

# Three Essays on Monetary Policy and Learning

A Dissertation

Presented to the Graduate School

of

University of Exeter

in Candidacy for the Degree of

Doctor of Philosophy

in

Economics

by

Šarūnas Girdėnas

Dissertation Advisors:

Tatiana Damjanovic

Joao Madeira

October 2014

This thesis is available for Library use on the understanding that it is copyright material and that no quotation from the thesis may be published without proper acknowledgement.

I certify that all material in this thesis which is not my own work has been identified and that no material has previously been submitted and approved for the award of a degree by this or any other University.

Signature: .....

## ACKNOWLEDGEMENTS

First and foremost I would like to thank my advisor Tatiana Damjanovic who's patience and guidance helped when I needed most. I would also like to thank to Joao Madeira for the useful comments and discussion. I am very grateful to Tatiana Kirsanova for the mentoring and support during the period of writing the thesis.

I am greatly indebted to ESRC for the financial support which was necessary to pursue this venture.

I would also like to thank my family for their patience and support during the period of my doctoral studies. I am very grateful to Ploy Thumavichai for inspiration and showing me the other side of everything.

In addition, for various things (some of them should not be mentioned in the polite company) I am thankful to my colleagues and dear friends (in random order): Jordan Large, Giuseppe Bova, Dooruj Rambaccussing, Angele Urbonavičiute, Lawrence Choo, Hyun Chang Yi, Keqing Liu, Sumedh Dalwai, Bing Chiao, Christoph Himmels, Yuting Bai, Jack Rupert Rogers, Yen Huang, Jonas Oskinis, Olafur Margeirsson, Rokas Tamosiunas, Lindsay Stringfellow, Kestutis Baltakys, Seonhwi Lee, Evelina Radzivonaite and Sabrina Kmitaite.

I am grateful to the participants of the following workshops and conferences who's comments improved my work: Macroeconomics Workshop at University of Warwick 2014, Search and Matching 2014 Conference in Edinburgh, XVIII Workshop in Dynamic Macroeconomics in Vigo 2013, Money and Macro 2013 Conference 2013, CEF 2014 Conference in Oslo, 7th CGBCR Conference in Manchester 2014, Royal Economic Society 2014 meeting and the Conference on Banking, Finance, Money and Institutions: The Post Crisis Era 2013 and various seminars at the University of Exeter Business School (2011-2014).

## ABSTRACT

**The first chapter** co-authored with Tatiana Damjanovic studies optimal monetary policy in a New Keynesian model at the zero bound interest rate where households use cash alongside house equity borrowing to conduct transactions. The amount of borrowing is limited by a collateral constraint. When either the loan to value ratio declines or house prices fall, we observe a decrease in the money multiplier. We argue that the central bank should respond to the fall in the money multiplier and therefore to the reduction in house prices or the loan to collateral value ratio. We also find that optimal monetary policy generates a large and persistent fall in the money multiplier in response to the drop in the loan to collateral value ratio.

**The second chapter** is focused on a macroeconomic model with sticky prices, firms' borrowing market and the labour market frictions. We study connection between monetary policy and labour market under the negative financial and the positive productivity shocks. We have found that the interest rate rule with inflation and labour market targeting performs better than the rules with the aggregate consumption and debt targeting and is closest to the optimal policy as compared to the other regimes in terms of the welfare measure. We demonstrate too that the sign of the coefficient next to unemployment in the policy rule depends on the value of workers bargaining power.

**The third chapter** co-authored with Tatiana Damjanovic and Keqing Liu uses the classical cobweb model framework to investigate properties of the transition matrix in the bounded memory econometric OLS-type learning. We define memory length as the number of past observations used to form a forecast and analytically prove that for any length, the eigenvalues of the transition matrix lie within the unit circle. In addition, we sketch the proof of stationarity of the cobweb model under

bounded memory learning. Furthermore, we investigate the relationship between the volatility of forecasts and the length of memory and find that shorter memory causes higher variance in both forecasts and estimates of the OLS parameters.

## TABLE OF CONTENTS

Acknowledgements . . . . .	2
Table of Contents . . . . .	4
<b>1 Introduction</b>	<b>10</b>
<b>2 Quantitative Easing and the Loan to Collateral Value Ratio</b>	<b>17</b>
2.1 Introduction . . . . .	17
2.2 Model . . . . .	20
2.2.1 Households . . . . .	21
2.2.2 Final Good Producers . . . . .	24
2.2.3 Government's Optimization Problem . . . . .	25
2.2.4 Negative Credit Shock and ZLB . . . . .	26
2.2.5 Linear Quadratic Model . . . . .	28
2.3 Optimal Monetary Policy . . . . .	30
2.3.1 Reaction to Credit Shock . . . . .	30
2.3.2 Cost of Inactive Government . . . . .	33
2.3.3 Credit Shocks and the Money Multiplier . . . . .	34
2.3.4 Credit Shock and House Prices . . . . .	36
2.4 Loan to Value Ratio . . . . .	39
2.4.1 Expected Collateral Inflation . . . . .	39
2.4.2 Liquidity, Transaction Costs and the Value of Collateral . . . . .	40
2.5 Preference for Money . . . . .	41
2.5.1 Labour Supply . . . . .	42
2.5.2 Credit Constraint . . . . .	43
2.6 Conclusion . . . . .	44
2.7 Appendix . . . . .	47
2.7.1 First-Order Conditions for Household Optimization . . . . .	47
2.7.2 Optimal Steady State . . . . .	48
2.7.3 Linear Approximation to the Constraints of Private Behavior . . . . .	49
2.7.4 Second-Order Approximation . . . . .	51
2.7.5 Solution to Central Bank LQ Problem . . . . .	54
2.7.6 Proof of Proposition 2.2.1 . . . . .	54
2.7.7 Proof of Proposition 2.3.2 . . . . .	55
2.7.8 Money in the Utility Function . . . . .	55
2.7.9 Proof of Proposition 2.5.1 . . . . .	56
2.7.10 Parameter Values . . . . .	57
<b>Bibliography</b>	<b>58</b>

<b>3</b>	<b>Monetary Policy in the Model with Financial and Labour Market Frictions</b>	<b>63</b>
3.1	Introduction <sup>1</sup>	63
3.1.1	Theoretical Approach	65
3.2	Model	67
3.2.1	Households	67
3.2.2	Firms	69
3.2.3	Final Good Producers and Price Setting	76
3.2.4	Steady State Efficiency Analysis	77
3.2.5	Calibration and Solution	78
3.3	Monetary Policy	80
3.3.1	Monetary Policy Rules	81
3.3.2	Relation Between Financial and Labour Market	82
3.3.3	Response to Financial Market Shock	84
3.3.4	Response to Productivity Shock	86
3.4	Welfare Implications	88
3.4.1	Should Response to Labour Market be Positive?	93
3.4.2	Policy Maker's Preferences and Welfare	95
3.5	Conclusion	96
3.6	Appendix	98
3.6.1	Benchmark Model	98
3.6.2	Policy Maker's Optimization Problem	100
3.6.3	Derivation of Wage Equation	106
3.6.4	Derivation of Final Good Producers Price Setting Relation	108
	<b>Bibliography</b>	<b>112</b>
<b>4</b>	<b>Stationarity of Econometric Learning with Bounded Memory</b>	<b>114</b>
4.1	Introduction <sup>2</sup>	114
4.2	Some Properties of Bounded Memory Learning	117
4.2.1	Learning with Expanding Memory	117
4.2.2	Learning with Bounded Memory	119
4.2.3	Stationarity of Price with Bounded Memory Learning	121
4.2.4	Additional Properties of Bounded Memory Learning	122
4.2.5	Cycles in the Bounded Memory Learning	125
4.2.6	Bounded Memory and Constant Gain Algorithm	128
4.3	Numerical Evaluation of Mean and Variance	129
4.3.1	Explicit Numerical Computation	129
4.3.2	Implicit Numerical Computation	131
4.4	Conclusion	133

<sup>1</sup>Codes used in this chapter could be seen at [https://github.com/Sarunas-Girdenas/2nd\\_chapter](https://github.com/Sarunas-Girdenas/2nd_chapter).

<sup>2</sup>Matlab codes used in this chapter could be seen at [https://github.com/Sarunas-Girdenas/Bounded\\_Memory](https://github.com/Sarunas-Girdenas/Bounded_Memory).

4.5	Appendix . . . . .	135
4.5.1	Proof of proposition 4.2.1 . . . . .	135
4.5.2	Proof of $E[S_{i,t}] > 0$ . . . . .	135
4.5.3	Proof of $E[S_i S_1] > 0$ . . . . .	137
4.5.4	Proof of $E[S_i S_j] > 0$ . . . . .	138
4.5.5	Proof of lemma 4.2.6 . . . . .	141
4.5.6	Spectral radius of $E[M_t \otimes M_t]$ . . . . .	142
	<b>Bibliography</b>	<b>150</b>
	<b>5 Conclusion</b>	<b>152</b>

## LIST OF TABLES

Table 3.1. Parameter Values .....	81
Table 3.2. Monetary Policy Rules and Parameter Values .....	83
Table 3.3. Monetary Policy Rules and Welfare .....	92
Table 3.4. Reduced Form $VAR(1)$ Estimation .....	108



## LIST OF FIGURES

Figure 1.1 Individual Borrowing in the UK .....	10
Figure 2.1 Output and Inflation .....	32
Figure 2.2. Money Multiplier after the 2008 Crisis .....	34
Figure 2.3. Optimal Transfers .....	35
Figure 2.4. Collateral Price and Financial Shock .....	36
Figure 2.5. Simulation of Money Multiplier .....	37
Figure 3.1. Firms' Borrowing and Unemployment in the UK in 1983-2012 .....	64
Figure 3.2. $VAR(1)$ Model Response to Borrowing Equation Shock .....	65
Figure 3.3. $VAR(1)$ Model Response to Output Equation Shock .....	65
Figure 3.4. Impulse Response Functions to Repayment Probability Shock (I) .....	84
Figure 3.5. Impulse Response Functions to Repayment Probability Shock (II) .....	85
Figure 3.6. Impulse Response Functions to Repayment Productivity Shock (I) .....	86
Figure 3.7. Impulse Response Functions to Repayment Productivity Shock (II) .....	87
Figure 3.8. Policy Rules for Various Parameter Values .....	93
Figure 3.9. Welfare and Bargaining Power .....	95
Figure 3.10. Rule 2, Inflation and Unemployment .....	96
Figure 4.1. Cycles of $p$ for Short and Long Memory .....	127
Figure 4.2. Power of Cycles with Bounded Memory .....	128
Figure 4.3. Simulation of Mean (I) .....	130
Figure 4.4. Simulation of Variance .....	131
Figure 4.5. Simulation of Mean (II) .....	133

# CHAPTER 1

## INTRODUCTION

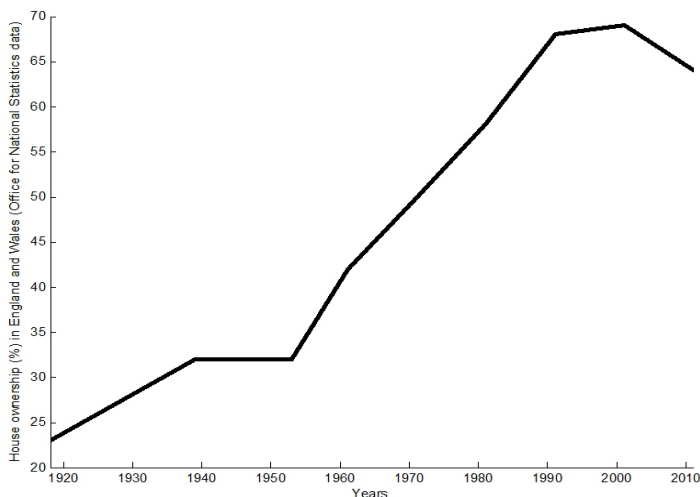
In the second chapter of this thesis (already published in *Journal of Economic Dynamics and Control*), motivated by the empirical evidence of the fall in households borrowing in the UK after the economic crisis in 2008 (shown in Figure 1.1.), we investigate the relationship between private credit and monetary policy under a negative financial market shock and a positive cost-push shock.

Figure 1.1. Individual Borrowing in the UK



It is worth noting that in the model we assume 100% house ownership. However, as we can see from figure 1.2 it is not the case in England and Wales where house ownership was only 69% at its peak in 2001. At the same year households borrowing also was at its highest value as it is shown in figure 1.1. To make our model more tractable we assume that all the houses in the economy are owned by the agents. In the future research we would like to relax this assumption to investigate what impact does partial ownership might have on the dynamic properties of the model.

Figure 1.2. Individual Houses Ownership in England and Wales



As it could be seen from Figure 1.1, the level of total consumer lending (loans) has dropped significantly around the year 2008 in the UK. We ask whether central bank should take into account loan to value (LTV) ratio for households when designing monetary policy.

We find that when LTV ratio declines, central bank can stabilize the economy by reducing interest rate according to the policy rule that includes households borrowing. When the drop in LTV is too large and interest rate hits ZLB, the government has to implement transfers policy.

The motivation for this assumption is clear: both in the US and in the UK Central Banks had kept interest rate at zero (to be precise, the current rate of the Bank of England is 0.5%, often referred to as 'the lowest efficient value of interest') to ignite the economic growth. When the interest rate is at the ZLB, central bank can use other policy instrument - transfers - to conduct monetary policy. In some sense transfers in this case could be seen as a negative taxation, because it enters households budget constraint directly. However, for the sake of simplicity we use the term 'Quantitative Easing' to denote the transfers rule. It is worth noting, that

this is a huge simplification because Quantitative Easing was not implemented in this way for example in the UK or the US. As mentioned before this simplification helps to focus our attention on the transmission channel of the newly issued money.

Our results are in favor of including the LTV ratio in the policy rule in both cases of interest rate. That is, when interest rate has some small positive value and financial market collapses, Central Bank by taking into account LTV can adjust the interest rate in such a way that the economy will be perfectly stabilized in terms of inflation and output fluctuations. In the case of ZLB, policy maker can again perfectly stabilize the economy if the transfers rule is defined by taking into account the LTV ratio. Further to it, we show that this rule results in the highest value of the welfare measure and hence it is the optimal response to the negative credit market shock.

We have also considered the case when the economy is at the ZLB and the positive cost-push shock occurs. We have found that the policy maker cannot avoid inflation and output deviations from their steady state values, but he can stabilize relative price of collateral by setting transfers rule to zero. The purpose and key findings of the second chapter are summarized below.

**Research Question:** Should central bank react to the changes in the households borrowing and if so, what is the optimal reaction in terms of the welfare measure?

**Key Findings:** When the economy is at the ZLB, central bank should take into account changes in households' borrowing whilst designing the optimal transfers rule, when the economy is not at the ZLB, central bank should take into account households' borrowing rule while setting interest rate. In both of these cases reaction to the changes in borrowing is optimal.

In the third chapter of the thesis we investigate the relation between finan-

cial and labor markets. That is, we are seeking to find out if a fall in the firms borrowing could increase unemployment and if so, through which channel. The motivation for such research question comes from the fact that after the crisis in 2008 unemployment reached the highest level in almost 20 years (at least in the UK) and firms borrowing has dropped significantly. More detailed discussion and figures illustrating the drop in firms borrowing and increase in unemployment are provided in chapter 3.

We set up a standard New-Keynesian macroeconomic model with financial and labor market frictions to investigate through which channel the fall in firms credit market could result in the increase in the aggregate level of unemployment. We have also analyzed possible monetary policy reactions to the negative productivity shock.

We have found that the Taylor-type interest rate rule with response to inflation and unemployment is the closest one to the optimal policy in terms of the welfare measure. It is consistent with the previous literature, for instance Faia (2008 or 2008a). Further to this, we find that the sign next to the unemployment response parameter in the policy rule depends on the strength of the workers bargaining power. The reason for it is intuitive: if workers have a lot of bargaining power they can negotiate a large part of the firms profit in the form of wage and thus firms' ability to borrow by using its expected future profit as a collateral deteriorates. Therefore policy maker in order to counteract this negative effect reduces interest rate much more (that is, reacts negatively to unemployment) than in the case of a low value of workers bargaining.

This paper has two policy implications. Firstly, it provides some justification for the increased concern of policy makers towards the unemployment targeting (for instance, Bank of England or Federal Reserve). The reason why unemployment

targeting is beneficial could be summed up as follows: the greatest instability in the model comes from the labor market fluctuations and inflation (price stickiness) therefore the rule which reacts on both of these variables is closest to the optimal policy as it is shown in Faia (2008).

Secondly, we find that the drop in firms borrowing market could increase the level of unemployment by reducing the aggregate demand. That is, in our model firms borrow to pay the wage for workers. If for some unknown reason lenders reduce the amount of credit, firms are forced to lay off workers and as result unemployment increases. This is reinforced by the drop in demand since workers have less money to finance their consumption. As it was mentioned earlier, in this case the policy maker could step in and use Taylor-type interest rate rule to counteract the drop in unemployment.

**Research Question:** Which Taylor-type policy rule is the closest one to the optimal policy in terms of welfare measure in the model with labor and credit market frictions?

**Key Findings:** Our model, motivated by the empirical evidence, suggests that we can indeed explain the rise in the aggregate level of unemployment by looking at the changes in the firms financial market. Furthermore, Taylor-type interest rate rule that reacts to unemployment and inflation is the closest one to the optimal policy in terms of the welfare measure. Lastly, we find that the sign of the unemployment reaction parameter in the interest rate rule depends on the workers bargaining power.

In the last chapter of the thesis we investigate econometric learning with bounded memory which is defined as the agents ability to use only some particular number  $T$  of past observations (instead of the whole sample) to form their forecasts. The main difference between our approach and the one presented in

the previous literature (Honkapohja and Mitra 2003, Evans and Honkapohja 2003 to name a few) is the fact that we do not use the recursive formulation of the learning algorithm with either exponentially decaying ( $\frac{1}{t}$ ) or constant weight as in Recursive Least Squares (RLS) or Constant Gain (CG) learning. Instead we make use of the standard non-recursive OLS algorithm and set weight of 1 on the each of observations that lie within the length of memory  $T$  and 0 on all the other observation. In pure econometric terms, bounded memory learning could be seen as a rolling window estimate with the window length set to  $T$ . This approach allows us to investigate the impact of length of memory  $T$  on the dynamic properties of the economic model.

Our results show that for any memory length  $T$ , the transition matrix of the learning algorithm has eigenvalues that lie within the unit circle. We use this finding to sketch the proof of stationarity of bounded memory learning. Using computational methods have shown too that the mean and variance of the actual price are finite and equal to rational expectations equilibrium. However, at this stage we were not able to obtain the analytical proof for such results regarding mean and variance.

It is interesting to note, that we have also discovered that bounded memory causes cycles. That is, short memory results in stronger cycles and long memory tends to reduce the strength of the cycles but increases its length. The reason for it is intuitive. As agents can use more information (longer  $T$ ), they have more information and hence their forecasts are more accurate. In the short memory case, agents have much less information and hence they are not able to predict the price level very well. Furthermore, when memory is short the arrival of a new information has much more impact than in the case of long memory. We use discrete Fourier transform to measure the cyclicity of the model.

Bounded memory learning, economically speaking, could be interpreted as follows: agents, as time passes, add one more and remove the last observation from their sample which has the same size as their memory. This could be seen as agents inability to use all the available information for forecasting prices which results in cycles of the parameter estimates that in turn imply cycles in the variables in the model. Moreover, we believe that letting agents have bounded memory is more realistic than employing the assumption of ever-expanding memory.

**Research Question:** What are the properties of the bounded memory econometric learning in the cobweb model setting?

**Key Findings:** Eigenvalues of the transition matrix lie within the unit circle for any length of memory. Actual price and expected price have the finite mean. Results from the computational exercise suggest that the mean of the actual price and parameter estimates of the bounded memory learning are equal to the rational expectations values.



CHAPTER 2  
QUANTITATIVE EASING AND THE LOAN TO COLLATERAL  
VALUE RATIO

## 2.1 Introduction

During the recent economic crisis credit availability has significantly decreased (Dell Ariccia et al., 2008), house prices and output has fallen sharply. Responding to worsening credit conditions, many developed countries significantly expanded their monetary bases. Several central banks engaged in quantitative easing (QE) taking “unconventional” assets onto their balance sheets (Gambacorta et al., 2012). In this paper, we provide a justification for QE and argue that monetary expansion is necessary for stabilizing price and output fluctuations when there is a marked tightening of credit conditions.

The general idea of connecting financial markets and business cycles can be traced back to Fisher (1933), Bernanke (1983) and Bernanke and Gertler (1989) who show that a contraction in the financial sector can lead to an economic slowdown. In this paper, we investigate whether, to what extent and how the monetary authorities should respond to worsening financial conditions in order to avoid an economic recession.

This question is not new to the academic literature.<sup>1</sup> On the one hand, in his review of recent monetary policy developments, Clarida (2012) argues that financial variables are not target variables and should not be included in monetary policy rules. The same opinion is also shared by Bernanke and Gertler (2001) and Iacoviello (2005) who claim that the government should not react to changes in asset prices as this does not improve the economy in terms of inflation and output

---

<sup>1</sup>For a comprehensive survey on macroeconomics with financial frictions, see Brunnermeier et al. (2012).

stabilization.

On the other hand, Mishkin (2011) argues that after the 2007-2009 economic crisis, monetary policy makers understood that the financial sector has a considerably greater impact on economic activity than has previously been realized. Further to this, Svensson (2009) recognizes that credit capacity and asset prices may have a potentially negative impact on inflation and resource utilization and, therefore, including them in the monetary policy rule is entirely consistent with the stabilization of inflation and output gaps. We will also observe this particular feature in our model.

We study optimal monetary policy in a New Keynesian economy with sticky prices where households use cash alongside equity borrowing to conduct transactions. The amount of borrowing is limited by a collateral constraint as in Kiyotaki and Moore (1997) or Iacoviello (2005). We simply assume that competitive financial intermediaries can costlessly create as much credit as they want. However, due to the lack of contract enforcement, each loan has to be collateralized.

We follow Iacoviello (2005) and Midrigan and Philippon (2011) and assume that households use houses as collateral. The households' credit capacity can deteriorate for two reasons: a reduction in collateral value or an exogenous shock which causes a decline in the average recovery rate of collateral. In the remainder of the paper, this is referred to as a "credit shock". When the loan to collateral value (LTV) declines, the credit capacity falls. Less inside money reduces nominal expenditure and, thus, nominal demand. In a flexible price economy, producers adjust their prices accordingly and recession is avoided. However, when prices are sticky, only incomplete adjustment is possible, and credit tightening results in both deflation and recession, unless an expansionary monetary policy is implemented.

The principal difference of our model to Iacoviello (2005), Monacelli (2009)

and Carlstrom and Fuerst (1995) is our assumption that the economy may reach a liquidity trap. When the interest rate is at its zero lower bound, the monetary authorities cannot reduce it any further and are hence forced to stimulate the economy by providing direct monetary transfers to households. Unconventional monetary expansion at a zero bound interest rate has been advocated by Friedman (2000, 2006) and Bernanke et al. (2004). When the interest rate is at zero bound, direct monetary targeting cannot be criticized in the sense of McCallum (1985), because it does not cause any volatility in the short-term interest rate. To our knowledge, direct monetary expansion when the interest rate is at its zero lower bound (ZLB) has not been formally modeled and it is not considered in recent academic publications (see Adam and Billi, 2007; Coibion, Gorodnichenko and Wieland, 2012). The only exception, perhaps, is the recent paper by Benigno and Nistico (2013), which studies a similar shock to us and gives similar policy recommendations.

We also find that monetary policy can ensure perfect stabilization of output and prices when a credit shock hits the economy. When the shock is small, the monetary policy maker can reduce the interest rate. However, a large negative credit shock can drive the interest rate to the ZLB. At that point monetary policy has to inject liquidity into the system.

The expansion of the monetary base in response to the deterioration in credit availability is necessary because of the fall in the money multiplier. The importance of the money multiplier has been discussed in Bernanke and Blinder (1988), Freeman and Kydland (2000) and recently in Goodhart (2009) and Abrams (2011). Since the money multiplier reflects monetary transmission, optimal monetary policy should respond to changes in it. Our model shows how the multiplier depends on the LTV ratio and the relative price of collateral. Hence, if houses are used as

collateral, monetary policy should respond to changes in house prices.

To evaluate monetary policy rules at the ZLB, we construct a second-order approximation as in Benigno and Woodford (2012)<sup>2</sup> and obtain a social loss function as in the conventional New Keynesian model (Benigno and Woodford, 2005). In our model, optimal monetary policy generates the same impulse responses to the cost-push shock for output and inflation as in the standard New Keynesian model. In order to achieve optimal dynamics, central bank should conduct monetary expansion when there is a fall in the LTV ratio or if the relative price of collateral declines.

This chapter is structured as follows. In Section 2.2, we present the model and define dynamic equations for the economy. In Section 2.3, we discuss optimal monetary policy and some other policy issues. We include a short discussion of what may happen if central bank ignores changes in the credit constraint or fluctuations in house prices. We also underline the importance of the money multiplier and its connection to credit constraints and the relative price of collateral. In Section 2.4 we provide a short discussion of the factors which can affect the LTV ratio. In Section 2.5 we investigate an economy when money and loans are not perfect substitutes. Section 2.6 concludes the chapter.

## 2.2 Model

In this section, we present a stylized New Keynesian economy with collateral constraint. Our model consists of several ingredients. First, we have households who consume final goods, provide labor to final good producers, earn wages, share the profit of the firms and take loans against collateral. Second, there are intermediate and final goods producers. The latter operate in a sticky price environment in the

---

<sup>2</sup>See also Levine et al. (2008).

spirit of Calvo (1980). There are also monetary authorities which make decisions about interest rate and money supply.

Finally, we have a financial sector which creates inside money through lending. We simply assume that a financial intermediary can costlessly create as much loans as is safe. That is a principal difference of our model compared to Benigno and Nistico (2013) or Iacoviello (2005), where financial intermediaries do not participate in the creation of inside money. To hedge against the risk of default, the lender issues debt against valuable collateral, represented by houses in our model. If borrowers repudiate their debt, the lender can repossess the borrowers' assets and recover a fixed proportion of their value. The only interesting outcome of financial intermediation for our model is the loan to value ratio. Micro modelling of the financial sector could explain/endogenise LTV. However, for simplicity, we treat it as exogenous but stochastic.

### 2.2.1 Households

A representative household has a utility function that includes the consumption of goods,  $Y_t$ , valuable collateral (house),  $h_t$ , and labor,  $L_t$ ,

$$U_t = E_t \sum_{t=0}^{\infty} \beta^t u(Y_t, h_t, L_t) = E_t \sum_{t=0}^{\infty} \beta^t \left( \log Y_t + \eta \log h_t - \lambda \frac{L_t^{v+1}}{v+1} \right), \quad (2.1)$$

where  $v$  is the labor supply elasticity parameter,  $\eta$  captures individual household preferences towards units of housing and  $\lambda$  defines the value of leisure.

For their transactions, households can use cash, i.e. outside money,  $M_t$ , and the money created by the banking system, i.e. inside money,  $B_t$ . The broad money can be used to buy consumption goods and invest in collateral

$$P_t Y_t + Q_t (h_t - h_{t-1}) \leq M_t + B_t, \quad (2.2)$$

where  $P_t$  is the price of final goods,  $Q_t$  is the price of collateral, and  $h_t - h_{t-1}$  is

investment in collateral. This is cash-in-advance constraint where credit has to be collateralized by housing. It is very similar to Lucas and Stokey (1987) setup.

The amount of private credit is subject to a collateral constraint

$$B_t \leq \theta_t Q_t h_t, \quad (2.3)$$

which implies that households cannot borrow more than a fraction  $\theta_t$  of their collateral value  $Q_t h_t$ . Parameter  $\theta_t$  denotes the tightness of the borrowing constraint. A smaller value of  $\theta_t$  implies a smaller loan size, whereas a high value means that a household may obtain a relatively large loan. The government implements monetary policy by printing new bills and distributing them across households as a lump-sum transfer

$$M_t^s = M_{t-1} + T_t. \quad (2.4)$$

The loan must be repaid immediately after households obtain their wage and dividend income. Let  $W_t$  be the nominal wage and  $\Pi_t$  be the profit of firms owned by households and paid in the form of dividends. Then, at the end of the period, the liquidity position of the household is

$$M_t^d = W_t L_t + \Pi_t + T_t - (1 + r_{t-1}) B_{t-1}. \quad (2.5)$$

### Household's Optimization

In the appendix, we show that maximization of household utility (3.1) subject to constraints (2.2, 2.3 and 2.5) results in the following Euler equation

$$U'_{ht} + \theta_t \frac{Q_t}{P_t} \left( U'_{ct} - \beta E_t U'_{ct+1} \frac{P_t}{P_{t+1}} (1 + r_t) \right) = U'_{ct} \frac{Q_t}{P_t} - \beta E_t U'_{ct+1} \frac{Q_{t+1}}{P_{t+1}}. \quad (2.6)$$

The left-hand side of the equation shows the marginal benefit from an extra unit of collateral: it consists of a direct boost to utility,  $U'_{ht}$ , as well as an effect due to the possibility of using collateral to secure a loan. The value of the second

source is proportional to credit tightness  $\theta_t$ . In other words, a smaller  $\theta_t$  reduces the loan size and, as a result, there is a fall in the benefits from using a house as a collateral .

Moreover, constraint (2.2) is always binding, while constraint (2.3) is binding when

$$U'_{ct} - \beta E_t U'_{ct+1} \frac{P_t}{P_{t+1}} (1 + r_t) > 0. \quad (2.7)$$

In a deterministic steady state the credit constraint will be binding if and only if  $\beta \frac{(1+r_t)}{\pi_{t+1}} < 1$ . As we see, either a reduction of the borrowing rate or an increase in inflation can move the economy from the state when the credit constraint is not important to the state when it is binding. In our main presentation we assume the interest rate to be at the zero bound,  $r_t = 0$ . In this case, if the price level and the consumption level are stable, inequality (2.7) is satisfied. Following Iacoviello (2005), we will consider that the uncertainty is "small enough" to preserve inequality (2.7).

Using the particular functional form of utility (3.1), the assumption of the zero interest rate, and normalizing the units of housing,  $h_t = 1$ , we transform equation (2.6) into the following form

$$\eta + \beta E_t [q_{t+1}] = \left( 1 - \theta_t \left( 1 - \beta E_t (1 + r_t) \left[ \frac{P_t Y_t}{P_{t+1} Y_{t+1}} \right] \right) \right) q_t, \quad (2.8)$$

where  $q_t$  is relative housing expenditure, which is defined as

$$q_t = \frac{Q_t h_t}{P_t c_t}.$$

Finally, the first-order condition with respect to  $L_t$  defines labor supply

$$-\frac{U'_L(C_t, L_t)}{U'_C(C_t, L_t)} = \lambda L_t^v Y_t = \frac{W_t}{P_t}. \quad (2.9)$$

## 2.2.2 Final Good Producers

We assume that final good producers take intermediary goods and sells it to households in the sticky prices environment. Furthermore, intermediary goods producers operate in the perfectly competitive market. Aggregated final good, denoted as  $Y_t^f$  could be defined as

$$Y_t^f = \left( \int_0^1 y_t^\varepsilon(i) di \right)^{\frac{1}{\varepsilon}} \quad (2.10)$$

We assume that final goods are imperfect substitutes and that consumption is defined over the Dixit-Stiglitz (1977) basket of goods,  $Y_t^f = \left[ \int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$ . The average price-level,  $P_t$ , is known to be  $P_t = \left[ \int_0^1 p_t(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$ . The demand for each good is given by  $Y_t(i) = \left( \frac{p_t(i)}{P_t} \right)^{-\varepsilon} Y_t^f$ , where  $p_t(i)$  is the nominal price of the final good produced in industry  $i$  and  $Y_t$  denotes aggregate demand. Each good is produced according to a linear technology using labor as the only input,  $Y_t(i) = L_t(i)$ .

There is an economy-wide labor market so that all firms pay the same wage for the same labor,  $w_t(i) = w_t, \forall i$ . All households provide the same share of labor to all firms, so that the total labor supply in (3.1) is defined as  $L_t = \int_0^1 L_t(i) di$ , which in combination with the production function and demand relates output to labor income.  $L_t = \int L_t(i) di = Y_t \int \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} di = Y_t \Delta_t$ , where  $\Delta_t$  is the measure of price dispersion:  $\Delta_t \equiv \int_0^1 \left( \frac{P_t(i)}{P_t} \right)^{-\varepsilon} di$ . The real wage,  $w_t = \frac{W_t}{P_t}$ , is then  $w_t = \beta \lambda Y_t^{v+1} \Delta_t^v$ .

### Price Setting of a Representative Firm

We will model price stickiness according to Calvo (1983). A fixed proportion of firms adjusts prices in each period. These firms choose the nominal price which maximizes their expected profit given that they have to charge the same price in  $k$  time periods with probability  $\alpha^k$ . The real profit can be written as  $\Pi(i) = \frac{p_t(i)}{P_t} Y_t(i) - \phi_t w_t L_t(i)$ , where  $\phi_t$  is a cost-push shock. We assume that firms are



price takers and cannot affect any aggregate variables. Let  $p'_t$  denote the choice of the nominal price by a firm that is permitted to re-price in period  $t$ . Then the firm's objective is to choose  $p'_t$  to maximize the following sum

$$\max E_t \sum_{k=0}^{\infty} \alpha^k \beta^k \left[ \frac{p'_t}{P_{t+k}} \left( \frac{p'_t}{P_{t+k}} \right)^{-\varepsilon} Y_{t+k} - \phi_t w_t \left( \frac{p'_t}{P_{t+k}} \right)^{-\varepsilon} Y_{t+k} \right]. \quad (2.11)$$

The first-order condition implies

$$\left( \frac{p'_t}{P_t} \right) = \frac{\left( \frac{\varepsilon}{\varepsilon-1} \right) E_t \sum_{k=0}^{\infty} (\alpha\beta)^k [\phi_{t+k} w_{t+k} Y_{t+k} (P_t/P_{t+k})^{-\varepsilon}]}{E_t \sum_{k=0}^{\infty} (\alpha\beta)^k (P_t/P_{t+k})^{1-\varepsilon} Y_{t+k}}. \quad (2.12)$$

It is useful to introduce new variables,  $X_t$  and  $Z_t$ , for the discounted expected real revenue and costs of the firm. We define them as  $X_t = E_t \sum_{k=0}^{\infty} (\alpha\beta)^k (P_t/P_{t+k})^{1-\varepsilon} Y_{t+k}$ ,  $Z_t = E_t \sum_{k=0}^{\infty} (\alpha\beta)^k [\phi_{t+k} w_{t+k} Y_{t+k} (P_t/P_{t+k})^{-\varepsilon}]$ . The price index will evolve according to the following law of motion,  $P_t = [(1-\alpha)p_t^{1-\varepsilon} + \alpha P_{t-1}^{1-\varepsilon}]^{\frac{1}{1-\varepsilon}}$ , which can be rewritten as  $\frac{p'_t}{P_t} = \left[ \frac{1-\alpha\pi_t^{\varepsilon-1}}{1-\alpha} \right]^{\frac{1}{1-\varepsilon}}$ .

### 2.2.3 Government's Optimization Problem

The policy maker maximizes the household's utility function with the awareness that the supply of houses is constant and normalized to 1,  $h_t = 1$ ,

$$U_t = E_t \sum_{t=0}^{\infty} \beta^t \left( \log Y_t - \frac{Y_t^{v+1} \Delta_t^{v+1}}{v+1} \right), \quad (2.13)$$

subject to a set of constraints imposed by private agents' behavior (2.14-2.21), where a block of three equations (2.15-2.17) represents formula (2.12) in a VAR form and (2.18) is the law of motion for the measure of price dispersion.

$$\eta + \beta E_t [q_{t+1}] = \left( 1 - \theta_t \left[ 1 - \beta E_t (1 + r_t) \left( \frac{Y_t}{\pi_{t+1} Y_{t+1}} \right) \right] \right) q_t; \quad (2.14)$$

$$X_t = Y_t + a \beta E_t \pi_{t+1}^{\varepsilon-1} X_{t+1}; \quad (2.15)$$

$$Z_t = \phi_t \beta \lambda Y_t^{v+2} \Delta_t^v + a \beta E_t \pi_{t+1}^\varepsilon Z_{t+1}; \quad (2.16)$$

$$X_t \left( \frac{1 - \alpha \pi_t^{\varepsilon-1}}{1 - a} \right)^{\frac{1}{1-\varepsilon}} = \frac{\varepsilon}{\varepsilon - 1} Z_t; \quad (2.17)$$

$$\Delta_t = \alpha \Delta_{t-1} \pi_t^\varepsilon + (1 - \alpha) \left( \frac{1 - \alpha \pi_t^{\varepsilon-1}}{1 - a} \right)^{\frac{\varepsilon}{\varepsilon-1}}; \quad (2.18)$$

$$\frac{M_t}{P_t Y_t} = 1 - \theta_t q_t; \quad (2.19)$$

$$T_t = M_t - M_{t-1}; \quad (2.20)$$

$$\pi_t = \frac{P_t}{P_{t-1}}. \quad (2.21)$$

As in Midrigan and Philippon (2011), we obtain that, in the steady-state output,  $Y$ , does not depend on the credit constraint,  $\theta$ . However, the value of  $\theta$  will positively affect relative housing expenditure,  $q = \frac{\eta}{(1-\theta)(1-\beta)}$  and, therefore, equilibrium real house price  $\frac{Q}{P} = qY$ . It will also define the broad money multiplier,  $m = \frac{M+B}{M}$ . Since broad money,  $M_t + B_t$ , equals total expenditure, we can compute the money multiplier from (2.19)

$$m_t = \frac{M_t + B_t}{M_t} = \frac{P_t Y_t}{M_t} = \frac{1}{1 - \theta_t q_t}. \quad (2.22)$$

This positive relation between the money multiplier,  $m_t$ , the credit constraint,  $\theta_t$ , and the relative collateral value,  $q_t$ , will drive our results.

## 2.2.4 Negative Credit Shock and ZLB

When a small negative credit shock hits an economy, the monetary authority can easily mitigate it by reducing the interest rate with or without applying direct monetary transfers. Consider a simple case when an economy is initially in an

optimal steady state. Suddenly, the Loan to Value ratio,  $\theta$ , declines. If the interest rate is positive and the credit shock is relatively small, the government can use conventional interest rate policy only, without providing monetary transfers. Precisely, we formulate it in the following proposition

**Proposition 2.2.1** *When interest rate is positive,  $r_t > 1$ , and the negative credit shock is small, there exists such stabilization policy that  $T_t = T_{t+1} = 0$ ,  $\pi_t = \pi_{t+1} = 1$ ;  $Y_{t+1} = Y_t = Y_{t-1}$ . That policy implies the following dynamics for interest rate*

$$r_t = r_{t-1} - \frac{1}{\beta} \left( \frac{1}{\theta_t} - \frac{1}{\theta_{t-1}} \right) - \left( E_{t-1} \frac{1}{\theta_t} - E_t \frac{1}{\theta_{t+1}} \right) \quad (2.23)$$

**Proof.** See appendix. ■

The exact amount of the interest rate reduction depends on three factors: *i*) the direct negative effect from the difference in inverse LTV ratio,  $\frac{1}{\beta} \left( \frac{1}{\theta_t} - \frac{1}{\theta_{t-1}} \right)$ ; *ii*) the surprise effect or the difference between actual and expected value  $s_t = \left( \frac{1}{\theta_t} - E_{t-1} \frac{1}{\theta_t} \right)$ , and *iii*) the expected change,  $e_t = \left( E_t \frac{1}{\theta_{t+1}} - \frac{1}{\theta_t} \right)$

$$r_t = r_{t-1} - \frac{1}{\beta} \left( \frac{1}{\theta_t} - \frac{1}{\theta_{t-1}} \right) + s_t + e_t. \quad (2.24)$$

For example, in an extremely myopic case when the fall in LTV is completely unexpected,  $E_{t-1} \frac{1}{\theta_t} = \frac{1}{\theta_{t-1}}$ , the surprise effect is big,  $s_t = \frac{1}{\theta_t} - \frac{1}{\theta_{t-1}}$ , however it is smaller than a negative direct effect. In a less extreme case the surprise effect should be smaller. The expectation effect depends on how permanent the fall in LTV is expected to be. Consider the extreme case when the public expects the LTV to return back to its original value,  $E_t \frac{1}{\theta_{t+1}} = \frac{1}{\theta_{t-1}}$ . In this case the expectation and surprise effects cancel each other and the optimal change in interest is

$$r_t = r_{t-1} - \frac{1}{\beta} \left( \frac{1}{\theta_t} - \frac{1}{\theta_{t-1}} \right). \quad (2.25)$$

While policy (2.23) stabilizes the economy perfectly, it is subject to the zero lower bound constraint on interest. Consider case (2.25) with the data of Sep-

tember 2008. The LTV ratio before the subprime mortgage crisis was above 95%, and then it sudden dropped below 85%. According to (2.25) the corresponding reduction of the interest rate would exceed 12%. However, the pre-crisis interest rate was below 5% and the optimal reduction was impossible without hitting the zero lower bound.

When the fall of the LTV is significant the interest rate adjustment can be insufficient to stabilize the economy due to the ZLB constraint. In that case, additional expansionary monetary policy is required through non-conventionanl methods.

In the next section we will show that the monetary authorities can mitigate a negative credit shock with direct monetary expansion,  $T_t > 0$ . The conventional interest rate policy is easier to implement, however it can be restricted by ZLB condition. The non-conventional policy of QE raises lots of questions about its technical implementation: Which assets the central bank should purchase? How much risk it should put on the central bank balance sheet? how easy will it be to reverse QE in the future? In the opposite situation, when the LTV ratio increases, the direct monetary contraction could raise even more implementation issues. Whether the government should increase the interest rate or sell unconventional assets in this case is an interesting topic for future research.

### 2.2.5 Linear Quadratic Model

In order to make our work easier, we linearize the constraints of private behavior presented in equations (2.14-2.15)<sup>3</sup>. In the appendix, we show that the optimal steady state is achieved under price stability, i.e.  $\pi = 1$ . Applying the Benigno and Woodford (2012) method to the non-linear problem (2.13, 2.14-2.21), we receive a pure quadratic approximation of the social objective. As we can see, it consists of

---

<sup>3</sup>See appendix 2.7.2-2.7.4.

the squares of output and inflation gaps similar to those of Benigno and Woodford (2005),

$$U_t = -\frac{1}{2}E_t \sum_{s=1}^{\infty} \beta^s (\alpha_C (\widehat{Y}_{t+s} + \alpha_\phi \widehat{\phi}_{t+s})^2 + \alpha_\pi \widehat{\pi}_{t+s}^2) + O3 + tip, \quad (2.26)$$

where  $\alpha_C$  and  $\alpha_\pi$  are the policy maker's preferences towards the output gap and inflation, respectively,  $-\alpha_\phi \widehat{\phi}_t$  is the target level of output, which is inversely related to the cost push shock and *tip* denotes the terms that are independent of the policy maker's choices. Coefficients  $\alpha_C$ ,  $\alpha_\pi$  and  $\alpha_\phi$  are all positive and computed in the Appendix.

### Private Sector Behavior Constraints

Linearized versions of equations (2.15-2.17) can be combined to form a New Keynesian Phillips Curve (2.27). A linearization of equation (2.18) shows that the relative price dispersion term is of second-order importance and can be ignored. Finally, (2.19-2.21) can be combined into (2.29), which relates the monetary policy instrument,  $T_t$ , to inflation and output. Therefore, as a result of simplification and linearization, we obtain a reduced system of three equations

$$\widehat{\pi}_t = \frac{1-a}{a}(1-\alpha\beta) \left[ (v+2)\widehat{Y}_t + \widehat{\phi}_t \right] + \beta E_t \widehat{\pi}_{t+1}; \quad (2.27)$$

$$\begin{aligned} \beta \widehat{q}_{t+1} &= \widehat{q}_t (1 - \theta(1 - \beta(1+r))) - \widehat{\theta}_t \theta(1 - \beta(1+r)) \\ &\quad + r\beta\theta \widehat{r}_t + \beta\theta(1+r) (\widehat{Y}_t - E_t \widehat{\pi}_{t+1} - E_t \widehat{Y}_{t+1}); \end{aligned} \quad (2.28)$$

$$\widehat{\pi}_t = \widehat{Y}_{t-1} - \widehat{Y}_t + \frac{\theta q}{1-\theta q} (\widehat{\theta}_t - \widehat{\theta}_{t-1} + \widehat{q}_t - \widehat{q}_{t-1}) + \widehat{T}_t, \quad (2.29)$$

where variables with hats denote percentage deviations from the steady state, and variables without time subscript denote the steady state values.

Expression (2.29) is the essence of the paper: whatever is the target for inflation and output dynamics, one cannot neglect the fluctuations in relative house value,  $\widehat{q}_t$ , or credit availability,  $\widehat{\theta}_t$ . In other words, for given dynamics of  $\widehat{\pi}_t$  and  $\widehat{Y}_t$ ,

monetary policy,  $\widehat{T}_t$ , should be adjusted to the shock in the credit constraint and the change in relative house expenditure ( $\widehat{q}_t$ ).

Our objective is to find the first-order approximation to the optimal policy reaction function. We will allow two shocks to perturb our economy: a cost-push shock,  $\widehat{\phi}_t$ , and a credit shock,  $\widehat{\theta}_t$ . We assume that  $\widehat{\phi}_t$  and  $\widehat{\theta}_t$  follow two independent  $AR(1)$  processes

$$\widehat{\theta}_t = \rho_\theta \widehat{\theta}_{t-1} + \xi_{\theta t}; \quad (2.30)$$

$$\widehat{\phi}_t = \rho_\phi \widehat{\phi}_{t-1} + \xi_{\phi t}. \quad (2.31)$$

The linear approximation to optimal policy can be found by maximizing the second-order approximation to social welfare (3.55), subject to linear constraints (2.27)-(2.29).

## 2.3 Optimal Monetary Policy

### 2.3.1 Reaction to Credit Shock

Consider the case when only credit shocks hit the economy, that is when the price markup is constant,  $\widehat{\phi}_t = 0$ . If the credit shock is the only source of instability, the government can achieve zero losses, perfectly stabilizing both output and inflation.

We formalize this statement in Proposition 2.3.1.

**Proposition 2.3.1** *In the absence of the cost-push shock, credit market contraction can be perfectly neutralized. Indeed, the policy maker can achieve perfect price and output stabilization, i.e.  $\widehat{\pi}_t = 0$  and  $\widehat{Y}_t = 0$ .*

**Proof.** If  $\widehat{\phi}_t = 0$ , output and price stability are not in contradiction with the system of constraints (2.27)-(2.29). ■

Note that complete price and output stabilization delivers the maximum value of social welfare (3.55). Hence, the corresponding policy is optimal and the optimal monetary policy rule in this case follows from equation (2.29) if inflation and output deviations are set to zero.

$$\widehat{T}_t = -\frac{\theta q}{1 - \theta q} \left( \widehat{\theta}_t - \widehat{\theta}_{t-1} + \widehat{q}_t - \widehat{q}_{t-1} \right). \quad (2.32)$$

As a result, we obtain an example where the credit constraint,  $\widehat{\theta}_t$ , and the collateral price,  $\widehat{q}_t$ , are not directly targeted by the government. However, they are the only arguments in the government's reaction function. This is exactly the case discussed in Svensson (2009). Government does not care about the financial sector *per se*, but since it affects inflation and output volatilities, the policy maker must consider the change in the financial environment when it implements its monetary policy.

It is worth noting that in our model fluctuations of  $\widehat{\theta}_t$  are exogenous and therefore suboptimal. We assume that LTV is set exogenously in the model and investigate how policy maker could react to its fluctuations.

The optimal monetary policy rule (2.32) has a straightforward interpretation. Recalling the collateral constraint (2.3) and taking price and output stability into consideration, it may be written as

$$\widehat{T}_t = k \left( \widehat{B}_{t+1} - \widehat{B}_t \right), \quad (2.33)$$

where we define  $k = \frac{\theta q}{1 - \theta q}$ . Coefficient  $k$  has an important economic meaning. In the steady state, the collateral constraint (2.3) implies that  $\theta q PY = B$ , while the cash-in-advance constraint (2.2) implies that  $PY = M + \theta q PY$ . Combining those two expressions, the debt to money ratio can be computed

$$B = kM. \quad (2.34)$$

Therefore,  $k$  is the marginal effect on loans of a change in the base money. In other words, a 1 dollar expansion of the monetary base will create  $k$  dollars of loans:  $k = \frac{dB}{dM}$ . Equation (2.33) tells us how much the central bank should expand its monetary base. The expansion should just be sufficient to offset the reduction in debt capacity.

It is noteworthy that optimal policy in the simple framework without cost-push shocks can be reduced to two equations

$$\beta E_t \widehat{q}_{t+1} = \widehat{q}_t(1 - \theta(1 - \beta)) - \widehat{\theta}_t \theta(1 - \beta); \quad (2.35)$$

$$\widehat{\theta}_{t+1} = \rho_\theta \widehat{\theta}_t - \xi_{\theta t+1}. \quad (2.36)$$

Blanchard and Kahn (1980) formulated the necessary conditions for the dynamic linear R.E. system to have a unique solution. It states that there must be the same number of eigenvalues larger than 1 in modulus as there are forward looking variables. To satisfy this condition, the following relation is necessary and sufficient,  $\frac{1-\theta(1-\beta)}{\beta} > 1$ , which is true if and only if  $\theta < 1$ .

In case  $\theta > 1$ , we will have indeterminacy and sunspot equilibria. In practice,  $\theta > 1$  was observed in 2006-2007, when new mortgages were available with up to a 110% loan to value ratio. Such a high LTV might have been partly responsible for the house bubble and the subsequent financial crisis. Thus, according to Korteweg and Sorensen (2012), LTV significantly contributed to the probability of foreclosure sales. In this light, the suggestion of the Hong Kong Monetary Authority (2011) to use LTV as a policy tool for the macroprudential regulation seems to have some justification. For the sake of stability, we calibrate the steady-state value of  $\theta$  to be less than 1.



### 2.3.2 Cost of Inactive Government

In this section, we will numerically assess the value of monetary policy. For this purpose, we will compare optimal policy generating with a policy that neglects changes in the credit market. Our alternative policy is

$$\widehat{T}_t = 0. \tag{2.37}$$

If a negative credit shock hits the economy and the government does not provide any monetary response to that shock,  $\widehat{T}_t = 0$ , both deflation and a significant fall in GDP would be expected.

Figure 2.1. Output and Inflation

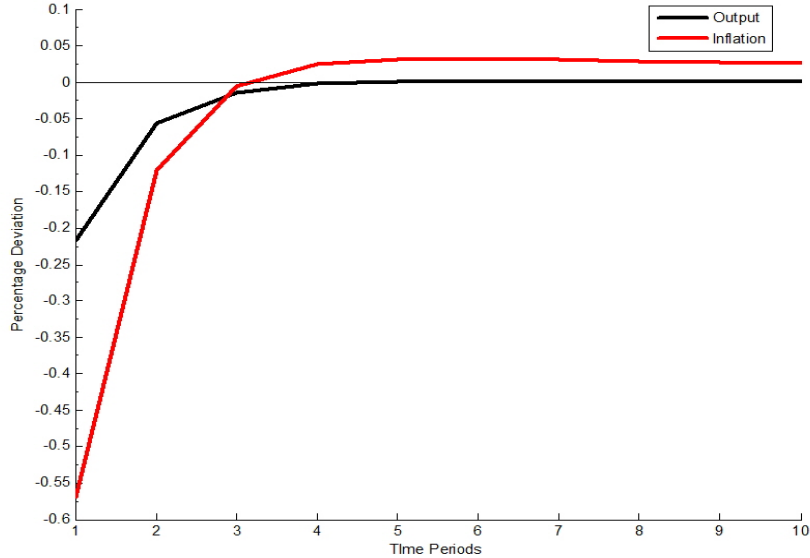


Figure 1 shows that a 1% drop in the loan to collateral value ratio reduces output by 0.2%. If our model is reasonably calibrated, a 20% drop in the mortgage LTV ratio will result in a 4% fall in GDP in the absence of quantitative easing. This would be even more damaging for consumer prices. In the absence of monetary transfers, the model economy will experience a 12% deflation.

### 2.3.3 Credit Shocks and the Money Multiplier

It is well known that the money multiplier fell dramatically after the recent financial crisis. Now, the monetary authorities have to expand the monetary base to a much larger extent in order to achieve the same expansion of broad money. The importance of the money multiplier is discussed in Goodhart (2009) who criticizes the macroeconomic literature for ignoring the money multiplier and for failing to formally model it. This criticism is not entirely fair, since the behavior of the money multiplier was a popular research topic in the 1990s. See, for example, Bernanke (1983), Bernanke and Blinder (1988), Beenstock (1989) and, more recently, Freeman and Kydland (2000). However, as the money multiplier was relatively stable for more than 20 years, it became a concern of second-order importance. Although the model we consider is very simple, it manages to identify two variables which may explain the fluctuation in the money multiplier as it is computed in equation (2.22). First there is  $\theta_t$ , the household's borrowing constraint. If we simply consider mortgage contracts offered before the crisis, the loan to value ratio was up to 110% in the UK. After the crisis, it fell to 90% or by almost 20%.

The second variable,  $q_t$ , is defined as

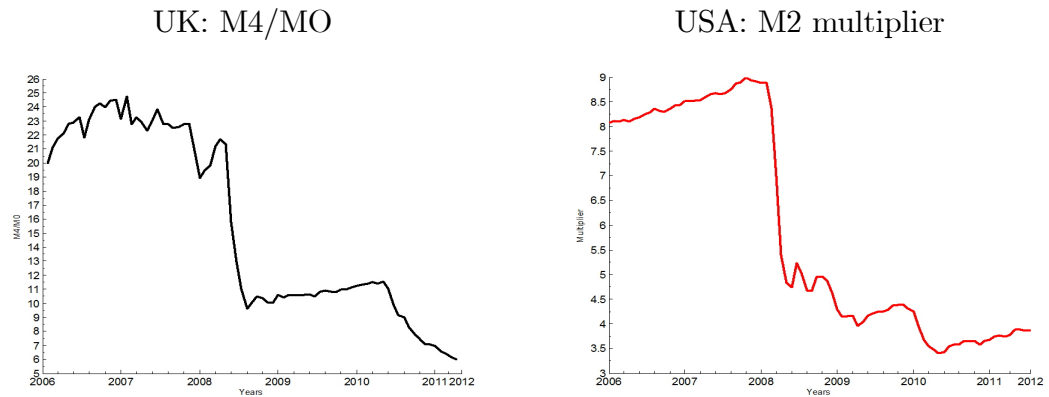
$$q_t = \frac{Q_t h_t}{P_t Y_t}. \quad (2.38)$$

In a relatively stable economy, where  $h$  and  $Y$  do not change, the proxy for  $q$  will be the real price for collateral. If we refer to the mortgage market, the collateral is houses and the real house price index will be a proxy for  $q$ . Therefore, the fall in house prices should reduce the money multiplier. As the money multiplier is significant for the transmission of monetary policy (Bernanke and Blinder, 1988; Goodhart, 2009; Abrams, 2011), its fluctuation should definitely be taken into account when monetary policy is designed. As house prices and the loan to value ratio affect the money multiplier, they cannot be neglected by the monetary

authorities.

As we have previously noted, the money multiplier experienced a significant fall after the last financial crisis. Figure 2 shows the dynamics of the M2 multiplier in the US and the M4 multiplier in the UK.

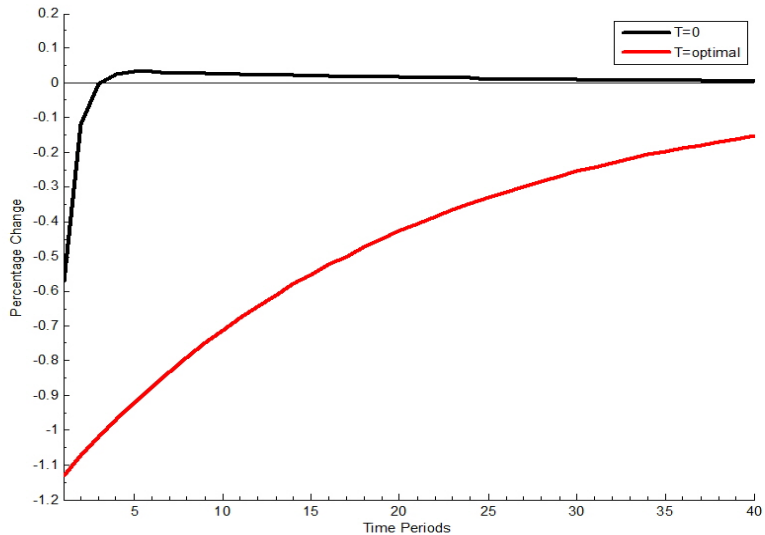
Figure 2.2. Money Multiplier after the 2008 Crisis



*Source: FRED and BoE Database*

The slump of the money multiplier is consistent with our model. Although optimal monetary policy can stabilize output and price fluctuations, it causes an even stronger and much more persistent decrease in the money multiplier than a policy of inaction. Both policies imply a dramatic fall in the money multiplier as we can see from Figure 3, but the stabilization policy almost doubles the size of the fall and causes a much slower recovery.

Figure 2.3. Optimal Transfers



We draw Figure 3 based on a linear approximation to (2.22)

$$\widehat{m}_t = k(\widehat{\theta}_t + \widehat{q}_t). \tag{2.39}$$

It can be easily seen that the optimal policy rule in the absence of a cost-push shock (2.32) implies that optimal transfers should be equal to the negative of the change in the money multiplier

$$\widehat{T}_t = -(\widehat{m}_t - \widehat{m}_{t-1}). \tag{2.40}$$

Therefore, Figure 3 shows that optimal policy entails large expansion immediately after the reduction in LTV and then a gradual contraction in all subsequent periods.

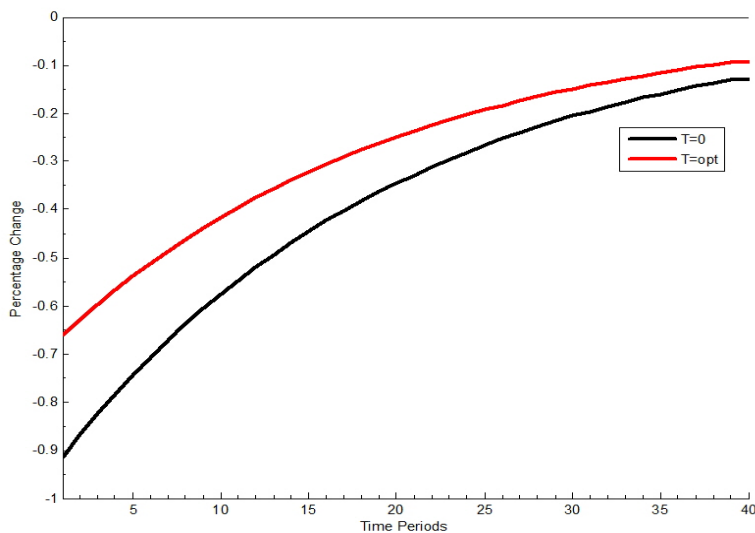
### 2.3.4 Credit Shock and House Prices

It is very intuitive that the relative price of collateral should react to the worsening in the loan to value ratio. The value of collateral in our model has two components. The first comes directly from the utility function (such as housing, for example).

The second is indirect and associated with the use of collateral for borrowing purposes. The larger is LTV  $\theta_t$ , the larger is the indirect component of the collateral value and therefore, the higher is the price of collateral. Formally, this can be observed from equation (2.6). This is why the negative shock to  $\theta_t$  should result in falling house prices.

Lower prices for collateral, in turn, further reduce the amount of available credit. As a consequence, households have less money to finance their consumption and purchase additional housing units. Figure 4 shows how house prices react to the tightening of the households' borrowing constraint in two different cases. The first case is when the government implements the optimal policy rule. The second case is when it keeps the monetary base constant,  $\widehat{T}_t = 0$ . When a negative credit market shock hits the economy, house prices decline in both cases, but optimal policy helps reduce the fall by approximately 20%.

Figure 2.4. Collateral Price and Financial Shock



## Cost-Push Shock, House Prices and Inflation

As expected, the optimal policy stabilizes inflation. However it would be interesting to see how the cost-push shock will affect house prices and relative housing expenditure. When the loan to value ratio is constant and only cost-push shocks hit the economy, there is a trade off between inflation and house price stability. The policy of inaction,  $\widehat{T}_t=0$ , in the absence of credit shocks implies constant house prices as stated in Proposition 2.3.2.

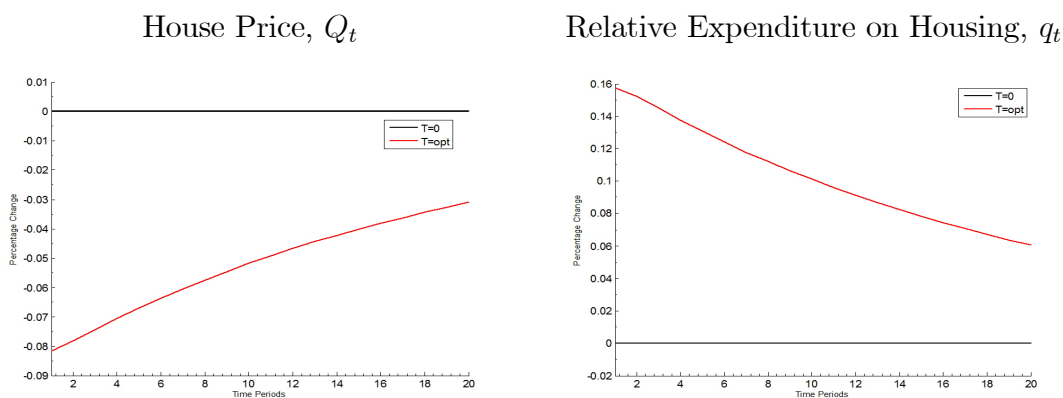
**Proposition 2.3.2** *If  $\widehat{\theta}_t = 0$ , policy  $\widehat{T}_t = 0$  results in house price stability and the stability of the relative collateral value,  $\widehat{q}_t = 0$ .*

**Proof.** *Provided in the Appendix.* ■

Figure 5 shows that in the absence of credit shocks, the stabilization of house prices will result in a higher volatility of CPI inflation. This result is consistent with the findings of Iacoviello (2005).

It is interesting to see that the money multiplier, which is proportional to relative housing expenditure, increases with the cost push shock, when optimal policy is implemented. However, house prices decline in this case, but to a smaller extent than the corresponding decline in output.

Figure 2.5. Simulation of Money Multiplier



## 2.4 Loan to Value Ratio

We have shown that the government ought to include the loan to value ratio ( $q_t$ ) in its policy rule because it delivers the highest welfare measure. In that discussion, we considered  $\theta_t$  to be exogenously given. Perhaps one of the most important questions is to identify the factors which explain the fluctuation in  $\theta_t$ .

Endogenising the loan to value ratio can have a number of very important policy implications, for example if the LTV ratio equals the effective recovery rate of mortgages, so that direct lending to households compromises the balance sheet of the central bank. In that case, Help to Buy and Start Up loans will result in budget losses and a Funding for Loans scheme could in this case result in yet more non-performing loans on the central bank's balance. We are not aware of any model which can assess these consequences.<sup>4</sup>

Although the recovery rate is highly correlated with default risk (Mora, 2012), there are some other explanatory factors which can be influenced by the government.

### 2.4.1 Expected Collateral Inflation

One of the explanatory variables of changes in LTV may be expected house price inflation ( $Q_t$ ) as in Iacoviello (2005). This can easily be modelled by substitution of (2.30).

$$\hat{\theta}_t = \delta E_t \left( \hat{Q}_{t+1} - \hat{Q}_t \right) + u_t, \quad (2.41)$$

where  $\delta < 1$ , and  $u_t$  is a persistent shock unrelated to expected changes in house prices.

Shocks  $u_t$  can in this case be regarded as a shock to expected future house

---

<sup>4</sup>In our model, the central bank simply increases its liabilities in the form of outstanding cash without any back-up on the asset side.

prices which is another factor for consideration for the monetary authorities. In some cases, shocks to expectations do not reflect the changes in fundamentals. The central bank will not compromise its balance sheet by buying collateral and keep it for a longer time period until the negative shock dies out.

In this modified model, we received a very similar impulse response function to the unexpected change in  $u_t$  and still we found that an optimal policy can completely stabilize output and inflation when the credit shock affects the economy. Similarly, in response to cost-push shocks, policy  $\hat{T}_t = 0$  stabilizes house prices but causes a positive and relatively large response of the consumer price inflation.

## **2.4.2 Liquidity, Transaction Costs and the Value of Collateral**

An increase in the collateral value is an effective way to raise the money multiplier. The attractiveness of collateral may increase with liquidity. A positive relation between the liquidity of collateral and the availability of funding is discussed in Brunnermeier and Pedersen (2009). Securitization of collateral is one way of enhancing liquidity. Different types of securitization were used for the American housing market. Thus, according to Frame and White (2005), the Federal Home Loan Mortgage Corporation – commonly known as Freddie Mac – was created to support the mortgage markets by securitizing mortgages. That arrangement worked successfully for at least 30 years before the last crisis. The Funding for Lending scheme is in line with that reasoning. The possibility of a collateral swap for T-Bills should increase the value of collateral which can be used by commercial banks to secure liquidity.

However, collateral securitization could also add to risk and even generate additional moral hazard problems (Ashcraft and Schuermann, 2008). In this context,



the Freddie Mac arrangements were safer than the new measures proposed in the UK. The mortgages that were eligible for securitization with Freddie Mac usually required a 20% downpayment, while the Funding for Lending Scheme does not specify the quality of the loans that can be used in the scheme (Bank of England, 2012). British "Help to Buy" may be even riskier, such that the government will "loan up to 20% of the value of your new build home and "mortgage guarantee" where lenders will be incentivised to make more mortgages available for people with small deposits." (HM Treasury, 2013). It could mean that the government intends to provide for the risk margin charged by the lender as an insurance against construction risks.

Apart from the expected value of collateral, the loan to value ratio should depend on the recovery rate for non-performing loans. The recovery rate negatively depends on the transaction costs associated with selling the repossessed assets. Any taxes collected during that process would negatively contribute to LTV. One straightforward recommendation can be to abolish stamp duties for repossessed properties.

The other way of increasing the value of collateral is to encourage the construction and consumption of housing. According to Frame and White (2005), the US government uses a tax deduction of the mortgage interest and accelerated depreciation on rental housing for that purpose.

## **2.5 Preference for Money**

Michael Woodford (2012) has suggested that the assumption about perfect substitutability between financial instruments could exclude from the model some very important economic realities. One way to make money and loans imperfectly

substitutable is to model real money as a part of the preference function<sup>5</sup>. The intuition is standard: people may prefer to hold money as it is a liquid asset that can be used for certain transactions where it may be difficult to use credit; i.e. rent payment or home repairs (See Telyukova and Wright, 2008). This would justify an additional positive value of cash money compared to credit. However, there are other occasions when money may be less desired. For example, in electronic transactions, especially when payment is made to an unknown supplier, credit cards are safer to use than debit cards<sup>6</sup>. This is just one example to demonstrate that as financial market develops the preference for cash may decline.

To capture imperfect substitutability between cash and credit, we modify household preference (3.1) in the following way

$$U_t = E_t \sum_{t=0}^{\infty} \beta^t u(Y_t, h_t, L_t) = E_t \sum_{t=0}^{\infty} \beta^t \left( \log Y_t + \eta \log h_t + \xi \log \left( \frac{M_t}{P_t} \right) - \lambda \frac{L_t^{v+1}}{v+1} \right), \quad (2.42)$$

where parameter  $\xi$  reflects the preference of using money. As before, the household maximizes its utility subject to constraints (2.2, 2.3 and 2.5). The first order conditions will be slightly different.

### 2.5.1 Labour Supply

First, the labour supply equation (2.9) of the main model will become (2.43)

$$-U'_{L,t} = \frac{W_t}{P_t} \left( U'_{m,t} + U'_{c,t} \right), \quad (2.43)$$

where  $U'_m$  is the marginal utility from real money. That can not be derived from a conventional money in utility model without credit constraint<sup>7</sup>, where labour supply equation (2.9) is usual. The difference is that in our model households can not

<sup>5</sup>We are grateful to the referee for that interesting idea.

<sup>6</sup>Section 75 of the Consumer Credit Act 1974 states that a cardholder's credit card company jointly liable with the merchant for any purchases made on a credit card between £100 and £30,000.

<sup>7</sup>See for example Walsh (2010), ch 2.

borrow cash. The household can only increase money holding by supplying more labour. As money is used for consumption, it gives double benefits: through utility from consuming good and through pleasure from possessing money. Therefore, households supply more labour when they have a greater preference for cash<sup>8</sup>.

## 2.5.2 Credit Constraint

It is even more interesting see how the condition when the credit constraint is binding will change. Equation (2.7) is becomes

$$\mu_{2t}P_t = U'_{ct} - \frac{\beta(1+r_t)}{\pi_{t+1}} (U'_{m,t+1} + U'_{c,t+1}) > 0, \quad (2.44)$$

where  $\mu_{2t}$  is the Lagrange multiplier to constraint (2.3). The probability that the constraint is binding declines with the value of  $U'_m$ . In particular, if  $U'_m$  is large the constraint can be non-binding, in contrast to the case when  $U'_m = 0$ .

Now we will investigate when the credit constraint is binding in a steady state. Condition (2.44) becomes

$$\left(1 - \frac{\beta(1+r)}{\pi}\right) U'_c - \frac{\beta(1+r)}{\pi} U'_m > 0. \quad (2.45)$$

Consider the case when  $r$  is sufficiently small,  $1 - \frac{\beta(1+r)}{\pi} > 0$ . Then, as we noticed above, inequality (2.45) is satisfied if  $U'_m = 0$ . As  $U'_m$  increases, the constraint may become non-binding. We can compute a threshold such that if preferences towards money grow stronger than that value, the credit constraint becomes non-binding.

**Proposition 2.5.1** *If the real interest rate is smaller than the inverse discount factor,  $\frac{(1+r)}{\pi} < \frac{1}{\beta}$ , There exists a value  $\xi^*$ , such that inequality (2.45) is satisfied*

---

<sup>8</sup>For a particular preference form (2.42) equation (2.43)

$$\lambda L_t^v = w_t (\xi / (M_t / P_t) + 1 / Y_t)$$

and the labour supply increases with  $\xi$  and declines with real money.

for preferences as in (2.42), when  $0 \leq \xi < \xi^*$ , where

$$\xi^* = \left(1 - \frac{\theta\eta}{1 - \beta}\right) \frac{1 - \frac{\beta(1+r)}{\pi}}{\frac{\beta(1+r)}{\pi}}. \quad (2.46)$$

**Proof.** See appendix. ■

In Proposition 2.5.1 we define the value of parameters under which the credit constraint is binding in steady state. The larger is  $\xi^*$  the higher is the probability that the constraint is binding. From formula (2.46) we directly obtain Corollary 2.5.2.

**Corollary 2.5.2**  $\xi^*$  increases with i) inflation,  $\pi$ ; and

$\xi^*$  declines with ii) nominal interest rate,  $r$ ; iii) real interest rate,  $\frac{(1+r)}{\pi}$ ; iv) household patience,  $\beta$ ; v) Loan to value ratio,  $\theta$ ; vi) intrinsic value of housing,  $\eta$ .

Corollary 2.5.2 helps to understand the evolution of the importance of the credit constraint over time. It could have been that the LTV ratio were not so important in the past because the real interest rate was high and thus  $\xi^*$  was low. At the same time, the financial market was less developed and borrowing practice was less socially acceptable, hence  $\xi$  was high. Perhaps, it was a time when  $\xi$  was larger than  $\xi^*$  and the borrowing constraint was non-binding. As a consequence the loan to value ratio was not so important. According to Peñaloza and Barnhart (2011) the culture of debt/credit payment became more acceptable overtime and, with further development of the financial market, hence  $\xi$  might have fallen below  $\xi^*$  making the LTV ratio so vital.

## 2.6 Conclusion

We have shown that in contrast to some other findings presented in the literature, such as Bernanke and Gertler (2001) and Iacoviello (2005), a simple stylized macro

model could yield results which are in favor of including a credit market variables into the optimal monetary policy rule. In particular, monetary authorities should adjust their rule to unfavorable changes in loan conditions, such as a fall in the loan to value ratio or the relative price of collateral, i.e. the real house price index.

We derived our results in a model where social welfare consists of output and inflation gaps and thus, credit market variables are not part of the government direct objective. However, as mentioned in Svensson (2009), credit capacities affect output and inflation through households' behavioral constraints. A lower households' borrowing rate reduces demand and, as a result, causes deflation. We have shown that an exogenous decrease in the loan to value ratio can be offset by expansionary monetary policy in such a way that credit tightening will neither affect output nor the consumer price inflation.

We connected our results to the money multiplier which is the most important variable in propagation of monetary policy, as discussed in Bernanke and Blinder (1988) and recently in Abrams (2011). Indeed, it is very intuitive that the expansion of the monetary base should be larger when the money multiplier falls. And since the multiplier depends on the loan to value ratio and real house prices, optimal monetary policy should react to their fluctuations. Finally, we have shown that optimal policy generates a large and persistent fall in the money multiplier in response to credit shocks.

Although our model is helpful in providing some justification for quantitative easing and explaining the fall in the money multiplier, there are a number of important extensions that should be addressed. First, the volatility of the loan to value ratio requires an economic explanation. This would allow for a better assessment of a number of currently proposed or adopted policy measures. Second, the assets of the central bank are not modelled directly and, therefore, it is impossible to see

how the risk taken on the central bank balance sheet will affect the economy. If assets are risky, the central bank will face difficulties when it decides to implement a monetary contraction.

Finally, the money multiplier per se does not generate any value in our model, and the steady-state output value does not depend on any financial variables. This is not the case according to King and Levine (1993) and Freeman and Kydland (2000) who found that total borrowing by the non-financial sector has a positive effect on economic growth.

## 2.7 Appendix

### 2.7.1 First-Order Conditions for Household Optimization

The household maximizes the expected discounted sum of future utility (3.1) subject to constraints (2.2, 2.3, 2.5). The corresponding Lagrangian is:

$$\begin{aligned}\mathcal{L} = & E_t \sum_{t=0}^{\infty} \beta^t [(U(C_t, L_t, h_t) \\ & + \mu_{1t}(-P_t C_t - Q_t(h_t - h_{t-1}) + B_t + M_t) \\ & + \mu_{2t}(-B_t + \theta_t Q_t h_t,) \\ & + \mu_{3t}(-M_t + W_t L_t + \Pi_t + T_t - (1 + r_{t-1}) B_{t-1})];\end{aligned}$$

where  $\mu_{it}$  are Lagrange multipliers. The first-order conditions with respect to consumption ( $C_t$ ), housing quantity ( $h_t$ ), debt ( $B_t$ ) and money ( $M_{t+1}$ ) are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = U'_c(C_t) - \mu_{1t} P_t = 0; \quad (2.47)$$

$$\frac{\partial \mathcal{L}}{\partial h_t} = U'_h - \mu_{1t} Q_t + \beta \mu_{1t+1} Q_{t+1} + \mu_{2t} \theta_t Q_t = 0; \quad (2.48)$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = U'_L + \mu_{3t} W_t \quad (2.49)$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = \mu_{1t} - \mu_{2t} - \beta(1 + r_t) \mu_{3t+1} = 0; \quad (2.50)$$

$$\frac{\partial \mathcal{L}}{\partial M_t} = \mu_{1t} - \mu_{3t} = 0. \quad (2.51)$$

First we use (2.47) and (2.49) to claim that constraints (2.2) and (2.5) are binding and for computing Lagrange multipliers  $\mu_{1t}$  and  $\mu_{3t}$ . That and (2.51) immediately give

$$\frac{U'_c(C_t)}{P_t} = -\frac{U'_L}{W_t},$$

as the labour supply equation. Furthermore, equation (2.50) becomes

$$\mu_{2t} = \frac{U'_c(C_t)}{P_t} - (1 + r_t) \beta \frac{U'_c(C_{t+1})}{P_{t+1}}.$$

We need to check if  $\mu_{2t}$  is strictly positive. Consider a deterministic steady state with price stability and zero interest rate. In this case,

$$\mu_2 = \frac{U'_c(C)}{P} (1 - \beta) > 0.$$

That is exactly the steady state that we will investigate; assuming that the shocks are not too large, the Lagrange multiplier should be positive and therefore constraint (2.3) should be binding.

Finally, combining (2.48) and (2.50) we get

$$U'_h - U'_c(C_t) \frac{Q_t}{P_t} + \beta U'_c(C_{t+1}) \frac{Q_{t+1}}{P_{t+1}} + U'_c(C_t) \frac{Q_t}{P_t} \theta_t - \beta \frac{U'_c(C_{t+1})}{P_{t+1}} (1 + r) \theta_t Q_t = 0. \quad (2.52)$$

This is the same as (2.6) in the main text.

## 2.7.2 Optimal Steady State

Following Benigno and Woodford (2012), we will find the best steady state for the optimal commitment policy from a timeless perspective. The policy maker will maximize household utility (2.13) subject to constraints (2.14-2.21). It is easy to see that constraints (2.14) and (2.19-2.21) are only used to define  $q_t$ ,  $M_t$ ,  $P_t$  and  $T_t$ . Therefore, we write a Lagrangian to reduced model (2.15-2.18 ).

$$\mathcal{L}_t = E_t \sum_{t=0}^{\infty} \beta^t \left\{ \log Y_t - \lambda \frac{Y_t^{v+1} \Delta_t^{v+1}}{v+1} \right. \quad (2.53)$$

$$\left. + \Lambda_{2t} (-X_t + Y_t + a\beta E_t \pi_{t+1}^{\varepsilon-1} X_{t+1}) \right. \quad (2.54)$$

$$\left. + \Lambda_{3t} \left( -Z_t + \phi_t \lambda Y_t^{v+2} \Delta_t^v + a\beta E_t \pi_{t+1}^{\varepsilon} Z_{t+1} \right) \right. \quad (2.55)$$

$$\left. + \Lambda_{4t} \left[ X_t \left( \frac{1 - \alpha \pi_t^{\varepsilon-1}}{1 - a} \right)^{\frac{1}{1-\varepsilon}} - \frac{\varepsilon}{\varepsilon - 1} Z_t \right] \right. \quad (2.56)$$

$$\left. + \Lambda_{5t} \left[ -\Delta_t + \alpha \Delta_{t-1} \pi_t^{\varepsilon} + (1 - \alpha) \left( \frac{1 - \alpha \pi_t^{\varepsilon-1}}{1 - a} \right)^{\frac{\varepsilon}{\varepsilon-1}} \right] \right\}. \quad (2.57)$$



The first-order conditions are

$$\begin{aligned}
\frac{\partial \mathcal{L}_t}{\partial \pi_t} \pi_t &= +(\varepsilon - 1) \Lambda_{2t-1} a \pi_t^{\varepsilon-1} X_t \\
&+ \varepsilon \Lambda_{3t-1} a \pi_t^{\varepsilon-1} Z_t + \Lambda_{4t} X_t \left( \frac{1 - \alpha \pi_t^{\varepsilon-1}}{1 - a} \right)^{\frac{1}{1-\varepsilon}} \frac{\alpha \pi_t^{\varepsilon-1}}{1 - \alpha \pi_t^{\varepsilon-1}} \\
&+ \Lambda_{5t} \varepsilon \Delta_{t-1} \alpha \left( \pi_t^\varepsilon - \pi_t^{\varepsilon-1} \left( \frac{1 - \alpha \pi_t^{\varepsilon-1}}{1 - a} \right)^{\frac{1}{\varepsilon-1}} \right); \tag{2.58}
\end{aligned}$$

$$\frac{\partial \mathcal{L}_t}{\partial Y_t} Y_t = 1 - \lambda Y_t^{v+1} \Delta_t^{v+1} + \Lambda_{2t} Y_t + \Lambda_{3t} \phi_t \lambda Y_t^{v+2} \Delta_t^v (v+2); \tag{2.59}$$

$$\frac{\partial \mathcal{L}_t}{\partial \Delta_t} = -\lambda Y_t^{v+1} \Delta_t^v + \Lambda_{3t} v \phi_t \lambda Y_t^{v+2} \Delta_t^{v-1} - \Lambda_{5t} + \beta \alpha \Lambda_{5t+1} \pi_t^\varepsilon; \tag{2.60}$$

$$\frac{\partial \mathcal{L}_t}{\partial X_t} = -\Lambda_{2t} + a \pi_t^{\varepsilon-1} \Lambda_{2t-1} + \Lambda_{4t} \left( \frac{1 - \alpha \pi_t^{\varepsilon-1}}{1 - a} \right)^{\frac{1}{1-\varepsilon}}; \tag{2.61}$$

$$\frac{\partial \mathcal{L}_t}{\partial Z_t} = -\Lambda_{3t} + a \pi_t^\varepsilon \Lambda_{3t-1} - \Lambda_{4t} \frac{\varepsilon}{\varepsilon - 1}. \tag{2.62}$$

We can easily verify that prices are stable in steady state, that is  $\pi = 1$ . From constraints (2.14-2.21) and first-order conditions (2.58-2.62), we compute the steady-state values for endogenous variables and Lagrange multipliers

$$\begin{aligned}
1 &= \frac{\varepsilon}{\varepsilon-1} \phi \lambda Y^{v+1}; & \Lambda_2 &= \frac{1 - \lambda Y^{v+1}}{Y^{(v+1)}}; \\
X &= \frac{Y}{1 - a\beta}; & \Lambda_3 &= -\Lambda_2 \frac{\varepsilon}{\varepsilon-1}; \\
Z &= \frac{\varepsilon-1}{\varepsilon} X; & \Lambda_4 &= \Lambda_2 (1 - a); \\
\Delta &= 1; & \Lambda_5 &(1 - \beta\alpha)(v+1) = -\lambda Y^{v+1} - v.
\end{aligned} \tag{2.63}$$

### 2.7.3 Linear Approximation to the Constraints of Private Behavior

To make our model more tractable and comparable to the standard New Keynesian version, we log linearize equations (2.14)-(2.21) around the zero inflation steady state<sup>9</sup>.

---

<sup>9</sup>All variables with hats **are here** expressed in terms of percentage deviations from **the** steady state.

$$\beta \widehat{q}_{t+1} = \widehat{q}_t(1 - \theta(1 - \beta(1 + r))) - \widehat{\theta}_t \theta(1 - \beta(1 + r)) \quad (2.64)$$

$$+ r\beta\theta\widehat{r}_t + \beta\theta(1 + r)(\widehat{Y}_t - E_t\widehat{\pi}_{t+1} - E_t\widehat{Y}_{t+1})$$

$$\widehat{X}_t = (1 - \alpha\beta)\widehat{Y}_t + \alpha\beta E_t(\widehat{X}_{t+1} + (\varepsilon - 1)\widehat{\pi}_{t+1}) \quad (2.65)$$

$$\widehat{Z}_t = (1 - \alpha\beta) \left[ (v + 2)\widehat{Y}_t + \widehat{\phi}_t \right] + \alpha\beta E_t(\varepsilon\widehat{\pi}_{t+1} + \widehat{Z}_{t+1}) \quad (2.66)$$

$$\widehat{Z}_t = \widehat{X}_t + \frac{a}{1 - a}\widehat{\pi}_t \quad (2.67)$$

$$\widehat{\Delta}_t = \alpha\widehat{\Delta}_{t-1} \quad (2.68)$$

$$\widehat{M}_t = \widehat{P}_t + \widehat{Y}_t - \frac{\theta q}{1 - \theta q}(\widehat{\theta}_t + \widehat{q}_t) \quad (2.69)$$

$$\widehat{M}_t = \widehat{M}_{t-1} + \widehat{T}_t \quad (2.70)$$

$$\widehat{\pi}_t = \widehat{P}_t - \widehat{P}_{t-1}. \quad (2.71)$$

We start with the log linear approximation to constraints (2.14-2.21) around the optimal steady state. As shown in Benigno and Woodford (2005), constraint (2.18) implies that  $\widehat{\Delta}_t = \alpha\widehat{\Delta}_{t-1} + O2$ , and the log deviation of the relative price dispersion is of second-order importance when price stability is optimal. Therefore, the log linearization of (2.14-2.17) gives (2.64-2.67). In turn, (2.65-2.67) can be combined into one equation which represents the Phillips Curve. First, combine equations (2.66) and (2.67):

$$\widehat{X}_t + \frac{a}{1 - a}\widehat{\pi}_t = (1 - \alpha\beta) \left[ (v + 2)\widehat{Y}_t + \widehat{\phi}_t \right] + \alpha\beta \left( \varepsilon\widehat{\pi}_{t+1} + \widehat{X}_{t+1} + \frac{a}{1 - a}\widehat{\pi}_{t+1} \right). \quad (2.72)$$

We subtract expression (2.65) and simplify to obtain the New Keynesian Phillips curve

$$\widehat{\pi}_t = \frac{1 - a}{a}(1 - \alpha\beta) \left[ (v + 2)\widehat{Y}_t + \widehat{\phi}_t \right] + \beta E_t\widehat{\pi}_{t+1}. \quad (2.73)$$

## 2.7.4 Second-Order Approximation

Applying the Benigno and Woodford (2012) algorithm, we will get the social welfare function which consists of the sum of squares of the output and inflation gaps. In particular, Benigno and Woodford (2012) show that the second-order approximation to social welfare can be computed as a sum of pure second-order terms.

$$U_t = E_t \sum_{t=0}^{\infty} \beta^t u_t = E_t \sum_{t=0}^{\infty} \beta^t \left[ S(u_t) + \sum_i \Lambda_i S(F_i) \right], \quad (2.74)$$

where  $F_i$  is the dynamic constraint imposed by household behavior and  $\Lambda_i$  is the value of the corresponding Lagrange multiplier in steady state. Furthermore,  $S(\cdot)$  is a functional defined on twice differentiable functions of multiple arguments  $F(X_t)$ ,  $X_t = [X_{1t}, \dots, X_{nt}]$  as follows

$$S(F(X_t)) = \widehat{X}'_t X' \nabla^2 F(X) X \widehat{X}_t = \sum_{jk} \frac{\partial^2 F(X)}{\partial X_j \partial X_k} X_k X_j \widehat{X}_{kt} \widehat{X}_{jt},$$

where  $\widehat{X}_{kt}$  is the log deviation of variable  $X_{kt}$  from its steady-state value  $X_k$ . To implement that algorithm, we need to apply functional  $S$  to constraints (2.15-2.18) since all other constraints are not binding and the corresponding Lagrange multipliers have zero values in the optimal steady state

$$S_u = S\left(\log Y_t - \lambda \frac{Y_t^{v+1} \Delta_t^{v+1}}{v+1}\right) = -\widehat{Y}_t^2 - v\lambda Y^{v+1} \widehat{Y}_t^2; \quad (2.75)$$

$$\begin{aligned} S_2 &= S(-X_t + Y_t + a\beta E_t \pi_{t+1}^{\varepsilon-1} X_{t+1}) \\ &= a\beta(\varepsilon - 1)(\varepsilon - 2) X E_t \widehat{\pi}_{t+1}^2 + 2a\beta(\varepsilon - 1) X E_t \widehat{\pi}_{t+1} \widehat{X}_{t+1}; \end{aligned} \quad (2.76)$$

$$\begin{aligned} S_3 &= S(-Z_t + \phi_t \lambda Y_t^{v+2} \Delta_t^v + a\beta E_t \pi_{t+1}^\varepsilon Z_{t+1}) \\ &= (v+2)(v+1) \phi \lambda Y^{v+2} \widehat{Y}_t^2 + 2(v+2) \phi \lambda Y^{v+2} \widehat{Y}_t \phi_t \\ &\quad + 2a\beta Z \varepsilon \widehat{\pi}_{t+1} \widehat{Z}_{t+1} + a\beta Z \varepsilon (\varepsilon - 1) \widehat{\pi}_{t+1}^2; \end{aligned} \quad (2.77)$$

$$\begin{aligned} S_4 &= S\left(X_t \left(\frac{1 - \alpha \pi_t^{\varepsilon-1}}{1-a}\right)^{\frac{1}{1-\varepsilon}} - \frac{\varepsilon}{\varepsilon-1} Z_t\right) \\ &= 2\frac{a}{1-a} X \widehat{\pi}_t \widehat{X}_t + \frac{a}{1-a} X \left(\varepsilon \frac{a}{1-a} + \varepsilon - 2\right) \widehat{\pi}_{t+1}^2; \end{aligned} \quad (2.78)$$

$$\begin{aligned} S_5 &= S\left(-\Delta_t + \alpha \Delta_{t-1} \pi_t^\varepsilon + (1-\alpha) \left(\frac{1 - \alpha \pi_t^{\varepsilon-1}}{1-a}\right)^{\frac{\varepsilon}{\varepsilon-1}}\right) \\ &= \frac{a}{1-a} \left( (1-a) \varepsilon (\varepsilon - 1) + \varepsilon \frac{a}{1-a} + (\varepsilon - 2) \right) \widehat{\pi}_t^2. \end{aligned} \quad (2.79)$$

Using steady-state values (2.63), we can compute the welfare approximation to (2.74)

$$W = S_u + \Lambda_2 S_2 + \Lambda_3 S_3 + \Lambda_4 S_4 + \Lambda_5 S_5. \quad (2.80)$$

First, we simplify  $S_3$ . We will use (2.67) to substitute for the  $\widehat{\pi}_{t+1} \widehat{Z}_{t+1}$  term:

$$\widehat{Z}_{t+1} \widehat{\pi}_{t+1} = \widehat{X}_{t+1} \widehat{\pi}_{t+1} + \frac{a}{1-a} \widehat{\pi}_{t+1}^2 + O3.$$

$$S_u = -(1 + v\lambda Y^{v+1}) \widehat{Y}_t^2; \quad (2.81)$$

$$\Lambda_2 S_2 = \Lambda_2 a \beta (\varepsilon - 1) \frac{Y}{1 - a\beta} \left( (\varepsilon - 2) E_t \widehat{\pi}_{t+1}^2 + 2 E_t \widehat{\pi}_{t+1} \widehat{X}_{t+1} \right); \quad (2.82)$$

$$\begin{aligned} \Lambda_3 S_3 &= -\Lambda_2 (v + 2) (v + 1) Y \widehat{Y}_t^2 + -\Lambda_2 2(v + 2) Y \widehat{Y}_t \widehat{\phi}_t; \\ &- \Lambda_2 2a\beta \frac{Y}{1 - a\beta} \varepsilon \left( \widehat{X}_{t+1} \widehat{\pi}_{t+1} + \frac{a}{1 - a} \widehat{\pi}_{t+1}^2 \right) \\ &- \Lambda_2 a \beta \frac{Y}{1 - a\beta} \varepsilon (\varepsilon - 1) \widehat{\pi}_{t+1}^2; \end{aligned} \quad (2.83)$$

$$\Lambda_4 S_4 = 2\Lambda_2 a \frac{Y}{1 - a\beta} \widehat{\pi}_t \widehat{X}_t + \Lambda_2 a \frac{Y}{1 - a\beta} \left( \varepsilon \frac{a}{1 - a} + \varepsilon - 2 \right) \widehat{\pi}_{t+1}^2; \quad (2.84)$$

$$\Lambda_5 S_5 = \frac{-a}{1 - a} \frac{\lambda Y^{v+1} + v}{(1 - \beta\alpha)(v + 1)} \widehat{\pi}_t^2 \left( (1 - a) \varepsilon (\varepsilon - 1) + \varepsilon \frac{a}{1 - a} + (\varepsilon - 2) \right) \quad (2.85)$$

Now we use that for any dynamic variable  $x_t$ ,

$$\sum_{t=0}^{+\infty} \beta^t x_{t+1} = \frac{1}{\beta} \sum_{t=0}^{+\infty} \beta^t x_t - \frac{1}{\beta} x_0 = \frac{1}{\beta} \sum_{t=0}^{+\infty} \beta^t x_t + tip,$$

where  $x_0$  is considered a "precommitted" variable which cannot be changed because of commitment and therefore, it is regarded as "term independent of policy (*tip*)".

Finally, we can compute the welfare approximation to (2.74).

$$U_t + O3 = E_t \sum_{t=0}^{+\infty} \beta^t W = -E_t \sum_{t=0}^{+\infty} \beta^t \left[ \alpha_C \left( \widehat{Y}_t + \alpha_\phi \widehat{\phi}_t \right)^2 + \alpha_\pi \widehat{\pi}_t^2 \right], \quad (2.86)$$

where the parameters are defined as follows

$$\begin{aligned} \alpha_C &= (1 + v - 2\lambda Y^{v+1}); \\ \alpha_\phi &= \frac{1 - \lambda Y^{v+1}}{1 + v - 2\lambda Y^{v+1}} \frac{v + 2}{v + 1}; \\ 1 &= \frac{\varepsilon}{\varepsilon - 1} \phi \lambda Y^{v+1}; \\ \alpha_\pi &= \frac{a \left[ (1 - \lambda Y^{v+1}) \varepsilon + (\lambda Y^{v+1} + v) \left( (1 - a) \varepsilon (\varepsilon - 1) + \frac{\varepsilon - 2 + 2a}{1 - a} \right) \right]}{(1 - a) (1 - \beta\alpha) (v + 1)}. \end{aligned}$$

### 2.7.5 Solution to Central Bank LQ Problem

Central bank maximizes (2.86) subject to constraint (2.73) only, since all other constraints are non-binding,

$$J_t = -\frac{1}{2}E_t \sum_{t=1}^{\infty} \beta^t \left[ \begin{array}{c} \alpha_C \left( \widehat{Y}_t - \alpha_\phi \widehat{\phi}_t \right)^2 + \alpha_\pi \widehat{\pi}_t^2 \\ + \lambda_t \left( -\widehat{\pi}_t + \frac{1-a}{a}(1-\alpha\beta) \left[ (v+2)\widehat{Y}_t + \widehat{\phi}_t \right] + \beta\widehat{\pi}_{t+1} \right) \end{array} \right]. \quad (2.87)$$

The first-order conditions imply the optimal inflation dynamics

$$\widehat{\pi}_t = \rho \left( \widehat{Y}_{t-1} - \widehat{Y}_t - \alpha_\phi \left( \widehat{\phi}_{t-1} - \widehat{\phi}_t \right) \right), \quad (2.88)$$

where  $\rho = \frac{\alpha_C}{\alpha_\pi} \frac{a}{(1-\alpha\beta)(v+2)(1-a)}$ .

### 2.7.6 Proof of Proposition 2.2.1

In this appendix we will show that a moderate fall in the LTV ratio can be neutralized by lowering the interest rate and without transfers. Consider model (2.14)-(2.21) and a policy which aims to achieve a complete stabilization of the credit shock by the means of interest rate. That is  $T_t = T_{t+1} = 0$ ,  $\pi_t = \pi_{t+1} = 1$ ;  $Y_{t+1} = Y_t = Y_{t-1}$ . That implies constant velocity of cash and from (2.19) it follows that product  $\theta_t q_t$  is also constant.

$$\theta_{t+1} q_{t+1} = \theta_t q_t = \theta_{t-1} q_{t-1} = A; \quad (2.89)$$

To compute the interest rate dynamics for stabilizing policy we will combine (2.14) for periods  $t$  and  $t-1$  with (2.89) and on the assumption that prices and output are stable,  $\frac{Y_t}{\pi_{t+1} Y_{t+1}} = 1$ , we derive

$$r_t = r_{t-1} + \frac{1}{\beta} \left( \frac{1}{\theta_{t-1}} - \frac{1}{\theta_t} \right) - \left( E_{t-1} \frac{1}{\theta_t} - E_t \frac{1}{\theta_{t+1}} \right)$$

### 2.7.7 Proof of Proposition 2.3.2

**Proposition 2.3.2** *If  $\hat{\theta}_t = 0$ , a policy of inaction,  $\hat{T}_t = 0$ , results in house price stability and the stability of the relative collateral value.*

**Proof.** Consider dynamics (2.28)-(2.29) with  $\hat{T}_t = 0$ ,  $\hat{\theta}_t = 0$ .

$$\hat{\pi}_t = \frac{1-a}{a}(1-\alpha\beta) \left[ (v+2)\hat{Y}_t + \hat{\phi}_t \right] + \beta E_t \hat{\pi}_{t+1}; \quad (2.90)$$

$$\beta E_t \hat{q}_{t+1} = \hat{q}_t(1-\theta(1-\beta)) + \beta\theta(\hat{Y}_t - E_t \hat{\pi}_{t+1} - E_t \hat{Y}_{t+1}); \quad (2.91)$$

$$\hat{\pi}_t = \hat{Y}_{t-1} - \hat{Y}_t + \frac{\theta q}{1-\theta q} (\hat{q}_t - \hat{q}_{t-1}). \quad (2.92)$$

Plugging Equation (2.92) with one lead into (2.91), we would get that the dynamic for relative house expenditure does not depend on a shock or any other variable in the system.

$$\beta E_t \hat{q}_{t+1} = \hat{q}_t(1-\theta(1-\beta)) - \beta\theta \frac{\theta q}{1-\theta q} (E_t \hat{q}_{t+1} - \hat{q}_t),$$

which implies complete stability,  $\hat{q}_t = 0$ . Moreover, by definition,  $Q_t = q_t P_t Y_t$  inflation of house prices is

$$\hat{\pi}_{Q_t} := \hat{Q}_t - \hat{Q}_{t-1} = \hat{\pi}_t + \hat{Y}_t - \hat{Y}_{t-1} + (\hat{q}_t - \hat{q}_{t-1}).$$

Combining it with (2.92) and  $\hat{q}_t = 0$ , we get zero house price inflation in every period. ■

### 2.7.8 Money in the Utility Function

The Lagrangian to the household problem when there is a preference over the means of payment is giving by:

$$\begin{aligned} \mathcal{L} = & E_t \sum_{t=0}^{\infty} \beta^t \left[ \left( U(C_t, \frac{M_t}{P_t}, L_t, h_t) \right) \right. \\ & + \mu_{1t} (-P_t C_t - Q_t (h_t - h_{t-1}) + B_t + M_t) \\ & + \mu_{2t} (-B_t + \theta_t Q_t h_t, ) \\ & \left. + \mu_{3t} (-M_t + W_t L_t + \Pi_t + T_t - (1 + r_{t-1}) B_{t-1}) \right]; \end{aligned}$$

where  $\mu_{it}$  are Lagrange multipliers. The first-order conditions with respect to consumption ( $C_t$ ), housing quantity ( $h_t$ ), debt  $B_t$  and money ( $M_{t+1}$ ) are:

$$\frac{\partial \mathcal{L}}{\partial C_t} = U'_c(C_t) - \mu_{1t}P_t = 0; \quad (2.93)$$

$$\frac{\partial \mathcal{L}}{\partial h_t} = U'_{h,t} - \mu_{1t}Q_t + \beta\mu_{1t+1}Q_{t+1} + \mu_{2t}\theta_tQ_t = 0; \quad (2.94)$$

$$\frac{\partial \mathcal{L}}{\partial L_t} = U'_{L,t} + \mu_{3t}W_t = 0; \quad (2.95)$$

$$\frac{\partial \mathcal{L}}{\partial B_t} = \mu_{1t} - \mu_{2t} - \beta(1+r_t)\mu_{3t+1} = 0; \quad (2.96)$$

$$\frac{\partial \mathcal{L}}{\partial M_t} = \frac{U'_m}{P_t} + \mu_{1t} - \mu_{3t} = 0. \quad (2.97)$$

It changes the labor supply equation

## 2.7.9 Proof of Proposition 2.5.1

**Proof.** Condition (2.45) for preference function (2.42) implies

$$\frac{M}{PC} > \xi \frac{\frac{\beta(1+r)}{\pi}}{1 - \frac{\beta(1+r)}{\pi}}. \quad (2.98)$$

From the first order conditions (2.93), (2.94), (2.96) and (2.97), we get the Euler equation for investment in the housing market

$$U'_{h,t} - U'_{c,t} \frac{Q_t}{P_t} + \beta U'_{c,t+1} \frac{Q_{t+1}}{P_{t+1}} + \theta_t \frac{Q_t}{P_t} \left[ U'_{ct} - \frac{\beta(1+r_t)}{\pi_{t+1}} (U'_{m,t+1} + U'_{c,t+1}) \right] = 0;$$

which in steady state is the same as (2.99)

$$\eta - \frac{Q}{PC} (1 - \beta) + \theta \frac{Q}{PC} \left( \left[ 1 - \frac{\beta(1+r)}{\pi} \right] - \xi \frac{\beta(1+r)}{\pi} \frac{PC}{M} \right) = 0. \quad (2.99)$$

Recall that if the constraint is binding, then  $\theta \frac{Q}{CP} = 1 - \frac{M}{CP}$ . We use letter  $u$  to denote the velocity of money,  $u = \frac{M}{CP}$ ; equation (2.99) becomes

$$\eta - \frac{(1-u)}{\theta} (1 - \beta) + (1-u) \left( \left[ 1 - \frac{\beta(1+r)}{\pi} \right] - \xi \frac{\beta(1+r)}{\pi} \frac{1}{u} \right) = 0.$$



Consider the function  $f(u) := \eta - \frac{(1-u)}{\theta} (1-\beta) + (1-u) \left( \left[ 1 - \frac{\beta(1+r)}{\pi} \right] - \xi \frac{\beta(1+r)}{\pi} \frac{1}{u} \right)$ . It is easy to see that when  $\xi > 0$ , there exists a unique solution  $u^* \in [0, 1]$  such that  $f(u^*) = 0$ . To prove it we need to note that,  $f(u)u$  is a quadratic function with positive first coefficient and  $\lim_{u \rightarrow 0} f(u) = -\infty$ ,  $f(1) = \eta > 0$ . That implies that for any  $u \in [0, u^*)$ ,  $f(u) < 0$ ; and for any  $u > u^*$ , it follows that  $f(u) > 0$ . To prove that the credit constraint is binding, we simply need to show that  $u^* > \xi \frac{\frac{\beta(1+r)}{\pi}}{1 - \frac{\beta(1+r)}{\pi}}$ , which in the case of positive  $\xi$  and small interest rate,  $\frac{\beta(1+r)}{\pi} < 1$ , is equivalent to  $f\left(\xi \frac{\frac{\beta(1+r)}{\pi}}{1 - \frac{\beta(1+r)}{\pi}}\right) < 0$ . That is the same as condition  $\xi < \left(1 - \frac{\theta\eta}{1-\beta}\right) \frac{1 - \frac{\beta(1+r)}{\pi}}{\frac{\beta(1+r)}{\pi}}$ . ■

### 2.7.10 Parameter Values

We use the following parameter values:  $q = 0.5$ ,  $v = 2$ ,  $\beta = 0.95$ ,  $\theta = 0.8$ ,  $a = 0.5$ ,  $\varepsilon = 6$ ,  $\phi = 1.1$ ,  $\rho_\theta = 0.95$ ,  $\rho_\phi = 0.95$ ,  $\eta = 0.005$ .

## BIBLIOGRAPHY

- [1] Abrams, B., A., 2011. Financial-sector shocks in a credit-view model, *Economics Letters*, vol. 112, pp. 256-258.
- [2] Adam, K., Roberto, M., B, 2007. "Discretionary Monetary Policy and the Zero Lower Bound on Nominal Interest Rates, *Journal of Monetary Economics*, vol. 54, pp. 728-752.
- [3] Ashcraft, B., Schuermann, T., 2008. Understanding the Securitization of Subprime Mortgage Credit, Staff Reports 318, Federal Reserve Bank of New York.
- [4] Bank of England, 2012. Funding for Lending Scheme, Market Notice.
- [5] Beenstock M., 1989. The Determinants of the Money Multiplier in the United Kingdom, *Journal of Money, Credit and Banking*, vol. 21, pp. 464-480.
- [6] Benigno, P., Nisticò, S. 2013. Safe Assets, Liquidity and Monetary Policy, London, Centre for Economic Policy Research, DP9767.
- [7] Benigno, P., Woodford, M. 2005. Inflation Stabilization And Welfare: The Case Of A Distorted Steady State, *Journal of the European Economic Association*, vol. 3, pp. 1185-1236.
- [8] Benigno, P. and Woodford, M. 2012. Linear-Quadratic Approximation of Optimal Policy Problems, *Journal of Economic Theory*, vol. 147, pp. 1-42.
- [9] Bernanke, B., S., 1983. Nonmonetary Effects of the Financial Crisis in Propagation of the Great Depression, *American Economic Review*, vol. 73, pp. 257-76.
- [10] Bernanke, B., S., Reinhart, R., V., Sack, B., P., 2004. Monetary Policy Alternatives at the Zero Bound: An Empirical Assessment, *Brookings Papers on Economic Activity*, vol. 35, pp. 1-100.
- [11] Bernanke, B., S, Blinder, A., S, 1988. Credit, Money, and Aggregate Demand, *American Economic Review*, vol. 78, pp. 435-39.
- [12] Bernanke, B., Gertler, M. 1989. Agency Costs, Net Worth and Business Fluctuations, *American Economic Review*, vol. 79, pp. 14-31.
- [13] Bernanke, B., Gertler, M. 2001. Should Central Banks Respond to Movements in Asset Prices?, *American Economic Review*, vol. 91, pp. 253-257.

- [14] Bernanke, B., Mishkin, F. 1997. Inflation Targeting: A New Framework for Monetary Policy?, *Journal of Economic Perspectives*, vol. 11, pp. 97-116.
- [15] Blanchard, O., Kahn, C., M. 1980. The Solution of Linear Difference Models under Rational Expectations, *Econometrica*, vol. 48, pp. 1305-1313.
- [16] Brunnermeier M., K., Eisenbach, T., M., Sannikov, Y., 2012, Macroeconomics with Financial Frictions: A Survey, *NBER Working Papers* 18102.
- [17] Brunnermeier, M., K., Pedersen, L., H., 2009, Market Liquidity and Funding Liquidity, *The Review of Financial Studies*, vol. 22, pp. 2201-2238.
- [18] Calvo, G., A. 1983. Staggered Prices in an Utility-Maximizing Framework, *Journal of Monetary Economics*, vol. 12, pp. 383-398.
- [19] Carlstrom, C., T., Fuerst, T., S., 1995. Interest Rate Rules vs. Money Growth Rules a Welfare Comparison in a Cash-in-Advance Economy, *Journal of Monetary Economics*, vol. 36, pp. 247-267.
- [20] Clarida, R., H. 2012. What Has - And Has Not - Been Learned About Monetary Policy in a Low-Inflation Environment?, *Journal of Money, Credit and Banking*, vol. 44, pp. 130-140.
- [21] Clarida, R., G., J., Gertler, M. The Science of Monetary Policy: A New Keynesian Perspective, *Journal of Economic Literature*, vol. 37, pp. 1661-1707.
- [22] Coibion, O., Gorodnichenko, Y., Wieland, J., 2012. The Optimal Inflation Rate in New Keynesian Models: Should Central Banks Raise Their Inflation Targets in Light of the Zero Lower Bound?, *Review of Economic Studies*, vol. 79, pp. 1371-1406.
- [23] Christiano, L., Eichenbaum, M., 1992, Liquidity Effects and the Monetary Transmission Mechanism, *NBER Working Paper No. 3974*.
- [24] Dell Ariccia, G., Detragiache, E., Rajan, R., 2008. The Real Effect of Banking Crises, *Journal of Financial Intermediation*, vol. 17, pp. 89-112.
- [25] Dixit, A., Stiglitz, J. 1977. Monopolistic Competition and Optimum Product Diversity, *American Economic Review*, vol. 67, pp. 297-308.

- [26] Fisher, I., 1933. The Debt-Deflation Theory of Great Depressions, *Econometrica*, vol. 1, pp. 337-357.
- [27] Frame, W., S, Lawrence J., W., 2005. Fussing and Fuming over Fannie and Freddie: How Much Smoke, How Much Fire?, *Journal of Economic Perspectives*, vol. 19, pp. 159-184.
- [28] Freeman, S., Kydland, F., E. 2000. Monetary Aggregates and Output, *American Economic Review*, vol. 90, pp. 1125-1135.
- [29] Friedman, M., 2000. Canada and Flexible Exchange Rates, the keynote speech on "Revisiting the Case for Flexible Exchange Rates - 2000 conference", Bank of Canada.
- [30] Friedman, M., 2006. Why Money Matters", Wall Street Journal, November 17th.
- [31] Gambacorta, L., Hofmann, B., G. Peersman, 2012, The Effectiveness of Unconventional Monetary Policy at the Zero Lower Bound: A Cross-Country Analysis , *BIS Working Papers* 384, Bank for International Settlements.
- [32] Goodhart C. A. E. , 2009, The Continuing Muddles of Monetary Theory: A Steadfast Refusal to Face Facts, *Economica*, vol. 76, pp. 821-830.
- [33] Hildebrand, M., P. 2006. Monetary Policy and Financial Markets, *Financial Markets and Portfolio Management*, vol. 20, pp. 7-18.
- [34] HM Treasury, 2013, Budget 2013.
- [35] Hong Kong Monetary Authority, 2011, Loan-to-value ratio as a macroprudential tool - Hong Kong SAR's experience and cross-country evidence, BIS Papers chapters, in: Bank for International Settlements (ed.), Capital flows, commodity price movements and foreign exchange intervention, volume 57, pages 163-178.
- [36] Iacoviello, M. 2005. House Prices, Borrowing Constraints and Monetary Policy in the Business Cycle, *American Economic Review*, vol. 95, pp. 739-764.
- [37] King, R., G., Levine, R., 1993, Finance and Growth: Schumpeter Might Be Right, *The Quarterly Journal of Economics*, vol. 108, pp. 717-37.

- [38] Korteweg, A., Sorensen, M., 2012, Estimating Loan-to-Value and Foreclosure Behavior, *NBER Working Papers*, No. 17882.
- [39] Kosuke, A., Proudman, J., Vliegbe, J. 2004. House Prices, Consumption and Monetary Policy: A Financial Accelerator Approach, *Journal of Financial Intermediation*, vol. 13, pp. 414-435.
- [40] Levine, P., Pearlman, J., Pierser, R., 2008. Linear-Quadratic Approximation, External Habit and Targeting Rules, *Journal of Economic Dynamics and Control*, vol. 32, pp. 3315-3349.
- [41] Lucas, R., Stokey, N., 1987, Money and Interest Rate in a Cash-in-Advance Economy, *Econometrica*, vol. 55, pp. 491-513.
- [42] Monacelli, T., 2009, New Keynesian Models, Durable Goods, and Collateral Constraints, *Journal of Monetary Economics*, vol. 56, pp. 242-254.
- [43] Mora, N., 2012., What determines creditor recovery rates?, *Economic Review*, Federal Reserve Bank of Kansas City, issue Q II.
- [44] McCallum, B., T., 1985, On Consequences and Criticisms of Monetary Targeting, *Journal of Money, Credit and Banking*, vol. 17, pp. 570-97.
- [45] Midrigan, V., Philippon, T. 2011. Household Leverage and the Recession, *NYU Working Paper*, No FIN-11-038.
- [46] Mishkin, S., F. 2011. Monetary Policy Strategy: Lessons from the Crisis, *NBER Working Paper*, No. 16755.
- [47] Peersman, G., Smets, F. 1999. The Taylor Rule: A Useful Monetary Policy Benchmark for the Euro Area?, *International Finance*, vol. 2, pp. 85-116.
- [48] Peñaloza L., Barnhar, M., Living U.S. Capitalism: The Normalization of Credit/Debt, *Journal of Consumer Research*, Vol. 38, pp. 743-762
- [49] Telyukova, I., A., Wright, R., 2008. A Model of Money and Credit, with Application to the Credit Card Debt Puzzle, *Review of Economic Studies*, vol. 75, pp. 629-647.
- [50] Taylor, J., B. 1993. Discretion versus Policy Rules in Practice, *Carnegie-Rochester Conference Series on Public Policy*, December 1993, vol. 39, pp. 195-214.

- [51] Thorbecke, W. 2012. On Stock Market Returns and Monetary Policy, *The Journal of Finance*, vol. 52, pp. 635-654.
- [52] Svensson L. 2009. Flexible Inflation Targeting: Lessons from the Financial Crisis, Speech Given at the Workshop “Towards a New Framework for Monetary Policy? Lessons from the Crisis.” Amsterdam: Organized by De Nederlandse Bank.
- [53] Walsh, C. 2010, , Monetary Theory and Policy, MIT Press, 3d edition, pp. 594.
- [54] Woodford, M. 2001. The Taylor Rule and Optimal Monetary Policy, *American Economic Review*, vol. 91, pp. 232-237.

## CHAPTER 3

# MONETARY POLICY IN THE MODEL WITH FINANCIAL AND LABOUR MARKET FRICTIONS

### 3.1 Introduction<sup>1</sup>

In earlier works (Ravenna and Walsh (2011), Walsh (2005), Blanchard and Gali (2010) to name a few) the relation between monetary policy and labor market was analyzed. It was found, that reaction on unemployment in the Taylor-type interest rate rule is optimal (for more details, see Faia (2008) or Faia (2008a)). In this note we extend previous analysis by adding firms financial market frictions and show that the same result holds, that is, reaction to unemployment is still optimal. We also find that coefficient on unemployment in Taylor-type interest rate rule depends on the extent of workers bargaining power.

As a starting point we use model considered in Monacelli, Trigari and Quadrini (2011) and extend it in several ways. Firstly, we introduce sticky prices by using Calvo (1983) framework and derive New-Keynesian Phillips curve. Secondly, we compute optimal monetary policy and use it as a benchmark to compare all other regimes. Thirdly, we use Benigno and Woodford (2005) algorithm to asses different monetary policy rules performance in terms of the welfare measure.

The main novelty of our paper as compared to the previous literature is the connection between financial and labour markets. For instance, works by Trigari (2004), Walsh (2005), Thomas (2008), Faia (2008, 2009) to name a few did not have either any financial frictions or a central bank who would be responsible for monetary policy.

The key idea of financial and labor market connection could be summed up as

---

<sup>1</sup>Codes used in this chapter could be seen at [https://github.com/Sarunas-Girdenas/2nd\\_chapter](https://github.com/Sarunas-Girdenas/2nd_chapter).

follows. In our model firms have incentive to borrow because of the wage bargaining process. That is, they are issuing debt to reduce bargaining surplus which in turn reduces wage paid to workers. For any given value of workers bargaining power, higher debt results in lower wage. This is different channel than Petrosky-Nadeau (2014) and Petrosky-Nadeau and Wasmer (2013) where firms participate in a credit market to finance vacancy costs. In their works the source of frictions in the economy is the evolving condition of credit market (tightness of borrowing constraint). As credit becomes more easily accessible to firms during the boom, they create more vacancies and thus unemployment falls.

Our approach is also different from Mumtaz and Zanetti (2014) where entrepreneurs are borrowing to finance production cost. In their model, when financial shock hits firms cut their production because of the shortage of capital to keep the production level. In our model financial shock reduces firms credit thus reducing surplus and as a result firms bargaining ability deteriorates resulting in an increase in wage and unemployment.

In addition to financial market, our model also includes central bank that could affect borrowing by changing interest rate thus indirectly influencing wage. Consider the case of central bank lowering interest rate. In such case borrowing would become cheaper and firms would increase their debt. By doing so they would reduce bargaining surplus thus lowering wage. This particular mechanism is the reason why reaction to unemployment in interest rate rule depends on the extent of workers bargaining power.

The chapter is structured as follows. Firstly, we present model with imperfect financial market similar to Monacelli, Trigari and Quadrini (2011). Then we introduce labour market frictions using Pissarides (1987) framework. In the third section we derive optimal monetary policy and discuss the relation between interest



rate and bargaining power. Then we compute the welfare measure using Benigno and Woodford (2005) algorithm. The last section concludes the chapter.

To investigate this we are going to use the New-Keynesian DSGE model with imperfect financial and labour markets. But before dwelling deeply into the specifications of the model we are going to provide some empirical evidence and intuition.

### **3.1.1 Theoretical Approach**

Despite the fact of unemployment being one of the target variables of central banks, in academic literature there are still only a handful of papers which assess monetary policy in the framework with labour and financial market frictions. We aim to contribute to this field by providing a simple macroeconomic model with credit and labour market frictions which could be used to investigate the relationship between the financial and labour markets.

As a starting point we use paper by Monacelli, Trigari and Quadrini (2011). We extend it in several ways. Firstly, we introduce sticky prices by using Calvo (1983) framework and derive New-Keynesian Phillips curve. Secondly, we compute optimal monetary policy reaction to both financial and labour market frictions and use it as a benchmark to compare all other policies. Thirdly, we use Benigno and Woodford (2005) algorithm to assess different monetary policy rules' performance in terms of the unconditional welfare measure. These extensions allow us to investigate monetary policy regimes that could be used to counteract negative effects of financial market shock to unemployment.

The main novelty of our paper as compared to previous literature is the connection between firms' financial market and the labour market with the search and matching frictions. For instance, works by Trigari (2004), Walsh (2005), Thomas (2008), Faia (2008, 2009) to name a few did not have either any financial frictions

or a central planner who would be responsible for monetary policy. By having these features we can investigate how unanticipated decrease in firms' loans' repayment probability may affect unemployment and other labour market characteristics. Zanneti (2012) has showed that financial variables affect labour market through aggregated demand channel.

Since the interest rate policy which is solely based on the current or forward looking rule is not sufficient to generate local stability when the labour market frictions are present (Zanneti 2006), we investigate monetary policy regimes that also feed back on frictions in financial and labour markets. In particular, we choose 3 different interest rate rules in spirit of Taylor (1993) to account for policy maker's preferences towards firms' borrowing and variables which define labour market such as aggregate level of unemployment.

We have found that the interest rate rule with response to inflation and labour market frictions (unemployment) performs closest to the optimal policy in terms of the unconditional welfare measure. This might be explained by the fact that our economy is distorted by labour market frictions, therefore reaction on it (along with inflation) gives the best result. It is somehow similar to Faia (2008 and 2009) where it was showed that the level of unemployment should be included in the policy maker's rule. In addition, we show that the sign of the unemployment reaction parameter in the interest rate rule depends on the bargaining power of workers.

The chapter is structured as follows. Firstly, we present model with imperfect financial market similar to Monacelli, Trigari and Quadrini (2011). Then we introduce labour market frictions using Pissarides (1987) framework. In the third section we compare 3 monetary policy regimes to the optimal policy. Then we compute the unconditional welfare measure using Beningo and Woodford (2005)

algorithm. The last section concludes the chapter.

## 3.2 Model

There are four key ingredients that form the model. Firstly, households consume final goods, make a choice to be employed or unemployed and save in riskless assets which yield interest rate and shares of firms which pay dividends. Secondly, we have intermediary goods (wholesale) producers who participate in the labour market by posting vacancies, employing agents and borrowing against their expected future profit. Thirdly, there are final good producers who are taking the wholesale good and selling it to households in the sticky prices (Calvo (1983)) environment. Lastly, there is government which aims to minimize the negative effects of unanticipated shocks by conducting monetary policy according to Taylor (1993) type rules. We begin our presentation with households.

### 3.2.1 Households

There is a continuum of agents of total mass 1 with a lifetime utility. At any point in time agents can be employed or unemployed. They save in two types of assets: shares of firms and assets. Households provide labour and get paid for it. They also could be employed with probability  $p_t^E$ . We follow Ravenna and Walsh (2011) and use CRRA functional form for the households utility

$$U_t^H = E_t \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} \right], \quad (3.1)$$

where  $\sigma$  is a relative risk aversion. Household's consume two kinds of goods: market goods  $C_t^m$  and domestically produced goods by unemployed agents  $a(1 - N_t)$ :

$$C_t = C_t^m + a(1 - N_t), \quad (3.2)$$

where  $a$  is the unemployment benefit (in real terms) and  $N_t$  is the number of employed members of household. We assume that everything what is produced is also consumed in our economy. Therefore we can write the following equation

$$Y_t = C_t^m + kv_t. \quad (3.3)$$

It says that all goods in the economy is consumed by households ( $C_t^m$ ) and firms ( $kv_t$ ) where  $k$  is the cost of posting vacancy and  $v_t$  is the number of available vacancies. Now we define measure of the labour market tightness by introducing a ratio of available vacancies to a number of unemployed agents  $\theta_t = \frac{v_t}{u_t}$ . Then equation (3.3) becomes

$$Y_t = C_t^m + k\theta_t u_t. \quad (3.4)$$

Recall, that we defined households consumption as being the sum of domestically produced goods  $a(1 - N_t)$  and final goods  $C_t^m$ . Therefore we can use (3.2) to express  $C_t^m$  as  $C_t^m = C_t - a(1 - N_t)$ . By substituting it into (3.4) we obtain economy's resource constraint

$$Y_t = C_t - a(1 - N_t) + k\theta_t u_t. \quad (3.5)$$

where  $a$  is value of labour in home production. Household's liquidity position at time  $t$  is defined as the sum of income received from employed and unemployed members

$$I_t = w_t N_t + a(1 - N_t), \quad (3.6)$$

where  $w_t$  real wage defined as  $w_t = \frac{W_t}{P_t}$ . We also let households to hold assets which could be used as savings. Next period assets  $A_t^h$  (in nominal terms) is defined as follows

$$A_{t+1}^h = A_t^h i_t - P_t C_t + W_t N_t + a(1 - N_t)P_t + \Pi_t, \quad (3.7)$$

where  $W_t$  is the nominal wage and the nominal profit of firms is denoted by  $\Pi_t$ .

Now we can state household optimization problem

$$\begin{aligned} \max E_t \sum_{t=0}^{\infty} \beta^t \left[ \frac{C_t^{1-\sigma}}{1-\sigma} \right], \\ \text{s.t. } A_{t+1}^h = A_t^h i_t - P_t C_t + W_t N_t + a(1 - N_t) P_t. \end{aligned}$$

which leads to the following definitions. Marginal utility of consumption (in real terms)

$$\Lambda_t = -\frac{C_t^{-\sigma}}{P_t}, \quad (3.8)$$

and interest rate

$$\frac{1}{i_t} = \beta E_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} \right]. \quad (3.9)$$

Combining two equations above we obtain relation between interest rate  $i_t$  and inflation  $\pi_t$  (Euler equation)

$$\frac{1}{i_t} = \beta E_t \left[ \frac{C_{t+1}^{-\sigma}}{C_t^{-\sigma} \pi_{t+1}} \right]. \quad (3.10)$$

### 3.2.2 Firms

Our model is populated by two kinds of firms: intermediary good producers as in Monacelli, Quadrini and Trigari (2011) who participate in the labour and financial markets by posting vacancies and borrowing and final good producers who take intermediary good and sell it to households in sticky prices (Calvo (1983)) environment.

#### Intermediary good producers

As in Monacelli, Quadrini and Trigari (2011) there are two types of intermediary goods producers: young and old firms.

Young firms are those which are created at the same period once the posted vacancy is filled with the worker. At this period the only decision they make is how

much to borrow. Capital obtained from the financial market in the first period is distributed to shareholders of the firm. Wage is paid only in the subsequent period if a match is not separated.

Old firms are the matches, which were not separated during the last period. At this stage old firms must bargain with the current worker and pay wage, make a decision about the future borrowing and produce.

It is important to notice that firms in our model have incentive to borrow up to the limit because high debt reduces bargaining surplus which in turn reduces the wage paid to workers. Compared to Monacelli, Quadrini and Trigari (2011) our production sector is more complicated because we require final good producers (which operate in a sticky prices environment) to derive Phillips curve.

#### - Old Firms

We begin our presentation by defining the production function of old wholesale firms which only input is labour:

$$Y_t = zN_t, \tag{3.11}$$

with  $z$  being productivity parameter. Then we can intuitively define number of unemployed agents (matches) by normalizing total mass to 1 and subtracting those who are employed

$$u_t = 1 - (1 - \lambda) N_{t-1}, \tag{3.12}$$

where  $\lambda$  is exogenously given probability that a match will be separated. Now we can immediately define number of employed workers

$$N_t = (1 - \lambda) N_{t-1} + m(u_t, v_t). \tag{3.13}$$

It is equal to matches from the last periods which were not separated  $(1 - \lambda) N_{t-1}$  and the newly created matches captured by CRS matching function (Pissarides

1987) with  $v_t$  denoting number of available (posted) vacancies,  $\varsigma$  being a matching parameter and  $\alpha$  being vacancy elasticity of matches

$$m(u_t, v_t) = \varsigma v_t^\alpha u_t^{1-\alpha}. \quad (3.14)$$

Probability that the posted vacancy is going to be filled ( $q_t$ ) is a ratio of available vacancies to filled vacancies

$$q_t = \frac{m(u_t, v_t)}{v_t}. \quad (3.15)$$

In the same way probability that worker will find a job,  $p_t^E$ , (the match will be created) is equal to

$$p_t^E = \frac{m(u_t, v_t)}{u_t}. \quad (3.16)$$

Using the two equations above and a particular form of the matching function (3.14) we obtain the relationship between the probability that vacancy will be filled and the probability that worker will find a job

$$p_t^E = (q_t)^{-\frac{\alpha}{1-\alpha}} \varsigma^{\frac{\alpha}{1-\alpha}}. \quad (3.17)$$

Making use of definition of labour market tightness,  $\theta_t = \frac{v_t}{u_t}$ , we can rewrite equation (3.17) as

$$p_t^E = \varsigma \theta_t^\alpha. \quad (3.18)$$

Now we want to derive the relation between the level of employment and the probability that worker will find a job. This can be achieved by substituting equations (3.12), (3.18) and (3.13)

$$N_t = (1 - \lambda) N_{t-1} + \varsigma \theta_t^\alpha (1 - (1 - \lambda) N_{t-1}). \quad (3.19)$$

Lastly, we may also show how level of employed agents ( $N_t$ ) affects output. For this purpose we use equations (3.12) and (3.5)

$$Y_t = C_t - a(1 - N_t) + k\theta_t(1 - (1 - \lambda) N_{t-1}). \quad (3.20)$$

This equations shows that the higher the number of employed agents ( $N_t$ ) the greater is the level of output. Now we are going to look at workers bargaining in the old firms since young firms are not negotiating with workers in the first period.

Firstly, we set up an equation which defines wholesale firms' equity value in terms of productivity, real wage and borrowing

$$J_t = z_t - w_t + b_t - \frac{b_{t-1}}{\pi_t} i_{t-1} + \beta(1 - \lambda) E_t [J_{t+1}]. \quad (3.21)$$

The above equation states that the firms value depends on a wage paid to workers and on the size of the loan which could be obtained from the bank. Hence, we can see the same result as in Monacelli, Quadrini and Trigari (2011): firms choice of a new debt does not depend on the past debt and current wage  $w_t$ . This could be seen by differentiating firms equity value (3.21) with respect to the new debt  $b_t$  :  $\frac{\partial J_t}{\partial b_t} = 1 + \beta(1 - \lambda) \frac{\partial J_{t+1}}{\partial b_t}$ . That is, wage does not affect choice of borrowing. As we will see later, this result will significantly simplify the description and analysis of the firms financial market.

Now we can define bargaining surplus ( $S_t$ ) which is split between worker and firm and is defined as  $S_t = J_t + V_t - U_t$ . The expression of surplus is as follows

$$S_t = \frac{z}{X_t} - a + b_t - \frac{b_{t-1}}{\pi_t} i_{t-1} + (1 - \lambda) \beta E_t [S_{t+1}] - \eta \beta p_t^E E_t [S_{t+1}], \quad (3.22)$$

where  $X_t$  is a markup defined as the price ratio  $\frac{p_t^w}{P_t}$  with  $p_t^w$  denoting the wholesale good price and  $P_t$  is the aggregate price of the final good.

When the worker is employed in a wholesale firm, his value is

$$V_t = w_t + \beta E_t [(1 - \lambda) V_{t+1} + \lambda U_{t+1}], \quad (3.23)$$

where  $w_t$  is the wage paid to workers (in real terms),  $(1 - \lambda)$  is probability of a match separation and finally,  $U_t$  is the value of being unemployed which we define as follows

$$U_t = a + \beta E_t [p_t^E V_{t+1} + (1 - p_t^E) U_{t+1}], \quad (3.24)$$



where the term in brackets denote trade-off between being employed and unemployed.

Finally, wage is given by

$$w_t = (1 - \eta) a + \eta \left( \frac{z}{X_t} - \frac{b_{t-1}}{\pi_t} i_{t-1} \right) + \frac{\eta k \left[ \frac{\pi_{t+1}}{i_t} \phi_t + p_t^E \beta \right]}{q_t \left( \phi_t \left( \frac{\pi_{t+1}}{i_t} \right)^2 + \beta \right)}, \quad (3.25)$$

where  $\phi_t$  is a borrowing parameter subject to a negative shock. As we can see, it negatively depends on  $b_{t-1}$ . That is, the higher is the borrowing, the lower is the current wage since worker gets a fraction of output *net* of debt. Therefore, for the given value of bargaining power  $\eta$  higher borrowing reduces wage. Furthermore, it is clear that policy maker could indirectly affect wage by changing interest. See appendix for details of wage derivation.

### - Young Firms

A new firm is created when the new match occurs as defined in (3.14). Then the new vacancy which is filled by the worker holds the following value:

$$Q_t = \frac{b_t}{i_t} \pi_{t+1} + \beta(1 - \eta) E_t [S_{t+1}]. \quad (3.26)$$

Firms post vacancies as long as the value of vacancy is positive. In other words, value of the filled vacancy  $Q_t$  must be greater than than the incurred cost  $k$ . Thus we can write the condition under which firms will keep on posting vacancies as

$$q_t Q_t = k, \quad (3.27)$$

where  $k$  is the cost of posting vacancy and  $q_t$  is the probability that posted vacancy is going to be filled. Rearranging two equations above we obtain the following expression which links cost of labour market (probability that posted vacancy will be filled) with financial market (firms' value which depends on borrowing):

$$\frac{k}{q_t} = b_t + \beta(1 - \eta) E_t [S_{t+1}]. \quad (3.28)$$

In the next period young firm will become old and implement the bargaining as we have previously described.

### Intermediary Goods Producers Financial Market

We begin by introducing the key assumptions. Firstly, we do not explicitly model the financial institutions which are providing loans to firms. We simply assume that they can costlessly issue the required amount of credit. Secondly, if a match is separated and as a result firm defaults, then its value is equal to 0 and the lender does not get any compensation. Because of this, lenders impose borrowing constraint which tightness is defined by the parameter  $\phi_t$ . This ensures that firms do not default when the match is not separated and thus lender is willing to issue loan as long as borrowing constraint is binding.

The reason why we can work with only one *representative* firm and use the same borrowing constraint is the fact that choice of debt does not depend on wage<sup>2</sup>. Because of this both young and old firms face the same borrowing constraint.

If we keep in mind that the real value of firms debt is  $b_t$ , then the borrowing constraint can be expressed as

$$b_t i_t E_t \left[ \frac{1}{\pi_{t+1}} \right] = \phi_t (1 - \eta) E_t [S_{t+1}]. \quad (3.29)$$

Here  $\phi_t$  is borrowing parameter that is subject to shock,  $\eta \in (0; 1)$  is workers' bargaining power. We can clearly see that the amount which firm could borrow depends on its expected bargaining surplus (denoted as  $E_t [S_{t+1}]$ ).

Economically speaking, borrowing constraint has a couple of important implications. Firstly, it implies that firms are borrowing against their future profit (denoted as their part of total bargaining surplus) which depends on workers bargaining power  $\eta$ . Secondly, we can think of  $\phi_t$  as being the probability that firm

---

<sup>2</sup>For more details about this result see Monacelli, Quadrini, Trigari (2011).

will repay its debt hence it must be between 0 and 1. If it drops, then it means that the larger number of firms are not able to meet their financial commitments and as a result the aggregate level of borrowing falls.

We specify  $AR(1)$  process which perturbs financial market (firms' repayment probability)

$$\phi_t = \alpha_\phi \phi_{t-1} - \varepsilon_{\phi,t}, \quad (3.30)$$

where  $\varepsilon_{\phi,t}$  is Gaussian white noise processes with 0 mean and variance  $\sigma^2$  and  $\alpha_\phi$  is persistency parameters of shock. It is noteworthy, that shock is negative in  $\phi_t$ .

### - Choice of Debt

It is clear, that firms will choose the maximum level of debt possible only when the borrowing (enforcement) constraint is binding. To see this, let firms maximize their equity value (3.21) subject to the borrowing constraint (3.29). If we denote corresponding Lagrange multiplier by  $\varphi_t$  and substitute  $E_t [J_{t+1}]$  using  $J_t = (1 - \eta)S_t$  maximization exercise will become

$$\begin{aligned} \max z_t - w_t + b_t - \frac{b_{t-1}}{\pi_t} i_{t-1} + \beta(1 - \lambda)(1 - \eta)E_t [S_{t+1}], \quad (3.31) \\ \text{s.t. } b_t i_t E_t \left[ \frac{1}{\pi_{t+1}} \right] \leq \phi_t (1 - \eta) E_t [S_{t+1}]. \end{aligned}$$

First order conditions yields the following expression of the Lagrange multiplier

$$\varphi_t = \frac{\left( \frac{\pi_{t+1}}{i_t} \right) - \beta(1 - \lambda)(1 - \eta)}{1 + \phi_t(1 - \eta)}. \quad (3.32)$$

Since in steady state we have that  $\frac{\pi}{i} = \beta$ , equation (3.32) could be rearranged to

$$\varphi = \frac{\beta - \beta(1 - \lambda)(1 - \eta)}{1 + \phi(1 - \eta)}. \quad (3.33)$$

From the above expression we can clearly see, that  $\varphi$  is positive if we restrict parameters  $\lambda$ ,  $\eta$  and  $\beta$  being in the interval between 0 and 1. This is claim is formally summarized below.

**Lemma 3.2.1** *Borrowing constraint is binding in the steady state if workers have any bargaining power ( $\eta \in (0; 1)$ ) and the match separation probability is positive ( $\lambda \in (0; 1)$ ).*

**Proof.** Follows directly from first order conditions of firms optimization problem.

■

The above proposition shows that borrowing constraint is indeed binding<sup>3</sup>.

### 3.2.3 Final Good Producers and Price Setting

We assume that the final good producers takes intermediary good and sells it to households in the sticky prices (Calvo 1983) framework. They produce according to the following production function where  $\varepsilon$  is the elasticity of substitution and  $Y_t^f$  is the production of final goods

$$Y_t^f = \left( \int_0^N y_t^\varepsilon(i) di \right)^{\frac{1}{\varepsilon}}. \quad (3.34)$$

We assume that final goods are imperfectly substituted and that consumption is defined over Dixit-Stiglitz (1977) basket of goods,  $Y_t^f = \left[ \int_0^1 Y_t(i)^{\frac{\varepsilon-1}{\varepsilon}} di \right]^{\frac{\varepsilon}{\varepsilon-1}}$ . The average price-level of the final good  $P_t$ , is known to be  $P_t = \left[ \int_0^1 p_t(i)^{1-\varepsilon} di \right]^{\frac{1}{1-\varepsilon}}$ . The demand for each retailers' final good is given by

$$Y_t(i) = \left( \frac{p_t(i)}{P_t} \right)^{-\varepsilon} Y_t^f, \quad (3.35)$$

where  $p_t(i)$  is the nominal price of the final good produced in industry  $i$  and  $Y_t(i)$  denotes aggregate demand for a good produced in  $i$ -th sector. Each retailer chooses sale price  $p_t(i)$  and buys intermediary good at a price  $p_t^w$  which is taken as given. The final good sale price  $p_t(i)$  could be changed only with probability  $1 - \varpi$ . Let

---

<sup>3</sup>If the constraint were not binding, then firms borrowing would be either zero or infinity.

us call the new price  $p_t^*(i)$  and corresponding demand  $Y_{t+k}^* = \left(\frac{p_t^*(i)}{P_{t+k}}\right)^{-\varepsilon} Y_t^f$ . The final good producers' profit maximization problem could be written as follows

$$\Pi_t(i) = \left(\frac{p_t^*(i)}{P_t} - \frac{P_t^w}{P_t}\right) \left(\frac{p_t^*(i)}{P_t}\right)^{-\varepsilon} Y_t^f. \quad (3.36)$$

For the sake of simplicity we can substitute  $\frac{p_t^*(i)}{P_{t+\tau}} = \frac{p_t^*(i)}{P_t} \frac{P_t}{P_{t+\tau}}$ . Now we rewrite the optimization problem

$$\max_{p_t^*(i)} E_t \sum_{\tau=0}^{\infty} \varpi^\tau \beta^\tau \frac{\Lambda_{t+\tau}}{\Lambda_t} \left[ \frac{p_t^*(i)}{P_t} \frac{P_t}{P_{t+\tau}} - \frac{P_{t+\tau}^w}{P_{t+\tau}} \right] \left(\frac{p_t^*(i)}{P_t} \frac{P_t}{P_{t+\tau}}\right)^{-\varepsilon} Y_{t+\tau}^f. \quad (3.37)$$

Because of the assumption that only proportion of firms could change the prices every period we know that price index evolves according to the following law of motion

$$P_t = \left(\varpi P_{t-1}^{1-\varepsilon} + (1-\varpi) p_t^*(i)^{1-\varepsilon}\right)^{\frac{1}{1-\varepsilon}}. \quad (3.38)$$

Solving final good producers optimization problem and using the law of motion of prices we derive standard New-Keynesian Phillips curve:

$$\beta \hat{\pi}_{t+1} = \iota \hat{X}_t + \hat{\pi}_t, \quad (3.39)$$

where variables with hats denote deviations from the steady state and  $\iota = \frac{(1-\varpi)(1-\varpi\beta)}{\varpi}$ .

The formal derivation is provided in the appendix.

### 3.2.4 Steady State Efficiency Analysis

To derive optimal policy reaction we let policy maker maximize households' utility (3.1) subject to the behavioral constraints presented in equations 3.22, 3.29, 3.28, 3.84, 3.85, 3.86, 3.10, ??, 3.18, ??, 3.11 and 3.19. The formal derivation of optimal monetary policy is showed in appendix.

In the previous literature there was reported (for example Thomas (2008) or Faia (2008 and 2008a)) that in the optimal steady state prices are stable, that is

$\pi = 1$ , where  $\pi$  is the steady state inflation. Indeed, we can confirm this finding by looking at the policy maker's optimization problem. Hence we can formally put it as follows.

**Lemma 3.2.2** *Optimal steady state implies price stability, that is  $\pi = 1$ .*

**Proof.** Follows from policy maker's optimization problem provided in appendix.

■

Further to this, we employ Hosios (1990) condition that workers bargaining power ( $\eta$ ) is equal to the elasticity of the matching function with respect to the available vacancies ( $\alpha$ ). This condition implies that unemployment in steady state is Pareto efficient. This assumption is needed to ensure that our economy evolves around the efficient steady state because it is needed for correct welfare ranking (Faia 2008a). However, in Benigno and Woodford (2005) it is reported that if the model is approximated to up to the second order<sup>4</sup>, the correct welfare ranking could be preserved even if steady state is inefficient.

In addition, since Hosios (1990) condition holds, we can conclude that the unemployment level in steady state is efficient (as it is showed in Ravenna and Walsh (2011) too). Hence the maximum welfare of the economy is achieved when the original steady state level of unemployment is maintained. This observation directly affects monetary policy, because strength of the response to unemployment in the interest rate rule depends on how far the level of unemployment is from its steady state value. Indeed, we will see this particular feature later in the paper.

### 3.2.5 Calibration and Solution

We analytically solve for the steady state (showed in appendix) since we have the same number of variables and equations and use those values for simulating the

---

<sup>4</sup>For review of the welfare approximation techniques see Kim and Kim (2003).

model. We also obtained closed form solution for the optimal monetary policy regime.

We assume the period to be one quarter long and set the discount factor to  $\beta = 0.99$ . It is the same as showed in Ravenna and Walsh (2011). The bargaining power parameter  $\eta$  is set to be equal to 0.5 because there are no direct evidence of a different estimate. Then we follow Monacelli, Quadrini and Trigari (2011) and set matching parameters  $\varsigma = 0.76$  and  $\alpha = 0.5$  to satisfy Hosios condition. Value of constant ( $\varsigma$ ) in the Cobb-Douglas matching function is consistent with Hall (2003) estimate. Steady state productivity is normalized to 1 and we take the values of firms repayment probability  $\phi$ , cost of posting vacancy  $k$  and an unemployment benefit  $a$  from Monacelli, Quadrini and Trigari (2011) calibration. Price adjustment probability  $\varpi$  is taken to be standard and equal to 0.25 together with price elasticity parameter  $\varepsilon = 6$  (Basu and Fernald (1997)) and we take this value from Ravenna and Walsh (2011). We take relative risk aversion  $\sigma$  to have its usual value (for instance Ravenna and Walsh (2011)) and set it be equal to 2. Finally, we set persistence parameters of shocks being 0.75 to smooth the impulse response functions. All the values are summarized in the table 1.

Table 3.1. Parameter Values

Parameter	Description	Value
$\beta$	Households' discount factor	0.99
$\eta$	Workers' bargaining power	0.5
$\varepsilon$	Price elasticity parameter	6
$\varsigma$	Matching parameter	0.76
$\lambda$	Probability of match separation	0.05
$k$	Cost of posting vacancy	0.598
$a$	Workers unemployment benefit	0.5
$z$	Productivity	1.0
$\phi$	Probability that firm will repay its debt	0.86
$\varpi$	Probability that price will not change	0.25
$\sigma$	Relative risk aversion	2.0
$\alpha$	Matching parameter (elasticity)	0.5
$\alpha_\phi$	Financial market shock persistence	0.75
$\alpha_z$	Productivity shock persistence	0.75

We solve the model by taking log-linear approximation around the local and deterministic steady state. We also assume that shocks are small enough and they occur in the neighborhood of the steady state and therefore equilibrium conditions are satisfied.

### 3.3 Monetary Policy

In this section we investigate possible policy maker's reaction to the financial and labour market shocks. It is structured as follows. Firstly, we present the benchmark model without policy, secondly, we introduce a couple of policy rules in spirit of



Taylor (1993), then we investigate the optimal monetary policy and lastly we rank monetary policy regimes according to the welfare measure in Benigno and Woodford (2005) framework.

### 3.3.1 Monetary Policy Rules

We consider the following monetary policy (Taylor 1993 type) rules: (1) Inflation and consumption response, (2) Inflation and unemployment response, (3) Inflation and firms borrowing response. For comparison purposes we also consider the optimal monetary policy regime. The reason, why in the first policy rule we have chosen consumption response instead of output (as in the case of the usual Taylor rule) is the fact that from the production function (3.11) we can see that output is equal to  $N_t$  - number of employed agents. Hence by targeting output we would be indirectly targeting unemployment which is the same as the second policy regime. For this reason we substituted output with consumption in the first policy rule.

There are a couple of reasons why we have chosen these particular policy rules. Firstly, similar regimes were considered in the literature<sup>5</sup> hence by using similar expressions we can compare findings. Secondly, we consider rules with labour and financial markets targeting. These are the main sources of instability in our model and there is some empirical evidence<sup>6</sup> of currently adopted aggregate level of unemployment targeting.

To investigate the central bank preferences towards the target variables in the monetary policy rules, we search numerically for the values which would give the highest welfare. We consider all the parameters in the intervals where Blanchard-Kahn (1980) conditions are satisfied. That is we set interest rate smoothing pa-

---

<sup>5</sup>For instance, Faia (2008 and 2009), Thomas (2008) and more recently Proano (2012).

<sup>6</sup>For example, 'forward guidance' concept introduced by Bank of England which includes unemployment level into policy targets.

parameter  $\alpha_i$  in the interval of  $(0; 1)$ , inflation preference parameter in the interval of  $(1.1; 8)$ , unemployment preference parameter in the interval of  $(0; 8)$  and firms borrowing preference parameters in the range of  $(0; 2)$ .

In the case of all three policy rules we searched for the parameter values that would give the highest welfare (introduced in the next chapter). Similarly to Faia (2008) we have found that interest rate smoothing does not increase welfare, that is, highest welfare in all three regimes is obtained when  $\alpha_i = 0$ .

Other parameter values that we received from searching the highest welfare are provided in the table below. We use them to compute impulse response functions and compare monetary policy regimes. Intuitively we can see that in any case policy maker can set his preference parameter to zero and come back to the inflation only targeting rule. However, this is not optimal choice.

Table 3.2. Monetary Policy Rules and Parameter Values

Policy	Functional Form	Parameter Values
1	$\hat{i}_t = (1 - \alpha_i) \left( \alpha_\pi \hat{\pi}_t + \alpha_Y \hat{C}_t \right) + \alpha_i \hat{i}_{t-1}$	$\alpha_i = 0, \alpha_\pi = 3.99, \alpha_C = 6.03,$
2.	$\hat{i}_t = (1 - \alpha_i) \left( \alpha_\pi \hat{\pi}_t + \alpha_u \hat{u}_t \right) + \alpha_i \hat{i}_{t-1}$	$\alpha_i = 0, \alpha_\pi = 1.69, \alpha_u = 7.37,$
3.	$\hat{i}_t = (1 - \alpha_i) \left( \alpha_\pi \hat{\pi}_t + \alpha_b \hat{b}_t \right) + \alpha_i \hat{i}_{t-1}$	$\alpha_i = 0, \alpha_\pi = 1.1, \alpha_b = 0.44.$

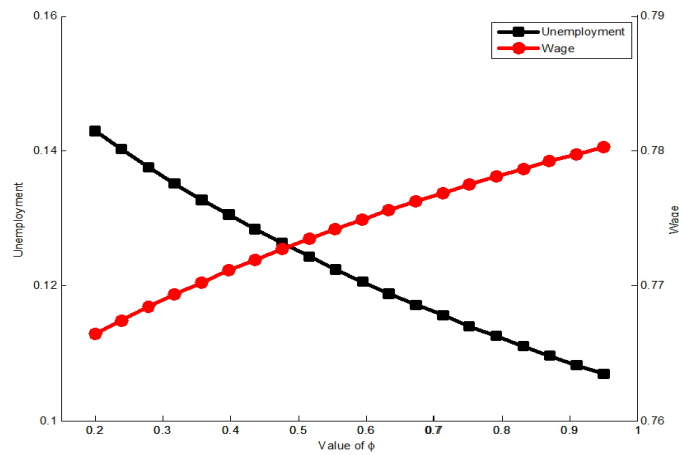
### 3.3.2 Relation Between Financial and Labour Market

To illustrate the channel through which financial market influences labour market we computed steady state wage and unemployment for different values of borrowing parameter  $\phi$ . As we can see, higher value of  $\phi$  reduces unemployment and increases wage. We can distinguish two cases. Firstly, if  $\phi$  has a very high value, then firms can easily issue more debt thus decreasing bargaining surplus and wage. As a result, unemployment falls (as we see in figure 3). However, due to decrease in unemployment it is harder for firms to find the right match and thus they are

forced to increase wage.

Secondly, if  $\phi$  has very low value, firms cannot access financial market and thus borrowing is very low and therefore bargaining surplus is large. Because of this, wages paid to workers are high too. As a result of high wage, unemployment increase as we see from figure 3. Since unemployment is high, firms can reduce the wage because it is easy to find a match. This is also shown in figure 3.

Figure 3.4 Financial and Labour Markets



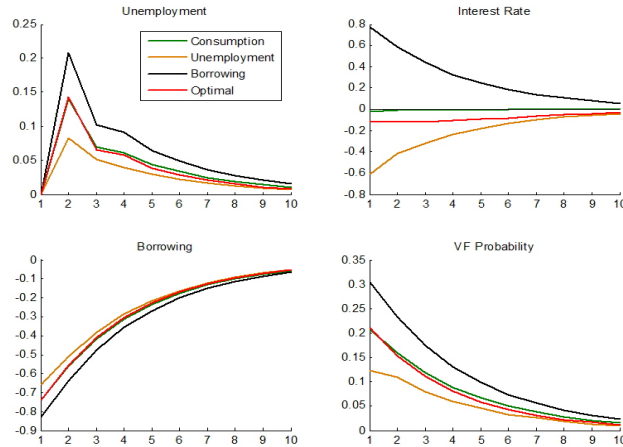
It is noteworthy that such relation between wage and unemployment is also found in some empirical studies, such as Apergis (2008) which examines 10 different OECD countries in period 1995-2005 and find that wages decrease as unemployment increases. In addition, our result suggest that less stringent firms borrowing constraint (higher value of  $\phi$ ) reduces unemployment which agrees with Acemoglu (2001) remark that easier access to financial market may reduce unemployment. In the next section we compute impulse response functions to illustrate how this mechanism work in more detail.

### 3.3.3 Response to Financial Market Shock

To begin with, we can see that the shock to the repayment probability causes a significant increase in unemployment and a huge drop in firms' borrowing. This is intuitive, because as the repayment probability drops, less loans are issued. Vacancy filling probability ( $q_t$ ) is affected through the same channel. As borrowing drops, firms' start laying off labour force. This results in an increase in the vacancy filling probability because there are more available workers than vacancies hence it is easier for the firm to find the right match for the posted vacancy.

We can see that the optimal monetary policy is the least volatile as compared to the other regimes. The closest one to it is the policy rule with response to the labour market. As expected, it results in the lowest volatility in the labour market variables (such as unemployment and probability that vacancy will be filled).

Figure 3.5. Impulse Response Functions to Repayment Probability Shock (I)

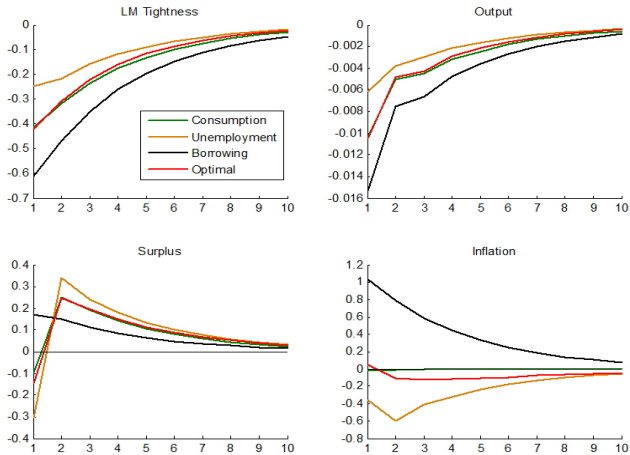


From Figure 3.5 we can see that the drop in borrowing makes labour market more flexible, because there are more empty vacancies than before the shock (in the top left panel in Figure 3.5 we can see the drop in the labour market tightness). It is straightforward to see that output also falls because the decrease in borrowing

causes the drop in production and also demand for final goods. Bargaining surplus, however, increases because of an increase in the level of unemployment.

It is worth noting, that interest rate expose only minor downward trend in all the regimes except the case with the financial market response (as seen from Figure's 3.5 top right panel) but inflation, in all of the cases (except optimal policy and borrowing targeting), falls. The reason for it is intuitive: as borrowing falls, there are less money in the economy and hence inflation falls. The sharpest decrease in inflation is seen in case of the policy rule with response to unemployment (Figure 3.6 bottom right panel). The explanation for that is the fall in the level of borrowing (Figure 3.5 bottom left panel) due to the negative financial shock.

Figure 3.6. Impulse Response Functions to Repayment Probability Shock (II)



To put it in a nutshell, it is clear that the optimal monetary policy is the least volatile regime as compared to the others. Moreover, movements in interest rate and inflation directly depends on the changes in borrowing caused by the shock in  $\phi_t$ .

### 3.3.4 Response to Productivity Shock

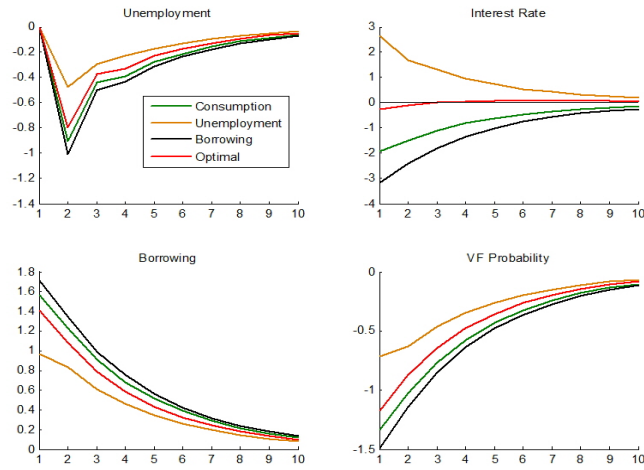
Contrary to the response to the financial shock, model's reaction to the change in productivity causes an increase in output and employment, as it could be seen from Figures 3.7 and 3.8 below.

However, different policies cause different behavior of the key variables. For instance, in Figure 3.7 we can see that the policy rule with response to unemployment and inflation causes the lowest volatility of employment as compared to the other rules. We can also see that it increases interest rate. The intuitive reason for that is the policy maker's willingness to reduce a jump in borrowing (bottom left panel in Figure 3.7) by raising interest rates.

For example, consider the rule with the response to inflation and consumption. We see that interest rate decrease because of the drop in vacancy filling probability (the bottom right panel in Figure 3.7). This, as a result causes downward trend in inflation which we see in Figure 3.8.

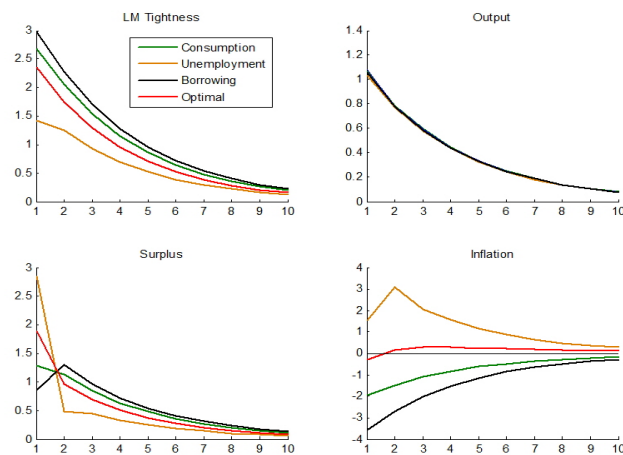
It is clear, that an increase in productivity causes decrease in unemployment because firms require more labour force to meet an increase in the level of aggregate demand. Further to it, as there are more firms taking loans (to finance their expansion and hire more workers), borrowing becomes cheaper and interest rate falls (except the case of response to inflation and unemployment). This is somehow counter intuitive, because in this case borrowing should become more expensive. However, this effect is mitigated by the drop in interest rate. Lastly, firms start hiring more workers and therefore vacancy filling probability falls as there are less unemployed agents and firms have smaller number of unemployed agents to choose from.

Figure 3.7. Impulse Response Functions to Productivity Shock (I)



From Figure 3.8 we can see that the productivity shock causes an increase in the labour market tightness (because increase in employment decreased number of available vacancies) and a jump in both output and bargaining surplus. It is consistent with the jump in the financial market because in our model firms borrow against their expected profit (surplus) which has increased and therefore firms are able to obtain more credit.

Figure 3.8. Impulse Response Functions to Productivity Shock (II)



To summarize, we can see that the rule with response to inflation and unemployment causes the smallest volatility of most of the variables and in this sense is the

closest one to optimal policy. Our model generates very similar dynamic responses to those showed in Faia (2008, 2009), Proano (2012) or Ravenna and Walsh (2011). In the following chapter we are going to compute unconditional welfare measure and rank policy rules accordingly and investigate how policy maker's preferences may affect welfare.

### 3.4 Welfare Implications

In this section we compute the welfare measure using Benigno and Woodford (2005) methodology and rank the policy rules according to their performance as it is suggested in Damjanovic et. al (2011).

We assume that the policy maker commits to the one monetary policy regime and never changes his mind. It is noteworthy, that welfare is computed in steady state therefore it does not depend on the realization of shocks. Welfare could be calculated by using a matrix of the second order moments. This is exactly what we are going to do.

To compute unconditional welfare, certain steps must be followed. Firstly, we have to write our model in VAR form. Then, we apply Benigno and Woodford (2005) algorithm and compute second order approximation (see Kim and Kim (2003) for review of spurious welfare calculations based on first-order approximation) around the steady state. Thirdly, we compute covariance matrix which is used to obtain the welfare measure.

We present linearized model equations of interest in vector autoregressive form below. Our system consists of 12 constraints and the household utility function (equation (3.1)). In order to simplify computations we substitute Phillips curve with the three additional equations as showed in 2.6 subsection. The constraints are denoted as equations 3.22, 3.29, 3.28, 3.84, 3.85, 3.86, 3.10, ??, 3.18, ??, 3.11



and 3.19. The system could be written as

$$\begin{aligned}
F_1 &= S\widehat{S}_t - S\widehat{S}_{t+1} \left( (1-\lambda)\beta - \eta\beta p^E \right) + bi \left( \widehat{b}_{t-1}^F + \widehat{i}_{t-1} - \widehat{\pi}_t \right) \\
&\quad - \widehat{b}b_t + \eta\beta S\widehat{p}_t^E p^E - \frac{z}{X} \left( z_t - \widehat{X}_t \right), \\
F_2 &= \left( \widehat{b}_t + \widehat{i}_t - \widehat{\pi}_{t+1} \right) - \phi_t + \widehat{S}_{t+1}, \\
F_3 &= -\frac{k}{q} (\widehat{q}_t) - \widehat{b}_t b - \beta(1-\eta)S\widehat{S}_{t+1}, \\
F_4 &= \widehat{i}_t - \widehat{\pi}_{t+1} + \sigma\widehat{C}_t - \sigma\widehat{C}_{t+1}, \\
F_5 &= Y\widehat{Y}_t - C\widehat{C}_t - aN\widehat{N}_t - k(1 - (1-\lambda)N)\theta\widehat{\theta}_t + k\theta(1-\lambda)N\widehat{N}_{t-1}, \\
F_6 &= \widehat{p}_t^E + \frac{\alpha}{1-\alpha}\widehat{q}_t, \\
F_7 &= \widehat{p}_t^E - \alpha\widehat{\theta}_t, \\
F_8 &= \widehat{Y}_t - \left( \widehat{N}_t + z_t \right), \\
F_9 &= \widehat{N}_t - \widehat{N}_{t-1} (p^E - 1) (1-\lambda) + a\lambda\widehat{\theta}_t, \\
F_{10} &= \varpi\beta\widehat{i}_t + \varpi\beta\widehat{K}_{t+1} + \varpi\beta(\varepsilon - 1)\widehat{\pi}_{t+1} + \widehat{Y}_t(1 - \varpi\beta) - \widehat{K}_t, \\
F_{11} &= (1 - \varpi\beta) \left( \widehat{Y}_t - \widehat{X}_t \right) + \varpi\beta\widehat{Z}_{t+1} + \varpi\beta\varepsilon\widehat{\pi}_{t+1} + \varpi\beta\widehat{i}_t - \widehat{Z}_t, \\
F_{12} &= \frac{\varpi}{1-\varpi}\widehat{\pi}_t + \widehat{K}_t - \widehat{Z}_t.
\end{aligned}$$

More concisely we can write the system in matrix form as

$$Z_{t+1}^W = AZ_t^W + B\varepsilon_{t+1}, \quad (3.40)$$

where  $Z_{t+1}^W$  is a vector of endogenous variables,  $\varepsilon_{t+1}$  is a vector of exogenous shocks and matrices  $A$  and  $B$  are coefficient matrices. Our aim is to construct variance-covariance matrix which could be denoted as

$$S_t^W = E_t \sum_{t=0}^{\infty} Z_{t+1}^W Z_{t+1}^{W'}. \quad (3.41)$$

Let's assume, that  $Z_t^W = 0$ , then equation (3.40) immediately becomes

$$Z_{t+1}^W = B\varepsilon_{t+1} \quad (3.42)$$

substituting it in our variance-covariance matrix expression we obtain

$$S_t^W = E_t B \varepsilon_{t+1} \varepsilon'_{t+1} B' \quad (3.43)$$

Now we denote covariance matrix of exogenous shocks as  $\Omega$

$$\Omega = \varepsilon_{t+1} \varepsilon'_{t+1}. \quad (3.44)$$

Then we continue in the same manner and obtain expression for  $Z_{t+2}^W$

$$Z_{t+2}^W = A Z_{t+1}^W + B \varepsilon_{t+2}. \quad (3.45)$$

Computing  $S_{t+2}^W$  yields

$$S_{t+2}^W = (A Z_{t+1}^W + B \varepsilon_{t+2}) (A Z_{t+1}^W + B \varepsilon_{t+2})', \quad (3.46)$$

which could be expanded to

$$S_{t+2}^W = A Z_{t+1}^W Z_{t+1}^{W'} A' + 2 A Z_{t+1}^{W'} \varepsilon'_{t+2} B' + B \varepsilon_{t+2} \varepsilon'_{t+2} B'. \quad (3.47)$$

Since the endogenous variables and the exogenous shocks are uncorrelated,  $2 A Z_{t+1}^{W'} \varepsilon'_{t+2} B'$  is equal to zero. Thus we can write it as

$$S_{t+2}^W = A Z_{t+1}^W Z_{t+1}^{W'} A' + B \Omega B'. \quad (3.48)$$

We assume that  $Z^W$  is always equal to  $B \varepsilon$  because we have started at the point  $Z_t^W = 0$ . Therefore we can always substitute  $Z_t^W$  with  $B \varepsilon$ . Then our variance-covariance matrix for the subsequent periods becomes sum of  $A$  which is a matrix of coefficients

$$S_{t+k+1}^W = \sum_{s=0}^{\infty} A^s B \Omega B' A'^s. \quad (3.49)$$

To compute the sum, recall that  $S_t = E_t \sum_{t=0}^{\infty} \beta^{t+1} Z_{t+1}^W Z_{t+1}^{W'}$ . Therefore rewriting equation above yields

$$S_t^W = E_t \sum_{t=0}^{\infty} Z_{t+1}^W Z_{t+1}^{W'} = E_t \sum_{t=0}^{\infty} \beta^{t+1} \left( \sum_{s=0}^t A^s B \Omega B' A'^s \right). \quad (3.50)$$

Hence our  $S_t^W$  is equal to the double sum

$$S_t^W = E_t \left( \sum_{s=0}^{\infty} A^s B \Omega B' A'^s \right) \sum_{t=s}^{\infty} \beta^{t+1}. \quad (3.51)$$

Since we know that  $\sum_{t=s}^{\infty} \beta^{t+1} = \frac{1}{(1-\beta)}$ <sup>7</sup>, simplification of the above equation yields

$$S_t^W = E_t \left( \sum_{s=0}^{\infty} A^s B \Omega B' A'^s \right) \frac{\beta^{s+1}}{(1-\beta)}. \quad (3.52)$$

Lastly, for the sake of simplicity we define matrix  $D = B \Omega B' \frac{\beta}{(1-\beta)}$  and matrix  $C = A \sqrt{\beta}$ . Therefore the final sum becomes

$$S_t^W = E_t \sum_{s=0}^{\infty} C^s D C'^s. \quad (3.53)$$

When matrix  $S_t^W$  is computed, we substituted its values to the second order welfare approximation to the linear system (second order approximation is provided in the appendix) and sum it all up. The general expression (Benigno and Woodford 2005) is given by

$$W_t = \sum_{t=0}^{\infty} F_t + t.i.p. + O^3. \quad (3.54)$$

where  $F_t$  is the objective function (approximated to the second order), *t.i.p.* are terms independent of policy and  $O^3$  are terms of the higher order which can be disregarded. In our case policy maker aims to minimize loss of the welfare, therefore the objective function is given by the following sum

$$F_t = U_t + \sum_{t=0}^{\infty} \Gamma_t S_t^W. \quad (3.55)$$

---

<sup>7</sup>Firstly, we multiply both sides by  $(1-\beta)$ . This gives us  $(1-\beta) \sum_{t=s}^{\infty} \beta^{t+1} = (1-\beta)(1 + \beta^2 + \beta^3 + \dots + \beta^{t+1})$ . Expanding it we get  $\sum_{t=s}^{\infty} \beta^{t+1} (1-\beta) = (1 + \beta^2 + \beta^3 + \dots + \beta^{t+1}) - \beta(1 + \beta^2 + \beta^3 + \dots + \beta^{t+1})$ , simplifying  $\sum_{t=s}^{\infty} \beta^{t+1} (1-\beta) = 1 - \beta^{t+2}$ . Thus the sum is equal to  $\sum_{t=s}^{\infty} \beta^{t+1} = \frac{1-\beta^{t+2}}{(1-\beta)}$ . If we take a limit of  $\beta$  as  $t \rightarrow \infty$ , we obtain  $\sum_{t=s}^{\infty} \beta^{t+1} = \frac{1}{(1-\beta)}$  since  $\lim_{t \rightarrow \infty} \beta^{t+2} = 0$ .

Household utility is denoted by  $U_t$ ,  $S_t^W$  are constraints presented in the linear system and  $\Gamma_t$  are corresponding Lagrange multipliers from the policy maker's optimization problem showed in appendix.

We use this algorithm to compute welfare under four different monetary policy regimes and rank policy rules according to the welfare loss. In addition, we compare each monetary regime with the optimal policy in terms of consumption equivalence. Since the consumption equivalence shows how much consumption when in the optimal policy regime agents are willing to give up to be as good as in other regime, we can write the following definition based on the utility function (3.1)

$$W^0 = x^{1-\sigma} W^P, \quad (3.56)$$

where  $W^O$  is welfare measure in optimal policy regime,  $W^P$  is welfare in any other regime,  $x$  is defined as consumption equivalence. Then  $x$  is given by

$$x = \left( \frac{W^O}{W^P} \right)^{\frac{1}{1-\sigma}}. \quad (3.57)$$

And this is the consumption equivalence we used in the table below expressed in the percentage terms.

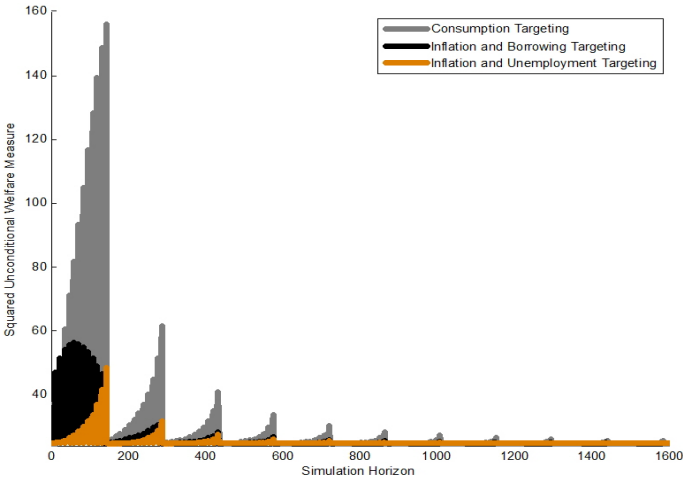
Table 3.3. Monetary Policy Rules and Welfare

<b>Policy</b>	<b>Welfare</b>	<b>Consumption Equivalence</b>
Inflation and Consumption Targeting	-4.949	1.104%
Inflation and Unemployment Targeting	-4.925	0.612%
Inflation and Borrowing Targeting	-4.951	1.137%
Optimal policy	-4.895	—

It is clear that the best performance in terms of welfare is delivered by the optimal policy. The closest one to it is the rule with response to inflation and unemployment.

Now we can use results from the grid search algorithm and plot the all three policy rules against squared welfare measure on  $y$  axis and number of all the possible combinations of parameter values on  $x$  axis. We can see from the figure 3.8 that the policy rule with response to unemployment causes smallest losses for any reasonable parameter values. In other words, if the policy maker decides to use interest rate rule with unemployment targeting, it will give higher welfare for any reasonable parameter values. This confirms the result provided in table 3.

Figure 3.8. Policy Rules for Various Parameter Values



### 3.4.1 Should Response to Labour Market be Positive?

In the previous literature (for example Faia (2008) and Proano (2012)) it was pointed out that the aggregate level of unemployment should be positively targeted in the interest rate rule. In this section we show that sign next to unemployment targeting in the Taylor rule (as showed in Table 2) depends on the size of bargaining power.

To begin with, we conducted a numerical experiment where we have set parameter  $\alpha_u$  to be equal either to 2 or  $-2$  for the same value of  $\eta$ . As we can see from Figure 11 below, if workers bargaining power is moderate that is, if  $\eta \in (0.2, 0.55)$ <sup>8</sup> then the positive response gives higher welfare, whereas in the case of  $\eta \in (0.55, 0.99)$  the negative response gives higher welfare. Now we can formalize this result.

**Result:** Sign next to the unemployment reaction coefficient (assuming the same numerical value) in the Taylor rule depends on the size of the workers bargaining power. That is, if  $\eta \in (0.2, 0.55)$  reaction should be positive, if  $\eta \in (0.55, 0.99)$  reaction should be negative to obtain the highest level of the welfare measure.

From equation (??) we know that  $p_t^E = (q_t)^{-\frac{\alpha}{1-\alpha}} \varsigma^{\frac{\alpha}{1-\alpha}}$ , which we can substitute into surplus equation (3.22):

$$S_t = \frac{z_t}{X_t} - a + b_t - \frac{b_{t-1}}{\pi_t} i_{t-1} + (1 - \lambda)\beta E_t S_{t+1} - \eta\beta (q_t)^{-\frac{\alpha}{1-\alpha}} \varsigma^{\frac{\alpha}{1-\alpha}} E_t S_{t+1}.$$

Differentiate it with respect to  $S_{t+1}$

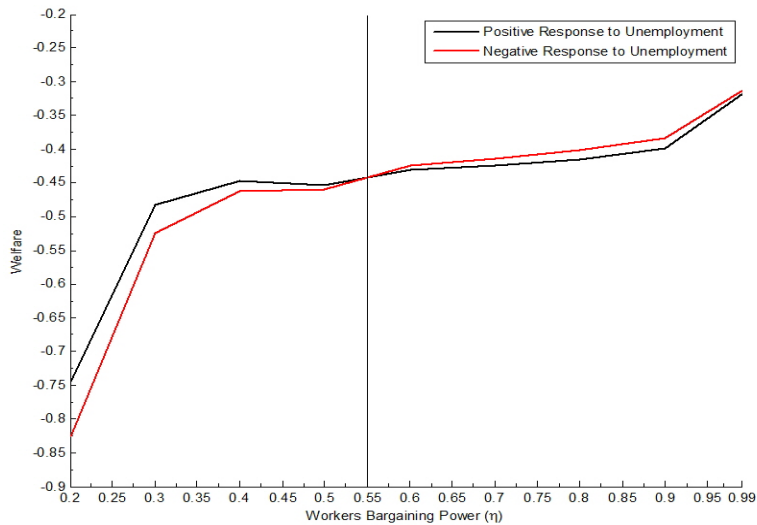
$$\frac{\partial S_t}{\partial S_{t+1}} = (1 - \lambda)\beta - \eta\beta (q_t)^{-\frac{\alpha}{1-\alpha}} \varsigma^{\frac{\alpha}{1-\alpha}}. \quad (3.58)$$

Expression above shows that the effect of the labour market changes ( $q_t$ ) on surplus depends on the size of workers bargaining power. If  $\eta$  is small, then the effect will be very little since  $q_t$  falls because of the labour market contraction. If  $\eta$  is large, then the negative effect on surplus will be much higher, i.e. decrease in surplus is much larger when  $\eta$  is large. Therefore in the case of  $\eta \in (0.2, 0.55)$  even the positive response to unemployment can increase welfare. When  $\eta \in (0.55, 0.99)$  response should be negative because in this case raise in unemployment will lower interest rates and borrowing will become cheaper.

---

<sup>8</sup>Workers bargaining power parameter has the values between 0 and 1 but if it is lower than 0.2, Blanchard-Kahn (1980) conditions are not satisfied. Therefore we restrict it to be in range of 0.2 : 1.

Figure 3.9. Welfare and Bargaining Power



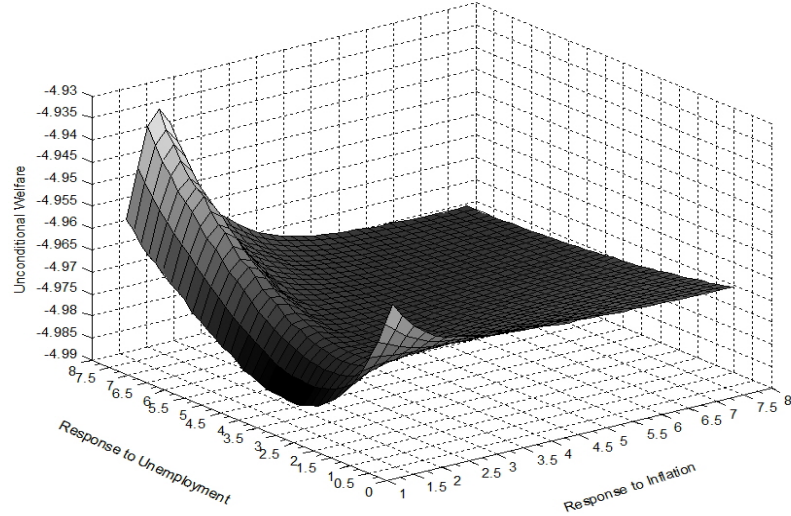
Intuition of this result could be summarized as follows. To begin with, we see that firms borrow against their future profit (surplus) as shown in equation (3.29). When the financial shock occurs, borrowing drops. Surplus, however, is affected through expected surplus and borrowing. As we have showed earlier, if workers have a lot of bargaining power, then they take most of the surplus. When the financial shock is present, the drop in surplus is reinforced and hence future borrowing is affected. Therefore the policy maker should significantly reduce interest rate to make borrowing much cheaper. Then it follows that higher bargaining power implies negative reaction to unemployment in the Taylor rule and in the case of low bargaining power - positive response to unemployment.

### 3.4.2 Policy Maker's Preferences and Welfare

In each of the monetary policy regimes (except the optimal policy) government faces a trade-off because it aims to minimize variance of at least two variables. Since we have found out that the interest rate smoothing is welfare detrimental, we set parameter  $\alpha_i$  to zero and compute welfare measure for each value of parameters

$\alpha_\pi$  and  $\alpha_u$  in the second policy regime (unemployment targeting). The resulting shape is provided below.

Figure 3.10. Rule 2, Inflation and Unemployment



As we can see, the highest welfare is obtained when policy maker has very strong response to unemployment. However, moderate response could be welfare detrimental as compared to very minor response which could give higher welfare. Interestingly we see that high response to inflation does not guarantee high welfare as it is usually the case in New-Keynesian models. The reason for that could be source of frictions. In our model they come from imperfect labour market. Therefore monetary policy which reacts to both inflation and unemployment is better than just plain inflation targeting.

### 3.5 Conclusion

In this paper we have presented economy with labour and financial market frictions. We have used matching model (Pissarides 1987) to account for the imperfections in the labour market and introduced firms' financial market which let us investigate



the relation between firms' borrowing and the aggregate level of unemployment. Further to this, we have derived the optimal policy reaction to the repayment probability and productivity shocks. Then we have introduced 3 different Taylor type monetary policy rules and ranked them according to the welfare measure which was computed using Benigno and Woodford (2005) methodology. The key findings could be summarized as follows.

Firstly, we have found that the monetary policy rule with response to inflation and labour market frictions (the aggregate level of unemployment) performs closest to the optimal rule in terms of the welfare measure and the consumption equivalence. Further to this, we also showed that targeting the aggregate level of borrowing (firms financial market) may slightly increase welfare too.

Secondly, we have showed that the sign of the labour market reaction parameter in the interest rate rule depends on the size of workers bargaining power. That is, if workers have very little bargaining power then reaction should be positive, if workers have high bargaining power, then reaction should be negative to obtain the highest value of welfare measure.

Thirdly, we have investigated how presence of the financial frictions (changes in firms' repayment probability) and changes in productivity can affect labour market and the level of unemployment and thus output. We have showed that the fall in firms borrowing can directly affect labour market and cause an increase in unemployment.

## 3.6 Appendix

Linearization of model equations around zero inflation steady state yields the following results.

### 3.6.1 Benchmark Model

#### Linearized Model without Policy

Bargaining surplus

$$S\widehat{S}_t = \frac{z}{X} (z_t - \widehat{X}_t) + S\widehat{S}_{t+1} ((1 - \lambda)\beta - \eta\beta p^E) + b\widehat{b}_t - \eta\beta S\widehat{p}_t^E p^E - bi (\widehat{b}_{t-1} + \widehat{i}_{t-1} - \widehat{\pi}_t). \quad (3.59)$$

Firms' borrowing constraint

$$\widehat{S}_{t+1} = \phi_t - (\widehat{b}_t + \widehat{i}_t - \widehat{\pi}_{t+1}). \quad (3.60)$$

Firms' borrowing and labour market ( $q_t$ ) relation

$$\widehat{b}_t b = -\frac{k}{q} (\widehat{q}_t) - \beta(1 - \eta)S\widehat{S}_{t+1}. \quad (3.61)$$

Interest rate and consumption relation

$$\widehat{i}_t = \widehat{\pi}_{t+1} + \sigma\widehat{C}_{t+1} - \sigma\widehat{C}_t. \quad (3.62)$$

Phillips curve

$$\beta\widehat{\pi}_{t+1} = \iota\widehat{X}_t + \widehat{\pi}_t.$$

Output and Employment

$$Y\widehat{Y}_t = C\widehat{C}_t + aN\widehat{N}_t + k(1 - (1 - \lambda)N)\theta\widehat{\theta}_t - k\theta(1 - \lambda)N\widehat{N}_{t-1}. \quad (3.63)$$

Probability that worker will find a job

$$\widehat{p}_t^E = -\frac{\alpha}{1-\alpha}\widehat{q}_t. \quad (3.64)$$

Employment and labour market tightness

$$\widehat{p}_t^E = \alpha\widehat{\theta}_t. \quad (3.65)$$

Final good production

$$\widehat{Y}_t = \widehat{N}_t + z_t. \quad (3.66)$$

Number of employed agents

$$\widehat{N}_t = \widehat{N}_{t-1} (p^E - 1) (1 - \lambda) - a\lambda\widehat{\theta}_t. \quad (3.67)$$

### Corresponding Steady State

$$\begin{aligned} \frac{z}{X} &= a + ((1 - \beta)\phi(1 - \eta) + 1 + \eta\beta p^E - (1 - \lambda)\beta) S, \\ \frac{k}{q} &= \beta(\phi + 1)(1 - \eta)S, \\ b &= \beta\phi(1 - \eta)S, \\ K &= \frac{\varepsilon}{\varepsilon - 1}Z, \\ K &= \frac{Nz}{1 - \varpi\beta}, \\ X &= \frac{\varepsilon}{\varepsilon - 1}, \\ N(z - a + k\theta(1 - \lambda)) &= C - a + k\theta, \\ p^E &= (q)^{-\frac{\alpha}{1-\alpha}} \varsigma^{\frac{1}{1-\alpha}}, \\ p_t^E &= \varsigma(\theta_t)^\alpha, \\ (\lambda + p^E(1 - \lambda))N &= p^E. \end{aligned}$$

### 3.6.2 Policy Maker's Optimization Problem

Government maximizes household utility subject to behavioral constraints. It could be formally written as Lagrange optimization problem:

$$\mathcal{L} = E_t \sum_{t=0}^{\infty} \beta^t \left[ \begin{array}{l} \frac{C_t^{1-\sigma}}{1-\sigma} \\ +\Gamma_{1,t} \left( S_t - \frac{z_t}{X_t} + a + b_{t-1} \frac{i_{t-1}}{\pi_t} - b_t - (1-\lambda)\beta S_{t+1} + \eta\beta p_t^E S_{t+1} \right) \\ +\Gamma_{2,t} \left( b_t \frac{i_t}{E\pi_{t+1}} - \phi(1-\eta)S_{t+1} \right) \\ +\Gamma_{3,t} \left( \frac{k}{q_t} - b_t - \beta(1-\eta)S_{t+1} \right) \\ +\Gamma_{4,t} \left( \left[ \frac{1-\varpi\pi_t^{\varepsilon-1}}{1-\varpi} \right]^{\frac{1}{1-\varepsilon}} K_t - \frac{\varepsilon}{\varepsilon-1} Z_t \right) \\ +\Gamma_{5,t} \left( K_t - Y_t - \varpi \frac{1}{i_t} \pi_{t+1}^{\varepsilon-1} K_{t+1} \right) \\ +\Gamma_{6,t} \left( Z_t - Y_t \frac{1}{X_t} - \varpi \frac{1}{i_t} \pi_{t+1}^{\varepsilon} Z_{t+1} \right) \\ +\Gamma_{7,t} \left( \beta i_t - \frac{C_t^{-\sigma}}{C_{t+1}^{-\sigma}} \pi_{t+1} \right) \\ +\Gamma_{8,t} (Y_t - C_t + a(1-N_t) - k\theta_t(1-(1-\lambda)N_{t-1})) \\ +\Gamma_{9,t} \left( p_t^E - (q_t)^{-\frac{\alpha}{1-\alpha}} \varsigma^{\frac{1}{1-\alpha}} \right) \\ +\Gamma_{10,t} (p_t^E - \varsigma(\theta_t)^\alpha) \\ +\Gamma_{11,t} (Y_t - N_t z_t) \\ +\Gamma_{12,t} (N_t - (1-\lambda)N_{t-1} - \varsigma\theta_t^\alpha(1-(1-\lambda)N_{t-1})) \end{array} \right].$$

which has the following first order conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial i_t} &= \beta \Gamma_{1,t+1} \frac{b_t}{\pi_{t+1}} i_t + \Gamma_{2,t} \frac{b_t}{\pi_{t+1}} i_t + \Gamma_{5,t} \varpi \frac{1}{i_t} \pi_{t+1}^{\varepsilon-1} K_{t+1} + \Gamma_{6,t} \varpi \frac{1}{i_t} \pi_{t+1}^\varepsilon Z_{t+1} + \beta \Gamma_{7,t} i_t, \\
\frac{\partial \mathcal{L}}{\partial Z_t} &= -\Gamma_{4,t} \frac{\varepsilon}{\varepsilon - 1} + \Gamma_{6,t} - \Gamma_{6,t-1} \varpi \frac{1}{\beta i_{t-1}} \pi_t^\varepsilon, \\
\frac{\partial \mathcal{L}}{\partial N_t} &= -\Gamma_{8,t} a + \Gamma_{8,t+1} \beta (1 - \lambda) k \theta_{t+1} - \Gamma_{11,t} z_t + \Gamma_{12,t} \\
&\quad - \Gamma_{12,t+1} \beta (1 - \lambda) + \Gamma_{12,t+1} p_{t+1}^E (1 - \lambda) \beta, \\
\frac{\partial \mathcal{L}}{\partial S_t} &= \Gamma_{1,t} - \Gamma_{1,t-1} (1 - \lambda) + \eta p_{t-1}^E \Gamma_{1,t-1} - \Gamma_{2,t-1} \frac{\phi_t}{\beta} (1 - \eta) - \Gamma_{3,t-1} (1 - \eta), \\
\frac{\partial \mathcal{L}}{\partial C_t} C_t &= C_t^{1-\sigma} + \Gamma_{7,t} \sigma \frac{C_t^{-\sigma}}{C_{t+1}^{-\sigma}} \pi_{t+1} - \Gamma_{7,t-1} \frac{\sigma}{\beta} \frac{C_{t-1}^{-\sigma}}{C_t^{-\sigma}} \pi_t - \Gamma_{8,t} C_t, \\
\frac{\partial \mathcal{L}}{\partial X_t} X_t &= \Gamma_{1,t} \frac{z_t}{X_t} + \Gamma_{6,t} \frac{Y_t}{X_t}, \\
\frac{\partial \mathcal{L}}{\partial b_t^F} &= \beta \Gamma_{1,t+1} \frac{i_t}{\pi_{t+1}} - \Gamma_{1,t} + \Gamma_{2,t} \frac{i_t}{\pi_{t+1}} - \Gamma_{3,t}, \\
\frac{\partial \mathcal{L}}{\partial p_t^E} &= \Gamma_{1,t} \eta \beta S_{t+1} + \Gamma_{9,t} + \Gamma_{10,t}, \\
\frac{\partial \mathcal{L}}{\partial q_t} q_t &= -\Gamma_{3,t} \frac{k}{q_t} + \Gamma_{9,t} \left( \frac{\alpha}{1 - \alpha} \right) p_t^E, \\
\frac{\partial \mathcal{L}}{\partial Y_t} &= -\Gamma_{5,t} - \Gamma_{6,t} \frac{1}{X_t} + \Gamma_{8,t} + \Gamma_{11,t}, \\
\frac{\partial \mathcal{L}}{\partial K_t} &= \Gamma_{4,t} \left[ \frac{1 - \varpi \pi_t^{\varepsilon-1}}{1 - \varpi} \right]^{\frac{1}{1-\varepsilon}} + \Gamma_{5,t} - \Gamma_{5,t-1} \varpi \frac{1}{\beta i_{t-1}} \pi_t^{\varepsilon-1}, \\
\frac{\partial \mathcal{L}}{\partial \pi_t} \pi_t &= -\Gamma_{1,t} b_{t-1} \frac{i_{t-1}}{\pi_t} - \Gamma_{2,t-1} \frac{b_{t-1} i_{t-1}}{\beta \pi_t} - \Gamma_{5,t-1} \frac{\varpi}{\beta i_{t-1}} (\varepsilon - 1) \pi_t^{\varepsilon-1} K_t - \Gamma_{6,t-1} \frac{\varpi}{\beta} \frac{\varepsilon}{i_{t-1}} \pi_t^\varepsilon Z_t \\
&\quad - \Gamma_{7,t-1} \frac{1}{\beta} \frac{C_{t-1}^{-\sigma}}{C_t^{-\sigma}} \pi_t + \Gamma_{4,t} \left( \frac{\varpi}{1 - \varpi} \right) \pi_t^{\varepsilon-1} \left[ \frac{1 - \varpi \pi_t^{\varepsilon-1}}{1 - \varpi} \right]^{\frac{\varepsilon}{1-\varepsilon}} K_t, \\
\frac{\partial \mathcal{L}}{\partial \theta_t} \theta_t &= -\Gamma_{8,t} k (1 - (1 - \lambda) N_{t-1}) \theta_t - \alpha \Gamma_{10,t} p_t^E - \Gamma_{12,t} \alpha p_t^E (1 - (1 - \lambda) N_{t-1}).
\end{aligned}$$

## Linearization of Policy Maker's FOC

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial i_t} i_t &= \Gamma_1 b \left( \widehat{\Gamma}_{1,t+1} + \widehat{b}_t - \widehat{\pi}_{t+1} + \widehat{i}_t \right) + \frac{1}{\beta} \Gamma_2 b \left( \widehat{\Gamma}_{2,t} + \widehat{b}_t - \widehat{\pi}_{t+1} + \widehat{i}_t \right) \\
&\quad + \beta \Gamma_6 \varpi Z \left( \widehat{\Gamma}_{6,t} - \widehat{i}_t + \varepsilon \widehat{\pi}_{t+1} + \widehat{Z}_{t+1} \right) + \Gamma_7 \left( \widehat{\Gamma}_{7,t} + \widehat{i}_t \right) \\
&\quad + \beta \Gamma_5 \varpi K \left( \widehat{\Gamma}_{5,t} - \widehat{i}_t + (\varepsilon - 1) \widehat{\pi}_{t+1} + \widehat{K}_{t+1} \right), \\
\frac{\partial \mathcal{L}}{\partial Z_t} &= -\Gamma_4 \frac{\varepsilon}{\varepsilon - 1} \widehat{\Gamma}_{4,t} + \Gamma_6 \widehat{\Gamma}_{6,t} - \Gamma_6 \varpi \left( \Gamma_{6,t-1} - \widehat{i}_{t-1} + \varepsilon \widehat{\pi}_t \right), \\
\frac{\partial \mathcal{L}}{\partial N_t} &= -\Gamma_8 a \widehat{\Gamma}_{8,t} + \Gamma_8 \beta (1 - \lambda) k \left( \widehat{\theta}_{t+1} + \widehat{\Gamma}_{8,t+1} \right) - z \Gamma_{11} \left( \widehat{\Gamma}_{11,t} + \widehat{z}_t \right) + \Gamma_{12} \widehat{\Gamma}_{12,t} \\
&\quad - \Gamma_{12} \beta (1 - \lambda) (1 - p_{t+1}^E) \widehat{\Gamma}_{12,t+1} + \Gamma_{12} \beta (1 - \lambda) p^E \widehat{p}_{t+1}^E, \\
\frac{\partial \mathcal{L}}{\partial S_t} &= \Gamma_1 \widehat{\Gamma}_{1,t} - \Gamma_1 \left[ (1 - \lambda) - \eta p_{t-1}^E \right] \widehat{\Gamma}_{1,t-1} + \Gamma_1 \eta p^E \widehat{p}_{t-1}^E \\
&\quad - \frac{\phi}{\beta} (1 - \eta) \Gamma_2 \left( \widehat{\Gamma}_{2,t-1} + \widehat{\phi}_t \right) - \Gamma_3 (1 - \eta) \widehat{\Gamma}_{3,t-1}, \\
\frac{\partial \mathcal{L}}{\partial C_t} C_t &= C^{1-\sigma} (1 - \sigma) \widehat{C}_t + \Gamma_7 \sigma \left( \widehat{\Gamma}_{7,t} + \sigma \left( \widehat{C}_{t+1} - \widehat{C}_t \right) + \widehat{\pi}_{t+1} \right) \\
&\quad - \Gamma_7 \frac{\sigma}{\beta} \left( \widehat{\Gamma}_{7,t-1} + \sigma \left( \widehat{C}_t - \widehat{C}_{t-1} \right) + \widehat{\pi}_t \right) - C \Gamma_8 \left( \widehat{\Gamma}_{8,t} + \widehat{C}_t \right), \\
\frac{\partial \mathcal{L}}{\partial X_t} X_t &= \left( \widehat{\Gamma}_{1,t} + \widehat{z}_t \right) - \left( \widehat{\Gamma}_{6,t} + \widehat{Y}_t \right), \\
\frac{\partial \mathcal{L}}{\partial b_t^F} &= \Gamma_1 \left( \Gamma_{1,t+1} + i_t - \pi_{t+1} \right) - \Gamma_1 \widehat{\Gamma}_{1,t} + \frac{\Gamma_2}{\beta} \left( \widehat{\Gamma}_{2,t} - \widehat{\pi}_{t+1} + \widehat{i}_t \right) - \Gamma_3 \widehat{\Gamma}_{3,t}, \\
\frac{\partial \mathcal{L}}{\partial p_t^E} &= \Gamma_1 \eta \beta S \left( \widehat{\Gamma}_{1,t} + \widehat{S}_{t+1} \right) + \Gamma_9 \widehat{\Gamma}_{9,t} + \Gamma_{10} \widehat{\Gamma}_{10,t}, \\
\frac{\partial \mathcal{L}}{\partial K_t} &= \Gamma_4 \left( \widehat{\Gamma}_{4,t} + \frac{\varpi}{1 - \varpi} \widehat{\pi}_t \right) + \Gamma_5 \widehat{\Gamma}_{5,t} - \Gamma_5 \varpi \left( \widehat{\Gamma}_{5,t-1} - \widehat{i}_{t-1} + (\varepsilon - 1) \widehat{\pi}_t \right), \\
\frac{\partial \mathcal{L}}{\partial \pi_t} \pi_t &= -b \frac{1}{\beta} \widehat{\Gamma}_{1,t-1} \Gamma_1 + \left( -\Gamma_1 \frac{1}{\beta} - \Gamma_2 \right) b \widehat{b}_{t-1} + \left( -\Gamma_1 b - \Gamma_2 \frac{b}{\beta} + \Gamma_6 \varpi \beta \varepsilon Z \right) \widehat{i}_{t-1} \frac{1}{\beta} \\
&\quad + \left( -\Gamma_5 \varpi (\varepsilon - 1) + \Gamma_4 \left( \frac{\varpi}{1 - \varpi} \right) \right) K \widehat{K}_t - \Gamma_6 \varpi \varepsilon Z \widehat{Z}_t \\
&\quad + \left( \frac{\varpi}{1 - \varpi} \right) K \widehat{\Gamma}_{4,t} \Gamma_4 - \frac{1}{\beta} \widehat{\Gamma}_{7,t-1} \Gamma_7 - \varpi \varepsilon Z \widehat{\Gamma}_{6,t-1} \Gamma_6 - \Gamma_7 \frac{\sigma}{\beta} \widehat{C}_t \\
&\quad + \Gamma_7 \frac{1}{\beta} \sigma \widehat{C}_{t-1} - \varpi (\varepsilon - 1) K \widehat{\Gamma}_{5,t-1} \Gamma_5 - b \widehat{\Gamma}_{2,t-1} \Gamma_2 \\
&\quad + \left( \begin{array}{l} \Gamma_1 b \frac{1}{\beta} + \Gamma_2 b \frac{1}{\beta} - \Gamma_5 \varpi (\varepsilon - 1) (\varepsilon - 1) K - \Gamma_6 \varpi \varepsilon^2 Z - \Gamma_7 \frac{1}{\beta} \\ + \Gamma_4 \left( \frac{\varpi}{1 - \varpi} \right) (\varepsilon - 1) K - \Gamma_4 \left( \frac{\varpi}{1 - \varpi} \right) \varepsilon K \frac{\varpi}{1 - \varpi} \end{array} \right) \widehat{\pi}_t,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial \theta_t} \theta_t &= -\Gamma_8 k \theta \left( (1 - (1 - \lambda) N) \left( \widehat{\Gamma}_{8,t} + \widehat{\theta}_t \right) - N(1 - \lambda) \widehat{N}_{t-1} \right) - \alpha \Gamma_{10} p^E \left( \widehat{\Gamma}_{10,t} + \widehat{p}_t^E \right) \\
&\quad - \Gamma_{12} \alpha p^E \left[ (1 - (1 - \lambda) N) \left( \widehat{\Gamma}_{12,t} + \widehat{p}_t^E \right) - N(1 - \lambda) \widehat{N}_{t-1} \right], \\
\frac{\partial \mathcal{L}}{\partial Y_t} &= -\Gamma_5 \widehat{\Gamma}_{5,t} - \frac{\Gamma_6}{X} \left( \widehat{\Gamma}_{6,t} - \widehat{X}_t \right) + \Gamma_8 \widehat{\Gamma}_{8,t} + \Gamma_{11} \widehat{\Gamma}_{11,t}, \\
\frac{\partial \mathcal{L}}{\partial q_t} q_t &= - \left( \widehat{\Gamma}_{3,t} - \widehat{q}_t \right) + \left( \widehat{\Gamma}_{9,t} + \widehat{p}_t^E \right).
\end{aligned}$$

### Steady State Solution for Policy Maker's FOC

Firstly, we rewrite FOCs with steady state values (obtained from steady state constraints solution), that is we use  $\pi = 1$  and  $i = \frac{1}{\beta}$  :

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial i_t} i_t &= \Gamma_1 b - \Gamma_2 b / \beta + \beta \Gamma_5 \varpi X Z + \beta \Gamma_6 \varpi Z + \Gamma_7, \\
\frac{\partial \mathcal{L}}{\partial Z_t} &= -\Gamma_4 X + \Gamma_6 (1 - \varpi), \\
\frac{\partial \mathcal{L}}{\partial N_t} &= \Gamma_8 (\beta (1 - \lambda) k \theta - a) - \Gamma_{11} z + \Gamma_{12} (1 - \beta (1 - \lambda) + (1 - \lambda) \beta p^E), \\
\frac{\partial \mathcal{L}}{\partial S_t} &= (\lambda + \eta p^E) \Gamma_1 - \Gamma_2 \frac{\phi}{\beta} (1 - \eta) - \Gamma_3 (1 - \eta), \\
\frac{\partial \mathcal{L}}{\partial C_t} C_t &= C^{1-\sigma} + \Gamma_7 \sigma \left( 1 - \frac{1}{\beta} \right) - \Gamma_8 C,
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial X_t} X_t &= z\Gamma_1 + Y\Gamma_6, \\
\frac{\partial \mathcal{L}}{\partial b_t} &= \Gamma_2 \frac{1}{\beta} - \Gamma_3, \\
\frac{\partial \mathcal{L}}{\partial p_t^E} &= \Gamma_1 \eta \beta S + \Gamma_9 + \Gamma_{10}, \\
\frac{\partial \mathcal{L}}{\partial \pi_t} \pi_t &= -\Gamma_1 b / \beta + \Gamma_2 b / \beta^2 - \Gamma_5 \varpi (\varepsilon - 1) K - \Gamma_6 \varpi \varepsilon Z \\
&\quad - \Gamma_7 \frac{1}{\beta} + \Gamma_4 \left( \frac{\varpi}{1 - \varpi} \right) K, \\
\frac{\partial \mathcal{L}}{\partial \theta_t} \theta_t &= -\Gamma_8 k (1 - (1 - \lambda) N) \theta - \alpha \Gamma_{10} p^E - \Gamma_{12as} p^E \alpha (1 - (1 - \lambda) N), \\
\frac{\partial \mathcal{L}}{\partial K_t} &= \Gamma_4 + \Gamma_5 (1 - \varpi), \\
\frac{\partial \mathcal{L}}{\partial Y_t} &= -\Gamma_5 - \Gamma_6 \frac{1}{X} + \Gamma_8 + \Gamma_{11}, \\
\frac{\partial \mathcal{L}}{\partial q_t} q_t &= -\Gamma_3 \frac{k}{q} + \Gamma_9 \left( \frac{\alpha}{1 - \alpha} \right) p^E.
\end{aligned}$$

Then we use the following identities to simplify FOC

$$\begin{aligned}
\Gamma_2 \frac{1}{\beta} &= \Gamma_3, \\
-\frac{\Gamma_1}{N} &= \Gamma_6, \\
K &= XZ, \\
\Gamma_4 &= -\frac{\Gamma_1}{NX} (1 - \varpi).
\end{aligned}$$



Which results in the reduced system

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial i_t} &= \Gamma_1 b - \Gamma_3 b + \beta \Gamma_5 \varpi X Z - \beta \frac{\Gamma_1}{N} \varpi Z + \Gamma_7, \\
\frac{\partial \mathcal{L}}{\partial N_t} &= \Gamma_8 (\beta (1 - \lambda) k \theta - a) - \Gamma_{11} z + \Gamma_{12} (1 - \beta (1 - \lambda) + (1 - \lambda) \beta p^E), \\
\frac{\partial \mathcal{L}}{\partial S_t} &= (\lambda + \eta p^E) \Gamma_1 + \Gamma_3 (1 - \eta) (\phi + 1), \\
\frac{\partial \mathcal{L}}{\partial C_t} C_t &= C^{1-\sigma} + \Gamma_7 \sigma \left(1 - \frac{1}{\beta}\right) - \Gamma_8 C, \\
\frac{\partial \mathcal{L}}{\partial p_t^E} &= \Gamma_1 \eta \beta S + \Gamma_9 + \Gamma_{10}, \\
\frac{\partial \mathcal{L}}{\partial q_t} q_t &= -\Gamma_3 \frac{k}{q} + \Gamma_9 \left(\frac{\alpha}{1 - \alpha}\right) p^E, \\
\frac{\partial \mathcal{L}}{\partial Y_t} &= -\Gamma_5 + \Gamma_1 \frac{1}{NX} + \Gamma_8 + \Gamma_{11}, \\
\frac{\partial \mathcal{L}}{\partial K_t} &= -\frac{\Gamma_1}{NX} (1 - \varpi) + \Gamma_5 (1 - \varpi), \\
\frac{\partial \mathcal{L}}{\partial \pi_t} \pi_t &= -\Gamma_1 b / \beta + \Gamma_3 b / \beta - \Gamma_5 \varpi \varepsilon Z + \frac{\Gamma_1}{N} \varpi \varepsilon Z - \Gamma_7 \frac{1}{\beta} - \frac{\Gamma_1 \varpi}{N} Z, \\
\frac{\partial \mathcal{L}}{\partial \theta_t} \theta_t &= -\Gamma_8 k (1 - (1 - \lambda) N) \theta - \alpha \Gamma_{10} p^E - \Gamma_{12} p^E \alpha (1 - (1 - \lambda) N).
\end{aligned}$$

To simplify it further we use

$$\Gamma_5 = \frac{\Gamma_1}{NX},$$

and then observe that FOC  $\frac{\partial \mathcal{L}}{\partial \pi_t} \pi_t$  becomes tautology. From  $\frac{\partial \mathcal{L}}{\partial Y_t}$  then we can deduce

$$\Gamma_{11} = -\Gamma_8.$$

So our FOC becomes

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial i_t} i_t &= \Gamma_1 b - \Gamma_3 b + \Gamma_7, \\
\frac{\partial \mathcal{L}}{\partial N_t} &= \Gamma_8 (\beta (1 - \lambda) k \theta - a) + \Gamma_8 z + \Gamma_{12} (1 - \beta (1 - \lambda) + (1 - \lambda) \beta p^E), \\
\frac{\partial \mathcal{L}}{\partial S_t} &= (\lambda + \eta p^E) \Gamma_1 + \Gamma_3 (1 - \eta) (\phi + 1), \\
\frac{\partial \mathcal{L}}{\partial C_t} C_t &= C^{1-\sigma} + \Gamma_7 \sigma \left(1 - \frac{1}{\beta}\right) - \Gamma_8 C, \\
\frac{\partial \mathcal{L}}{\partial p_t^E} &= \Gamma_1 \eta \beta S + \Gamma_9 + \Gamma_{10}, \\
\frac{\partial \mathcal{L}}{\partial q_t} q_t &= -\Gamma_3 \frac{k}{q} + \Gamma_9 \left(\frac{\alpha}{1 - \alpha}\right) p^E, \\
\frac{\partial \mathcal{L}}{\partial \theta_t} \theta_t &= -\Gamma_8 k (1 - (1 - \lambda) N) \theta - \alpha \Gamma_{10} p^E - \Gamma_{12} p^E \alpha (1 - (1 - \lambda) N).
\end{aligned}$$

This system is solvable because it has the same number of equations as number of unknown Lagrange multipliers.

### 3.6.3 Derivation of Wage Equation

Firstly, consider equation (3.26) that defines value of the filled vacancy. By using borrowing constraint (3.29) to eliminate  $b_t$  and by employing vacancy posting condition  $Q_t = \frac{k}{q_t}$  we obtain the following expression

$$E_t [S_{t+1}] = \frac{k}{q_t (1 - \eta) \left( \phi_t \left( \frac{\pi_{t+1}}{i_t} \right)^2 + \beta \right)}. \quad (3.68)$$

Now we use expression for bargaining surplus (3.22) and substitute  $b_t$  from (3.29):

$$S_t = \frac{z}{X_t} - a - \frac{b_{t-1}}{\pi_t} i_{t-1} + \left( \frac{\pi_{t+1}}{i_t} \phi_t (1 - \eta) + ((1 - \lambda) - \eta p_t^E) \beta \right) E_t [S_{t+1}].$$

Substitution of  $E_t [S_{t+1}]$  from (3.68) yields

$$S_t = \frac{z}{X_t} - a - \frac{b_{t-1}}{\pi_t} i_{t-1} + \frac{k \left( \frac{\pi_{t+1}}{i_t} \phi_t (1 - \eta) + ((1 - \lambda) - \eta p_t^E) \beta \right)}{q_t (1 - \eta) \left( \phi_t \left( \frac{\pi_{t+1}}{i_t} \right)^2 + \beta \right)}. \quad (3.69)$$

Next, consider the net value of a worker given by  $V_t - U_t$  (equations (3.23) and (3.24)):

$$V_t - U_t = w_t - a + \beta (1 - \lambda - p_t^E) (V_{t+1} - U_{t+1}), \quad (3.70)$$

since we know that  $V_t - U_t = \eta S_t$ , we can use it to simplify the expression above

$$\eta S_t = w_t - a + \beta (1 - \lambda - p_t^E) \eta E_t [S_{t+1}].$$

Now, use equation (3.68) to eliminate  $E_t [S_{t+1}]$

$$\eta S_t = w_t - a + \frac{k\beta (1 - \lambda - p_t^E) \eta}{q_t(1 - \eta) \left( \phi_t \left( \frac{\pi_{t+1}}{i_t} \right)^2 + \beta \right)}. \quad (3.71)$$

Lastly, combining expressions 3.69 and 3.71 and solving for wage we obtain

$$w_t = (1 - \eta) a + \eta \left( \frac{z}{X_t} - \frac{b_{t-1}}{\pi_t} i_{t-1} \right) + \frac{\eta k \left[ \frac{\pi_{t+1}}{i_t} \phi_t + p_t^E \beta \right]}{q_t \left( \phi_t \left( \frac{\pi_{t+1}}{i_t} \right)^2 + \beta \right)}, \quad (3.72)$$

which is the same as provided in the text.

## Second Order Welfare Approximation

To compare welfare under different monetary policy regimes we compute second order approximation using Benigno and Woodfor (2005). It could be done by applying the following formula to policy maker's optimization problem:

$$S(X_t, Y_t) = \frac{\partial S^2}{\partial^2 X} X^2 \widehat{X}_t^2 + \frac{\partial S^2}{\partial^2 Y} Y^2 \widehat{Y}_t^2 + 2 \frac{\partial S^2}{\partial Y \partial X} Y X \widehat{Y}_t \widehat{X}_t, \quad (3.73)$$

where  $S_t$  is the constraint in a policy maker's Lagrange equation. Then welfare could be represented as a sum of approximated constraints and Lagrange multipliers

$$W = \sum_{t=0}^{\infty} \beta^t \Gamma_t S_t. \quad (3.74)$$

Applying Benigno and Woodford algorithm we obtain the following system

$$\begin{aligned}
S_0^W &= -\sigma C^{-1-\sigma} \widehat{C}_t^2, \\
S_1^W &= -\frac{2z}{X} \widehat{X}_t^2 + \frac{2}{X} \widehat{X}_t z \widehat{z}_t + 2bi\widehat{\pi}_t^2 + 2\eta\beta Sp^E \widehat{S}_{t+1} \widehat{p}_t^E \\
&\quad + 2bi\widehat{b}_{t-1} \widehat{i}_{t-1} - 2bi\widehat{b}_{t-1} \widehat{\pi}_t - 2bi\widehat{i}_{t-1} \widehat{\pi}_t, \\
S_2^W &= 2bi\widehat{\pi}_{t+1}^2 + 2bi\widehat{b}_t \widehat{i}_t - 2ib\widehat{b}_t \widehat{\pi}_{t+1} - 2bi\widehat{i}_t \widehat{\pi}_{t+1} - 2\phi S(1-\eta) \widehat{\phi}_t \widehat{S}_{t+1}, \\
S_3^W &= \frac{2k}{q} \widehat{q}_t^2, \\
S_4^W &= \frac{\varpi}{1-\varpi} K \left( \varepsilon \left( \frac{\varpi}{1-\varpi} \right) + (\varepsilon - 2) \right) \widehat{\pi}_t^2 + 2\frac{\varpi}{1-\varpi} K \widehat{K}_t \widehat{\pi}_t, \\
S_5^W &= -\varpi \frac{(\varepsilon-1)(\varepsilon-2)}{i} K \widehat{\pi}_{t+1}^2 - 2\varpi \frac{1}{i} K \widehat{i}_t^2 + 2\varpi \frac{(\varepsilon-1)}{i} K \widehat{i}_t \widehat{\pi}_{t+1} \\
&\quad + \varpi \frac{2}{i} \widehat{i}_t \widehat{K}_{t+1} K - 2\varpi \left( \frac{\varepsilon-1}{i} \right) K \widehat{K}_{t+1} \widehat{\pi}_{t+1}, \\
S_6^W &= -\varpi \frac{\varepsilon(\varepsilon-1)}{i} Z \widehat{\pi}_{t+1}^2 - 2\varpi \frac{1}{i} Z \widehat{i}_t^2 + 2\frac{Y}{X} \widehat{Y}_t \widehat{X}_t - \frac{2Y}{X} \widehat{X}_t^2, \\
&\quad + 2\varpi \frac{\varepsilon \widehat{i}_t}{i} \widehat{\pi}_{t+1} Z - 2\varpi \frac{\varepsilon}{i} \widehat{Z}_{t+1} Z \widehat{\pi}_{t+1} + 2\varpi \frac{1}{i} \widehat{Z}_{t+1} \widehat{Z}_t, \\
S_7^W &= \sigma(-\sigma-1) \widehat{C}_t^2 - \sigma(\sigma-1) \widehat{C}_{t+1}^2 + 2\sigma^2 \widehat{C}_t \widehat{C}_{t+1} + 2\sigma \widehat{C}_t \widehat{\pi}_{t+1} - 2\sigma \widehat{C}_{t+1} \widehat{\pi}_{t+1}, \\
S_8^W &= 2k(1-\lambda) N \theta \widehat{\theta}_t \widehat{N}_{t-1}, \\
S_9^W &= -\frac{\alpha}{1-\alpha} \left( \frac{1}{1-\alpha} \right) q^{\frac{\alpha-2}{1-\alpha}+2} \varsigma^{\frac{1}{1-\alpha}} \widehat{q}_t^2, \\
S_{10}^W &= -\varsigma \alpha (\alpha-1) (\theta)^\alpha \widehat{\theta}_t^2, \\
S_{11}^W &= -2Nz \widehat{N}_t \widehat{z}_t, \\
S_{12}^W &= -\varsigma \alpha (\alpha-1) \theta^\alpha (1-(1-\lambda)N) \widehat{\theta}_t^2 + 2\varsigma \alpha \theta^\alpha N(1-\lambda) \widehat{\theta}_t \widehat{N}_{t-1}.
\end{aligned}$$

which we use to compare different policy rules.

### 3.6.4 Derivation of Final Good Producers Price Setting Relation

Here we show derivation of Phillips curve. The final good producers' profit maximization problem could be written as follows

$$\Pi_t(i) = \left( \frac{p_t^*(i)}{P_t} - \frac{P_t^w}{P_t} \right) \left( \frac{p_t^*(i)}{P_t} \right)^{-\varepsilon} Y_t^f. \quad (3.75)$$

For the sake of simplicity we can substitute  $\frac{p_t^*(i)}{P_{t+\tau}} = \frac{p_t^*(i)}{P_t} \frac{P_t}{P_{t+\tau}}$ . Now we rewrite the optimization problem as

$$\max_{p_t^*(i)} E_t \sum_{\tau=0}^{\infty} \varpi^\tau \beta^\tau \frac{\Lambda_{t+\tau}}{\Lambda_t} \left[ \frac{p_t^*(i)}{P_t} \frac{P_t}{P_{t+\tau}} - \frac{P_{t+\tau}^w}{P_{t+\tau}} \right] \left( \frac{p_t^*(i)}{P_t} \frac{P_t}{P_{t+\tau}} \right)^{-\varepsilon} Y_{t+\tau}^f. \quad (3.76)$$

Differentiating it with respect to  $\frac{p_t^*(i)}{P_{t+\tau}}$  gives us the following expression

$$\sum_{\tau=0}^{\infty} \varpi^\tau \beta^\tau \frac{\Lambda_{t+\tau}}{\Lambda_t} \left( \begin{array}{c} (1 - \varepsilon) \left( \frac{p_t^*(i)}{P_t} \right)^{-\varepsilon} \left( \frac{P_t}{P_{t+\tau}} \right)^{1-\varepsilon} Y_{t+\tau}^f \\ + \varepsilon \frac{P_{t+\tau}^w}{P_{t+\tau}} \left( \frac{p_t^*(i)}{P_t} \right)^{-\varepsilon-1} \left( \frac{P_t}{P_{t+\tau}} \right)^{-\varepsilon} Y_{t+\tau}^f \end{array} \right). \quad (3.77)$$

Then we multiply it by  $\left( \frac{p_t^*(i)}{P_t} \right)^{\varepsilon+1}$  to obtain

$$\sum_{\tau=0}^{\infty} \varpi^\tau \beta^\tau \frac{\Lambda_{t+\tau}}{\Lambda_t} \left( (1 - \varepsilon) \left( \frac{p_t^*(i)}{P_t} \right) \left( \frac{P_t}{P_{t+\tau}} \right)^{1-\varepsilon} Y_{t+\tau}^f + \varepsilon \frac{P_{t+\tau}^w}{P_{t+\tau}} \left( \frac{P_t}{P_{t+\tau}} \right)^{-\varepsilon} Y_{t+\tau}^f \right). \quad (3.78)$$

Now can define total revenue ( $K_t$ ) and total cost ( $Z_t$ ) as follows

$$K_t \equiv \sum_{\tau=0}^{\infty} \varpi^\tau \beta^\tau \frac{\Lambda_{t+\tau}}{\Lambda_t} \left( \frac{P_t}{P_{t+\tau}} \right)^{1-\varepsilon} Y_{t+\tau}^f, \quad (3.79)$$

$$Z_t \equiv \sum_{\tau=0}^{\infty} \varpi^\tau \beta^\tau \frac{\Lambda_{t+\tau}}{\Lambda_t} \frac{P_{t+\tau}^w}{P_{t+\tau}} \left( \frac{P_t}{P_{t+\tau}} \right)^{-\varepsilon} Y_{t+\tau}^f. \quad (3.80)$$

Now we substitute these definitions into our derivative and arrive at the expression

$$(1 - \varepsilon) \left( \frac{p_t^*(i)}{P_t} \right) K_t + \varepsilon Z_t = 0. \quad (3.81)$$

Because of the assumption that only proportion of firms could change the prices every period we know that price index evolves according to the following law of motion

$$P_t = \left( \varpi P_{t-1}^{1-\varepsilon} + (1 - \varpi) p_t^*(i)^{1-\varepsilon} \right)^{\frac{1}{1-\varepsilon}}. \quad (3.82)$$

Rearranging it gives us

$$\frac{p_t^*(i)}{P_t} = \left[ \frac{1 - \varpi \pi_t^{\varepsilon-1}}{1 - \varpi} \right]^{\frac{1}{1-\varepsilon}}. \quad (3.83)$$

Then combining it with equation (2.6) and keeping in mind that  $\varepsilon > 1$

$$\left[ \frac{1 - \varpi \pi_t^{\varepsilon-1}}{1 - \varpi} \right]^{\frac{1}{1-\varepsilon}} K_t = \frac{\varepsilon}{1 - \varepsilon} Z_t. \quad (3.84)$$

We need equation above to derive the new Keynesian curve which we are going to do in the next section.

### Phillips Curve

Now let's look again at the equations of total revenue and cost of the final good producer. Notice that we define ratio of average price to wholesale price  $\frac{P_t}{P_t^w}$  as a markup and called it  $X_t$ . Recall, that we can rewrite equations (3.79) and (3.80) as follows

$$K_t = Y_t + \varpi \beta E_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} \pi_{t+1}^{\varepsilon-1} K_{t+1} \right], \quad (3.85)$$

and

$$Z_t = \frac{Y_t}{X_t} + \varpi \beta E_t \left[ \frac{\Lambda_{t+1}}{\Lambda_t} \pi_{t+1}^{\varepsilon} Z_{t+1} \right]. \quad (3.86)$$

Then we linearize two equations above and (3.84) around the steady state to arrive at the following expressions:

$$\widehat{K}_t = \widehat{Z}_t - \frac{\varpi}{1 - \varpi} \widehat{\pi}_t, \quad (3.87)$$

$$\widehat{K}_t = \widehat{Y}_t^f (1 - \varpi \beta) + \varpi \beta \widehat{K}_{t+1} + \varpi \beta (\varepsilon - 1) \widehat{\pi}_{t+1} + \varpi \beta \widehat{\Lambda}_{t+1} - \varpi \beta \widehat{\Lambda}_t, \quad (3.88)$$

$$\widehat{Z}_t = (1 - \varpi \beta) \widehat{Y}_t^f - (1 - \varpi \beta) \widehat{X}_t \quad (3.89)$$

$$+ \varpi \beta \varepsilon \widehat{\pi}_{t+1} + \varpi \beta \widehat{\Lambda}_{t+1} - \varpi \beta \widehat{\Lambda}_t + \varpi \beta \widehat{Z}_{t+1}, \quad (3.90)$$

where variables with hats denote deviations from the steady state. Lastly, we substitute them to derive final goods Phillips curve

$$\beta \widehat{\pi}_{t+1} = \iota \widehat{X}_t + \widehat{\pi}_t, \quad (3.91)$$

where  $\iota = \frac{(1-\varpi)(1-\varpi\beta)}{\varpi}$ . And this is exactly the same expression as it is in the text.

Table 1. List of Parameter Values

<b>Parameter</b>	<b>Description</b>	<b>Value</b>
$\beta$	Households' discount factor	0.99
$\eta$	Workers' bargaining power	0.5
$\varepsilon$	Price elasticity parameter	6
$\varsigma$	Matching parameter	0.76
$\lambda$	Probability of match separation	0.05
$k$	Cost of posting vacancy	0.598
$a$	Workers unemployment benefit	0.5
$z$	Productivity	1.0
$\phi$	Probability that firm will repay its debt	0.86
$\varpi$	Probability that price will not change	0.25
$\sigma$	Relative risk aversion	2.0
$\alpha$	Matching parameter (elasticity)	0.5
$\alpha_\phi$	Financial market shock persistence	0.75
$\alpha_z$	Productivity shock persistence	0.75

## BIBLIOGRAPHY

- [1] Acemoglu, D. 2001. Credit Market Imperfections and Persistent Unemployment, *European Economic Review*, vol. 45, pp 665-679.
- [2] Apergis, N., 2008. The Employment-Wage Relationship: Was Keynes Right after all?, *American Review of Political Economy*, vol. 6, pp. 40-50.
- [3] Basu, S., Fernald, J., 1997. Returns to Scale in U.S. Production: Estimates and Implications, *Journal of Political Economy*, vol. 105, pp. 249-283.
- [4] Benigno, P., Woodford, M. 2005. Inflation Stabilization And Welfare: The Case Of A Distorted Steady State, *Journal of the European Economic Association*, vol. 3, pp. 1185-1236.
- [5] Blanchard, O., Gali, J. 2010. Labour Markets and Monetary Policy: A New Keynesian Model with Unemployment, *American Economic Journal: Macroeconomics*, vol 2 pp. 1-30.
- [6] Blanchard, O., J., Kahn, C., M. 1980. The Solution of Linear Difference Models Under Rational Expectations, *Econometrica*, vol. 48, pp. 1305-1311.
- [7] Calvo, G., A. 1983. Staggered Prices in an Utility-Maximizing Framework, *Journal of Monetary Economics*, vol. 12, pp. 383-398.
- [8] Damjanovic, T., Damjanovic, V. Nolan, C. 2011. Ordering Policy Rules with an Unconditional Welfare Measure, *CDMA Working Paper Series No. 1205*.
- [9] Dixit, A., Stiglitz, J. 1977. Monopolistic Competition and Optimum Product Diversity, *American Economic Review*, vol. 67, pp. 297-308.
- [10] Faia, E. 2008. Optimal Monetary Policy Rules with labour Market Frictions, *Journal of Economic Dynamics and Control*, vol. 32, pp. 1600-1621.
- [11] Faia, E. 2009. Ramsey Monetary Policy with Labour Market Frictions, *Journal of Monetary Economics*, vol. 56, pp. 570 - 581.
- [12] Faia, E. 2008a. Comments on: Search and Matching Frictions and Optimal Monetary Policy, *Journal of Monetary Economics*, vol. 55, pp. 957-960.
- [13] Gatti, D., Rault, C., Vaubourg, A. 2012. Unemployment and Finance: How Do Financial and Labour Market Factors Interact?, *Oxford Economic Papers*, vol. 64, pp. 464-489.



- [14] Hosios, A. 1990. On the Efficiency of Matching and Related Models of Search and Unemployment, *Review of Economic Studies*, vol. 57, pp. 279-298.
- [15] Kim, J., Kim, K.,S. 2003. Spurious Welfare Reversals in International Business Cycle Models, *Journal of International Economics*, vol. 60 pp. 471-500.
- [16] Monacelli, T., Quadrini, R., Trigari, A. 2011. Financial Markets and Unemployment, *NBER Working Paper* 17389, National Bureau of Economic Research.
- [17] Mumtaz, H., Zanetti, F., 2014, Labor and Financial Frictions and Aggregate Fluctuations, *Macroeconomic Dynamics*, *forthcoming*.
- [18] Petrosky-Nadeau, N., 2014, Credit, Vacancies and Unemployment Fluctuations, *Review of Economic Dynamics*, vol. 17, pp. 191-205.
- [19] Petrosky-Nadeau, N., Wasmer, E., 2013, The Cyclical Volatility of Labor Markets under Frictional Financial Markets, *American Economic Journal: Macroeconomics*, vol. 5, pp. 193-221.
- [20] Pissarides, C.,A. 1987. Search, Wage Bargains and Cycles, *Review of Economic Studies*, vol. 54, pp. 473-483.
- [21] Proano, R., C. 2012. Gradual Wage-Price Adjustments, Labour Market Frictions and Monetary Policy Rules, *Journal of Economic Behavior & Organization*, vol 82, pp. 220-235.
- [22] Ravenna, F., Walsh, C., E. 2011. Welfare-Based Optimal Monetary Policy with Unemployment and Sticky Prices: A Linear Quadratic Framework, *American Economic Journal: Macroeconomics*, vol. 3, pp. 130-162.
- [23] Taylor, J., B. 1993. Discretion versus Policy Rules in Practice, *Carnegie-Rochester Conference Series on Public Policy*, vol. 39, pp. 195-214.
- [24] Thomas, C. 2008. Search and Matching Frictions and Optimal Monetary Policy, *Journal of Monetary Economics*, vol. 55, pp. 936 - 956.
- [25] Trigari, A. 2009. Equilibrium Unemployment, Job Flow and Inflation Dynamics, *Journal of Money, Credit and Banking*, vol. 41, pp. 1 - 33.
- [26] Walsh, C. 2005. Labour Market Search, Sticky Prices and Interest Rate Rules, *Review of Economic Dynamics*, vol. 8, pp. 829-849.

CHAPTER 4  
STATIONARITY OF ECONOMETRIC LEARNING WITH  
BOUNDED MEMORY

## 4.1 Introduction<sup>1</sup>

Econometric learning in macroeconomics is one of the possible extensions of the rational expectations hypothesis which allows for *irrational* behavior. However, certain conditions are imposed. For instance, it is assumed that agents (who are learning) have access to all the information about some variables evolution over time. That is, they have an ever-expanding memory. One possible way to tackle this issue is to put exponentially decreasing weights (the so-called *gain*) on each of the observations as in the Recursive Least Squares (RLS) algorithm or constant weight as in the Constant Gain (CG) learning rule as shown in Honkapohja and Mitra (2003), Evans and Honkapohja (2003) or Barucci (2000 and 2001). However, in each of these cases it is still assumed that agents have access to every past observation. In this paper we relax this assumption and consider the bounded memory OLS learning algorithm with weights of 1 and 0 put on each observation which is also a special case of the Constant Gain algorithm.

The idea here is simple: agents form their forecasts for the next period by using only some number  $T$  of the past observations. In that sense agents put weight of 1 on each observation within the length of  $T$  and 0 on all the other past observations. Then it follows, that each observation within the range  $T - 1$  is equally important for the next periods forecast and each observation that is outside the range is not accessible (has 0 weight). This is the very similar to the ordinary CG algorithm.

To the best of our knowledge, the role of memory has been only moderately

---

<sup>1</sup>Matlab codes used in this chapter could be seen at [https://github.com/Sarunas-Girdenas/Bounded\\_Memory](https://github.com/Sarunas-Girdenas/Bounded_Memory).

analyzed in the framework of OLS learning. Holmes and Manning (1988) find that in the non-linear cobweb model with OLS type learning, memory increases stability and prevents chaotic behavior. However, memory in their case is defined as all the past information that agents can access. Further to this, Canning (1992) showed that if agents employ the OLS type of learning and the past 'does not matter too much' in the sense that each observation is discounted by  $\frac{1}{t}$ , then the model converges to a unique equilibrium that does not depend on the initial conditions. A very similar result was provided in Bray and Savin (1986). Lastly, LeBaron (2012) presents a heterogenous agent model where each agent is using CG learning to obtain the forecast but with a different realization of the gain parameter  $\gamma$  and argues that it helps to explain a lot of features of the real stock market.

In this paper we aim to study the length of memory in a slightly different way. That is, we use classical cobweb model setting and allow agents to access only a limited amount of observations  $T$  instead of all of them with discounting as in the CG or RLS types of learning. In purely econometric terms the OLS algorithm with bounded memory could be seen as a rolling window estimate. This setup allows us to investigate the impact of the memory length on the stability and stationarity of agents' forecasts.

As shown in Berardi and Galimberti (2013), the OLS type learning algorithm with the rolling window (bounded memory) could be seen as a special case of the RLS and CG algorithms with weights of 1 or 0 placed on each observation. The theoretical discussion of the RLS type of learning with decaying weights is presented in Evans and Honkapohja (2003), Berardi and Galimberti (2013). For more detailed survey of econometric learning and cycles see Chevillon and Mavroeidis (2014). Also there have been some empirical attempts to obtain the numerical value of the gain parameter using the data, for instance Berardi and Galimberti

(2014) or Carceles-Poveda and Giannitsarou (2008). In this paper we show that the so-called weight can be derived by assuming that agents are not able to process all the information from  $t$  to  $t - 1$  time periods.

The key question which we are aiming to answer in this paper is what are the properties of the transition matrix of the OLS-type learning with bounded memory in the cobweb model setting. If the properties were known, then we could investigate stationarity and convergence of the model. So far we have proved that for any length of agents memory  $T$  the eigenvalues of the transition matrix lie in the unit circle. We have used this finding to construct analytical proof of stationarity of price under bounded memory learning. Furthermore, by employing numerical computations we find that OLS parameter estimates  $\alpha_2$ ,  $\beta_2$ , price  $p_t$  and expected price  $p_t^e$  have the same mean as rational expectations equilibrium value. However, at this stage we are unable to provide analytical proof to confirm this finding.

As in Carceles-Poveda and Giannitsarou (2008) we find that bounded memory increases volatility. For illustrative purposes we use the classical cobweb model (Kaldor 1934, Ezekiel 1938) framework. However, our results could be easily extended to any macroeconomic model where agents use the OLS learning algorithm.

This paper is structured as follows. Firstly, we briefly introduce the cobweb model with OLS learning and expanding memory as in Evans and Honkapohja (2003). Secondly, we formalize the problem of the bounded memory learning. Thirdly, we investigate properties of the transition matrix of the OLS algorithm with bounded memory. Lastly, using numerical methods we show that the OLS estimates are stationary and provide some results. The final section concludes the paper.

## 4.2 Some Properties of Bounded Memory Learning

In this section we firstly introduce the classical cobweb model with the OLS type of learning. Then we relax the assumption that agents are able to use all the observations to form their forecasts.

### 4.2.1 Learning with Expanding Memory

The cobweb model, as presented by Kaldor (1934) and Ezekiel (1938) defines a market for one type of good with demand and supply. Ezekiel (1938) stated that the market price will converge to its equilibrium (REE) if the ratio of the demand and supply slopes is less than 1. Otherwise it will diverge. In this case we will analyze only the stable case of the cobweb model which consists of the two key equations.

Demand:

$$d_t = m_I - m_p p_t + v_{1,t}. \quad (4.1)$$

Supply:

$$s_t = r_I + r_p p_t^e + r_w w_{t-1} + v_{2,t}. \quad (4.2)$$

where  $d_t$  is quantity demanded,  $s_t$  is quantity supplied,  $m_I$ ,  $r_I$  are intercept terms, parameters  $m_p$  and  $r_p$  are greater than 0,  $v_{1,t}$  and  $v_{2,t}$  capture white noise unobserved shocks. Here  $p_t^e$  denotes expectations of price formed at time  $t$ . Notice, that here we use the same notation as in Evans and Honkapohja (2003). Furthermore, we follow Bray and Savin (1986) and assume that  $w_t \sim i.i.d. (0, \sigma_w^2)$  and is observed by the agents. We require this assumption in the latter part of the paper where we use matrix permutation to show the stationarity of forecasts. By employing the market clearing condition and solving for the aggregate price level  $p_t$  we obtain the

actual law of motion (ALM) of the economy:

$$p_t = \alpha + \beta p_t^e + \delta w_{t-1} + v_t, \quad (4.3)$$

where  $\alpha = \frac{m_I - r_I}{m_p}$ ,  $\delta = -\frac{1}{m_p} r_w$ ,  $\alpha = -\frac{r_p}{m_p}$ ,  $v_t = \frac{v_{1,t} - v_{2,t}}{m_p}$  and we know that  $v_t \sim i.i.d. (0, \sigma_\eta^2)$ . For now we assume that they are formed using rational expectations, in the next part of the paper we will relax this assumption.

Rational Expectations Equilibrium (REE)  $p_t^e = E_{t-1} p_t$  is given by

$$E_{t-1} p_t = \frac{\alpha}{1 - \beta} + \frac{\delta}{1 - \beta} w_{t-1}, \quad (4.4)$$

and is unique because it does not depend on the future price level.

Now we are going to consider the classical OLS-type learning. Assume that agents (in this case - firms which are trying to forecast the future price level) are not aware of the ALM (equation 4.3). They form expectations based on the following Perceived Law of Motion (PLM):

$$p_t = \alpha_{2,t-1} + \beta_{2,t-1} w_{t-1} + \eta_t, \quad (4.5)$$

but they have no information about the value of coefficients  $\alpha_{2,t-1}$  and  $\beta_{2,t-1}$  and therefore they are using an OLS regression to estimate these values. In the classical case it is assumed that firms have the data from periods  $t = 0, 1, \dots, t-1$  which is summarized in the information set  $\{p_t, w_t\}_{t=0}^{t-1}$ . In other words, firms use all the information available up to the current period. Then using the fact that  $p_t - E_{t-1} p_t = \eta_t$  and (4.5) we obtain the price forecast at time period  $t$ :

$$p_t^e = \alpha_{2,t-1} + \beta_{2,t-1} w_{t-1}. \quad (4.6)$$

We can clearly see, that the current period forecast of price level  $p_t^e$  depends on the parameter estimates  $\alpha_{2,t-1}$  and  $\beta_{2,t-1}$  which are obtained using all the information up to  $t-1$ . Corresponding learning rule (OLS equations) reads as follows:

$$\widehat{\beta}_{2,t-1} = \frac{\sum_{i=1}^{t-1} [(w_{t-i} - \bar{w}_{t-1})(p_{t-i} - \bar{p}_{t-1})]}{\sum_{i=1}^{t-1} [(w_{t-i} - \bar{w}_{t-1})^2]}, \quad (4.7)$$

$$\widehat{\alpha}_{2,t-1} = \bar{p}_{t-1} - \widehat{\beta}_{2,t-1} \bar{w}_{t-1}. \quad (4.8)$$

Bray and Savin (1986) show that under these assumptions the model is stable and converges to REE. Furthermore, Evans and Honkapohja (2003) use a Recursive Stochastic Algorithm (RSA) to show that the estimates almost surely converge to REE, that is

$$\begin{bmatrix} \widehat{\alpha}_2 \\ \widehat{\beta}_2 \end{bmatrix} \rightarrow \begin{bmatrix} \frac{\alpha}{1-\beta} \\ \frac{\delta}{1-\beta} w_{t-1} \end{bmatrix}. \quad (4.9)$$

In the following section we will investigate stationarity of the price forecast when agents have bounded memory, that is they are able to use only some amount of information instead of all of it up to period  $t - 1$ . We will show that forecasts are stationary and parameters converge in distribution to REE values even if we relax the assumption of the number of periods agents use for forecast formation.

## 4.2.2 Learning with Bounded Memory

One of the first attempts to investigate bounded memory in terms of econometric learning are works by Honkapohja and Mitra (2003 and 2006) and Mitra (2005). However, in all of these cases the CG algorithm was used instead of OLS. We extend this literature by investigating bounded memory learning in the OLS framework because it is not possible to represent bounded memory learning in a recursive manner (RLS) as noted in Honkapohja and Mitra (2003).

To begin with, we rewrite the cobweb model equations for the bounded memory case. Evolution of the economy is given by:

$$p_t = \alpha + \beta p_t^e + \delta w_{t-1} + v_t. \quad (4.10)$$

The firms' learning rule is defined as:

$$p_t^e = \alpha_{2,t-1} + \beta_{2,t-1}w_{t-1}. \quad (4.11)$$

Two OLS estimates:

$$\widehat{\beta}_{2,t-1} = \frac{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})(p_{t-i} - \bar{p}_{t-1})]}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]}, \quad (4.12)$$

$$\widehat{\alpha}_{2,t-1} = \bar{p}_{t-1} - \widehat{\beta}_{2,t-1}\bar{w}_{t-1}, \quad (4.13)$$

Notice, that here agents can use only a limited number of past observations  $T$  to form their expectations by using OLS. This is the key difference from the conventional least squares learning and it will drive all our results. Variables with bar denote the sample mean for  $T$  periods as it demonstrated in (4.16).

Our aim is to show that  $p_t$  is stationary for all  $T > 0$ .

The core idea of the proof is simple. Firstly, we have to represent the cobweb model with bounded memory in terms of a Random Coefficient Model (RCM), secondly, we use some conditions specified in Conlisk (1974) to prove the stationarity of such a model and hence price forecasts (in terms of the cobweb model).

**Proposition 4.2.1** *Consider an economy where expectations are formed as adaptive learning with bounded memory as in (4.10), (4.11), (4.12), (4.13). Then the actual price follows an autoregressive process of order  $T$  with random coefficients as in RCM:*

$$p_t = \alpha + \beta \left( \sum_{i=1}^T S_{i,t} p_{t-i} \right) + \delta w_{t-1} + v_t, \quad (4.14)$$

where

$$S_i = \frac{1}{T} + \frac{(w_{t-1} - \bar{w}_{t-1})(w_{t-i} - \bar{w}_{t-1})}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]}, \quad (4.15)$$



and

$$\bar{w}_{t-1} = \frac{1}{T} \sum_{i=1}^T w_{t-i}. \quad (4.16)$$

### 4.2.3 Stationarity of Price with Bounded Memory Learning

To investigate stationarity of the process  $y_t$  (which captures realized prices  $p_t$ ) we will use some conditions presented by Conlisk (1974), but the method shall be modified since some of them are not satisfied in our case. In particular, Conlisk (1974) does not allow for autocorrelation of matrix  $M_t$  which is defined below, but in our case autocorrelation exists because of the assumptions of the cobweb model. It is clear that the cobweb model can be represented as RCM of order 1:

$$y_t = \varepsilon_t + M_t y_{t-1}, \quad (4.17)$$

$$\text{where } y_t = \begin{pmatrix} p_t \\ p_{t-1} \\ \dots \\ p_{t-T+1} \end{pmatrix}, M_t = \begin{pmatrix} \beta S_{1,t} & \beta S_{2,t} & \dots & \beta S_{T,t} \\ 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \text{ and } \varepsilon_t = \begin{pmatrix} a + \delta w_{t-1} + v_t \\ 0 \\ \dots \\ 0 \\ 0 \end{pmatrix}.$$

We begin our investigation of stationarity of the model presented in (4.17) by setting up an additional property of coefficients  $S_{i,t}$ .

**Lemma 4.2.2** *For any realization of  $w_t$  and  $v_t$ ,  $\sum_{i=1}^T S_{i,t}^2 < 1$ .*

**Proof.** See Appendix. ■

We need lemma (4.2.2) for the following proposition where we show that the product of matrices  $M_t$  is bounded from above.

**Proposition 4.2.3** *For any memory length  $T$ , there exists a natural number  $n_T$  and a boundary  $B_T$  such that for any  $n > n_T$ , every elements of the product of  $n$  matrices is bounded by  $B_T\beta^n$  :*

$$\left| \prod_{i=1,n} (M_{t-i}) \right|_{ij} < B_T\beta^n \quad (4.18)$$

**Proof.** See Appendix. ■

Having established these results, we could investigate the stationarity of  $y_t$  by looking at the unconditional expectations  $E[y_t y_t']$ .

**Conjecture 4.2.4** *Process (4.17) is stationary.*

**Proof.** At the moment we are still working on the full proof, but we have a very strong feeling based on the properties we have already investigated that the process  $y_t$  is stationary. We are in the process of finalizing the proof that  $E[y_t y_t'] < \infty$  and  $E[y_t y_{t-j}'] < \infty$  which would lead to a conclusion that the process is indeed stationary. ■

## 4.2.4 Additional Properties of Bounded Memory Learning

In this section we present some additional properties of coefficients  $S_{i,t}$  and matrix  $M_t$ . All of the following proposition hold under the assumption that the exogenous variable  $w_t$  and shock  $v_t$  are *i.i.d.* processes.

The autoregressive coefficients  $S_{i,t}$  have the properties as described in Proposition 4.2.5:

**Proposition 4.2.5** *Random coefficients  $S_{i,t}$  defined in (4.15) have the following properties:*

- i)  $\sum_{i=1}^T S_{i,t} = 1$ ;
- ii)  $E[S_i] > 0$ ;
- iii)  $E[S_i S_j] \geq 0$ .

**Proof.** See appendix ■

It is interesting to note, the first observation ( $S_1$ ) is the most important one (has the highest value) whereas all the others have almost the equal size. To formalize this statement, we firstly have to compute unconditional expectations of  $S_1$  and  $S_2$  and then extend the result to a more general case of  $S_i$  for  $i > 1$ .

**Lemma 4.2.6**  $E[S_1 - S_2] = \frac{1}{T-1}$ .

**Proof.** See appendix. ■

Now we show that for  $i > 1$  all  $S_i$  have the same unconditional expectations.

**Lemma 4.2.7** *Unconditional expectations of  $S_1$  take the form of  $E[S_1] = \frac{2}{T}$  and unconditional expectations of  $S_i, \forall i > 1$  are  $E[S_i] = \frac{T-2}{T(T-1)}$ .*

**Proof.** Firstly we notice from equation (4.15) that due to symmetry all  $S_i$  are identically distributed for  $i > 1$ . Therefore we have

$$E[S_1] + (T - 1) E[S_2] = 1, \quad (4.19)$$

solving for  $E[S_2]$  yields

$$E[S_2] = \frac{1 - E[S_1 - S_2]}{T}, \quad (4.20)$$

which we can rewrite using the result of lemma (4.2.6) as

$$E[S_i] = E[S_2] = \frac{T - 2}{T(T - 1)}. \quad (4.21)$$

■

The results from lemmas 4.2.6 and 4.2.7 provide intuition on why bounded memory learning can be seen as a special case of the CG learning. Firstly, we can clearly see, that after the first observation each of the  $S_{i,t}$  for any length of  $T$  has the same unconditional expectation (which can be interpreted as a weight) which falls in the interval between 0 and 1. And this is the same interval where gain

parameter  $\gamma$  is defined in the case of CG learning. Therefore, parameters  $S_{i,t}$  in the loose sense could also be interpreted as the dynamic weights which depend on the length of memory  $T$ . In the latter parts of the paper we will illustrate this point more explicitly.

**Proposition 4.2.8** *Eigenvalues of the matrix  $E[M_t]$  are smaller than 1 in modulus.*

**Proof.** See appendix, section 4.5.6. ■

**Proposition 4.2.9**  *$E[M_{0,t} \otimes M_{0,t}]$  is a stochastic matrix.*

**Proof.** It follows immediately from Proposition 4.2.5. Firstly, as  $\sum_{i=1,T} S_{i,t} = 1$ , we have  $M_{0,t} \times e = e$ , where  $e$  is a column vector of 1s. Secondly, we use the mixed-product property rule  $(M_{0,t} \times e) \otimes (M_{0,t} \times e) = (M_{0,t} \otimes M_{0,t})(e \otimes e)$ . Therefore  $(M_{0,t} \otimes M_{0,t})(e \otimes e) = (e \otimes e)$ , and thus  $E[M_{0,t} \otimes M_{0,t}](e \otimes e) = (e \otimes e)$ . Finally, all elements of the matrix  $E[M_{0,t} \otimes M_{0,t}]$  are from the set  $\{ES_{i,t}; ES_{i,t}S_{j,t}, 1, 0\}$  and therefore they are non-negative. ■

**Proposition 4.2.10** *All eigenvalues of matrix  $E[M_t \otimes M_t]$  are strictly smaller than 1 in their absolute value.*

**Proof.** In the appendix we proved that  $(E[M_t \otimes M_t])^T < \beta(E[M_{0,t} \otimes M_{0,t}])^T$  (see Lemma 4.5.4). We have also shown that matrix  $(E[M_t \otimes M_t])^T$  is strictly positive (see Lemma 4.5.3). As  $E[M_{0,t} \otimes M_{0,t}]$  is a stochastic matrix  $(E[M_{0,t} \otimes M_{0,t}])^T$  is also a stochastic matrix, and all its eigenvalues are not bigger than 1 in absolute value. Therefore we can apply Perron–Frobenius theorem to deduce that the spectral radius of  $(E[M_t \otimes M_t])^T$  is smaller or equal to  $\beta$ . ■

### 4.2.5 Cycles in the Bounded Memory Learning

In this section we are going to use the Discrete Fourier Transformation (DFT) to measure the cyclicity of the actual price  $p_t$  occurring when agents are learning using the bounded memory algorithm in the cobweb model setting.

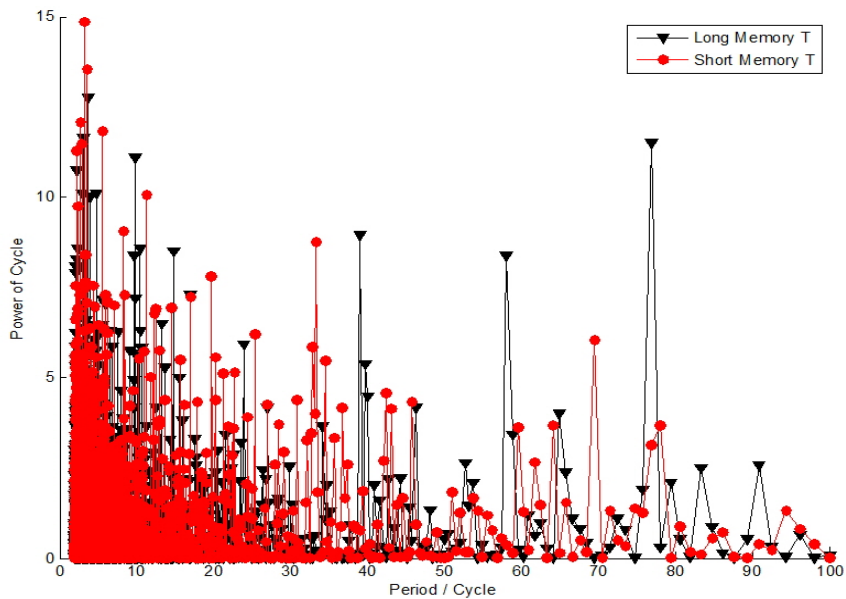
It is known that the DFT of a discrete sequence  $a = [a_0, a_1, \dots, a_{n-1}]$  is another discrete sequence  $b = [b_0, b_1, \dots, b_{n-1}]$  given by

$$b_k = \frac{1}{\sqrt{n}} \sum_{j=0}^{n-1} a_j e^{i \frac{2\pi}{n} jk}, \quad (4.22)$$

where  $a_j$  is the original sequence,  $b_k$  denotes the transformed series and  $j$ ,  $k$  and  $n$  are indices of sequences. By using Euler's formula  $\cos \varphi + i \sin \varphi = e^{i\varphi}$  where  $\varphi$  is an angle given in radians we can see that by applying the above expression we will represent the sequence as a sum of sine and cosine functions. Hence we can measure the cyclicity of the sequence.

Furthermore, we can relate the cycles to the memory length. To do that we apply DFT (4.22) to the  $p_t$  generated using the cobweb model. We analyze two extreme cases: one is very short memory,  $T = 10$ , the other one is very long memory,  $T = 4900$  and the model horizon is 5000 periods. We simulated the model with each window length for 5000 times using different realizations of shocks and exogenous variable  $w_t$ . Then we applied formula (4.22) to the time series and took the mean. Lastly, we plotted the results as shown in the figure below.

Figure 4.1. Cycles  $p$  for Short and Long Memory

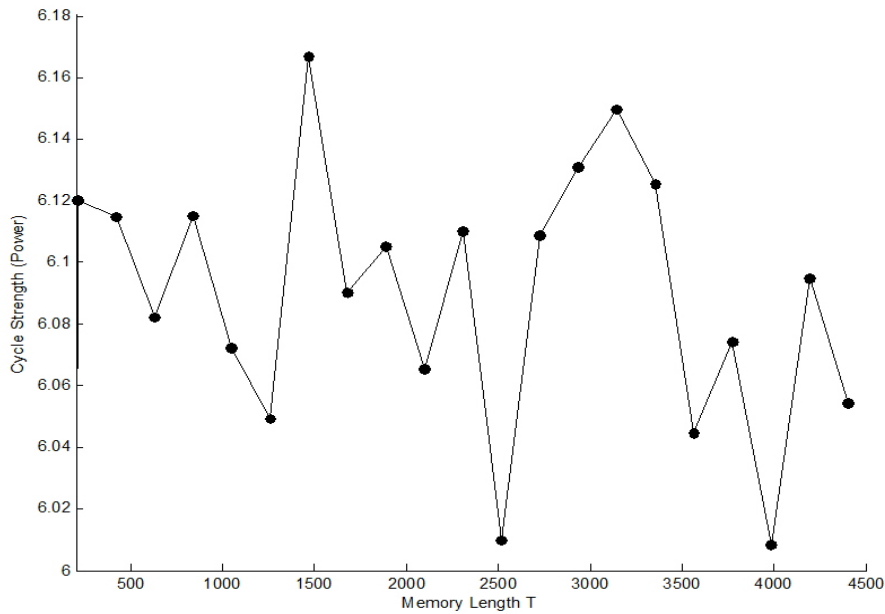


On the  $X$  axis we show the length of period and on the  $Y$  axis we depict the power,  $f$ , of the cycle which is obtained by this formula

$$f_i = x_i^2 + y_i^2, \quad (4.23)$$

where  $i = 1, \dots, \frac{n}{2}$  and  $n$  is the length of the sequence,  $x_i$  and  $y_i$  (obtained by substituting Euler's formula into DFT) denote the real and imaginary parts of  $b_k$  obtained from DFT. As we can see from the figure, short memory (red line) causes much shorter cycles (red line is above the black one for a small length of a period) and long memory  $T$  results in long cycles (black line is above the red one for long periods). To investigate the relation between the length of cycles and the length of memory  $T$  we compute DFT for various lengths of memory shown on the  $X$  axis in the figure below. To produce figure 4.2 we have simulated the cobweb model with bounded memory and time horizon 5000 for 1000 times using different length of memory in the interval 200, 4200 with the step size 200.

Figure 4.2. Power of Cycles with Bounded Memory



As we can see, the relation between the strength of cycles and the memory length seems to be a little ambiguous. On the one hand, we can see that cycles look to become less strong as we increase memory, on the other hand it is clear that there are some fluctuations and the trend of decreasing strength is not that clear. However, as we increase memory  $T$  the strength of cycles tend to be very similar (for instance, cycles for  $T = 3500$  and  $T = 3700$  are almost the same) and for short memory the differences between cycles is huge (for example,  $T = 1200$  and  $T = 1400$  cycles are very different).

The reasons why short memory causes strong cycles are: 1) The fact that it uses only a small sample of observations to obtain OLS parameters and hence the estimates are very inaccurate. 2) Parameters change more sensitively when new information is obtained when a small sample is used, which makes the price cycle strength bigger. Therefore as we expand memory, cycles tend to decrease. It is interesting to note that the cobweb model itself has instability in it because of the

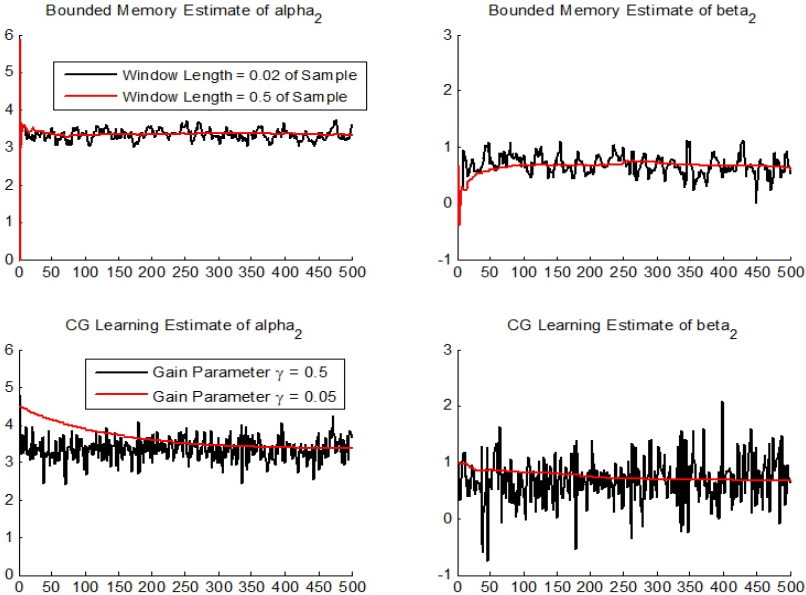
stochastic shock in the price equation, hence the strength of cycles is not tending to zero.

To put it in a nutshell, we see that there is some connection between the length of memory and the strength of cycles. While it is clear that shorter memory has stronger cycles, as  $T$  increases this relation becomes less and less obvious.

### 4.2.6 Bounded Memory and Constant Gain Algorithm

In this section we compare learning in the cobweb model using constant gain (CG) algorithm and bounded memory. We have simulated the model with each learning type using the same realization of shocks and endogenous variable  $w_t$ . Results are provided in the figure below.

Figure 4.3. Bounded Memory and Constant Gain Algorithms



As we can see from the top panel figure 4.3 bounded memory learning algorithm can generate smooth or volatile behavior of  $\alpha_{2,t}$  and  $\beta_{2,t}$  (parameter estimates in



OLS equation) dependent on the length of memory. That is, as noted previously, the shorter memory length causes higher volatility than the short one.

In the bottom panel in figure 4.3 we show that constant gain algorithm can generate very similar behavior dependent on the value of gain parameter  $\gamma$ . The smaller the value of  $\gamma$  is, the closer the CG algorithm is to OLS and the more smooth the estimates are as could be seen from the figure 4.3.

### 4.3 Numerical Evaluation of Mean and Variance

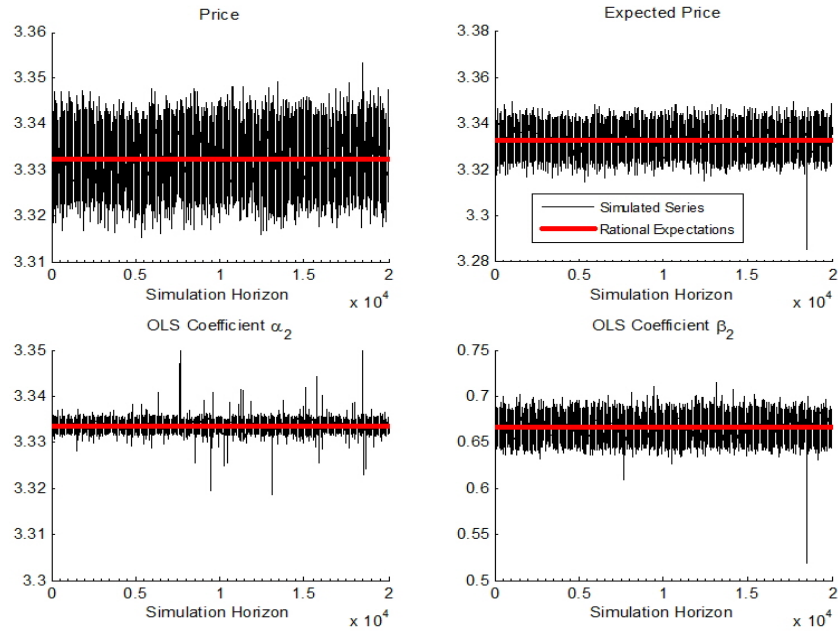
In this section we evaluate numerically the mean and variance of both OLS parameters and price expectations  $p_t^e$ . We use two different methods: explicit numerical computation where we simply simulate the model and take the mean and variance of the variables of interest and implicit numerical computation where we use the indirect method to numerically evaluate the mean.

#### 4.3.1 Explicit Numerical Computation

##### Mean

In this case we simulate the model shown in equations (4.10, 4.11, 4.12, 4.13) 20.000 times with a time horizon of 5000 in each simulation. We also take 20.000 draws of  $w_t$  and  $v_t$  from the standard normal distribution. Then we calculate the mean of price, expected price and parameters  $\alpha_2$  and  $\beta_2$ . It is worth noting that we would get the same results for any size of memory  $T$  because we have shown that the transition matrix has the same properties for any  $T$ . As expected, results show that the mean of all variables and parameters is equal to the REE value. We illustrate this finding in the figure below.

Figure 4.4. Simulation of Mean (I)



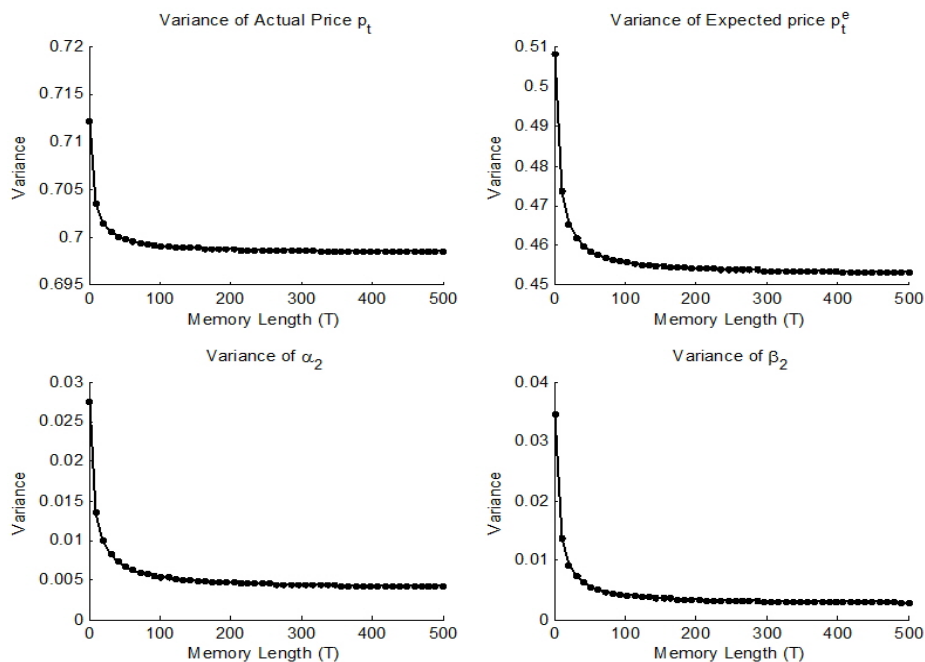
It is clear that the mean is stationary around the REE values. Economically speaking, fluctuations in parameter values could be interpreted as cycles generated by the bounded memory. Since agents have only a limited number of observations to forecast the next period price, they always fluctuate around REE but do not converge to it. This property is also shared by the classical CG learning algorithm.

### Variance

We use the same method to compute the variance of the variables. Results from 20,000 simulations are provided in the figure below. It is intuitive, that variance decreases as the memory length increases because agents have more information and hence their forecasts are more accurate. However, we can see that after  $T$  reaches approximately 100 observations the variance does not decrease any more. This is due to the fact that variance in our case could be split into two components: one is stochastic and depends on the memory length  $T$ , the other is constant and is caused by the shocks in the cobweb model itself. Hence we can see that after

increasing memory length to 100 or more observations we almost eliminate the change in variance which occurs due to small  $T$ .

Figure 4.5. Simulation of Variance



In terms of economics, this finding could be interpreted as a decreasing returns to scale of information. It means that it is very beneficial for the agent to increase his memory as much as possible until it reaches some threshold value which in our case is approximately 100. After that, increasing the length of memory does not decrease variance any more.

### 4.3.2 Implicit Numerical Computation

In this section we use another way to compute the mean of the variables of interest to confirm the results from the previous section. Recall, that our model has the following form:

$$p_t = \alpha + \beta \left( \sum_{i=1}^T S_{it} p_{t-i} \right) + \delta w_{t-1} + v_t, \quad (4.24)$$

where  $S_{i,t} = \frac{1}{T} + \frac{(w_{t-1} - \bar{w}_{t-1})(w_{t-i} - \bar{w}_{t-1})}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]}$ . Now we take unconditional expectations of the equation above

$$E[p] = \alpha + \beta E[p^e], \quad (4.25)$$

where  $p^e = \sum_{i=1}^T S_{it} p_{t-i}$ .

Now we have to evaluate  $E[p^e]$ . We can express it as

$$E[p^e] = E \sum_{i=1}^T [S_{i,t} p_{t-i}], \quad (4.26)$$

and<sup>2</sup>

$$E[p^e] = \sum_{i=1}^T [E[S_{i,t}] E[p] + cov(S_{i,t}, p_{t-i})]. \quad (4.27)$$

Since from proposition 4.2.5 we know that  $\sum_{i=1}^T S_{i,t} = 1$  the above expression simplifies to

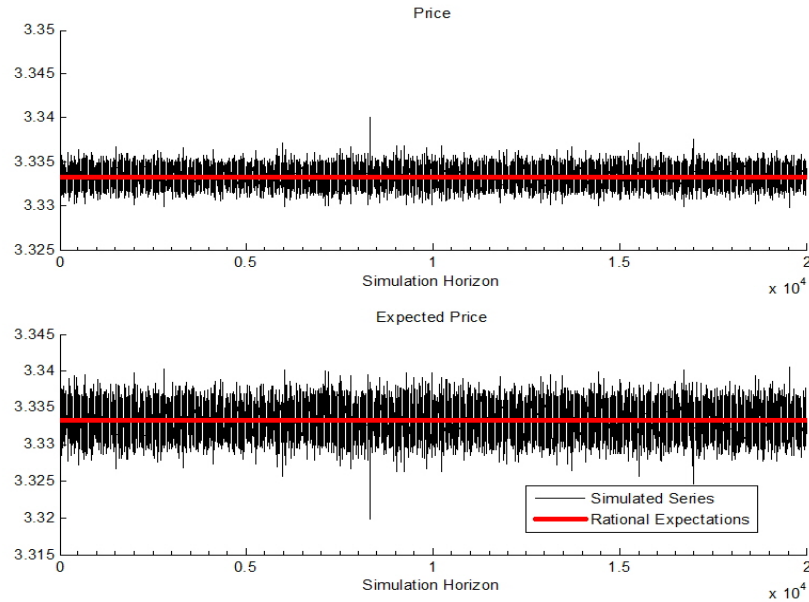
$$E[p^e] = \sum_{i=1}^T [E[p] + cov(S_{i,t}, p_{t-i})]. \quad (4.28)$$

Therefore two equations, namely 4.28 and 4.25 can be used to solve for the variance of price  $p_t$  and expected price  $p_t^e$ . We simulate this system numerically 20.000 times with time horizon of 5000 in each observation. As in the previous case, we use the fact that the mean of forecasts and  $p_t$  does not depend on the memory length and hence we arbitrarily pick  $T = 5$  too. The result is provided in the figure below.

---

<sup>2</sup>To obtain expression for  $E[p^e]$  we use the covariance formula:  $cov[X, Y] = E[XY] - E[X]E[Y]$

Figure 4.6. Simulation of Mean (II)



It is clear that mean is stationary around the RE equilibrium. Hence we have confirmed by two different numerical computations that agents forecasts  $p_t^e$  and the actual price  $p_t$  have a mean which is equal to the rational expectations value.

#### 4.4 Conclusion

In this chapter we have investigated properties of econometric (OLS) learning with bounded memory in the cobweb model setting. We have shown that the eigenvalues of the transition matrix lie within the unit circle for any length of memory  $T$ . Furthermore, we have sketched the proof of stationarity of price  $p_t$ . In addition, using computational techniques we find that price and both OLS estimates have the mean which is the same as REE.

Secondly, we find that cycles in our algorithm occur as a result of bounded memory. In this sense bounded memory learning could be seen as a special case of Constant Gain learning where instability results from the constant gain parameter

$\gamma$ .

Thirdly, there is a relation between the length of memory and the strength of cycles occurring in the actual price  $p_t$ . Whilst it is clear that shorter memory results in stronger cycles, as memory length increases the strength of cycles seems to be decreasing.

Lastly, we have shown that there is a negative relationship between the variance of forecasts and parameter estimates and the length of memory. That is, shorter memory implies greater variance of forecasts and estimates. The reason for this is intuitive. If agents have short memory, they are unable to compute accurate forecasts of price and thus the variance increases. In the case of long memory, forecasts are much more accurate and thus the variance is lower.

## 4.5 Appendix

### 4.5.1 Proof of proposition 4.2.1

Combining (4.10), (4.11), (4.12), (4.13) we get

$$p_t = \alpha + \beta \left( \bar{p}_{t-1} + (w_{t-1} - \bar{w}_{t-1}) \frac{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})(p_{t-i} - \bar{p}_{t-1})]}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \right) + \delta w_{t-1} + v_t. \quad (4.29)$$

Let us simplify it and compute  $S_i$  :

$$\begin{aligned} p_t^e &= \bar{p}_{t-1} + (w_{t-1} - \bar{w}_{t-1}) \frac{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})(p_{t-i} - \bar{p}_{t-1})]}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]}, \quad (4.30) \\ &= \bar{p}_{t-1} \left( 1 - (w_{t-1} - \bar{w}_{t-1}) \frac{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})]}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \right) \\ &\quad + (w_{t-1} - \bar{w}_{t-1}) \frac{\sum_{i=1}^T (w_{t-i} - \bar{w}_{t-1}) p_{t-i}}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]}. \quad (4.31) \end{aligned}$$

Note that  $\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})] = 0$ . Therefore it simplifies further to:

$$p_t^e = \sum_{i=1}^T \frac{1}{T} p_{t-i} + (w_{t-1} - \bar{w}_{t-1}) \frac{\sum_{i=1}^T (w_{t-i} - \bar{w}_{t-1}) p_{t-i}}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \quad (4.32)$$

$$= \sum_{i=1}^T p_{t-i} \left[ \frac{1}{T} + \frac{(w_{t-1} - \bar{w}_{t-1}) (w_{t-i} - \bar{w}_{t-1})}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \right]. \quad (4.33)$$

Hence

$$S_i = \frac{1}{T} + \frac{(w_{t-1} - \bar{w}_{t-1}) (w_{t-i} - \bar{w}_{t-1})}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]}.$$

### 4.5.2 Proof of $E[S_{i,t}] > 0$

Now we wish to prove that  $E[S_i] > 0$ . First, we see that  $S_1 > 0$  as it is a sum of squares. However other coefficients can take both positive and negative values.

We will use permutation trick to prove that  $E[S_i]$  is positive. Recall, that  $w_t$  is

*i.i.d.* Now consider a possible realization of  $w$  such that

$$(w_{t-1} = x_1, \dots, w_{t-T} = x_T). \quad (4.34)$$

Then for any permutation  $\sigma$  the following realization is equally possible

$$\Pr(w_{t-1} = x_{\sigma(1)}, \dots, w_{t-T} = x_{\sigma(T)}) = \Pr(w_{t-1} = x_1, \dots, w_{t-T} = x_T). \quad (4.35)$$

To proceed with the proof, we will split the space of events between those generated by the one set and all of its possible permutations. Then we will show that expectations is positive on every such set.

We consider a set  $X = \{x_1, \dots, x_T\}$ , and define  $\bar{x} = \sum_{i=1}^T \frac{1}{T} x_i$  as its mean. Lets compute a sum  $\sum_{\sigma(x)} S_i$  over all the permutations of set  $X$ . The formal definition of this sum is

$$\sum_{\sigma(X)} f(w_{t-1}, \dots, w_{t-T}) = \sum_{w_{t-1} \in X} \left[ \left[ \left[ \begin{array}{c} \sum_{w_{t-2} \in X \setminus w_{t-1}} \\ \sum_{w_{t-3} \in X \setminus w_{t-1} \setminus w_{t-2}} \dots \\ \left[ \sum_{w_{t-T} \in X \setminus w_{t-1} \setminus w_{t-2} \setminus \dots \setminus w_{t-T+1}} (w_{t-1}, \dots, w_{t-T}) \right] \end{array} \right] \right] \right]. \quad (4.36)$$

For set  $X$ ,  $\bar{w}_{t-1} = \bar{x}$

$$\begin{aligned} \sum_{\sigma(x)} S_i &= \sum_{\sigma(x)} \left[ \frac{1}{T} + \frac{(w_{t-1} - \bar{x})(w_{t-i} - \bar{x})}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \right] \\ &= \frac{T!}{T} + (T-2)! \sum_{w_{t-1} \in X} \left( \frac{(w_{t-1} - \bar{x})}{\sum_{i=1}^T [(w_{t-i} - \bar{x})^2]} \sum_{w_{t-i} \in X \setminus w_{t-1}} [(w_{t-i} - \bar{x})] \right) \end{aligned} \quad (4.37)$$

we use the fact that  $\sum_{w_{t-i} \in X \setminus w_{t-1}} [(w_{t-i} - \bar{x})] = \left[ \sum_{w_{t-i} \in X} (w_{t-i} - \bar{x}) \right] - (w_{t-1} - \bar{x}) = -(w_{t-1} - \bar{x})$ .

Therefore

$$\begin{aligned} &\frac{T!}{T} + (T-2)! \sum_{w_{t-1} \in X} \left( \frac{(w_{t-1} - \bar{x})}{\sum_{i=1}^T [(w_{t-i} - \bar{x})^2]} \sum_{w_{t-i} \in X \setminus w_{t-1}} [(w_{t-i} - \bar{x})] \right) \\ &= \frac{T!}{T} - (T-2)! \sum_{w_{t-1} \in X} \left[ \frac{(w_{t-1} - \bar{x})^2}{\sum_{i=1}^T (w_{t-i} - \bar{x})^2} \right] = \frac{T!}{T} - (T-2)! = (T-2)(T-2)! > 0. \end{aligned}$$



### 4.5.3 Proof of $E[S_i S_1] > 0$

Recall:

$$S_i S_1 = \left[ \frac{1}{T} + \frac{(w_{t-1} - \bar{w}_{t-1})(w_{t-i} - \bar{w}_{t-1})}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \right] \left[ \frac{1}{T} + \frac{(w_{t-1} - \bar{w}_{t-1})(w_{t-1} - \bar{w}_{t-1})}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \right]. \quad (4.38)$$

As before, we will use permutations to compute conditional expectations.

$$\begin{aligned} \sum_{\sigma(x)} S_i S_1 &= \sum_{\sigma(x)} \frac{1}{T^2} + \frac{1}{T} \sum_{\sigma(x)} \frac{(w_{t-1} - \bar{w}_{t-1})^2}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \\ &\quad + \frac{1}{T} \sum_{\sigma(x)} \frac{(w_{t-1} - \bar{w}_{t-1})(w_{t-i} - \bar{w}_{t-1})}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} + \sum_{\sigma(x)} \frac{(w_{t-1} - \bar{w}_{t-1})^3 (w_{t-i} - \bar{w}_{t-1})}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{w}_{t-1})^2 \right]^2}, \end{aligned}$$

we will compute each component of the sum separately. The first component is:

$$\sum_{\sigma(x)} \frac{1}{T^2} = \frac{(T-1)!}{T}. \quad (4.39)$$

The second component is:

$$\sum_{\sigma(x)} \frac{1}{T} \frac{(w_{t-1} - \bar{w}_{t-1})^2}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} = \frac{(T-1)!}{T}. \quad (4.40)$$

The third component is a little more complex:

$$\begin{aligned} &\sum_{\sigma(x)} \frac{1}{T} \frac{(w_{t-1} - \bar{w}_{t-1})(w_{t-i} - \bar{w}_{t-1})}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \\ &= \frac{(T-2)!}{T} \sum_{w_{t-i} \in X} \left( \frac{(w_{t-1} - \bar{x})}{\sum_{i=1}^T [(w_{t-i} - \bar{x})^2]} \sum_{w_{t-i} \in X \setminus w_{t-1}} (w_{t-i} - \bar{x}) \right) \\ &= -\frac{(T-2)!}{T} \sum_{w_{t-i} \in X} \left( \frac{(w_{t-1} - \bar{x})^2}{\sum_{i=1}^T [(w_{t-i} - \bar{x})^2]} \right) = -\frac{(T-2)!}{T}. \end{aligned} \quad (4.41)$$

Finally, we compute the last and the most complex component:

$$\begin{aligned}
& \sum_{\sigma(x)} \frac{(w_{t-1} - \bar{x})^3 (w_{t-i} - \bar{x})}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \\
&= (T-2)! \sum_{w_{t-i} \in X} \left( \frac{(w_{t-1} - \bar{x})^3}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \sum_{w_{t-i} \in X \setminus w_{t-1}} (w_{t-i} - \bar{x}) \right) \\
&= -(T-2)! \sum_{w_{t-i} \in X} \left( \frac{(w_{t-1} - \bar{x})^4}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \right) \\
&= -(T-2)! \frac{\sum_{i=1}^T (w_{t-1} - \bar{x})^4}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2}. \tag{4.42}
\end{aligned}$$

The total sum becomes

$$\begin{aligned}
\sum_{\sigma(x)} S_i S_1 &= \frac{2(T-1)!}{T} - \frac{(T-2)!}{T} - (T-2)! \frac{\sum_{i=1}^T (w_{t-1} - \bar{x})^4}{\sum_{i=1}^T (w_{t-i} - \bar{x})^2} \\
&= \frac{(T-2)!}{T} \left[ T-3 + T \frac{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2 - \sum_{i=1}^T (w_{t-1} - \bar{x})^4}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \right] \tag{4.43}
\end{aligned}$$

#### 4.5.4 Proof of $E[S_i S_j] > 0$

Recall:

$$S_i S_j = \left[ \frac{1}{T} + \frac{(w_{t-1} - \bar{x})(w_{t-i} - \bar{x})}{\sum_{i=1}^T [(w_{t-i} - \bar{x})^2]} \right] \left[ \frac{1}{T} + \frac{(w_{t-1} - \bar{x})(w_{t-j} - \bar{x})}{\sum_{i=1}^T [(w_{t-i} - \bar{x})^2]} \right]. \tag{4.44}$$

Like in the previous proof, we will use permutations to compute conditional expectations.

$$\begin{aligned}
\sum_{\sigma(x)} S_i S_j &= \sum_{\sigma(x)} \frac{1}{T^2} + \sum_{\sigma(x)} \frac{1}{T} \frac{(w_{t-1} - \bar{x})(w_{t-i} - \bar{x})}{\sum_{i=1}^T [(w_{t-i} - \bar{x})^2]} + \sum_{\sigma(x)} \frac{1}{T} \frac{(w_{t-1} - \bar{x})(w_{t-j} - \bar{x})}{\sum_{i=1}^T [(w_{t-i} - \bar{x})^2]} \\
&\quad + \sum_{\sigma(x)} \frac{(w_{t-1} - \bar{x})^2 (w_{t-i} - \bar{x})(w_{t-j} - \bar{x})}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2}. \tag{4.45}
\end{aligned}$$

We have already computed the first 3 components of the sum in (4.39) and (4.42). Now we will compute the last component

$$\begin{aligned} & \sum_{\sigma(x)} \frac{(w_{t-1} - \bar{x})^2 (w_{t-i} - \bar{x})(w_{t-i} - \bar{x})}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \tag{4.46a} \\ &= (T-3)! \sum_{w_{t-i} \in X} \left( \frac{(w_{t-1} - \bar{x})^2}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \sum_{w_{t-i} \in X \setminus w_{t-1}} (w_{t-i} - \bar{x}) \left[ \sum_{w_{t-j} \in X \setminus w_{t-1} \setminus w_{t-i}} (w_{t-j} - \bar{x}) \right] \right). \end{aligned}$$

To proceed we use

$$\begin{aligned} \sum_{w_{t-j} \in X \setminus w_{t-1} \setminus w_{t-i}} (w_{t-j} - \bar{x}) &= \left[ \sum_{w_{t-j} \in X} (w_{t-j} - \bar{x}) \right] - (w_{t-i} - \bar{x}) - (w_{t-1} - \bar{x}) \\ &= -[(w_{t-i} - \bar{x}) + (w_{t-1} - \bar{x})] \tag{4.47} \end{aligned}$$

Therefore (4.46a) is the same as

$$\begin{aligned} & - (T-3)! \sum_{w_{t-i} \in X} \left( \frac{(w_{t-1} - \bar{x})^2}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \sum_{w_{t-i} \in X \setminus w_{t-1}} (w_{t-i} - \bar{x}) [(w_{t-i} - \bar{x}) + (w_{t-1} - \bar{x})] \right) \\ &= - (T-3)! \sum_{w_{t-i} \in X} \left( \frac{(w_{t-1} - \bar{x})^2}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \sum_{w_{t-i} \in X \setminus w_{t-1}} (w_{t-i} - \bar{x})^2 \right) \\ & \quad - (T-3)! \sum_{w_{t-i} \in X} \left( \frac{(w_{t-1} - \bar{x})^3}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \sum_{w_{t-i} \in X \setminus w_{t-1}} (w_{t-i} - \bar{x}) \right) \\ &= - (T-3)! \sum_{w_{t-i} \in X} \left( \frac{(w_{t-1} - \bar{x})^2}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \left[ \sum_{w_{t-i} \in X} (w_{t-i} - \bar{x})^2 - (w_{t-1} - \bar{x})^2 \right] \right) \\ & \quad + (T-3)! \sum_{w_{t-i} \in X} \left( \frac{(w_{t-1} - \bar{x})^4}{\left[ \sum_{i=1}^T (w_{t-i} - \bar{x})^2 \right]^2} \right). \tag{4.48} \end{aligned}$$

Hence the total sum is

$$\begin{aligned} \sum_{\sigma(x)} S_i S_j &= \frac{(T-1)!}{T} - \frac{2(T-2)!}{T} - (T-3)! + (T-3)! \frac{2 \sum_{i=1}^T (w_{t-1} - \bar{x})^4}{\left[ \sum_{i=1}^T (w_{t-1} - \bar{x})^2 \right]^2} \\ &= \frac{(T-3)!}{T} \left[ (T-3)(T-2) - T + T \frac{2 \sum_{i=1}^T (w_{t-1} - \bar{x})^4}{\left[ \sum_{i=1}^T (w_{t-1} - \bar{x})^2 \right]^2} \right] \gg 0. \end{aligned}$$

Now we need two statements. The first one is  $\sum_{\sigma(x)} S_2 S_3 = 0$ , for  $T = 3$ . We will prove it in lemma 4.5.1. Finally we will use lemma 4.5.2 to complete the proof for  $T > 3$ .

**Lemma 4.5.1** For any  $x_1, x_2, x_3$   $2 \sum_{i=1}^3 (x_i - \bar{x})^4 = \left[ \sum_{i=1}^3 (x_i - \bar{x})^2 \right]^2$

**Proof.** It can be proved by direct multiplication and opening all the brackets. ■

**Lemma 4.5.2** For any  $x_1, x_2, x_T, T > 3$

$$((T-3)(T-2) - T) \left[ \sum_{i=1}^T (x_i - \bar{x})^2 \right]^2 + 2T \sum_{i=1}^3 (x_i - \bar{x})^4 > 0 \quad (4.49)$$

Expression 4.49 can be written as the quadratic form  $Y^T Q Y$ , where  $Y =$

$$\begin{pmatrix} (x_1 - \bar{x})^2 \\ \dots \\ (x_T - \bar{x})^2 \end{pmatrix}, Q = \begin{pmatrix} d & z & z & z \\ z & d & z & z \\ z & z & d & z \\ z & z & z & d \end{pmatrix} \text{ and}$$

$$d = (T-3)(T-2) - T + 2T, \quad (4.50)$$

$$z = (T-3)(T-2) - T. \quad (4.51)$$

This quadratic form  $Q$  has only two eigenvalues,

$$\lambda_1 = d - z = 2T > 0, \quad (4.52)$$

$$\lambda_2 = (T-1)z + d = Tz + 2T = T(z + 2), \quad (4.53)$$

$$= T(T^2 - 6T + 8). \quad (4.54)$$

$\lambda_2$  is positive and increasing for  $T \geq 4$ .

### 4.5.5 Proof of lemma 4.2.6

It is easy to see that by definition (4.15)

$$S_1 - S_2 = \frac{(w_{t-1} - w_{t-2})(w_{t-1} - \bar{w}_{t-1})}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]}. \quad (4.55)$$

To compute the expected value we will split all the possible realizations of  $w$  into subsets, generated by permutations. Let  $X$  be the set of realizations and  $\sigma(X)$  set of all the possible permutations over  $T$  of set  $X$ . Furthermore let  $p(X) = Pr(w \in X)$ . Then  $p(\sigma(X)) = T!p(X)$ . Unconditional expectations of  $E[S_1 - S_2]$  could be computed as

$$E[S_1 - S_2] = \int (S_1(w) - S_2(w)) dp(w) \quad (4.56)$$

$$= \int dp(\sigma(X)) \frac{1}{T!} \left[ \sum_{w \in \sigma(X)} (S_1(w) - S_2(w)) \right]. \quad (4.57)$$

Now we consider set  $X$  and sum up over all the possible permutations

$$\begin{aligned} \sum_{w \in \sigma(X)} (S_1 - S_2) &= \sum_{\sigma(X)} \frac{(w_{t-1} - w_{t-2}) w_{t-1}}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \\ &= (T-1)! \sum_{w_{t-1}} \frac{w_{t-1}^2}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \\ &\quad - (T-2)! \sum_{w_{t-1}} \frac{w_{t-1}}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \sum_{w_{t-2} \neq w_{t-1}} w_{t-2} \\ &= (T-1)! \sum_{w_{t-1}} \frac{w_{t-1}^2}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} \\ &\quad - (T-2)! \sum_{w_{t-1}} \frac{w_{t-1}}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} (T\bar{w}_{t-1} - w_{t-1}) \end{aligned} \quad (4.58)$$

The above expression could be simplified to

$$\sum_{w \in \sigma(X)} (S_1 - S_2) = [T(T-2)!] \sum_{w_{t-1}} \frac{w_{t-1}^2 - w_{t-1}\bar{w}_{t-1}}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]}. \quad (4.59)$$

Now we will show that  $\sum_{w_{t-1}} [w_{t-1}^2 - w_{t-1}\bar{w}_{t-1}] = \sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]$ . Suppose that the realization of  $w$  is  $(x_1, x_2, \dots, x_T)$ , then, using permutations we can show

that

$$\begin{aligned}
\sum_{w_{t-1}} [w_{t-1}^2 - w_{t-1}\bar{w}_{t-1}] &= \sum_{i=1}^T [x_i^2 - x_i\bar{x}] = \sum_{i=1}^T [x_i^2 - 2x_i\bar{x} + x_i\bar{x}] = \sum_{i=1}^T [x_i^2 - 2x_i\bar{x}] + \sum_{i=1}^T x_i\bar{x} \\
&= \sum_{i=1}^T [x_i^2 - 2x_i\bar{x}] + T\bar{x}^2 = \sum_{i=1}^T [x_i^2 - 2x_i\bar{x} + \bar{x}^2] \\
&= \sum_{i=1}^T [x_i - \bar{x}]^2 = \sum_{i=1}^T [w_{t-i} - \bar{w}_{t-1}]^2. \tag{4.60}
\end{aligned}$$

Hence we know that  $\sum_{w_{t-1}} \frac{w_{t-1}^2 - w_{t-1}\bar{w}_{t-1}}{\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})^2]} = 1$ . Then computing conditional expectations yields

$$E \mid_{w \in \sigma(X)} (S_1 - S_2) = \frac{1}{T!} \left[ \sum_{w \in \sigma(X)} (S_1(w) - S_2(w)) \right] = [T(T-2)!] \frac{1}{T!} = \frac{1}{T-1}. \tag{4.61}$$

Lastly, we know that unconditional expectations will be the same since  $\int dp(\sigma(X)) = 1$ .

#### 4.5.6 Spectral radius of $E [M_t \otimes M_t]$

As it is sketched above, to accomplish the proof we need some preliminary steps. First we will prove that matrix  $E [M_{t,0} \otimes M_{t,0}]$  is irreducible. In particular we will show that all elements of matrix  $(E [M_{t_0} \otimes M_{t_0}])^T$  are strictly positive

**Lemma 4.5.3** *All elements of matrix  $(E [M_t \otimes M_t])^T$  are strictly positive.*

**Proof.** We will consider the first case when  $T \geq 4$ . Then we know that  $E [S_{i,t}S_{j,t}]$  are strictly positive and therefore there exists a positive number  $z$ , such that  $z^2 < \min(\beta^2 E [S_{i,t}S_{j,t}], \beta E [S_i])$ . Consider matrix  $Z$  with elements defined as following

$$Z_{1,j} = z > 0; \text{ for any } j; \tag{4.62}$$

$$Z_{i,i-1} = 1, \text{ if } i > 2; \tag{4.63}$$

$$Z_{ij} = 0 \text{ otherwise.} \tag{4.64}$$

$$Z = \begin{pmatrix} z & z & \dots & z & z \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad (4.65)$$

All the elements of matrix  $Z$  are non-negative and  $(Z \otimes Z)_{i,j} \leq (E[M_{t,0} \otimes M_{t,0}])_{i,j}$ . Therefore,  $(Z \otimes Z)^T \leq (E[M_{t,0} \otimes M_{t,0}])^T$ . Because  $(Z \otimes Z)^T = Z^T \otimes Z^T$ , it would be sufficient to show that all elements of  $Z^T$  are strictly positive. Matrix  $Z$  can be represented as a sum of shift matrix  $S$  and the a matrix with positive elements on its first row,  $Z_1$ .

$$Z_1 = \begin{pmatrix} z & z & \dots & z \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \quad (4.66)$$

$$S = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}; \quad (4.67)$$

By induction we can prove that  $Z_1^m = \begin{pmatrix} z^m & z^m & \dots & z^m \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}$ . It is also easy to

verify that  $S^k Z_1^{T-k}$  is a matrix which has  $z^{T-K}$  as its  $(k+1, j)$  element. Therefore, we have proven that all elements of  $Z^{T-1}$  are positive since  $Z^T = (S + Z_1)^T \geq$

$$\sum_{k=0, T-1} S^k Z_1^{T-k} > 0 .$$

We proved the lemma for  $T > 3$ . For  $T = 3$  it is also true, we can easily verify it by computing  $Z^3$ . ■

**Lemma 4.5.4** *Any element of  $(E[M_t \otimes M_t])^T$  is smaller than  $\beta(E[M_{0,t} \otimes M_{0,t}])^T$*

**Proof.** As before we present matrix  $E[M_{0,t} \otimes M_{0,t}]$  as a sum of two matrices

$$E[M_{0,t} \otimes M_{0,t}] = S \otimes S + M_1, \quad (4.68)$$

where  $S$  is a shift matrix defined in (4.67). Then

$$E(M_t \otimes M_t) = S \otimes S + (B \otimes B)M_1, \quad (4.69)$$

where  $(B \otimes B)M_1 \leq \beta M_1$ . It is easy to show that

$$E[M_t \otimes M_t]^K = S^k \otimes S^k + Q^k((B \otimes B)M_1)$$

where  $Q((B \otimes B)M_1)$  is a sum of products of the elements  $((B \otimes B)M_1)$  and  $(S^k \otimes S^k)$ , such that  $((B \otimes B)M_1)$  comes to it at least once. For example,  $Q^1 = (B \otimes B)M_1$ . And it is easy to compute a recursive formula for  $Q^{k+1}$

$$Q^{k+1} = (B \otimes B)M_1(S^k \otimes S^k) + (S \otimes S)Q^k((B \otimes B)M_1) + (B \otimes B)M_1Q^k((B \otimes B)M_1). \quad (4.70)$$

Moreover, it is also true that  $E[M_{0,t} \otimes M_{0,t}]^K = S^k \otimes S^k + Q^k(M_1)$ . As  $((B \otimes B)M_1) \leq \beta M_1$ , it follows that

$$\begin{aligned} Q^k((B \otimes B)M_1) &\leq \beta Q^K(M_1) E[M_t \otimes M_t]^T \\ &= S^T \otimes S^T + Q^k((B \otimes B)M_1) \leq S^T \otimes S^T + \beta Q^T(M_1). \end{aligned}$$

However,  $S^T$  is a shift matrix and therefore,  $S^T = NULL$ . Hence we proved that

$$E[M_t \otimes M_t]^T = Q^k((B \otimes B)M_1) \leq \beta Q^T(M_1) \quad (4.71)$$

$$= \beta Q^T(M_1) \beta(E[M_{0,t} \otimes M_{0,t}])^T. \quad (4.72)$$

■



### Spectral Radius of Matrix $E [M_t]$

We wish to prove that all eigenvalue of matrix  $E [M]$  are smaller than one. That matrix is

$$E [M] = BE [M_0] = \begin{pmatrix} \beta ES_{1t} & \beta ES_{2t} & \dots & \beta ES_{T-1t} & \beta ES_{Tt} \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Consider a characteristic polynomial:

$$\det(E [M_t] - xI) = \det \begin{pmatrix} \beta ES_1 - x & \beta ES_2 & \dots & \beta ES_{T-1} & \beta ES_T \\ 1 & -x & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & -x \end{pmatrix}$$

$$\begin{aligned} \det(E [M_t] - xI) &= (-x)^T + \beta E [S_{1t}(-x)^{T-1}] - \beta E [S_{2t}(-x)^{T-2}] \\ &+ \beta E [S_{3t}(-x)^{T-3}] + \dots (-1)^{k+1} \beta E [S_{kt}(-x)^{T-k}] \dots (-1)^{T+1} \beta E [S_{Tt}] = 0 \end{aligned} \quad (4.73)$$

$$Q(x) = (-1)^T \det(M_t - xI) = x^T - \beta E [S_{1t}x^{n-1}] - \beta E [S_{2t}x^{n-2}] + \dots - \beta E [S_{kt}x^{T-k}] \dots - \beta E [S_{Tt}] \quad (4.74)$$

We will prove that this polynomial does not have any roots higher than or equal than 1 in its absolute value. Consider  $x$ , such that  $|x| \geq 1$ , then  $|x|^{-k} \leq 1$ .

$$\left| \frac{Q(x)}{x^T} \right| = \left| 1 - \beta \sum_{i=1}^T E [S_{i,t}x^{-i}] \right| \geq 1 - \beta \sum_{i=1}^T E [S_{i,t}|z|^{-i}] \geq 1 - \beta \sum_{i=1}^T E [S_{i,t}] = 1 - \beta \quad (4.75)$$

Therefore,  $z$  can not be an eigenvalue of  $E [M]$  as it is not a root of its characteristic polynomial.

### Proof of lemma 4.2.2

Let us define  $k_{i,t}$  as

$$k_{i,t} = \frac{(w_{t-i} - \bar{w}_{t-1})}{\left(\sum_{i=1}^T [(w_{t-i} - \bar{w}_{t-1})]\right)^{\frac{1}{2}}}, \quad (4.76)$$

then according to (4.15)

$$S_{i,t} = \frac{1}{T} + k_{i,t}k_{1,t}, \quad (4.77)$$

where  $k_{i,t}$  can be any number with the following restrictions

$$\sum_{i=1}^T k_{i,t} = 0, \quad (4.78)$$

$$\sum_{i=1}^T k_{i,t}^2 = 1. \quad (4.79)$$

Now we can compute  $S_{i,t}^2$ :

$$\sum_{i=1}^T S_{i,t}^2 = \sum_{i=1}^T \left(\frac{1}{T} + k_{i,t}k_{1,t}\right)^2 = \frac{1}{T} + k_{1,t}^2. \quad (4.80)$$

Lets maximize (4.80) subject to constraints (4.78) and (4.79). The Lagrangian is

$$\max_{k_i} L = \frac{1}{T} + k_1^2 - \lambda_0 \sum_{i=1}^T k_i - \lambda_1 \left(\sum_{i=1}^T k_i^2 - 1\right) \quad (4.81)$$

The first order conditions imply

$$\frac{\partial L}{\partial k_1} = 2k_1 - \lambda_0 - 2\lambda_1 k_1 = 0; \quad (4.82)$$

$$\frac{\partial L}{\partial k_i} = -\lambda_0 - 2\lambda_1 k_i = 0; \quad (4.83)$$

summing up all FOC and combining it with (4.78), we obtain  $\lambda_0 = \frac{2k_1}{T}$ . Similarly  $\sum_{i=1}^T \frac{\partial L}{\partial k_i} k_i = 2k_1^2 - 2\lambda_1 = 0$ , and implies  $\lambda_1 = k_1^2$ . Finally, from (4.82), we obtain

$$k_1 \left(1 - \frac{1}{T} - k_1^2\right) = 0;$$

which implies that  $k_1^2 = 1 - \frac{1}{T}$  at its maximum ( and  $k_1 = 0$  at its minimum).

Therefore

$$\sum_{i=1}^T S_{i,t}^2 = \frac{1}{T} + k_{1t}^2 \leq \frac{1}{T} + \left(1 - \frac{1}{T}\right) = 1 \quad (4.84)$$

### Proof of proposition 4.2.3

For any memory length  $T$ , there exist a natural number  $n_T$  and boundary  $B_T$  such that for any  $n > n_T$ , every elements of the product of  $n$  matrices is bounded by  $B_T \beta^n$  :

$$\left| \prod_{i=1,n} (M_{t-i}) \right|_{ij} < B_T \beta^n. \quad (4.85)$$

To prove this statement we will represent matrix  $M_t$  as follows:

$$M_t = Z_t + S \quad (4.86)$$

where we define  $Z_t$  as

$$Z_t = \begin{pmatrix} \beta S_{1,t} & \beta S_{2,t} & \dots & \beta S_{T-1,t} \\ 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix}, \quad (4.87)$$

and  $S$  is the shift matrix

$$S = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \quad (4.88)$$

We claim that any matrix  $X$ , multiplied by  $B$  row of  $BX$ , is zero. Moreover is the first  $k$  rows of  $X$  are zeros, than the first of  $k + 1$  row of  $BX$  are zeroes as

well. Lets compute the product:

$$\prod_{i=1,n} (M_{t-i}) = (Z_{t-1} + S) (Z_{t-2} + S) \dots (Z_{t-n} + S) \quad (4.89)$$

To compute the operation we need to sum up the matrix products of length  $n$  consisting of  $Z$  and  $S$ . However, if  $S$  appears more than  $T - 1$  times, the product is zero. Therefore we can restrict our attention only to those cases which will have  $S$  less than  $T$  times. The number of products with  $S$  being exactly on  $k$  places is  $n!/k!/(n - k)!$  and the total number of non-zero products is less than  $n!/((T - 1)!(n - T + 1)!) * (T - 1)$ .

Moreover we can claim that every product is a matrix with elements less than  $(\beta z)^{n-T}$ , where  $z = \max_i |S_{i,t}| \leq 1$ . Therefore, every element of

$$[(Z_{t-1} + S) (Z_{t-2} + S) \dots (Z_{t-n} + S)]_{ij} < \frac{n!}{(T - 2)!(n - T + 1)!} \beta^{n-T} < n^T \beta^{n-T}.$$

Consider the sequence  $\{a_n\}$ , where

$$a_n = n^T \beta^{n-T}$$

then

$$\frac{a_{n+1}}{a_n} = \left( \frac{n+1}{n} \right)^T \beta$$

Let  $\tilde{\beta} \in (\beta, 1)$  then we can find  $n^*(\tilde{\beta}, T, \beta)$ , such that for any  $n > n^*$ ,

$$\frac{a_{n+1}}{a_n} = \left( \frac{n+1}{n} \right)^T \beta < \tilde{\beta}, \quad (4.90)$$

in that case for any  $k$

$$a_{n^*+k} < \tilde{\beta}^k a_{n^*} = \tilde{\beta}^k n^{*T} \beta^{n^*-T}. \quad (4.91)$$

## BIBLIOGRAPHY

- [1] Arifovic, J., 1994, 'Genetic Algorithm Learning and the Cobweb Model', *Journal of Economic Dynamics and Control*, vol. 18, pp. 3-28.
- [2] Barucci, E., 2000, 'Exponentially Fading Memory Learning in Forward-Looking Economic Models', *Journal of Economic Dynamics and Control*, vol. 24, pp. 1027-1046.
- [3] Barucci, E., 2001, 'Fading Memory Learning in a Class of Forward-Looking Models with an Application to Hyperinflation Dynamics', *Economic Modelling*, vol. 18, pp. 233-252.
- [4] Bray, M., M., Savin, N., E., 'Rational Expectations Equilibria, Learning and Model Specification', *Econometrica*, vol. 54, pp. 1129-1160.
- [5] Berardi, M., Galimberti, J., K., 2014, 'On the Determination of Adaptive Learning Gains', mimeo.
- [6] Berardi, M., Galimberti, J., K., 2013, 'A note on Exact Correspondence Between Adaptive Learning Algorithms and the Kalman Filter', *Economics Letters*, vol. 118, pp. 139-142.
- [7] Canning, D. 1992, 'Average Behavior in Learning Models', *Journal of Economic Theory*, vol. 57, pp. 442-472.
- [8] Carceles-Poveda, E., Giannitsarou, C., 2008, 'Asset Pricing and Adaptive Learning', *Review of Economic Dynamics*, vol. 11, pp.629-651.
- [9] Chevillon, G., Mavroeidis, S., Learning Can Generate Long Memory, mimeo.
- [10] Conlisk, J., 1974, 'Stability in a Random Coefficient Model', *International Economic Review*, Vol. 15, pp. 529-533.
- [11] Dufrense, D., Vazquez-Abad, F., 2013, 'Cobweb Theorems with Production Lags and Price Forecasting', *Economics: The Open-Access, Open-Assement E-Journal*, 7, pp. 1-49.
- [12] Evans, G., W, Honkapohja, S., 'Learning and Expectations in Macroeconomics', 2001, Princeton University Press, Princeton, 419 p.
- [13] Ezekiel, M., 1938, 'The Cobweb Theorem', *Quarterly Journal of Economics*, vol. 52, pp. 255-280.

- [14] Holmes, M., J., Manning, R., 1988, 'Memory and Market Stability: The Case of the Cobweb', *Economics Letters*, vol. 28, pp. 1-7.
- [15] Honkapohja, S., Mitra, K., 2003, 'Learning with Bounded Memory in Stochastic Models', *Journal of Economic Dynamics and Control*, vol. 27, pp. 1437-1457.
- [16] Honkapohja, S., Mitra, K., 2006, 'Learning Stability in Economies with Heterogeneous Agents', *Review of Economic Dynamics*, vol. 9, pp. 284-309.
- [17] Kaldor, N., 1934, 'A Classificatory Note on the Determinateness of Equilibrium', *The Review of Economic Studies*, vol. 1, pp. 122-136.
- [18] LeBaron, B., 2012, 'Heterogeneous Gain Learning and the Dynamics of Asset Prices', *Journal of Economic Behavior & Organization*, vol. 83, pp. 424-445.
- [19] Mitra, K., 2005, 'Is More Data Better?', *Journal of Economic Behavior & Organization*, vol. 56, pp. 263-272.
- [20] Mitra, K., Evans, G, W., Honkapohja, S., 2013, 'Policy Change and Learning in the RBC Model', *Journal of Economic Dynamics and Control*, vol. 37, pp. 1947-1971.
- [21] Sargent, T., Williams, N., Zha, T., 'Shocks and Government Beliefs: the Rise and Fall of American Inflation', *American Economic Review*, vol. 96, pp. 1193-1224.

## CHAPTER 5

### CONCLUSION

In this thesis we have looked at the two broad topics in macroeconomics: monetary policy design and learning.

In the second chapter we considered a standard New-Keynesian model with sticky prices, collateralized borrowing as in Iacoviello (2005) and interest rate at the ZLB. We have investigated dynamic properties of the economy under negative financial and positive productivity shocks. The results suggests that under the presence of financial market shock the government could stabilize the economy by implementing transfers according to the rule that takes into account households borrowing. If interest rate is not at the ZLB and has some small positive value, economy could be stabilized by reducing it. When the productivity shock occurs, central bank cannot keep the economy at the steady state any more in terms of inflation and output deviations. However, he could stabilize the price of collateral by setting the transfers rule to zero. We also find that reaction to households borrowing in the policy rules (interest rate and transfers) is optimal because it yields the highest value of the welfare measure.

In the third chapter we investigated the relation between firms borrowing and unemployment in the framework of the standard new Keynesian model with firms borrowing market and search and matching frictions. Our results show that in the case of a negative financial and a positive productivity shocks Taylor-type policy rule which reacts on inflation and unemployment is the closest one to the optimal policy in terms of the welfare measure. This finding is consistent with the existing literature (for instance, Faia (2008) or Thomas (2008)). Furthermore, we find that the sign of the unemployment reaction parameter in the policy rule depends on the value of workers bargaining power.

In the last chapter of thesis we have investigated properties of the bounded memory learning algorithm using the classical cobweb model. We have sketched the analytical proof for the stationarity of price for any length of memory  $T$ . Furthermore, by making use of the computational techniques we found that mean and variance of the parameter estimates and price have the same mean as rational expectations values. However, we are still working on the analytical proof of this property. In addition, we discovered that bounded memory causes cycles in agents forecasts. If agents have long memory, then cycles in their forecasts are tend to be longer than in the case of a short memory. This finding also implies, that there exists some threshold value of memory length  $T$  after which adding more observation to the sample does not improve the accuracy of forecast.

In terms of future research, firstly, we would like to finalize the analytical proof of stationarity of bounded memory learning. Secondly, we would like to try to apply our learning algorithm for the analysis of the Central Bank policy, in particular, Bank of England. We are motivated by the success of Sargent, Williams and Zha (2004) who applied very similar model (with Kalman filter learning instead of bounded memory) to investigate FED's policy towards inflation and unemployment. The reason why we believe that bounded memory algorithm could be useful in this case is the fact that central planner updates his policy based on the recent economic performance. Lastly, once the properties of bounded memory learning algorithm are known, it would be very interesting to look at the economy where agents have different memory lengths because this might give a suggestion of an optimal (in terms of mean square error, for instance) memory length or a convergence to multiple equilibria that may depend on the calibration of the model.