MNRAS **448,** 456–463 (2015) doi:10.1093/mnras/stu2759

An exact solution for arbitrarily rotating gaseous polytropes with index unity

Dali Kong,[1] Keke Zhang[1]★ and Gerald Schubert[2]

[1]*Department of Mathematics and Center for Geophysical and Astrophysical Fluid Dynamics, University of Exeter, Exeter EX4 4QF, UK*
[2]*Department of Earth, Planetary and Space Sciences, University of California, Los Angeles, CA 90095-1567, USA*

Accepted 2014 December 29. Received 2014 December 25; in original form 2014 September 14

ABSTRACT

Many gaseous planets and stars are rapidly rotating and can be approximately described by a polytropic equation of state with index unity. We present the first exact analytic solution, under the assumption of the oblate spheroidal shape, for an arbitrarily rotating gaseous polytrope with index unity in hydrostatic equilibrium, giving rise to its internal structure and gravitational field. The new exact solution is derived by constructing the non-spherical Green's function in terms of the oblate spheroidal wavefunction. We then apply the exact solution to a generic object whose parameter values are guided by the observations of the rapidly rotating star α Eridani with its eccentricity $\mathcal{E}_\alpha = 0.7454$, the most oblate star known. The internal structure and gravitational field of the object are computed from its assumed rotation rate and size. We also compare the exact solution to the three-dimensional numerical solution based on a finite-element method taking full account of rotation-induced shape change and find excellent agreement between the exact solution and the finite-element solution with about 0.001 per cent discrepancy.

Key words: planets and satellites: gaseous planets – planets and satellites: interiors – stars: interiors.

1 INTRODUCTION

A fully compressible polytropic gas with index unity obeying the polytropic equation of state (EOS)

$$p^* = K\,(\rho^*)^2, \tag{1}$$

where $p*$ is the pressure, K is a constant and $\rho*$ is the density, has been widely employed to study the physical properties of gaseous planets, exoplanets and stars (see for example, Chandrasekhar 1933; Roberts 1962; Hubbard 1973; Stevenson 1982; Dintrans & Ouyed 2001; Horedt 2004; Kong et al. 2014). In this paper, the superscript $*$ is adopted to represent a dimensional variable and its corresponding dimensionless variable is denoted without the superscript. Many astrophysical gaseous bodies are rapidly rotating, causing significant departure from sphericity: the eccentricity at the one-bar surface is $\mathcal{E}_S = 0.4316$ for Saturn (Seidelmann et al. 2007) while the star α Eridani is marked by a much larger departure from sphericity with the approximate eccentricity $\mathcal{E}_\alpha = 0.7454$ (Carciofi et al. 2008).

The rotational effect on the shape and physical structure of a slowly rotating polytrope was first studied by Chandrasekhar (1933) using a perturbation analysis. For an isolated, non-rotating and self-gravitating body, the density distribution $\rho*$ within the interior of the polytrope is spherically symmetric and described by the Lane–Emden equation, a second-order ordinary differential equation that can be readily solved to determine the one-dimensional density distribution. For a polytropic body that is slowly rotating with small angular velocity Ω such that its departure from sphericity is slight, Chandrasekhar (1933) introduced a small parameter $\epsilon \sim \Omega^2$ and was able to solve for the density distribution $\rho*$ of the slowly rotating polytrope via a perturbation method in terms of the small expansion parameter ϵ. Without developing a small parameter expansion, Roberts (1962) proposed a numerical method – which is based on a variational principle that minimizes the sum of the internal energy, the kinetic energy of rotation and the gravitational energy through selection of the best trial function – for obtaining an approximate solution for a rapidly rotating polytrope. Recently, particular attention is being paid to the highly accurate solution for rapidly rotating giant planets, which is largely motivated by the ongoing Juno mission that will make high-precision measurements of the gravitational field of Jupiter (see for example, Hubbard 1999; Helled, Schubert & Anderson 2009; Kaspi et al. 2010; Kong, Zhang & Schubert 2012). Interpretation of these gravitational measurements requires a highly accurate description of the shape and internal density structure of the planet and, hence, the effect of rotational distortion can no longer be treated as a small perturbation on a spherically symmetric state. Kong et al. (2013) proposed a hybrid inverse numerical method using a finite-element formulation for calculating the non-spherical shape and physical structure

★ E-mail: kzhang@ex.ac.uk

of rapidly rotating gaseous bodies valid for arbitrary angular velocity Ω, and Hubbard (2013) developed a radially discontinuous numerical model using concentric Maclaurin spheroids valid for a moderate angular velocity.

This paper reports a breakthrough in this classical problem by obtaining the first exact analytic solution, under the assumption of the oblate spheroidal shape, for a rapidly rotating gaseous polytrope with index unity valid for arbitrary angular velocity. There exist at least three inherent mathematical difficulties in studying the shape and physical structure of a rapidly rotating polytrope marked by a large deviation from sphericity: (i) a Lane–Emden-type of differential equation describing the one-dimensional density distribution no longer exists in non-spherical geometry; (ii) non-spherical coordinates such as oblate spheroidal coordinates need to be employed in the mathematical analysis dramatically increasing the complexity of the analysis and, more significantly, (iii) practically useful mathematical tools for analysing the problem of a rapidly rotating polytrope were not available. An essential mathematical tool in deriving the exact solution for a rapidly rotating gaseous polytrope is the oblate spheroidal wavefunction (Chu & Stratton 1941; Morse & Feshbach 1953; Flammer 1957) which has had many applications in non-astrophysical problems. For example, the spheroidal wavefunction was applied to the problem of designing a spheroidal antenna (Chu & Stratton 1941); the problem of acoustic scattering by a solid spheroid was solved using the spheroidal wavefunction (Spence & Granger 1951); and Asano & Yamamoto (1975) employed the spheroidal wavefunction for solving the problem of light scattering by a spheroidal object. By contrast, the spheroidal wavefunction has never been applied to any astrophysical problem. This is perhaps due to the fact that the solution of an astrophysical problem, as discussed in the present investigation, requires the accurate computation of all the eigenvalues of the spheroidal wavefunction in an oblate spheroid of arbitrary eccentricity, representing a mathematically nearly intractable problem because the eigenvalues involve the transcendental equation of an infinitely continued fraction. It was not until a decade ago that Van Buren & Boisvert (2002, 2004) found a practical way of accurately computing all the eigenvalues of the spheroidal wavefunction which is adopted in this investigation. It represents the first application of the oblate spheroidal wavefunction to an astrophysical problem.

In what follows, we begin in Section 2 by presenting the model and governing equations for rapidly rotating polytropes with index unity in hydrostatic equilibrium. This is followed in Section 3 by deriving, via use of the spheroidal wavefunction, the exact solution based on the non-spherical Green's function. In Section 4, we calculate the exact solution for a generic object whose parameter values are guided by the observations of the rapidly rotating star α Eridani, the most oblate body observed so far, and then compare the exact solution to the three-dimensional finite-element solution. While Section 5 discusses the virial test of the equilibrium solution, the paper closes with a summary and some remarks in Section 6.

2 MODEL AND GOVERNING EQUATIONS

Our polytropic model for rapidly rotating gaseous planets and stars employs the widely used three assumptions (Chandrasekhar 1933; Roberts 1962; Hubbard 1999): (i) the star/planet with mass M is isolated and rotating uniformly about the z-axis with angular velocity $\Omega\hat{z}$ with $\Omega > 0$ whose hydrostatic equilibrium is in the state of rigid-body rotation; (ii) the star/planet is axially symmetric with respect to the rotation axis and its shape can be described by an oblate spheroid of eccentricity $\mathcal{E}$ with polar radius $R_{\rm p}$ and equatorial radius $R_{\rm e}$ ($R_{\rm e} > R_P$) that is still within the stable limit; and (iii) the EOS for the gaseous star/planet is given by the polytropic law with index unity. The assumption of an oblate spheroidal shape has been generally adopted in studying the hydrostatic equilibrium of rapidly rotating gaseous planets and stars (see for example, Roberts 1962; Kong et al. 2013). James (1964) found that the result without making use of this assumption is consistent with that of Roberts (1962) who used the assumption.

In an inertial frame of reference, the hydrostatic equilibrium of a rapidly rotating polytrope is governed by the dimensional equations

$$\boldsymbol{u}^* \cdot \nabla \boldsymbol{u}^* = -\frac{1}{\rho^*}\nabla p^* - \nabla V_{\rm g}^*, \tag{2}$$

$$\nabla^2 V_{\rm g}^* = 4\pi G_{\rm g}\rho^*, \tag{3}$$

$$p^* = K(\rho^*)^2, \tag{4}$$

$$\nabla \cdot \left(\boldsymbol{u}^*\rho^*\right) = 0, \tag{5}$$

where the velocity $\boldsymbol{u}^*$ is given by $\Omega\hat{z} \times \boldsymbol{r}^*$ with $\boldsymbol{r}^*$ being the position vector, $p*$ is the pressure, $\rho*$ is the density of the polytropic gas, K is a constant, $V_{\rm g}^*$ is the gravitational potential and $G_{\rm g} = 6.673\,84 \times 10^{-11}\ {\rm m^3\,kg^{-1}\,s^{-2}}$ is the universal gravitational constant. Equations (2)–(5) are solved subject to the two boundary conditions

$$p^* = 0 \tag{6}$$

$$V_{\rm g}^* + V_{\rm c}^* = \text{constant}, \tag{7}$$

at the outer bounding surface $\mathcal{S}$ of the rapidly rotating polytrope, where $V_{\rm c}^* = -|\Omega\hat{z} \times \boldsymbol{r}^*|^2/2$ is the centrifugal potential. The second boundary condition (7) represents the free surface condition required in the hydrostatic equilibrium of the rotating polytrope. Note that equation (5) is automatically satisfied because $\partial\rho^*/\partial\phi^* = 0$, where $\phi*$ is the azimuthal angle.

With the length-scale $R_{\rm e}$ and the density scale $M/R_{\rm e}^3$, there arises the two dimensionless parameters,

$$\alpha = \frac{2\pi G_{\rm g} R_{\rm e}^2}{K}, \quad \beta = \frac{\Omega^2 R_{\rm e}^5}{MK},$$

where $\alpha > 0$ and $\beta > 0$, that characterize the physical properties of a rapidly rotating spheroidal polytrope. The rotation parameter β provides a measure of the centrifugal force. It is mathematically convenient (Zhang, Liao & Earnshaw 2004; Kong, Zhang & Schubert 2010) to adopt oblate spheroidal coordinates, (ξ, η, ϕ), defined by the coordinate transformation with Cartesian coordinates

$$x = f\sqrt{(1+\xi^2)(1-\eta^2)}\cos\phi,$$

$$y = f\sqrt{(1+\xi^2)(1-\eta^2)}\sin\phi,$$

$$z = f\xi\eta,$$

where $\xi =$ constant represents a confocal oblate spheroid, $\eta =$ constant describes confocal hyperboloids and $f = \sqrt{R_{\rm e}^2 - R_{\rm p}^2}/R_{\rm e} > 0$ is the common focal length of an oblate spheroidal polytrope whose bounding surface $\mathcal{S}$ is described by

$$\xi = \xi_o = \sqrt{\frac{1}{\mathcal{E}^2} - 1} \quad \text{with} \quad 0 < \mathcal{E} < 1.$$

In the oblate spheroidal system, the interior domain $\mathcal{D}$ of a rapidly rotating spheroidal polytrope is defined by

$$-1 \le \eta \le 1, \quad 0 \le \xi \le \sqrt{\frac{1}{\mathcal{E}^2} - 1}, \quad 0 \le \phi \le 2\pi.$$

In the present problem, the shape parameter, the size of eccentricity $\mathcal{E}$, is generally unknown and can be determined by making use of the free surface condition (7).

By combining the equations (2)–(4) into a single equation together with oblate spheroidal coordinates (ξ, η, ϕ), we derive a single dimensionless equation governing the density distribution of the rotating polytrope,

$$\frac{\partial}{\partial \xi}\left[\left(1+\xi^2\right)\frac{\partial \rho}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[\left(1-\eta^2\right)\frac{\partial \rho}{\partial \eta}\right] + \left[\frac{\xi^2+\eta^2}{(1+\xi^2)(1-\eta^2)}\right]\frac{\partial^2 \rho}{\partial \phi^2} + c^2\left(\xi^2+\eta^2\right)\rho = \beta f^2(\xi^2+\eta^2), \tag{8}$$

where ϕ denotes the dimensionless density and $c = f\sqrt{\alpha}$. The boundary conditions (6) and (7) become

$$0 = \rho, \tag{9}$$

$$\text{constant} = \int_{\mathcal{D}} \frac{\rho\left(\boldsymbol{r}'\right) \mathrm{d}^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{\beta \mathcal{E}^2}{4\alpha}(1+\xi^2)(1-\eta^2) \tag{10}$$

on the bounding surface $\mathcal{S}$ of the rotating polytrope, where $\int_{\mathcal{D}} \mathrm{d}^3\boldsymbol{r}'$ represents the volume integral over the oblate spheroidal domain $\mathcal{D}$. We shall first use the iterative method (Kong et al. 2012) to search for the hydrostatic equilibrium solution that satisfies (10).

In order to find an equilibrium solution satisfying (10), we introduce an auxiliary function, $||\,\mathrm{d}V_\mathrm{t}/\,\mathrm{d}\eta||_2$, defined as

$$\left|\left|\frac{\mathrm{d}V_\mathrm{t}}{\mathrm{d}\eta}\right|\right|_2 = \frac{1}{2\pi}\int_{\mathcal{S}}\left|\frac{\partial}{\partial \eta}\left[\int_{\mathcal{D}} \frac{\rho\left(\boldsymbol{r}'\right)\mathrm{d}^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|} + \frac{\beta c^2}{4\alpha}(1+\xi^2)(1-\eta^2)\right]_{\xi=\xi_o}\right|^2 \mathrm{d}\mathcal{S}, \tag{11}$$

where $\int_{\mathcal{S}} \mathrm{d}\mathcal{S}$ represents the surface integral over the bounding surface $\mathcal{S}$ of the oblate spheroid. We state that the system is at the equilibrium when the auxiliary function $||\,\mathrm{d}V_\mathrm{t}/\,\mathrm{d}\eta||_2$ reaches a minimum that is small but usually non-zero (Kong et al. 2012). Another way of searching for the equilibrium solution of rapidly rotating bodies is through the virial criterion (Chandrasekhar 1981; Espinosa Lara & Rieutord 2007), which will also be discussed. Different criteria will lead to slightly different shapes of a rotating body.

The main object of our analysis is to find an exact solution ρ to equation (8) satisfying the boundary condition (11) with an arbitrary shape parameter $\mathcal{E}$ and rotation parameter β. This is because the solution satisfying both the boundary conditions (11) and (10) can be found by performing an appropriate iterative process (Kong et al. 2012).

3 THE EXACT SOLUTION

An exact solution ρ of equation (8) subject to condition (11) can be expressed in the form

$$\rho(\boldsymbol{r}') = f^3\beta\int_0^{\xi_0}\int_{-1}^{+1}\int_0^{2\pi}(\xi^2+\eta^2)G(\boldsymbol{r};\boldsymbol{r}')\,\mathrm{d}\phi\,\mathrm{d}\eta\,\mathrm{d}\xi, \tag{12}$$

where $G(\boldsymbol{r};\boldsymbol{r}') = G(\xi,\eta,\phi;\xi',\eta',\phi')$ denotes the Green function that satisfies the equation

$$\frac{\partial}{\partial \xi}\left[\left(1+\xi^2\right)\frac{\partial G}{\partial \xi}\right] + \frac{\partial}{\partial \eta}\left[\left(1-\eta^2\right)\frac{\partial G}{\partial \eta}\right] + \left[\frac{\xi^2+\eta^2}{(1+\xi^2)(1-\eta^2)}\frac{\partial^2 G}{\partial \phi^2}\right] + c^2\left(\xi^2+\eta^2\right)G = \frac{1}{f}\,\delta(\xi-\xi')\delta(\eta-\eta')\delta(\phi-\phi'), \tag{13}$$

where $\delta(x)$ denotes the standard delta function, subject to the boundary condition

$$G(\xi=\xi_0,\eta,\phi;\xi',\eta',\phi') = 0. \tag{14}$$

The main task of our analysis is then to derive the Green function G for the interior domain $\mathcal{D}$ of a rapidly rotating polytrope satisfying both (13) and (14).

We begin the analysis from the general Green's function G that satisfies only (13)

$$G(\boldsymbol{r};\boldsymbol{r}') = \frac{\mathrm{e}^{i\sqrt{\alpha}|r-r'|}}{|\boldsymbol{r}-\boldsymbol{r}'|},$$

which can be expanded in terms of the oblate spheroidal radial function R_{mn} and the oblate spheroidal angle function S_{mn} (Flammer 1957):

$$G(\boldsymbol{r};\boldsymbol{r}') = \sum_{n=0}^{\infty}\sum_{m=0}^{n} A_{mn}S_{mn}(\eta)S_{mn}(\eta')\cos m(\phi-\phi') \begin{cases} R^{(1)}_{mn}(i\xi)R^{(3)}_{mn}(i\xi') & \text{when } \xi<\xi', \\ R^{(1)}_{mn}(i\xi')R^{(3)}_{mn}(i\xi) & \text{when } \xi>\xi', \end{cases} \tag{15}$$

where A_{mn} are the complex coefficients to be determined, $i = \sqrt{-1}$, m is the azimuthal wavenumber, n is the angle wavenumber, $R^{(1)}_{mn}(i\xi)$ denotes the spheroidal radial function of the first kind while $R^{(3)}_{mn}(i\xi)$ is the spheroidal radial function of the third kind. We need to define the spheroidal angle function $S_{mn}(\xi)$ and the spheroidal radial functions $R^1_{mn}(i\xi)$ and $R^3_{mn}(i\xi)$ in the expansion (15). First, the spheroidal angle function $S_{mn}(\eta)$ is the solution of the ordinary differential equation

$$0 = \frac{\mathrm{d}}{\mathrm{d}\eta}\left[(1-\eta^2)\frac{\mathrm{d}}{\mathrm{d}\eta}S_{mn}(\eta)\right] + \left(\lambda_{mn} + c^2\eta^2 - \frac{m^2}{1-\eta^2}\right)S_{mn}(\eta), \tag{16}$$

which defines an eigenvalue problem in the interval $-1 \le \eta \le 1$ with the real and generally non-integer eigenvalues λ_{mn}. Equation (16) is derived using the standard procedure of separation of variables from the homogeneous part of equation (13). As discussed by Flammer (1957), the angle function $S_{mn}(\eta)$ can be expressed in terms of an infinite sum of associated Legendre functions P^m_l in the form

$$S_{mn}(\eta) = {\sum_{r=0,1}^{\infty}}' d^{mn}_r P^m_{m+r}(\eta), \tag{17}$$

where the symbol $\sum'$ means that the summation starts from $r = 0$ over even subscripts if $(n-m)$ is even but it begins from $r = 1$ over odd subscripts if $(n-m)$ is odd. The expansion coefficients d_r^{mn} are determined by the recurrence relation

$$-\frac{(2m+r+2)(2m+r+1)c^2}{(2m+2r+3)(2m+2r+5)}d_{r+2}^{mn}+\left[(m+r)(m+r+1)\right.$$
$$\left.-\lambda_{mn}-\frac{2(m+r)(m+r+1)-2m^2-1}{(2m+2r-1)(2m+2r+3)}c^2\right]d_r^{mn}$$
$$-\frac{r(r-1)c^2}{(2m+2r-3)(2m+2r-1)}d_{r-2}^{mn}=0 \quad (18)$$

for $r \geq 2$. In (18), λ_{mn} is the eigenvalue given by a solution of the transcendental equation

$$\gamma_r^m-\lambda_{mn}-\cfrac{\beta_r^m}{\gamma_{r-2}^m-\lambda_{mn}-\cfrac{\beta_{r-2}^m}{\gamma_{r-4}^m-\lambda_{mn}\cdots}}$$
$$=\cfrac{\beta_{r+2}^m}{\gamma_{r+2}^m-\lambda_{mn}-\cfrac{\beta_{r+4}^m}{\gamma_{r+4}^m-\lambda_{mn}-\cfrac{\beta_{r+6}^m}{\gamma_{r+6}^m-\lambda_{mn}\cdots}}}, \quad (19)$$

with γ_r^m defined as

$$\gamma_r^m=(m+r)(m+r+1)-\frac{c^2}{2}$$
$$\times\left[1-\frac{4m^2-1}{(2m+2r-1)(2m+2r+3)}\right], \quad \text{when} \quad r\geq 0,$$

and β_r^m being

$$\beta_r^m=\frac{r(r-1)(2m+r)(2m+r-1)c^4}{(2m+2r-1)^2(2m+2r-3)(2m+2r+1)},$$

when $r \geq 2$. While the continued fraction on the left side of equation (19) is finite, the right-hand side contains an infinitely continued fraction. The value of d_0^{mn} or d_1^{mn} required in the recurrence relation (18) is chosen such that

$$\int_{-1}^{1}S_{mn}(\eta)S_{mn'}(\eta)\,\mathrm{d}\eta=\delta_{nn'}.$$

We have adopted an accurate method proposed by Van Buren & Boisvert (2002, 2004) for computing the eigenvalues λ_{mn} and the spheroidal angle function $S_{mn}(\eta)$. Secondly, the spheroidal radial function $R_{mn}(i\xi)$ represents a solution of the ordinary differential equation

$$0=\frac{\mathrm{d}}{\mathrm{d}\xi}\left[(1+\xi^2)\frac{\mathrm{d}}{\mathrm{d}\xi}R_{mn}(i\xi)\right]$$
$$-\left(\lambda_{mn}-c^2\xi^2-\frac{m^2}{1+\xi^2}\right)R_{mn}(i\xi). \quad (20)$$

Equation (20) is also derived using the standard procedure of separation of variables from the homogeneous part of equation (13). There exist two different kinds of spheroidal radial functions satisfying equation (20), denoted by $R_{mn}^1(i\xi)$ and $R_{mn}^2(i\xi)$, that can be expressed as (Flammer 1957)

$$R_{mn}^{(1)}(i\xi)=\frac{1}{\sum_{r=0,1}^{\infty}{}'d_r^{mn}\frac{(2m+r)!}{r!}}\left(\frac{1+\xi^2}{\xi^2}\right)^{(1/2)m}$$
$$\times\sum_{r=0,1}^{\infty}{}'i^{r+m-n}d_r^{mn}\frac{(2m+r)!}{r!}j_{m+r}(ci\xi),$$

$$R_{mn}^{(2)}(i\xi)=\frac{1}{\sum_{r=0,1}^{\infty}{}'d_r^{mn}\frac{(2m+r)!}{r!}}\left(\frac{1+\xi^2}{\xi^2}\right)^{(1/2)m}$$
$$\times\sum_{r=0,1}^{\infty}{}'i^{r+m-n}d_r^{mn}\frac{(2m+r)!}{r!}n_{m+r}(ci\xi),$$

where $j_l(ci\xi)$ and $n_l(ci\xi)$ are the modified spherical Bessel and Neumann functions. Thirdly, the spheroidal radial function of the third kind $R^{(3)}(i\xi)$ is obtained simply through a combination of $R_{mn}^{(1)}$ and $R_{mn}^{(2)}$

$$R^{(3)}(i\xi)=R^{(1)}(i\xi)+iR^{(2)}(i\xi). \quad (21)$$

The Wronskian of $R_{mn}^{(1)}$ and $R_{mn}^{(2)}$ is

$$W\left[R_{mn}^{(1)},R_{mn}^{(2)}\right]=R_{mn}^{(1)}(i\xi)\frac{\mathrm{d}R_{mn}^{(2)}}{\mathrm{d}\xi}-R_{mn}^{(2)}(i\xi)\frac{\mathrm{d}R_{mn}^{(1)}}{\mathrm{d}\xi}$$
$$=-\frac{1}{c(1+\xi^2)}, \quad (22)$$

which is always non-zero within the oblate spheroid and will be needed in deriving the Green function.

By substituting the expansion (15) into (13), integrating the resulting equation with respect to ξ over the infinitesimal interval $[\xi'-0,\xi'+0]$ and making use of the Wronskian (22), we obtain the coefficients A_{mn} given by

$$A_{mn}=-\frac{i\sqrt{\alpha}(2-\delta_{0m})}{2\pi}. \quad (23)$$

In other words, the general Green's function G is of the form

$$G(\boldsymbol{r};\boldsymbol{r}')=\sum_{n=0}^{\infty}\sum_{m=0}^{n}\frac{-i\sqrt{\alpha}(2-\delta_{0m})}{2\pi}S_{mn}(\eta)S_{mn}(\eta')$$
$$\begin{cases}\cos m(\phi-\phi')R_{mn}^{(1)}(i\xi)R_{mn}^{(3)}(i\xi') & \text{when} \quad \xi<\xi',\\ \cos m(\phi-\phi')R_{mn}^{(1)}(i\xi')R_{mn}^{(3)}(i\xi) & \text{when} \quad \xi>\xi'.\end{cases}$$

By assuming that a rotating polytrope is axially symmetric in hydrostatic equilibrium, only the axisymmetric component marked by $m=0$ is needed, leading to the simplified Green's function

$$G(\xi,\eta;\xi',\eta')=\sum_{n=0}^{\infty}\frac{-i\sqrt{\alpha}}{2\pi}S_{0n}(\eta)S_{0n}(\eta')$$
$$\times\begin{cases}R_{0n}^{(1)}(i\xi)R_{0n}^{(3)}(i\xi') & \text{when} \quad \xi<\xi',\\ R_{0n}^{(1)}(i\xi')R_{0n}^{(3)}(i\xi) & \text{when} \quad \xi>\xi',\end{cases} \quad (24)$$

that satisfies equation (13) but not the boundary condition (14).

In order that the Green function G satisfies both equations (13) and (14), the expression (24) needs to be modified by adding an extra term involving the boundary condition (14), which yields

$$G(\xi,\eta;\xi',\eta')=-\frac{i\sqrt{\alpha}}{2\pi}\sum_{n=0}^{\infty}S_{0n}(\eta)S_{0n}(\eta')$$
$$\times\left[R_{0n}^{(1)}(i\xi)R_{0n}^{(3)}(i\xi')-\frac{R_{0n}^{(3)}(i\xi_0)}{R_{0n}^{(1)}(i\xi_0)}R_{0n}^{(1)}(i\xi)R_{0n}^{(1)}(i\xi')\right]$$

Table 1. Several parameters used in our calculation for a generic object whose values are guided by the observations of the rapidly rotating star α Eridani.

Parameter	Value	Source
M	9.7466×10^{30} kg	Levenhagen & Leister (2006)
$V_{\rm eq} \sin i$	225 km s^{-1}	Slettebak (1982)
i	65 deg	Carciofi et al. (2008)
$R_{\rm e}$	8.3520×10^{9}m	Domiciano de Souza et al. (2003)
Ω	2.9725×10^{-5} s^{-1}	Derived

when $\xi < \xi'$, and

$$G(\xi,\eta;\xi',\eta') = -\frac{i\sqrt{\alpha}}{2\pi}\sum_{n=0}^{\infty} S_{0n}(\eta)S_{0n}(\eta')$$

$$\times\left[R^{(1)}_{0n}(i\xi')R^{(3)}_{0n}(i\xi) - \frac{R^{(3)}_{0n}(i\xi_0)}{R^{(1)}_{0n}(i\xi_0)}R^{(1)}_{0n}(i\xi')R^{(1)}_{0n}(i\xi))\right]$$

when $\xi > \xi'$. The above expressions for G can be further simplified by taking their real part

$$G(\xi,\eta;\xi',\eta') = \frac{\sqrt{\alpha}}{2\pi}\sum_{n=0}^{\infty} S_{0n}(\eta)S_{0n}(\eta')$$

$$\times\left[R^{(1)}_{0n}(i\xi)R^{(2)}_{0n}(i\xi') - \frac{R^{(2)}_{0n}(i\xi_0)}{R^{(1)}_{0n}(i\xi_0)}R^{(1)}_{0n}(i\xi)R^{(1)}_{0n}(i\xi')\right] \tag{25}$$

when $\xi < \xi'$, and

$$G(\xi,\eta;\xi',\eta') = \frac{\sqrt{\alpha}}{2\pi}\sum_{n=0}^{\infty} S_{0n}(\eta)S_{0n}(\eta')$$

$$\times\left[R^{(1)}_{0n}(i\xi')R^{(2)}_{0n}(i\xi) - \frac{R^{(2)}_{0n}(i\xi_0)}{R^{(1)}_{0n}(i\xi_0)}R^{(1)}_{0n}(i\xi')R^{(1)}_{0n}(i\xi))\right] \tag{26}$$

when $\xi > \xi'$. Expressions (25) and (26) represent the Green function that is derived for the first time for the interior of an oblate spheroid of arbitrary eccentricity satisfying both (13) and (14). With the Green function given by equations (25) and (26), the exact solution ρ for an axisymmetric rotating polytrope is

$$\rho = 2\pi f^3\beta\int_0^{\xi_0}\int_{-1}^{+1}(\xi^2+\eta^2)G(\xi,\eta;\xi',\eta')\,\mathrm{d}\eta\,\mathrm{d}\xi, \tag{27}$$

where the two-dimensional integration can be readily performed.

4 THE EXACT AND NUMERICAL SOLUTIONS FOR AN OBJECT WITH CHARACTERISTICS SIMILAR TO α ERIDANI

We now apply the exact solution (27) to a generic object whose parameter values are guided by the observations of the rapidly rotating star α Eridani, the most oblate body observed so far (Carciofi et al. 2008; Domiciano de Souza et al. 2014). It should be noted that massive star α Eridani is likely to be fully radiative and, hence, the size of its polytropic index would be larger than unity. Nevertheless, we chose parameter values for a generic object guided by the observations of the rapidly rotating star α Eridani as a simple example for the application of our new exact solution method. The parameter values used in our calculation are given in Table 1, where $V_{\rm eq}$ denotes the rotational speed projected on the equator of the object, i is the inclination angle between the rotational axis and the line-of-sight and Ω is the angular velocity of rotation. The

Table 2. Several values of the accurate eigenvalues λ_{0n} of the spheroidal wavefunction computed, following the work of Van Buren & Boisvert (2002, 2004), for the oblate spheroid of eccentricity $\mathcal{E} =$ 0.745 356 with $c = 3.053\,955$.

n	The eigenvalues λ_{0n}
0	−4.547 720
1	−4.129 856
2	2.147 617
3	7.457 724
4	15.430 830
5	25.394 476
6	37.376 726
7	51.366 121
8	67.359 275
9	85.354 588

perturbation theories based on an expansion using a small rotation parameter around sphericity or the discontinuous numerical method using concentric Maclaurin spheroids are inapplicable to the present problem because of the large derived eccentricity $\mathcal{E} = 0.745\,356$. In our calculation, we regard the equatorial radius $R_{\rm e}$, the mass M and the rotation rate Ω as given while the density $\rho(\xi,\eta)$, the pressure $p(\xi,\eta)$ and the sizes of K and $\mathcal{E}$ are to be determined by an iterative method (Kong et al. 2012).

With the eccentricity $\mathcal{E} = 0.745\,356$, we first compute the eigenvalues λ_{0n} and the corresponding spheroidal wave functions $S_{0n}(\eta)$ and $R_{0n}(\xi)$ required in the Green function G (25) and (26). Several typical values of the eigenvalue λ_{0n} for $\mathcal{E} = 0.745\,356$ are listed in Table 2. With the availability of the Green function G together with the parameters provided in Table 2, we can, starting from an initial trial value $K = K_0$, compute an exact solution (27) for the density $\rho(\xi,\eta)$ that satisfies the boundary condition (11) on the bounding surface $\mathcal{S}$ of the body. In general, the solution $\rho(\xi,\eta)$ at $K = K_0$ is inconsistent with the equilibrium condition (10) on $\mathcal{S}$. It is then straightforward to repeat this process with different values of K, through a proper iterative scheme, to determine a particular value $K = K_\alpha$ such that the hydrostatic equilibrium condition (10) is also satisfied. The requirement of hydrostatic equilibrium (10) yields $K = K_\alpha = 1.7500 \times 10^9$ Pa m^6 kg^{-2} which, along with the parameters in Table 1, yields $\alpha = 16.787\,95$ and $\beta = 2.114\,48$ and $c = 3.053\,955$. Fig. 1(a) depicts the two-dimensional density distribution $\rho(\eta,\xi)$ in hydrostatic equilibrium for a model of the rapidly rotating object computed from the exact solution (27). When the internal density distribution $\rho(\eta,\xi)$ is available, we can compute its external gravitational potential $V_{\rm g}$ in the dimensionless form

$$V_{\rm g}(\boldsymbol{r}) = -\int_{\mathcal{D}}\frac{\rho(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\quad\text{for}\quad|\boldsymbol{r}|\geq 1, \tag{28}$$

which, using spherical polar coordinates (r,θ,ϕ) with $\theta = 0$ at the axis of rotation, can be further expanded in terms of the zonal gravitational coefficients J_n,

$$V_{\rm g} = -\frac{1}{r}\left[1-\sum_{n=2}^{\infty}J_n\left(\frac{1}{r}\right)^n P_n(\cos\theta)\right]\quad\text{for}\quad r\geq 1, \tag{29}$$

where P_n is normalized such that

$$\int_0^{\pi}P_n^2(\cos\theta)\,\sin\theta\,\mathrm{d}\theta = \frac{2}{2n+1}.$$

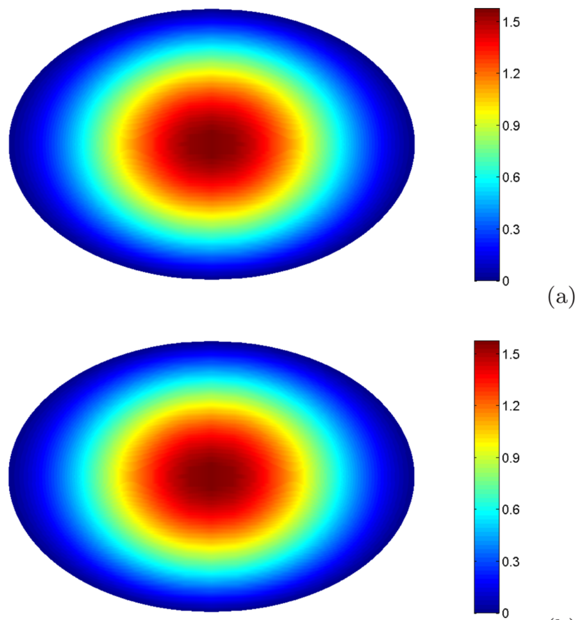

Figure 1. The density distribution $\rho(\xi, \eta)$ in a meridional plane for a polytropic object with characteristics similar to the rapidly rotating star α Eridani, representing the solution of (8) for $\mathcal{E} = 0.745\,356$, $\alpha = 16.787\,95$, $\beta = 2.114\,48$ and $c = 3.053\,955$: (a) the density distribution ρ obtained from the exact solution (27) and (b) the density distribution ρ from the finite-element solution.

Table 3. Gravitational zonal coefficients J_n in the expansion (29) obtained from the exact solution, denoted as $(J_n)_{\rm exact}$, and from the finite-element solution, denoted as $(J_n)_{\rm num}$, for a polytropic model of a highly rotating object. The solution ρ for equation (8) used in evaluating J_n in (30) is obtained using $\mathcal{E} = 0.745\,356$, $\alpha = 16.787\,95$, $\beta = 2.114\,48$ and $c = 3.053\,955$.

n	$(J_n)_{\rm exact} \times 10^6$	$(J_n)_{\rm num} \times 10^6$
2	57 385.71	57 385.72
4	−8427.06	−8427.10
6	1788.66	1788.58
8	−467.38	−467.54
10	140.23	139.95

The zonal gravitational coefficients J_n in equation (29) are computed, via the exact solution $\rho(\eta, \xi)$ given by equation (27), by integrating

$$J_n = \frac{2n+1}{2} \int_0^{\pi} \left[-\int_{\mathcal{D}} \frac{\rho(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'}{|\boldsymbol{r} - \boldsymbol{r}'|} \right]_{|\boldsymbol{r}|=1} P_n(\cos\theta)\ \sin\theta\, \mathrm{d}\theta. \quad (30)$$

Several values of the zonal gravitational coefficients J_n, up to $n = 10$, computed from the exact solution through (30), are listed in Table 3 for a polytropic object with properties similar to the rapidly rotating star α Eridani.

Another objective of the present investigation is to check the accuracy of the numerical solution that is based on a three-dimensional finite-element method (Kong et al. 2013). Of course, the finite-element solution can also be used to validate the exact solution given by equation (27). Since the detail of the finite-element method is discussed in Kong et al. (2013), only a brief description is presented here.

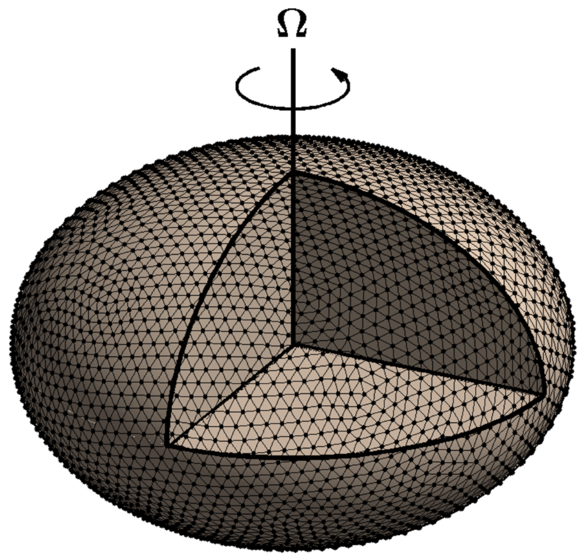

Figure 2. Sketch of the three-dimensional tetrahedral mesh for an oblate spheroid of eccentricity $\mathcal{E} = 0.745\,356$ containing the 30 000 tetrahedral elements. In our actual numerical computation, 32×10^6 tetrahedral elements are used.

We first construct a three-dimensional finite-element mesh by making a tetrahedralization of an oblate spheroid of eccentricity $\mathcal{E} = 0.745\,356$. A sketch of the tetrahedral finite-element mesh for the oblate spheroid is illustrated in Fig. 2. In comparison to a spectral or finite difference method, the finite-element method is free of the pole and central numerical singularities. We use a Galerkin weighted residual approach in the finite-element formulation of equation (8). Although only 30 000 tetrahedral elements are displayed in Fig. 2, 32×10^6 tetrahedral elements are used in our actual numerical computation for the result reported in this paper. Fig. 1(b) shows the two-dimensional hydrostatic density distribution $\rho(\eta, \xi)$ for a polytropic model of the highly rotating object computed from the three-dimensional finite-element method. Several values of the zonal gravitational coefficients $(J_n)_{\rm num}$, up to $n = 10$, computed from the finite-element solution, are listed in Table 3. There are no noticeable differences between the exact and numerical solutions depicted in Fig. 1 and, moreover, the values of the zonal coefficients in Table 3, $(J_n)_{\rm exact}$ and $(J_n)_{\rm num}$, show an excellent agreement with about 0.001 per cent discrepancy for the leading coefficients. Furthermore, we may introduce Δ defined as

$$\Delta = \frac{||\rho_{\rm exact}(\boldsymbol{r}) - \rho_{\rm num}(\boldsymbol{r})||_2}{||\rho_{\rm exact}(\boldsymbol{r})||_2},$$

where

$$||g(\boldsymbol{r})||_2 = \left[\int_{\mathcal{D}} |g(\boldsymbol{r})|^2 \mathrm{d}^3\boldsymbol{r} \right]^{1/2},$$

to measure the difference between the exact and numerical solution. It is found that $\Delta = 1.50 \times 10^{-5}$, which is consistent with the values of the gravitational zonal coefficients $(J_n)_{\rm exact}$ and $(J_n)_{\rm num}$ given in Table 3.

5 AN EQUILIBRIUM SOLUTION BASED ON THE VIRIAL CRITERION

Equations (2)–(4) can be written in the dimensionless form

$$0 = -\nabla p - \rho\nabla V_{\rm g} - \rho\nabla V_{\rm c}, \quad (31)$$

where

$$p = \frac{2\pi}{\alpha}\rho^2,$$

$$V_{\rm g} = -\int_{\mathcal{D}} \frac{\rho(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|},$$

$$V_{\rm c} = -\frac{1}{2}\frac{2\pi\beta}{\alpha}s^2,$$

and where s is the distance from the rotation axis. At the hydrostatic equilibrium, the virial criterion (Chandrasekhar 1981; Eriguchi & Mueller 1985; Espinosa Lara & Rieutord 2007),

$$0 = \int_{\mathcal{D}} -\boldsymbol{r}\cdot\nabla p\mathrm{d}^3\boldsymbol{r} + \int_{\mathcal{D}} \rho(\boldsymbol{r})\boldsymbol{r}\cdot\nabla\left(\int_{\mathcal{D}}\frac{\rho(\boldsymbol{r}')\mathrm{d}^3\boldsymbol{r}'}{|\boldsymbol{r}-\boldsymbol{r}'|}\right)\mathrm{d}^3\boldsymbol{r} + \frac{\pi\beta}{\alpha}\int_{\mathcal{D}}\rho\boldsymbol{r}\cdot\nabla s^2\mathrm{d}^3\boldsymbol{r},$$

must be satisfied, where $\mathcal{D}$ denotes the three-dimensional non-spherical domain of a rapidly rotating gaseous body. Each integral in the virial identity can be further simplified, yielding

$$0 = \frac{6\pi}{\alpha}\int_{\mathcal{D}}\rho^2\mathrm{d}^3\boldsymbol{r} + \frac{1}{2}\int_{\mathcal{D}}\rho V_{\rm g}\mathrm{d}^3\boldsymbol{r} + \frac{2\pi\beta}{\alpha}\int_{\mathcal{D}}\rho s^2\mathrm{d}^3\boldsymbol{r}, \quad (32)$$

where we have used the boundary condition required at the bounding surface of the body. It is expected that each integral in equation (32) is of order unity.

With the equilibrium solution given by $\rho(\xi,\eta)$, $\mathcal{E} = 0.751\,600$, $\alpha = 16.787\,95$ and $\beta = 2.114\,48$ derived from the virial criterion, we can readily compute each integral in equation (32):

$$\frac{6\pi}{\alpha}\int_{\mathcal{D}}\rho^2\mathrm{d}^3\boldsymbol{r} = -1.331\,545\,3005,$$

$$\frac{1}{2}\int_{\mathcal{D}}\rho V_{\rm g}\mathrm{d}^3\boldsymbol{r} = 0.231\,643\,799\,15,$$

$$\frac{2\pi\beta}{\alpha}\int_{\mathcal{D}}\rho s^2\mathrm{d}^3\boldsymbol{r} = 1.099\,964\,8116,$$

where $-1.331\,545\,3005 + 0.231\,643\,799\,15 + 1.099\,964\,8116 = 6.3310\times10^{-5}$. With a more accurate solution given by $\rho(\xi,\eta)$, $\mathcal{E} = 0.751\,688\,72$, $\alpha = 16.787\,95$ and $\beta = 2.114\,48$, each integral in equation (32) becomes

$$\frac{6\pi}{\alpha}\int_{\mathcal{D}}\rho^2\mathrm{d}^3\boldsymbol{r} = -1.333\,367\,4317,$$

$$\frac{1}{2}\int_{\mathcal{D}}\rho V_{\rm g}\mathrm{d}^3\boldsymbol{r} = 0.231\,803\,621\,61,$$

$$\frac{2\pi\beta}{\alpha}\int_{\mathcal{D}}\rho s^2\mathrm{d}^3\boldsymbol{r} = 1.101\,563\,8106,$$

which give rise to $-1.333\,367\,4317 + 0.231\,803\,621\,61 + 1.101\,563\,8106 = 5\times10^{-10}$. The satisfaction of the virial identity with a very high accuracy $\times\,10^{-10}$ at the hydrostatic equilibrium not only indicates the accuracy of the solution but also reconfirms that an oblate spheroid provides an excellent approximation to the shape of rotating gaseous polytropes with index unity (Roberts 1962).

6 SUMMARY AND REMARKS

The present investigation derives the first exact solution for an arbitrarily rotating gaseous polytrope with index unity in hydrostatic equilibrium, which represents important progress on this classical problem since Chandrasekhar (1933) derived the first approximate solution for slowly rotating polytropic planets and stars. We apply the exact solution, as an example, to a polytropic object with properties similar to the rotating star α Eridani, and compute its internal structure and gravitational field from the observed rotation rate and size. Comparison between the exact solution and the three-dimensional finite-element solution shows an excellent agreement with about 0.001 per cent discrepancy. The different criteria, given by equations (11) and (32), lead to slightly different shapes of the object marked by $\Delta\mathcal{E} = 0.006$, representing less than 1 per cent change. In the present problem, we have solved the inhomogeneous partial differential equation (8) whose right-hand side is constant, corresponding to the shape and physical structure of rotating polytropic planets and stars in hydrostatic equilibrium. With little modification, the analytical method proposed in this study can also be applied to other astrophysical or planetary physical problems that are governed by an inhomogeneous partial differential equation whose right-hand side is a function of the spatial variables ξ and η.

Finally, we provide some remarks on the state of rigid-body rotation in hydrostatic equilibrium which has usually been assumed in the figure theory of rotating gaseous bodies (see for example, Chandrasekhar 1933; Roberts 1962; Hubbard 2013). It is important that the realistic fluid in rotating gaseous planets or stars is slightly viscous: the viscosity ν may be small but $\nu \neq 0$. A state of the hydrostatic equilibrium may be regarded as the final state of a force-free initial value problem (Greenspan 1968). Suppose that a gaseous planet or star is rotating with uniform angular velocity $\Omega\hat{\boldsymbol{z}}$ along with an arbitrary but physically acceptable initial velocity $\widehat{\boldsymbol{u}}_0^*$ and density profile ρ_0^*. We may write, in an inertial frame of reference, the total velocity $\widehat{\boldsymbol{u}}(\boldsymbol{r}^*,t)$ and the total density $\widehat{\rho}(\boldsymbol{r}^*,t)$ in the tensor notation

$$\widehat{u}_i(\boldsymbol{r}^*,t)^* = u_i^*(\boldsymbol{r}^*) + v_i^*(\boldsymbol{r}^*,t) \quad\text{and}\quad \widehat{\rho}(\boldsymbol{r}^*,t) = \rho^*(\boldsymbol{r}^*) + \widetilde{\rho}^*(\boldsymbol{r}^*,t),$$

where $\boldsymbol{u}^* = \Omega\hat{\boldsymbol{z}}\times\boldsymbol{r}^*$ while $\boldsymbol{v}^*(\boldsymbol{r}^*,t)$ and $\widetilde{\rho}^*(\boldsymbol{r}^*,t)$ denote the time-dependent parts of the solution for the initial value problem. In the inertial frame of reference using the tensor notation, the evolution of the initial value problem towards the state of the hydrostatic equilibrium is governed by the equations

$$\frac{\partial v_i^*}{\partial t^*} + (u_j^*+v_j^*)\frac{\partial(u_i^*+v_i^*)}{\partial x_j^*} = -\frac{1}{(\rho^*+\widetilde{\rho}^*)}\frac{\partial p^*}{\partial x_i^*} - \frac{\partial V_{\rm g}^*}{\partial x_i^*} + \nu\left\{\frac{\partial^2(u_i^*+v_i^*)}{\partial x_j^*\partial x_j^*} + \frac{2}{3}\frac{\partial}{\partial x_i^*}\left[\frac{\partial(u_k^*+v_k^*)}{\partial x_k^*}\right]\right\}, \quad (33)$$

$$\frac{\partial^2 V_{\rm g}^*}{\partial x_j^*\partial x_j^*} = 4\pi G(\rho^*+\widetilde{\rho}^*), \quad (34)$$

$$p^* = K(\rho^*+\widetilde{\rho}^*)^2, \quad (35)$$

$$\frac{\partial\widetilde{\rho}^*}{\partial t^*} = \frac{\partial}{\partial x_j^*}\left[(\rho^*+\widetilde{\rho}^*)(u_j^*+v_j^*)\right]. \quad (36)$$

It is well known (Greenspan 1968) that the time-dependent part of the solution $v_i^*(\boldsymbol{r}^*, t)$ and $\widetilde{\rho}^*(\boldsymbol{r}^*, t)$, starting from an arbitrary initial state, will always approach zero after a sufficiently long time, leading to the equations given by (2)–(4). In other words, the state of hydrostatic equilibrium is always stationary in the inertial frame of reference and marked by $v_i^*(\boldsymbol{r}^*, t) = 0$ and $\widetilde{\rho}^*(\boldsymbol{r}^*, t) = 0$ with

$$\nu \neq 0 \quad \text{and} \quad \nu \left\{ \frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} + \frac{2}{3} \frac{\partial}{\partial x_i^*} \left[\frac{\partial u_k^*}{\partial x_k^*} \right] \right\} = 0.$$

This means that in writing the governing equations (2)–(4) for hydrostatic equilibrium of a rotating gaseous polytrope, we have implicitly used the fact that the fluid is viscous ($\nu \neq 0$) and, hence, that rigid-body rotation represents the only state of hydrostatic equilibrium. The viscous term is not included in equation (2) simply because it vanishes exactly in hydrostatic equilibrium.

ACKNOWLEDGEMENTS

DK is supported by the NSFC under grant 11473014, KZ is supported by the UK STFC under grant ST/J001627/1 and the HKRGC grant(Project 14306814), and GS is supported by the National Science Foundation under grant NSF AST-0909206. The computation made use of the high-performance computing resources in the Core Facility for Advanced Research Computing at Shanghai Astronomical Observatory, Chinese Academy of Sciences. We would like to thank Dr Van Buren for helpful discussions about the spheroidal wavefunction.

REFERENCES

Asano S., Yamamoto G., 1975, Appl. Opt., 14, 29
Carciofi A.C., Domiciano de Souza A., Magalhes A. M., Bjorkman J. E., Vakili F., 2008, ApJ, 676, L41
Chandrasekhar S., 1933, MNRAS, 93, 390
Chandrasekhar S., 1981, Hydrodynamic and Hydromagnetic Stability. Dover Press, New York
Chu L. J., Stratton J. A., 1941, J. Math. Phys., 20, 259
Dintrans B., Ouyed R., 2001, A&A, 375, L47
Domiciano de Souza A., Kervella P., Jankov S., Abe L., Vakili F., di Folco E., Paresce F., 2003, A&A, 407, L47
Domiciano de Souza A. et al., 2014, A&A, 569, A10
Eriguchi Y., Mueller E., 1985, A&A, 146, 260
Espinosa Lara F., Rieutord M., 2007, A&A, 470, 1013
Flammer C., 1957, Spheroidal Wave Functions. Standform Univ. Press, Stanford, CA
Greenspan H. P., 1968, The Theory of Rotating Fluids. Cambridge Univ. Press, Cambridge
Helled R., Schubert G., Anderson J. D., 2009. Planet. Space Sci., 57, 1467
Horedt G. P., 2004, Polytropes. Applications in Astrophysics and Related Fields. Kluwer, Dordrecht
Hubbard W. B., 1973, Annu. Rev. Earth Planet. Sci., 1, 85
Hubbard W. B., 1999, Icarus, 137, 357
Hubbard W. B., 2013, ApJ, 768, 43
James R. A., 1964, ApJ, 140, 552
Kaspi Y., Hubbard W. B., Showman A. P., Flierl G. R., 2010, Geophys. Res. Lett., 37, L01204
Kong D., Zhang K., Schubert G., 2010, J. Geophys. Res., 115, E12003
Kong D., Zhang K., Schubert G., 2012, ApJ, 748, 2012
Kong D., Zhang K., Schubert G., Anderson J., 2013, ApJ, 763, 116
Kong D., Liao X., Zhang K., Schubert G., 2014, MNRAS, 445, L26
Levenhagen R. S., Leister N. V., 2006, MNRAS, 371, 252
Morse P. M., Feshbach H., 1953, Methods of Mathematical Physics. McGraw-Hill, New York
Roberts I. P. H., 1962, ApJ, 136, 1108
Seidelmann P. K. et al., 2007, Celest. Mech. Dyn. Astron., 98, 155
Slettebak A., 1982, ApJS, 50, 55
Spence R. D., Granger S., 1951, J. Acoust. Soc. Am., 23, 701
Stevenson D. J., 1982, Ann. Rev. Earth Planet. Sci., 10, 257
Van Buren A. L., Boisvert J. E., 2002, Q. Appl. Math., 60, 589
Van Buren A. L., Boisvert J. E., 2004, Q. Appl. Math., 62, 493
Zhang K., Liao X., Earnshaw P., 2004, J. Fluid Mech., 504, 1.

This paper has been typeset from a TEX/LATEX file prepared by the author.