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Neutral Optima in Informed Principal Problems with Common Values*

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Abstract

In a class of informed principal problems with common values often used in applications we define a particular mechanism which we call the assured allocation. It is always undominated, i.e. efficient among the different types of the principal. We show it is a perfect Bayesian equilibrium allocation of the three-stage game studied in Maskin and Tirole (1992) that coincides with the Rothschild-Stiglitz-Wilson allocation when the latter is undominated. Under familiar conditions on hazard rates we show that the assured allocation is a neutral optimum in the sense of Myerson (1983).

Keywords: Neutral optimum, mechanism design, informed principal
JEL classification: D82, D86

1 Introduction

Informed principal problems are adverse selection problems where the principal, who proposes a contract to an agent, has private information. Important examples are given by firms with private information about projects who seek finance from competing lenders or by managers with private information who have bargaining power when dealing with shareholders. Informed principal problems are conceptually harder to understand than

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models where only the agent has private information because the contract proposed by the principal could potentially be a signal about the principal's knowledge.

The purpose of this paper is to draw connections between two major contributions on this subject, namely the work by Myerson (1983) and by Maskin and Tirole (1992). Myerson characterizes a "neutral" optimum, which is constructed in such a way that no private information can be revealed by the proposal itself. His approach is in part cooperative. In a more restrictive setting Maskin and Tirole (1992) and Maskin and Tirole (1990) characterize the perfect Bayesian equilibria of a contract-proposal game where the agent does not have any private information.

We are primarily interested in the case of common values. We will work here with the framework of Maskin and Tirole (1992), assuming, in addition, quasi-linear payoffs and certain sorting conditions. We construct in an inductive fashion, resembling the construction of the RSW (Rothschild-Stiglitz-Wilson) allocation in Maskin and Tirole (1992), Proposition 2, a unique mechanism which we call the assured allocation. The mechanism can be understood as a solution to a principle-agent problem where the outside option varies with the type, as in Jullien (2000). In our case the outside options are, however, defined endogenously.

The assured mechanism is shown to weakly Pareto dominate for the different types of the principal the RSW allocation. In consequence, it is always a perfect Bayesian equilibrium outcome of their contract-proposal game. When the RSW is undominated, i.e. efficient among the different types of the principal, the assured mechanism is the RSW, otherwise it is one of many perfect Bayesian equilibrium outcomes.

When the RSW is undominated it is a strong solution in the sense of Myerson (1983), and, hence, strong solution, neutral optima, the RSW allocation and the assured mechanism all coincide. We also show that the assured mechanism is undominated (in the terminology of Myerson) or interim efficient (in the sense of Maskin and Tirole).

We do not know how the assured mechanism and neutral optima relate in the presence of bunching. If bunching occurs then our iterative approach may possibly be inappropriate to capture neutral optima, which are defined as solutions to limits of rather involved fixed point problems. However, under familiar conditions on hazard rates that rule out bunching or generally if there are only two types, we can show that the assured mechanism is indeed a neutral optimum. Whether there are other neutral optima when there are more than two types is an open question.

Assured mechanisms may not be robust as they rely on the principal knowing the prior beliefs of the agent about the principal's type. In this paper the emphasis is, following Myerson, on the efficiency properties of the mechanisms studied. Obviously an understanding of the robustness properties of mechanisms and how this interacts with efficiency is very important. However, it is out of the scope of the current work and must be left for future research. It is worth emphasizing that such a task may not be merely a direct application of the results recently derived in Bergemann and Morris (2005), Bergemann and Morris (2008) and the literature they cite. The reason is that in our framework the designer herself has private information. Therefore, on the one hand, an offered mechanism may have some informational content and, on the other hand, the type space of the agent will include her prior beliefs about the principal's payoff type, and the type space of the principal will include the prior beliefs of the principal about the agent's prior beliefs.

In the case of independent values (see Maskin and Tirole (1990)) it is typically possible to find undominated mechanism which are also ex-post efficient, at least when utilities are quasi-linear. For this case we refer to the recent work of Mylovanov and Troeger (2007) and the literature they cite. We know only of a few other applications of the neutral optimum (Myerson (1983)) or the neutral bargaining solution (Myerson (1984)). Both the discussion of the lemon problem in Myerson (1985) and the extended liability problem discussed in Balkenborg (2006) are common value problems. Both problems do not fall into our framework. Applications to bargaining problems with two-sided incomplete information are given in Myerson (1984) and Darrough and Stoughton (1989).

The organization of the paper is as follows. In Section 2 we provide in a representative example a simple geometric analysis of the RSW, the neutral optimum and the assured allocation for the case of two types. In Section 3 we describe the general model and the weakest assumptions for which we know our analysis to hold. Section 4 discusses a number of central concepts, including interim efficiency, the Rothschild-Stiglitz-Wilson allocation, the neutral optimum and the assured allocation. The first part of Section 4 is devoted to a detailed study of assured allocations, in particular their uniqueness. In the second part we state our main results. Section 5 provides the proof that the assured allocation is a neutral optimum under no-bunching conditions. Section 6 concludes.

2 An illustration for the two-types case

We will be focusing on a model of bilateral trade between a producer and a buyer, when the producer's cost-efficiency is her private information. While being seemingly very special, this model is representative of a wide range of models used in the literature to investigate the problems of monopoly regulation (see Baron and Myerson (1982)) and nonlinear pricing (see Maskin and Riley (1984)), and study financial contracts (see, for instance, Freixas and Laffont (1990)) and quality and price discrimination (see, for instance, Mussa and Rosen (1978)). Maskin and Tirole (1992) also give many examples of informed principal problems. The modeling used here fits for their case of managerial compensation. It is, in principle, the model used in the textbook Laffont and Martimort (2002), except that we also consider the case of common values. The defining characteristics of the framework that encompasses our and the aforementioned models will be outlined shortly.

For concreteness, let us consider here the example of an uninformed firm owner and an informed manager. For this section only, we assume specific revenue and cost functions to fix ideas. However, the analysis remains valid for arbitrary revenue and cost functions satisfying the assumptions in Section 3.

The owner and the manager are assumed to be risk neutral. The manager produces an output $q \geq 0$ which yields to the owner a revenue

$$S(q,\theta) = \sqrt{\theta^{\beta} q}$$

where $\beta \geq 0$ and $\theta > 0$ are scalars. The production cost, incurred by the manager, is

$$C(q,\theta) = \frac{1}{\theta} q.$$

Thus, θ denotes the cost-efficiency of the manager, which is private information to him. $\beta = 0$ captures the case of private values, while $\beta > 0$ captures the case of common values. In the latter case, cost-efficiency is positively related with the owner's valuation.

Assume that there are only two types $\theta \in \{\theta_1, \theta_2\}$ with $\theta_2 > \theta_1$, and let $s_i = \Pr(\theta = \theta_i)$ be the prior probability of type θ_i. s_1 (and hence $s_2 = 1 - s_1$) is assumed to be common knowledge. We denote the surplus from producing q when the cost-type is θ_i ($i \in \{1, 2\}$) by

$$W_i(q) = S(q, \theta_i) - C(q, \theta_i).$$

Let the *first-best production level* for type $\theta = \theta_i$ be

$$q_i^o \equiv \arg\max_{q \geq 0} W_i(q) = \frac{1}{4}\theta_i^{2+\beta}.$$

The owner and the manager have to agree on a compensation $t \geq 0$ and a production level q. Because of the concavity assumptions we are making we can ignore randomization in the contract. It suffices hence to consider only option contracts where the owner and the manager agree on a pair of options $((t_1, q_1), (t_2, q_2))$. Hereby t_i denotes the compensation to the manager for producing output q_i; $i = 1, 2$. Once the contract is agreed, the manager can select any of the two options: either the option (t_1, q_1) designed for type 1 or the option (t_2, q_2) designed for type 2. We assume that the owner cannot renege on the contract after he has observed which option the manager takes. Under such a contract, if both types select the option designed for them, then the payoff of the owner upon facing a manager of type θ_i will be

$$\Pi_i = S(q_i, \theta_i) - t_i,$$

and the payoff of the manager of type θ_i will be

$$U_i = t_i - C(q_i, \theta_i).$$

We will assume that the owner and all types of the manager have the same outside option, which is normalized to zero.[1]

Let now $\hat{s}_i$ be the posterior probability of type θ_i as it is perceived by the owner given the observed behavior of the manager during the negotiation of the contract. To be (*interim*) individually rational (given posterior beliefs about the manager's type), the agreed contract must satisfy the participation constraints $U_1 \geq 0, U_2 \geq 0$ and

$$\sum_{i=1}^{2} \hat{s}_i \Pi_i = \sum_{i=1}^{2} \hat{s}_i (W_i(q_i) - U_i) \geq 0.$$

That is, the manager and the owner prefer to reach an agreement rather than take up their outside option, given the posterior beliefs.

The Revelation Principle allows us to restrict attention to incentive-compatible contracts. Define

$$U_i^j = t_j - C(q_j, \theta_i)$$

[1] We assume that all types of the manager have the same outside option. Therefore, our model cannot be used to investigate the problem of a monopolist insurer (see Stiglitz (1977)) or of franchising (see Maskin and Tirole (1992)). We also assume that the owner is a monopolist and hence our model cannot be used to investigate the problem of a competitive insurance market (see Rothschild and Stiglitz (1976)).

for all $i, j \in \{1, 2\}$. Then incentive-compatibility requires

$$U_i^i \geq U_i^j$$

to hold for any $i, j = 1, 2$, $i \neq j$. Equivalently, since $U_i^i = U_i$, this can be expressed as

$$\begin{aligned} U_1 &\geq U_2 - \psi_1(q_2), \text{ and} \quad (1) \\ U_2 &\geq U_1 + \psi_1(q_1), \quad (2) \end{aligned}$$

where

$$\psi_1(q) \equiv C(q, \theta_1) - C(q, \theta_2) = \frac{\theta_2 - \theta_1}{\theta_1 \theta_2} q \equiv \Delta\theta q$$

is the cost-gain from being of efficiency-type θ_2 and not of θ_1. The first condition simply says that the high-cost type prefers the option (t_1, q_1) - which ensures a payoff U_1 - over the option (t_2, q_2) - which ensures a payoff $U_2 - \psi_1(q_2)$. Similarly, the second condition is the low-cost type's incentive-compatibility constraint. We call an option contract *feasible relative to beliefs* $\hat{s}$ if it is individually rational given beliefs $\hat{s}$ and incentive compatible. When beliefs coincide with priors, i.e. $\hat{s} = s$, then we will be referring to such contracts simply as *feasible* contracts.

For later use, let us define here the owner's *ex post* individual-rationality constraints as

$$\Pi_i = W_i(q_i) - U_i \geq 0, \text{ for } i = 1, 2. \quad (3)$$

For the analysis it is convenient to work with a "reduced" form of the model where the transfers t_i are eliminated. We redefine an option contract as

$$m \equiv (m_1, m_2) = ((U_1, q_1), (U_2, q_2))$$

with the implicit understanding that in equilibrium each type selects the contract designed for himself and obtains the transfer $t_i = U_i + C(q_i, \theta_i)$.

At this stage, we can highlight why our model is representative of the models we mentioned above. First, in all these models, the surplus and the first-best production levels are both either increasing or decreasing in the type of the informed party. In our case, we have $W_2(q) > W_1(q)$ for $q > 0$ and $q_2^o > q_1^o$. Second, if the option contract dictates first-best production for each type and that all types break even, then the informed party does not have an incentive to mimic a type with higher first-best surplus, while she does have an incentive to mimic a type with lower first-best surplus. That is, in terms of our model here, when $U_1 = U_2 = 0$ we have that $U_2 - U_1 < \psi_1(q_1^o)$ and $U_2 - U_1 \leq \psi_1(q_2^o)$. Note that this is true in our model.

In this paper we analyze the case where one party, namely the manager, has full bargaining power, and hence offers to the other party an option contract m in a take-or-leave-it manner. It will be instructive, however, to compare this case with the standard case where the principal is the uninformed party.

We are hence comparing two versions of the following 3-stage game (see Maskin and Tirole (1992)). After nature has determined the type of the manager, the principal (the party with the full bargaining power) proposes an option contract. Then the agent (the party without bargaining power) can accept or reject the contract. If it is rejected, the parties get their outside options. Otherwise the manager chooses an option. In Version 1, the standard principal-agent model with hidden information (see, for instance, Laffont and Martimort (2002) Chapter 2), the principal is the uninformed owner and hence $\hat{s}_i = s_i$, $i = 1, 2$. In Version 2 the principal is the informed manager.

For Version 1, it is well-known that the owner-optimal, full-information contract

$$m^O = \left((U_1^O, q_1^O), (U_2^O, q_2^O)\right) = ((0, q_1^o), (0, q_2^o))$$

is not incentive-compatible. The reason is, as we have seen above, that the manager prefers to choose the option $(0, q_1^o)$ regardless of her cost-type: in this way, the low-cost manager will attain information rents $U_2^1 = \psi_1(q_1^o)$. For any given β, the *feasible* mechanism which maximizes the profit of the owner, $(\check{m}_1, \check{m}_2)$, leaves the high-cost manager with zero profits, i.e. $\check{U}_1 = 0$, and gives the low-cost manager an informational rent which leaves him indifferent between the two options, i.e. $\check{U}_2 = \psi_1(\check{q}_1)$. Furthermore, the low-cost manager produces the first-best level, $\check{q}_2 = q_2^o$, while the high-cost manager underproduces, i.e. $\check{q}_1 < q_1^o$. The high-cost manager's output balances the trade-off, in terms of the owner's expected (given prior beliefs) payoff, between lower information rents for the low-cost manager and higher output distortion.[2]

The analysis changes dramatically in Version 2 when the informed party becomes the principal. Start with the case of private values, $\beta = 0$, where $S(q, \theta)$ does not depend on θ. In this case, the manager-optimal, full-information contract

$$\begin{aligned} m^M &= \left((U_1^M, q_1^M), (U_2^M, q_2^M)\right) \\ &= ((W_1(q_1^o), q_1^o), (W_2(q_2^o), q_2^o)) \end{aligned}$$

is *always* incentive-compatible. Effectively the contract allows each type to choose the output he or she wants to produce and to keep all the proceeds. Note that under this contract the owner breaks even regardless of the type he faces. This is the unique perfect Bayesian equilibrium outcome in the three-stage game where the manager is the proposer (see Maskin and Tirole (1990), Maskin and Tirole (1992)). The contract is illustrated in the figure below.

Consider next the case of common values, $\beta > 0$, which is the focus of our work. Still, as long as $W_1(q_1^o) \geq W_2(q_2^o) - \psi_1(q_2^o)$ holds, the manager does not have an incentive to pretend that she is of a different type.[3] m^M remains *the* perfect Bayesian equilibrium outcome.

We consider now the case $W_1(q_1^o) < W_2(q_2^o) - \psi_1(q_2^o)$ that arises for sufficiently high β and sufficiently low ratio of cost-types $\frac{\theta_2}{\theta_1}$. In this case, the manager-optimal, first-best contract is no longer incentive compatible. Now, there are two different subcases, depending on the prior probabilities s_i, $i = 1, 2$. To discuss this case, we need a number

[2] Formally, $\check{q}_1 = \arg\max_{q \geq 0}\{s_1 W_1(q_1) - s_2 \psi(q)\}$. That is, in our example, $\check{q}_1 = \frac{\frac{1}{4}\theta_1^\beta}{(\frac{s_2}{s_1}\Delta\theta + \frac{1}{\theta_1})^2}$.

[3] The inequality in the main text guarantees that the high-cost type does not want to mimic the low-cost type. Note also that $W_2(q_2^o) \geq W_1(q_1^o) + \psi(q_1^o)$ due to $S(\theta_2, q_2^o) \geq S(\theta_2, q_1^o) > S(\theta_1, q_1^o)$ and $C(\theta_2, q_2^o) \leq C(\theta_2, q_1^o)$. So, the low-cost manager prefers the option designed for her type, $\left(U_2^M, q_2^M\right)$.

of additional concepts (see Myerson (1983) and Maskin and Tirole (1992)).

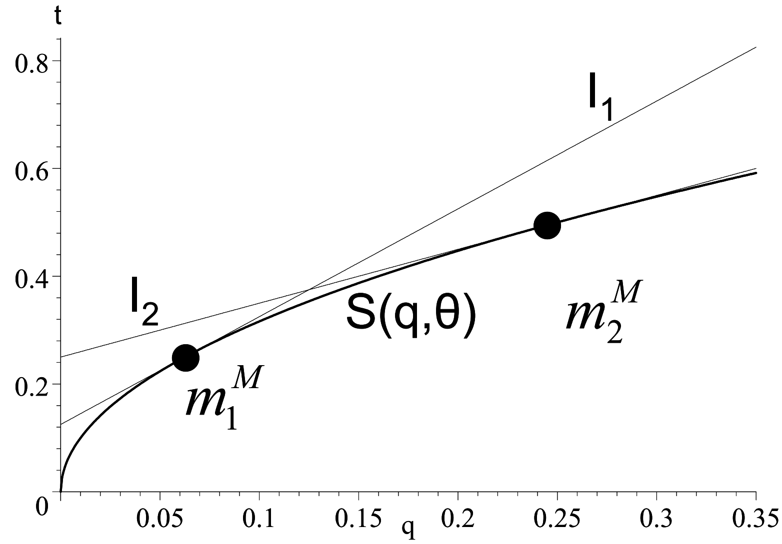

Figure 1.

In the private value case, $S(q,\theta)$ is independent of θ ($\beta = 0$). The manager-optimal, first-best contract is incentive compatible, as is indicated by the linear indifference curves I_1 and I_2 for the two types. (Each type would prefer contracts above his indifference curve.)

A feasible contract is *undominated* if it is not dominated in the Pareto sense, from the point of view of the two types of the manager, by another feasible contract. A mechanism is *safe* if it is incentive compatible for the manager and ex post individually rational for the owner. Note that a safe mechanism is also feasible. A mechanism is a *strong solution* if it is safe and undominated. For instance, the manager-optimal, full-information contract is a strong solution if and only if it is incentive compatible. A mechanism is an *RSW (Rothschild-Stiglitz-Wilson) allocation* (relative to the outside options) if it maximizes for each type of the manager his utility subject to the incentive-compatibility constraints of all types and subject to the ex post participation constraints of the owner. Note that the RSW allocation is a strong solution if and only if it is undominated.[4]

In our set-up here we have directly that the RSW allocation is given by the full information contract m^M if the latter is incentive-compatible. So, if the manager-optimal, full-information contract is incentive-compatible, then this contract coincides with the RSW allocation. Therefore, the latter is also a strong solution.

However, if m^M is not incentive-compatible, then the RSW allocation m^{RSW} is given by

$$m_1^{RSW} = \left(U_1^{RSW}, q_1^{RSW}\right) = (W_1(q_1^o), q_1^o)$$

[4]Note also that the notion of an undominated contract and, as we will see shorty, of the neutral optimum depends, in contrast to the notion of an RSW allocation, on the prior probabilities s_i.

and

$$m_2^{RSW} = \left(U_2^{RSW}, q_2^{RSW}\right) = \left(U_1^{RSW} + \psi_1\left(q_2^{RSW}\right),\ W_2^{-1}\left(U_2^{RSW}\right)\right)$$

(see Maskin and Tirole (1992), Proposition 2). That is, the high-cost manager attains her best full-information option, while the low-cost manager's option is such that the *high-cost* type is indifferent between the two options and the owner makes zero profits from *each and every* type. Note that at the option m_2^{RSW} we have $q_2^{RSW} > q_2^o$. The following Figure illustrates.

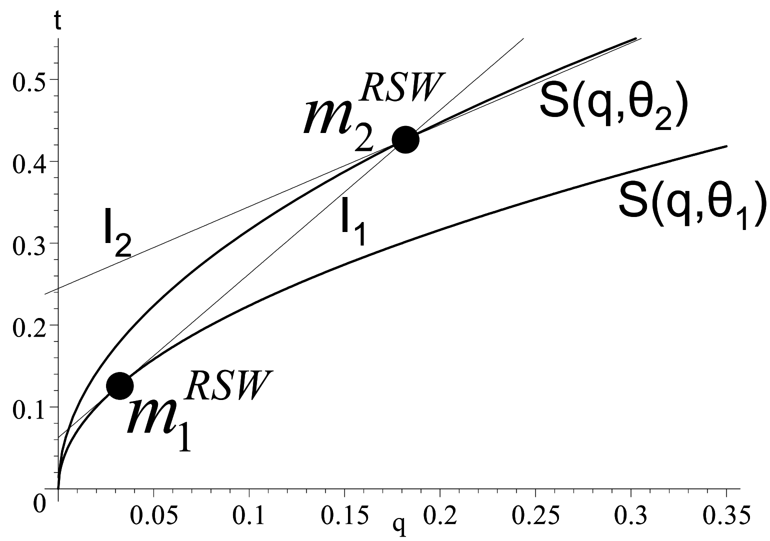

Figure 2 The RSW allocation

> In this case with common values the manager-optimal first-best contract is not incentive compatible.

The RSW is an important allocation. Maskin and Tirole (1992) Theorem 1 show, for our set-up, that a feasible mechanism is a perfect Bayesian equilibrium (PBE) outcome *if and only if* it gives each type of the manager at least his payoff in the RSW allocation.

Therefore, if the RSW allocation is undominated, then it is the only perfect Bayesian equilibrium allocation. However, the RSW allocation can be dominated. In this case, then, there is multiplicity of PBE.

We show that a *neutral optimum*, as defined by Myerson (1983), is a feasible mechanism that weakly dominates the RSW allocation, and hence it is always a PBE. By definition, the neutral optimum is undominated.

To find the neutral optimum we must first determine the set of all undominated feasible allocations. For each of these allocations Myerson (1983) defines so-called "warranted claim" vectors that are essential in the calculation of the neutral optimum. These are determined simultaneously with the undominated mechanisms.

We can find the undominated option contracts by maximizing for given weights $\tau_1, \tau_2 \geq 0$, $\tau_1 + \tau_2 > 0$, the weighted sum of utilities

$$\tau_1 U_1^1 + \tau_2 U_2^2$$

subject to the incentive constraints for the two types of the manager and the participation constraint for the owner. Without loss of generality we can assume $\tau_1 + \tau_2 = 1$.

The Lagrangian for this problem is, with the appropriate Lagrange multipliers μ_{12}, μ_{21} and γ,

$$\begin{aligned} \mathcal{L} &= \tau_1 U_1^1 + \tau_2 U_2^2 + \mu_{12}\left(U_1^1 - U_1^2\right) + \mu_{21}\left(U_2^2 - U_2^1\right) + \gamma\left(s_1 \Pi_1 + s_2 \Pi_2\right) \\ &= \gamma s_1 VS_1 + \gamma s_2 VS_2 \end{aligned}$$

where, for $\{i, j\} = \{1, 2\}$, $j \neq i$,

$$VS_i = \frac{1}{\gamma s_i}\left(\left(\tau_i + \mu_{ij}\right) U_i^i - \mu_{ji} U_j^i + \gamma s_i \Pi_i\right)$$

are the so-called *virtual surpluses* for the two types. The relevance of the virtual surpluses becomes clear if we write them explicitly, using the definitions for U_i^j, $i, j = 1, 2$, and that $\Pi_i = W_i(q_i) - U_i$, as

$$\begin{aligned} VS_1 &= \frac{1}{\gamma s_1}\left(\left(\tau_1 + \mu_{12} - \mu_{21} - \gamma s_1\right) U_1 + \left(\gamma s_1 W_1\left(q_1\right) - \mu_{21}\psi_1\left(q_1\right)\right)\right) \\ VS_2 &= \frac{1}{\gamma s_2}\left(\left(\tau_2 + \mu_{21} - \mu_{12} - \gamma s_2\right) U_2 + \left(\gamma s_2 W_2\left(q_2\right) + \mu_{12}\psi_1\left(q_2\right)\right)\right) \end{aligned}$$

Thus VS_1 depends only on U_1 and q_1, and VS_2 only on U_2 and q_2. Maximizing the Lagrangian is (for given Lagrange multipliers!) the same as maximizing each virtual surplus separately. The first-order conditions for an optimum are hence

$$\gamma s_1 \frac{\partial VS_1}{\partial U_1} = \tau_1 + \mu_{12} - \mu_{21} - \gamma s_1 = 0 \tag{4}$$

$$\gamma s_2 \frac{\partial VS_2}{\partial U_2} = \tau_2 + \mu_{21} - \mu_{12} - \gamma s_2 = 0 \tag{5}$$

$$\gamma s_1 \frac{\partial VS_1}{\partial q_1} = \gamma s_1 W_1'\left(q_1\right) - \mu_{21}\psi_1'\left(q_1\right) = 0 \tag{6}$$

$$\gamma s_2 \frac{\partial VS_2}{\partial q_2} = \gamma s_2 W_2'\left(q_2\right) + \mu_{12}\psi_1'\left(q_2\right) = 0 \tag{7}$$

As usual, the complementarity conditions must hold and the Lagrangian multipliers must be non-negative. Summing the first-order conditions (4) and (5) gives, since $\tau_1 + \tau_2 = 1$, that $\gamma = 1$. This implies, by the complementarity conditions, that the participation constraint of the owner must be binding.

Before continuing with the more detailed description of the optimal solution we note that Myerson (1983) associates with each pair of *strictly positive* utility weights (τ_1, τ_2) a vector of warranted claims (ω_1, ω_2) defined by

$$\left(\tau_1 + \mu_{12}\right)\omega_1 - \mu_{21}\omega_2 = \gamma s_1 VS_1^* \tag{8}$$

$$\left(\tau_2 + \mu_{21}\right)\omega_2 - \mu_{12}\omega_1 = \gamma s_2 VS_2^* \tag{9}$$

where VS_i^* $(i = 1, 2)$ is the virtual surplus, and μ_{12}, μ_{21} are the Lagrange multipliers, in the optimal solution. Clearly, this simultaneous system of equations has a unique solution.

We discuss the interpretation of the warranted claims further below. At the moment our aim is to determine them simultaneously with the optimal solution for the given weights. Notice that

$$\tau_1\omega_1 + \tau_2\omega_2 = \gamma s_1 VS_1^* + \gamma s_2 VS_2^* = \mathcal{L}^* = \tau_1 U_1^* + \tau_2 U_2^* \tag{10}$$

where $\mathcal{L}^*$ is the value of the Lagrangian at the optimum and hence, by the complementarity conditions, equal to the value of the objective function at the optimum, $\tau_1 U_1^* + \tau_2 U_2^*$. The following Figure 3 illustrates the Pareto frontier, i.e. the utility allocations in undominated mechanisms, for the two types of manager. Several tangents corresponding to different utility weights are drawn. The two dotted lines intersect in the manager-optimal, first-best allocation, which is not feasible in the example shown. The intersections of the tangents with the two dotted halflines is here the set H of warranted claim vectors.

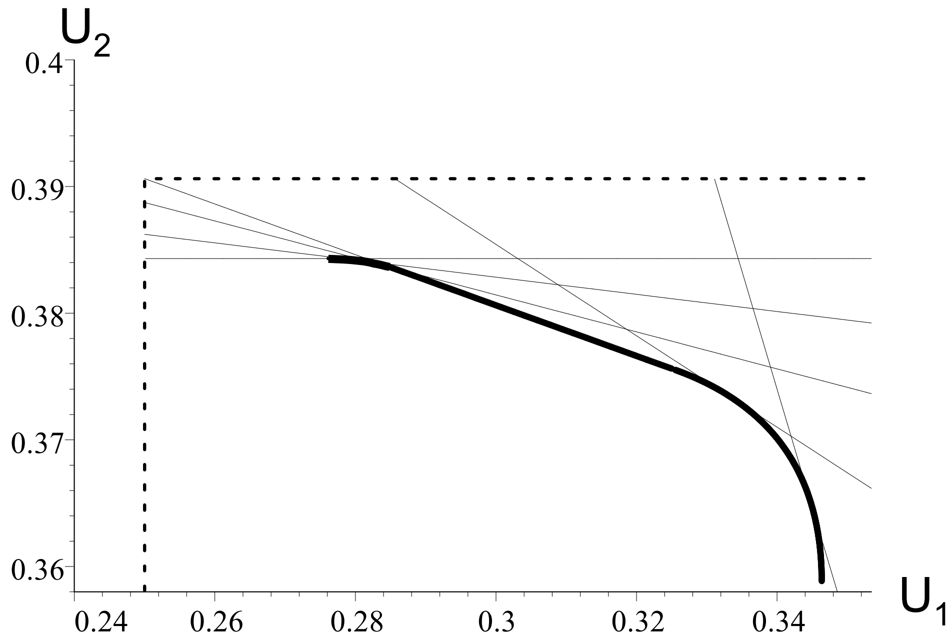

Figure 3

The warranted claim allocations for a particular Pareto frontier.

While the participation constraint of the owner is always binding in an undominated mechanism, we must distinguish three cases for the incentive constraints of the two types.[5] To discuss these cases denote with q_i^* the output of type θ_i at optimum. The following

[5] There are four cases a priori concerning which incentive constraint is binding or not. Note that the incentive constraints can be written as

$$\psi(q_1) \leq U_2 - U_1 \leq \psi(q_2)$$

Suppose that both incentive constraints were binding, so $\psi(q_1) = U_2 - U_1 = \psi(q_2)$ and hence $q_1 = q_2 = q$ (the option contract is a pooling contract). This is seen to lead immediately to a contradiction with the first-order conditions (6) and (7) given that $\arg\max_{q\geq 0} VS_1 \leq q_1^o < q_2^o \leq \arg\max_{q\geq 0} VS_2$.

Figure 4 shows the *extended* Pareto frontier associated with the undominated contracts, as explained below.

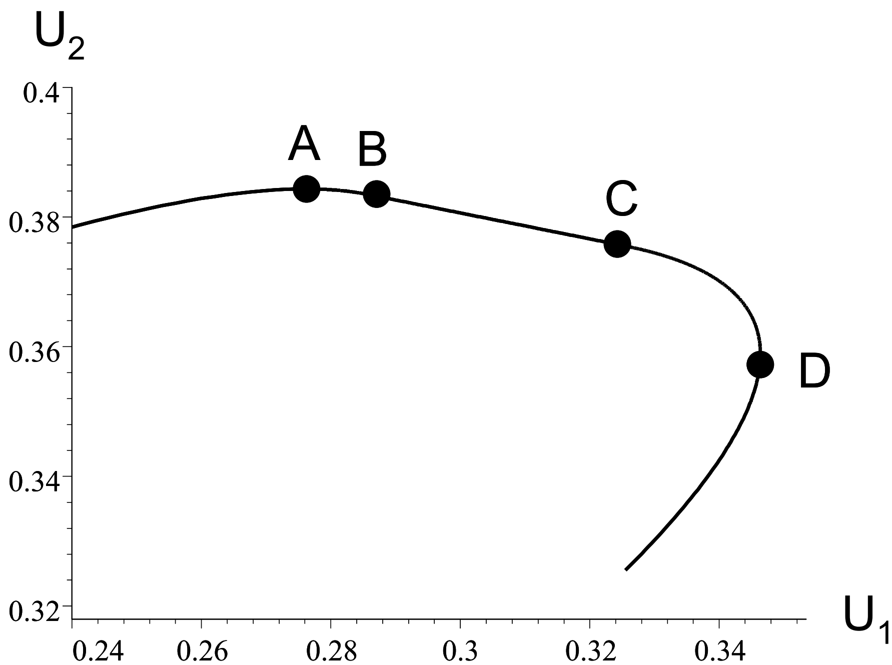

Figure 4

The extended Pareto frontier given a prior.

CASE 1. No incentive is binding and hence $\mu_{12} = \mu_{21} = 0$. (4) and (5) imply that $\tau_1 = s_1$ and $\tau_2 = s_2$. (6) and (7) imply that both types produce at their first-best levels, $q_1^* = q_1^o$ and $q_2^* = q_2^o$. The pair of optimal utilities (U_1^*, U_2^*) is on the line segment with slope $-s_1/s_2$ given by the binding participation constraint

$$s_1 U_1^* + s_2 U_2^* = s_1 W_1 (q_1^o) + s_2 W_2 (q_2^o)$$

and the incentive constraints, which can be written as,

$$\psi_1 (q_1^o) \leq U_2^* - U_1^* \leq \psi_1 (q_2^o)$$

One checks immediately that this is a proper line segment of positive, finite length. It is the line segment BC in Figure 4.

Since $\mu_{12} = \mu_{21} = 0$ the virtual surpluses and hence the warranted claims are just the first-best surpluses of the two types, $\omega_i = W_i (q_i^o)$ $(i = 1, 2)$. This case includes the one we already studied above, where the manager–optimal, full-information is incentive-compatible and, hence, coincides with the RSW allocation.

CASE 2. Only the incentive constraint $U_2^2 \geq U_2^1$ is binding and hence $\mu_{12} = 0$. (4) and (5) imply that $\tau_1 = s_1 + \mu_{21} \geq s_1$ and $\tau_2 = s_2 - \mu_{21} \leq s_2$. Thus, this case is relevant when a relatively high utility weight is given on the high-cost type. By (7) the low type's effort level is at its first-best: $q_2^* = q_2^o$. By (6), and given that $\psi_1(q)$ is increasing, the high type's output level is weakly lower than the first-best: $q_1^* = \arg\max_{q\geq 0}\{s_1 W_1(q) - \mu_{21}\psi_1(q)\}$. Given $\mu_{21} = s_2 - \tau_2$, we can find the various optima for various utility weights $0 \leq \tau_2 \leq s_2$ by gradually increasing μ_{21} from $\mu_{21} = 0$ until $\mu_{21} = s_2$. Point C in Figure 4 corresponds to the utility allocation at the optimal solution with $\mu_{21} = 0$, where the low-cost type still produces first-best. By increasing μ_{21} (decreasing τ_2), high-cost output falls below the first-best level. Since the incentive constraint

$$U_2^* - U_1^* = \psi_1(q_1^*)$$

and the participation constraint are binding and since $\psi_1(q)$ is increasing, as we increase μ_{21} we necessarily decrease U_2^* (since $U_2^* = s_1 W_1(q_1^*) + s_2 W_2(q_2^0) + s_1\psi_1(q_1^*)$ and q_1^* is decreasing in μ_{21}). U_1^*, on the other hand, is increasing insofar $\mu_{21} \leq s_2$ (since $U_1^* = s_1 W_1(q_1^*) + s_2 W_2(q_2^0) - s_2\psi_1(q_1^*)$). This yields segment CD in the Pareto frontier in Figure 4. However, precisely from $\mu_{21} = s_2$ onwards U_1^* is decreasing. Any further decrease in type 2's utility U_2^* is necessarily associated with a reduction in U_1^*. We now have a negative utility weight on the low type and are in the backward bending range below point D of what we call the extended Pareto frontier in Figure 4. These contracts are, of course, dominated. Interestingly, note that if $\mu_{21} = s_2$, then $q_1^* = \check{q}_1$.

Associated with an undominated option contract in range CD is the following warranted claims vector. Since $\mu_{12} = 0$ and $\tau_2 + \mu_{21} = s_2$ we obtain $\omega_2 = VS_2^* = W_2(q_2^0)$. A simple geometric argument using (10) shows that $\omega_1 \geq W_1(q_1^o)$. This means that (ω_1, ω_2) is on the horizontal half line in Figure 3 (and 5).

CASE 3. Only the incentive constraint $U_1^1 \geq U_1^2$ is binding and hence $\mu_{21} = 0$. The analysis is mostly symmetric to CASE 2. (4) and (5) imply that $\tau_1 = s_1 - \mu_{12} \leq s_1$ and $\tau_2 = s_2 + \mu_{12} \geq s_2$. Now, a relatively high weight is given on the low-cost type. By (6) the low type's effort level is at its first-best: $q_1^* = q_1^o$. By (7), and given that $\psi_1(q)$ is increasing, the high type's output level is weakly higher than the first-best: $q_2^* = \arg\max_{q\geq 0}\{s_2 W_2(q) + \mu_{12}\psi_1(q)\}$. Given $\mu_{12} = s_1 - \tau_1$, we can find the various optima for various utility weights by gradually increasing μ_{12} from $\mu_{12} = 0$ until $\mu_{12} = s_1$. Point B in Figure 4 corresponds to the utility allocation at the optimal solution with $\mu_{12} = 0$, where the low-cost type still produces first-best. By increasing μ_{12} (decreasing τ_1), low-cost output increases beyond the first-best level. Alongside the owner's participation constraint, the incentive constraint

$$U_2^* - U_1^* = \psi_1(q_2^*)$$

is binding. This implies that as we increase μ_{12} we necessarily decrease U_1^* (since $U_1^* = s_1 W_1(q_1^o) + s_2 W_2(q_2^*) - s_2\psi_1(q_2^*)$ and q_2^* is increasing in μ_{12}). Initially, insofar $\mu_{12} \leq s_1$, as we increase μ_{12} we increase U_2^* (since $U_2^* = s_1 W_1(q_1^o) + s_2 W_2(q_2^*) + s_1\psi_1(q_2^*)$). This yields segment AB in the Pareto frontier in Figure 4. From $\mu_{12} = s_1$ onwards the utility weight on the low type is negative and U_2^* is decreasing. We are in the backward bending range to the left of Point A of the extended Pareto frontier in Figure 4. The corresponding contracts are dominated.

Associated with an undominated option contract in range AB is the following warranted claims vector. Since $\mu_{21} = 0$ and $\tau_1 + \mu_{12} = s_1$ we obtain $\omega_1 = VS_1^* = W_1(q_1^0)$. This means that (ω_1, ω_2) is on the vertical line $U_1 = W_1(q_1^0)$ in Figure 3 (and 5).

Figure 5 shows how the extended Pareto frontier varies as we vary the probability on the low cost type from $s_1 = 1/6$ to $s_1 = 2/3$. The latter prior corresponds to the steeper curve. Because the RSW allocation does not depend on this distribution it is the intersection of these curves. For $s_1 = 1/6$ the point N is a neutral optimum, as will be discussed shortly.

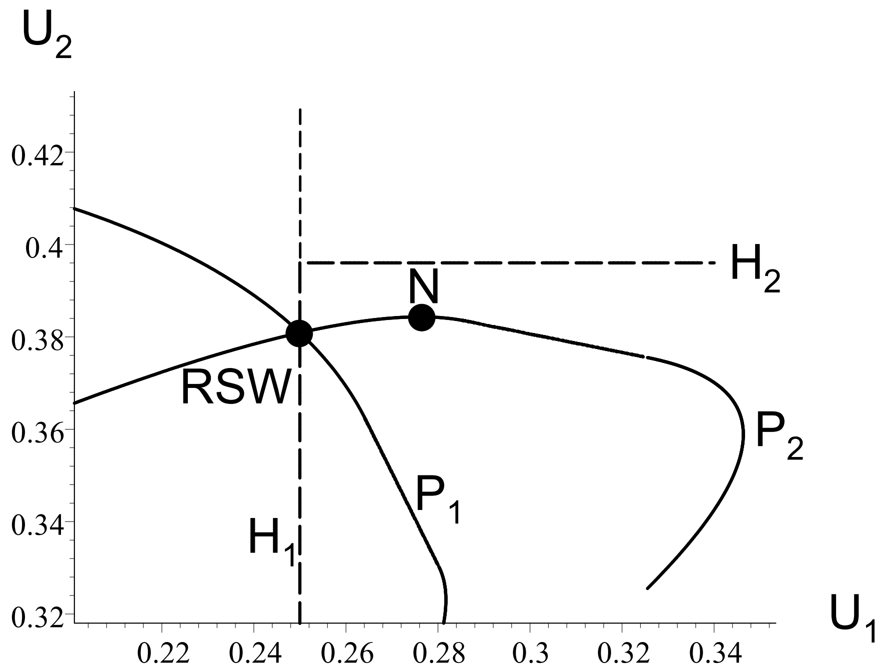

Figure 5

Two extended Pareto frontiers corresponding to different priors. The RSW intersects both frontiers. The neutral optimum coincides with the RSW for the steeper curve, but is different (point N) for the other.

We are now ready to discuss the neutral optimum.

Let P be the Pareto frontier for the two types of the manager, i.e. the set of utility pairs (U_1^*, U_2^*) in undominated option contracts. Define for $i = 1, 2$

$$U_i^{\max} = \max_{(U_1^*, U_2^*) \in P} U_i^*.$$

As we have seen, the set H of all warranted claims allocations is located on the union of the two line half lines

$$\begin{aligned} H_1 &= \left\{ \left(W_1 \left(q_1^0 \right), U_2 \right) | U_2 \leq \max \left\{ U_2^{\max}, W_2 \left(q_2^0 \right) \right\} \right\} \\ H_2 &= \left\{ \left(U_1, W_2 \left(q_2^0 \right) \right) | U_1 \geq W_1 \left(q_1^0 \right) \right\} \end{aligned}$$

Let $\bar{H}$ be the closure of H. Myerson (1983) characterizes a neutral optimum as an undominated mechanism that dominates a utility allocation in $\bar{H}$, i.e. that dominates a limit of a sequence of warranted claims.

Notice that the RSW allocation is on the half-line H_1. It is either on segment CA of the Pareto frontier or on the backward-bending part of the extended frontier. Correspondingly, there are two cases:

a) H_1 intersects the Pareto-frontier P, as for the frontier P_1 in Figure 5. The point of intersection is necessarily the RSW allocation, which, hence, coincides with the neutral optimum.

b) H_1 intersects the extended Pareto-frontier in the downward-bending part to the left of Point A in Figure 4. In this case the RSW allocation is also on the backward-bending part of the extended Pareto frontier. Here, the neutral optimum is the option contract that gives the low-cost type the highest payoff, $U_2^{\max}$. In Figure 5 point N gives the neutral optimum for frontier P_2.

Before summarizing our results we motivate briefly the notion of warranted claims in Myerson (1983): Start with strictly positive utility weights τ_1 and τ_2 and the associated optimal contract $((U_1^*, q_1^*), (U_2^*, q_2^*))$. Let us jump out of the model and consider for the same players and types in the abstract a mechanism that is also optimal with respect to the same utility weights for the same Lagrange multipliers. Whatever the more general model is exactly, such a mechanism must define the numbers

$$\left(\left(v_1^1, v_1^2\right), \left(v_2^2, v_1^2\right), (\pi_1, \pi_2)\right)$$

where v_i^j is the expected utility to type i if he "mimics" type j and π_i is the expected payoff of the owner conditional on facing type i. If the mechanism is optimal with respect to the same Lagrange multipliers, then, by the complementarity conditions and $\gamma > 0$, we have

$$s_1\pi_1 + s_2\pi_2 = 0$$

which means that the participation constraint of the owner is binding. Moreover, $\mu_{ij}v_i^i = \mu_{ij}v_i^j$, $i \neq j, i, j = 1, 2$. The virtual surpluses associated with the abstract contract would be

$$VS_i = \frac{1}{\gamma s_i}\left(\left(\tau_i + \mu_{ij}\right)v_i^i - \mu_{ji}v_j^i + \gamma s_i\pi_i\right) = \frac{1}{\gamma s_i}\left(\left(\tau_i + \mu_{ij}\right)v_i^i - \mu_{ji}v_j^j + \gamma s_i\pi_i\right)$$

We now ask: what would the utilities (v_1^1, v_2^2) have to be in an abstract mechanism like the one above to yield the same *virtual* surpluses as the original optimal mechanism $((U_1^*, q_1^*), (U_2^*, q_2^*))$ *and* be a *strong* solution? Since $\pi_1 = \pi_2 = 0$ in a strong solution, (v_1^1, v_2^2) has to satisfy the system of equations

$$\left(\tau_i + \mu_{ij}\right)v_i^i - \mu_{ji}v_j^j = \gamma s_i VS_i^* \text{ for } i \in \{1, 2\}$$

The unique solution to this system is called a vector of *warranted claims* by Myerson (1983). In that work, he makes it precise in which sense the warranted claims vector defines a strong solution in an artificial, extended model where there are more mechanism available than in the original model. His axiomatic characterization describes the neutral optimum as a mechanism that dominates the limit of a sequence of payoff vectors in strong solutions of extended models. His main characterization result describes this condition in terms of the warranted claim vectors only.

Define for the two-type case the *assured allocation* as the mechanism which maximizes among all feasible option contracts the expected payoff of low-cost type subject to high-cost type getting at least his first-best surplus $W^1\left(q_1^0\right)$. This definition will be extended for an arbitrary number of types in Section 4. We can now summarize our analysis as follows.

Proposition 1 *The neutral optimum is the assured allocation..*

Corollary 1 *If the RSW allocation is undominated, it is a strong solution and coincides with the neutral optimum. Otherwise a neutral optimum dominates the RSW.*

We now turn to the case of at least two types and general value and cost functions. With more than two types the possibility of bunching arises, in which case we do not know how neutral optima look like for our environment. Moreover, in the more general set-up, we do not know whether neutral allocations are unique.

3 The Model

A manager (the *principal*) produces q units of a product at the total cost $C\left(q,\theta\right)$ which depends on his "type" described by the scalar θ belonging to some interval Θ. Costs and marginal costs are non-decreasing in q, i.e.

$$\frac{\partial C}{\partial q} \geq 0, \ \frac{\partial^2 C}{\partial q^2} \geq 0$$

for all θ and $q \geq 0$. Total and marginal costs are decreasing in θ, i.e.

$$\frac{\partial C}{\partial \theta}\left(q,\theta\right) < 0, \text{ if } q > 0, \text{ and } \qquad \frac{\partial^2 C}{\partial q \partial \theta}\left(q,\theta\right) < 0$$

holds for all θ. The latter is a sorting condition that ranks types according to their marginal utility from trade. It states that higher types value trading with the agent more.

We restrict attention to the case of finitely many types $\theta_1 < \theta_2 < \cdots < \theta_N$ taken from Θ. We denote the probability of type θ_i, $1 \leq i \leq N$, by s_i. Let $f_i \equiv \sum_{j=1}^{i} s_j$, with the convention that $f_0 = 0$. The type of the manager is his private knowledge, and the distribution of types is common knowledge.

The value of the product to the owner (the *agent*) is $S\left(q,\theta\right)$. The value of the product is non-decreasing and the marginal value non-increasing in the level of output:

$$\frac{\partial S}{\partial q} \geq 0 \qquad \frac{\partial^2 S}{\partial q^2} \leq 0$$

for all θ and $q \geq 0$. In the case of independent values $S\left(q,\theta\right)$ is independent of θ. We are here primarily interested in the case of *common values* and assume that the value of the product is non-decreasing in θ and its marginal value not decreasing too fast in θ and too slow in q. Moreover, we assume that the value of the product is not decreasing too slow in q :

$$\frac{\partial S}{\partial \theta} \geq 0 \qquad \frac{\partial^2 S}{\partial q \partial \theta} > \frac{\partial^2 C}{\partial q \partial \theta} \qquad \frac{\partial^2 S}{\partial q^2} < \frac{\partial^2 C}{\partial q^2}$$

and

$$\lim_{q \to \infty} \{\frac{\partial S}{\partial q} - \frac{\partial C}{\partial q}\} < 0$$

for all θ and $q \geq 0$. We also assume that there are gains from trade and fixed costs are not too high:

$$\frac{\partial S(0, \theta_1)}{\partial q} > \frac{\partial C(0, \theta_1)}{\partial q} \qquad S(0, \theta_1) \geq C(0, \theta_1)$$

These conditions imply that the participation constraints for the various types of the principal are never binding in the RSW allocation and the assured allocation discussed below because high types will not want to mimic lower types and the lowest type will get at least the first best surplus. We can therefore ignore them in the following.

Remark 1 We could drop the assumption that Θ is an interval and assume that $S(q, \theta)$ and $C(q, \theta)$ are defined for $\theta = \theta_1, \cdots, \theta_N$ only. We would just need to express the monotonicity conditions without partially differentiating with respect to θ.

Remark 2 The model satisfies the Sorting Assumption on page 5 of Maskin and Tirole (1992), except that we restrict q to be nonnegative. However, even with this restriction, Propositions 2, 3, 4(a) and 5 of Maskin and Tirole (1992) still hold. Thus, the hypothesis of their Theorem 1 is satisfied (see their Remark 3 after their Theorem 1). Proposition 2 and Theorem 1 of Maskin and Tirole (1992) will be used here to yield Propositions 2 and 3 below.

Throughout we will use the notations for the surplus generated from trade between the agent and a principal of type θ_i and the cost difference, respectively,

$$W_i(q) := S(q, \theta_i) - C(q, \theta_i) \text{ for } 1 \leq i \leq N$$
$$\psi_i(q) := C(q, \theta_i) - C(q, \theta_{i+1}) \text{ for } 1 \leq i < N$$

Our assumptions imply that each W_i is a strictly concave function, and that $q_i^o \equiv \arg\max_{q \geq 0} W_i(q)$ exists and is unique and strictly positive and increasing in i. In addition, $W_i(q_i^0) > 0$ for any $1 \leq i \leq N$ and $W_i(q_i^0)$ is strictly increasing in i.

Moreover, each $\psi_i(q)$ is strictly positive, if $q > 0$, and a strictly increasing function. We assume that it is also concave.

A1 : $\psi_i''(q) \leq 0$ for all $1 \leq i < N$ and $q \geq 0$.

We need A1 in order to guarantee that the problem $X_n(y)$ discussed below is convex for every n and $y \geq 0$, and hence has a convex solution set which is described by the Lagrangian.

We also assume that

A2 : $\lambda \psi_i(q) + W_i(q)$ has a unique maximizer for all scalars $\lambda \geq 0$ and all $1 \leq i < N$.

Given A2, we obtain that $\frac{f_{i-1} - \delta}{s_i} \psi_{i-1}(q) + W_i(q)$ has a unique maximizer for all $\delta \in [0, f_{i-1}]$, denoted by $q_i(\delta)$.

Remark 3 Notice that $q_i(f_{i-1}) = q_i^o$. Notice also that $q_i(\delta)$ is strictly decreasing in δ whenever $q_i(\delta) > 0$.

We will also use for Theorem 2 the following assumption:

A3: $q_i(\delta)$ is a strictly increasing function of i for all $\delta \in [0, f_{N-1}]$ if $q_i(\delta) > 0$.

This condition prevents bunching in certain important mechanisms that will be defined later. A sufficient condition for A3 is that

$$\frac{\partial W_i(q)/\partial q}{\partial \psi_{i-1}(q)/\partial q}\ ,\ \frac{f_{i-1}}{s_i} \text{ and } \frac{f_{i-1} - f_{N-1}}{s_i} \text{ are strictly increasing in } i. \tag{11}$$

Remark 4 These conditions are very similar to those in Jullien (2000).

Remark 5 To understand the above conditions on the probability distribution function, note that these are sufficient for[6] $\frac{f_{i-1}-\delta}{s_i}$ to be strictly increasing in i for any $\delta \in [0, f_{N-1}]$. The latter condition in conjunction with the above condition on $\frac{\partial W_i(q)/\partial q}{\partial \psi_{i-1}(q)/\partial q}$ and the concavity properties of $W_i(q)$ and $\psi_{i-1}(q)$ ensure[7] in turn A3.

The principal and the agent are involved in the following three-stage game Γ_3: First the principal offers a contract/mechanism to the agent in a take-or-leave-it manner. A mechanism is a set of announcements for the principal, M^p, and a set of announcements for the agent M^a and an allocation-rule that maps announcements to (possibly lotteries over) transfer-output pairs. The agent then accepts or rejects the offered contract. If the agent accepts the contract then the contract is executed: the players choose their announcements and the associated allocation is implemented. We assume that the principal's and agent's reservation payoffs are zero.

Following Maskin and Tirole (1992) we focus on finite simultaneous-actions mechanisms. By the revelation principle for Bayesian games, we have that for any mechanism offered by the principal and for given beliefs *after* the contract has been accepted, any equilibrium of the mechanism corresponds to a truthful equilibrium of a direct revelation mechanism (DRM). In such a mechanism, the principal simply announces her type, and given an announcement θ_i an allocation κ_i is implemented. This allocation is the same with the one from the general mechanism. We therefore focus on DRMs.

Given our convexity assumptions we also focus on deterministic mechanisms. A deterministic DRM is then an option contract: $\kappa \equiv (\kappa_i)_{1 \le i \le N} \equiv ((t_i, q_i))_{1 \le i \le N}$. t_i denotes the net[8] transfer from the owner to the manager of type θ_i. For expositional simplicity, let us refer, hereafter, with some abuse of terminology, to a deterministic DRM as, simply, a mechanism/contract. Under such a contract, denote the payoff of the agent upon facing a principal of type θ_i by

$$\Pi_i = S(q_i, \theta_i) - t_i, \tag{12}$$

[6] To see this consider $\delta \in [0, f_{N-1}]$ and fix i, j with $j > i$. Suppose that $s_j \ge s_i$. We then have, by $\delta \ge 0$ and the assumed monotonicity of $\frac{f_{i-1}}{s_i}$, that $\frac{f_{j-1}-\delta}{s_j} - \frac{f_{i-1}-\delta}{s_i} > 0$. Suppose now that $s_j < s_i$. We then have, by $\delta \le f_{n-1}$ and the assumed monotonicity of $\frac{f_{i-1}-f_{N-1}}{s_i}$, that $\frac{f_{j-1}-\delta}{s_j} - \frac{f_{i-1}-\delta}{s_i} = \frac{f_{j-1}-f_{N-1}}{s_j} - \frac{f_{i-1}-f_{N-1}}{s_i} + (f_{N-1} - \delta)(\frac{1}{s_j} - \frac{1}{s_i}) > 0$.

[7] This is a direct consequence of the fact that $q_i(\delta)$ is given by $\frac{\delta - f_{i-1}}{s_i} = \frac{\partial W_i(q)/\partial q}{\partial \psi_{i-1}(q)/\partial q}$ if $q_i(\delta) > 0$.

[8] Though we allow for *net* transfers to be negative in the optimal allocations all transfers will be positive.

and the payoff of the principal of type θ_i by

$$U_i = t_i - C(q_i, \theta_i). \tag{13}$$

The agent's participation constraint is

$$\sum_{i=1}^{N} \hat{s}_i \Pi_i \geq 0, \tag{FPC}$$

where $(\hat{s}_i)_{1 \leq i \leq N}$ are the posterior beliefs of the agent after the principal has offered an option contract κ. Offered a mechanism κ, the agent will accept it if and only if the participation constraint is satisfied by the offered contract. We will refer to a mechanism that satisfies the participation constraint as an *individually-rational* mechanism. An ex post individually-rational mechanism is a contract that satisfies:

$$S(q_i, \theta_i) - t_i \geq 0 \text{ for } 1 \leq i \leq N. \tag{14}$$

Moreover, a mechanism κ satisfies truth-telling if the following incentive-compatibility constraints hold:

$$t_i - C(q_i, \theta_i) \geq t_j - C(q_j, \theta_i) \text{ for all } 1 \leq i, j \leq N. \tag{IC$_{i,j}$}$$

We will refer to a mechanism that satisfies truth-telling as an *incentive-compatible* mechanism.

Notice that, after eliminating t_i from the agent's payoff upon facing a principal of type θ_i, we have that $\Pi_i = W_i(q_i) - U_i$. After using the definition of U_i, the incentive-compatibility and participation constraints can be re-written, respectively, as[9]

$$U_i \geq U_j - \sum_{v=i}^{j-1} \psi_v(q_j) \text{ for } 1 \leq i < N,\ i < j \leq N, \tag{15}$$

$$U_i \geq U_j + \sum_{v=j}^{i-1} \psi_v(q_j) \text{ for } 1 < i \leq N, 1 \leq j < i, \tag{16}$$

$$\sum_{i=1}^{N} \hat{s}_i (W_i(q_i) - U_i) \geq 0. \tag{17}$$

We have:

Lemma 1 *For a mechanism to be incentive-compatible, it is necessary and sufficient that*

$$U_i \geq U_{i+1} - \psi_i(q_{i+1}) \textit{ for } 1 \leq i < N, \tag{18}$$

$$U_{i+1} \geq U_i + \psi_i(q_i) \textit{ for } 1 \leq i < N, \tag{19}$$

$$q_i \leq q_{i+1} \textit{ for } 1 \leq i < N. \tag{20}$$

[9] To see how these incentive-compatibility constraints arise note that $t_i - C(q_i, \theta_i) \geq t_j - C(q_j, \theta_i)$ is re-written as $U_i \geq U_j + C(q_j, \theta_j) - C(q_j, \theta_i)$. Notce then that for $j > i$ we have $C(q_j, \theta_j) - C(q_j, \theta_i) = C(q_j, \theta_j) - C(q_j, \theta_{j-1}) + C(q_j, \theta_{j-1}) - \cdots + C(q_j, \theta_{i+1}) - C(q_j, \theta_i) = -\psi_{j-1}(q_j) \cdots - \psi_i(q_j)$. Similarly if $j < i$.

Proof. It follows by usual arguments. Specifically, the necessary part is a direct consequence of the definition of incentive-compatible mechanisms, the properties of $\psi_i(q)$ and that the local incentive-compatibility constraints for types i and $i+1$ imply

$$\psi_i(q_{i+1}) \geq U_{i+1} - U_i \geq \psi_i(q_i).$$

The sufficiency part follows from: (a) after forward iteration of $U_i \geq U_{i+1} - \psi_i(q_{i+1})$ we have $U_i \geq U_j - \sum_{v=i}^{j-1} \psi_v(q_{v+1})$, with $j > i$, which, given $\psi'_v > 0$ and, from $q_i \leq q_{i+1}$ for any $1 \leq i < N$, that $q_{v+1} \leq q_j$ for $i \leq v < j-1$, implies $U_i \geq U_j - \sum_{v=i}^{j-1} \psi_v(q_j)$, (b) after backward iteration of $U_i + \psi_i(q_i) \leq U_{i+1}$ we have $U_j + \sum_{v=j}^{i} \psi_v(q_v) \leq U_{i+1}$, with $j < i$, which, given $\psi'_v > 0$ and, from $q_i \leq q_{i+1}$ for any $1 \leq i < N$, that $q_v \geq q_j$ for $j < v \leq i$, implies $U_j + \sum_{v=j}^{i} \psi_v(q_j) \leq U_{i+1}$. ■

From now on we will, unless stated otherwise, use the substitution $U_i = t_i - C(q_i, \theta_i)$ and describe a contract as an N-tuple $((U_i, q_i))_{1 \leq i \leq N} \equiv (m_i)_{1 \leq i \leq N} \equiv m$.

4 Definitions and Concepts

4.1 Strong, undominated, interim efficient

In Myerson's framework all constraints are rewritten as incentive constraints. Moreover, in his model there can be several agents, who can have their own private information. See page 1772 of Myerson (1983) for how to interpret the given model in his framework. Adjusting his definitions to the way we set up the model here, we have:

Definition 1 *A mechanism is* feasible *if it is incentive-compatible and individually-rational relative to prior beliefs (i.e. if $\hat{s}_i = s_i$ for all $1 \leq i \leq N$).*

Definition 2 *A mechanism is* interim efficient *if it is feasible and there does not exist another feasible mechanism which gives every type* of every player *at least the same utility and some types a strictly higher utility.*

Definition 3 *A mechanism is* undominated *if it is feasible and there does not exist another feasible mechanism which gives every type* of the principal *at least the same utility and some of his types a strictly higher utility.*

Remark 6 An undominated mechanism is called an interim efficient allocation in Maskin and Tirole (1992). For the purpose of this paper it will be more convenient to work with Myerson's terminology throughout.

Definition 4 *A mechanism is* safe *if it is incentive-compatible and ex post individually rational.*

Remark 7 A safe mechanism is feasible because ex-post implies (interim) individual rationality.

Remark 8 Myerson's definition is more general because it also applies to models with several agents who can have their own private information. He requires that the mechanism would have to remain feasible even if the type of the principal would become common knowledge.

Definition 5 *A mechanism is a* strong solution *if it is safe and undominated.*

Remark 9 The assured allocation, we define shortly, is an undominated mechanism which, under certain conditions, is not safe.

4.2 Perfect Bayesian Equilibria and the RSW allocation

We will restrict attention to perfect Bayesian equilibria (for a formal definition, see page 8 of Maskin and Tirole (1992)) of the three-stage game Γ_3.

Definition 6 *A mechanism* $((U_i, q_i))_{1 \le i \le N}$ *is an RSW (Rothschild-Stiglitz-Wilson) allocation if it maximizes for each type of the principal his utility subject to the incentive constraints of all types and subject to the ex post participation constraints of the agent.*

Given our assumptions (recall our Remark 2) we can apply Proposition 2 in Maskin and Tirole (1992) to yield:

Proposition 2 *The RSW allocation* $\left(\left(U_i^{RSW}, q_i^{RSW}\right)\right)_{1 \le i \le N}$ *is the solution to Programs* (RSW_n) *defined inductively for* $n = 1, \cdots, N$ *as follows:*

$$\max_{(U_n, q_n)} U_n$$

subject to

$$\begin{aligned} U_{n-1}^{RSW} &\ge U_n - \psi_{n-1}(q_n) \ \ \textit{provided } n > 1 \\ W_n(q_n) &\ge U_n \end{aligned}$$

The following proposition is an application of Theorem 1 in Maskin and Tirole (1992) to our setting (recall our Remark 2).

Proposition 3 *A feasible mechanism* $((U_i, q_i))_{1 \le i \le N}$ *is a Perfect Bayesian equilibrium allocation of the three-stage game* Γ_3 *if and only if it gives each type of the principal at least his utility in the RSW allocation. In particular, if the RSW allocation is undominated, then it is the only Perfect Bayesian equilibrium allocation.*

From the above Proposition, it follows directly that

Corollary 2 *The following statements are equivalent:*

1. *The RSW allocation is undominated.*

2. *The RSW allocation is a strong solution.*

3. *The 3-stage game has a unique perfect Bayesian equilibrium allocation, namely the RSW allocation.*

However, if the RSW allocation is dominated, then, following from the above Proposition, an equilibrium selection problem arises. In this paper, we characterize one PBE allocation that dominates the RSW allocation. This allocation is undominated, and, in particular, a neutral allocation as introduced by Myerson (1983).

4.3 The Assured Allocation

This work introduces the *assured* allocation, which is defined in an inductive manner.

We consider a sequence of inductively defined optimization problems. Let $1 \leq n \leq N$. Suppose the numbers $V_1, \cdots, V_{n-1}$ have been defined and let $y \geq 0$. We define a mechanism $\left(U_i^n(y), q_i^n(y)\right)_{1 \leq i \leq n}$ as the solution to, and the scalar $V_n(y)$ as the maximal value of, the following constrained optimization problem, referred to as $X_n(y)$:

$$V_n(y) \equiv \max_{(U_i, q_i)_{1 \leq i \leq n}} U_n$$

subject to

$$U_i \geq U_{i+1} - \psi_i(q_{i+1}) \qquad \text{for } 1 \leq i \leq n-1 \tag{IC$_i$}$$

$$q_{i+1} \geq q_i \qquad \text{for } 1 \leq i \leq n-1 \tag{MC$_i$}$$

$$U_i \geq V_i \qquad \text{for } 1 \leq i \leq n-1 \tag{WC$_i$}$$

$$\sum_{i=1}^{n} s_i\left(W_i(q_i) - U_i\right) + s_n y \geq 0 \tag{PC}$$

Of particular importance is the case $y = 0$. We call $\left(U_i^n(0), q_i^n(0)\right)_{1 \leq i \leq n}$ the *n-assured allocation*. For $n = N$ we speak for short of the *assured allocation*. We call $V_n \equiv V_n(0)$ the *assured claim* of type n and use it in the definition of problems $X_j(y)$, $n+1 \leq j \leq N$. The assured utility levels V_i are, as we will see, closely related to the warranted claims in Myerson (1983). We hence call the constraints $U_i \geq V_i$ the *warranted claim* constraints.

One way to interpret the assured allocation is to think of the various types of the principal as players and that the total group of N players is formed gradually by starting with type/player $i = 1$ and adding each time a type/player $i+1$. Type $i+1$ is more productive than every lower type. Therefore, conditional on, first, all lower types receiving at least as much as they would if they were the most productive types in the group and, second, incentive-compatibility being maintained, type $i+1$ extracts the maximum possible surplus from what he has generated. The difference, in our setting, of the assured allocation from the RSW allocation is that in the latter it is the agent's ex post, instead of the interim, participation constraint(s) that must be satisfied.

For further reference we introduce two related optimization problems defined for each $1 \leq i \leq n$. First, let $X_n^*(y)$ be the more constrained optimization problem which is derived from $X_n(y)$ after adding the downward incentive constraints

$$U_{i+1} \geq U_i + \psi_i(q_i) \qquad \text{for } 1 \leq i \leq n-1. \tag{DC$_i$}$$

The significance of this problem comes from Lemma 1 which ensures that a solution to $X_n^*(y)$ is a feasible mechanism for the informed principal problem with n types and beliefs $s_i^n = \frac{s_i}{s_1+\cdots+s_n}$ for $1 \leq i \leq n$ and $s_i^n = 0$ for $n < i$.[10] Secondly, let $\tilde{X}_n(y)$ be the less constrained problem which is derived from $X_n(y)$ after dropping all the monotonicity constraints MC_i. In both the latter problems the numbers V_i used in the warranted claim constraints are the maximal values $U_i^i(0)$ from the problem $X_i(0)$.

The Lagrangian for the problem $X_n(y)$, with the appropriate multipliers σ_i, μ_i, ρ_i and γ, is (with $\sigma_n + \mu_n > 0$)

$$\begin{aligned}\mathcal{L} = &(\sigma_n + \mu_n) U_n + \sum_{i=1}^{n-1} \sigma_i (U_i - V_i) \\ &+ \sum_{i=1}^{n-1} \mu_i (U_i - U_{i+1} + \psi_i(q_{i+1})) \\ &+ \sum_{i=1}^{n-1} \rho_i (q_{i+1} - q_i) \\ &+ \gamma \left(\sum_{i=1}^{n} s_i (W_i(q_i) - U_i) + s_n y \right).\end{aligned} \tag{21}$$

Notice that $(\sigma_n + \mu_n)$ is a weight on the objective function in the Lagrangian and not, strictly speaking, a Lagrange multiplier. Normally one would set $(\sigma_n + \mu_n) = 1$. However, if one multiplies in a solution to the first-order conditions for the Lagrangian problem all multipliers, including $(\sigma_n + \mu_n)$, by the same constant, the optimum is not changed. Rather than fixing $(\sigma_n + \mu_n)$ we can hence fix any positive Lagrange multiplier at a suitable value. For us it will be convenient to set $\gamma = 1$, once we have shown that γ must always be positive. This will ease the comparison between the solutions to the problems $X_n(y)$ and $X_k(0)$ for $k < n$ below. Note thus that σ_k and μ_k are determined as Lagrange multipliers for the problem $X_n(y)$ while only their sum is determined in the problem $X_k(0)$.

Using $\mu_0 \equiv \rho_0 \equiv \rho_n \equiv 0$ we can rewrite the Lagrangian as:

$$\begin{aligned}\mathcal{L} = &\sum_{i=1}^{n} \left[\sigma_i + (\mu_i - \mu_{i-1}) - \gamma s_i\right] U_i \\ &+ \sum_{i=1}^{n} \left[\mu_{i-1}\psi_{i-1}(q_i) + \gamma s_i W_i(q_i) + (\rho_{i-1} - \rho_i) q_i\right] \\ &- \sum_{i=1}^{n-1} \sigma_i V_i + \gamma s_n y.\end{aligned} \tag{22}$$

[10] Note that (PC) can be rewritten as

$$\sum_{i=1}^{n} s_i^n (W_i(q_i) - U_i) + s_n^n y \geq 0.$$

The first-order conditions for $i = 1, \cdots, n$ are:

$$\sigma_i + \left(\mu_i - \mu_{i-1}\right) - \gamma s_i = 0$$
$$\mu_{i-1}\psi'_{i-1}(q_i) + \gamma s_i W'_i(q_i) + \rho_{i-1} - \rho_i = 0.$$

Addition of the former over i yields $0 < \sum_{i=1}^{n} \sigma_i + \mu_n = \gamma \sum_{i=1}^{n} s_i$. Hence $\gamma > 0$ and so the participation constraint of the agent must be binding

$$\sum_{i=1}^{n} s_i \left(W_i(q_i) - U_i\right) + s_n y = 0 \tag{23}$$

From now on we set $\gamma = 1$.

Hence the first-order conditions for $i = 1, \cdots, n$ become:

$$\sigma_i + \left(\mu_i - \mu_{i-1}\right) - s_i = 0$$
$$\mu_{i-1}\psi'_{i-1}(q_i) + s_i W'_i(q_i) + \rho_{i-1} - \rho_i = 0. \tag{24}$$

Let $g_i = \sum_{j=1}^{i} \sigma_j$, with $g_0 \equiv 0$. The first-order conditions with respect to U_i imply, using $f_i = \sum_{j=1}^{i} s_j$, that

$$\mu_i = f_i - g_i. \tag{25}$$

We will denote a solution to optimization problem $X_n(y)$ by $\left(U_i^n(y), q_i^n(y)\right)_{1 \leq i \leq n}$ and corresponding multipliers by $\sigma_i^n(y)$, $\mu_i^n(y)$, and $\rho_i^n(y)$. We also write $g_i^n(y) = \sum_{j=1}^{i} \sigma_j^n(y)$. Notice that $g_i^n(y)$ is non-decreasing in i.

When there is no danger of confusion we will often drop the (y) or even the superscript n in the solution.

4.4 The neutral optimum

We use here Theorem 7 in Myerson (1983) to define the neutral optima because our framework does not allow us to introduce his axiomatic characterization. The notion of a neutral optimum extends that of a strong solution.

Fix *weights* $\vec{\tau} = (\tau_1, \cdots, \tau_N)$ with $\tau_i > 0$ and $\sum_{i=1}^{N} \tau_i = 1$ for the different types of the principal. We can then obtain an undominated mechanism by maximizing the weighted utility

$$\sum_{i=1}^{N} \tau_i U_i$$

subject to the participation constraint (17) for the full model in Section 3 and subject to all incentive constraints $\mathrm{IC}_{i,j}$ which can be rewritten as

$$U_i \geq U_j - \psi_{i,j}(q_j)$$

for all i, j, where $\psi_{i,j}(q) = C(q, \theta_j) - C(q, \theta_i)$.[11] Let γ be the Lagrange multiplier on the participation constraint and let $\mu_{i,j}$ be the Lagrange multiplier for the incentive constraint $\mathrm{IC}_{i,j}$.

[11] We only ignore randomization here for simplicity of the exposition. However, it should be allowed here to obtain a convex optimization problem. The critical reader may hence prefer to assume that all $\psi_{i,j}$ are linear in this section.

Myerson shows that on the hyperplane

$$\left\{ (u_i)_{1\leq i\leq N} \left| \sum_{i=1}^{N} \tau_i u_i = \sum_{i=1}^{N} \tau_i U_i \right. \right\}$$

one can uniquely identity a point $(\omega_i)_{1\leq i\leq N}$, called the warranted claim allocation, by the system of equations

$$\left(\tau_i + \sum_j \mu_{i,j} \right) \omega_i - \sum_j \mu_{ji} \omega_j = \gamma s_i V S_i$$

where

$$VS_i = \frac{1}{\gamma s_i} \left(\gamma s_i W_i (q_i) - \sum_j \mu_{ji} \psi_{ji} (q_i) \right)$$

are the virtual surpluses of each type, where $U_i, q_i, \mu_{ij}, 1 \leq i, j \leq n$, and γ are evaluated at the optimal solution of the problem that determines the undominated allocations for given weights τ_i, $1 \leq i \leq n$.[12] In case of a fixed point, $(\omega_i)_{1\leq i\leq n} = (U_i)_{1\leq i\leq n}$, the mechanism $(U_i, q_i)_{1\leq i\leq n}$ is a strong solution of the given model and hence, according to Myerson (1983), a neutral optimum. However, to have hope for existence of a fixed point one must allow for some of the utility weights τ_i to be zero. If some of the τ_i are zero the above definition of warranted claims no longer guarantees a unique solution for the warranted claims. Myerson therefore proceeds by considering a sequence of strictly positive utility weights $\vec{\tau}^{\nu}$ converging to $\vec{\tau}$. For each element in the sequence one can calculate the undominated mechanism $\left((U_i^{\nu}, q_i^{\nu})_{1\leq i\leq N} \right)_{\nu\geq 1}$ which maximizes the weighted utility with respect to these weights and the corresponding warranted claims $\left((\omega_i^{\nu})_{1\leq i\leq N} \right)_{\nu\geq 1}$. If there is an undominated mechanism $(U_i, q_i)_{1\leq i\leq N}$ such that

$$\lim_{\nu\to\infty} \sup \omega_i^{\nu} \leq U_i$$

then $(U_i, q_i)_{1\leq i\leq N}$ is called a *neutral mechanism*, regardless of whether it is a strong solution or not.

Myerson shows that a "fixed point" defined in this way always exists in his model with finite action spaces. He does this by combining the method to prove existence of a perfect equilibrium with duality theory and Kakutani's fixed point theorem. Whether neutral optima exist for our framework is unknown a priori. We avoid the problem by constructing explicit candidates and then show that they have the above properties.

5 Properties of assured allocations

In this Section we characterize some properties of the assured allocations. These properties will be crucial in proving our main results, Theorems 1 and 2, at the end of this Section.

[12] The warranted claim can be shown to yield a strong solution in an extension of the given model, which yields the vector of virtual surpluses associated with an undominated allocation.

5.1 Basic properties

Lemma 2 *The participation constraint of the agent is binding in a solution to the problem* $X_n(y)$*. Moreover, for each type* $1 \leq i < n$ *either the incentive constraint* IC_i *and/or the warranted claim constraint* WC_i *is binding.*

Proof. Suppose the participation constraint was not binding in an optimum. By increasing all utilities in this contract slightly by the same amount for all types $1 \leq i \leq n$, we would obtain another admissible contract for the problem $X_n(y)$ which would give strictly higher utility to type n than in the proposed optimum.

Suppose for some type $1 \leq i \leq n-1$ both IC_i and WC_i are slack. Then one can obtain a new admissible contract by reducing type i's utility slightly and increasing the utility of all types $i+1, \cdots, n$ such that the participation constraint is not violated and they all receive the same amount more. This contradicts the optimality of the presumed solution. ■

Lemma 3 *Suppose type* n *produces* q_n^n *in a solution to* $X_n(0)$*. Then*

$$V_{n+1} \geq V_n + \psi_n(q_n^n)$$

and

$$W_n(q_n^n) \geq V_n$$

Proof. Let $((U_i^n, q_i^n))_{1\leq i\leq n}$ be a solution to the program $X_n(0)$. We have $U_n^n = V_n$. Consider now the contract $\left(\left(\hat{U}_i^{n+1}, \hat{q}_i^{n+1}\right)\right)_{1\leq i\leq n+1}$ for the types $1 \leq i \leq n+1$ defined by $\left(\hat{U}_i^{n+1}, \hat{q}_i^{n+1}\right) = (U_i^n, q_i^n)$ for $1 \leq i \leq n$ and $\left(\hat{U}_{n+1}^{n+1}, \hat{q}_{n+1}^{n+1}\right) = (V_n + \psi_n(q_n^n), q_n^n)$. We show now that this contract is admissible for the problem $X_{n+1}(0)$ and hence $V_{n+1} \geq V_n + \psi_n(q_n^n)$ follows from the definition of V_{n+1}. To show admissibility notice first that the contract satisfies by construction all incentive constraints IC_i, the monotonicity constraints MC_i and the warranty constraints WC_i.

The participation constraint for the agent is also satisfied. To see the latter, we notice first that $W_n(q_n^n) \geq V_n$ must hold. For $n = 1$ this is clear. For $n > 1$ we would otherwise obtain from the participation constraint for the problem $X_n(0)$ that

$$\sum_{i=1}^{n-1} s_i\left(W_i(q_i^n) - U_i^n\right) > 0.$$

So $((U_i^n, q_i^n))_{1\leq i\leq n-1}$ would be admissible for the problem $X_{n-1}(0)$, give utility at least V_{n-1} to type $n-1$, due to $U_{n-1}^n \geq V_{n-1}$ by WC_{n-1}, and have a slack participation constraint for the agent. However, this means that we have a solution for problem $X_{n-1}(0)$ in which the participation constraint is slack. This contradicts Lemma 2.

We have furthermore that for any q

$$\begin{aligned}
W_{n+1}(q) &= S(q, \theta_{n+1}) - C(q, \theta_{n+1}) \geq S(q, \theta_n) - C(q, \theta_{n+1}) \\
&= (S(q, \theta_n) - C(q, \theta_n)) + (C(q, \theta_n) - C(q, \theta_{n+1})) \\
&= W_n(q) + \psi_n(q)
\end{aligned}$$

and therefore

$$\sum_{i=1}^{n+1} s_i \left(W_i \left(\hat{q}_i^{n+1} \right) - \hat{U}_i^{n+1} \right) =$$

$$\sum_{i=1}^{n} s_i \left(W_i \left(q_i^n \right) - U_i^n \right) + s_{n+1} \left(W_{n+1} \left(q_n^n \right) - \left(V_n + \psi_n \left(q_n^n \right) \right) \right) \geq$$

$$s_{n+1} \left(W_n \left(q_n^n \right) + \psi_n \left(q_n^n \right) - \left(V_n + \psi_n \left(q_n^n \right) \right) \right) \geq 0$$

■

The above Lemma implies, clearly, that assured claims are weakly increasing.

Lemma 4 *Suppose that in a solution* $(U_i^n, q_i^n)_{1 \leq i \leq n}$ *to* $X_n(y)$ *the* k*-warranted claim constraint holds with equality for* $1 \leq k < n$. *Then*

$$\sum_{i=1}^{k} s_i \left(W_i \left(q_i^n \right) - U_i^n \right) = 0. \tag{26}$$

and the restricted solution $(U_i^n, q_i^n)_{1 \leq i \leq k}$ *is a solution to* $X_k(0)$. *If in addition the warranted claim constraint is not binding at* $k+1$, *then we must have* $q_k^n < q_{k+1}^n$, *i.e. there cannot be bunching between types* k *and* $k+1$.

Proof. Suppose first that

$$\sum_{i=1}^{k} s_i \left(W_i \left(q_i^n \right) - U_i^n \right) > 0$$

Then the restriction of the solution $((U_i^n, q_i^n))_{1 \leq i \leq k}$ would satisfy all constraints of the problem $X_k(0)$ with the participation constraint being slack. Since type k receives V_k in this solution, $(U_i^n, q_i^n)_{1 \leq i \leq k}$ is an optimal solution of $X_k(0)$ in which the participation constraint is slack. This contradicts Lemma 2.

Suppose next that

$$\sum_{i=1}^{k} s_i \left(W_i \left(q_i^n \right) - U_i^n \right) < 0 \tag{27}$$

Take a solution $\left(\left(U_i^k, q_i^k \right) \right)_{1 \leq i \leq k}$ for the problem $X_k(0)$. Consider the new contract defined by

$$\left(\hat{U}_i^n, \hat{q}_i^n \right) = \begin{cases} \left(U_i^k, q_i^k \right) & \text{for} \quad i \leq k \\ \left(U_i^n, q_i^n \right) & \text{for} \quad i > k \end{cases}$$

We have $U_k^n = V_k = U_k^k = \hat{U}_k^n$. Since the participation constraint for $X_n(y)$ is binding in the optimal contract $(U_i^n, q_i^n)_{1 \leq i \leq n}$ and since the participation constraint for $X_k(0)$ is binding in the optimal contract $\left(U_i^k, q_i^k \right)_{1 \leq i \leq k}$, inequality (27) implies

$$\sum_{i=1}^{n} s_i \left(W_i \left(\hat{q}_i^n \right) - \hat{U}_i^n \right) + s_n y > 0.$$

By construction all warranted claim constraints and all incentive constraints IC_i are satisfied as well as the monotonicity constraints MC_i for $1 \le i \le n$ possibly with the exception of $\hat{q}_{k+1}^n \ge \hat{q}_k^n$, are satisfied. However, also the latter inequality holds.

To see this, note that, using IC_k, and WC_{k+1} if $\kappa+1 < n$ or $\hat{U}_n^n = U_n^n \ge V_n$ if $\kappa+1 = n$, and Lemma 3,

$$V_k + \psi_k\left(\hat{q}_{k+1}^n\right) \ge \hat{U}_{k+1}^n \ge V_{k+1} \ge V_k + \psi_k\left(q_k^k\right)$$

which implies by the monotonicity of ψ_k that $\hat{q}_{k+1}^n \ge q_k^k = \hat{q}_k^n$. Thus $\left(\left(\hat{U}_i, \hat{q}_i\right)\right)_{1 \le i \le n}$ is admissible for the problem $X_n(y)$ with a slack participation constraint. Since $\hat{U}_n^n = U_n^n$ it is also an optimal solution for this problem, again in contradiction to Lemma 2.

Thus the equality (26) must hold. In consequence, $(U_i^n, q_i^n)_{1 \le i \le k}$ is admissible for the problem $X_k(0)$ and since $U_k^n = V_k$ it is an optimal solution for this problem.

Finally, suppose that the warranted claim constraint is not binding at $k+1$, i.e. $U_{k+1}^n > V_{k+1}$, and $q_k^n = q_{k+1}^n$. Since q_k^n is part of a solution to $X_k(0)$ Lemma 3 implies

$$U_{k+1}^n > V_{k+1} \ge V_k + \psi_k\left(q_k^n\right) = U_k^n + \psi_k\left(q_{k+1}^n\right)$$

which contradicts the incentive constraint IC_k. ■

Lemma 5 *The solution $(U_i^n, q_i^n)_{1 \le i \le n}$ to the optimization problem $X_n(y)$ is unique. Moreover, the corresponding mulitpliers σ_i^n, μ_i^n and ρ_i^n as defined above (see formula (21)) are unique up to the choice of the terms in $\sigma_n^n + \mu_n^n$.*

Proof. The proof is by induction. The claim is clearly true for $n = 1$. Suppose it holds for $n-1 \ge 1$. In the problem $X_n(y)$ let $k < n$ be the largest index for which the warranted claim constraint is binding. (Set $k = 0$ if no warranted claim constraint is binding. The remaining statements in this paragraph are then vacuously true.) Let $\left(U_i^k, q_i^k\right)_{1 \le i \le k}$ be the solution to the problem $X_k(0)$, which is by assumption unique. Also the Lagrange multipliers σ_i^k, μ_i^k, and ρ_i^k of the latter problem are unique, up to the choice of the terms in $\sigma_k^k + \mu_k^k$. Let $(U_i^n, q_i^n)_{1 \le i \le n}$ be the solution to the problem $X_n(y)$, and let σ_i^n, μ_i^n, and ρ_i^n be corresponding Lagrange mulitpliers. By Lemma 4 $(U_i^n, q_i^n)_{1 \le i \le k}$ is a solution to $X_k(0)$ and by the assumed uniqueness we have $U_i^n = U_i^k$ and $q_i^n = q_i^k$ for $1 \le i \le k$. From Lemma 4 we have $\rho_k^n = 0$ $(= \rho_k^k)$ by the definition of k. The first-order conditions for the problem $X_k(0)$ are hence a subset of the set of first order conditions for the problem $X_n(y)$. Thus $\sigma_i^k = \sigma_i^n$, $\mu_i^k = \mu_i^n$ and $\rho_i^k = \rho_i^n$ for $i < k$ and $\sigma_k^k + \mu_k^k = \sigma_k^n + \mu_k^n$ by the induction assumption.

We now show the uniqueness of μ_k^n, σ_k^n, $(U_i^n, q_i^n, \sigma_i^n, \mu_i^n, \rho_i^n)_{k+1 \le i \le n-1}$, U_n^n, q_n^n and $\sigma_n^n + \mu_n^n$ for a given 'bunching pattern'. By the latter we mean that the set B of indices i for which $q_i^n = q_{i+1}^n$ holds is fixed. The binding incentive constraints IC_i for $k < i < n$ (recall Lemma 2) give

$$U_i^n = U_n^n - \sum_{j=i}^{n-1} \psi_j\left(q_{j+1}^n\right)$$

where U_n^n is by definition the same in all solutions of $X_n(y)$. Since $\sigma_i^n = 0$ for $k < i < n$ by assumption, μ_i^n for $k \le i < n$ and $\sigma_n^n + \mu_n^n$ are uniquely determined by formula (25) and depend only on g_k^n or, equivalently, σ_k^n. For any $i \notin B$ we have $\rho_i^n = 0$. Partition the set of indices $\{k+1, \cdots, n\}$ into maximally connected sets J such that the monotonicity constraint is binding for any two adjacent indices i, $i+1$ in J. Thus a set $J = \{i_1 \leq i \leq i_2\}$ is in the partition if $i_1 - 1, i_2 \notin B$ and for all $i \in J$ it holds that $(i \in B \Leftrightarrow i+1 \in J)$. (Notice that $J = \{i\}$ is a set in the partition if neither $q_{i-1}^n = q_i^n$ nor $q_i^n = q_{i+1}^n$.) Summing the first order conditions (24) over all $i \in J$ gives the equation

$$q_i^n \equiv \arg\max_{q \geq 0} \{\sum_{j \in J} \left(\mu_{j-1}^n \psi_{j-1}(q) + s_j W_j(q)\right)\}.$$

in which no non-zero ρ_i^n occur and from which q_i^n for any $i \in J$ can be inferred uniquely from $\left\{\mu_{i-1}^n\right\}_{i \in J}$ by the implicit function theorem. Starting with the lowest index in $J = \{i_1 \leq i \leq i_2\}$ one can then infer the $\rho_{i_1}^n, \rho_{i_1+1}^n, \cdots, \rho_{i_2-1}^n$ inductively from the first-order conditions (24).

We now claim that the left-hand term in the participation constraint (23) is, with the variables determined as just described, strictly increasing in g_k^n. To show this we prove that its derivative with respect to g_k^n is strictly positive almost everywhere. The term on the left-hand side in (23), which depends on g_k^n, is, using again Lemma 4,

$$\begin{aligned}
\sum_{i=k+1}^{n} s_i \left(W_i(q_i^n) - U_i\right) &= \sum_{i=k+1}^{n} s_i \left(W_i(q_i^n) - U_n^n + \sum_{j=i}^{n-1} \psi_j\left(q_{j+1}^n\right)\right) \\
&= \sum_{i=k+1}^{n} \left(s_i \left(W_i(q_i^n) - U_n^n\right) + \left(f_{i-1} - f_k\right) \psi_{i-1}(q_i^n)\right)
\end{aligned}$$

Differentiating and using the equations (24) and (25) yields (recall $g_i^n = g_k^n$ for $k \leq i < n$.)

$$\begin{aligned}
& \sum_{i=k+1}^{n} \left(s_i W_i'(q_i^n) + \left(f_{i-1} - f_k\right) \psi_{i-1}'(q_i^n)\right) \frac{dq_i^n}{dg_k^n} \\
= & \sum_{i=k+1}^{n} \left(s_i W_i'(q_i^n) + \left(\mu_{i-1} + g_k^n - f_k\right) \psi_{i-1}'(q_i^n)\right) \frac{dq_i^n}{dg_k^n} \\
= & \sum_{i=k+1}^{n} \left(\left(g_k^n - f_k\right) \psi_{i-1}'(q_i^n) + \rho_i^n - \rho_{i-1}^n\right) \frac{dq_i^n}{dg_k^n} \\
= & -\mu_k^n \sum_{i=k+1}^{n} \psi_{i-1}'(q_i^n) \frac{dq_i^n}{dg_k^n} - \sum_{i=k}^{n-1} \rho_i^n \left(\frac{dq_{i+1}^n}{dg_k^n} - \frac{dq_i^n}{dg_k^n}\right) \\
= & -\mu_k^n \sum_{i=k+1}^{n} \psi_{i-1}'(q_i^n) \frac{dq_i^n}{dg_k^n} > 0
\end{aligned}$$

for $\mu_k^n > 0$ because $\frac{dq_i^n}{dg_k^n} < 0$ by the monotonicity properties of the ψ_i and W_i. Thus the derivative is strictly positive except when $g_k^n = f_k$, which proves our claim. Thus there can be at most one value of g_k^n for which the participation constraint is satisfied. Thus there can only be one solution (including the Lagrange multipliers) for each bunching pattern.

Suppose next that we have two different solutions to the optimization problem $X_n(y)$ (excluding the Lagrange multipliers). Any convex combination of the two solutions is also a solution because the optimization problem is convex. Since there are infinitely many convex combinations and only finitely many bunching patterns, we can find two different solutions with the same bunching pattern, which contradicts the above finding. This concludes the proof. ■

In particular, we have shown the following.

Remark 10 In the same fashion it can be shown that the optimization problem $\tilde{X}_n(y)$ has a unique solution.

Corollary 3 *The assured allocation is unique.*

Lemma 6 *The solution* $((U_i^n, q_i^n))_{1\leq i\leq n}$ *for problem* $X_n(0)$ *is also a solution to the complete problem* $X_n^*(0)$*, which must hence be unique.*

Proof. For type $1 \leq i < n$ we distinguish two cases: i) The incentive constraint IC_i is binding

$$U_i^n = U_{i+1}^n - \psi_i\left(q_{i+1}^n\right)$$

Then by the monotonicity constraint MC_i we have

$$U_i^n = U_{i+1}^n - \psi_i\left(q_{i+1}^n\right) \leq U_{i+1}^n - \psi_i\left(q_i^n\right)$$

and hence (DC_i) is satisfied.

ii) If IC_i is not binding, then by Lemma 2 WC_i is binding. Then, by Lemma 3 and WC_{i+1} and the definition of V_n we have

$$U_{i+1}^n \geq V_{i+1} \geq V_i + \psi_i\left(q_i^i\right) = U_i^n + \psi_i\left(q_i^i\right)$$

where we can set $q_i^i = q_i^n$ by Lemma 4 and Lemma 5. Thus the incentive constraint (DC_i) holds. ■

Lemma 7 *The n-assured allocation is undominated.*

Proof. Suppose the n-assured allocation is dominated by an undominated mechanism. Then the undominated allocation is also a solution to $X_n(0)$ and hence identical to the n-assured allocation. ■

We are now ready to prove the following important result.

Proposition 4 *The assured allocation weakly dominates the RSW allocation.*

Proof. For $n = 1, \ldots, N$ consider the following allocations. Let $\left(\left(U_i^{RSW}, q_i^{RSW}\right)\right)_{1\leq i\leq n}$ be the RSW allocation for the restricted typeset $\{1, \ldots n\}$. Let $((U_i^n, q_i^n))_{1\leq i\leq n}$ be a solution to the problem $X_n(0)$. Let $\left(\left(\hat{U}_i^n, \hat{q}_i^n\right)\right)_{1\leq i\leq n}$ be the contract which satisfies $\left(\hat{U}_i^n, \hat{q}_i^n\right) = \left(U_i^{n-1}, q_i^{n-1}\right)$ for any $1 \leq i < n$ and where $\left(\hat{U}_n^n, \hat{q}_n^n\right)$ solves the following optimization problem Z_n:

$$\max_{(U_n, q_n)} U_n$$

subject to

$$\begin{aligned} U_{n-1}^{n-1} &\geq U_n - \psi_{n-1}(q_n) \text{ for } n > 1 \\ W_n(q_n) &\geq U_n \end{aligned}$$

We prove by induction over n that a) $\left(U_n^{RSW}, q_n^{RSW}\right)$ is admissible for the problem Z_n and hence satisfies $U_n^{RSW} \leq \hat{U}_n^n$, and b) $\left(\left(\hat{U}_i^n, \hat{q}_i^n\right)\right)_{1\leq i\leq n}$ is admissible for the problem $X_n(0)$ and hence satisfies that $\hat{U}_n^n \leq U_n^n = V_n$. Thus, $U_n^{RSW} \leq V_n$, which proves our proposition.

For $n = 1$ all three solutions coincide and so our claims hold. Suppose that they hold for $n - 1 \geq 1$. To prove claim a) for type n, notice that $U_{n-1}^{RSW} \leq \hat{U}_{n-1}^{n-1} \leq U_{n-1}^{n-1}$ by our induction assumption. Hence $\left(U_n^{RSW}, q_n^{RSW}\right)$ is feasible for problem Z_n, due to $U_{n-1}^{RSW} \geq U_n^{RSW} - \psi_{n-1}(q_n^{RSW})$. Since $\hat{U}_n^n$ is the optimal value for problem Z_n we have therefore $U_n^{RSW} \leq \hat{U}_n^n$.

We show next that claim b) holds for type n. We notice that all warranty constraints and all incentive constraints for Problem $X_n(0)$ hold (by construction of $\left(\hat{U}_n^n, \hat{q}_n^n\right)$ and because $\left(\left(\hat{U}_i^n, \hat{q}_i^n\right)\right)_{1\leq i\leq n-1}$ is a solution to $X_{n-1}(0)$). The ex-ante participation constraint from problem $X_{n-1}(0)$ for $\left(\left(\hat{U}_i^n, \hat{q}_i^n\right)\right)_{1\leq i\leq n-1}$ and the ex post participation constraint for $\left(\hat{U}_n^n, \hat{q}_n^n\right)$ from Problem Z_n imply that the participation constraint from the Problem $X_n(0)$ also holds for $\left(\left(\hat{U}_i^n, \hat{q}_i^n\right)\right)_{1\leq i\leq n}$. As in the proof of Proposition 2 (Appendix A, top of page 38) in Maskin and Tirole (1992), using that $W_{n-1}\left(q_{n-1}^{n-1}\right) \geq V_{n-1} = U_{n-1}^{n-1}$ by Lemma 3, one can show for the problem Z_n that $\hat{q}_{n-1}^n < \hat{q}_n^n$. This implies in particular that all monotonicity constraints of the problem $X_n(0)$ are satisfied by this contract. Hence $\left(\left(\hat{U}_i^n, \hat{q}_i^n\right)\right)_{1\leq i\leq n}$ is admissible for $X_n(0)$. Since $U_n^n = V_n$ is the optimal value for problem $X_n(0)$ we have therefore $\hat{U}_n^n \leq U_n^n = V_n$. This completes the proof. ■

Corollary 4 *The RSW allocation is undominated if and only if it is the assured allocation.*

The previous properties will be used to prove Theorem 1. The next two will be used to prove Theorem 2.

Lemma 8 *Suppose that* $\left(U_i^n(y), q_i^n(y)\right)_{1\leq i\leq n}$ *is a solution to the optimization problem* $\tilde{X}_n(y)$. *Then*

$$\sum_{i=1}^{k} s_i\left(W_i\left(q_i^n(y)\right) - U_i^n(y)\right) \leq 0$$

for any $k < n$ *and the restricted solution* $\left(U_i^n(y), q_i^n(y)\right)_{1\leq i\leq k}$ *is a solution to* $\tilde{X}_k(y')$, *where*

$$y' \equiv -\sum_{i=1}^{k} \frac{s_i}{s_k}\left(W_i\left(q_i^n(y)\right) - U_i^n(y)\right).$$

Proof. The inequality follows as in the first part of the proof of Lemma 4. $(U_i^n(y), q_i^n(y))_{1\le i\le k}$ is admissible for $\tilde{X}_k(y')$. If it is a solution, our claim is proved. Otherwise there is a solution to $\tilde{X}_k(y')$ where type k's utility $\check{U}_k^k$ is higher than $U_k^n(y)$. Therefore

$$\check{U}_k^k > U_k^n(y) \ge U_{k+1}^n(y) - \psi_k\left(q_{k+1}^n(y)\right)$$

By the maximum theorem we can find a solution $\left(\left(U_i^k(y'-\varepsilon), q_i^k((y'-\varepsilon))\right)\right)_{1\le i\le k}$ to problem $\tilde{X}_k(y'-\varepsilon)$ with ε small and $U_k^k(y'-\varepsilon) > U_k^n(y) \ge U_{k+1}^n(y) - \psi_k\left(q_{k+1}^n(y)\right)$. The new contract defined by

$$\left(\hat{U}_i^n, \hat{q}_i^n\right) = \begin{cases} \left(U_i^k(y'-\varepsilon), q_i^k(y'-\varepsilon)\right) & \text{for} \quad i \le k \\ (U_i^n(y), q_i^n(y)) & \text{for} \quad i > k \end{cases}$$

is admissible for problem $\tilde{X}_n(y)$ and has a slack participation constraint. The former is by construction, while the latter follows directly from the definitions of y' and $(U_i^n(y), q_i^n(y))_{1\le i\le N}$ and $\left(U_i^k(y'-\varepsilon), q_i^k(y'-\varepsilon)\right)_{1\le i\le N}$. The new contract is also optimal since it gives utility $U_n^n(y)$ to type n. This contradicts Lemma 2. ■

Proposition 5 *Under Assumption A3 there is no bunching in a solution for the problem $X_n(y)$, $y \ge 0$, i.e.*

$$q_1^n(y) < q_2^n(y) < \cdots < q_n^n(y)$$

Thus a solution to the problem $\tilde{X}_n(y)$ is also a solution to the problem $X_n(y)$.

Proof. The proof of the first part is by induction on n. For $n = 1$ there is nothing to show. Suppose the claim holds for all $1 \le i < n$. Let $(U_i^n(y), q_i^n(y))_{1\le i\le n}$ be a solution of $X_n(y)$

Suppose first that none of the warranted claim constraints WC_i, $1 \le i < n$, are binding. Then $g_i^n(y) = \sum_{j=1}^{i} \sigma_j^i(y) = 0$ for all $1 \le i \le n-1$. Therefore $q_i^n(y) = q_i(0)$ for all $1 \le i \le n$, with $q_i(\delta)$ as defined prior to Assumption A3. By Assumption A3 we have

$$q_1(0) < q_2(0) < \cdots < q_n(0).$$

Hence $(U_i^n(y), q_i^n(y))_{1\le i\le n}$ is the solution of the relaxed problem $\tilde{X}^n(y)$ where the monotonicity constraints of problem $X_n(y)$ are ignored.

Otherwise there exists a largest $1 \le k < n$ for which WC_i is binding. By Lemma 4 $(q_i^n(y))_{1\le i\le k}$ is part of a solution for problem $X_k(0)$ and hence for $\tilde{X}_k(0)$ because

$$q_1^n(y) < q_2^n(y) < \cdots < q_k^n(y)$$

by the induction assumption. By definition of k, none of the warranted claim constraints WC_i, $k+1 \le i < n$, are binding. Thus $g_i^n(y) = g_k^n(y)$ for all $k+1 \le i \le n-1$ and therefore $q_i^n(y) = q_i(g_k^n(y))$ for all $k+1 \le i \le n$. By Assumption A3

$$q_k(g_k^n(y)) < q_{k+1}(g_k^n(y)) < q_{k+2}(g_k^n(y)) < \cdots < q_n(g_k^n(y))$$

Thus, $(U_i^n(y), q_i^n(y))_{1\le i\le n}$ is a solution of the relaxed problem $\tilde{X}^n(y)$. This completes the proof. ■

5.2 The main results

Theorem 1 *The assured allocation is a perfect Bayesian equilibrium of the three stage game* Γ_3.

Proof. This follows from Proposition 3 and Proposition 4. ■

Theorem 2 *Under assumption A3 the assured allocation is a neutral optimum.*

The proof is given below.

6 Proof of Theorem 2

We assume that Assumption A3 holds. Hence, by Proposition 5, any solution to a problem $X_n(y)$ for any y features output $q_i^n(y)$ which is increasing in i, and so we can ignore the monotonicity constraints $q_i < q_{i+1}$ in the following. Thus, we can assume that all multipliers ρ_i are zero, and only the multipliers σ_i and μ_i appear in the Lagrangian for the derivation of the assured allocation.

The virtual surplus associated with this solution is

$$VS_i^n(y) = \frac{1}{s_i}\left[\mu_{i-1}^n(y)\,\psi_{i-1}^n(q_i^n(y)) + s_i W_i(q_i^n(y))\right]. \tag{28}$$

Corresponding to Equation (8.8) in Myerson (1983) we can define the warranted claims $\omega_i^n(y)$ for problem $X_n(y)$ inductively over i by

$$(\sigma_i^n(y) + \mu_i^n(y))\,\omega_i^n(y) - \mu_{i-1}^n(y)\,\omega_{i-1}^n(y) = s_i VS_i^n(y). \tag{29}$$

Notice that the $\omega_i^n(y)$ are uniquely determined by Lemma 5. A solution to the problem $X_n(y)$ can now overall be described by

$$(U_i^n(y), q_i^n(y), \sigma_i^n(y), \mu_i^n(y))_{1\le i\le n}$$

and has the warranted claims $(\omega_i^n(y))_{1\le i\le n}$ associated with it.

Suppose the following holds for all $l < n$ and $y \ge 0$:

1. $$\omega_i^l(y) \le U_i^l(y) \qquad \text{for } 1 \le i \le l$$
2. We have $\omega_l^l(0) = V_l$ and for all $1 \le i < l$ and $y \ge 0$ we have $\omega_i^l(y) = V_i$ whenever $\sigma_i^l(y) > 0$.
3. A solution to $X_l(0)$ is a neutral optimum for the restricted type set $\{1, \cdots, l\}$.

Note that all these claims are true for any $y \ge 0$ when $n = l = 1$. $X_1(0)$ is feasible and the first best, hence a strong solution and therefore a neutral optimum according to Myerson (1983). Therefore (3) holds. Since $\omega_1^1(y) = W_1(q_1^1(y)) \le W_1(q_1^1(y)) + y = U_1^1(y)$ also (1) holds.

We prove them now for $l = n > 1$ and $y \geq 0$. By Lemma 8 the solution to $X_n(0)$ induces the solution $X_{n-1}(y)$ with y determined by

$$s_{n-1}y = -\sum_{i=1}^{n-1} s_i \left(W_i \left(q_i^n(0)\right) - U_i^n(0)\right) \geq 0$$

We thus have $\omega_i^n(0) = \omega_i^{n-1}(y) \leq U_i^{n-1}(y) = U_i^n(0)$ for all $i < n$, after using also claim (1) of the induction assumption and because we can choose the same Lagrange multipliers in both problems. This proves claim (1) for $y = 0$ and all $i < l = n$. We also have $\sigma_i^n(0) = \sigma_i^{n-1}(y)$. Hence, we have by a similar argument $\omega_i^n(0) = \omega_i^{n-1}(y) = V_i$ for all $1 \leq i < n-1$ which satisfy $\sigma_i^{n-1}(y) > 0$ by claim (2) of our induction assumption. If $\sigma_{n-1}^n(0) > 0$, then necessarily $y = 0$ by Lemma 4 and hence $\omega_{n-1}^n(0) = \omega_{n-1}^{n-1}(0) = V_{n-1}$ again by the induction assumption. This proves the second part of claim (2) for $y = 0$ and $l = n$. Per construction, from the definition of warranted claims for $y = 0$,

$$\sum_{i=1}^{n-1} \sigma_i^n(0)\,\omega_i^n(0) + \left(\sigma_n^n(0) + \mu_n^n(0)\right)\omega_n^n(0) = \sum_{i=1}^{n} s_i VS_i^n(0)\,. \tag{30}$$

Since $U_n^n(0) = V_n$ by definition, and since $U_i^n(0) = V_i$ whenever $\sigma_i^n(0) > 0$, for $i < n$, we have

$$\sum_{i=1}^{n-1} \sigma_i^n(0)\,U_i^n(0) + \left(\sigma_n^n(0) + \mu_n^n(0)\right)U_n^n(0) = \sum_{i=1}^{n-1} \sigma_i^n(0)\,V_i + \left(\sigma_n^n(0) + \mu_n^n(0)\right)V_n. \tag{31}$$

Comparing expressions (21) and (22), and using the complementarity and the first-order conditions yields

$$\sum_{i=1}^{n} s_i VS_i^n(0) = \sum_{i=1}^{n-1} \sigma_i^n(0)\,U_i^n(0) + \left(\sigma_n^n(0) + \mu_n^n(0)\right)U_n^n(0).$$

Thus, we have form (30) and (31)

$$\sum_{i=1}^{n-1} \sigma_i^n(0)\,\omega_i^n(0) + \left(\sigma_n^n(0) + \mu_n^n(0)\right)\omega_n^n(0) = \sum_{i=1}^{n-1} \sigma_i^n(0)\,V_i + \left(\sigma_n^n(0) + \mu_n^n(0)\right)V_n$$

We obtain, after using the second part of claim (2) for $y = 0$ and $i < n$, overall that $\omega_n^n(0) = V_n = U_n^n(0)$. This proves claims (1) and the first part of (2) for $y = 0$ and $i = l = n$.

Next, we construct a sequence

$$\left(U_i^n(0,\varepsilon_\nu), q_i^n(0,\varepsilon_\nu), \sigma_i^n(0,\varepsilon_\nu), \mu_i^n(0,\varepsilon_\nu), \omega_i^n(0,\varepsilon_\nu)\right)_{1\leq i\leq n}$$

for $\varepsilon_\nu > 0$, $\lim_{\nu\to\infty} \varepsilon_\nu = 0$ which converges to

$$\left(U_i^n(0), q_i^n(0), \sigma_i^n(0), \mu_i^n(0), \omega_i^n(0)\right)_{1\leq i\leq n}$$

such that the first-order conditions for the Lagrangian and the complementarity conditions always hold, where the $\omega_i^n(0,\varepsilon_\nu)$ are defined as above with respect to the "virtual surpluses" $VS_i^{\varepsilon_\nu}$ and where $\sigma_i^n(0,\varepsilon_\nu), \mu_i^n(0,\varepsilon_\nu) \geq \varepsilon_\nu$. Namely, set $\varepsilon_\nu = 1/\nu$ for any integer ν sufficiently large and set $\sigma_i^n(0,\varepsilon_\nu) = \sigma_i^n(0)$ if $\sigma_i^n(0) > 0$ and $\sigma_i^n(0,\varepsilon_\nu) = \varepsilon_\nu$ otherwise. Set $g_i^{\varepsilon_\nu} = \sum_{j=1}^{i} \sigma_j^n(0,\varepsilon_\nu)$. Then $\mu_i^n(0,\varepsilon_\nu) = f_i - g_i^{\varepsilon_\nu}$ is non-negative for sufficiently large ν. The $q_i^n(0,\varepsilon_\nu)$ are then uniquely determined by $\mu_{i-1}^n(0,\varepsilon_\nu)$ from the condition (24) using $\rho_i = \rho_{i-1} = 0$. The "virtual surpluses" $VS_i^{\varepsilon_\nu}$ are then derived from the formulae (28) and the "warranted claims" $\omega_i^n(0,\varepsilon_\nu)$ from (29). Continuity and the uniqueness of the solution implies as $\nu \to \infty$ that $\omega_i^n(0,\varepsilon_\nu) \to \omega_i^n(0)$, $\mu_{i-1}^n(0,\varepsilon_\nu) \to \mu_{i-1}^n(0)$ etc. The $\sigma_i^n(0,\varepsilon_\nu)$, $\sigma_i^n(0)$ and $\omega_i^n(0)$ etc. play hereby the role of the utility weights τ_i^ν, τ_i, and the warranted allocation ω_i, etc. as in the characterization of the neutral bargaining solution discussed in Section 4.4 when the type space is restricted to the types $i = 1, \cdots, n$ and the prior for each type is s_i/f_i. Thus, from claim (1) with $y = 0$ the assured allocation for the first n types is a neutral optimum by Theorem 7 in Myerson (1983) and so claim (3) holds.

It remains to show claims (1) and (2) for $l = n$ and $y > 0$.

Suppose that $\sigma_i^n(0) = 0$ for any $i < n$ in the solution to $X_n(0)$. It is immediately seen that

$$\left(U_i^n(0) + \frac{s_n y}{n}, q_i^n(0), 0, \mu_i^n(0)\right)_{1 \leq i \leq n}$$

is a solution to the problem $X_n(y)$ for all $y > 0$ because $U_i^n(y) = U_i^n(0) + \frac{s_n y}{n} > U_i^n(0) \geq V_i$ (and hence $\sigma_i^n(y) = 0$) for all $i < n$ and $(q_i^n(0), \mu_i^n(0))_{1 \leq i \leq n}$ satisfy the first-order conditions of the problem $X_n(y)$. Clearly, then, claim (2) is trivially satisfied and, furthermore, $\omega_i^n(y) = \omega_i^n(0)$. Thus, given $\omega_i^n(0) \leq U_i^n(0)$ by claim (1) for $y = 0$, we have $U_i^n(y) > \omega_i^n(y)$ for any $y > 0$.

Suppose next that there is some $i < n$ such that $\sigma_i^n(0) > 0$ in the solution to $X_n(0)$ and let $k < n$ be the largest such index. By continuity, as we increase y we will have $\sigma_k^n(y) > 0$ in some maximal interval $0 \leq y < \bar{y}$, which can easily be shown to be of finite length. In this interval we will first show that all $U_i^n(y)$ for $i > k$ are strictly increasing in y. Hence the warranted claim constraints $U_i^n(y) \geq V_i$ cannot become binding for $k < i < n$. At $\bar{y}$ the largest index k' for which $\sigma_{k'}^n(\bar{y}) > 0$ is thus necessarily smaller than k. We will also show that claims (1) and (2) hold for all $0 \leq y < \bar{y}$. One can now apply exactly the same arguments on the maximal interval $\bar{y} \leq y < \bar{y}'$ where $\sigma_{k'}(y) > 0$ as on the interval $0 \leq y < \bar{y}$.

Proceeding by induction in this way one will eventually arrive at a level of $\tilde{y}$ from which onwards all $\sigma_i^n(y)$, $i < n$, are zero. From there onwards a further increase in y does not affect the multipliers $\mu_i^n(y)$, $i < n$, anymore, which are now at their maximal value $\mu_i^n(y) = f_i$. Hence, neither $q_i^n(y)$ nor the virtual surpluses nor $\omega_i^n(y)$ change as y increases. Only the $U_i^n(y)$ are increased, all in the same way because all incentive constraints IC_i are binding (recall Lemma 2). Thus, once all $\sigma_i^n(y)$ are zero for $i < n$ they remain so for all $y' \geq y$, and if $\omega_i^n(y) \leq U_i^n(y)$ holds in addition, this remains so for all $y' \geq y$. Hence, claims (1) and (2) hold for all $y \geq \bar{y}$ if they hold for $y < \bar{y}$.

To complete, given this outline, the proof, we now show (a) that $U_i^n(y)$, for $i > k$, are strictly increasing in y, and (b) that claims (1) and (2) hold for all $0 \leq y < \bar{y}$.

Since $U_k^n(y) = V_k$ in the solution to $X_n(y)$, for all $0 \leq y < \bar{y}$, the solution in question induces by Lemma 4 the solution to $X_k(0)$, and so we have $\omega_i^n(y) = \omega_i^k(0) \leq U_i^k(0) = U_i^n(y)$ for all $0 \leq y < \bar{y}$ and $i \leq k$. Moreover, $U_i^n(y) = U_i^k(0) = V_i = \omega_i^k(0) = \omega_i^n(y)$ for all $i \leq k$ which satisfy $\sigma_i^k(0) = \sigma_i^n(y) > 0$, in particular for $i = k$. Thus, claims (1) and (2) are proved in the interval for all $i \leq k$. None of these variables change as we vary y in the interval. Since, by Lemma 4, $\mu_l^n(y) = f_l - g_k^n(y) = f_l - g_{k-1}^n(0) - \sigma_k^n(y)$, a marginal change in y affects $\sigma_k^n(y)$ and thereby affects all $\mu_l^n(y)$, $l \geq k$, in the same way, i.e.

$$\frac{d\mu_l^n}{dy} = \frac{d\mu_j^n}{dy} = -\frac{d\sigma_k^n}{dy} \text{ for all } l, j \geq k$$

We abbreviate $\frac{d\mu}{dy} = \frac{d\mu_l^n}{dy}$ for $l \geq k$. All incentive constraints IC_l must be binding for $l > k$ by Lemma 2. Therefore, $U_l^n(y) = U_k^n(y) + \sum_{i=k+1}^{l} \psi_{i-1}(q_i^n(y))$. When we now slightly increase y we obtain

$$\frac{dU_l^n(y)}{dy} = \sum_{i=k+1}^{l} \psi'_{i-1}(q_i^n(y)) \frac{dq_i^n(y)}{dy}$$

The first-order condition for $q_i^n(y)$ is $\mu_{i-1}^n(y)\psi'_{i-1}(q_i^n(y)) + s_i W'_i(q_i^n(y)) = 0$. Differentiation yields

$$\begin{aligned}
\frac{d\mu}{dy}\psi'_{i-1}(q_i^n(y)) + \left[\mu_{i-1}^n(y)\psi''_{i-1}(q_i^n(y)) + s_i W''_i(q_i^n(y))\right]\frac{dq_i^n(y)}{dy} &= 0 \\
\frac{dq_i^n(y)}{dy} &= -\frac{\psi'_{i-1}(q_i^n(y))}{\mu_{i-1}^n(y)\psi''_{i-1}(q_i^n(y)) + s_i W''_i(q_i^n(y))}\frac{d\mu}{dy},
\end{aligned}$$

which has the same sign as $d\mu/dy$ by the second-order conditions, and hence

$$\frac{dU_l^n(y)}{dy} = \left(-\sum_{i=k+1}^{l} \frac{\left(\psi'_{i-1}(q_i^n(y))\right)^2}{\mu_{i-1}^n(y)\psi''_{i-1}(q_i^n(y)) + s_i W''_i(q_i^n(y))}\right)\frac{d\mu}{dy}$$

where the term in brackets is positive. For $l = n$ we obtain $\frac{d\mu}{dy} > 0$ since an increase in y slackens the participation constraint and hence $U_n^n(y)$ must increase. (Formally $\frac{dU_n^n(y)}{dy} > 0$ follows by applying the envelope theorem to the Lagrangian.) We see, in turn, that all $U_l^n(y)$ are strictly increasing for all $0 \leq y < \bar{y}$ and $k < l < n$. Therefore, all warranted claim constraints remain slack. In particular, $\sigma_i^n(y) = 0$ for all $i > k$ and, hence, claim (2) is proved for all $0 \leq y < \bar{y}$ and $1 \leq i < l = n$.

We continue with the proof of claim (1). For $l > k$ we have for the warranted claims[13]

$$\mu_l^n(y)\omega_l^n(y) - \mu_k^n(y)\omega_k^n(y) = \sum_{i=k+1}^{l}\left[\mu_{i-1}^n(y)\psi_{i-1}(q_i^n(y)) + s_i W_i(q_i^n(y))\right]$$

Since $q_i^n(y)$ maximizes the virtual surplus and since $\omega_k^n(y) = V_k$ constant we obtain

$$\begin{aligned}
\frac{d}{dy}\left[\mu_l^n(y)\omega_l^n(y) - \mu_k^n(y)\omega_k^n(y)\right] &= \frac{d\mu}{dy}\left(\omega_l^n(y) - \omega_k^n(y)\right) + \mu_l^n(y)\frac{d\omega_l^n}{dy} \\
&= \left[\sum_{i=k+1}^{l}\psi_{i-1}(q_i^n(y))\right]\frac{d\mu}{dy}
\end{aligned}$$

[13] For this calculation it is convenient to set $\sigma_n^n(y) = 0$.

and so

$$\frac{d\omega_l^n}{dy} = \left[\sum_{i=k+1}^{l} \psi_{i-1}\left(q_i^n(y)\right) + \omega_k^n\left(y\right) - \omega_l^n\left(y\right)\right] \frac{d\mu}{dy} / \mu_l^n\left(y\right)$$

Since $\omega_k^n(y) = U_k^n(y)$ and $U_l^n(y) = U_k^n(y) + \sum_{i=k+1}^{l} \psi_{i-1}\left(q_i^n(y)\right)$ by the incentive constraints, it follows that

$$\frac{d\omega_l^n}{dy} = \left(U_l^n\left(y\right) - \omega_l^n\left(y\right)\right) \frac{d\mu}{dy} / \mu_l^n\left(y\right).$$

The proof is now concluded by the following statement:

Consider the maximal interval $[0, \bar{y})$ of all values of y for which $\sigma_k^n(y) > 0$. Then $\omega_l^n(y) \leq U_l^n(y)$ for all $l > k$ and for all y in this interval.

The proof of the statement is by contradiction. Suppose $U_l^n(y) < \omega_l^n(y)$ for some $0 \leq y < \bar{y}$. Let $\hat{y} = \inf\{y | U_l^n(y) < \omega_l^n(y)\}$. For $y = \hat{y}$ we have $U_l^n(y) = \omega_l^n(y)$ because both functions are continuous and so $\frac{d\omega_l^n}{dy}_{|y=\hat{y}} = 0$. Since $\frac{dU_l^n}{dy} > 0$ we have $\frac{d\left(U_l^n - \omega_l^n\right)}{dy}_{|y=\hat{y}} > 0$. It follows that $U_l^n(y) > \omega_l^n(y)$ for all small $y > \hat{y}$, in contradiction to the definition of $\hat{y}$.

7 Conclusions

In this paper we introduced a specific mechanism which we called the assured allocation. We showed that it always weakly dominates the RSW and coincides with the RSW only when the latter is undominated. If the assured allocation is fully separating it is a neutral optimum. Thus the assured allocation sheds new light on the connection between the papers by Myerson (1983) and Maskin and Tirole (1992).

Here, we impose quasi-linearity primarily to simplify the analysis. We believe that our analysis extends to more general utility functions. However, many questions still remain open. We do not know whether the assured allocation is a neutral optimum when there is bunching. We also do not know whether other neutral optima exist. These issues are left for future research.

It would also be interesting to add ex post participation constraints for the agent (i.e. he can ex post refuse a contract if the principal decides to take a particular option). Such model modifications, which relax the commitment assumptions used here, could be handled within the Myerson framework and would alter the nature of the neutral optima. A very important task in such a future extension would be to investigate the ex post efficiency and robustness properties of these mechanisms. We conjecture that such mechanisms will involve bunching which might be in conflict with ex-post efficiency. Such mechanisms will be less robust than mechanisms that can be implemented in ex post equilibria, studied, for instance, in Bergemann and Morris (2005), but might have better efficiency properties and will be more robust than the assured allocation studied here. Moreover, they may be more appropriate in certain environments such as when ex post equilibria do not exist or are extremely inefficient.

One could also allow for type-dependent outside options. This would enable the investigation of insurance and franchise contracts. Our analysis, and the one in Maskin and Tirole (1992), does not allow us to handle cases where there is a trade-off between high quality and high costs. For our analysis, it is important that higher types are unambiguously preferred by the agent. However, interesting applications, such as in the procurement of public services, might require purchaser who prefers a producer of lower cost-efficiency and higher quality of produced services. Finally, the relationship to refinement concepts for signalling games remains to be studied. All these are very interesting future research projects.

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