Spatial Kramers–Kronig relations and the reflection of waves

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Abstract

When a planar dielectric medium has a permittivity profile that is an analytic function in the upper or lower half of the complex position plane $x = x' + \mathrm{i}x''$ then the real and imaginary parts of its permittivity are related by the spatial Kramers-Kronig relations. We find that such a medium won't reflect radiation incident from one side, whatever the angle of incidence. Using the spatial Kramers-Kronig relations, one can derive a real part of a permittivity profile from some given imaginary part (or vice versa) such that the reflection is guaranteed to be zero. This result is valid for both scalar and vector wave theories, and may have relevance for designing materials that either efficiently absorb radiation, or for the creation of a new type of anti-reflection surface.

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A wave propagating through an inhomogeneous medium is usually partially reflected, which is often practically undesirable. Although the reflection from an sharp interface can be suppressed through applying an anti–reflection coating [1], less seems to be understood about what is required for a generic inhomogeneous medium not to reflect any radiation. There are some well–known examples of non–reflecting material profiles, such as the hyperbolic secant profile which can be found in Landau and Lifshitz [2] and has been very clearly discussed by Lekner [3] (see [4] for an experimental realisation). More recently the design technique of transformation optics [5, 6] has been a significant development, giving us a recipe for finding inhomogeneous, anisotropic materials (transformation media) that reflect no radiation whatever the incident field [5, 7, 8]. In the same vein, perfectly matched layers [9] are a family of anisotropic lossy media that are often used in computer simulations to mimic an infinitely extended system and are closely connected to transformation media, absorbing a wave without producing any reflection [10–12]. The last few years have seen increasing interest in the property of PT–symmetry in optics [13–15], partly because these media can suppress reflection [16, 17]. For complex permittivities this requires regions of gain ($\text{Im}[\epsilon(x)] < 0$) as well as loss ($\text{Im}[\epsilon(x)] > 0$), and can be related to the use of complex coordinates in transformation optics [18]. Realising any of these non–reflecting materials is challenging, but suitably structured metamaterials [19] are a promising method where a wide range of material parameters can be obtained through the use of specially designed sub–wavelength elements. In particular recent work on so–called '*dispersion engineering*' [20] has seen the simultaneous control of the real and imaginary parts of the permittivity and permeability which is necessary to realise PT–symmetric media as well as the materials proposed in this work.

Here we find a new and very general relation between the real and imaginary parts of a (locally isotropic) planar permittivity profile $\epsilon(x)$ that guarantees zero reflection. The result makes use of the properties of $\epsilon(x)$ at complex values of the spatial coordinate $x = x' + \text{i}x''$ to predict what happens to the wave when x is real. In short: when the permittivity profile $\epsilon(x' + \text{i}x'')$ is an analytic function (i.e. without poles or branch cuts) in the upper or lower half of the complex position plane—and therefore obeys the Kramers–Kronig relations in space—the reflection from respectively the left or the right of the profile vanishes, whatever the angle of incidence. We note at the outset that this condition is only sufficient and not necessary for zero reflection. As a corollary of our finding, if the real part of such a non-reflecting permittivity profile is symmetric about some point in space then the corresponding imaginary part always turns out to be antisymmetric about this point, thus exhibiting PT–symmetry. We stress, however, that the spatial Kramers-Kronig relations do not require any definite symmetry under inversion of space or reversal of time, and thus they lead to an enormous class of reflectionless metamaterials. Furthermore, while needing an appropriate dispersion engineering in space the requisite materials are locally isotropic, non–magnetic, and do not rely in general on the presence of gain or negatively refracting media. This makes our findings quite distinct from those that are based on PT–symmetry, which always require gain, or make use of transformation optics (e.g. [18]), which nearly always require anisotropic magnetic materials.

Consider a monochromatic electromagnetic wave of frequency ω propagating in the x–y plane within a medium with an inhomogeneous permittivity $\epsilon(x)$ that tends to a constant positive value ϵ_b as $x \to \pm\infty$. A schematic of this situation is shown in figure 1a. The two polarizations are TE (electric field along z) and TM (magnetic field along z). For the TE

FIG. 1: We consider a wave propagating in the x–y plane in an inhomogeneous medium with permittivity $\epsilon(x)$ (indicated by the blue shading in panel (a)). The TE polarization has an electric field pointing only along z and the TM polarization has a magnetic field pointing only along z. When the real (blue) and imaginary (red) parts of the permittivity are related to one another by the Kramers–Kronig relations as in panel (b) then the reflection vanishes for all angles of incidence. Panel (b) shows the permittivity profile $\epsilon(x)$ given by equation (11) for the parameters $A = 2.0$ and $\xi = 0.1\lambda$ with $\lambda = 2\pi/k_0$.

polarization we can write the electric field as

$$E_z(x, y) = e_z(x)\mathrm{e}^{\mathrm{i}k_y y} \tag{1}$$

where k_y determines the angle of incidence, and the x–dependent amplitude e_z obeys the 1D Helmholtz equation

$$\frac{d^2 e_z}{dx^2} + \left[K^2 + k_0^2\alpha(x)\right] e_z = 0. \tag{2}$$

In the above equation $K = \pm(\epsilon_b k_0^2 - k_y^2)^{1/2}$, and the permittivity has the assumed form of the positive background contribution ϵ_b plus a spatially varying part

$$\epsilon(x) = \epsilon_b + \alpha(x), \tag{3}$$

with $k_0 = \omega/c$. The spatially varying part of the permittivity $\alpha(x)$ vanishes at large distances from the origin, where the field is made up of plane waves $\exp(\pm\mathrm{i}Kx)$. Now suppose that a right–going wave comes from infinity $x = -\infty$, onto the inhomogeneous permittivity profile. The effect is to produce a scattered field e_s, and we can write the total field as

$$e_z(x) = E_0\mathrm{e}^{\mathrm{i}Kx} + e_s(x). \tag{4}$$

where $K > 0$. Inserting (4) into (2) we find the inhomogeneous differential equation that governs the scattered field

$$\left[\frac{d^2}{dx^2} + K^2 + k_0^2\alpha(x)\right] e_s(x) = -k_0^2\alpha(x)E_0\mathrm{e}^{\mathrm{i}Kx}. \tag{5}$$

One well–known way to solve equation (5) is to expand e_s as a series $e_s = \sum_n e_s^{(n)}$ where the n^{th} term is proportional to the n^{th} power of α. The first term in this series—known as the Born approximation in scattering calculations—can be found immediately and is

$$e_s^{(1)}(x) = -E_0 k_0^2 \int \frac{dk}{2\pi} G(k)\tilde{\alpha}(k - K)\mathrm{e}^{\mathrm{i}kx}, \tag{6}$$

where $\tilde{\alpha}$ is the spatial Fourier transform of $\alpha(x)$ and $G(k)$ is the retarded Green function (η is an infinitesimal positive number)

$$G(k) = \frac{1}{(K + \mathrm{i}\eta)^2 - k^2} \tag{7}$$

FIG. 2: To test the reflection for all angles of incidence we simulated a line source (pointing out of the page) placed within the inhomogeneous permittivity profile shown in figure 1b. The simulations were performed using Comsol Multiphysics, and panels (a) and (b) show the absolute value of the electric field of a line source placed at respective positions $x = -5\lambda$ and $x = +5\lambda$ in the permittivity profile. The region between the vertical dashed lines in (a) indicates the region plotted in figure 1b. Panels (c) and (d) are for identical parameters, but we have taken only the real and imaginary parts of the permittivity respectively. The absence of any oscillations in panel (a) shows that the reflection is completely suppressed for incidence from the left, for all incident angles.

Notice that if $\tilde{\alpha}(k < 0) = 0$ then the Born approximation to the scattered field (6) is made up of only right–going waves, whatever the value of K (i.e. whatever the angle of incidence). This means that to first order in $\alpha(x)$ there is no backscattering from such a permittivity profile. As a first order result this is not all that remarkable, but through examining all the other terms in the series expression for e_s we can see that there is actually no backscattering to any order. To prove this consider the n^{th} term in the scattering series

$$e_s^{(n)}(x) = -k_0^2 \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} G(k)\tilde{\alpha}(k-k')\tilde{e}_s^{(n-1)}(k')\mathrm{e}^{\mathrm{i}kx}. \tag{8}$$

This term is also made up of only right–going waves if (i) the Fourier components of the scattered electric field $\tilde{e}^{(n-1)}$ are zero for left–going waves $\tilde{e}^{(n-1)}(k < 0) = 0$, and (ii) the Fourier components of the permittivity profile are also zero for left–going waves $\tilde{\alpha}(k < 0) = 0$. We have already established that $e_s^{(1)}$ is made up of entirely right–going waves when $\tilde{\alpha}(k < 0) = 0$, and therefore every successive term also contains only right–going waves. There is thus zero back–scattering to every order when $\tilde{\alpha}(k < 0) = 0$. One way to understand this result is to think that when a wave scatters multiple times from an object, for each scattering event there is a momentum change Δk that occurs with an amplitude proportional to $\tilde{\alpha}(\Delta k)$. A permittivity profile that has only positive Fourier components therefore cannot convert a right–going wave to a left–going one. It might appear that this argument relies on a smallness condition for $\alpha(x)$, but in the Supplementary Material we give an alternative argument that does not rely on a series expansion of the electric field, as well as deriving two exact solutions for propagation in such profiles that confirm the effect. The Supplementary Material also contains a numerical investigation to show that an order of magnitude increase in $\alpha(x)$ does not disturb the non–reflecting behaviour.

In light of these properties, such non–reflecting permittivity profiles can be generally written as

$$\epsilon(x) = \epsilon_b + \int_0^{\infty} \frac{dk}{2\pi} \tilde{\alpha}(k)\mathrm{e}^{\mathrm{i}kx}, \tag{9}$$

which is necessarily a complex function of position. The spatial distribution of the reactive and dissipative parts of such a material response *together* completely suppress reflection. To make use of this finding, we note that equation (9) is the same in form as the relationship between the susceptibility in the frequency and time domains which embodies the causality principle [21], and one need only make the replacements $k \to t$ and $x \to \omega$ in (9) in order to recover this well known formula. As a consequence [22, 23], the non–reflecting permittivity profile $\alpha(x)$ is an analytic function in the upper half complex position plane and satisfies the Kramers–Kronig relations in *space*

$$\text{Re}[\alpha(x)] = \frac{1}{\pi}\text{P}\int_{-\infty}^{\infty}\frac{\text{Im}[\alpha(s)]}{s-x}ds. \tag{10}$$

where 'P' indicates the principal part of the integral. Therefore if we were given some $\text{Im}[\alpha(s)]$, say as a (square integrable) function of position, a corresponding real part can be constructed from (10) such that the reflection from the complex susceptibility profile is zero. We note that if the imaginary part of $\alpha(x)$ is symmetric about $x = 0$, then the real part calculated from (10) will be antisymmetric, and vice versa. Therefore the Kramers–Kronig relations generate a whole family of permittivity profiles that exhibit PT–symmetry ($\alpha(-x) = \alpha^*(x)$). Likewise, we also have a whole family of non–reflecting profiles where $\alpha(x)$ exhibits PT–*antisymmetry* ($\alpha(-x) = -\alpha^*(x)$), a property that has already been associated with zero back–scattering in optics [24], just as PT–symmetry [25]. Actually, even purely lossy periodic media can be engineered [26] such that their Bragg reflection from one side vanishes when the real and imaginary parts of their susceptibility are spatially out of phase, which is a characteristic property of Hilbert transform pairs. However, our findings are more general than these known results, as they are also compatible with non–reflecting profiles exhibiting no definite PT–symmetry at all.

As an initial illustration of this finding, we consider the simplest example: a permittivity profile with a single pole in the lower half position plane

$$\epsilon(x) = \epsilon_b + A\frac{\text{i} - x/\xi}{1 + (x/\xi)^2} \tag{11}$$

where ξ sets the spatial scale of the profile, and A the amplitude. Equation (11) is plotted in figure 1 and takes a form which would be very familiar if the x–axis represented frequency rather than space. The non–reflecting behaviour is demonstrated in figure 2 which shows the absolute value of the electric field for a point source (a line source in 2D) placed either side of $x = 0$, and compares the behaviour of the full profile (11), versus its real and imaginary parts separately.

We note that, as is well known, the Helmholtz equation (2) is equivalent to a Schrödinger equation in which $-\alpha(x)$ plays the role of a potential profile. Thus, the spatial Kramers–Kronig relations generate a large family of *complex* non-reflecting potential profiles. Needless to say, as the relation given by Eq.(9) is a sufficient, but not necessary condition, real non–reflecting profiles also exist which are perfectly transparent (for example, as mentioned in the introduction the potential $V(x) = U_0\,\text{sech}^2(x/a)$ is known to be non-reflecting for quantum particles when U_0 takes specific values [2, 3]).

The above analysis was carried out for TE polarization, but how is TM polarized radiation affected by the permittivity profile (3)? This polarization obeys a different wave equation

$$\boldsymbol{\nabla}\cdot[\epsilon^{-1}(x)\boldsymbol{\nabla}H_z] + k_0^2 H_z = 0. \tag{12}$$

Writing $\epsilon^{-1}(x) = \epsilon_b^{-1} + \beta(x)$ and $H_z = h_z(x)e^{ik_y y}$ equation (12) becomes

$$\frac{d^2h_z(x)}{dx^2} + K^2h_z(x) = -\epsilon_b\left[\frac{d}{dx}\left(\beta(x)\frac{dh_z(x)}{dx}\right) - k_y^2\beta(x)h_z(x)\right] \tag{13}$$

FIG. 3: In cases where the permittivity has neither poles nor zeros in the upper half position plane, then the reflection also vanishes for the TM polarization. Panel (a) shows the field of a dipole source aligned along the y axis (so that the field is TM polarized), and placed at $x = -5\lambda$ in front of the permittivity profile shown in panel (e). Panel (b) shows a dipole radiating in the same profile but at $x = +5\lambda$. Panels (c–d) show the reflection from just the real and imaginary parts of the profile respectively. Panel (e) shows the permittivity profile, the imaginary part of which is given by (15) with $\epsilon_b = 2$, $L = 1.5\lambda$, $\xi = 0.1\lambda$, and $h = 4.5$. The real part of the profile was numerically calculated using (10).

Comparing this with (2) we can show that the equivalent of (8) is given by

$$h_s^{(n)}(x) = \epsilon_b \int \frac{dk}{2\pi} \int \frac{dk'}{2\pi} G(k)\tilde{\beta}(k-k')(k_y^2 + kk')\tilde{h}_s^{(n-1)}(k')\mathrm{e}^{\mathrm{i}kx} \tag{14}$$

Therefore if $\tilde{\beta}(k<0) = 0$ then reflection of the TM polarization is also suppressed. However, it is not necessarily the case that both $\tilde{\alpha}(k<0) = 0$ and $\tilde{\beta}(k<0) = 0$. For both equations to hold we need both $\epsilon(x) - \epsilon_b$ and $\epsilon^{-1}(x) - \epsilon_b^{-1}$ to be analytic functions in the upper half complex position plane. In particular, if $\epsilon(x)$ satisfies the spatial Kramers-Kronig relations it will be free of zeros in the upper half plane when $\mathrm{Im}[\epsilon(x)] > 0$ along the real axis [27]. Therefore lossy media obeying the Kramers–Kronig relations in space will not reflect radiation of either polarization for any angle of incidence. Meanwhile for profiles exhibiting gain (or more precisely, zeros in the upper half position plane) reflection will be suppressed for only one of the two polarizations.

As example to demonstrate the general applicability of (3) for eliminating the reflection of both polarizations we take a more complicated permittivity profile with an imaginary part given by

$$\mathrm{Im}[\epsilon(x)] = \frac{h(L-x)}{4L}[1 + \mathrm{erf}(x/\xi)][1 + \mathrm{erf}((L-x)/\xi)] \tag{15}$$

which represents a smoothed triangle function, where 'erf' is an error function, h is the height, L the length, and ξ characterizes the smoothness of the corners. Numerically calculating the integral (10) we obtain the real part of the permittivity that, when added to 'i' times (15) is necessary to reduce the reflection to zero. The full function is shown in figure 3(e), and unlike (11) has no definite parity symmetry. Figures 3(a–d) then show that the combination of the real and imaginary parts of the resulting profile suppress the reflection of TM polarized waves incident from the left (we have also verified that this profile is non–reflecting for TE waves).

A general result in the theory of reflection is that when waves are incident onto a generic planar interface, at angles close to grazing ($k_y \sim k_0$) the reflectivity usually approaches unity (see e.g. [28]). The above findings at first sight appear to contradict this result. However, strictly speaking the profiles we have calculated are of infinite extent so that there is no 'interface' to speak of, as can be seen from the long tails evident in the real parts of $\epsilon(x)$ shown in figures 1 and 3. We could however confine these infinite profiles to a finite region of space through truncating these tails, multiplying the profile by an envelope function $U(x)$

$$\alpha'(x) = \alpha(x)U(x)$$
$$\to \tilde{\alpha}'(k) = \int_0^\infty \frac{dk'}{2\pi}\tilde{U}(k'-k)\tilde{\alpha}(k')$$

where $U(x) = 0$ at some distance from the centre of the profile. The price of prematurely truncating the long tails is that $\alpha'(k<0) \neq 0$. However if $\tilde{U}(\Delta k)$ is sharply peaked around $\Delta k = 0$, $\tilde{\alpha}'$ will only be non–zero for negative k of a small magnitude. For example if we use $U(x) = \exp(-(x/a)^2)$, then $\tilde{U}(\Delta k) = (a/2\sqrt{\pi})\exp(-a^2(\Delta k)^2/4)$, which rapidly goes to zero beyond around $\Delta k = -2/a$. The consequence of this is that waves close to grazing will now be reflected by the profile, but the angle at which this is significant approaches $\pi/2$ as a is made ever larger. A numerical investigation of this effect can be found in the supplementary material.

While the non-reflecting property of specific classes of potential profiles have long been studied [29], more recently the analogous behaviour of PT–symmetric complex susceptibility profiles have been considered in optics. Yet, to the best of our knowledge the simple and much more general relation here discussed between the one-sided absence of reflection and the analytic extension of the spatially dependent susceptibility to one half of the complex position plane has not been pointed out before. We have shown how the corresponding Kramers-Kronig relations in space can be used to generate a very large family of non-reflecting profiles. If the profile is also free of zeros in the upper or lower half complex position plane (the half plane being determined by whether reflection vanishes from the left or the right) then the profile is also non–reflecting for both polarizations. In practice the catch is that the profiles have very long tails which must be truncated, and where one chooses to perform the truncation determines the range of angles that are not reflected. Nevertheless, the advantage of this method is that it does not require magnetic or anisotropic media, and only requires control over the real and imaginary parts of an isotropic permittivity (without necessarily having gain or taking negative values). The method is also in principle applicable to any wave equation, including the Schrödinger equation. While the Kramers-Kronig relations in the frequency domain are a cornerstone of optics, it is hoped that the spatial Kramers-Kronig relations will provide at least some guidance and insight in the development of metamaterials based on judiciously chosen susceptibility profiles.

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Author contributions

S.A.R.H. devised the theory, performed the simulations and wrote the manuscript. M.A. and G.C.L.R. worked on the theory and co–wrote the manuscript.

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[1] Macleod, H. A. *Thin–Film Optical Filters*, Institute of Physics Publishing, London (2001).

[2] Landau, L. D. & Lifshitz, E. M. *Quantum Mechanics*, Butterworth-Heinemann, Oxford (2003).

[3] Lekner, J. Reflectionless eigenstates of the sech^2 potential *Am. J. Phys.* **75**, 1151 (2007).

[4] Thekkekara, L. V., Achanta, V. G. & Gupta, S. D. Optical reflectionless potentials for broadband, omnidirectional antireflection *Opt. Exp.* **22**, 17382 (2014).

[5] Pendry, J. B., Schurig, D. & Smith, D. R. Controlling Electromagnetic Fields *Science* **312**, 1780 (2006).

[6] Leonhardt, U. & Philbin, T. G. General Relativity in Electrical Engineering *New J. Phys.* **8**, 247 (2006).

[7] Valentine, J., Li, J., Zentgraf, T., Bartal, G. & Zhang, X. An optical cloak made of dielectrics *Nature Mat.* **8**, 568 (2009).

[8] Pendry, J. B. Negative Refraction Makes a Perfect Lens *Phys. Rev. Lett.* **85**, 3966 (2000).

[9] Berenger, J.-P. A Perfectly Matched Layer for the Absorption of Electromagnetic Waves *Journal of Computational Physics* **114**, 185 (1994).

[10] Teixeira, F. L. & Chew, W. C. Differential Forms, Metrics, and the Reflectionless Absorption of Electromagnetic Waves *J. Electro. Wav. Appl.* **13**, 665 (1999).

[11] Popa, B.-I. & Cummer, S. A. Complex coordinates in transformation optics *Phys. Rev. A* **84**, 063837 (2011).

[12] Odabasi, H., Teixeira, F. L. & Chew W. C. Impedance–matched absorbers and optical pseudo black holes *J. Opt. Soc. Am. B*, **28**, 1317 (2011).

[13] Kottos, T. Optical physics: broken symmetry makes light work *Nature Phys.* **6**, 166 (2010).

[14] Makris, K. G., El–Ganainy, R., Christodoulides, D. N. & Musslimani, Z. H. PT–Symmetric Periodic Optical Potentials *Int. J. Theor. Phys.* **50**, 1019 (2011).

[15] Longhi, S. Optical Realization of Relativistic Non-Hermitian Quantum Mechanics *Phys. Rev. Lett.* **105**, 013903 (2010)

[16] Lin, Z. et al. Unidirectional Invisibility Induced by PT-Symmetric Periodic Structures *Phys. Rev. Lett.* **106**, 213901 (2011).

[17] Regensburger, A. et al. Paritytime synthetic photonic lattices *Nature*, **488**, 167 (2012).

[18] Castaldi, G., Savoia, S., Galdi, V., Alù, A. & Engheta, N. PT Metamaterials via Complex-Coordinate Transformation Optics *Phys. Rev. Lett.* **110**, 173901 (2013).

[19] Cai, W. & Shalaev, V. *Optical Metamaterials*, Springer, New York (2010).

[20] Ye, D. et al. Ultrawideband Dispersion Control of a Metamaterial Surface for Perfectly-Matched-Layer-Like Absorption *Phys. Rev. Lett.* **111**, 187402 (2013)

[21] Landau, L. D. & Lifshitz, E. M. *Electrodynamics of Continuous Media*, Butterworth–Heinemann, Oxford (2004).

[22] Titchmarsh, E. *Introduction to the theory of Fourier integrals*, Clarendon Press, Oxford (1986).

[23] Lucarini, V., Bassani, F., Peiponen, K.-E. & Saarinen, J. J. Dispersion theory and sum rules in linear and nonlinear optics *La Rivista del Nuovo Cimento* **26**, 12, 1 (2003).

[24] Ge, L. & Türeci, H. E. Antisymmetric PT-photonic structures with balanced positive- and negative-index materials *Phys. Rev. A* **88**, 053810 (2013).

[25] L. Feng et al. Experimental demonstration of a unidirectional reflectionless parity-time metamaterial at optical frequencies *Nature Mat.* **12**, 108-113 (2012).

[26] Wu, J.-H., Artoni, M. & La Rocca, G. C. Non-Hermitian Degeneracies and Unidirectional Reflectionless Atomic Lattices *Phys. Rev. Lett.* **113**, 123004 (2014).

[27] Landau, L. D. & Lifshitz, E. M. *Statistical Physics (Part 1)*, Butterworth–Heinemann, Oxford (2003).

[28] Lekner, J. *Theory of Reflection*, Martinus Nijhoff Publishers, Dordrecht (1987).

[29] Kay, I. & Moses, H. E. Reflectionless Transmission through Dielectrics and Scattering Potentials *J. Appl. Phys.* **27**, 1503 (1956).