Robustness of funnel control in the gap metric

Achim Ilchmann and Markus Mueller

***Abstract*— For m-input, m-output, finite-dimensional, linear systems satisfying the assumptions (i) minimum phase, (ii) relative degree one and (iii) positive high-frequency gain), the funnel controller achieves output regulation in the following sense: all states of the closed-loop system are bounded and, most importantly, transient behaviour of the tracking error is ensured such that its evolution remains in a performance funnel with prespecified boundary. As opposed to classical adaptive high-gain output feedback, system identification or internal model is not invoked and the gain is not monotone.**

Invoking the conceptual framework of the nonlinear gap metric we show that the funnel controller is robust in the following sense: the funnel controller copes with bounded input and output disturbances and, more importantly, it may even be applied to a system not satisfying any of the classical conditions (i)–(iii) as long as the initial conditions and the disturbances are "small" and the system is "close" (in terms of a "small" gap) to a system satisfying (i)–(iii).

***Index Terms*— funnel control, gap metric, robust stabilization, tracking, output feedback control**

I. INTRODUCTION

Adaptive control without identifying the entries of the system being controlled is known for almost 30 years. Pioneering contributions to the area include [1], [10], [11], [13], [17] (see also the survey [7] and the textbook [15] and references therein). The classical assumptions on such a system class – rather than a single system – of linear m-input, m-output systems are: (i) minimum phase, (ii) strict relative degree one and (iii) positive-definite high-frequency gain matrix. Then the simple output feedback $u(t) = -k(t)\, y(t)$ stabilizes each system belonging to the above class and $k(\cdot)$ adapted by $\dot{k}(t) = \|y(t)\|^2$ and variations thereof. Two drawbacks of the latter strategy, i.e. that first $k(t)$ is, albeit bounded, monotonically increasing which might finally become too large whence amplifying measurement noise, and secondly, that, whilst asymptotic performance is guaranteed, transient behaviour is not taken into account (apart from [12], where the issue of prescribed transient behaviour is successfully addressed), can be overcome with a different approach, the so-called "funnel controller", introduced in [8]. This controller ensures prespecified transient behaviour of the tracking error, has a non-monotone gain, is simpler than the above adaptive controller (actually it is not adaptive in so far the gain is not dynamically generated) and does not invoke any internal model.

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The contribution of the present paper is to show that the funnel controller is *robust* in the sense that the control objectives (bounded signals and tracking within a prespecified performance funnel) are still met if the funnel controller is applied to any system "close" (in terms of the gap metric) to a system satisfying the classical assumptions (i)–(iii) and if initial conditions and disturbances are "sufficiently small". This will be achieved by exploiting the concept of (nonlinear) gap metric and graph topology from [4], [2].

We present an example which suggests that there is a tight trade-off between uncertainty and allowable initial condition and disturbances: initial conditions and disturbances might be "very small" in some cases.

A. System class

We consider the class of linear n-dimensional, m-input m-output systems ($n, m \in \mathbb{N}$ with $n \geq m$)

$$\left.\begin{aligned} \dot{x}(t) &= A\,x(t) + B\,u_1(t)\,, \qquad x(0) = x^0 \in \mathbb{R}^n, \\ y_1(t) &= C\,x(t)\,, \end{aligned}\right\} \quad (1)$$

which satisfy the classical assumptions in high-gain adaptive control, that is minimum phase with relative degree one and positive definite high-frequency gain matrix, i.e. they belong to

$$\widetilde{\mathcal{M}}_{n,m} := \left\{ \begin{array}{c|c} \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{n\times n} \\ \quad \times \mathbb{R}^{n\times m} \\ \quad \times \mathbb{R}^{m\times n} \end{array} & \begin{array}{l} CB + (CB)^T > 0\,, \\ \forall\, s \in \overline{\mathbb{C}}_+ : \\ \quad \det \begin{bmatrix} sI_n - A & B \\ C & 0 \end{bmatrix} \neq 0 \end{array} \end{array} \right\}.$$

Note, that only structural assumptions are required but the system entries may be completely unknown. For $(A,B,C) \in \widetilde{\mathcal{M}}_{n,m}$ with $\det CB \neq 0$ we may choose an invertible $T \in \mathbb{R}^{n\times n}$ such that

$$T^{-1}AT = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ T^{-1}B = \begin{bmatrix} CB \\ 0 \end{bmatrix},\ CT = \begin{bmatrix} I_m & 0_{m\times(n-m)} \end{bmatrix}.$$

Moreover, if (A,B,C) is minimum-phase, then A_4 has spectrum in the open left half complex plane $\mathbb{C}_-$. Therefore, we replace $\widetilde{\mathcal{M}}_{n,m}$ by

$$\mathcal{M}_{n,m} := \left\{ \begin{array}{c|c} \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{n\times n} \\ \quad \times \mathbb{R}^{n\times m} \\ \quad \times \mathbb{R}^{m\times n} \end{array} & \begin{array}{l} A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix},\ B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \\ C = [I,\ 0],\ B_1, A_1 \in \mathbb{R}^{m\times m}, \\ \text{spec}(A_4) \subset \mathbb{C}_-,\ B_1 + B_1^T > 0 \end{array} \end{array} \right\},$$

and restrict our attention to systems $(A,B,C) \in \mathcal{M}_{n,m}$ in Byrnes–Isidori normal form, see for example [9, Sec. 4], i.e.

$$\left.\begin{aligned} \dot{y}_1 &= A_1 y_1 + A_2 z + CB\,u_1, & y_1(0) &= y_1^0 \in \mathbb{R}^m, \\ \dot{z} &= A_3 y_1 + A_4 z, & z(0) &= z^0 \in \mathbb{R}^{n-m}. \end{aligned}\right\} \quad (2)$$

We will study the initial value problem (1) or (2) as *plant* P mapping the interior input signal u_1 to the interior output signal y_1, in conjunction with the *controller* C (the funnel controller (4) in our setup), mapping the interior output-signal y_2 to the interior input signal u_2, and in the presence of additive input/output disturbances u_0, y_0 so that

$$u_0 \;=\; u_1 + u_2, \qquad y_0 \;=\; y_1 + y_2\,, \tag{3}$$

as depicted in Figure 1.

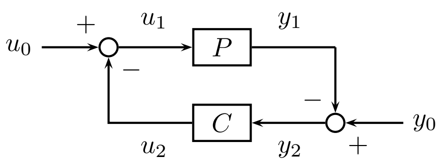


Fig. 1. The closed-loop system $[P, C]$.

B. Performance funnel and funnel control

The control objective, defined in the following sub-section, will be captured in terms of the *performance funnel*

$$\mathcal{F}_\varphi := \{(t,e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|e\| < 1\}\,,$$

determined by $\varphi(\cdot)$ belonging to

$$\Phi := \left\{ \varphi\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0} \;\middle|\; \begin{array}{l} \varphi \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}), \\ \varphi(0) = 0,\ \forall\, t > 0 \,:\, \varphi(t) > 0, \\ \liminf_{t\to\infty} \varphi(t) > 0, \\ \forall\, \varepsilon > 0 \,:\, \varphi|_{[\varepsilon,\infty)}(\cdot)^{-1} \text{ is} \\ \text{globally Lipschitz continuous} \end{array} \right\}.$$

Note that the funnel boundary is given by $1/\varphi(t)$, $t > 0$; see Figure 2. The concept of performance funnel had been introduced by [8]. There it is not assumed that $\varphi(\cdot)$ has the Lipschitz condition as given in Φ; we incorporate this mild assumption for technical reasons. The assumption $\varphi(0) = 0$ allows to start with arbitrarily large initial conditions x_0 and output disturbances y_0. If for special applications the initial value and y_0 are known, then $\varphi(0) = 0$ may be relaxed by $\varphi(0)\|y_0(0) - Cx^0\| < 1$, see also the simulations in Example 3.5.

The funnel controller, for prespecified $\varphi \in \Phi$, is given by

$$u_2(t) = -k(t) y_2(t)\,, \qquad k(t) = \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|} \tag{4}$$

and will be applied to (1) or (2). Note that the funnel controller (4) is actually not an adaptive controller in the sense that it is not dynamic. The gain $k(t)$ is the reciprocal of the distance between $y_2 = y_0 - y_1$ (i.e. the difference of a reference signal y_0 and the output of (1)) and the funnel boundary $1/\varphi(t)$; and, loosely speaking, if the error approaches the funnel boundary, then $k(t)$ becomes large, thereby exploiting the high-gain properties of the system and precluding boundary contact.

C. Control objectives

We will study properties of the closed-loop system generated by the application of the funnel controller (4) to systems (1) of class $\mathcal{M}_{n,m}$ or of class $\mathcal{P}_{n,m}$ (see below) in the presence of disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ satisfying the interconnection equations (3).

If, for prespecified $\varphi \in \Phi$ determining the funnel boundary, the funnel controller (4) is applied to any system (1), belonging to the class $\mathcal{M}_{n,m}$, in the presence of disturbances (u_0, y_0) satisfying the interconnection equations (3), then the closed-loop system (2), (4), (3) is supposed to meet the following control objectives:

- all signals are bounded;
- the output error $y_2(t) = y_0(t) - y_1(t)$ of the output disturbance and the output of the linear system evolves in the funnel, in other words

$$\begin{aligned} \forall\, t \geq 0 \;:\; (t, y_2(t)) \in \mathcal{F}_\varphi = \\ \{(t,y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}\,. \end{aligned}$$

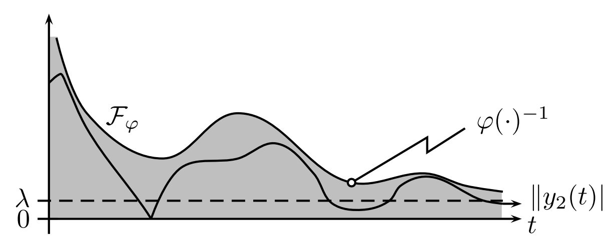


Fig. 2. Funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi$ and $\inf_{t>0} \varphi(t)^{-1} = \lambda$

D. Main result: robustness

The main result of the present paper is to show robustness of the funnel controller in the following sense: The control objectives should still be met if $(A, B, C) \in \mathcal{M}_{n,m}$ is replaced by some system $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ belonging to the system class

$$\mathcal{P}_{q,m} := \left\{ \begin{array}{l} (A,B,C) \\ \in \mathbb{R}^{q\times q} \\ \quad \times \mathbb{R}^{q\times m} \\ \quad \times \mathbb{R}^{m\times q} \end{array} \;\middle|\; \begin{array}{l} (A,B,C) \text{ is} \\ \text{stabilizable} \\ \text{and detectable} \end{array} \right\} \supsetneq \mathcal{M}_{q,m}$$

where $q, m \in \mathbb{N}$ with $q \geq m$, and $(\widetilde{A}, \widetilde{B}, \widetilde{C})$ is close (in terms of the gap metric) to a system belonging to $\mathcal{M}_{n,m}$ and the initial conditions and the disturbances are "small".

For the purpose of illustration, we will further show that a minimal realization $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ of the transfer function

$$s \mapsto \frac{N(M-s)}{(s-\alpha)(s+N)(s+M)}\,, \qquad \alpha, N, M > 0\,, \tag{5}$$

(which obviously does not satisfy any of the classical assumptions since it is not minimum phase, has relative degree 2 and negative high-frequency gain) becomes arbitrarily close to a system belonging to $\mathcal{M}_{n,m}$ as N and M tend to infinity.

II. Funnel control

In this section we show that the funnel controller (4) applied to any linear system (A, B, C) of class $\mathcal{M}_{n,m}$ achieves, in presence of input/output disturbances $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, the control objectives: y_2 is forced to evolve within a performance funnel $\mathcal{F}_\varphi$ for prespecified $\varphi \in \Phi$ and all signals and states of the closed-loop (2), (3), (4) remain essentially bounded. Moreover, it is shown that the derivatives of the output signals y_1, y_2 and the state $\binom{y_1}{\eta}$ are essentially bounded, too. Write, for $n, m \in \mathbb{N}$, $n \geq m$,

$$\mathcal{D}_{n,m} := \mathcal{M}_{n,m} \times (\mathbb{R}^m \times \mathbb{R}^{n-m}) \times \Phi \times L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m).$$

Proposition 2.1: Let $n, m \in \mathbb{N}$, $n \geq m$ and $\varphi \in \Phi$. Then there exists a map $\nu \colon \mathcal{D}_{n,m} \to \mathbb{R}_{\geq 0}$ such that, for all $d = \left(\left[\begin{smallmatrix} A_1 & A_2 \\ A_3 & A_4 \end{smallmatrix} \right], B, C, (y_1^0, \eta^0), \varphi, u_0, y_0 \right) \in \mathcal{D}_{n,m}$, the associated closed-loop initial value problem (2), (3), (4) satisfies

$$\|(k, u_2, y_2, \eta)\|_{\substack{L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^{1+m}) \\ \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{m+n-m})}} \leq \nu(d), \quad (6)$$

and

$$\forall\, t \geq 0 \,:\, (t, y_2(t)) \in \mathcal{F}_\varphi = \{(t, y) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \mid \varphi(t)\|y\| < 1\}. \quad (7)$$

Proof: See proof of [6, Prop. 2.1]. ∎

III. Robustness of the funnel controller

In this section we show that the funnel controller (4) are robust in the sense that one may apply these controllers to any stabilizable and detectable system which is "close" (in terms of a "small" gap) to any system in $\mathcal{M}_{m,n}$, as long as the initial conditions and the disturbances are "small".

A. The concept of the gap metric

We refer the reader to [3, Sec. 2], [6, Sec. 3] and mainly [14, Ch. 6] for a detailed outline of all required definitions for extended and ambient spaces, well posedness, the nonlinear gap, gain-functions and gain-function stability, which are required for the results on robust stability.

However, we recall some basic concepts. Let, for signal spaces $\mathcal{U}, \mathcal{Y}$ and $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, and for plant and controller operators $P \colon \mathcal{U}_a \to \mathcal{Y}_a$, $u_1 \mapsto y_1$, and $C \colon \mathcal{Y}_a \to \mathcal{U}_a$, $y_2 \mapsto u_2$, resp., the closed-loop

$$[P, C] \,:\, y_1 = Pu_1,\ u_2 = Cy_2,\ \begin{array}{l} u_0 = u_1 + u_2 \\ y_0 = y_1 + y_2 \end{array} \quad (8)$$

correspond to Figure 1, and introduce the closed-loop operator

$$H_{P,C} \colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a,\ w_0 \mapsto (w_1, w_2).$$

The closed-loop system $[P, C]$, given by (8), is said to be:

- *locally well posed* if, and only if, it has the existence and uniqueness properties and the operator $H_{P,C} \colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$, $w_0 \mapsto (w_1, w_2)$, is causal;
- *globally well posed* if, and only if, it is locally well posed and $H_{P,C}(\mathcal{W}) \subset \mathcal{W}_e \times \mathcal{W}_e$;
- *regularly well posed* if, and only if, it is locally well posed and

$$\forall\, w_0 \in \mathcal{W} \,:\, \big[\, \omega_{w_0} < \infty \implies \big\| (H_{P,C} w_0)|_{[0,\tau)} \big\|_{\mathcal{W}_\tau \times \mathcal{W}_\tau} \to \infty \text{ as } \tau \to \omega_{w_0} \,\big]. \quad (9)$$

To measure the distance between two plants P and P_1 it is necessary to find sets associated with the plant operators within some space where one may define a map which identifies the gap. These sets are the *graphs* of the operators: for the plant operator $P \colon \mathcal{U}_a \to \mathcal{Y}_a$ define the *graph* $\mathcal{G}_P$ as

$$\mathcal{G}_P := \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \,\middle|\, u \in \mathcal{U},\ Pu \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

Robust stability is the property that the stability properties of a globally well posed closed-loop system $[P, C]$ persists under "sufficiently small" perturbations of the plant. In other words, robust stability is the property that $[P_1, C]$ inherits the stability properties of $[P, C]$, when the plant P is replaced by any plant P_1 sufficiently "close" to P. In the present context, plants P and P_1 are deemed to be close if, and only if, their respective graphs are *close* in the gap sense of [4]: the nonlinear gap $\vec{\delta}(P, P_1)$ is small (see [6, Sec. 3.3] for a definition of the gap).

We close this sub-section with an example. Define, for $\alpha, N, M > 0$, $x^0 \in \mathbb{R}$, $\widetilde{x}^0 \in \mathbb{R}^3$ and for signal spaces $\mathcal{U}$ and $\mathcal{Y}$ as in Prop. 2.1, the plant operator $P((\alpha, 1, 1), x^0) \colon \mathcal{U}_e \to \mathcal{Y}_e$,

$$u_1 \mapsto y_1 = x,\ \dot{x} = \alpha x + u_1,\ x(0) = x^0, \quad (10)$$

and, for a minimal realization $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ of (5), the plant operator $P\big((\widetilde{A}, \widetilde{b}, \widetilde{c}), \widetilde{x}^0\big) \colon \mathcal{U}_e \to \mathcal{Y}_e$,

$$\widetilde{u}_1 \mapsto \widetilde{y}_1 = \widetilde{c}\, x,\ \dot{x} = \widetilde{A}\, x + \widetilde{b}\, \widetilde{u}_1,\ x(0) = \widetilde{x}^0. \quad (11)$$

In [5, Sec. 3] it is shown that, for sufficiently large $M > 0$ and $N = 2M$, $P((\alpha, 1, 1), 0)$ is close to $P((\widetilde{A}, \widetilde{b}, \widetilde{c}), 0)$ in the sense

$$\limsup_{M \to \infty} \vec{\delta}\big(P((\alpha, 1, 1), 0), P((\widetilde{A}, \widetilde{b}, \widetilde{c}), 0)\big) = 0. \quad (12)$$

B. Well posedness of the nominal closed-loop system

For normed signal spaces $\mathcal{U}$ and $\mathcal{Y}$ and $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$, where $\theta = (A, B, C)$ is the plant and $x^0 \in \mathbb{R}^n$ is the initial value of a linear system (1), we associate the causal plant operator

$$P(\theta, x^0) \colon \mathcal{U}_a \to \mathcal{Y}_a, \quad u_1 \mapsto P(\theta, x^0)(u_1) := y_1\,, \quad (13)$$

where, for $u_1 \in \mathcal{U}_a$ with $\mathrm{dom}(u_1) = [0, \omega)$, we have $y_1 = cx$, x being the unique solution of (1) on $[0, \omega)$. Consider, for $\varphi \in \Phi$, the control strategy (4) and associate the causal control operator, parameterized by φ, i.e.

$$C(\varphi) \colon \mathcal{Y}_a \to \mathcal{U}_a, \qquad y_2 \mapsto C(\varphi)(y_2) := u_2\,. \quad (14)$$

Proposition 3.1: Let $n, m \in \mathbb{N}$ with $n \geq m$, $\varphi \in \Phi$, $(\theta, x^0) \in \mathcal{M}_{n,m} \times \mathbb{R}^n$ and $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Then, for plant operator $P(\theta, x^0)$ and funnel control operator $C(\varphi)$, given by (13) and (14), resp., the closed-loop initial value problem $[P(\theta, x^0), C(\varphi)]$, given by (2), (3), (4), is globally well posed and moreover $[P(\theta, x^0), C(\varphi)]$ is $\big(L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)\big)$-stable.

Proof: The statement is a consequence of Prop. 2.1. ■

C. Well posedness of the general closed-loop system

For $(A, B, C) \in \mathcal{P}_{n,m}$, $x^0 \in \mathbb{R}^n$ and $\varphi \in \Phi$, the closed-loop initial value problem (1), (3), (4) may be written as

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B[u_0(t) - u_2(t)], \quad x(0) = x^0 \in \mathbb{R}^n,\\ k(t) &= \frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|},\\ y_2(t) &= y_0(t) - Cx(t),\\ u_2(t) &= -k(t)y_2(t).\end{aligned} \tag{15}$$

Proposition 3.2: Let $n \in \mathbb{N}$ with $n \geq m$, $\varphi \in \Phi$, $(\theta, x^0) \in \mathcal{P}_{n,m} \times \mathbb{R}^n$ and $(u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$. Then, for plant operator $P(\theta, x^0)$ and funnel control operator $C(\varphi)$, given by (13) and (14), resp., the closed-loop initial value problem $[P(\theta, x^0), C(\varphi)]$, given by (15), has the following properties:

(i) there exists a unique solution $x\colon [0, \omega) \to \mathbb{R}^n$, for some $\omega \in (0, \infty]$, and the solution can be maximally extended;

(ii) if $(u_2, y_2) \in L^\infty([0, \omega) \to \mathbb{R}^m) \times W^{1,\infty}([0, \omega) \to \mathbb{R}^m)$, then $\omega = \infty$, $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and y_2 is uniformly bounded away from the funnel boundary $1/\varphi(\cdot)$;

(iii) $[P(\theta, x^0), C(\varphi)]$ is regularly well posed.

Proof: Set, for $\varphi \in \Phi$ and $y_0 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$,

$$\mathcal{H}_{\varphi, y_0} := \left\{ (t, x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^n \,\middle|\, \varphi(t)\|y_0(t) - C\,x\| < 1 \right\}.$$

(i): The initial value problem (15) may be written as

$$\dot{x} = g(t, x),\ x(0) = x^0,\ (0, y_0(0) - C\,x_0) \in \mathcal{H}_{\varphi, y_0}, \tag{16}$$

where $g\colon \mathcal{H}_{\varphi, y_0} \to \mathbb{R}^n$, $(t, x) \mapsto Ax + Bu_0(t) + \frac{\varphi(t)}{1 - \varphi(t)\|y_0(t) - Cx\|} B(y_0(t) - Cx)$, satisfies a local Lipschitz condition on the relatively open set $\mathcal{H}_{\varphi, y_0}$ as required to apply [16, Th. III.11.III], which yields that (16), and therefore (15), has an absolutely continuous solution $x\colon [0, \omega) \to \mathbb{R}^n$ for some $\omega \in (0, \infty]$, and the graph of the solution is not completely contained in any subset of $\mathcal{H}_{\varphi, y_0}$, i.e. the solution can be maximally extended, as required.

(ii): Suppose $(u_2, y_2) \in L^\infty([0, \omega) \to \mathbb{R}^m) \times W^{1,\infty}([0, \omega) \to \mathbb{R}^m)$ and, for contradiction, $\omega < \infty$. By boundedness of φ it follows that there exists $\lambda > 0$ such that $\varphi(t) \leq 1/\lambda$ for all $t \in [0, \omega)$. Thus

$$\begin{aligned} 1 - \varphi(t)\|y_2(t)\| \leq \tfrac{1}{2} \quad &\Rightarrow \quad \tfrac{1}{2} \leq \varphi(t)\|y_2(t)\| \leq \tfrac{\|y_2(t)\|}{\lambda}\\ &\Rightarrow \quad \|y_2(t)\| \geq \tfrac{\lambda}{2}\end{aligned}$$

for all $t \in [0, \omega)$, which yields, in view of $y_2 \in L^\infty([0, \omega) \to \mathbb{R}^m)$ and $\frac{-\varphi}{1 - \varphi\|y_2\|} y_2 = u_2 \in L^\infty([0, \omega) \to \mathbb{R})$, that

$$\begin{aligned}\forall\, t \in [0, \omega) \ :\ 1 - \varphi(t)\|y_2(t)\| \leq \tfrac{1}{2}\\ \Rightarrow \ \|u_2\|_\infty \geq \tfrac{\varphi(t)\,\|y_2(t)\|}{1 - \varphi(t)\|y_2(t)\|} \geq \tfrac{\lambda\varphi(t)}{2(1 - \varphi(t)\|y_2(t)\|)},\end{aligned}$$

thus $\frac{\varphi}{1 - \varphi\|y_2\|}$ is bounded on $\left\{ t \in [0, \omega) \,\middle|\, 1 - \varphi(t)\|y_2(t)\| \leq 1/2 \right\}$. Moreover, for all $t \in [0, \omega)$ with $1 - \varphi(t)\|y_2(t)\| > 1/2$ holds $\frac{\varphi(t)}{1 - \varphi(t)\|y_2(t)\|} \leq 2/\lambda$. Thus $k = \frac{\varphi}{1 - \varphi\|y_2\|} \in L^\infty([0, \omega) \to \mathbb{R})$. Hence, by continuity of the solution

$$\exists\, \varepsilon > 0\ \forall\, t \in [0, \omega) \ :\ 1 - \varphi(t)\|y_2(t)\| \geq \varepsilon. \tag{17}$$

Then, Variation of Constants applied to (15) yields the existence of constants $c_0 = c_0(B, \lambda, \varepsilon)$, $c_1 = c_1(A) > 0$ such that

$$\|x(t)\| \leq c_0 \left(e^{c_1 \omega} + \int_0^\omega e^{c_1(\omega - s)} \left(\|u_0(s)\| + \|y_2(s)\| \right) \mathrm{d}s \right) \tag{18}$$

for all $t \in [0, \omega)$. Since $y_2 \in L^\infty([0, \omega) \to \mathbb{R}^m)$ and $u_0 \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, it follows that the right hand side of (18) is bounded by $c_3 = c_0\big(e^{c_1\omega} + (e^{c_1\omega} + 1)(\|u_0\|_{L^\infty([0,\omega)\to\mathbb{R}^m)} + \|y_2\|_{L^\infty([0,\omega)\to\mathbb{R}^m)})/c_1\big) > 0$ on $[0, \omega)$ which gives that $\mathcal{K} := \{(t, x) \in \mathcal{H}_{\varphi, y_0} \,|\, t \in [0, \omega],\ \|x\| \leq c_3\}$ is a compact subset of $\mathcal{H}_{\varphi, y_0}$ with $(t, x(t)) \in \mathcal{K}$ for all $t \in [0, \omega)$, which contradicts the fact that the solution can be maximally extended, see (i). Therefore, $\omega = \infty$ and in view of (17) we have k bounded and y_2 is uniformly bounded away from the funnel boundary $\varphi(\cdot)^{-1}$.

(iii): By (i), the closed-loop initial value problem $[P(\theta, x^0), C(\varphi)]$ is locally well posed. It suffices to show that (9) holds. For $w_0 = (u_0, y_0) \in \mathcal{W}$ consider $(w_1, w_2) = H_{P(\theta, x^0), C(\varphi)}(w_0)$ where $\mathrm{dom}(w_1, w_2) = [0, \omega)$ is maximal. Suppose, contrary to the right hand side of (9), $\big\|(w_1, w_2)|_{[0,\omega)}\big\|_{\mathcal{W}_\omega \times \mathcal{W}_\omega} < \infty$. Then $(u_2, y_2) \in L^\infty([0, \omega) \to \mathbb{R}^m) \times W^{1,\infty}([0, \omega) \to \mathbb{R}^m)$, which, in view of (ii), yields $\omega = \infty$, i.e. the contrary of the left hand side of (9), hence $[P(\theta, x^0), C(\varphi)]$ is regularly well posed. ■

D. Robustness of funnel control

Theorem 3.3: Let $n, q, m \in \mathbb{N}$ with $n, q \geq m$, $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$, $\varphi \in \Phi$ and $\theta \in \mathcal{M}_{n,m}$. For $(\widetilde{\theta}, \widetilde{x}^0) \in \mathcal{P}_{q,m} \times \mathbb{R}^q$ consider the associated operators $P(\widetilde{\theta}, \widetilde{x}^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C(\varphi)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by (13) and (14), resp., and the closed-loop initial value problem (1), (3), (4). Then there exist a continuous function $\eta\colon (0, \infty) \to (0, \infty)$ and a function $\psi\colon \mathcal{P}_{q,m} \to (0, \infty)$ such that the following holds:

$$\forall \big(\widetilde{\theta}, \widetilde{x}^0, w_0, r\big) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0, \infty) \ :$$

$$\left.\begin{aligned} \psi(\widetilde{\theta})\|\widetilde{x}^0\| + \|w_0\|_{\mathcal{W}} \leq r\\ \vec{\delta}\big(P(\theta, 0), P(\widetilde{\theta}, 0)\big) \leq \eta(r) \end{aligned}\right\} \Rightarrow \begin{cases} \forall\, t \geq 0 \ :\ (t, y_2(t)) \in \mathcal{F}_\varphi\\ k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})\\ x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q), \end{cases} \tag{19}$$

where (x, k) and y_2 satisfy (15).

Loosely speaking, the main result shows that funnel control achieves the control objectives if applied to a system $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ as long as this system is sufficiently close – in the terms of the gap metric – to a system $(A, B, C) \in \widetilde{\mathcal{M}}_{n,m}$ and the initial value $\widetilde{x}^0 \in \mathbb{R}^q$ for $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big)$ and the input/output disturbances (u_0, y_0) are sufficiently small. As a consequence $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ may not even satisfy any of the classical assumptions: minimum phase, relative degree one and positive high-frequency gain.

To establish gap margin results, we show gain-function stability of the so-called augmented closed-loop system, i.e. the closed-loop of extensions of P and C be incorporation the system class, see [6, Prop. 4.3]. This leads to:

Proposition 3.4: Let $n, q, m \in \mathbb{N}$ with $n, q \geq m$, $\mathcal{U} = L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{Y} = W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$, $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ and $\theta \in \mathcal{M}_{n,m}$. For $(\widetilde{\theta}, \widetilde{x}^0, \varphi) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \Phi$, consider $P(\widetilde{\theta}, \widetilde{x}^0)\colon \mathcal{U}_a \to \mathcal{Y}_a$, and $C(\varphi)\colon \mathcal{Y}_a \to \mathcal{U}_a$ defined by (13) and (14), resp. Then there exist a continuous function $\eta\colon (0,\infty) \to (0,\infty)$ and a function $\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ such that the following holds:

$$\forall \left(\widetilde{\theta}, \widetilde{x}^0, w_0, r\right) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \ : \ \left.\begin{array}{l} \psi(\widetilde{\theta})|\widetilde{x}^0| + \|w_0\|_{\mathcal{W}} \leq r \\ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \leq \eta(r) \end{array}\right\} \Rightarrow \begin{array}{r} H_{P(\widetilde{\theta},\widetilde{x}^0),C(\varphi)}(w_0) \\ \in \mathcal{W} \times \mathcal{W}. \end{array}$$

See [6, Prop. 4.4] for a proof.

Proof of Thm. 3.3. Step 1: We show

$$\big((u_1, y_1), (u_2, y_2)\big) = H_{P(\widetilde{\theta},\widetilde{x}^0),C(\varphi)}(w_0) \in \mathcal{W} \times \mathcal{W}. \quad (20)$$

Choose functions $\eta\colon (0,\infty) \to (0,\infty)$ and $\psi\colon \mathcal{P}_{q,m} \to (0,\infty)$ from Prop. 3.4. Let

$$\left(\widetilde{\theta}, \widetilde{x}^0, w_0, r\right) \in \mathcal{P}_{q,m} \times \mathbb{R}^q \times \mathcal{W} \times (0,\infty) \ : \ \psi(\widetilde{\theta})|\widetilde{x}^0| + \|w_0\|_{\mathcal{W}} \leq r \ \wedge \ \vec{\delta}\big(P(\theta,0), P(\widetilde{\theta},0)\big) \leq \ \eta(r).$$

Then Prop. 3.4 gives (20).

Step 2: By Prop. 3.2 it follows that (15) has a unique solution $x\colon [0,\omega) \to \mathbb{R}^q$ on a maximal interval of existence $[0,\omega)$ for some $\omega \in (0,\infty]$. Prop. 3.2(iii) yields $\omega = \infty$ and $k = \frac{\varphi}{1-\varphi\|y_2\|} \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$, the second assertion of (19).

Step 3: By Step 2 we have $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ which, in view of continuity of $1 - \varphi\|y_2\|$ on $(0,\infty)$, yields $1 - \varphi(t)\|y_2(t)\| \geq \|\varphi\|_\infty \|k\|_\infty^{-1} > 0$. Thus, for all $t \geq 0$, $\varphi(t)\|y_2(t)\| < 1$, which yields the first assertion of (19).

Step 4: It remains to show that $x \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$.

Let $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big) \in \mathcal{P}_{q,m}$ associated with (1). Detectability of $\big(\widetilde{A}, \widetilde{B}, \widetilde{C}\big)$ yields the existence of $F \in \mathbb{R}^{q \times m}$ such that $\operatorname{spec}(\widetilde{A} + F\widetilde{C}) \subset \mathbb{C}_-$. Setting $g := -\big[F + k\,\widetilde{B}\big]\,(y_0 - y_2) + \widetilde{B}\,u_0 + \widetilde{B}\,k y_0$ gives

$$\dot{x} = \big[\widetilde{A} - k\,\widetilde{B}\widetilde{C}\big]x + \widetilde{B}\,u_0 + \widetilde{B}\,k y_0 = \big[\widetilde{A} + F\widetilde{C}\big]x + g\,. \quad (21)$$

By Prop. 3.4 and Step 3 we have $y_2 \in W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and $k \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R})$ and since $w_0 = (u_0, y_0) \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times W^{1,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ it follows that $g \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. Hence, by (21) and Variation of Constants we obtain $x \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$. The first equation in (15) then gives $\dot{x} \in L^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ which shows the third assertion in (19) and the proof is complete. ■

Example 3.5: We revisit, for $\alpha, N, M > 0$, the plant operators (short for convenience) $P^{\mathrm{n}}_{x^0} := P((\alpha,1,1), x^0)$ and $P^{\mathrm{g}}_{\widetilde{x}^0} := P((\widetilde{A}, \widetilde{b}, \widetilde{c}), \widetilde{x}^0)$ defined by (10) and (11), resp. These plants will be studied in conjunction with the control operator $C(\varphi)$ defined by (14).

In passing, note that $P^{\mathrm{n}}_{x^0}$ has transfer functions $s \mapsto \frac{1}{s-\alpha}$; the plant $P^{\mathrm{g}}_{\widetilde{x}^0}$ with transfer function (5) has a minimal realization in normal form

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi_1\\ \xi_2\\ z\end{pmatrix} = \begin{bmatrix} 0 & 1 & 0\\ r_1 & r_2 & r_3\\ -1 & 0 & M\end{bmatrix}\begin{pmatrix}\xi_1\\ \xi_2\\ z\end{pmatrix} + \begin{pmatrix}0\\ -N\\ 0\end{pmatrix}u_1, \quad y_1 = \xi_1, \quad (22)$$

with $r_1 = \alpha N + 2M(\alpha - M - N)$, $r_2 = \alpha - 2M - N$ and $r_3 = 2M(NM + M^2 - \alpha M - \alpha N)$.

Recall from (12) that for zero initial conditions the gap between the system $\big(\widetilde{A}, \widetilde{b}, \widetilde{c}\big) \in \mathcal{P}_{3,1} \setminus \mathcal{M}_{3,1}$ and $(\alpha, 1, 1) \in \mathcal{M}_{1,1}$ tends to zero as $N = 2M$ and M tend to infinity.

Note that Thm. 3.3 shows only existence of the functions ψ and η which guarantee the robust stability result; it is not straightforward to find these functions. We now discuss simulations for various values of $N, M > 0$, initial values $\widetilde{x}^i$, and input disturbances u_0^j (we consider $y_0 = 0$ for convenience: with $y_0 \neq 0$ systems become extremely stiff and MATLAB's solvers fail to provide a numerical solution); all simulations are performed by MATLAB for $\alpha = 1$ and funnel boundary $\varphi(\cdot)^{-1}\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$, $t \mapsto \begin{cases} 15.31 - 7.8\,t + t^2, & \text{if } t \in [0, 3.9)\\ 0.1, & \text{if } t \geq 3.9.\end{cases}$

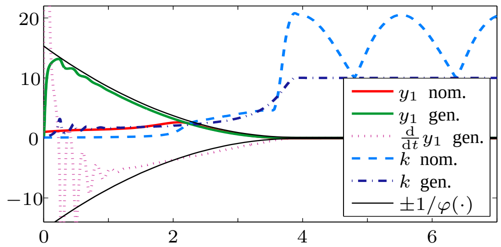
The variables y_1 and k of the nominal closed-loop system (10), (4), (3) are depicted in Figure 3(a) for initial value $x^0 = 1$ and $u_0 = \sin(2\cdot)$.

Consider the closed-loop system (22), (4), (3) for $N = 2M = 100$. In Figure 3(a) we depict the simulations for initial value $\widetilde{x}^1 = (0.1, 0.1, 0.08)^\top$, which is sufficiently small to guarantee funnel control: all components of the solution $(\xi(\cdot)^\top, z(\cdot)) = (y_1(\cdot), \dot{y}_1(\cdot), z(\cdot))$ and $k(\cdot)$ and $u_1(\cdot)$ are bounded. However, a slight increase of the third component of the initial value to $\widetilde{x}^2 = (0.1, 0.1, 0.1)^\top$ leads to a finite escape time: the output y_1 tends to the funnel boundary in finite time $t_1 > 0$ and therefore $u_1(t)$ tends to infinity as $t \to t_1$, see Figure 3(b).

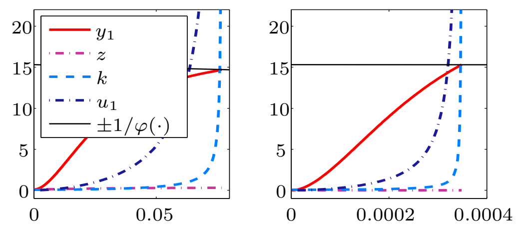
Consider the closed-loop system (22), (4), (3) for $N = 2M = 10,000$. Then the gap is very small, and $r > 0$ may be large such that the second inequality of the left hand side of (19) holds. However, the system has very unstable zero dynamics; this indicates that $\psi(\widetilde{A}, \widetilde{b}, \widetilde{c})$ might be very large. Therefore, the initial value must be very small so that the first inequality of the left hand side of (19) holds. Since ψ maps any system $(\widetilde{A}, \widetilde{b}, \widetilde{c})$ into $(0,\infty)$, then in view of (19) and given that the second inequality holds for r and $(\widetilde{A}, \widetilde{b}, \widetilde{c})$, it is always possible to choose a sufficiently small initial value not equal to zero such that the first inequality holds.

Figure 4(a) shows that funnel control is achieved in case of the initial value $\widetilde{x}^3 = (0.001, 0.001, 0.001)^\top$, whereas funnel control is not achieved in case of the initial value $\widetilde{x}^4 = (0.001, 0.001, 0.0015)^\top$, see Figure 3(b).

Finally we consider the general system with zero initial conditions but non-zero input disturbance. Figures 4(b) and 4(c) show that for $N = 2M = 100$ and $u_0^1 = \sin(2\cdot)$ funnel control is achieved, that for $N = 2M = 100$ and $u_0^2 = 2\sin(2\cdot)$ the controller fails to stabilize the system, however, that for $N = 2M = 400$ and $u_0^2 = 2\sin(2\cdot)$ the funnel controller achieves the control objectives. These simulations show that, for large disturbances, the first inequality in (19) gives large r and therefore the gap $\vec{\delta}\big(P_0^{\mathrm{g}}, P_0^{\mathrm{g}}\big)$ has to be smaller than for small disturbances.



(a) y_1, k, of $[P^{\mathrm{n}}_{x^0=1}, C(\varphi)]$ for $u_0 = \sin(2\cdot)$; y_1, $\dot{y}_1$, k of $[P^{\mathrm{g}}_{\widetilde{x}^1=(0.1,0.1,0.08)^\top}, C(\varphi)]$ for $N = 2M = 100$ and $u_0 = 0$



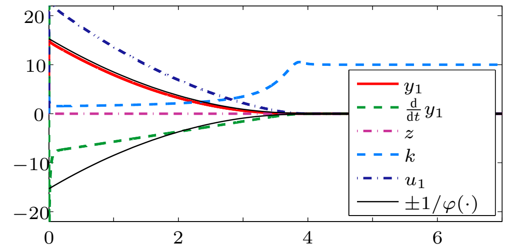
(b) y_1, k of $[P^{\mathrm{g}}_{\widetilde{x}^2=(0.1,0.1,0.1)^\top}, C(\varphi)]$ for $N = 2M = 100$ and $u_0 = 0$; y_1, k of $[P^{\mathrm{g}}_{\widetilde{x}^4=(0.001,0.001,0.0015)^\top}, C(\varphi)]$ for $N = 2M = 10,000$ and $u_0 = 0$

Fig. 3. Funnel control simulations: nominal system $P((\alpha, 1, 1), x^0)$ with $u_0 = \sin(2\cdot)$ and general system $P((\widetilde{A}, \widetilde{b}, \widetilde{c}), \widetilde{x}^i)$, $i = 1, 2, 4$, with $N = 2M = 100$, $N = 2M = 10,000$, resp., and $u_0 = 0$.

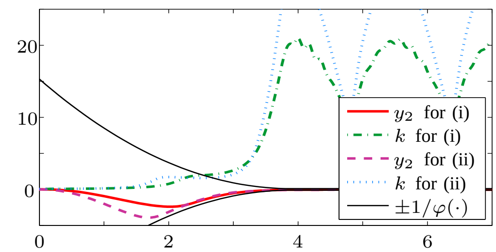
The simulations show that funnel control may be applied to system (11) despite the fact that it has unstable zero dynamics, relative degree two and negative high-frequency gain. Restrictions are that the zero is "far" in the right half complex plane, the initial condition $\widetilde{x}^0$ is "small" and the $L^\infty \times W^{1,\infty}$ input/output disturbances u_0 and y_0 are "small".

IV. Conclusions

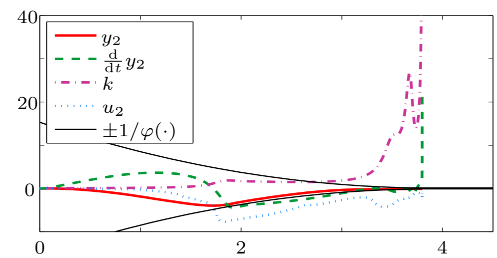
We have shown robustness of the funnel controller (4) for a class of linear systems which are close in the gap metric to minimum phase systems with (strict) relative degree one; moreover, funnel control copes with certain bounded input/output disturbances.



(a) y_1, z, k and u_1 of $[P^{\mathrm{g}}_{\widetilde{x}^3=(0.001,0.001,0.001)^\top}, C(\varphi)]$ for $N = 2M = 10,000$ and $u_0 = 0$



(b) y_2, k of $[P^{\mathrm{g}}_0, C(\varphi)]$ for (i) $N = 2M = 100$ and u_0^1; (ii) $N = 2M = 400$ and u_0^2



(c) y_2, $\dot{y}_2$, k and u_2 of $[P_{N,M,\alpha;\widetilde{x}^5}, C(\varphi)]$ for $[P^{\mathrm{g}}_0, C(\varphi)]$ for $N = 2M = 100$ and u_0^2

Fig. 4. Funnel control simulations: general system $[P^{\mathrm{g}}_{\widetilde{x}^3}, C(\varphi)]$, with $N = 2M = 10,000$ and $u_0 = 0$; $[P^{\mathrm{g}}_0, C(\varphi)]$ with u_0^j, $j = 1, 2$.

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