A Fault Tolerant Control Allocation Scheme with Output Integral Sliding Modes

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Abstract

In this paper a new fault tolerant control scheme is proposed, where only measured system outputs are assumed to be available. The scheme ensures closed-loop stability throughout the entire closed-loop response of the system even in the presence of certain actuator faults/failures. This is accomplished by incorporating ideas of integral sliding modes, unknown input observers and a fixed control allocation scheme. A rigorous closed-loop stability analysis is undertaken, and in fact a convex representation of the problem is created in order to synthesize the controller and observer gains. The efficacy of the proposed scheme is tested by applying it to a benchmark civil aircraft model.

Keywords: Fault tolerant control (FTC), integral sliding mode (ISM) control, linear matrix inequalities (LMI).

1. Introduction

Fault tolerant control (FTC) systems are designed to be able to handle emergency situations arising from actuator and/or sensor faults, and improve the reliability of the overall system. In the existing literature many control schemes have been proposed to tackle this problem based on different design paradigms, and it remains an open area of research. In most engineering systems, not all states are measurable and therefore output feedback schemes are more desirable. This also applies to FTC systems. In the FTC literature, methods such as $\mathcal{H}_\infty$ control (see for example Ganguli et al. (2002)) and eigenstructure assignment (Duan (2003)) inherently deal with the output feedback situation, and do not require observers for estimating the unmeasured states[1]. Other FTC methods such as the Pseudo Inverse Method (e.g. Konstantopoulos and Antsaklis (1999)) use static output feedback to deal with actuator faults and the model-following approach of Tao and Joshi (2008) exploits an adaptive output feedback framework which does not require state estimation.

In many FTC schemes, a fault detection and isolation (FDI) component is an important part of the overall system and is used to trigger controller reconfiguration. Some FDI schemes have the capability not only to detect faults, but to simultaneously provide estimates of the states – for example Zhang and Jiang (2002) which uses a Kalman filter to estimate the efficiency of the actuators. Papers such as Zhang and Jiang (2002, 2001) (which use Kalman filters and eigenstructure assignment based control design) use an integrated FDI/FTC structure and take advantage of the observer to provide state estimates to use in the control. Similarly Kanev (2004) uses a Kalman filter in conjunction with a finite-horizon MPC formulation (which assumes state availability) in a single optimization approach.

Redundancy is a key element in any FTC system, and exploiting it in an efficient way, in any over-actuated system, is important. Control allocation (CA) is one approach which can effectively manage this redundancy (Boskovic and Mehra, 2002; Harkegard and Glad, 2005). By using CA methods, the virtual control effort produced by the nominal controller can be distributed among the actuators to achieve the desired performance. One of the benefits of using CA methods is that they can be used with other control design techniques to handle faults/-failures: for details see for example (Buffington, 1997).

Sliding mode control (SMC) (Edwards and Spurgeon, 1998; Utkin et al., 1999) has attracted much recent attention in the field of FTC, due to its inherent robustness properties against matched uncertainties. As argued in Alwi et al. (2011), SMC has a natural capability for dealing with faults (i.e. a reduction in the effectiveness) in actuators, but cannot deal directly with total failures. However an appropriate combination of SMC and CA can achieve tolerance to a wide class of total actuator failures (Alwi and Edwards, 2008a; Shtessel et al., 2002). In all these earlier schemes it is assumed that full state information is available for the controller design. In order to eliminate the *reaching phase* associated with traditional SMC techniques, and to ensure a sliding mode throughout the entire closed-loop response of the system, the idea of integral sliding mode (ISM) control was initially proposed in Utkin and Shi (1996). More recently Hamayun et al. (2012) considered the combination of ISM control with CA to accommodate the potential faults/failures associated with the actuators, but the ideas were developed under the assumption that all states are available.

The main contribution of this paper is to relax the assumption

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[1] Although certain $\mathcal{H}_\infty$ problems yield dynamical controllers which can be interpreted in a feedback/observer paradigm.

associated with (Alwi and Edwards, 2008a; Hamayun et al., 2012) that state information is known, and to consider instead the situation where only measured outputs are available. Early work on ISM control (not in the context of FTC) also assumed state information (Utkin and Shi, 1996; Castanos and Fridman, 2006), but this has more recently been extended to the situation where only measured outputs are available (Bejarano et al., 2009, 2007). In (Bejarano et al., 2007), an ISM controller using output information was used to compensate for matched uncertainties, and a hierarchical sliding mode observer was proposed to estimate the states. In (Chang, 2009) the state dependent methods from (Cao and Xu, 2004), were developed into an output feedback framework by introducing a dynamic output dependent sliding surface employing a full order compensator.

In this paper, a new fault tolerant control scheme is proposed for the case when only the system outputs are measured. A full order linear unknown input observer (UIO) is employed to estimate the system states used in the underlying (virtual) controller. In the proposed FTC scheme, actuator faults are modelled using an effectiveness gain matrix, whereas component faults are modelled as parametric uncertainty in the system matrix. The proposed scheme does not attempt to estimate the actuator faults/failures (using an FDI scheme), instead, the robustness properties of the UIO coupled with the ISM are relied upon. Compared to Alwi and Edwards (2008a); Hamayun et al. (2012), a fixed control allocation scheme is used to translate the virtual control signals into physical actuator demands. An LMI synthesis procedure is proposed in order to synthesize the observer gains and the controller parameters in a tractable way. An aircraft benchmark model from the literature is used to investigate the feasibility of the scheme (Edwards et al., 2010).

2. Problem Formulation

Consider an uncertain system with actuator and component faults or failures written as

$$\dot{x}(t) = (A + A^{\delta})x(t) + Bu(t) - BK(t)u(t) \qquad (1)$$

$$y(t) = Cx(t) \qquad (2)$$

where $A \in \mathbb{R}^{n\times n}$ is the state matrix, A^{δ} is parametric uncertainty in the system matrix arising from imprecisely known parameters and possible faults at a component level, $B \in \mathbb{R}^{n\times m}$ is the input distribution matrix and $C \in \mathbb{R}^{p\times n}$ is the output distribution matrix where $p \geq m$. The diagonal weighting matrix $K(t) = diag\{k_1(t), .., k_m(t)\}$, where the scalars $k_1(t), .., k_m(t)$, model the effectiveness level of the actuators. If $k_i(t) = 0$, the corresponding ith actuator is fault free and working perfectly, whereas if $1 > k_i(t) > 0$, an actuator fault is present. The value $k_i(t) = 1$ indicates the ith actuator has completely failed. It is assumed that controlled outputs for the system are given by

$$y_c(t) = C_c x(t) \qquad (3)$$

where $C_c \in \mathbb{R}^{l\times n}$ and $l < m$. This implies there is redundancy in the system in terms of the number of control inputs. This will be exploited in the sequel to achieve fault tolerance. To resolve this redundancy, as in Alwi and Edwards (2008a), it is assumed the input distribution matrix can be partitioned such that

$$B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \qquad (4)$$

where $B_1 \in \mathbb{R}^{(n-l)\times m}$ and $B_2 \in \mathbb{R}^{l\times m}$ is of rank $l < m$. After partitioning, by appropriate scaling of the last l states via a change in the state-space coordinates, it can be ensured $B_2B_2^T = I_l$, which in turn implies $\|B_2\| = 1$. As argued in Alwi and Edwards (2008a), it is assumed that $\|B_1\| \ll \|B_2\|$, so that the control action predominantly acts in the last l channels of the system. Using (4), the system in (1) can be written as

$$\dot{x}(t) = (A + A^{\delta})x(t) + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \underbrace{(I - K(t))}_{W(t)} u(t) \qquad (5)$$

where the matrix $W(t)$ is diagonal and its diagonal elements $w_i(t)$ satisfy $0 \leq w_i(t) \leq 1$.

The objective of the paper is to develop a control scheme, based only on output measurements, which can maintain closed loop stability in the face of a class of actuator faults and failures. The physical control law $u(t)$ is realized by a so-called ‘fixed’ control allocation scheme of the form

$$u(t) = B_2^T \nu(t) \qquad (6)$$

where $\nu(t) \in \mathbb{R}^l$ is the ‘virtual control’ effort produced by the control law, which will be described in the sequel.

Remark 1: The control allocation structure in (6) is different to the ones in Alwi and Edwards (2008a) and Hamayun et al. (2012), since both require $W(t)$ (or a good estimate of $W(t)$ to be known). The fixed CA/ISM scheme developed in this paper will be *independent* of $W(t)$ and will not require an FDI scheme. By using (6), equation (5) can be written as

$$\dot{x}(t) = (A + A^{\delta})x(t) + \underbrace{\begin{bmatrix} B_1W(t)B_2^T \\ B_2W(t)B_2^T \end{bmatrix}}_{B_w(t)} \nu(t) \qquad (7)$$

In the nominal case when there is no fault (i.e. when $W(t) = I$ and $A^{\delta} = 0$), equation (7) simplifies to

$$\dot{x}(t) = Ax(t) + \underbrace{\begin{bmatrix} B_1B_2^T \\ I_l \end{bmatrix}}_{B_\nu} \nu(t) \qquad (8)$$

because $B_2B_2^T = I_l$ by design. The following assumption will be made and used in the remainder of the paper.

$A1$**:** The pair (A, B_ν) is controllable.

3. ISM Controller Design

An integral sliding mode strategy (Utkin and Shi, 1996) will be adopted for synthesizing the virtual control signal $\nu(t)$. The virtual control signal $\nu(t)$ will use estimated states $\hat{x}(t)$, obtained from an observer, because it is assumed only outputs are measured. As a first step, an output and state-estimate dependent integral switching function is proposed of the form

$$\sigma(t) = Gy(t) - Gy(0) + \int_0^t F\hat{x}(\tau)d\tau \tag{9}$$

where $G \in \mathbb{R}^{l\times p}$ and $F \in \mathbb{R}^{l\times n}$ are design feedback gains, selected to specify nominal closed-loop performance.
In order to create the state estimate $\hat{x}(t)$, a full-order unknown input observer UIO developed in Chen and Patton (1999) is used. The expression $BK(t)u(t)$ in (1) is treated as an unknown input since by assumption $K(t)$ is unknown. Consequently the distribution matrix associated with the unknown input signal to be rejected is chosen as B. Necessary and sufficient conditions[2] for a linear UIO to exist for the system in (1)-(2), to provide insensitivity with respect to the term $BK(t)u(t)$, are

$A2$**:** $rank(CB) = rank(B) = m$

$A3$**:** the triple (A, B, C) is minimum phase

The structure of the full-order UIO from Chen and Patton (1999) in this particular case

$$\dot{z}(t) = A_0 z(t) + Ly(t) \tag{10}$$

$$\hat{x}(t) = z(t) + Hy(t) \tag{11}$$

where $\hat{x}(t)$ is the estimated state, and A_0, L and H are design parameters of appropriate dimension chosen in order to decouple the unknown inputs. In particular the matrix $H \in \mathbb{R}^{n\times p}$ must be chosen so that

$$(I - HC)B = 0 \tag{12}$$

As argued in Chen and Patton (1999), Assumption $A2$ is sufficient to solve (12), and $H := B((CB)^T CB)^{-1}(CB)^T$ is an appropriate choice. After computing H, the matrix

$$A_0 := \underbrace{A - HCA}_{A_h} - L_1 C \tag{13}$$

can be defined, where $L_1 \in \mathbb{R}^{n\times p}$ is design freedom which is exploited to make A_0 Hurwitz. Finally

$$L_2 := A_0 H \tag{14}$$

and the gain $L := L_1 + L_2$.
Remark 2: The ISM scheme in this paper can tolerate the presence of stable invariant zeros (assumption *A3*). However invariant zeros preclude the use of the strong observability approach in (Bejarano et al., 2009, 2007). However there is a price to be paid for this increase in applicability. In this paper the state estimate $\hat{x}(t) \to x(t)$ asymptotically, whereas the use of high-order sliding modes in (Bejarano et al., 2009, 2007) provides (arbitrarily small) finite time convergence of the state estimation error to zero for their observer.

If $e(t) := x(t) - \hat{x}(t)$, using (1) and (11), after some algebra and simplifications based on (12)-(14), the error dynamics

$$\dot{e}(t) = A_0 e(t) + (I - HC)A^\delta x(t) \tag{15}$$

The choice of the gain G in (9) suggested in this paper is

$$G := B_2((CB)^T CB)^{-1}(CB)^T \tag{16}$$

The existence of the inverse in expression (16) is guaranteed by assumption $A2$. As a result of this choice of G, generically $GCB_w(t) = B_2 W(t) B_2^T$, which is symmetric. The symmetry is important and simplifies much of the subsequent analysis and avoids the introduction of conservatism. Also nominally, when there are no faults and $W = I$, from the special properties of the matrix B_2, it follows that

$$GCB_w(t)|_{W=I} = B_2 B_2^T = I$$

This means, nominally, G has the 'pseudo inverse properties' which Castanos and Fridman (2006) argue are optimal from the point of view of minimizing the impact of unmatched uncertainties on the closed loop dynamics.

Suppose a control law can be designed to force a sliding motion for all time. The equivalent control $\nu_{eq}(t)$ necessary to maintain sliding is obtained from equating $\dot{\sigma} = 0$ (Utkin et al., 1999). The derivative of $\sigma(t)$ in (9) is

$$\dot{\sigma}(t) = G\dot{y}(t) + F\hat{x}(t) \tag{17}$$

Then substituting from equation (7) and equating $\dot{\sigma}(t) = 0$ yields

$$\nu_{eq}(t) = -(GCB_w(t))^{-1}(F\hat{x}(t) + GC(A + A^\delta)x(t)) \tag{18}$$

under the assumption $\det(GCB_w(t)) \neq 0$. With the choice of G in (16), it follows $GCB_w(t) = B_2 W(t) B_2^T$, and

$$\nu_{eq}(t) = -(B_2 W(t) B_2^T)^{-1}(F\hat{x}(t) + GC(A + A^\delta)x(t)) \tag{19}$$

Substituting (19) into (7), the sliding dynamics are given by

$$\dot{x}(t) = (A + A^\delta)x(t) - B_m(F\hat{x}(t) + GC(A + A^\delta)x(t)) \tag{20}$$

where

$$B_m(t) := \begin{bmatrix} B_1 W(t) B_2^T (B_2 W(t) B_2^T)^{-1} \\ I_l \end{bmatrix}$$

Adding and subtracting the term $B_\nu(F\hat{x}(t) + GC(A + A^\delta)x(t))$ to the right hand side of (20) and exploiting the fact that $e(t) := x(t) - \hat{x}(t)$, the sliding dynamics can be written as

$$\begin{aligned}\dot{x}(t) = {} & (A - B_\nu F - B_\nu GCA)x(t) + A^\delta x(t) - B_\nu GCA^\delta x(t) \\ & + B_\nu Fe(t) + \widetilde{B}\Phi(t)(Fx(t) - Fe(t) + GC(A + A^\delta)x(t))\end{aligned} \tag{21}$$

where

$$\widetilde{B} := \begin{bmatrix} I_{n-l} \\ 0 \end{bmatrix} \tag{22}$$

and

$$\Phi(t) = B_1 B_2^T - \underbrace{B_1 W(t) B_2^T (B_2 W(t) B_2^T)^{-1}}_{\psi(t)} \tag{23}$$

[2] In Chen and Patton (1999), the necessary and sufficient conditions stated for solving (12)-(14) are that $rank(CB) = rank(B)$ and (C, A_h) is detectable. As argued in Tan and Edwards (2003) these are equivalent to $A2$ and $A3$.

Combining equations (15) and (21), the closed-loop system dynamics can be written as

$$\begin{bmatrix} \dot{e}(t) \\ \dot{x}(t) \end{bmatrix} = \underbrace{\begin{bmatrix} A_0 & 0 \\ B_\nu F & A_c - B_\nu F \end{bmatrix}}_{A_a} \underbrace{\begin{bmatrix} e(t) \\ x(t) \end{bmatrix}}_{x_a} + B_a \Delta(t)\, C_a \begin{bmatrix} e(t) \\ x(t) \end{bmatrix} \quad (24)$$

where $A_c := (I - B_\nu GC)A$ and

$$B_a := \begin{bmatrix} (I - HC) & 0 & 0 \\ (I - B_\nu GC) & \widetilde{B} & \widetilde{B} \end{bmatrix} \qquad C_a := \begin{bmatrix} 0 & I \\ -F & GCA + F \\ 0 & I \end{bmatrix}$$

and the uncertainty term $\Delta(t)$ is

$$\Delta(t) := \mathrm{diag}\begin{bmatrix} A^\delta & \Phi(t) & \Phi(t)GCA^\delta \end{bmatrix} \quad (25)$$

It is convenient to analyze (24) in the $(e, \hat{x})$ coordinates.

$$\underbrace{\begin{bmatrix} e(t) \\ \hat{x}(t) \end{bmatrix}}_{\hat{x}_a} = \underbrace{\begin{bmatrix} I & 0 \\ -I & I \end{bmatrix}}_{\widetilde{T}} \begin{bmatrix} e(t) \\ x(t) \end{bmatrix} \quad (26)$$

then it follows in the new $(e, \hat{x})$ coordinates

$$\dot{\hat{x}}_a(t) = \widetilde{A}_a \hat{x}_a(t) + \widetilde{B}_a \Delta(t) \widetilde{C}_a \hat{x}_a(t) \quad (27)$$

where

$$\widetilde{A}_a := \widetilde{T} A_a \widetilde{T}^{-1} = \begin{bmatrix} A_0 & 0 \\ A_c - A_0 & A_c - B_\nu F \end{bmatrix} \quad (28)$$

$$\widetilde{B}_a := \widetilde{T} B_a = \begin{bmatrix} (I - HC) & 0 & 0 \\ HC - B_\nu GC & \widetilde{B} & \widetilde{B} \end{bmatrix} \quad (29)$$

$$\widetilde{C}_a := C_a \widetilde{T}^{-1} = \begin{bmatrix} I & I \\ GCA & GCA + F \\ I & I \end{bmatrix} \quad (30)$$

In order to ensure that the term $\Phi(t)$ in (25) is bounded, note that $\Phi(t) = B_1 B_2^T - \psi(t)$ and $\psi(t) = B_1 B_2^\dagger(t)$, where $B_2^\dagger(t)$ is a right pseudo inverse of B_2. Using the pseudo inverse properties in Stewart (1989), and arguing exactly as in Alwi and Edwards (2008a), there exists a scalar γ_0 such that $\|B_2^\dagger(t)\| := \|W(t)B_2^T(B_2 W(t) B_2^T)^{-1}\| < \gamma_0$ for all combinations of $(w_1(t), \ldots, w_m(t))$ such that $\det(B_2 W B_2^T) \neq 0$. Therefore $\|\psi(t)\| \leq \gamma_1\gamma_0$ and $\|\Phi(t)\| \leq \gamma_1(1 + \gamma_0)$.

*A*4**:** Assume that the parametric uncertainty A^δ is bounded, and therefore since $\|\Phi(t)\|$ is bounded, it follows

$$\|\Delta(t)\| < \gamma_a \quad (31)$$

for some positive scalar γ_a.

3.1. Closed-loop Stability Analysis

In the nominal case, (i.e. when $W(t) = I$, $A^\delta = 0$ and $\Delta(t) = 0$), equation (27) simplifies to $\dot{\hat{x}}_a(t) = \widetilde{A}_a \hat{x}_a(t)$. From (28) it is clear that the eigenvalues of $\widetilde{A}_a$ are given by the union of the eigenvalues of A_0 and $A_c - B_\nu F$. Both these matrices can be made Hurwitz by choice of the design freedom matrices L_1 from (13) and F respectively. Consequently, by design, $\widetilde{A}_a$ can be made Hurwitz, and hence nominally the closed loop system is stable. However for the fault/failure cases, stability needs to be proved. Define

$$\gamma_2 = \|\widetilde{G}_a(s)\|_\infty \quad (32)$$

where

$$\widetilde{G}_a(s) := \widetilde{C}_a(sI - \widetilde{A}_a)^{-1}\widetilde{B}_a \quad (33)$$

Proposition 1: In fault/failure conditions, for any combination of $(w_1(t), \ldots, w_m(t))$ such that $\det(B_2 W B_2^T) \neq 0$, the closed-loop system in (27) will be stable if:

$$\gamma_2\gamma_a < 1 \quad (34)$$

Proof. In order to establish closed-loop stability, the system defined in (27) can also be written as

$$\dot{\hat{x}}_a(t) = \widetilde{A}_a \hat{x}_a(t) + \widetilde{B}_a \widetilde{u}_a(t) \quad (35)$$
$$\widetilde{y}_a(t) = \widetilde{C}_a \hat{x}_a(t) \quad (36)$$

where $\widetilde{u}_a(t) := \Delta(t)\widetilde{y}_a(t)$. In this form, equation (27), is the feedback interconnection of the known linear system $\widetilde{G}_a(s)$, and the bounded uncertain gain $\Delta(t)$. According to the small gain theorem, the feedback interconnection of $\widetilde{G}_a(s)$ and $\Delta(t)$ will be stable if (34) is satisfied. ■

Remark 3: Note that by hypothesis, $\gamma_1 = \|B_1\|$ is assumed to be small. Furthermore if $A^\delta = 0$, then $\|\Delta(t)\| \to 0$ as $\|B_1\| \to 0$ and Proposition 1 is trivially satisfied.

3.2. LMI Synthesis

In this section, the observer gain L_1 and the controller gain F are synthesized, so that stability condition (34) is satisfied. For the triple $(\widetilde{A}_a, \widetilde{B}_a, \widetilde{C}_a)$, from the Bounded Real Lemma, $\|\widetilde{G}_a(s)\|_\infty < \gamma_2$ if and only if there exists a s.p.d matrix $X \in \mathbb{R}^{2n\times 2n}$ such that

$$\begin{bmatrix} \widetilde{A}_a X + X\widetilde{A}_a^{\mathrm{T}} & \widetilde{B}_a & X\widetilde{C}_a^{\mathrm{T}} \\ \widetilde{B}_a^T & -\gamma_2^2 I & 0 \\ \widetilde{C}_a X & 0 & -I \end{bmatrix} < 0 \quad (37)$$

Here it is assumed that $X = \mathrm{diag}(X_1, X_2)$, where the two sub-blocks $X_1, X_2 \in \mathbb{R}^{n\times n}$ are s.p.d. With this assumption

$$\widetilde{C}_a X = \begin{bmatrix} X_1 & X_2 \\ GCAX_1 & GCAX_2 + Y \\ X_1 & X_2 \end{bmatrix} \quad (38)$$

where $Y := FX_2$. The top left sub-block in (37)

$$\widetilde{A}_a X + X\widetilde{A}_a^{\mathrm{T}} = \begin{bmatrix} A_0 X_1 + X_1 A_0^{\mathrm{T}} & X_1 A_c^{\mathrm{T}} - X_1 A_0^{\mathrm{T}} \\ A_c X_1 - A_0 X_1 & \Theta \end{bmatrix} \quad (39)$$

where $\Theta = A_c X_2 + X_2 A_c^{\mathrm{T}} - B_\nu Y - Y^{\mathrm{T}} B_\nu^T$. Also write $A_0 = A_h - L_1 C$ where A_h is from (13). To create a convex representation, define the observer gain

$$L_1 := \beta BE \quad (40)$$

where β is a positive scalar and $E \in \mathbb{R}^{m\times p}$ is chosen so that (A_h, B, EC) is minimum phase. This is possible if (A, B, C) is minimum phase (Edwards et al., 2007). Then as argued in Edwards et al. (2007), it is possible to find an s.p.d matrix P which has a structure $P = N^{\mathrm{T}}\mathrm{diag}(P_1, P_2)N$ such that $PB = (EC)^{\mathrm{T}}$, where $N \in \mathbb{R}^{n\times n}$ is invertible (and depends on E) and the s.p.d. matrices $P_1 \in \mathbb{R}^{(n-m)\times(n-m)}, P_2 \in \mathbb{R}^{m\times m}$. The matrix N is associated with a change of coordinates to force the system triple (A_h, B, EC) into the canonical form proposed in Edwards et al. (2007). Define $X_{11} = P_1^{-1}$ and $X_{12} = P_2^{-1}$. It follows that $L_1C = \beta BEC = \beta BB^{\mathrm{T}}P$ and so if

$$X_1 := P^{-1} = N^{-1}\mathrm{diag}(X_{11}, X_{12})(N^{-1})^{\mathrm{T}} > 0 \tag{41}$$

then $L_1CX_1 = \beta BB^{\mathrm{T}}$ and $A_0X_1 = A_hX_1 - \beta BB^{\mathrm{T}}$. It follows that the matrix inequality in (37) is affine w.r.t. the decision variables $X_{11}, X_{12}, X_2, \beta, Y$, and so the synthesis problem is convex.

For the nominal system in (8), (i.e. in the case when $W(t) = I$ and $A^\delta = 0$) the gain F must stabilize $(A - B_\nu F)$. Since from *Assumption A1*, the pair (A, B_ν) is assumed to be controllable, an LQR formulation will be adopted where F is selected to minimize

$$J = \int_0^\infty (x^TQx + \nu^TR\nu)dt$$

where Q and R are symmetric positive definite design matrices. It is known this problem can be posed as an LMI optimization:

Minimize $trace(X_2^{-1})$ subject to

$$\begin{bmatrix} AX_2 + X_2A^T - B_\nu Y - Y^TB_\nu^T & (QX_2 - RY)^T \\ QX_2 - RY & -I \end{bmatrix} < 0 \tag{42}$$

For a given $\mathcal{L}_2$-gain γ_2, the overall optimization problem proposed in convex form becomes:
Minimize $trace(Z)$ w.r.t the decision variables $X_{11}, X_{12}, X_2, \beta, Y$ subject to

$$\begin{bmatrix} -Z & I_n \\ I_n & -X_2 \end{bmatrix} < 0 \tag{43}$$

together with (37), (42), (41) and (43). The matrix Z is a slack variable which satisfies $Z > X_2^{-1}$ and therefore $trace(Z) \geq trace(X_2^{-1})$. Finally the controller and observer gains can be recovered as $F = YX_2^{-1}$ and $L_1 = \beta BE$.

3.3. ISM Control Laws

A control law will be defined to ensure sliding is maintained from $t = 0$. Define the virtual control law in (8) as

$$\nu(t) = \nu_l(t) + \nu_n(t) \tag{44}$$

where the linear part, responsible for the nominal performance of the system, is

$$\nu_l(t) = -F\hat{x}(t) - GCA\hat{x}(t) \tag{45}$$

and the nonlinear part

$$\nu_n(t) = -\rho(t)\frac{\sigma(t)}{\|\sigma(t)\|} \qquad \text{for } \sigma(t) \neq 0 \tag{46}$$

where $\rho(t)$ is a modulation gain which will be defined later in the paper.

The following final assumption will be made:

A5: The plant initial conditions $x(0)$, although not perfectly known, are assumed to belong to a *known* hyper-sphere $\mathcal{S} = \{x \in \mathbb{R}^n \ : \ \|x - c_0\| \leq r_0\}$ for some given $c_0 \in \mathbb{R}^n$ and positive scalar r_0.

Define a time varying scalar $\epsilon(t)$ as the solution to

$$\dot{\epsilon}(t) = -m_0\epsilon(t) + m_1\|\hat{x}(t)\| \tag{47}$$

where m_0 and m_1 are positive scalars to be defined in the sequel and let $V_0 = e^TP_0e$ where P_0 is the s.p.d matrix obtained from solving

$$P_0A_0 + A_0^TP_0 = -I \tag{48}$$

Further, suppose that $\|A^\delta\|$ is sufficiently small so that P_0 also satisfies

$$2\|P_0\|\|(I - HC)A^\delta\| < 1 - \mu_o \tag{49}$$

where $1 > \mu_o > 0$. Then the following can be proved:
Proposition 2: Define the modulation gain $\rho(t)$ from (46) as

$$\rho(t) = \frac{\delta_a\|GC\|\|\hat{x}(t)\| + \|\nu_l(t)\| + \epsilon(t)(\|GCA\| + \delta_a\|GC\|)/p_0 + \eta}{(1 - \lambda_0)} \tag{50}$$

where $p_0 = \sqrt{\lambda_{min}(P_0)}$, the scalar $\delta_a > \|A^\delta\|$ and η is a positive design scalar. Assume the fault tuple $(k_1(t), \ldots, k_m(t))$ belongs to the set

$$\mathcal{D} = \{(k_1(t), ..., k_m(t)) : \lambda_{max}(B_2K(t)B_2^T) < \lambda_0 < 1\}$$

Also assume that by choice of $\hat{x}(0)$ and $\epsilon(0)$, the state estimation error $e(t) = x(t) - \hat{x}(t)$ at time $t = 0$, written $e(0)$, satisfies $\sqrt{e(0)^TP_0e(0)} < \epsilon(0)$. Then the integral sliding mode control law defined in (44)-(46), guarantees that the system trajectories remain on the sliding surface.

Proof. Equation (15) can be written as

$$\dot{e}(t) = (A_0 + (I - HC)A^\delta)e(t) + (I - HC)A^\delta\hat{x}(t) \tag{51}$$

then the derivative of the positive definite function $V_0 = e^TP_0e$ is given by

$$\begin{aligned} \dot{V}_0 &= -ee^T + 2e^TP_0(I - HC)A^\delta e + 2e^TP_0(I - HC)A^\delta\hat{x} \\ &\leq -\|e\|^2 + 2\|P_0\|\|(I - HC)A^\delta\|(\|e\|^2 + \|e\|\|\hat{x}\|) \end{aligned}$$

Therefore since by assumption $2\|P_0\|\|(I - HC)A^\delta\| < 1 - \mu_o$ where $\mu_o > 0$ it follows

$$\dot{V}_0 \leq -\mu_0\|e\|^2 + (1 - \mu_0)\|\hat{x}\|\|e\| \leq -\frac{\mu_0}{p_1}V_0 + \frac{1 - \mu_0}{\sqrt{p_1}}\|\hat{x}\|\sqrt{V_0} \tag{52}$$

where $p_1 = \lambda_{max}(P_0)$. By defining $\widetilde{V} = \sqrt{V_0}$, equation (52) can be written as

$$\dot{\widetilde{V}} \leq -\frac{\mu_0}{2p_1}\widetilde{V} + \frac{1 - \mu_0}{2\sqrt{p_1}}\|\hat{x}\| \tag{53}$$

which for notational convenience can be further written as

$$\dot{\widetilde{V}} \leq -m_0\widetilde{V} + m_1\|\hat{x}\| \tag{54}$$

where the positive scalars m_0, and m_1 are appropriately defined. Comparing (54) and (47), if $\epsilon(0) > \widetilde{V}(0)$, it is easy to show that $\epsilon(t) > \widetilde{V}(t)$ for all $t \geq 0$ and consequently

$$\epsilon(t) \geq p_0\|e(t)\| \quad \text{for} \quad t \geq 0 \tag{55}$$

Since $K(t) = I - W(t)$ and $B_2B_2^T = I_l$, equation (17) can be written as

$$\begin{aligned}\dot{\sigma}(t) &= GC(A{+}A^\delta)x(t){+}(B_2W(t)B_2^T)\nu(t){+}F\hat{x}(t)\\ &= GC(A{+}A^\delta)x(t){+}\nu(t){-}(I{-}B_2W(t)B_2^T)\nu(t){+}F\hat{x}(t)\\ &= GC(A{+}A^\delta)x(t){+}\nu(t){-}(B_2(I{-}W(t))B_2^T)\nu(t){+}F\hat{x}(t)\\ &= GC(A{+}A^\delta)x(t){+}\nu(t){-}B_2K(t)B_2^T\nu(t){+}F\hat{x}(t) \end{aligned}\tag{56}$$

Substituting the control law (44)-(46) into equation (56) and exploiting the fact that $e(t) = x(t) - \hat{x}(t)$ yields

$$\dot{\sigma} = GCA^\delta(\hat{x}(t){+}e(t)){+}GCAe(t){-}(B_2KB_2^T)(\nu_l{+}\nu_n){-}\rho\frac{\sigma}{\|\sigma\|} \tag{57}$$

Now consider the candidate Lyapunov function $V = \frac{1}{2}\sigma^T\sigma$. From (57) the time derivative

$$\begin{aligned}\dot{V} \leq \|\sigma\|\Big(&\|GCA^\delta\|\|\hat{x}\| + (\|GCA^\delta\| + \|GCA\|)\|e\|\\ &+ \|B_2K(t)B_2^T\|\|\nu_l\| - \rho(1-\lambda_0)\Big)\end{aligned}\tag{58}$$

for a fault $(k_1(t), \ldots k_m(t)) \in \mathcal{D}$. Then from the definition of $\rho(\cdot)$ and the fact that $\epsilon(t) \geq p_0\|e(t)\|$, inequality (58) can be written as $\dot{V} \leq -\eta\|\sigma\| = -\eta\sqrt{2V}$ which is a standard reachability condition, (Edwards and Spurgeon, 1998), and is sufficient to guarantee that a sliding motion is maintained for all time. ■

Finally the physical control law $u(t)$ is obtained by substituting equations (44)-(46) into (6) to obtain

$$u(t) = B_2^T(-F\hat{x}(t) - GCA\hat{x}(t) - \rho\frac{\sigma(t)}{\|\sigma(t)\|}) \tag{59}$$

Remark 4: Note the control law above does not provide global convergence since the assumptions of Proposition 2 are only satisfied if the initial conditions of the observer and of the filter are chosen so that $\sqrt{e(0)^TP_0e(0)} < \epsilon(0)$. However both $\hat{x}(0)$ and $\epsilon(0)$ are user defined parameters, and so provided that $x(0) \in \mathcal{S}$ defined in Assumption 5, this constraint can always be satisfied by choice of $\hat{x}(0)$ and $\epsilon(0)$.

4. Simulations

In order to demonstrate the efficacy of the proposed FTC scheme, a civil aircraft benchmark model from Alwi and Edwards (2008b) is used in the simulation. To design the controller in (45), the aircraft model has been linearized around an operating condition of straight and level flight with a mass of 263,000 Kg, 92.6m/s true airspeed, and at an altitude of 600m based on 25.6% of maximum thrust. In the simulations, only measured system outputs

$$y = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\end{bmatrix}}_{C_p}\begin{bmatrix} q\\ V_{tas}\\ \alpha\\ \theta\end{bmatrix}$$

are available for use in the control law, where θ is the pitch angle (rad), α is the angle of attack (rad), V_{tas} is the true airspeed (m/sec), and q is the pitch rate (rad/sec). The linearized state space model is

$$A_p = \begin{bmatrix} -0.4623 & 0.0004 & -0.5248 & 0\\ 0 & -0.0149 & 1.7171 & -9.8046\\ 1.1071 & -0.0021 & -0.5655 & 0\\ 1 & 0 & 0 & 0\end{bmatrix}$$

$$B_p = \begin{bmatrix} -0.6228 & -1.3578 & 0.0599\\ 0 & -0.1756 & 5.7071\\ -0.0352 & -0.0819 & -0.0085\\ 0 & 0 & 0\end{bmatrix}$$

The available control surfaces for the longitudinal control are $\delta_{long} = [\delta_e, \delta_s, \delta_{epr}]^T$ which represent elevator deflection (rad), horizontal stabilizer deflection (rad) and aggregated longitudinal EPR. In the simulations, a series of 3-deg flight path angle (FPA) commands are given to change the altitude of the aircraft, while the true airspeed V_{tas} is held constant by using a separate inner-loop Proportional Integral (PI) controller which creates an auto-throttle manipulating EPR. Throughout the simulations *it is assumed that the engines are fault free*. By splitting the input distribution matrix into matrices which are associated with $[\delta_e, \delta_s]^T$ and δ_{epr}, the linear model can be rewritten as

$$\dot{x}_p(t) = A_px_p(t) + B_su_1 + B_e\delta_{epr} \tag{60}$$

$$y = C_px_p(t) \tag{61}$$

where $u_1 = \begin{bmatrix} \delta_e & \delta_s\end{bmatrix}^{\mathrm{T}}$ and matrices $B_s \in \mathbb{R}^{4\times 2}$ and $B_e \in \mathbb{R}^{4\times 1}$. Define a new state in the PI controller for V_{tas} as

$$\dot{x}_r(t) = r_1(t) - C_1x_p(t)$$

where $r_1(t)$ is the reference signal for V_{tas} tracking and the matrix $C_1 = \begin{bmatrix} 0 & 1 & 0 & 0\end{bmatrix}$. The inner-loop PI control is

$$\delta_{epr} = K_p(r_1(t) - C_1x_p(t)) + K_ix_r(t)$$

where the PI gains are $K_p = 0.6$, and $K_i = 0.9$. Now augmenting the state $x_r(t)$ with the plant in (60) yields

$$\begin{bmatrix}\dot{x}_r\\ \dot{x}_p\end{bmatrix} = \underbrace{\begin{bmatrix} 0 & -C_1\\ B_eK_i & (A_p - B_eK_pC_1)\end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x_r\\ x_p\end{bmatrix}}_{x(t)} + \underbrace{\begin{bmatrix} 0\\ B_s\end{bmatrix}}_{B}u_1 + \underbrace{\begin{bmatrix} I\\ B_eK_p\end{bmatrix}}_{B_r}r_1 \tag{62}$$

Also it is assumed that the state $x_r(t)$ is available for the controller design, therefore $y = Cx(t)$ where $C = diag\{1, C_p\}$. In order to introduce steady state tracking for the controlled output $y_c(t)$, a feedforward term L_rr_2 is introduced where

$$L_r := -(C_c(A - B_\nu F - B_\nu GCA)^{-1}B_\nu)^{-1} \tag{63}$$

and the exogenous constant signal r_2 is the reference to be tracked (by the FPA). From assumption A1, F can always be chosen to ensure that $(A - B_\nu F - B_\nu GCA)$ is Hurwitz and therefore $\det(A - B_\nu F - B_\nu GCA) \neq 0$. Consequently the inverse in (63) is well defined. In the absence of faults and uncertainty it is easy to see the linear control law $u(t) = -Fx(t){-}GCAx(t){+}L_rr_2$

ensures that at steady state $y_c = r_2$. To accommodate this tracking requirement, the control law in (59) must be changed to

$$u(t) = B_2^T(-F\hat{x}(t) + L_r r_2 - GCA\hat{x}(t) - \rho \frac{\sigma(t)}{\|\sigma(t)\|}) \qquad (64)$$

and

$$\sigma(t) = Gy(t) - Gy(0) + \int_0^t (F\hat{x}(\tau) - L_r r_2) d\tau \qquad (65)$$

The fault tolerant controller will now be designed based on the system in (62) governed by the triple (A, B, C), using only the elevator and stabilizer as inputs. A further scaling of B is required to ensure that $B_2 B_2^T = I_l$ (where in this example $l = 1$). In this aircraft system (A, B, C) has one stable invariant zero. In can be verified that $rank(CB) = rank(B) = 2$, and therefore Assumption $A2$ holds. Since the objective is to track an FPA command, the controlled output is $y_c(t) = C_c x(t)$, where $C_c = \begin{bmatrix} 0 & 0 & 0 & -1 & 1 \end{bmatrix}$. The gain in equation (16) is $G = \begin{bmatrix} 0 & 0.6694 & 0 & 0 \end{bmatrix}$. In addition to actuator faults/failures, to introduce potential faults which cause changes to the aerodynamics of the aircraft, a 10% change in the aerodynamic coefficients (due to possible airframe damage) is considered: specifically

$$A^\delta = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0.0514 & 0 & 0.0583 & 0 \\ 0 & 0 & 0.0017 & 0 & 0 \\ 0 & 0.1006 & 0 & 0.0628 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Choosing

$$E = \begin{bmatrix} -12.4139 & -1.6056 & 12.4139 & -1.6056 \\ 5.6942 & 0 & -5.6942 & 0 \end{bmatrix}$$

gives $ECB = I$, and ensures (A_h, B, EC) is minimum phase with stable zeros at $\{-1.0000, -0.6451, -1.0000\}$. With the choice of $Q = \text{diag}(0.02, 0.5, 0.2, 0.1, 10)$ and $R = 1$ in (42), solving the LMIs in (37), (42) and (43), gives the feedback gain $F = \begin{bmatrix} -0.8142 & 9.9401 & -2.2095 & -0.3356 & 8.8802 \end{bmatrix}$. In the simulations, it is assumed that the engines are fault free. Based on this assumption, using a numerical search, it can be verified using (25) that $\gamma_a = 0.1597$. To satisfy the closed-loop stability condition in (34), the value of γ_2 must satisfy $\gamma_2 < \frac{1}{0.1597} = 6.2621$. This has been satisfied through the designed parameters L_1 and F.

4.1. Simulation Results

In this section the performance of the benchmark civil aircraft model is demonstrated by considering potential failures in the actuators. In the simulations, the discontinuity associated with the control signal in (46) is smoothed using a sigmoidal approximation $\frac{\sigma}{\|\sigma\|+\delta}$, where the value of the positive scalar δ is chosen as $\delta = 0.01$. The value of the modulation gain is chosen as $\rho = 2$. In the simulations, the aircraft undergoes a series of 3-deg FPA commands in order to increase the altitude of the aircraft, while the true airspeed V_{tas} is kept constant. The initial conditions for the plant and observer are taken as $x_0 = [0, 0, 0, 0]^T$, and $x_{0_{obs}} = [0, 0, 0, 0, 0.5(\pi/180)]^T$ respectively. In Figure 2(a), a failure is considered, where the elevator jams at some offset position. To maintain the performance close to nominal, the proposed FTC scheme invokes the horizontal stabilizer to counteract the failure as can be seen in Figure 2(b). In Figure 3(b), the stabilizer runs-away to a maximum position of 3-deg. Due to the availability of the redundant actuator (i.e. elevator) the scheme can still maintain good tracking as seen in Figure 3(a). In Figure 1, it can be seen that in both scenarios the observer error quickly converges to zero despite the faults.

5. Conclusion

In this paper, a new fault tolerant control scheme was proposed which assumes only output information is available and no information about the actuator faults or failures is available. To estimate the system states, a linear unknown input observer is employed. The estimated states are used in the virtual control law to produce signals which are then translated into the physical control signals (associated with the actuators) by using a fixed control allocation scheme. The closed-loop stability analysis allows for parameter uncertainty in the system matrix (due to airframe damage for example) in addition to actuator faults or failures. A convex representation of the synthesis problem is established in order to prove closed-loop stability by synthesizing appropriate observer and controller gains. The simulation results on a benchmark aircraft model show fast convergence of the observer output error, and demonstrate good FTC features of the proposed scheme.

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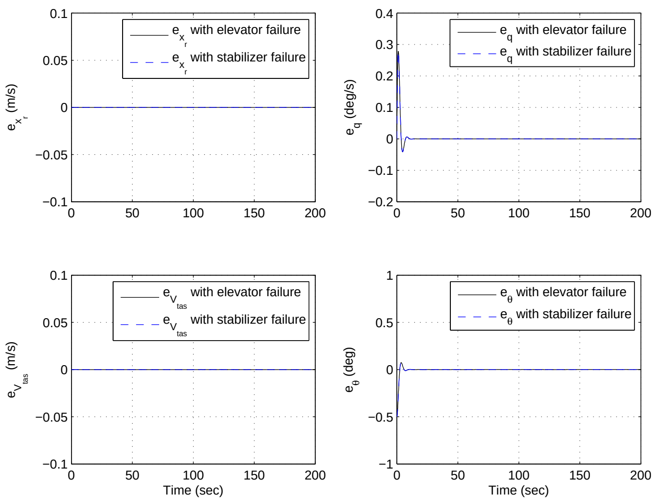
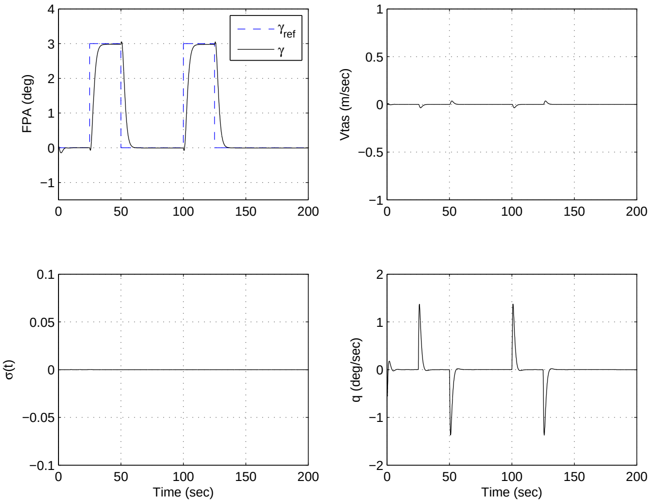
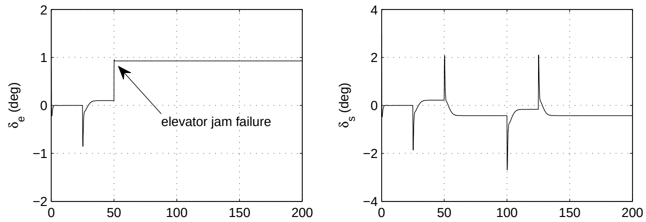


Figure 1: State estimation error signals (elevator jam and stabilizer runaway)

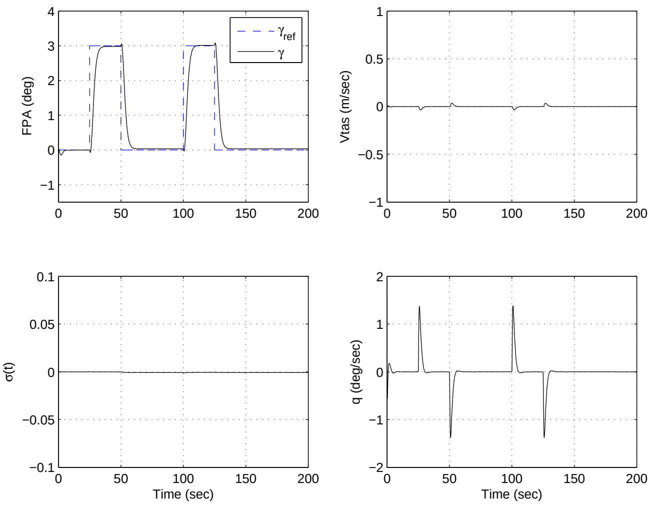


(a) states and switching functions

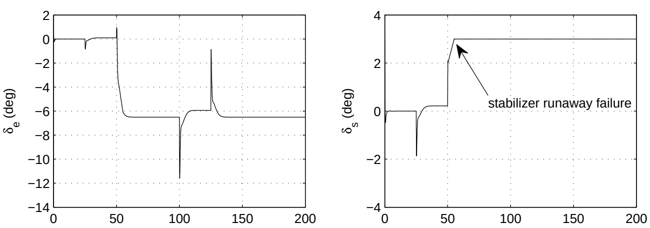


(b) control surface deflections

Figure 2: Stabilizer runaway failure



(a) states and switching functions



(b) control surface deflections

Figure 3: Stabilizer runaway failure