

Sharing Ambiguous Risks*

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Abstract

We analyse risk-sharing when individuals perceive ambiguity about future events. The main departure from previous work is that different individuals perceive ambiguity differently. We show that individuals fail to share risks for extreme events. This may provide an explanation why we do not observe individuals buying insurance for certain events like hurricanes or earthquakes and why many contracts contain an “act of God” clause, which allows non-performance if an unforeseen event occurs.

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1 INTRODUCTION

1.1 Background

“Most risk management is really just advanced contingency planning and disciplining yourself to realize that, given enough time, very low probability events not only can happen, but they absolutely will happen,” Goldman Sachs CEO, Lloyd Blankfein.¹

Individuals and organizations seek to insure themselves against risks which can cause significant harm. This includes extreme risks, which can be described as very low probability events, which can cause huge damages. Examples include natural disasters like hurricanes, and earthquakes, and man-made disasters arising from terrorism or rogue states. A relevant policy question is whether conventional economic institutions such as insurance are able to handle such risks. Otherwise it may be appropriate for government to provide insurance to cover extreme risks; see Torregrosa (2002). In this paper we ask whether extreme risks can be shared efficiently and if it is possible to trade assets whose pay-offs depend on such risks.

Finance theory implies that any risk uncorrelated with market fundamentals should be traded at its expected value. Hence one would expect that extreme risks like earthquake risks could be traded at a price close to the expected loss. In fact such risks are rarely traded on financial markets. In a few cases markets for catastrophe bonds have been established. For instance, the Chicago board of trade allows trade in disaster options. However these sell at very large premiums compared to estimates of the possible losses; see Jaffee and Russell (2003). Thus markets do not appear to be particularly effective at transferring extreme risk.

In law, the force majeure and the act of God clauses appear to be designed to avoid sharing extreme risks. Common law will often allow performance to be excused and not enforce the contract if an unforeseen and highly unlikely event occurs. As a result such clauses make the party, which has paid for performance, bear all the risk of the extreme event and the other

¹Katie Holliday, “Goldman CEO on risk: The worst ‘absolutely will happen’”, 25 Jul 2013. <http://www.cnbc.com/id/100915696>.

party bears none of the risk.² ³ Similarly in tort law, the court may not find an injurer liable for damage to a victim's property, if it was due to an event which could not reasonably have been foreseen.

At first sight, these features of financial markets and contracts do not appear compatible with our usual models of risk-sharing. Why do insurance contracts contain an act of God clause? Conventional theory suggests that this failure to write risk-sharing contracts could arise if the insurance company were considerably more risk averse than its customers, or there is market failure due to adverse selection.⁴ Adverse selection is possible, for example, people who live in known earthquake zones may be more likely to buy insurance. However since insurance companies have access to detailed actuarial tables, they are typically *better* informed than their customers; see Chiappori and Salanie (2000). The alternative hypothesis that insurance companies are very risk averse is also not highly compelling. Since non-performance clauses occur frequently it would be desirable to explain them.

In the present paper we show that ambiguity can be a barrier to risk-sharing and hence provides a potential explanation for failure to share extreme risks. Risks are said to be ambiguous when the relevant probabilities are impossible or difficult to determine. It is not implausible that extreme risks might be perceived to be ambiguous. There is currently too little data to formulate precise probabilities for events which occur once in every 250 years or 500 years. Even for events which occur once in every 100 years and 50 years, there are considerable uncertainties about their impact on the economy and society. In our analysis individuals buying and selling insurance perceive ambiguity differently. Each individual sees his/her own endowment as being less ambiguous than the endowments of others. This is going to be the main departure from the bulk of the existing literature. For example, Californians perceive earthquake risk as less ambiguous than hurricanes, while inhabitants of Florida have the opposite opinion. We show

²These non performance clauses are often used in construction contracts. For example, exceptionally high realised costs can be used to excuse non-performance by the contractor. As a result all the risk of unforeseen or extreme event is borne by the buyer and the contractor bears no risk. See, *Mineral Park Land Co. v. Howard* (1916) 172 C 289 where the costs were about ten to twelve times higher than anticipated and the contractor was excused.

³We were told by Southwest Water (UK) that the contract between Southwest Water (UK) and its regulator requires the firm to prevent pollution from entering rivers but allows for non-performance when 'exceptional' events happen.

⁴Moral hazard is highly unlikely for risks such as earthquakes or hurricanes.

that extreme risks may not be traded in equilibrium when there are asymmetric perceptions of ambiguity.

Organisation of the Paper In the remaining part of this section we discuss the relevant literature and ambiguity, in the next section we explain how we model ambiguity. The analysis of risk sharing and ambiguity can be found in Section 3. Our conclusions can be found in section 4. The Appendix contains the proofs of all results not proved in the text.

1.2 Related Literature

Chateauneuf, Dana, and Tallon (2000) study risk-sharing with Choquet expected utility (henceforth CEU) preferences; see Schmeidler (1989). They consider an economy with a single physical commodity and multiple states of nature. If all individuals have beliefs represented by *the same* convex capacity, they show that the equilibrium is identical with one which would be obtained if all individuals had subjective expected utility (henceforth SEU) preferences and a particular set of beliefs, $\hat{\pi}$. Since all competitive equilibria are Pareto optima, any competitive equilibrium coincides with an equilibrium of the economy with additive beliefs given by $\hat{\pi}$. Consequently when all individuals have the *same* beliefs, ambiguity does not appear to affect the risk-sharing properties of equilibrium. This was extended by Dana (2004) who analysed the comparative statics of changes in the endowment. Carlier and Dana (2003) study insurance against catastrophic losses. However they need to assume that the insurance company is more ambiguity-averse than its customers to explain why there is a limit on coverage in the event of disasters. A further extension can be found in Strzalecki and Werner (2011).

Recently, Rigotti and Shannon (2012) have shown that, with variational preferences, equilibrium risk sharing is equivalent to the case of standard expected utility and equilibrium price of such trades are generically determinant. This result is similar to that of Chateauneuf, Dana, and Tallon (2000) and Dana (2004).⁵

Billot, Chateauneuf, Gilboa, and Tallon (2000) study incentives to bet with MEU prefer-

⁵Dana and LeVan (2014) have also shown that with maxmin expected utility preferences and incomplete Bewley preferences efficient equilibrium can exist with common priors.

ences. If individuals follow SEU and have the same beliefs they will never bet, while if they have different beliefs they will always wish to bet. In fact betting is not so widespread, which they take as evidence against SEU. The authors show that with MEU preferences individuals with different beliefs will not wish to bet with one another provided they have at least one prior in common. If there is no macroeconomic risk they show that in any Pareto Optimum each individual has certain consumption. However the implications of this result for understanding risk-sharing are limited by the assumption that there is no macroeconomic risk.

Mukerji and Tallon (2001) use CEU to show that ambiguity can be a barrier to risk-sharing. There are securities which could, in principle, allow risk to be shared. However each security carries some idiosyncratic risk. The authors show that if this risk is sufficiently ambiguous, the security will not be traded. They establish a crucial result that ambiguity cannot be diversified in the same way as probabilistic risk. Hence firms, as well as individuals, may be ambiguity-averse.

Epstein (2001) shows that ambiguity can explain the consumption home bias paradox. This arises from failures of risk sharing between countries. Observed consumption is more affected by shocks to the domestic economy than can be explained by the theoretically predicted level of international risk sharing. Epstein explains this by hypothesising that individuals perceive the risks of the foreign economy to be more ambiguous than those to the domestic economy.

Chateauneuf, Dana, and Tallon (2000) poses a problem to those who wish to explain features of risk-sharing contracts by ambiguity. According to their result, risk-sharing contracts do not differ in any qualitative way with and without ambiguity. Rigotti and Shannon (2005) suggest a possible solution using a different model of ambiguity due to Bewley (2002). They are able to show with these preferences that ambiguity prevents risk-sharing. The key difference is that Bewley preferences have a kink at the endowment, while CEU preferences have a kink at certain consumption. Since there are more places in which kinks can potentially occur, it is easier to find failures of risk-sharing.

Although this result solves the problem posed by Chateauneuf, Dana, and Tallon (2000) it also creates some new problems. Bewley preferences are incomplete and moreover they

depend on using an exogenous reference point. In many economic applications it is not easy to determine where this reference point should be. Moreover incomplete preferences are difficult to use in conventional economic models.

The present paper uses ambiguity to explain failures of risk-sharing. It uses the CEU model, which does not rely on an exogenously given status quo point and generates complete preferences. We assume each individual perceives the risks affecting his/her own endowment as being less ambiguous than those affecting other people's endowments. This is not compatible with the representing beliefs by a common convex capacity. Hence the results of Chateauneuf, Dana, and Tallon (2000), which suggest that ambiguity does not prevent risk-sharing, do not apply. Combining their analysis with our own we may conclude that it is not ambiguity *per se* but differences in the perception of ambiguity which create barriers to risk-sharing.

1.3 The Ellsberg Paradox

One of the classic pieces of evidence for ambiguity-aversion is the three-ball Ellsberg paradox. In this thought experiment, a ball is drawn from an urn which contains 90 balls, 30 of which are red. The remaining balls are either blue or yellow in unknown proportion. There are four possible acts a, b, c and d , as described in the table below:

		30	60	
		R	B	Y
Choice 1	a	100	0	0
	b	0	100	0
Choice 2	c	100	0	100
	d	0	100	100

Subjects are offered two choices. In first they have to choose between acts a and b , while in the second they choose between acts c and d . It is found that most subjects prefer option a in the first choice and option d in the second. These preferences are not compatible with expected utility theory or indeed any other plausible decision theory in which decision-makers assign conventional subjective probabilities to events.

One way to understand the paradox is that in the second choice, the decision-maker effectively has an endowment of £100 if a yellow ball is drawn. He/she is then offered the choice of extra money if a red ball is drawn or extra money if a blue ball is drawn. Money in the event that a blue ball is drawn is complementary with the endowment, since it reduces the ambiguity of the endowment. No such complementarity exists for money in the event that a red ball is drawn, hence the decision-maker values money when a blue ball is drawn more highly. This complementarity is not present in the first choice, which is why the opposite decision is often made in that case.

1.4 Ambiguity

There is a substantial body of laboratory evidence which shows that individuals behave differently when faced with ambiguous risks (see, for instance, Camerer and Weber (1992) or Kilka and Weber (2001)). The dominant mode of behaviour observed in experiments is ambiguity-aversion. People will pay significant premiums to convert ambiguous risks into risks with known probabilities.

The traditional model of decision-making under uncertainty is SEU; see Savage (1954). According to SEU, individuals have beliefs represented by a subjective probability distribution and act to maximise the expected value of their utility with respect to that probability distribution. SEU is not able to model the distinction between risk and ambiguity, since all kinds of uncertainty are represented by a single subjective probability distribution. We model ambiguity with CEU, which represents beliefs as capacities (non-additive subjective probabilities). Preferences are represented by maximising the expected value of utility with respect to a capacity. The expectation is expressed as a Choquet integral; Choquet (1953-4). A more detailed description of CEU is provided later.

Next we provide an example of how different perceptions of ambiguity may cause failures of risk-sharing.

1.5 Example

This section presents an example, which illustrates the main points of our argument. Let $S_1 = \{s'_1, s''_1\}$, $S_2 = \{s'_2, s''_2\}$ and $S = S_1 \times S_2$. Assume that there are two individuals, 1 and 2, who have endowments $\omega^1(s'_1) = 2$, $\omega^1(s''_1) = 0$, and $\omega^2(s'_2) = 2$, $\omega^2(s''_2) = 0$ respectively.

1.5.1 Independent Beliefs

First we assume the two individuals have beliefs represented by independent product capacities on $S_1 \times S_2$. We shall show that they will not be prepared to share risk with such beliefs. Assume that individual 1 has beliefs ν^1 with additive marginal on S_1 given by $\pi_1(s'_1) = \pi_1(s''_1) = \frac{1}{2}$ and non-additive, marginal μ^1 on S_2 given by $\mu^1(s'_2) = \mu^1(s''_2) = \frac{1-\gamma}{2}$. Similarly individual 2 has beliefs ν^2 with additive marginal on S_2 given by $\pi_2(s'_2) = \pi_2(s''_2) = \frac{1}{2}$ and non-additive, marginal μ^2 on S_1 given by $\mu^2(s'_1) = \mu^2(s''_1) = \frac{1-\gamma}{2}$. (Under these assumptions the product capacities are unique; see Ghirardato (1997).)

Consider acts as follows:

	s'_1, s'_2	s'_1, s''_2	s''_1, s'_2	s''_1, s''_2
a	2	2	0	0
b	2	0	2	0
c	2	1	1	0

Act a (resp. b) is the pay-off which individual 1 (resp. 2) will receive if (s)he does not trade and consumes his/her endowment, while act c represents a situation where the two individuals share risks equally.

Assume that both individuals have the same concave utility function u . We may normalise by setting $u(0) = 0$, $u(1) = \eta$, and $u(2) = 1$. The larger is η the more concave is the utility function. Then with independent beliefs, individual 1's Choquet expected values of acts a and c are:

$$V^1(a) = \frac{1}{2} \cdot u(2) = \frac{1}{2},$$

$$V^1(c) = \frac{1}{2} \left[u(2) \cdot \frac{(1-\gamma)}{2} + \eta \left(1 - \frac{(1-\gamma)}{2} \right) \right] + \frac{1}{2} \eta \frac{(1-\gamma)}{2} = \frac{(1-\gamma)}{4} + \frac{\eta}{2}.$$

Thus risk-sharing is preferred if and only if

$$2\eta - 1 > \gamma. \quad (1)$$

Concavity of u implies $1 > 2\eta - 1 > 0$, thus if there is sufficient ambiguity, (i.e. γ is sufficiently large) then the risk sharing contract (act c) will be not accepted. In the special case where individual 1 is not risk-averse $\eta = \frac{1}{2}$, equation (1) can never be satisfied. If there is ambiguity there is no preference for risk sharing.

Example 1.1 *Assume that endowments are as above and both individuals have utility function $u(x) = \ln(1+x)$, then provided $0 \leq \gamma \leq \frac{1}{2}$, an interior competitive equilibrium exists and the equilibrium allocations and prices are as given in the table below.⁶*

	s'_1, s'_2	s'_1, s''_2	s''_1, s'_2	s''_1, s''_2
x^A	2	$1 + 2\gamma$	$1 - 2\gamma$	0
x^B	2	$1 - 2\gamma$	$1 + 2\gamma$	0
p	1	$\frac{3}{2(1-\gamma)}$	$\frac{3}{2(1-\gamma)}$	$\frac{3(1+\gamma)}{(1-\gamma)}$

These results are intuitive. If $\gamma = 0$ there is no ambiguity and risk sharing is complete. If $0 < \gamma < \frac{1}{2}$ there is partial risk sharing, while if $\gamma \geq \frac{1}{2}$ there is no risk sharing in equilibrium. Hence ambiguity can be a barrier to risk-sharing. Each individual has higher consumption in the state which is complementary with his/her endowment. This reduces the amount of ambiguity which (s)he faces. As ambiguity increases, the price of the consumption good increases in all states in which it is relatively scarce.⁷

This example illustrates some of the main points of the paper. Ambiguity can be a barrier to risk-sharing. In two cases we find there is no risk-sharing. With linear utility and independent beliefs, individuals do not wish to share ambiguous risks. There is also no risk-sharing if individuals express extreme ambiguity-aversion. With concave utility, individuals may wish to

⁶ A proof can be found in the Appendix.

⁷ At first sight it may look as if the prices tend to infinity as $\gamma \rightarrow 1$. However the prices are calculated under the assumption of an interior equilibrium. This is not valid for $\gamma \geq \frac{1}{2}$. In fact equilibrium prices are finite for all values of γ .

share ambiguous risks. However they will undertake less risk-sharing than in the absence of ambiguity. If utility is concave there is a trade-off. For the usual reasons diminishing marginal utility of wealth encourages people to share risks. However ambiguity-aversion is a barrier to risk-sharing. If there is sufficient ambiguity then there will be no risk-sharing in equilibrium.

1.5.2 Non-Independent Beliefs

We shall now modify the example to show that when beliefs are not independent, risk-sharing may occur even when individuals are ambiguity-averse. Suppose now that both individuals have beliefs defined by a capacity ν defined as follows:

$$\nu(\{S\}) = 1, \nu(\{s'_1, s'_2\}) = \nu(\{s''_1, s''_2\}) = \nu(\{s'_1, s'_2\}, \{s''_1, s''_2\}) = \nu(\emptyset) = 0$$

$$\nu(\{s'_1, s''_2\}, \{s''_2, s'_1\}) = \nu(\{s'_1, s'_2\}, \{s'_1, s''_2\}, \{s''_1, s'_2\}) = \nu(\{s'_1, s''_2\}, \{s''_1, s'_2\}, \{s''_1, s''_2\}) = 2\alpha + \epsilon$$

and all other events have capacity α , where $\alpha > 0, \epsilon > 0$ and $2\alpha + \epsilon < 1$. This capacity is, by construction, a belief function and hence convex. However it is not an independent product.

Capacity 0 is given to the event that both endowments have a high pay-off $\nu(s'_1, s'_2) = 0$.

However the event that the endowment of individual 1 yields a high pay-off is not believed to be impossible on its own ($\nu(\{s'_1, s'_2\}, \{s'_1, s''_2\}) = \alpha$). With these beliefs we can show that there

is a motive for risk-sharing despite the presence of ambiguity. The Choquet expected values of acts a, b and c as defined in the previous section are as follows:

$$V^1(a) = 2 \cdot \nu(\{s'_1, s'_2\}, \{s'_1, s''_2\}) = 2\alpha = V^2(b);$$

$$V^1(c) = V^2(c) = 2 \cdot \nu(\{s'_1, s'_2\}) + 1 \cdot [\nu(\{s'_1, s'_2\}, \{s'_1, s''_2\}, \{s''_1, s'_2\}) - \nu(\{s'_1, s'_2\})] = 2\alpha + \epsilon.$$

Hence both individuals would prefer sharing risks (i.e. act c) to consuming their endowment.

Thus when agents do not believe risks to be independent, they may wish to share ambiguous risks even if their utility is linear.

2 MODELLING AMBIGUITY

This section introduces CEU, which is our main theory of ambiguity. We consider a finite set of states of nature S . The set of outcomes is denoted by X . An act is a function from S to X . The set of all acts is denoted by $A(S)$. We shall restrict attention to the case where beliefs

are represented by convex capacities. A capacity assigns non-additive weights to subsets of S . Formally, they are defined as follows.

Definition 2.1 *A convex capacity on S is a real-valued function ν on the subsets of S which satisfies $A \subseteq B \Rightarrow \nu(A) \leq \nu(B)$; $\nu(\emptyset) = 0$, $\nu(S) = 1$; and $\nu(A) + \nu(B) \leq \nu(A \cup B) + \nu(A \cap B)$, for all $A, B \subseteq S$.*

Schmeidler (1989) argues that convex capacities represent ambiguity-aversion and Wakker (2001) has argued that convexity is implied by a generalised version of the Allais paradox.

Definition 2.2 *The degree of ambiguity of capacity ν is defined by:*

$$\gamma(\nu) = 1 - \max_{A \subseteq S} [\nu(A) + \nu(\neg A)].^8$$

If beliefs are represented by a capacity ν on S , the expected utility of a given act can be found using the Choquet integral, defined below.

Notation 2.1 *Since S is finite, one can order the utility from a given act a : $u(a^1) > u(a^2) > \dots > u(a^{r-1}) > u(a^r)$, where $u(a^1), \dots, u(a^r)$ are the possible utility levels yielded by act a . Denote by $A^k(a) = \{s \in S \mid u(a(s)) \geq u(a^k)\}$ the set of states that yield a utility at least as high as a^k . By convention, let $A^0(a) = \emptyset$.*

Definition 2.3 *The Choquet expected utility of u with respect to capacity ν is:*

$$\int u(a(s))d\nu(s) = \sum_{k=1}^r u(a^k) [\nu(A^k(a)) - \nu(A^{k-1}(a))].$$

Schmeidler (1989), Gilboa (1987) and Sarin and Wakker (1992) provide axioms for representing preferences by a Choquet integral of utility.

⁸This is not a perfect measure of ambiguity since it does not separate ambiguity and ambiguity-aversion. However the distinction is not so important for the issues discussed in the present paper. It is the combination of ambiguity and ambiguity-aversion, which can provide a barrier to risk sharing. Ambiguity would not matter if individuals were not averse to it. For more detailed theories of ambiguity see Epstein (1999), Epstein and Zhang (2001) and Ghirardato and Marinacci (2002).

Definition 2.4 Let ν be a capacity on S_{-i} . The core of ν , $\mathcal{C}(\nu)$ is defined by

$$\mathcal{C}(\nu) = \{p \in \Delta(S_{-i}); \forall A \subset S_{-i}, p(A) \geq \nu(A)\}.$$

The following result shows that for a convex capacity, the Choquet integral for a given act a is equal to the minimum over the core of the expected value of a . Hence convex capacities provide an attractive representation of pessimism. When a decision-maker does not know the true probabilities (s)he considers a set of probabilities to be possible and evaluates any given act by the least favourable of these probabilities. For a proof see the Proposition in Schmeidler (1989).

Proposition 2.1 If ν is an convex capacity on S_{-i} , then $\int a d\nu = \min_{p \in \mathcal{C}(\nu)} \mathbf{E}_p a$, where \mathbf{E} denotes the expected value of a with respect to the additive probability p .

3 RISK-SHARING WITH AMBIGUITY

In this section we argue that differences in the perception of ambiguity are a barrier to risk-sharing. We study a standard economy, where individuals would share risk in the absence of ambiguity.

3.1 The Economy

We consider an exchange economy with uncertainty. There are n individuals $1 \leq i \leq n$ and one physical commodity. The state space S , is a Cartesian product, $S = S_1 \times S_2 \times \dots \times S_n$. As usual we use S_{-i} to denote $\times_{j \neq i} S_j$ and s_{-i} denotes a typical element of S_{-i} . There are markets in all state contingent commodities. The endowment of individual i , $\omega_i(s_i)$ is independent of s_{-i} . Thus S_i is a set of factors which affect i 's endowment but not the endowment of any other individual. Each individual faces some risks, which can shock his/her endowment. The risks different individuals face are independent. It would be possible to add a common shock which affects all individuals, however this would not change our conclusions. The key feature of our model is that each individual regards the shocks to his/her own endowment as unambiguous

but perceives the shocks to the endowments of others as ambiguous. This is modelled by our assumptions on beliefs as described below.

Individual i has beliefs represented by a convex capacity ν^i on S with an additive marginal π_i on S_i . By Lemma 2.1 of Eichberger, Grant, and Kelsey (2005) this implies that

$$\int f(s_i, s_{-i}) d\nu^i(s) = \sum_{s_i \in S_i} \pi_i(s_i) \int f(s_i, s_{-i}) d\mu^i(s_{-i}),$$

where μ^i is a convex capacity on S_{-i} . Since individual i believes that the shocks to his/her own endowment are unambiguous π_i is assumed to be additive.⁹

3.2 Concave Utility

In this section we assume that individuals have concave utility, i.e. u_i is concave for $1 \leq i \leq n$. Thus, apart from ambiguity, our assumptions are standard in risk-sharing models.¹⁰ Now equilibrium is determined by the interaction of two effects. Ambiguity-aversion creates a barrier to sharing ambiguous risks, while diminishing marginal utility of wealth makes it desirable to smooth consumption across states for the usual reasons. In the resulting equilibrium there will be a trade-off between these two effects. First a preliminary lemma.

Lemma 3.1 *Let μ be a convex capacity on S then*

$$\int u(a) d\mu \leq (1 - \gamma) \max_{s \in S} u(a(s)) + \gamma \min_{s \in S} u(a(s)).$$

The next result shows that if there is sufficient ambiguity-aversion, each individual consumes his/her endowment plus a riskless lump sum tax or subsidy. It follows that competitive equilibrium is unique and involves no trade. This holds even though risk-sharing is desirable in the absence of ambiguity. These results are not restricted to competitive trade. They would also imply the absence of risk-sharing in situations where one party is able to make ‘take it

⁹In general there are many capacities on a product space, which are compatible with given marginal capacities on the components. However when one marginal is additive and the other is convex the product capacity is unique; see Eichberger and Kelsey (1996) and Ghirardato (1997).

¹⁰In this context, terms such as risk or risk-sharing should not be taken to imply the absence of ambiguity.

or leave it' offers to the other as is standard in principal-agent models. It would also apply to Coasian efficient bargaining.

Theorem 3.1 *If u_i is concave for $1 \leq i \leq n$, there exists $\bar{\gamma}$ such that if $1 \geq \gamma^i > \bar{\gamma}$, for $1 \leq i \leq n$, then an allocation $x = \langle x_1, \dots, x_n \rangle$ is Pareto optimal if and only if it has the form $x_i(s) = \omega_i(s_i) + \eta_i$, where $\sum_{i=1}^n \eta_i = 0$.*

3.3 Linear Utility

In this section we assume that the marginal utility of wealth is constant. This is to isolate the effects of ambiguity by removing any other reason for risk-sharing. Here we show that if individuals have linear utility and risks are independent then they will bear their own ambiguous risks in any Pareto optimum. Thus ambiguity creates a barrier to risk-sharing.

Let π be the additive probability distribution on S , which is the independent product of the marginals π_i for $1 \leq i \leq n$. Similarly let π_{-i} be the additive probability distribution on S_{-i} , which is the independent product of the marginals π_j for $j \neq i$. The following assumption is a maintained hypothesis for this section.

Assumption 3.1 $\pi_{-i} \in \text{int } \mathcal{C}(\mu^i)$.

We interpret $\mathcal{C}(\mu^i)$ as the set of probability distributions, which i believes are possible. This assumption implies that there is a limited amount of agreement between the beliefs of different individuals since they all believe the probability distribution π is possible. Moreover as $\pi_{-i} \in \text{int } \mathcal{C}(\mu^i)$, i believes that it is possible for the probability to deviate from π_{-i} in any direction. The shocks to other individuals' endowments appear ambiguous to individual i . He/she believes that these shocks may be described by any probability distribution in the set $\mathcal{C}(\mu^i)$.

Let L denote the vector space of real-valued functions on $\times_{i=1}^n S_i$. Let $\bar{h} \in L$ be defined by $\bar{h}(s) = \pi(s)$ and $h \in L^n$ by $h = \langle \bar{h}, \dots, \bar{h} \rangle$, (n times). Consider the hyperplane H defined by: $H = \left\{ z \in L^n : \sum_{j=1}^n \sum_{s \in S} \pi(s) (z_j(s) - \eta_j) = 0 \right\}$ or $H = \{z \in L^n : h \cdot (z - \eta) = 0\}$. We shall show that H is a separating hyperplane through the endowment points.

Lemma 3.2 Suppose that z is not independent of s_{-i} and $\bar{h} \cdot (z_i - \eta_i) \leq 0$ then $\omega_i(s_i) + \eta_i \succ^i \omega_i(s_i) + z_i(s)$.

The next result shows that with linear utility and ambiguity-aversion, each individual consumes his/her endowment plus a non-stochastic lump sum tax or subsidy.

Proposition 3.1 If u_i is linear, for $1 \leq i \leq n$, then an allocation $x = \langle x_1, \dots, x_n \rangle$ is Pareto optimal if and only if it has the form $x_i(s) = \omega_i(s_i) + \eta_i$, where $\sum_{i=1}^n \eta_i = 0$.¹¹

Proof. First we shall show that the allocation $\omega_i(s_i) + \eta_i, 1 \leq i \leq n$, is Pareto optimal. Suppose $y_i(s) = \omega_i(s_i) + \zeta_i(s) \succ^i \omega_i(s_i) + \eta_i$ for $1 \leq i \leq n$. Feasibility implies that $\sum_{i=1}^n \zeta_i(s) = 0$. By Lemmas A.1 and 3.2, if $\omega_i + \zeta_i$ is preferred to $\omega_i + \eta_i$ we must have $\bar{h} \cdot (\zeta_i - \eta_i) \geq 0$. Adding up implies that $\bar{h} \cdot (\zeta_i - \eta_i) = 0$ for $1 \leq i \leq n$. By Lemma 3.2, if there exists k such that $\zeta_k(s)$ is not independent of s_{-k} then $\omega_k(s_k) + \eta_k \succ^k \omega_k(s_k) + \zeta_k(s)$. Hence for $1 \leq i \leq n$, $\zeta_i(s)$ is independent of s_{-i} . However since $\sum_{i=1}^n \zeta_i(s) = 0$, this is only possible if ζ_i is constant for $1 \leq i \leq n$. Thus we must have $\zeta_i(s) = \eta_i$ for $1 \leq i \leq n$, which demonstrates that $\omega_i(s_i) + \eta_i, 1 \leq i \leq n$, is Pareto optimal.

To prove the converse, let $\tilde{x}_i = \omega_i(s_i) + \delta_i(s), 1 \leq i \leq n$, be a feasible allocation i.e. $\sum_{i=1}^n \delta_i(s) = 0$. We shall show that it can only be Pareto optimal if it has the stated form. Define $\eta_j \in \mathbb{R}$ by $\eta_j = \sum_{s \in S} \pi(s) \delta_j(s)$. By construction $h \cdot (\delta - \eta) = 0$. Suppose that δ_k is not independent of s_{-k} then by Lemma 3.2, $\omega_k(s_k) + \eta_k \succ^k \omega_k(s_k) + \delta_k(s)$. Thus if $\omega_i(s_i) + \delta_i(s)$ is Pareto Optimal, δ_i must be independent of s_{-i} . It is also not possible that for some k, δ_k is a non-trivial function of s_k . Together with $\sum_{i=1}^n \delta_i(s) = 0$ this implies that there exists $j \neq k$ such that δ_j is also a non-trivial function of s_k . However we have already argued that δ_j must be independent of s_{-j} . The only remaining possibility is that δ_i is constant for $1 \leq i \leq n$. The result follows. ■

We have shown that if all individuals have linear utility, ambiguous risks will not be shared. This holds even if individuals only display small degrees of ambiguity-aversion. In many contracts some of the parties are insurance companies or large firms, which may well be risk-neutral

¹¹Although Proposition 3.1 requires Assumption 3.1, this assumption is implied automatically by the hypotheses of Theorem 3.1.

in the sense of having linear utility. This result tells us that they would be expected to bear their own ambiguous risks. Our analysis would be applicable to the usual principal-agent model, where the principal has linear utility. In this model it is usual to assume that one individual is able to make take it or leave it offers to the others, which would result in a Pareto optimal allocation.

4 CONCLUSION

In this paper we have used the CEU model to show that ambiguity-aversion may reduce risk-sharing. There are two alternative models of ambiguity, the MEU model and the smooth model of ambiguity due to Klibanoff, Marinacci, and Mukerji (2005).¹² If the smooth model is used, our results would need to be modified. The proof that no risk is shared in equilibrium, depends on the fact that CEU preferences have a kink at certain consumption. Smooth preferences, as the name implies, do not have a kink. In this case ambiguity would reduce risk-sharing, however we could not conclude that no risk would be shared even for high levels of ambiguity-aversion. Since MEU preferences also have a kink at certain consumption we believe it would be possible to prove similar results for such preferences.

In our model each party perceives the risk to his/her own endowment as unambiguous, while the risk to the other party's endowment is seen as ambiguous. In practice force majeure and act of God clauses often apply to risks which both parties believe are ambiguous. We believe it would be possible to extend our analysis to this case. Suppose that there are two individuals. Each of them views the risk to his/her own endowment as *less ambiguous* than the risk to the other party's endowment. We conjecture that a similar result will hold, even though individuals agree which risks are ambiguous. Ambiguity will reduce risk-sharing. The difficulty with this approach is that, at present, there is no generally agreed definition of when one event is more ambiguous than another. Consequently we shall leave this extension for future research.

¹²As discussed earlier, in a recent paper Rigotti and Shannon (2012) using variational preferences show that as long as risks are correlated, risks can be shared even for ambiguous events.

In this model, all competitive equilibria are Pareto optima. Thus although there is no risk-sharing in equilibrium, there is no possible government intervention in markets which can bring about a Pareto improvement. It is possible that other kinds of intervention might be desirable. For instance, the government could act to reduce ambiguity about future events by providing information about the likelihood of extreme events. This can facilitate risk-sharing.

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A APPENDIX

Proof of Example 1.1: Assume that $x^A(s'_1, s'_2) \geq x^A(s'_1, s''_2)$ and $x^A(s''_1, s'_2) \geq x^A(s''_1, s''_2)$.

Then individual A 's preferences are given by the Choquet integral:

$$V^A(x^A) = \frac{1}{2} \left[\frac{1-\gamma}{2} u(x^A(s'_1, s'_2)) + \frac{1+\gamma}{2} u(x^A(s'_1, s''_2)) \right] \\ + \frac{1}{2} \left[\frac{1-\gamma}{2} u(x^A(s''_1, s'_2)) + \frac{1+\gamma}{2} u(x^A(s''_1, s''_2)) \right].$$

Similarly if we assume $u(x^B(s'_1, s'_2)) \geq u(x^B(s''_1, s'_2))$ and $u(x^B(s'_1, s''_2)) \geq u(x^B(s''_1, s''_2))$ individual B 's preferences can be represented by the Choquet integral:

$$V^B(x^B) = \frac{1}{2} \left[\frac{1-\gamma}{2} u(x^B(s'_1, s'_2)) + \frac{1+\gamma}{2} u(x^B(s''_1, s'_2)) \right] \\ + \frac{1}{2} \left[\frac{1-\gamma}{2} u(x^B(s'_1, s''_2)) + \frac{1+\gamma}{2} u(x^B(s''_1, s''_2)) \right].$$

Now specialise to the case where $u(x) = \ln(1+x)$. Checking the marginal conditions are satisfied at the proposed solution:

$$\text{MRS}_{\{s'_1, s'_2\}\{s'_1, s''_2\}}^A = \frac{2(1-\gamma)}{3} = \text{MRS}_{\{s'_1, s'_2\}\{s'_1, s''_2\}}^B = \frac{\frac{1}{3}}{\frac{1}{2-2\gamma}} = \frac{1}{p(s'_1, s''_2)}, \\ \text{MRS}_{\{s'_1, s'_2\}\{s''_1, s'_2\}}^A = \frac{2(1-\gamma)}{3} = \text{MRS}_{\{s'_1, s'_2\}\{s''_1, s'_2\}}^B = \frac{1}{p(s''_1, s'_2)}, \\ \text{MRS}_{\{s'_1, s'_2\}\{s'_1, s''_2\}}^A = \frac{(1-\gamma)}{3(1+\gamma)} = \text{MRS}_{\{s'_1, s'_2\}\{s'_1, s''_2\}}^B = \frac{1}{p(s'_1, s''_2)}.$$

By concavity, these conditions are sufficient for utility maximisation. One can check that supply equals demand in each state and that the two budget constraints are satisfied. ■

Proof of Lemma 3.1: $\int u(a) d\mu = u(a^1) \mu(A^1(a)) + \sum_{i=2}^{r-1} u(a^i) [\mu(A^i(a)) - \mu(A^{i-1}(a))] \\ + u(a^r) [1 - \mu(A^{r-1}(a))]^{13}$
 $\leq (1-\gamma) u(a^1) u + \gamma u(a^r)$, since $1 - \mu(A^{r-1}(a)) \geq 1 - \mu(S \setminus A^{r-1}(a)) - \mu(A^{r-1}(a)) \geq \gamma$.
 $= (1-\gamma) \max_{s \in S} u(a(s)) + \gamma \min_{s \in S} u(a(s))$. ■

Proof of Theorem 3.1: First we shall show that the allocation $\omega_i(s_i) + \eta_i, 1 \leq i \leq n$; is Pareto optimal. Let $h_i \in L$ be defined by

$$h_i = \langle \pi(s_i^1) u'_i(\omega_i(s_i^1) + \eta_i), \dots, \pi(s_i^{K_i}) u'_i(\omega_i(s_i^{K_i}) + \eta_i) \rangle,$$

where $S_i = \{s_i^1, \dots, s_i^{K_i}\}$. Define $h \in L^n$ by $h = \frac{1}{\rho} \langle h_1, \dots, h_n \rangle$, where

$\rho = \sum_{j=1}^n \sum_{S_j} \pi(s) u'_j(\omega_j(s_j) + \eta_j)$.¹⁴ Thus h is a normalised vector of marginal rates of substitution. Consider the hyperplane H in L^n defined by: $H = \{z \in L^n : h.z = 0\}$. We shall show that H is a separating hyperplane through the allocation $\omega_i(s_i) + \eta_i, 1 \leq i \leq n$ and hence

¹³See Notation 2.1.

¹⁴Recall L denotes the vector space of real-valued functions on $\times_{i=1}^n S_i$.

a Pareto improvement is not possible.

Lemma A.1 *Suppose that $h_i \cdot z_i \leq 0$ (resp. $h_i \cdot z_i < 0$) and z_i is independent of s_{-i} i.e. $z_i(s) = z_i(s_i)$, then $\omega_i(s_i) + \eta_i \succ^i \omega_i(s_i) + \eta_i + z_i(s_i)$ (resp. $\omega_i(s_i) + \eta_i \succ^i \omega_i(s_i) + \eta_i + z_i(s_i)$).*

Proof. $V^i(\omega_i(s_i) + \eta_i + z_i(s_i)) = \sum_{S_i} \pi_i(s_i) \int u_i(\omega_i(s_i) + \eta_i + z_i(s_i)) d\mu_i(s_{-i})$
 $\leq \sum_{S_i} \pi_i(s_i) u_i(\omega_i(s_i) + \eta_i) + \sum_{S_i} \pi_i(s_i) u'_i(\omega_i(s_i) + \eta_i) z_i(s_i)$
 $\leq \sum_{S_i} \pi_i(s_i) u_i(\omega_i(s_i) + \eta_i)$. The last inequality applies because $h_i \cdot z_i \leq 0$ implies that
 $\sum_{S_i} \pi_i(s_i) u'_i(\omega_i(s_i) + \eta_i) z_i(s_i) \leq 0$. The proof of the other case is similar. ■

Lemma A.2 *Suppose that z is not independent of s_{-i} and $h_i \cdot z_i \leq 0$ then if γ_i is sufficiently large $\omega_i(s_i) + \eta_i \succ^i \omega_i(s_i) + \eta_i + z_i(s)$.*

Proof. Define $m = \sum_{S_i} \pi_i(s_i) \min_{s_{-i} \in S_{-i}} u_i(\omega_i(s_i) + \eta_i + z_i(s_i, s_{-i}))$
 $\leq \sum_{S_i} \pi_i(s_i) \min_{s_{-i} \in S_{-i}} [u_i(\omega_i(s_i) + \eta_i) + u'_i(\omega_i(s_i) + \eta_i) z_i(s_i, s_{-i})]$
 $= \sum_{S_i} \pi_i(s_i) u_i(\omega_i(s_i) + \eta_i) + \sum_{S_i} \pi_i(s_i) \min_{s_{-i} \in S_{-i}} [u'_i(\omega_i(s_i) + \eta_i) z_i(s_i, s_{-i})]$
 $< \sum_{S_i} \pi_i(s_i) u_i(\omega_i(s_i) + \eta_i) + \sum_{S_i} \pi_i(s_i) u'_i(\omega_i(s_i) + \eta_i) z_i(s_i, s_{-i})$.

The inequality follows since a weighted average is greater than a minimum. The fact that z is not independent of s_{-i} , ensures that the inequality is strict.

Hence $m < \sum_{S_i} \pi_i(s_i) u_i(\omega_i(s_i) + \eta_i)$, since $h_i \cdot z_i \leq 0$ implies

$\sum_{S_i} \pi_i(s_i) u'_i(\omega_i(s_i) + \eta_i) (z_i(s_i, s_{-i})) \leq 0$. The result now follows from Lemma 3.1. ■

This conclusion only holds if i is sufficiently ambiguity-averse. The reason is that diminishing marginal utility of income generates a demand for risk-sharing in the usual way. It is only when ambiguity-aversion is sufficiently high, that this demand goes to zero.

Proof of Theorem 3.1 (continued) Suppose $\omega_i(s_i) + \zeta_i(s) + \eta_i \succ^i \omega_i(s_i) + \eta_i$ for $1 \leq i \leq n$. Feasibility implies that $\sum_{i=1}^n \zeta_i(s) = 0$. By Lemmas A.1 and A.2, if this preference holds for γ_i sufficiently large, $h_i \cdot \zeta_i \geq 0$. Since $0 = h \cdot \sum_{i=1}^n \zeta_i = \sum_{i=1}^n h_i \cdot \zeta_i$ we must have $h_i \cdot \zeta_i = 0$, for $1 \leq i \leq n$. By Lemma A.2, there cannot exist k such that $\zeta_k(s)$ is not independent of s_{-k} , since otherwise we would have $\omega_k(s_k) + \eta_k \succ^k \omega_k(s_k) + \eta_k + \zeta_k(s_k, s_{-k})$ for γ_k sufficiently large. Hence for $1 \leq i \leq n$, $\zeta_i(s)$ is independent of s_{-i} . However since $\sum_{i=1}^n \zeta_i(s) = 0$ this is

only possible if ζ_i is constant for $1 \leq i \leq n$. Thus we must have $\zeta_i(s) = \eta_i$ for $1 \leq i \leq n$, which demonstrates that $\omega_i(s_i) + \eta_i, 1 \leq i \leq n$, is Pareto optimal.

Thus it follows that for all vectors of net trades $\zeta = \langle \zeta_1, \dots, \zeta_n \rangle$ there exists $\gamma_\zeta, 0 \leq \gamma_\zeta < 1$, such that if $\gamma_\zeta < \gamma_i \leq 1$ for $1 \leq i \leq n$, then $\omega_i + \eta_i \succ^i \omega_i + \eta_i + \zeta_i$, with at least one strict preference. Let $Z = \{\zeta = \langle \zeta_1, \dots, \zeta_n \rangle : \|\zeta\| \leq 1; 1 \leq i \leq n; \sum_{i=1}^n \zeta_i = 0\}$ denote the set of net trades in the unit ball. Define $\bar{\gamma} = \max\{\gamma_\zeta : \zeta \in Z\}$. Since we are maximising a continuous function over a compact set, the maximum is attained and we must have $0 \leq \bar{\gamma} < 1$. If $1 \geq \gamma_i > \bar{\gamma}$, for $1 \leq i \leq n$, then there are no net trades in the unit ball, which will not make at least one individual worse off. By Lemma A.3 this implies that there are no Pareto improving net trades. Hence $\omega_i + \eta_i, 1 \leq i \leq n$, is Pareto Optimal.

To prove the converse, let $x_i = \omega_i(s_i) + \delta_i(s), 1 \leq i \leq n$, be a feasible allocation i.e. $\sum_{i=1}^n \delta_i(s) = 0$. We shall show that it is only Pareto optimal if it has the stated form. Define $\theta_j \in \mathbb{R}$ by $\theta_j = \sum_{s \in S} \pi(s) \delta_j(s)$. By Lemma A.2, there cannot exist k such that $\delta_k(s)$ is not independent of s_{-k} , since otherwise we would have $\omega_k(s_k) + \theta_k \succ^k \omega_k(s_k) + \delta_k(s)$ when γ_k is sufficiently large. Thus if $\omega_i(s_i) + \delta_i(s)$ is Pareto Optimal, δ_i must be independent of s_{-i} , for $1 \leq i \leq n$.

It is also not possible that for some k, δ_k is a non-trivial function of s_k . Together with $\sum_{i=1}^n \delta_i(s) = 0$ this implies that there exists $j \neq k$ such that δ_j is also a non-trivial function of s_k . However we have already argued that δ_j must be independent of s_{-j} . The only remaining possibility is that δ_i is constant for $1 \leq i \leq n$. The result follows. ■

Lemma A.3 *If $1 \geq \gamma_i > \bar{\gamma}$, for $1 \leq i \leq n$, and there are no Pareto improving net trades, ζ , such that $\|\zeta\| \leq 1$, then there are no Pareto improving net trades.*

Proof. Let z be a net trade such that $\|z\| > 1$. We may write $z = \hat{\lambda}\zeta$, where $\|\zeta\| \leq 1$ and $\hat{\lambda} > 1$. Since there are no Pareto improving net trades in the unit ball, there must exist j such that $\omega_j + \eta_j \succ^j \omega_j + \eta_j + \zeta_j$. Define

$$\phi(\lambda) = V^j(\omega_j + \eta_j + \lambda\zeta_j) = \sum_{s_j} \pi_j(s_j) \int u_j(\omega_j(s_j) + \eta_j + \lambda\zeta_j(s)) d\mu_j(s_{-j})$$

$= \sum_{S_j} \pi_j(s_j) \min_{p \in C(\mu_j)} \mathbf{E}_p u_j(\omega_j(s_j) + \eta_j + \lambda \zeta_j(s)) d\mu_j(s_{-j})$. The function ϕ is concave, since concavity is preserved by the min operation and by taking linear combinations.

By concavity, $V^j(\omega_j + \eta_j) = \phi(0) \leq \phi(1) - \phi'(1) = V^j(\omega_j + \eta_j + \zeta_j) - \phi'(1)$.¹⁵ Since $\omega_j + \eta_j \succ^j \omega_j + \eta_j + \zeta_j$, we must have $\phi'(1) < 0$. Concavity also implies, $V^j(\omega_j + \eta_j + z_j) = \phi(\hat{\lambda}) \leq \phi(1) + (1 - \hat{\lambda})\phi'(1) < \phi(1) = V^j(\omega_j + \eta_j + \zeta_j) \leq V^j(\omega_j + \eta_j)$. This demonstrates that the net trade z makes individual j worse off and is therefore not a Pareto improvement.

■

Proof of Lemma 3.2: Let $\bar{s}_{-i}(s_i) = \operatorname{argmax}_{z(s_i, s_{-i})}$ and $\underline{s}_{-i}(s_i) = \operatorname{argmin}_{z(s_i, s_{-i})}$.

Define a probability distribution $\tilde{\pi}_{s_i}$ on S_{-i} by $\tilde{\pi}_{s_i}(\bar{s}_{-i}(s_i)) = \pi_{-i}(\bar{s}_{-i}(s_i)) - \epsilon$, $\tilde{\pi}_{s_i}(\underline{s}_{-i}(s_i)) = \pi_{-i}(\underline{s}_{-i}(s_i)) + \epsilon$, $\tilde{\pi}_{s_i}(s_{-i}) = \pi_{-i}(s_{-i})$ otherwise. Since $\pi \in \operatorname{int} \mathcal{C}(\mu)$ for ϵ sufficiently small $\tilde{\pi}_{s_i} \in \mathcal{C}(\mu)$.

$$\begin{aligned} \text{Now } V^i(\omega_i(s_i) + z(s)) &= \sum_{S_i} \pi_i(s_i) \int \omega_i(s_i) + z_i(s_i) d\mu_i(s_{-i}) \\ &\leq \sum_{S_i} \pi_i(s_i) \sum_{S_{-i}} \tilde{\pi}_{s_i}(s_{-i}) (\omega_i(s_i) + z_i(s_i)), \text{ since } \tilde{\pi}_{s_i} \in \mathcal{C}(\mu), \\ &< \sum_{S_i} \pi_i(s_i) \sum_{S_{-i}} \pi_{-i}(s_{-i}) (\omega_i(s_i) + z_i(s_i)) \\ &= \sum_{S_i} \pi_i(s_i) (\omega_i(s_i) + \eta_i) = V^i(\omega_i(s_i) + \eta_i). \quad \blacksquare \end{aligned}$$

¹⁵Strictly speaking ϕ may not be differentiable at 1, in which case $\phi'(1)$ should be interpreted as a super-differential.