Sliding Mode Estimation Schemes for Incipient Sensor Faults ⋆

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Abstract

This paper proposes a new method for the analysis and design of sliding mode observers for sensor fault reconstruction. The proposed scheme addresses one of the restrictions inherent in other sliding mode estimation approaches for sensor faults in the literature (which effectively require the open-loop system to be stable). For open-loop unstable systems, examples can be found, for certain combinations of sensor faults, for which existing sliding mode and unknown input linear observer schemes cannot be employed, to reconstruct faults. The method proposed in this paper overcomes these limitations. Simulation results demonstrate the effectiveness of the design framework proposed in the paper.

Key words: Fault detection; Fault isolation; Observers; Sliding mode

1 Introduction

In active fault tolerant control (FTC), one of the important components is the fault detection and isolation (FDI) scheme [17]. The FDI scheme detects and isolates the faults that exist in the system and initiates controller reconfiguration to allow the faults/failures to be mitigated and to enable safe degraded closed-loop performance. Most model based FDI schemes are residual based and an analytical redundancy approach is adopted to compare the system measurements with a mathematical model of the system, and the difference provides residual signals from which the faults/failures can be detected and isolated. Work on residual based FDI is discussed extensively in the literature: see for example [3]. Some active fault tolerant control schemes however require more information regarding the faults, where estimates of the actuator efficiency are required to allow the FTC scheme to accommodate the faults/failures. This information can be provided by schemes such as those proposed in [23] which use the so-called modified two stage Kalman filter. In terms of sensor fault tolerant control, the reconstructions can be used directly to 'correct' the sensor measurements before the erroneous information is used by the controller [10].

In [8], the novel idea of using the 'equivalent output error injection signal' to reconstruct faults was introduced. Further work, aimed at reducing the system constraints associated with the result in [8], has recently appeared in the literature [11,12,5,16,2,14]. With the exception of [2,14] most of this work has focussed on an unknown input formulation, which from the viewpoint of fault detection, is associated with an actuator fault reconstruction problem. This paper is concerned with sensor fault reconstruction. Two methods for sensor fault reconstruction were proposed in [20]. However the problem setup is different to the one in this paper, and no uncertainty was considered. In both methods, two sliding mode observers are used in cascade. The first approach ignores the effects of the derivative of the fault (modelled as an additive perturbation) and requires the open loop system matrix to be nonsingular. The second method in [20] employs a different configuration but requires the open-loop plant to be stable. The latter approach was later improved to achieve robust sensor fault estimation by Tan & Edwards [21] using a Linear Matrix Inequalities (LMI) formulation where open loop stability is no longer a necessary condition. However for open loop unstable systems, with certain classes of faults, examples can be found such that the method in [21] is not applicable. This paper proposes a new observer design for sensor fault reconstruction which addresses this restriction. In particular the proposed observer designs are applicable for open-loop stable and unstable systems.

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2 Preliminaries

Consider a dynamical system affected by sensor faults described by

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (1)$$
$$y(t) = Cx(t) + Ff_o(t) \quad (2)$$

where $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}, C \in \mathbb{R}^{p\times n}$ and $F \in \mathbb{R}^{p\times q}$, and the matrices C and F have full row and column rank respectively. Also assume that the triple (A, B, C) is a minimal realization of the fault-free input/output behaviour of the system. The function $f_o : \mathbb{R}_+ \to \mathbb{R}^q$ is unknown but smooth and bounded so that

$$\|f_o(t)\| \le \alpha(t) \quad (3)$$

where $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is a known function. The signal $f_o(t)$ represents (additive) sensor faults and F represents a distribution matrix, which indicates which of the sensors are prone to possible faults.

Assumption 0: The dimensions of the state, output and fault vectors satisfy $n \ge p > q$.

Without loss of generality, it can be assumed that the outputs of the system have been reordered (and scaled if necessary) so that the matrix F has a structure

$$F = \begin{bmatrix} 0 \\ I_q \end{bmatrix} \quad (4)$$

Remark 1: The assumption that only certain sensors are fault prone is a limitation. However in practical situations, some sensors may be more vulnerable to damage or may be more sensitive or delicate in terms of construction than others, and so such a situation is not unrealistic. Also certain key sensors may have back-ups (hardware redundancy) and so essentially a fault free signal can be assumed from a certain subset of the sensors.

The objective is to design a *sliding mode* observer [22,6] in order to *reconstruct* the faults $f_o(t)$ using only measurements of $y(t)$ and $u(t)$. Suppose the signal f_o is smooth and so assume

$$\xi(t) := \dot{f}_o(t) \quad (5)$$

In this paper it is assumed that the sensor faults are incipient and so $\|\xi(t)\|$ is small in magnitude, but over time the effects of the fault compound, and become significant. Equations (1) and (5) can be combined to give a system with states $x_a := \text{col}(x, f_o)$ in the form

$$\begin{bmatrix} \dot{x}(t) \\ \dot{f}_o(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}}_{A_a} \begin{bmatrix} x(t) \\ f_o(t) \end{bmatrix} + \underbrace{\begin{bmatrix} B \\ 0 \end{bmatrix}}_{B_a} u(t) + \underbrace{\begin{bmatrix} 0 \\ I_q \end{bmatrix}}_{F_a} \xi(t) \quad (6)$$

$$y(t) = \underbrace{\begin{bmatrix} C & F \end{bmatrix}}_{C_a} \begin{bmatrix} x(t) \\ f_o(t) \end{bmatrix} \quad (7)$$

where $A_a \in \mathbb{R}^{(n+q)\times(n+q)}$, $B_a \in \mathbb{R}^{(n+q)\times m}$, $C_a \in \mathbb{R}^{p\times(n+q)}$ and $F_a \in \mathbb{R}^{(n+q)\times q}$. Equations (6) and (7) represent an unknown input problem for the triple (A_a, F_a, C_a) driven by the unmeasurable signal $\xi(t)$. If a good estimate of x_a can be computed, then f_o can be estimated as the last q states of x_a.

From (7), and based on the structure of F in (4),

$$C_a = \begin{bmatrix} C & F \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ C_2 & I_q \end{bmatrix} \quad (8)$$

where $C_1 \in \mathbb{R}^{p-q\times n}$ and $C_2 \in \mathbb{R}^{q\times n}$. Notice that the triple (A_a, F_a, C_a) is inherently relative degree one since $C_aF_a = F$ and $\text{rank}(F) = q$ by assumption.

Lemma 1 *The triple (A_a, F_a, C_a) is minimum phase if and only if (A, C_1) is detectable.*

Proof: Consider the Rosenbrock system matrix associated with (A_a, F_a, C_a):

$$R(s) = \begin{bmatrix} sI - A & 0 & 0 \\ 0 & sI & -I_q \\ C_1 & 0 & 0 \\ C_2 & I_q & 0 \end{bmatrix} \quad (9)$$

The invariant zeros of (A_a, F_a, C_a) are given by the values of $s \in \mathbb{C}$ where $R(s)$ loses normal rank. It is clear from (9) that

$$rank\ R(s) = rank \begin{bmatrix} sI - A & 0 \\ C_1 & 0 \\ C_2 & I_q \end{bmatrix} + q$$

and so $R(s)$ loses rank if and only if

$$rank \begin{bmatrix} sI - A \\ C_1 \end{bmatrix} < n$$

It follows from the PBH rank test that the invariant zeros of the triple (A_a, F_a, C_a) are the unobservable modes of (A, C_1). Consequently (A_a, F_a, C_a) is minimum phase if and only if (A, C_1) is detectable. ■

Lemma 2 *The pair (A_a, C_a) is observable if (A, C_1) does not have an unobservable mode at zero.*

Proof: From the PBH test and the definition of A_a and C_a in (6) and (7), the pair (A_a, C_a) is observable if and only if

$$rank \left[\begin{array}{c|c} sI - A & 0 \\ 0 & sI_q \\ \hline C_1 & 0 \\ C_2 & I_q \end{array}\right] = n + q, \quad \text{for all } s \in \mathbb{C} \quad (10)$$

For $s \neq 0$

$$\begin{bmatrix} sI-A & 0 \\ 0 & sI_q \\ C_1 & 0 \\ C_2 & I_q \end{bmatrix} \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix} = 0 \Rightarrow \eta_2 = 0 \Rightarrow \begin{bmatrix} sI-A \\ C_1 \\ C_2 \end{bmatrix} \eta_1 = 0 \Rightarrow \eta_1 = 0 \quad (11)$$

since (A, C) is observable, and so for $s \neq 0$, the rank of the PBH matrix in (10) is $n+q$. When $s = 0$,

$$rank \begin{bmatrix} sI-A & 0 \\ 0 & sI_q \\ C_1 & 0 \\ C_2 & I_q \end{bmatrix}_{s=0} = rank \begin{bmatrix} -A & 0 \\ C_1 & 0 \\ C_2 & I_q \end{bmatrix} = rank \begin{bmatrix} -A \\ C_1 \end{bmatrix} + q \quad (12)$$

Consequently (10) holds if and only if

$$rank \begin{bmatrix} -A \\ C_1 \end{bmatrix} = n$$

A sufficient condition for this is that (A, C_1) does not have an unobservable mode at $s = 0$. ■

Corollary 1 *If the open loop system in (1) is stable the pair (A_a, C_a) is observable.*

■

Assume without loss of generality that C from (2) is given as

$$C = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & I_{p-q} \\ 0 & I_q & 0 \end{bmatrix} \quad (13)$$

For any system with C of full row rank, this canonical form can be achieved by a change of coordinates in (1)–(2). Change coordinates in the augmented system in (6) and (7) according to $x_a \mapsto T x_a$ where

$$T = \begin{bmatrix} I_n & 0 \\ C_2 & I_q \end{bmatrix} \quad (14)$$

The system triple in the new coordinates is $(T A_a T^{-1}, T F_a, C_a T^{-1})$ where

$$T A_a T^{-1} = \begin{bmatrix} I_n & 0 \\ C_2 & I_q \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -C_2 & I_q \end{bmatrix} = \begin{bmatrix} A & 0 \\ C_2 A & 0 \end{bmatrix} \quad (15)$$

and

$$C_a T^{-1} = \begin{bmatrix} C_1 & 0 \\ C_2 & I_q \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -C_2 & I_q \end{bmatrix} = \begin{bmatrix} C_1 & 0 \\ 0 & I_q \end{bmatrix} = \begin{bmatrix} 0 & I_p \end{bmatrix} \quad (16)$$

from the definition of C_1 in (13). It is also easy to check that

$$T F_a = F_a = \begin{bmatrix} 0 \\ I_q \end{bmatrix} \quad (17)$$

where F_a is defined in (6).

In the original x_a coordinates, the states corresponding to f_o are given by the last q components i.e.

$$f_o(t) = C_f x_a(t) \quad (18)$$

where

$$C_f := \begin{bmatrix} 0_{q \times n} & I_q \end{bmatrix} \quad (19)$$

After the change of coordinates $x_a \mapsto T x_a$ the new matrix relating the states to the fault signals f_o is

$$C_f T^{-1} = \begin{bmatrix} 0 & I_q \end{bmatrix} \begin{bmatrix} I & 0 \\ -C_2 & I_q \end{bmatrix} = \begin{bmatrix} 0_{q \times (n-p)} & -I_q & 0_{q \times (p-q)} & I_q \end{bmatrix} \quad (20)$$

using C_2 as defined in (13).

Remark 2: Although the problem tackled here is similar to the one considered in [20,21], the approach is different. The work in [20] employed observers in cascade, and [20,21] both consider filtered output measurements as the basis of the observer design. The net effect is that, in both cases, the sensor signal estimation problem becomes an unknown input problem. This unknown input is then reconstructed using the concept of equivalent output error injection. In this paper, the robustness properties of sliding mode observers will be exploited. In this respect, the approach taken here is more akin to the unknown input approaches [4,19] whereby the fault signal to be estimated is augmented with the plant state vector, then the augmented state vector is robustly estimated using an observer.

Remark 3: The problem formulation in (6)-(7) constitutes a 'classical' unknown input observer situation. However the form in (6)-(7) is very specific, in that it is inherently relative degree one by construction (i.e rank$(C_a F_a) = q$). There has been extensive recent research into the use of sliding mode observers for unknown input problems – although the focus has been primarily directed at the situation where the relative degree one requirement is not met: see for example [11,12,5,16,2,14]. As far as the authors are aware, much less attention has been directed towards the problem of obviating minimum phase limitations [1]. Thus, although the main motivation in this paper is to tackle the problem of sensor fault reconstruction, the problem may also be viewed as one of unknown input reconstruction in nonminimum phase systems. It is important to note that classical linear unknown input observers (UIO) also cannot be employed in this situation [9,4,19].

3 Main Results

This section will consider a system, arising from the augmented sensor fault system (6)-(7), of the form

$$\dot{x}_a(t) = A_a x_a(t) + B_a u(t) + F_a \xi(t) \qquad (21)$$
$$y(t) = C_a x_a(t) \qquad (22)$$

where the faults $f_o(t) = C_f x_a(t)$ and C_f is defined in (19). Without loss of generality, (following the series of transformations described above) the matrices A_a, F_a, C_a and C_f are assumed to have the forms given in (15), (16), (17) and (20) respectively. Write

$$A_a = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{211} & \\ & A_{22} \\ A_{212} & \end{array}\right] \qquad (23)$$

where $A_{11} \in \mathbb{R}^{(n+q-p)\times(n+q-p)}$. Define A_{211} as the top $p-q$ rows of A_{21}. By construction, the unobservable modes of (A_{11}, A_{211}) are the invariant zeros of (A_a, F_a, C_a) [8]. Also define $F_2 \in \mathbb{R}^{p\times q}$ as the bottom p rows of F_a so from (17)

$$F_2 = \begin{bmatrix} 0_{(p-q)\times q} \\ I_q \end{bmatrix} \qquad (24)$$

Assumption 1: Assume that the system triple (A, B, C) is such that the new pair (A, C_1) resulting from the reordering and partitioning of the outputs as shown in (6)-(8), does not have any unobservable modes at the origin.

Remark 4: It follows from Assumption 1 and Lemma 1, that the pair (A_a, C_a) is observable. Using the results of Lemma 1, Assumption 1 is equivalent to the assumption that (A_a, C_a) is observable. It is then straightforward to show using the PBH test that the pair (A_{11}, A_{21}) from the partition in (23) is also observable.

3.1 Observer analysis

For the system in (6) - (7) a sliding mode observer of the form

$$\dot{z}(t) = A_a z(t) + B_a u(t) - G_l e_y(t) + G_n \nu \qquad (25)$$

will be considered. In (25) the discontinuous output error injection term

$$\nu = -\rho(t, y, u)\frac{P_o e_y}{\|P_o e_y\|} \qquad \text{if } e_y \neq 0 \qquad (26)$$

where $e_y(t) := C_a z(t) - y(t)$ is the output estimation error and P_o is a symmetric positive definite (s.p.d.) matrix. The matrix G_l is a traditional Luenberger observer gain used to make $(A_a - G_l C_a)$ stable. The scalar function $\rho(\cdot)$ must be an upper bound on the uncertainty and the faults; for details see [21].

An appropriate gain G_n for the nonlinear injection term ν in (25) has the structure

$$G_n = \begin{bmatrix} -L \\ I_p \end{bmatrix} \quad \text{where} \quad L = \begin{bmatrix} L_1 & L_2 \end{bmatrix} \qquad (27)$$

and $L_1 \in \mathbb{R}^{(n+q-p)\times(p-q)}$ and $L_2 \in \mathbb{R}^{(n+q-p)\times q}$ represent design freedom [7,22]. In particular the gain L must be chosen so that $A_{11} + LA_{21}$ is stable.

If $e := z - x_a$ is the estimation error, then from (21) and (25)

$$\dot{e}(t) = (A_a - G_l C_a)e(t) - F_a \xi + G_n \nu \qquad (28)$$

where ξ is defined in (5), and represents the derivative of the sensor fault signal. For an appropriate choice of $\rho(t, y, u)$ in (26), it can be shown using arguments similar to those used in [21], that an ideal sliding motion takes place on

$$\mathcal{S} = \{e : C_a e = 0\}$$

in finite time: for details see [21]. During the ideal sliding motion [22,6], $e_y = \dot{e}_y = 0$ and the discontinuous signal ν must take on average a value to compensate for ξ to maintain sliding. The average quantity, denoted by ν_{eq}, is referred to as the *equivalent output error injection term* (the natural analogue of the concept of equivalent control [22]). It follows from (28) that during sliding

$$\nu_{eq} = -(C_a G_n)^{-1}(C_a A_a e - C_a F_a \xi) \qquad (29)$$

Substituting from (29) into (28), it follows that the sliding motion is governed by

$$\dot{e} = (A_a - G_n(C_a G_n)^{-1} C_a A_a)e - (F_a - G_n(C_a G_n)^{-1} C_a F_a)\xi \qquad (30)$$

Ideally the effect of the unknown disturbance ξ on the state estimation, particularly on the states which correspond to estimates of f_o, needs to be minimized.

The effect of ξ on the estimate of f_o is given by $C_f e(t)$, where $e(t)$ evolves according to (30) since $\hat{f}_o - f_o = C_f e(t)$ if $\hat{f}_o(t) := C_f z(t)$. Therefore, the impact of ξ on the estimate of f_o can be expressed as $G(s)\xi$ where

$$G(s) := \left[\begin{array}{c|c} (A_a - G_n(C_a G_n)^{-1} C_a A_a) & (F_a - G_n(C_a G_n)^{-1} C_a F_a) \\ \hline C_f & 0 \end{array}\right] \qquad (31)$$

For accurate estimation of the faults f_o, the transfer function matrix $G(s)$ must be 'small' and for complete decoupling $G(s) = 0$. Here, the $\mathcal{H}_\infty$ norm of $G(s)$ will be minimized by choice of G_n.

Partition the state error vector e from (28) conformably with the canonical form in (23) as $\text{col}(e_1, e_y)$. One way to identify the reduced order sliding motion is to perform a further change of coordinates according to the nonsingular matrix

$$T_L = \begin{bmatrix} I_{n+q-p} & L \\ 0 & I_p \end{bmatrix} \qquad (32)$$

so that

$$e = (e_1, e_y) \mapsto (e_1 + Le_y, e_y) \equiv (\tilde{e}_1, e_y) =: \tilde{e} \tag{33}$$

It can be easily verified that in the coordinate system in (33), during the sliding motion, the error system i.e. (the reduced order sliding motion) can be written as

$$\dot{\tilde{e}}_1(t) = \big(A_{11} + L_1A_{211} + L_2A_{212}\big)\tilde{e}_1(t) + L_2\xi \tag{34}$$

$$\dot{e}_y(t) = e_y(t) = 0 \tag{35}$$

The gain matrices L_1 and L_2 needed to be chosen to ensure $A_{11} + LA_{211} + L_2A_{212}$ is stable for the sliding motion to be stable. Therefore the effect of ξ on the estimation $\hat{f}_o$ is given by $C_f e = \tilde{C}_f \tilde{e}$ where $\tilde{C}_f = C_f T_L^{-1}$ and C_f is given in (19). It can be verified

$$\tilde{C}_f = \begin{bmatrix} 0_{n-p\times q} & I_q & * \end{bmatrix} \tag{36}$$

where $*$ represents a matrix which plays no part in the subsequent analysis. During the sliding motion,

$$\tilde{C}_f\tilde{e} = \begin{bmatrix} 0_{n-p\times q} & I_q \,|\, * \end{bmatrix} \begin{bmatrix} \tilde{e}_1 \\ e_y \end{bmatrix} = \underbrace{\begin{bmatrix} 0_{n-p\times q} & I_q \end{bmatrix}}_{C_e} \tilde{e}_1 \tag{37}$$

since $e_y \equiv 0$ during sliding. Consequently,

$$G(s)\xi = \tilde{G}(s)\xi \tag{38}$$

where

$$\tilde{G}(s) := \left[\begin{array}{c|c} A_{11} + L_1A_{211} + L_2A_{212} & L_2 \\ \hline C_e & 0 \end{array}\right] \tag{39}$$

and C_e is defined in (37). As argued in Remark 4, the pair (A_{11}, A_{21}) is observable, and so from the partition of A_{21} in (23) to obtain A_{211} and A_{212}, it follows that there exist L_1 and L_2 so that $A_{11} + L_1A_{211} + L_2A_{212}$ is stable.

Proposition 1 *If (A_a, F_a, C_a) from (21)-(22) is minimum phase, then a sliding mode observer of the form in (25) exists such that $\hat{f}_o = C_f x_a \to f_o$ as $t \to \infty$.*

Proof: If (A_a, F_a, C_a) from (21)-(22) is minimum phase, then the pair (A_{11}, A_{211}) is detectable since it can be shown that the unobservable modes of (A_{11}, A_{211}) exactly correspond to the invariant zeros of (A_a, F_a, C_a) [8]. Consequently there exists an L_o such that $(A_{11} + L_oA_{211})$ is stable. Therefore the selection $L_1 = L_o$ and $L_2 = 0$ is a feasible choice which makes $A_{11} + L_1A_{211} + L_2A_{212} = A_{11} + L_oA_{211}$ stable. Since $L_2 = 0$, equation (34) collapses to $\dot{\tilde{e}}_1(t) = (A_{11} + L_oA_{211})\tilde{e}_1(t)$. Asymptotic tracking of the states takes place since $(A_{11} + L_oA_{211})$ is stable and therefore $\tilde{e}_1(t) \to 0$ as $t \to \infty$. It follows $\hat{f}_o(t) - f(t) = C_f e(t) \to 0$ since $e(t) \to 0$, and the fault is estimated asymptotically. ■

Proposition 2 *If the plant system matrix A from (1) is stable, then a sliding mode observer of the form in (25) exists such that $\hat{f}_o = C_f z_a \to f_o$ as $t \to \infty$.*

Proof: If the plant system matrix A from (1) is stable, then (A, C_1) is detectable and from Lemma 1, (A_a, F_a, C_a) is minimum phase. Therefore from Proposition 1, $\hat{f}_o = C_f z_a \to f_o$ since $e(t) \to 0$. ■

Remark 5: If A from (1) is unstable then for certain fault conditions, (A, C_1) may be unobservable and perfect reconstruction is not possible. An example of this is discussed in §4 in the sequel. Furthermore if (A, C_1) is undetectable then from Lemma 1, (A_a, F_a, C_a) is non-minimum phase. Then as argued in [9] classical unknown input observers UIOs also cannot be employed to reject the unknown input $\xi(t)$: see for example [19,4]. The next subsection considers the ramifications of this.

3.2 Observer Design

The observer described in this section embodies the same design philosophies as those proposed in [21]. In this paper, however, there is one major difference from the results described in [21]. The developments in [21] are completely predicated on the assumption that (A_a, F_a, C_a) is minimum phase, and as a consequence, the fault signal can be perfectly replicated if no uncertainty is present. As argued in the previous section, this situation can be recovered as the special case when L from (27) takes the form $[\, L_1 \;\; 0 \,]$. The full block structure in (27) considered in this paper allows the triple (A_a, F_a, C_a) to be non-minimum phase thus broadening the class of systems for which the results are applicable.

As in [21], define a Lyapunov matrix for the error system in (28) to have the form

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^{\mathrm{T}} & P_{22} \end{bmatrix} \tag{40}$$

where $P_{11} \in \mathbb{R}^{(n+q-p)\times(n+q-p)}$ is s.p.d. Let $G_l \in \mathbb{R}^{(n+q)\times p}$ be any matrix which satisfies

$$P\underbrace{(A_a - G_lC_a)}_{A_0} + (A_a - G_lC_a)^{\mathrm{T}}P < 0 \tag{41}$$

Here, the design of the linear gain G_l for the sliding mode observer from (25) will be chosen to satisfy

$$\begin{bmatrix} PA_0 + A_0^{\mathrm{T}}P & P(G_lD - B_d) & E^{\mathrm{T}} \\ (G_lD - B_d)^{\mathrm{T}}P & -\gamma_0 I_{p+q} & 0 \\ E & 0 & -\gamma_0 I_q \end{bmatrix} < 0 \tag{42}$$

The matrices $B_d \in \mathbb{R}^{(n+q)\times(p+q)}$, $D \in \mathbb{R}^{p\times(p+q)}$ in (42) are defined as

$$B_d := \begin{bmatrix} 0 & F_a \end{bmatrix} \tag{43}$$

$$D := \begin{bmatrix} D_1 & 0 \end{bmatrix} \tag{44}$$

where $D_1 \in \mathbb{R}^{p\times p}$, F_a is defined in (17), and

$$E := \begin{bmatrix} C_e & F_2^{\mathrm{T}} \end{bmatrix} \tag{45}$$

where C_e is defined in (37) and F_2 is defined in (24). From the Bounded Real Lemma, if (42) holds, then $\|\tilde{G}_a(s)\|_\infty < \gamma_0$, where the transfer function matrix $\tilde{G}_a(s) := E(sI - A_0)^{-1}(G_l D - B_d)$. This represents an $\mathcal{H}_\infty$ filtering problem [24] associated with the linear part of the observer from (25) obtained from setting $\rho = 0$. The matrix D_1 in (44) represents design freedom used to trade-off the speed of response of the observer versus the magnitude of the gain matrix G_l. As argued in [21], inequality (42) is feasible if and only if

$$\begin{bmatrix} PA_a + A_a^{\mathrm{T}}P - \gamma_0 C_a^{\mathrm{T}}(DD^{\mathrm{T}})^{-1}C_a & -PB_d & E^{\mathrm{T}} \\ -B_d^{\mathrm{T}}P & -\gamma_0 I_{(p+q)} & 0 \\ E & 0 & -\gamma_0 I_q \end{bmatrix} < 0 \quad (46)$$

in which case

$$G_l = \gamma_0 P^{-1} C_a^{\mathrm{T}}(DD^{\mathrm{T}})^{-1}C_a \quad (47)$$

is a choice of the Luenberger gain. Let

$$PA_a + A_a^{\mathrm{T}}P := \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \end{bmatrix} \quad (48)$$

where P is defined in (40) and $X_{11} \in \mathbb{R}^{(n+q-p)\times(n+q-p)}$ is defined as

$$X_{11} = P_{11}A_{11} + P_{12}A_{21} + (P_{11}A_{11} + P_{12}A_{21})^{\mathrm{T}} \quad (49)$$

From (43) it follows that

$$PB_d = P\begin{bmatrix} 0 & F_a \end{bmatrix} = \begin{bmatrix} 0 & P_{122} \\ 0 & P_{222} \end{bmatrix} \quad (50)$$

where P_{122} and P_{222} are the last q columns of P_{12} and P_{22} from (40) respectively. Using (48) and (50), inequality (46) can be written as

$$\begin{bmatrix} X_{11} & X_{12} & 0 & -P_{122} & C_e^{\mathrm{T}} \\ X_{12}^{\mathrm{T}} & X_{22} - \gamma_0^{\mathrm{T}}(DD^{\mathrm{T}})^{-1} & 0 & -P_{222} & F_2 \\ 0 & 0 & -\gamma_o I_p & 0 & 0 \\ -P_{122}^{\mathrm{T}} & -P_{222}^{\mathrm{T}} & 0 & -\gamma_o I_q & 0 \\ C_e & F_2^T & 0 & 0 & -\gamma_o I_q \end{bmatrix} < 0 \quad (51)$$

A necessary condition for this inequality to hold is

$$\begin{bmatrix} X_{11} & -P_{122} & C_e^{\mathrm{T}} \\ -P_{122}^{\mathrm{T}} & -\gamma_0 I_q & 0 \\ C_e & 0 & -\gamma_0 I_q \end{bmatrix} < 0 \quad (52)$$

If $L := P_{11}^{-1}P_{12}$ then $P_{11}L_2 = P_{122}$ and (52) can be re-written as

$$\begin{bmatrix} P_{11}(A_{11}+LA_{21})+(A_{11}+LA_{21})^T P_{11} & -P_{11}L_2 & C_e^{\mathrm{T}} \\ * & -\gamma_0 I_q & 0 \\ * & * & -\gamma_0 I_q \end{bmatrix} < 0 \quad (53)$$

which is the Bounded Real Lemma associated with $\tilde{G}(s) = C_e(sI - (A_{11} + LA_{21}))^{-1}L_2$ and implies $\|\tilde{G}(s)\|_\infty < \gamma_0$.

Formally the design problem is: for a given matrix D_1 and scalar γ_0, minimize γ with respect to P, subject to

$$\begin{bmatrix} X_{11} & -P_{122} & C_e^{\mathrm{T}} \\ -P_{122}^{\mathrm{T}} & -\gamma I_q & 0 \\ C_e & 0 & -\gamma I_q \end{bmatrix} < 0 \quad (54)$$

$$P > 0 \quad (55)$$

and (46). This is a convex optimization problem. Standard LMI software can be used to synthesize numerically γ and P. Once P has been determined, L can be determined as $L = P_{11}^{-1}P_{12}$. The observer gain G_l can be determined from (47) and G_n is determined from (27). As argued in [20] a possible choice of the s.p.d matrix P_0 associated with the unit-vector term (26) is $P_0 = P_{22} - P_{21}P_{11}^{-1}P_{12}$.

3.3 System Uncertainty

Suppose the system in (1) is subject to uncertainty:

$$\dot{x}(t) = Ax(t) + Bu(t) + M\psi(t,x) \quad (56)$$
$$y(t) = Cx(t) + Ff_o(t) \quad (57)$$

where $\psi(\cdot)$ represents a bounded unknown disturbance. The term $M\psi(t,x)$ is assumed to capture the mismatch between the model about which the observer is designed, and the real plant which is to be monitored. This model representation is common in the robust FDI literature [3], and significant effort has been made to develop practical methods to determine the distribution matrix M from measured input/output data [18]. Therefore the augmented system in (6) - (7) becomes

$$\dot{x}_a(t) = A_a x_a(t) + B_a u(t) + M_a\psi(t,x) + F_a\xi(t) \quad (58)$$
$$y(t) = C_a x_a(t) \quad (59)$$

where the term $M_a\psi(t,x)$ represents the effect of additive bounded uncertainty. Again the fault to be reconstructed is given by $f_o = C_f x_a$. The idea now is to represent (58) as

$$\dot{x}_a(t) = A_a x_a(t) + B_a u(t) + \begin{bmatrix} M_a & F_a \end{bmatrix}\begin{bmatrix} \psi(t,x) \\ \xi(t) \end{bmatrix} \quad (60)$$

and to minimize the effect of (ψ, ξ) on the reconstruction of f_o. In this new optimization problem, the disturbance matrix B_d from (43) must be augmented and becomes

$$\bar{B}_d = \begin{bmatrix} 0 & F_a & M_a \end{bmatrix} \quad (61)$$

and the matrix D from (44) becomes

$$\bar{D} = \begin{bmatrix} D_1 & 0 \end{bmatrix} \quad (62)$$

The new optimization problem becomes:

For a given matrix D_1 and γ_0, minimize with respect to γ and P, inequalities (54), (55) and

$$\begin{bmatrix} PA_a + A_a^{\mathrm{T}}P - \gamma_0 C_a^{\mathrm{T}}(\bar{D}\bar{D}^{\mathrm{T}})^{-1}C_a & -P\bar{B}_d & E^{\mathrm{T}} \\ -\bar{B}_d^{\mathrm{T}}P & -\gamma_0 I & 0 \\ E & 0 & -\gamma_0 I \end{bmatrix} < 0 \quad (63)$$

Again this represents a convex optimization problem and LMI solvers can be employed to synthesize γ and P. Note M_a needs to be pre-scaled appropriately so that ψ_a and ξ are of the same order – or suitably weighted to reflect the relative importance of rejection of uncertainty compared to the effect of the fault derivative.

4 Simulation Results

An unstable fighter aircraft will now be used to demonstrate the theory which has been developed in the earlier sections. The ADMIRE model represents a rigid small fighter aircraft with a delta-canard configuration based on a real fighter aircraft. Details of the model can be found in [13]. The linear model used here has been obtained at a low speed flight condition of Mach 0.22 at an altitude of 3000m [15]. The states are angle of attack (AoA) (rad), sideslip angle (rad), roll rate (rad/sec), pitch rate (rad/sec) and yaw rate (rad/sec). The outputs are roll rate (rad/sec), yaw rate (rad/sec) and pitch rate (rad/sec). The control surfaces represent the deflections (rad) of the canard, right elevon, left elevon and rudder respectively. The linear model is open-loop unstable, which is a typical characteristic of fighter aircraft to allow high manoeuvrability. It is assumed that the sensor for the pitch rate is prone to faults. This system is an example where the fault estimation scheme in [20,21] will not work because it can be shown that if

$$F = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^{\mathrm{T}}$$

in (2), then the associated augmented system (A_a, F_a, C_a) is non-minimum phase with a zero at $\{1.0769\}$. Note that the C matrix has been reordered to comply with the requirements in (4) where the sensors that are prone to faults are in the lower part of the C matrix. However, the approach proposed in §3 is applicable for this particular system. The design parameters for the observer were chosen as, $\gamma_0 = 10$ from (42) and $D_1 = I_3$ from (44). The LMI solver yields a γ and P such that $\|\tilde{G}(s)\|_\infty = 1.2212$. The nonlinear gain in (26) has been chosen as $\rho = 1$.

The simulations in Figures 1 and 2 have been obtained from the full nonlinear ADMIRE model with the aircraft undergoing a change in altitude (Figure 1). Figure 2 shows the results of the fault reconstruction using different sensor fault shapes, to show the effectiveness of the method.

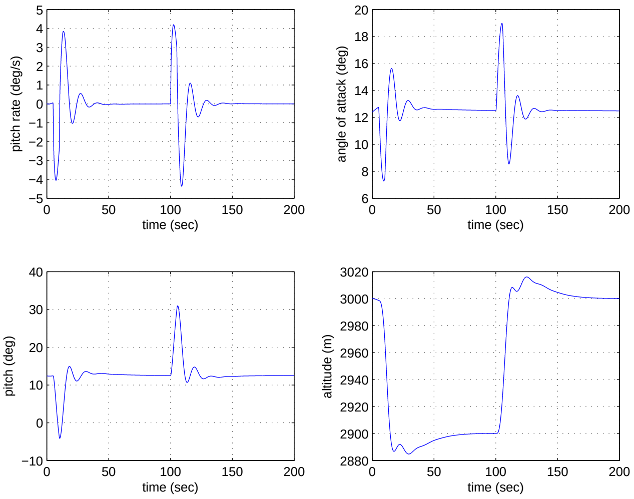
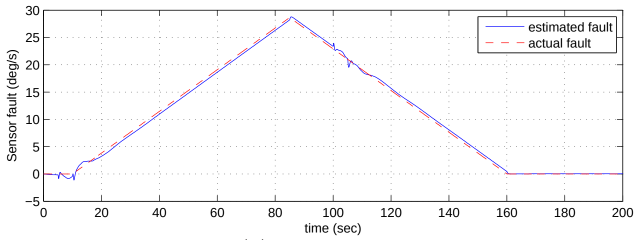
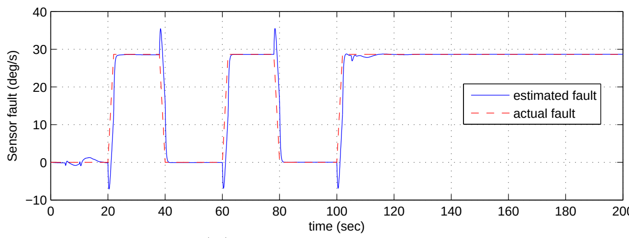


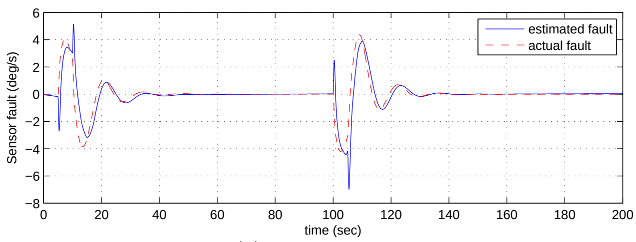
Fig. 1. Manoeuvre on ADMIRE full nonlinear model



(a) ramp fault



(b) smooth step fault



(c) total failure

Fig. 2. Sensor fault reconstruction on the pitch rate sensor on ADMIRE full nonlinear model

5 Conclusion

This paper has addressed one of the system restrictions in the literature for sensor fault reconstruction based on sliding mode observers as proposed in [21,20]. The existing literature guarantees that a sensor fault reconstruction observer exists for open loop stable systems. As shown in this paper, for an open-loop unstable system, for certain fault combinations, the methods in [21,20] are not applicable. In this paper, a sliding mode observer for fault reconstruction which is applicable for both open-loop stable and unstable systems has been proposed. Only one meaningful assumption is required: namely, that after the outputs have been partitioned into the fault-free and fault-prone subsets, the system associated with the fault-free measurements does not have an unobservable mode at the origin. Simulation results from an open-loop unstable system representing a fighter jet shows good fault estimation properties even when simulated on the full nonlinear model.

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