2730}, & B_{14} &= \frac{7}{6}, & B_{16} &= -\frac{3617}{510}. \end{aligned}$$

The first five Bernoulli polynomials are

$$B_0(u) = 1, \quad B_1(u) = u - \frac{1}{2}, \quad B_2(u) = u^2 - u + \frac{1}{6},$$
$$B_3(u) = u^3 - \frac{3}{2}u^2 + \frac{1}{2}u, \quad B_4(u) = u^4 - 2u^3 + u^2 - \frac{1}{30}.$$

In combinatorics, the Stirling numbers $S(n,k)$ of the second kind for $n \ge k \ge 1$ can be computed and generated by

$$S(n,k) = \frac{1}{k!} \sum_{\ell=1}^{k} (-1)^{k-\ell} \binom{k}{\ell} \ell^n \quad \text{and} \quad \frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n,k) \frac{x^n}{n!}$$

respectively. See [7, p. 206].

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2010 *Mathematics Subject Classification.* Primary 11B68, Secondary 11B73, 26A06, 26A09, 26C05.

*Key words and phrases.* closed form; Bernoulli polynomial; Bernoulli number; Stirling numbers of the second kind; determinant.

This paper was typeset using $\mathcal{A}_{\mathcal{M}}\mathcal{S}$-LaTeX.

It is easy to see that the generating function of $B_k(u)$ can be

$$\frac{ze^{uz}}{e^z-1}=\left[\frac{e^{(1-u)z}-e^{-uz}}{z}\right]^{-1}=\frac{1}{\int_{-u}^{1-u}e^{zt}\,\mathrm{d}\,t}=\frac{1}{\int_0^1 e^{z(t-u)}\,\mathrm{d}\,t}. \tag{1.1}$$

This expression will play important role in this paper. For related information on the integral expression (1.1), please refer to [12, 15, 17, 31, 32] and plenty of references cited in the survey and expository article [30].

In mathematics, a closed form is a mathematical expression that can be evaluated in a finite number of operations. It may contain constants, variables, four arithmetic operations, and elementary functions, but usually no limit.

The main aim of this paper is to find two closed forms for the Bernoulli polynomials and numbers $B_k(u)$ and $B_k$ for $k\in\mathbb{N}$.

The main results can be summarized as the following theorems.

**Theorem 1.1.** *The Bernoulli polynomials $B_n(u)$ for $n\in\mathbb{N}$ can be expressed as*

$$\begin{aligned}B_n(u)=\sum_{k=1}^{n}k!\sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}\frac{\ell!}{(\ell+r)!}\frac{m!}{(m+s)!}\Biggl[\sum_{i=0}^{r}\sum_{j=0}^{s}(-1)^{i+j}\binom{\ell+r}{r-i}\\ \times\binom{m+s}{s-j}S(\ell+i,i)S(m+j,j)\Biggr]u^{m+s}(1-u)^{\ell+r}.\end{aligned} \tag{1.2}$$

*Consequently, the Bernoulli numbers $B_k$ for $k\in\mathbb{N}$ can be represented as*

$$B_n=\sum_{i=1}^{n}(-1)^i\frac{\binom{n+1}{i+1}}{\binom{n+i}{i}}S(n+i,i). \tag{1.3}$$

**Theorem 1.2.** *Under the conventions that $\binom{0}{0}=1$ and $\binom{p}{q}=0$ for $q>p\ge0$, the Bernoulli polynomials $B_k(u)$ for $k\in\mathbb{N}$ can be expressed as*

$$B_k(u)=(-1)^k\left|\frac{1}{\ell+1}\binom{\ell+1}{m}\left[(1-u)^{\ell-m+1}-(-u)^{\ell-m+1}\right]\right|_{1\le\ell\le k,0\le m\le k-1}, \tag{1.4}$$

*where $|\cdot|_{1\le\ell\le k,0\le m\le k-1}$ denotes a $k\times k$ determinant. Consequently, the Bernoulli numbers $B_k$ for $k\in\mathbb{N}$ can be represented as*

$$B_k=(-1)^k\left|\frac{1}{\ell+1}\binom{\ell+1}{m}\right|_{1\le\ell\le k,0\le m\le k-1}. \tag{1.5}$$

## 2. Lemmas

For proving the main results, we need the following notation and lemmas.

In combinatorial mathematics, the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ are defined by

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^n i\ell_i=n\\ \sum_{i=1}^n\ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}$$

for $n\ge k\ge0$. See [7, p. 134, Theorem A].

**Lemma 2.1** ([1, Example 2.6] and [7, p. 136, Eq. [3n]]). *The Bell polynomials of the second kind* $\mathrm{B}_{n,k}$ *meets*

$$\begin{aligned}&\mathrm{B}_{n,k}(x_1+y_1,x_2+y_2,\dots,x_{n-k+1}+y_{n-k+1})\\&\qquad=\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\mathrm{B}_{\ell,r}(x_1,x_2,\dots,x_{\ell-r+1})\mathrm{B}_{m,s}(y_1,y_2,\dots,y_{m-s+1}).\end{aligned}\tag{2.1}$$

**Lemma 2.2** ([7, p. 135]). *For* $n\ge k\ge0$, *we have*

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big)=a^kb^n\mathrm{B}_{n,k}(x_1,x_n,\dots,x_{n-k+1}),\tag{2.2}$$

*where* $a$ *and* $b$ *are any complex numbers.*

**Lemma 2.3** ([13, 39]). *For* $n\ge k\ge1$, *we have*

$$\mathrm{B}_{n,k}\left(\frac12,\frac13,\dots,\frac{1}{n-k+2}\right)=\frac{n!}{(n+k)!}\sum_{i=0}^{k}(-1)^{k-i}\binom{n+k}{k-i}S(n+i,i).\tag{2.3}$$

**Lemma 2.4.** *Let* $f(t)=1+\sum_{k=1}^{\infty}a_kt^k$ *and* $g(t)=1+\sum_{k=1}^{\infty}b_kt^k$ *be formal power series such that* $f(t)g(t)=1$. *Then*

$$b_n=(-1)^n\begin{vmatrix}a_1&1&0&0&\cdots&0\\a_2&a_1&1&0&\cdots&0\\a_3&a_2&a_1&1&\cdots&0\\\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\a_{n-1}&a_{n-2}&a_{n-3}&a_{n-4}&\cdots&1\\a_n&a_{n-1}&a_{n-2}&a_{n-3}&\cdots&a_1\end{vmatrix}.$$

*Proof.* The identity $f(t)g(t)=1$ entails the matrix identity

$$\begin{pmatrix}1&0&0&\cdots&0\\b_1&1&0&\cdots&0\\b_2&b_1&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\b_n&b_{n-1}&b_{n-2}&\cdots&1\end{pmatrix}=\begin{pmatrix}1&0&0&\cdots&0\\a_1&1&0&\cdots&0\\a_2&a_1&1&\cdots&0\\\vdots&\vdots&\vdots&\ddots&\vdots\\a_n&a_{n-1}&a_{n-2}&\cdots&1\end{pmatrix}^{-1},$$

where $\big(\cdot\big)^{-1}$ stands for the inverse of an invertible matrix $\big(\cdot\big)$. Applying Cramer's rule for a system of linear equations proves Lemma 2.4. ∎

## 3. Proofs of Theorems 1.1 and 1.2

We are now in a position to prove our main results.

*Proof of Theorem 1.1.* In terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$, the Faà di Bruno formula for computing higher order derivatives of composite functions is described in [7, p. 139, Theorem C] by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}f\circ g(x)=\sum_{k=1}^{n}f^{(k)}(g(x))\mathrm{B}_{n,k}\big(g'(x),g''(x),\dots,g^{(n-k+1)}(x)\big).\tag{3.1}$$

By the integral expression (1.1), applying the formula (3.1) to the functions $f(y)=\frac{1}{y}$ and $y=g(x)=\int_0^1 e^{x(t-u)}\operatorname{d}t$ results in

$$
\begin{aligned}
&\frac{\operatorname{d}^n}{\operatorname{d}x^n}\biggl(\frac{xe^{ux}}{e^x-1}\biggr)=\frac{\operatorname{d}^n}{\operatorname{d}x^n}\biggl(\frac{1}{\int_0^1 e^{x(t-u)}\operatorname{d}t}\biggr)\\
&=\sum_{k=1}^n\frac{(-1)^k k!}{\bigl(\int_0^1 e^{x(t-u)}\operatorname{d}t\bigr)^{k+1}}\mathrm{B}_{n,k}\biggl(\int_0^1(t-u)e^{x(t-u)}\operatorname{d}t,\\
&\quad\int_0^1(t-u)^2e^{x(t-u)}\operatorname{d}t,\dots,\int_0^1(t-u)^{n-k+1}e^{x(t-u)}\operatorname{d}t\biggr)\\
&\to\sum_{k=1}^n(-1)^k k!\mathrm{B}_{n,k}\biggl(\int_0^1(t-u)\operatorname{d}t,\int_0^1(t-u)^2\operatorname{d}t,\dots,\int_0^1(t-u)^{n-k+1}\operatorname{d}t\biggr)\\
&=\sum_{k=1}^n(-1)^k k!\mathrm{B}_{n,k}\biggl(\frac{(1-u)^2-(-u)^2}{2},\frac{(1-u)^3-(-u)^3}{3},\dots,\frac{(1-u)^{n-k+2}-(-u)^{n-k+2}}{n-k+2}\biggr)
\end{aligned}
$$

as $x\to0$. Further employing (2.1), (2.2), and (2.3) acquires

$$
\begin{aligned}
\frac{\operatorname{d}^n}{\operatorname{d}x^n}\biggl(\frac{xe^{ux}}{e^x-1}\biggr)\bigg|_{x=0}&=\sum_{k=1}^n(-1)^k k!\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}\\
&\quad\times\mathrm{B}_{\ell,r}\biggl(\frac{(1-u)^2}{2},\frac{(1-u)^3}{3},\dots,\frac{(1-u)^{\ell-r+2}}{\ell-r+2}\biggr)\\
&\quad\times\mathrm{B}_{m,s}\biggl(-\frac{(-u)^2}{2},-\frac{(-u)^3}{3},\dots,-\frac{(-u)^{m-s+2}}{m-s+2}\biggr)\\
&=\sum_{k=1}^n(-1)^k k!\sum_{r+s=k}\sum_{\ell+m=n}\binom{n}{\ell}(1-u)^{\ell+r}\mathrm{B}_{\ell,r}\biggl(\frac12,\frac13,\dots,\frac{1}{\ell-r+2}\biggr)\\
&\quad\times u^s(-u)^m\mathrm{B}_{m,s}\biggl(\frac12,\frac13,\dots,\frac{1}{m-s+2}\biggr)\\
&=\sum_{k=1}^n k!\sum_{r+s=k}\sum_{\ell+m=n}(-1)^m\binom{n}{\ell}\frac{\ell!}{(\ell+r)!}\frac{m!}{(m+s)!}\\
&\quad\times\Biggl[\sum_{i=0}^r\sum_{j=0}^s(-1)^{i+j}\binom{\ell+r}{r-i}\binom{m+s}{s-j}S(\ell+i,i)S(m+j,j)\Biggr]u^{m+s}(1-u)^{\ell+r}.
\end{aligned}
$$

As a result, the formula (1.2) follows immediately.

Letting $u=0$ in (1.2), simplifying, and interchanging the order of sums lead to the formula (1.3). The proof of Theorem 1.1 is complete. $\square$

*The first proof of Theorem 1.2.* Let $u=u(z)$ and $v=v(z)\ne0$ be differentiable functions. In [3, p. 40], the formula

$$\frac{\mathrm{d}^k}{\mathrm{d}\,z^k}\biggl(\frac{u}{v}\biggr)=\frac{(-1)^k}{v^{k+1}}\begin{vmatrix} u & v & 0 & \dots & 0\\ u' & v' & v & \dots & 0\\ u'' & v'' & 2v' & \dots & 0\\ \hdotsfor{5}\\ u^{(k-1)} & v^{(k-1)} & \binom{k-1}{1}v^{(k-2)} & \dots & v\\ u^{(k)} & v^{(k)} & \binom{k}{1}v^{(k-1)} & \dots & \binom{k}{k-1}v' \end{vmatrix} \tag{3.2}$$

for the $k$th derivative of the ratio $\frac{u(z)}{v(z)}$ was listed. For easy understanding and convenient availability, we now reformulate the formula (3.2) as

$$\frac{\mathrm{d}^k}{\mathrm{d}\,z^k}\biggl(\frac{u}{v}\biggr)=\frac{(-1)^k}{v^{k+1}}\bigl|A_{(k+1)\times1}\quad B_{(k+1)\times k}\bigr|_{(k+1)\times(k+1)}, \tag{3.3}$$

where the matrices

$$A_{(k+1)\times1}=(a_{\ell,1})_{0\le\ell\le k}$$

and

$$B_{(k+1)\times k}=(b_{\ell,m})_{0\le\ell\le k,0\le m\le k-1}$$

satisfy

$$a_{\ell,1}=u^{(\ell)}(z)\quad\text{and}\quad b_{\ell,m}=\binom{\ell}{m}v^{(\ell-m)}(z)$$

under the conventions that $v^{(0)}(z)=v(z)$ and that $\binom{p}{q}=0$ and $v^{(p-q)}(z)\equiv0$ for $p<q$. See also [26, Section 2.2] and [38, Lemma 2.1]. By the integral expression (1.1), applying the formula (3.3) to $u(z)=1$ and $v(z)=\int_0^1 e^{z(t-u)}\,\mathrm{d}\,t$ yields $a_{1,1}=1$, $a_{\ell,1}=0$ for $\ell>1$,

$$\begin{aligned} b_{\ell,m}&=\binom{\ell}{m}\int_0^1(t-u)^{\ell-m}e^{z(t-u)}\,\mathrm{d}\,t\\ &\to\binom{\ell}{m}\int_0^1(t-u)^{\ell-m}\,\mathrm{d}\,t,\quad z\to0\\ &=\binom{\ell}{m}\frac{(1-u)^{\ell-m+1}-(-u)^{\ell-m+1}}{\ell-m+1} \end{aligned}$$

for $0\le\ell\le k$ and $0\le m\le k-1$ with $\ell\ge m$, and

$$\begin{aligned} \frac{\mathrm{d}^k}{\mathrm{d}\,z^k}\biggl(\frac{ze^{uz}}{e^z-1}\biggr)&=\frac{(-1)^k}{b_{0,0}^{k+1}}|b_{\ell,m}|_{1\le\ell\le k,0\le m\le k-1}\\ &\to(-1)^k\biggl|\binom{\ell}{m}\int_0^1(t-u)^{\ell-m}\,\mathrm{d}\,t\biggr|_{1\le\ell\le k,0\le m\le k-1},\quad z\to0\\ &=(-1)^k\biggl|\binom{\ell}{m}\frac{(1-u)^{\ell-m+1}-(-u)^{\ell-m+1}}{\ell-m+1}\biggr|_{1\le\ell\le k,0\le m\le k-1}. \end{aligned}$$

The formula (1.4) is proved.

The formula (1.5) follows readily from taking $u=0$ in (1.4). The first proof of Theorem 1.2 is complete. ∎

*The second proof of Theorem 1.2.* Applying Lemma 2.4 to $g(t)=\frac{te^{ut}}{e^t-1}$ and $f(t)=\frac{e^{(1-u)t}-e^{-ut}}{t}$ reveals that $b_n=\frac{B_n(u)}{n!}$ and $a_n=\frac{(1-u)^{n+1}-(-u)^{n+1}}{n!}$. Hence, by virtue of Lemma 2.4,

$$B_n(u)=(-1)^n n!\left|\frac{(1-u)^{\ell-m+1}-(-u)^{\ell-m+1}}{(\ell-m+1)!}\right|_{1\le\ell\le n,0\le m\le n-1}.$$

Multiplying the row $\ell$ of this determinant by $\ell!$ and dividing the row $m$ by $m!$ gives

$$B_n(u)=(-1)^n\left|\frac{1}{\ell+1}\binom{\ell+1}{m}\left[(1-u)^{\ell-m+1}-(-u)^{\ell-m+1}\right]\right|_{1\le\ell\le n,0\le m\le n-1}.$$

The formula (1.4) is thus proved. The second proof of Theorem 1.2 is complete. □

## 4. Remarks and comparisons

In this final section, we will remark on our main results and compare them with some known conclusions.

*Remark* 4.1. The formula (1.3) recovers the one appeared in [9, p. 48, (11)], [13, (6)], [19, p. 59], and [35, p. 140]. For detailed infirmation, please refer to [13, Remark 4]. There are also some other formulas and inequalities for the Bernoulli numbers and polynomials in [10, 11, 16, 18, 24, 25, 27, 29, 33] and references cited therein. Hence, Theorems 1.1 and 1.2 generalize those corresponding results obtained in these references.

*Remark* 4.2. Motivated by the idea in [31, 32], Guo and Qi generalized in [15] the Bernoulli polynomials and numbers. Hereafter, some papers such as [21, 22] were published.

*Remark* 4.3. The special values of the Bell polynomials of the second kind $\mathrm{B}_{n,k}$ are important in combinatorics and number theory. Recently, some special values for $\mathrm{B}_{n,k}$ were discovered and applied in [14, 26, 34, 39].

*Remark* 4.4. In [4, 20], several different approaches to the theory of Bernoulli polynomials $B_k(u)$ were surveyed. However, there is no any conclusion directly related to Theorems 1.1 and 1.2.

Let $\{a_n\}_{0\le n\le\infty}$ be a sequence of complex numbers and let $\{D_n(a_k)\}_{0\le n\le\infty}$ be a sequence of determinants such that $D_0(a_k)=1$ and

$$D_n(a_k)=\begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0\\ a_2 & a_1 & a_0 & \cdots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ a_{n-1} & a_{n-2} & a_{n-3} & \cdots & a_0\\ a_n & a_{n-1} & a_{n-2} & \cdots & a_1 \end{vmatrix}, \quad n\in\mathbb{N}.$$

In [37], two identities

$$B_{2n}=(-1)^n\frac{(2n)!}{2}\left\{\sum_{\ell=0}^{n}\frac{(-1)^\ell}{(2\ell)!}D_{n-\ell}\left(\frac{1}{(2k+1)!}\right)+D_n\left(\frac{1}{(2k+1)!}\right)\right\} \tag{4.1}$$

and

$$B_{2n}=(-1)^{n+1}\frac{(2n)!}{2(2^{2n-1}-1)}D_n\left(\frac{1}{(2k+1)!}\right) \tag{4.2}$$

for $n\in\mathbb{N}$ were established.

In [8], six approaches to the theory of Bernoulli polynomials were mentioned. Mainly, a determinantal approach was introduced in [8] by defining $B_0(x)=1$ and

$$B_n(x)=\frac{(-1)^n}{(n-1)!}\begin{vmatrix}1 & x & x^2 & x^3 & \cdots & x^{n-1} & x^n\\ 1 & \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & \frac{1}{n+1}\\ 0 & 1 & 1 & 1 & \cdots & 1 & 1\\ 0 & 0 & 2 & 3 & \cdots & n-1 & n\\ 0 & 0 & 0 & \binom{3}{2} & \cdots & \binom{n-1}{2} & \binom{n}{2}\\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & 0 & \cdots & \binom{n-1}{n-2} & \binom{n}{n-2}\end{vmatrix}, \quad n\in\mathbb{N}. \tag{4.3}$$

As a result, the Bernoulli numbers

$$B_n=\frac{(-1)^n}{(n-1)!}\begin{vmatrix}\frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \cdots & \frac{1}{n} & \frac{1}{n+1}\\ 1 & 1 & 1 & \cdots & 1 & 1\\ 0 & 2 & 3 & \cdots & n-1 & n\\ 0 & 0 & \binom{3}{2} & \cdots & \binom{n-1}{2} & \binom{n}{2}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & \binom{n-1}{n-2} & \binom{n}{n-2}\end{vmatrix}, \quad n\in\mathbb{N}. \tag{4.4}$$

In [6, Theorem 1.1], it was obtained that, if $A(z)=\sum_{n=0}^{\infty}a_nz^n$ and $B(z)=\sum_{n=0}^{\infty}b_nz^n$ are the ordinary generating functions of $\{a_n\}_{0\le n\le\infty}$ and $\{b_n\}_{0\le n\le\infty}$ such that $A(x)B(x)=1$, then $a_0\ne0$ and $b_n=(-1)^n\frac{D_n(a_k)}{a_0^{n+1}}$. Therefore, Lemma 2.4 is a special case of [6, Theorem 1.1]. In the paper [6], as applications of [6, Theorem 1.1], some properties of $D_n(a_k)$ were discovered and applied to give an elegant proof of (4.1) and (4.2), and to express the Genocchi numbers, the tangent numbers, higher order Bernoulli numbers, the Stirling numbers of the first and second kinds, the harmonic numbers, higher order Euler numbers, higher order Bernoulli numbers of the second kind, and so on, in terms of $D_n(a_k)$. Especially, the formulas

$$B_n=(-1)^nn!D_n\left(\frac{1}{(k+1)!}\right),$$

which recovers [5, Eq.(4)], and

$$B_n=n!D_n\left(\frac{(-1)^k}{(k+1)!}\right)$$

were derived.

In [2], it was mentioned that the formula

$$B_n=\frac{(-1)^{n-1}}{(n+1)!}\begin{vmatrix}1 & 2 & 0 & 0 & \cdots & 0\\ 1 & 3 & 3 & 0 & \cdots & 0\\ 1 & 4 & 6 & 4 & \cdots & 0\\ 1 & 5 & 10 & 10 & \cdots & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\ \binom{n+1}{0} & \binom{n+1}{1} & \binom{n+1}{2} & \binom{n+1}{3} & \cdots & \binom{n+1}{n-1}\end{vmatrix}$$

was traced back to the book [36]. The Bernoulli polynomials $B_n(x)$ were represented in [2] as

$$
\begin{aligned}
B_n(x)&=(-1)^n n!\begin{vmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0\\
\frac{x}{1!} & \frac{1}{2!} & 1 & 0 & 0 & 0 & \cdots & 0\\
\frac{x^2}{2!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & 0 & \cdots & 0\\
\frac{x^3}{3!} & \frac{1}{4!} & \frac{1}{3!} & \frac{1}{2!} & 1 & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
\frac{x^n}{n!} & \frac{1}{(n+1)!} & \frac{1}{n!} & \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \cdots & 1
\end{vmatrix}\\
&=(-1)^n\frac{n!}{\prod_{k=1}^{n}k!}\begin{vmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0\\
x & \frac{1}{2!} & 1 & 0 & 0 & 0 & \cdots & 0\\
x^2 & \frac{2!}{3!} & 1 & 2! & 0 & 0 & \cdots & 0\\
x^3 & \frac{3!}{4!} & 1 & \frac{3!}{2!} & 3! & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
x^n & \frac{n!}{(n+1)!} & 1 & \frac{n!}{(n-1)!} & \frac{n!}{(n-2)!} & \frac{n!}{(n-3)!} & \cdots & n!
\end{vmatrix}\\
&=(-1)^n\prod_{k=1}^{n-1}\frac{(k-1)!}{k!}\begin{vmatrix}
1 & 1 & 0 & 0 & 0 & 0 & \cdots & 0\\
x & \frac{1}{2!} & 1 & 0 & 0 & 0 & \cdots & 0\\
x^2 & \frac{2!}{3!} & 1 & 2! & 0 & 0 & \cdots & 0\\
x^3 & \frac{3!}{4!} & 1 & \frac{3!}{2!} & \frac{3!}{2!} & 0 & \cdots & 0\\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots\\
x^n & \frac{n!}{(n+1)!} & 1 & \frac{n!}{(n-1)!} & \frac{n!}{(n-2)!2!} & \frac{n!}{(n-3)!3!} & \cdots & \frac{n!}{2!(n-2)!}
\end{vmatrix}.
\end{aligned}
$$

Similar to the above representations for the Bernoulli polynomials $B_n(x)$, some determinantal expressions for the hypergeometric Bernoulli polynomials were further presented in [2].

*Remark* 4.5. The idea of Lemma 2.4 was used in [23, pp. 22–23] to express determinants of complete symmetric functions in terms of determinants of elementary symmetric functions.

*Remark* 4.6. This manuscript is a revision and extension of the first two versions of the preprint [28].

**Acknowledgements.** This manuscript was drafted while the first author visited Kyungpook National University and Hannam University between 25–31 May 2015 to attend the 2015 Conference and Special Seminar of the Jangjeon Mathematical Society. The first author appreciates Professors Dmitry V. Dolgy and Taekyun Kim at Kwangwoon University, Byung-Mun Kim at Dongguk University, Dae San Kim at Sogang University, Seog-Hoon Rim at Kyungpook National University, Jong Jin Seo at Pukyong National University, and many others for their invitation, support, and hospitality.

The first author thanks Professor James F. Peters at University of Manitoba for his recommendation of the reference [1] on 26 May 2015 through the ResearchGate website and thanks Professor Hieu D. Nguyen at Rowan University for his providing the reference [2] on 11 June 2015.

The authors appreciate the anonymous referee for his/her careful corrections to and valuable comments on the original version of this paper.

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