ON THE FORM OF THE VISCOUS TERM FOR TWO DIMENSIONAL NAVIER–STOKES FLOWS

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Summary

The form of the viscous term is discussed for incompressible flow on a two-dimensional curved surface S and for the shallow water equations. In the case of flow on a surface three versions are considered. These correspond to taking curl twice, to applying the Laplacian defined in terms of a metric, and to taking the divergence of a symmetric stress tensor. These differ on a curved surface, for example a sphere. The three terms are related and their properties discussed, in particular energy and angular momentum conservation.

In the case of the shallow water equations again three forms of dissipation are considered, the last of which involves the divergence of a stress tensor. Their properties are discussed, including energy conservation and whether the rotating bucket solution of the three-dimensional Navier–Stokes equation is reproduced. A derivation of the viscous term is also given based on shallow water equations as a truncation of the Navier–Stokes equation, with forces on a column determined by integration over the vertical. For both incompressible flow on a surface and for the shallow water equations, it is argued that a viscous term based on a symmetric stress tensor should be used as this leads to correct treatment of angular momentum.

1. Introduction

The aim of this paper is to discuss the form of viscous terms in some simplified models of the Navier–Stokes equation, in particular two-dimensional flow on a curved surface and the shallow water equations. The Navier–Stokes equation is easily written down for idealised flows on a flat two-dimensional surface. However when the surface is curved, for example flow on a sphere, one encounters the immediate difficulty that there are two different ways to define the Laplacian of a vector field and so define the viscous term. This was first noted in the fluid dynamics context by Il'in in (**1**). One definition is informally called the *topologists' Laplacian* and is the Laplace–de Rham operator in differential geometry, while the other is called the *analysts' Laplacian* as it is based on the metric (**2**, **3**). They differ because one cannot freely exchange derivatives of a vector quantity such as the flow velocity $\boldsymbol{u}$ in a curved space. This lack of commutation is precisely captured by the Riemann curvature tensor: exchanging two derivatives generates a term linear in $\boldsymbol{u}$ which vanishes only if the space is intrinsically flat.

Now for many purposes it may not matter which definition of the Laplacian is taken, for example in establishing results about the regularity of solutions of the Navier–Stokes equations, or in bounding attractor dimensions; see, for example, (**1**, **4**, **5**, **6**, **7**). These two Laplacians involve the same second and first order derivatives of velocity and only differ in terms that are linear in $\boldsymbol{u}$. However from the point of view of creating idealised models of more detailed geophysical or astrophysical phenomena, for example the formation of zonal jets in a thin-layer model of Jupiter, the question remains about which is the most appropriate choice.

To add to the possibilities and potential confusion, there is a third way to proceed, which is to define the viscous term as the divergence of a symmetric stress tensor, and several contributions argue that this is important for angular momentum conservation (**8**, **9**, **10**, **11**). Indeed an important part of the classical derivation of the Navier–Stokes equation is establishing that the stress tensor is symmetric by considering the net torque on a vanishingly small volume of fluid (**12**). This is a local calculation that is valid in curved as well as flat space (**8**). If the stress tensor is not symmetric, then there will generally be sources and sinks of angular momentum through viscous effects.

The aim of the paper is to set out these three possible definitions of the viscous term for flows on a general curved surface, to explain how and why they differ and, ultimately, to recommend that based on taking the divergence of a symmetric stress tensor. After recapping briefly ways of writing the usual viscous term in three-dimensional Euclidean space in section 2, we discuss the viscous term in the context of incompressible Navier–Stokes flow on an arbitrary curved surface in section 3. In section 4 we consider the form of the viscous term for the shallow water (or Saint Venant) equations. Again there are a number of possibilities with different properties as has been discussed in papers including (**9**, **10**, **13**, **14**, **15**), and most recently (**16**). We again write down three models of the dissipative term and discuss them. In particular we compare the models with a standard three-dimensional Navier–Stokes solution of rotating 'bucket' flow and find that this is not reproduced by one commonly used form of the viscous term.

Note that there is a philosophical question: what are we aiming to achieve with a reduced model for flow on a curved surface or a shallow water system? Given that all fluids are in reality three-dimensional, a reduction to incompressible flow in two dimensions or use of the shallow water equations has two possible motivations. One is to create a detailed, predictive model of a system that is approximately two-dimensional, for example Hele–Shaw flow, flow in a soap bubble, or flow on large scales in the atmosphere or oceans. In this case it is necessary to approximate the full equations and to determine effects that can arise, say from boundary layers or from surface tension, order by order (for example in the shallow water case see (**17**, **18**)). The other motivation is to explore a basic phenomenon (for example turbulence, waves, jets and mixing) in a simpler setting in order to establish scientific understanding. Here it is crucial that the reduced model has well-understood properties to isolate the phenomenon in question. For example, a model should not possess spurious or poorly controlled sources of energy or angular momentum even if in the real applications that motivate the model, there are such sinks and sources (for example from boundary layers in geophysical flows) unless these are the object of the study. Our motivation is the second one: we are interested in reductions of the Navier–Stokes equations that have clear properties such as conservation of angular momentum and energy, in common with the studies (**9**, **10**, **14**, **15**, **16**). In each case we observe that the formulation based on the divergence of a symmetric stress tensor (CS3 and SW3 below) has the best properties.

2. Navier–Stokes flow in three dimensions

The Navier–Stokes equation for incompressible fluid flow in three-dimensional Euclidean space is

$$\rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}) = -\nabla p + \mu\boldsymbol{d}, \quad \nabla\cdot\boldsymbol{u} = 0. \tag{2.1}$$

Here μ is the coefficient of viscosity and the stress vector $\boldsymbol{d}$ may be written in three equivalent forms:

Definition NS1: in terms of differential operators

$$\boldsymbol{d} = -\nabla \times (\nabla \times \boldsymbol{u}), \tag{2.2}$$

Definition NS2: as a Laplacian,

$$\boldsymbol{d} = \nabla^2 \boldsymbol{u} = \partial_i \partial_i \boldsymbol{u}, \tag{2.3}$$

Definition NS3: as the divergence of a stress tensor,

$$\boldsymbol{d} = \nabla \cdot \boldsymbol{t}, \quad \boldsymbol{t} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T = 2\boldsymbol{e}, \tag{2.4}$$

with $\boldsymbol{e}$ the rate-of-strain tensor. In components $\boldsymbol{t}$ is

$$t_{ij} = \partial_i u_j + \partial_j u_i = 2e_{ij}. \tag{2.5}$$

The tensor $\boldsymbol{t}$ is trace free as the divergence of $\boldsymbol{u}$ is zero. We have removed the pressure term from the usual definition of the stress tensor $\boldsymbol{t}$ as pressure does not form part of our discussion. We have also conveniently taken μ from the usual definition of $\boldsymbol{t}$ and placed it in (2.1).

Note that the advective term can also be written in two equivalent forms, with the standard result

$$\boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\boldsymbol{u} \times (\nabla \times \boldsymbol{u}) + \nabla \tfrac{1}{2} |\boldsymbol{u}|^2, \tag{2.6}$$

and for later reference we state the equivalent form of the Navier–Stokes equation,

$$\rho[\partial_t \boldsymbol{u} - \boldsymbol{u} \times (\nabla \times \boldsymbol{u}) + \nabla \tfrac{1}{2} |\boldsymbol{u}|^2] = -\nabla p + \mu \boldsymbol{d}, \quad \nabla \cdot \boldsymbol{u} = 0. \tag{2.7}$$

All these equivalent forms of the viscous term (NS1–3) lead to dissipation of energy,

$$K = \tfrac{1}{2} \rho \int_V \boldsymbol{u}^2 \, dV. \tag{2.8}$$

We use (2.2–2.4) in turn and integrate by parts in a finite volume V, with surface ∂V and unit normal $\boldsymbol{n}$. These give, respectively,

$$\mu^{-1} \partial_t K = -\int_V |\boldsymbol{\omega}|^2 \, dV + \int_{\partial V} \boldsymbol{n} \cdot \boldsymbol{u} \times \boldsymbol{\omega} \, dS, \tag{2.9}$$

$$\mu^{-1} \partial_t K = -\int_V (\partial_i u_j)(\partial_i u_j) \, dV + \int_{\partial V} \boldsymbol{n} \cdot \nabla \tfrac{1}{2} |\boldsymbol{u}|^2 \, dS, \tag{2.10}$$

$$\mu^{-1} \partial_t K = -\int_V e_{ij} t_{ij} \, dV + \int_{\partial V} n_i t_{ij} u_j \, dS, \tag{2.11}$$

with $\boldsymbol{\omega} = \operatorname{curl} \boldsymbol{u}$ as the vorticity. All three volume integrals are negative semi-definite, and the form of these suggests that the flow will relax to a state of zero vorticity $\boldsymbol{\omega}$, zero flow $\boldsymbol{u}$ or zero rate-of-strain tensor $\boldsymbol{e}$, respectively. Of course this discrepancy is settled by the different forms of the surface terms. For example if we take the solid body rotating flow in a cylinder of unit radius and unit height we have (in cylindrical polars (r, ϕ, z)),

$$\boldsymbol{u} = ar\hat{\boldsymbol{\phi}} = a(-y\boldsymbol{i} + x\boldsymbol{j}), \quad \boldsymbol{\omega} = 2a\boldsymbol{k}. \tag{2.12}$$

We take the boundary of the cylinder to have the same angular velocity a, so that this flow will be preserved in time. In this case the volume integrals in (2.9–11) become $-4\pi a^2$, $-2\pi a^2$ and 0 respectively. These are all cancelled by the respective surface terms, but note that it is only (2.11) that highlights the important physical content that the fluid can exist in a steady state with zero rate-of-strain tensor $\boldsymbol{e}$ in the volume, provided there are no stresses exerted at the boundary through the presence of $\boldsymbol{t}$ in the surface integral.

Although this is a well understood textbook calculation, it will be useful to refer to it below when we consider the case of flow on a closed surface or flow under the shallow water equations. In the case of a closed surface there is no boundary and all the boundary terms vanish, which already hints that the definitions generalising NS1, NS2 and NS3 will be fundamentally different.

3. Viscous term for flow on a curved surface

We now consider flow on a curved surface S, which we take to be closed to avoid consideration of boundary conditions. It is helpful to have in mind the unit sphere as an example we will discuss in more detail later. There are two distinct routes that may be followed.

3.1 *Navier-Stokes through embedding in* $\mathbb{R}^3$

The first approach is to think of S as being smoothly embedded as a surface in $\mathbb{R}^3$ with a unit normal vector field $\boldsymbol{n}$. We may then use operators in $\mathbb{R}^3$ to define the advective and viscous terms, and we outline the discussion of Il'in (**1**, **4**) in what follows. We start with a smooth flow $\boldsymbol{u}$ defined at points of S in $\mathbb{R}^3$ and everywhere tangential to S, $\boldsymbol{u}\cdot\boldsymbol{n}=0$. We can then define two operators based on the curl operator in $\mathbb{R}^3$, namely,

$$\mathrm{curl}_{\mathrm{s}}\,\chi = \nabla\times(\chi\boldsymbol{n}), \quad \mathrm{curl}_{\mathrm{v}}\,\boldsymbol{u} = \boldsymbol{n}\cdot\nabla\times\boldsymbol{u}, \tag{3.1}$$

where χ is any smooth scalar field defined on S. Here we use operators involving ∇ in $\mathbb{R}^3$ and reserve 'curl' and similar names for the induced operators acting on fields defined only on S. In order to define the operators in (3.1), one needs to extend the flow $\boldsymbol{u}$, the scalar field χ, and the normal $\boldsymbol{n}$, for points near to but not on the surface S. There are some subtleties in this, and we indicate the approach of Il'in (**4**) in appendix A. When this is done, the normal vector field $\boldsymbol{n}$ is defined for points not on S in such a way that

$$\nabla\times\boldsymbol{n} = 0, \tag{3.2}$$

and then the definitions above may also be written as

$$\mathrm{curl}_{\mathrm{s}}\,\chi = -\boldsymbol{n}\times\nabla\chi, \quad \mathrm{curl}_{\mathrm{v}}\,\boldsymbol{u} = -\nabla\cdot(\boldsymbol{n}\times\boldsymbol{u}). \tag{3.3}$$

The results of the operators $\mathrm{curl}_{\mathrm{s}}\,\chi$ and $\mathrm{curl}_{\mathrm{v}}\,\boldsymbol{u}$ depend only on the the values of the fields χ and $\boldsymbol{u}$ at points on S and not on how they are extended off S (**4**). (An alternative procedure is to simply redefine the curl operator for a field that is only defined on the surface.) Similar remarks apply to the use of the divergence operator on the flow $\boldsymbol{u}$: the two-dimensional divergence on the surface is given by applying the three-dimensional divergence operator in $\mathbb{R}^3$ to $\boldsymbol{u}$ when extended appropriately into $\mathbb{R}^3$ (see appendix A). We stress that the extension of vector and scalar fields to a neighbourhood of S is simply a means to define vector calculus operators for fields defined on S: the fields have no physical meaning at points off S. Once these operators are defined, we may write down the Navier–Stokes equation for two-dimensional flow confined to the surface by analogy with (2.7) (**4**).

Consider first the advective term (although our main concern is the dissipative term in this paper). Given the flow $\boldsymbol{u}$ tangential to S the quantity $\boldsymbol{u}\cdot\nabla\boldsymbol{u}$ will generally have a component perpendicular to S. This may be removed using the projection $\boldsymbol{\pi}$ orthogonal to S, as a surface embedded in $\mathbb{R}^3$, which is defined by

$$\boldsymbol{\pi}\boldsymbol{u} = \boldsymbol{u} - (\boldsymbol{n}\cdot\boldsymbol{u})\boldsymbol{n} = -\boldsymbol{n}\times(\boldsymbol{n}\times\boldsymbol{u}). \tag{3.4}$$

With this in hand we use (2.6) to write

$$\boldsymbol{\pi}\boldsymbol{u}\cdot\nabla\boldsymbol{u} = \boldsymbol{\pi}(-\boldsymbol{u}\times(\nabla\times\boldsymbol{u}) + \nabla\tfrac{1}{2}|\boldsymbol{u}|^2) = -\boldsymbol{u}\times\boldsymbol{n}\,\mathrm{curl_v}\,\boldsymbol{u} + \mathrm{grad}\,\tfrac{1}{2}|\boldsymbol{u}|^2, \tag{3.5}$$

Here we have followed (**4**) by defining 'grad' to give just the component of the gradient tangential to the surface,

$$\mathrm{grad}\,\chi \equiv \boldsymbol{\pi}\nabla\chi; \tag{3.6}$$

more formally we extend any scalar field into a neighbourhood of S as constant in the normal direction to S and then take the gradient in $\mathbb{R}^3$ (see appendix A).

This results in the Navier–Stokes equation written in a form similar to (2.7),

$$\rho(\partial_t\boldsymbol{u} - \boldsymbol{u}\times\boldsymbol{n}\,\mathrm{curl_v}\,\boldsymbol{u} + \mathrm{grad}\,\tfrac{1}{2}|\boldsymbol{u}|^2) = -\,\mathrm{grad}\,p + \mu\boldsymbol{d}, \quad \mathrm{div}\,\boldsymbol{u} = 0. \tag{3.7}$$

In this equation the advective term is clearly tangential to the surface S. Finally one uses the curl operators to give definition CS1 (3.9) below for the viscous term, which closely mirrors NS1 (2.2).

3.2 *Navier-Stokes through intrinsic geometry*

The second approach is to consider the intrinsic geometry of the surface S and use the machinery of differential geometry (see, for example, (**19**, **20**) for background and notation used here), starting with the metric g_{ij} (**4**, **21**). There is no need to consider S as embedded in any other space, although if it is embedded in $\mathbb{R}^3$ the Euclidean metric in $\mathbb{R}^3$ naturally induces a metric on S. We give the details in the case of a sphere below, but in general the Navier–Stokes equation now takes the form

$$\rho(\partial_t U^i + U^j\nabla_j U^i) = -\nabla^i p + \mu D^i, \quad \nabla_i U^i = 0. \tag{3.8}$$

Here we have used capital letters to denote contravariant components (with upper subscripts) and covariant components (with lower subscripts) of quantities such as velocity field $\boldsymbol{u}$ and stress vector $\boldsymbol{d}$. The metric can be used to raise and lower components and ∇_i is the covariant derivative. We retain lower case letters to denote the physical components (**3**, **8**) of quantities, that is components referred to an orthonormal basis; see section 3.4. Within this framework, one may define the Laplacian below by CS2 (3.10) which mirrors NS2 (2.3), or by taking the divergence of a stress tensor, definition CS3 (3.11) analogous to NS3 (2.4).

3.3 *Definitions of the viscous term*

In this way we obtain three distinct definitions of the viscous term on a curved surface ('CS') by analogy with NS1–3, which we label by Δ_c, Δ_g and Δ_T:

Definition CS1: $\boldsymbol{d} = \Delta_\mathrm{c}\boldsymbol{u}$ given in terms of curl operators by

$$\boldsymbol{d} = -\,\mathrm{curl_s}\,\mathrm{curl_v}\,\boldsymbol{u}, \tag{3.9}$$

(this is the topologist's Laplacian, or Laplace–de Rham operator, for a divergence-free field; see (**3**, **4**)).

Definition CS2: $\boldsymbol{d} = \Delta_{\mathrm{g}}\boldsymbol{u}$ defined using a metric

$$D^k = g^{ij}\nabla_i\nabla_j U^k, \tag{3.10}$$

(this is the analyst's Laplacian).

Definition CS3: $\boldsymbol{d} = \Delta_{\mathrm{T}}\boldsymbol{u}$ obtained in terms of a stress tensor

$$D^j = \nabla_i T^{ij}, \quad T^{ij} = \nabla^i U^j + \nabla^j U^i. \tag{3.11}$$

We will discuss these different cases in generality shortly, after we have given the example of a unit sphere.

3.4 *Viscous flow on a unit sphere*

To illustrate these definitions, and because this is the most important application, we consider flow on a unit sphere, with the usual spherical polar coordinates (θ, ϕ) (**6**, **9**, **10**). We write velocity $\boldsymbol{u}$ in terms of *physical components* defined by $\boldsymbol{u} = u\hat{\boldsymbol{\theta}} + v\hat{\boldsymbol{\phi}}$ where $\hat{\boldsymbol{\theta}}$ and $\hat{\boldsymbol{\phi}}$ are unit vectors; see (**3**, **8**), and (3.18) below). The two-dimensional Navier–Stokes equation from (3.7) are:

$$\rho(\partial_t u + u\partial_\theta u + s^{-1}v\partial_\phi u - s^{-1}c\,vv) = -\partial_\theta p + \mu d_\theta, \tag{3.12}$$

$$\rho(\partial_t v + u\partial_\theta v + s^{-1}v\partial_\phi v + s^{-1}c\,uv) = -s^{-1}\partial_\phi p + \mu d_\phi, \tag{3.13}$$

$$s^{-1}\partial_\theta(su) + s^{-1}\partial_\phi v = 0. \tag{3.14}$$

For convenience we have abbreviated $s = \sin\theta$ and $c = \cos\theta$. It is easiest to calculate the non-linear term as $\boldsymbol{\pi u}\cdot\nabla\boldsymbol{u}$ directly from standard equations for flow in three dimensional spherical polar coordinates.

Now we proceed to calculate the various viscous terms. The normal vector field to the sphere is simply $\boldsymbol{n} = \hat{\boldsymbol{r}}$ and following through the definition CS1 (3.9) the components of the stress vector are

$$d_\theta = (\Delta_{\mathrm{c}}\boldsymbol{u})_\theta = \Delta u - 2s^{-2}c\partial_\phi v - s^{-2}u, \tag{3.15}$$

$$d_\phi = (\Delta_{\mathrm{c}}\boldsymbol{u})_\phi = \Delta v + 2s^{-2}c\partial_\phi u - s^{-2}v. \tag{3.16}$$

(To obtain this in the above form it is convenient to add the vanishing term $\nabla\nabla\cdot\boldsymbol{u}$ to $\boldsymbol{d}$.) Here the scalar Laplacian is defined by

$$\Delta\chi = s^{-1}\partial_\theta(s\partial_\theta\chi) + s^{-2}\partial_\phi^2\chi = -\operatorname{curl}_{\mathrm{v}}\operatorname{curl}_{\mathrm{s}}\chi. \tag{3.17}$$

To calculate the viscous term for definitions CS2 (3.10) and CS3 (3.11) we need to define the flow in terms of components U^i or U_i with

$$u = U^1 = U_1, \quad v = sU^2 = s^{-1}U_2. \tag{3.18}$$

Here the *contravariant components* U^i are defined in standard fashion so that

$$\boldsymbol{u}\cdot\nabla\chi = (u\partial_\theta + s^{-1}v\partial_\phi)\chi = (U^1\partial_\theta + U^2\partial_\phi)\chi = U^i\partial_i\chi. \tag{3.19}$$

acting on any scalar field χ (with $\partial_\theta \equiv \partial_1$, $\partial_\phi \equiv \partial_2$). The *covariant components* are obtained using the metric to lower the index, resulting in (3.18) (see appendix B).

From (3.8) the Navier–Stokes equations become

$$\rho(\partial_t U + U\partial_\theta U + V\partial_\phi U - scVV) = -\partial_\theta p + \mu D^1, \tag{3.20}$$
$$\rho(\partial_t V + U\partial_\theta V + V\partial_\phi V + 2s^{-1}cUV) = -\partial_\phi p + \mu D^2, \tag{3.21}$$
$$\partial_\theta U + \partial_\phi V + s^{-1}cU = 0. \tag{3.22}$$

where we have abbreviated $U = U^1$, $V = U^2$ and where the stress vector has components

$$d_\theta = D^1 = D_1, \quad d_\phi = sD^2 = s^{-1}D_2. \tag{3.23}$$

We omit further details, which may be found in appendix B. The left-hand sides of equations (3.20–3.22) are then the same as (3.12–3.14) but we obtain for the right-hand side different definitions CS2 and CS3 of the viscous term given in (3.10) and (3.11). When these are now moved back into the physical components d_θ, d_ϕ we find

$$d_\theta = (\Delta_{\mathrm{g}}\boldsymbol{u})_\theta = \Delta u - 2s^{-2}c\partial_\phi v - s^{-2}c^2u, \tag{3.24}$$
$$d_\phi = (\Delta_{\mathrm{g}}\boldsymbol{u})_\phi = \Delta v + 2s^{-2}c\partial_\phi u - s^{-2}c^2v, \tag{3.25}$$

for CS2 and

$$d_\theta = (\Delta_{\mathrm{T}}\boldsymbol{u})_\theta = \Delta u - 2s^{-2}c\partial_\phi v + s^{-2}(s^2 - c^2)u, \tag{3.26}$$
$$d_\phi = (\Delta_{\mathrm{T}}\boldsymbol{u})_\phi = \Delta v + 2s^{-2}c\partial_\phi u + s^{-2}(s^2 - c^2)v \tag{3.27}$$

for CS3, obtained by Williams (**9**), by approximating the full three-dimensional stress tensor in a thin layer, and Becker (**10**). We may summarise the results for flow on a unit sphere very simply: the definitions CS1 (3.9), CS2 (3.10) and CS3 (3.11) are related by

$$\Delta_{\mathrm{T}}\boldsymbol{u} = \Delta_{\mathrm{g}}\boldsymbol{u} + \boldsymbol{u} = \Delta_{\mathrm{c}}\boldsymbol{u} + 2\boldsymbol{u}. \tag{3.28}$$

The right-hand equality is discussed in (**4**) and (**6**). To decide which of the dissipative terms is most physically sensible we should recall that when dissipation is treated in terms of the divergence of a stress tensor, for example in (**12**), it is the consideration of angular momentum of small fluid elements that requires symmetry of the stress tensor $\boldsymbol{t}$. If this is not enforced then angular momentum conservation will not generally hold, and this is the problem with definitions CS1 and CS2. To see this consider first a flow rotating with uniform angular velocity a about the polar axis, with velocity and vorticity $\omega = \mathrm{curl_v}\,\boldsymbol{u}$:

$$u = 0, \quad v = a\sin\theta, \quad \omega = 2a\cos\theta. \tag{3.29}$$

Here the only definition that gives zero dissipation is CS3 (**9**). In fact for this flow the stress tensor $\boldsymbol{t}$ vanishes, as indeed it should: in a locally flat region of the sphere each fluid element undergoes no net strain. The same applies for a more general flow consisting of solid body rotation about an arbitrary axis $\boldsymbol{a}$ having polar angles θ_0 and ϕ_0 (**10**). We set $\boldsymbol{u} = \boldsymbol{a}\times\boldsymbol{r}$ and obtain the velocity components and vorticity as

$$u = -a\sin\theta_0\sin(\phi - \phi_0), \tag{3.30}$$
$$v = a[\sin\theta\cos\theta_0 - \cos\theta\sin\theta_0\cos(\phi - \phi_0)], \tag{3.31}$$
$$\omega = 2a[\cos\theta\cos\theta_0 + \sin\theta\sin\theta_0\cos(\phi - \phi_0)]. \tag{3.32}$$

Again the dissipative term only vanishes for definition CS3 (**10**). Note that the approach of requiring that families of flows (here only solid body motions) are undamped can be used to constrain the form of a dissipative term, as in the design of the three-dimensional UK Met Office Unified Model (**22**). We also stress that of the forms CS1, CS2, CS3 of dissipation, it is only CS3 that is derived from the basic physics of a fluid flow: the force is the divergence of a stress tensor (from the appropriate application of the divergence theorem), and this is symmetric from consideration of the local angular momentum balance for fluid elements (**3**, **8**, **12**).

The vorticity on the sphere is given by

$$\omega = \mathrm{curl}_{\mathrm{v}}\, \boldsymbol{u} = s^{-1}[\partial_\theta(sv) - \partial_\phi u] \tag{3.33}$$

and satisfies

$$\rho(\partial_t \omega + \boldsymbol{u}\cdot\nabla\omega) = \mu(\Delta\omega + 2\omega) \tag{3.34}$$

in the case of CS3 (with the obvious replacement by $\Delta\omega$ for CS1, $\Delta\omega+\omega$ for CS2). Here the Laplacian is the scalar Laplacian (3.17) and the extra term 2ω ensures that these solid body motions are not viscously damped. To see that a general fluid motion is viscously damped towards a state of solid body rotation, we expand the vorticity field in spherical harmonics,

$$\omega(\theta,\phi) = \sum_{l,m} \hat{\omega}_{lm} Y_{lm}(\theta,\phi) \tag{3.35}$$

(noting that there is no $l = m = 0$ mode as ω necessarily has zero mean). Then, the effect of the viscous term in (3.34) is to damp all modes except for $l = 1$, $m = 0, \pm 1$, since $\Delta Y_{lm} = -l(l+1)Y_{lm}$. This is consistent with our earlier discussion as these are precisely the flows of solid body rotation on the sphere.

3.5 *Viscous flow on an arbitrary curved surface*

We have written down the three definitions CS1, CS2 and CS3 of the viscous term in generality in section 3.3 and noted that they take different forms on a sphere S in section 3.4, related by (3.28), simply the addition of a multiple of the velocity. On a more general surface the origin of this difference may be seen to arise because of an exchange of derivatives that introduces curvature terms. The standard definition we need is that

$$(\nabla_k\nabla_l - \nabla_l\nabla_k)U^i = R^i{}_{jkl}U^j, \tag{3.36}$$

where $R^i{}_{jkl}$ is the Riemann tensor, vanishing in a flat space. In this more general context the three definitions are related by

$$(\Delta_{\mathrm{T}}\boldsymbol{u})^i = (\Delta_{\mathrm{g}}\boldsymbol{u})^i + R^i{}_jU^j = (\Delta_{\mathrm{c}}\boldsymbol{u})^i + 2R^i{}_jU^j, \tag{3.37}$$

where $R_{jl} = R^i{}_{jil}$ is the (symmetric) Ricci tensor (**8**). On a sphere of radius r,

$$R^i{}_j = r^{-2}\delta^i{}_j, \tag{3.38}$$

and we recover the results for a unit sphere in section 3.4. In the planar limit $r\to\infty$ the three definitions agree, as they should, while for a general surface the Ricci tensor gives a term dependent on the local curvature described as the 'force due to intrinsic curvature' in (**8**).

To show that the left-hand identity in (3.37) holds is straightforward. Taking the divergence of $\boldsymbol{t}$ using (3.11) and using $\nabla_i U^i = 0$ yields

$$(\Delta_{\mathrm{T}}\boldsymbol{u})^j = \nabla_i T^{ij} = g^{ik}\nabla_i\nabla_k U^j + g^{jk}\nabla_i\nabla_k U^i \tag{3.39}$$

$$= (\Delta_{\mathrm{g}}\boldsymbol{u})^j + g^{jk}(\nabla_i\nabla_k - \nabla_k\nabla_i)U^i, \tag{3.40}$$

and the latter term is

$$g^{jk}(\nabla_i\nabla_k - \nabla_k\nabla_i)U^i = g^{jk}R^i{}_{lik}U^l = g^{jk}R_{lk}U^l = R_l{}^jU^l = R^j_{\ l}U^l. \tag{3.41}$$

This establishes the left-hand of the two identities in (3.37). For the right-hand of the identities, given in (**1**), we first define

$$\eta_{ij} = |g|^{1/2}(\delta^1_{\ i}\delta^2_{\ j} - \delta^2_{\ i}\delta^1_{\ j}), \quad g = \det(g_{ij}) \tag{3.42}$$

as the volume two-form on S, which satisfies $\nabla_i\eta_{jk} = 0$ and $\eta^{ij}\eta_{kl} = \delta^i_{\ k}\delta^j_{\ l}$. With this the two approaches may be linked together via (from appendix A)

$$\eta_{ij}\nabla^iU^j = \mathrm{curl_v}\,\boldsymbol{u}, \quad \eta^{ij}\nabla_j\chi = (\mathrm{curl_s}\,\chi)^i. \tag{3.43}$$

Then, in intrinsic coordinates the calculation beginning from (3.9) is

$$\begin{aligned}(\Delta_{\mathrm{c}}\boldsymbol{u})^k &= -\eta^{kl}\nabla_l(\eta_{ij}\nabla^iU^j) = -\eta^{kl}\eta_{ij}\nabla_l\nabla^iU^j\\ &= -(\delta^k_{\ i}\delta^l_{\ j} - \delta^k_{\ j}\delta^l_{\ i})\nabla_l\nabla^iU^j = -\nabla_j\nabla^kU^j + \nabla_i\nabla^iU^k\\ &= (\nabla^k\nabla_j - \nabla_j\nabla^k)U^j + \nabla_i\nabla^iU^k = -R^k_{\ l}U^l + \nabla_i\nabla^iU^k\\ &= (\Delta_{\mathrm{g}}\boldsymbol{u})^k - R^k_{\ l}U^l,\end{aligned} \tag{3.44}$$

and the contribution from the Ricci tensor naturally appears on the interchange of covariant derivatives. Finally we note that for the advective term, Il'in (**4**) shows that the left-hand sides of the two versions (3.7) and (3.8) are equivalent.

Staying with CS3, the energy

$$K = \tfrac{1}{2}\rho\int_S U^iU_i\,dV \tag{3.45}$$

has rate of change written in the standard form in terms of T_{ij} and the rate of strain tensor $E_{ij} = \frac{1}{2}T_{ij}$ (in this context of incompressible flow) as

$$\mu^{-1}\partial_t K = -\int_S E_{ij}T^{ij}\,dV \tag{3.46}$$

(there is no surface term given that S is closed). This quantity is negative semi-definite, only vanishing if E_{ij} is identically zero, which amounts to

$$\nabla_iU^j + \nabla_jU^i = 0 \tag{3.47}$$

and is satisfied when the vector field $\boldsymbol{u}$ is a Killing vector field (automatically divergence-free). In this case the flow $\boldsymbol{u}$ generates a one-parameter family of symmetries of S, that is metric-preserving diffeomorphisms. In the case of a sphere S there are two such fields, corresponding to uniform solid body motion about an arbitrary axis, while for an oblate ellipsoid there would only be one. For a general S possessing no symmetries there are no such rate-of-strain-free flows and any motion must come to zero under the action of dissipation CS3.

3.6 *Vorticity equation*

It is natural to seek a vorticity–stream function formulation of any two-dimensional flow. The problem of point vortex dynamics on spheres and other surfaces is well studied (**23**, **24**, **25**) but here our focus is on the viscous term. The first approach above, giving (3.7) with (3.9), leads very naturally to a vorticity equation (**1**, **4**). The divergence free condition for a flow $\boldsymbol{u}$ may be satisfied by the use of a stream function

$$\boldsymbol{u} = \operatorname{curl}_{\mathrm{s}} \psi \tag{3.48}$$

(though there may be flow components that cannot be represented in this fashion depending on the topology of the surface S, for example uniform flows on a torus (**1**, **3**)). We may also define a vorticity

$$\omega = \operatorname{curl}_{\mathrm{v}} \boldsymbol{u}. \tag{3.49}$$

In this formulation the vorticity–stream function link becomes

$$\omega = \operatorname{curl}_{\mathrm{v}} \operatorname{curl}_{\mathrm{s}} \psi = -\operatorname{div} \operatorname{grad} \psi \equiv -\Delta\psi. \tag{3.50}$$

Taking the $\operatorname{curl}_{\mathrm{v}}$ of (3.7) to eliminate the pressure and obtain the vorticity equation gives

$$\rho[\partial_t \omega - \operatorname{curl}_{\mathrm{v}}(\boldsymbol{u} \times \boldsymbol{n}\omega)] = \mu \operatorname{curl}_{\mathrm{v}} \boldsymbol{d}. \tag{3.51}$$

The advective term here can be rewritten as

$$-\operatorname{curl}_{\mathrm{v}}(\boldsymbol{u} \times \boldsymbol{n}\omega) = \boldsymbol{u} \cdot \nabla\omega, \tag{3.52}$$

or, introducing the stream function, as

$$J(\omega, \psi) = \boldsymbol{n} \cdot (\nabla\omega) \times (\nabla\psi). \tag{3.53}$$

We recover the usual material transport of vorticity in the first term. With the form of dissipation given in CS1 (3.9), the vorticity equation becomes

$$\rho[\partial_t \omega + J(\omega, \psi)] = -\mu \operatorname{curl}_{\mathrm{v}} \operatorname{curl}_{\mathrm{s}} \omega = \mu\Delta\omega, \tag{3.54}$$

which is attractive for analysis as what now appears on the right-hand side is simply the Laplacian applied to a scalar field. However for CS3, we now obtain an extra term involving the Ricci tensor R_{ij}, written here as $\boldsymbol{r}$ in physical coordinates (not to be confused with the position vector):

$$\rho[\partial_t \omega + J(\omega, \psi)] = \mu[\Delta\omega + 2 \operatorname{curl}_{\mathrm{v}}(\boldsymbol{r} \cdot \operatorname{curl}_{\mathrm{s}} \psi)]. \tag{3.55}$$

In intrinsic geometry the vorticity is given by

$$\omega = \eta_{ij} \nabla^i U^j, \tag{3.56}$$

whence

$$\omega = |g|^{1/2} (\nabla^1 U^2 - \nabla^2 U^1). \tag{3.57}$$

In spherical geometry this is easily checked to give (3.33) (noting that $g = s^2$ here). We can also generally define a stream function via

$$U_i = \eta_{ij} \nabla^j \psi, \tag{3.58}$$

so that $\omega = -\nabla^i \nabla_i \psi$ corresponding to (3.50).

We may apply $\eta_{li}\nabla^l$ to the equation (3.8) to eliminate the pressure giving, in parallel with the standard calculation in Euclidean space,

$$\partial_t\omega + \eta_{ij}(\nabla^i\omega)(\nabla^j\psi) = \mu\eta_{ij}\nabla^i D^j. \tag{3.59}$$

For CS3 the dissipative term results in

$$\begin{aligned}\partial_t\omega + \eta_{ij}(\nabla^i\omega)(\nabla^j\psi) &= \mu[\nabla_i\nabla^i\omega + 2\eta^{ij}\eta^{kl}\nabla_i(R_{jk}\nabla_l\psi)] &(3.60)\\ &= \mu[\nabla_i\nabla^i\omega + 2\nabla^i[(R_{ij} - Rg_{ij})\nabla^j\psi]] &(3.61)\end{aligned}$$

(cf. 3.55) with $R = R^j{}_j$. This gives the correct formula obtained above for the special case of a sphere.

4. Viscous term for shallow water equations on a flat surface

4.1 *Definitions*

We now consider the shallow water equations for an incompressible, constant density fluid on a (horizontal) plane, that is flat geometry. Conservation of mass gives the equation for the height $h(x, y, t)$ linked to the flow field $\boldsymbol{u}(x, y, t)$

$$\partial_t h + \nabla\cdot(h\boldsymbol{u}) = 0, \tag{4.1}$$

while the momentum equation may be written in the conservative form

$$\rho[\partial_t(h\boldsymbol{u}) + \nabla\cdot(h\boldsymbol{u}\otimes\boldsymbol{u})] + \rho g\nabla\tfrac{1}{2}h^2 = \mu h\boldsymbol{d}, \tag{4.2}$$

or as

$$\rho[\partial_t\boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u}] + \rho g\nabla h = \mu\boldsymbol{d}. \tag{4.3}$$

For later use, the components of the momentum equation in terms of polar coordinates are

$$\rho[\partial_t u + u\partial_r u + r^{-1}v\partial_\phi u - r^{-1}v^2] + \rho g\partial_r h = \mu d_r, \tag{4.4}$$

$$\rho[\partial_t v + u\partial_r v + r^{-1}v\partial_\phi v + r^{-1}uv] + \rho g r^{-1}\partial_\phi h = \mu d_\phi. \tag{4.5}$$

We consider three possible dissipative terms $\boldsymbol{d}$ from the literature:

Definition SW1: simply a Laplacian in $\boldsymbol{u}$:

$$\boldsymbol{d} = \nabla^2\boldsymbol{u}, \tag{4.6}$$

Definition SW2: a Laplacian but weighted by the fluid depth:

$$\boldsymbol{d} = h^{-1}\nabla\cdot(h\nabla\boldsymbol{u}), \tag{4.7}$$

Definition SW3: the divergence of a stress tensor, weighted by the fluid depth

$$\boldsymbol{d} = h^{-1}\nabla\cdot(h\boldsymbol{t}), \quad \boldsymbol{t} = \nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T - \varsigma\boldsymbol{I}(\nabla\cdot\boldsymbol{u}). \tag{4.8}$$

Here $\boldsymbol{I}$ is the 2×2 identity matrix and we have included a constant ς, whose value we will discuss, to allow for different forms of the stress tensor. In the case $\varsigma = 1$ the stress tensor $\boldsymbol{t}$ is divergence free; note that the rate of strain tensor $\boldsymbol{e}$ is still given by (2.5).

Note that for SW2 we have

$$\boldsymbol{d} = \nabla^2 \boldsymbol{u} + h^{-1}(\nabla h) \cdot \nabla \boldsymbol{u}, \tag{4.9}$$

while for SW3,

$$\boldsymbol{d} = \nabla^2 \boldsymbol{u} + (1 - \varsigma)\nabla(\nabla \cdot \boldsymbol{u}) + h^{-1}(\nabla h) \cdot \boldsymbol{t}. \tag{4.10}$$

The term SW2 is discussed by (**13**) (and references therein) who also considers a damping term in the height equation (4.1). The term SW3 is put forward by (**14**) with $\varsigma = 1$, by (**18**) with $\zeta = -2$, and discussed in generality by (**15**). All three viscous terms, (4.6), (4.7) and (4.8) with $\varsigma = 1$, are discussed from the point of view of energy and angular momentum by (**16**), and this paper also considers further generalisations.

4.2 *Dissipation of energy*

In the shallow water system, the total energy (kinetic plus potential) is given by

$$K = \tfrac{1}{2}\rho \int_S (h\boldsymbol{u}^2 + gh^2)\, dS \tag{4.11}$$

in any region S in the plane with normal $\boldsymbol{n}$ and boundary ∂S, and

$$\partial_t K = -\rho \int_{\partial S} \boldsymbol{n} \cdot \boldsymbol{u}(gh^2 + \tfrac{1}{2}h|\boldsymbol{u}|^2)\, ds + \mu \int_S h\boldsymbol{u} \cdot \boldsymbol{d}\, dS. \tag{4.12}$$

With suitable boundary conditions, for example $\boldsymbol{n} \cdot \boldsymbol{u} = 0$, and no dissipation $\mu = 0$ energy is conserved.

It is clearly an important property of any model for dissipation that the energy can only decay (or be constant) and this immediately enables us to eliminate SW1: for this form of dissipation the energy equation is

$$\partial_t K = \mu \int_{\partial S} \boldsymbol{n} \cdot h\nabla \tfrac{1}{2}|\boldsymbol{u}|^2\, ds - \mu \int_S h(\partial_i u_j)(\partial_i u_j)\, dS - \mu \int_S (\nabla h) \cdot \nabla \tfrac{1}{2}|\boldsymbol{u}|^2\, dS \tag{4.13}$$

(taking $\boldsymbol{n} \cdot \boldsymbol{u} = 0$ to eliminate the advective surface term in (4.12)). Now the second, surface integral term is negative definite but the third term could take either sign *a priori*. There would generally be sources (and sinks) of energy. Even if only local, they are unphysical. For this reason we eliminate SW1 from consideration, our discussion following that in (**13**) and (**14**). Note that we exhibit the boundary integrals (over ∂S) in (4.13–4.15) but we are concerned with dissipation in the bulk, and in any case these vanish under suitable conditions.

For SW2 and SW3 the corresponding energy equations are, respectively,

$$\partial_t K = \mu \int_{\partial S} \boldsymbol{n} \cdot h\nabla \tfrac{1}{2}|\boldsymbol{u}|^2\, ds - \mu \int_S h(\partial_i u_j)(\partial_i u_j)\, dS, \tag{4.14}$$

and

$$\partial_t K = \mu \int_{\partial S} n_i t_{ij} h u_j\, ds - \mu \int_S h e_{ij} t_{ij}\, dS. \tag{4.15}$$

In the case of SW2 the surface integral piece is clearly negative definite, driving the flow to a state of zero motion. For SW3 it is easily checked that

$$e_{ij}t_{ij} = (e_{11} - e_{22})^2 + (1 - \varsigma)(e_{11} + e_{22})^2 + 4e_{12}^2 \geq 0, \tag{4.16}$$

so the dissipative term is negative semi-definite provided $\varsigma \leq 1$ (**14**). If $\varsigma < 1$ the fluid will be driven to a state of zero e_{ij} and so solid body motion. If $\varsigma = 1$ then in addition pure compression $e_{11} = e_{22}$, $e_{12} = 0$ is possible without dissipation. For discussion of this and more general models see also (**16**).

4.3 *Example of rotating bucket flow*

Having eliminated SW1 from consideration, we consider rotating flow under SW2 and SW3. To see which of these is more physically sensible, we turn to flow under steady solid body rotation with a free surface, given by

$$u = 0, \quad v = ar, \quad h = h_0 + \tfrac{1}{2}a^2r^2/g, \tag{4.17}$$

in polar coodinates. This classical solution satisfies the full three-dimensional viscous Navier–Stokes equations (with vertical velocity $w = 0$). It also satisfies the inviscid shallow water equations (4.4, 4.5) with $\mu = 0$.

Now we consider the various viscous terms: the flow satisfies $\nabla^2 \boldsymbol{u} = 0$ so remains a steady flow under the viscous term SW1. However we have already ruled out that term on the basis of energy conservation. The term SW2 is

$$d_r = (\nabla^2 - r^{-2})u - 2r^{-2}\partial_\phi v + h^{-1}[(\nabla h)\cdot\nabla u - r^{-2}(\partial_\phi h)v], \tag{4.18}$$

$$d_\phi = (\nabla^2 - r^{-2})v + 2r^{-2}\partial_\phi u + h^{-1}[(\nabla h)\cdot\nabla v + r^{-2}(\partial_\phi h)u]. \tag{4.19}$$

in polar coordinates using now (u, v) as polar components of the flow. It does not vanish for this flow but has a non-zero ϕ-component of $d_\phi = h^{-1}a^3r$. Thus we rule out SW2 on the basis that we do not recover this solution of the full Navier–Stokes equation.

Fortunately SW3 has a vanishing viscous term, which is most easily checked since the stress tensor $\boldsymbol{t}$ vanishes for solid body motion. The stress tensor has components

$$t_{rr} = (1-\varsigma)2\partial_r u + \varsigma(\partial_r u - r^{-1}\partial_\phi v - r^{-1}u), \tag{4.20}$$

$$t_{\phi\phi} = (1-\varsigma)2(r^{-1}\partial_\phi v + r^{-1}u) + \varsigma(r^{-1}\partial_\phi v + r^{-1}u - \partial_r u), \tag{4.21}$$

$$t_{r\phi} = t_{\phi r} = \partial_r v + r^{-1}\partial_\phi u - r^{-1}v. \tag{4.22}$$

We conclude that the stress tensor formulation gives the most physically sensible approach to dissipation in this system, and so would argue for the use of SW3, although the value of ς remains to be discussed.

It is interesting to determine which flows are steady solutions under the viscous term SW2. For a steady flow with $u = 0$, $v = v(r)$, $h = h(r)$ we obtain from the u and v equations (4.4, 4.5) with (4.18, 4.19),

$$-r^{-1}v^2 + g\partial_r h = 0, \tag{4.23}$$

$$0 = \partial_r^2 v + r^{-1}\partial_r v - r^{-2}v + h^{-1}(\partial_r h)(\partial_r v). \tag{4.24}$$

With boundary conditions $h(0) = h_0$, $v'(0) = a$, $h'(0) = 0$ we can rescale to obtain

$$v = (gh_0)^{1/2}v^*(r^*), \quad h = h_0h^*(r^*), \quad r = (gh_0)^{1/2}a^{-1}r^* \tag{4.25}$$

where, dropping the stars,

$$-r^{-1}v^2 + \partial_r h = 0, \quad 0 = \partial_r^2 v + r^{-1}\partial_r v - r^{-2}v + h^{-1}(\partial_r h)(\partial_r v), \tag{4.26}$$

$$v(0) = 0, \quad v'(0) = 1, \quad h(0) = 1. \tag{4.27}$$

For small r the solution is

$$v = r - \tfrac{1}{8}r^3 + \tfrac{3}{96}r^5 + \cdots, \quad h = 1 + \tfrac{1}{2}r^2 - \tfrac{1}{16}r^4 + \cdots, \tag{4.28}$$

which may be compared with (4.17). To see the behaviour for larger, r figure 1 shows the solution for SW2 in solid and the correct solution for SW3 (4.17) dashed. It may be seen that SW2 leads to slower azimuthal and angular velocities as r increases, and with this the increase in the height h with distance is much reduced.

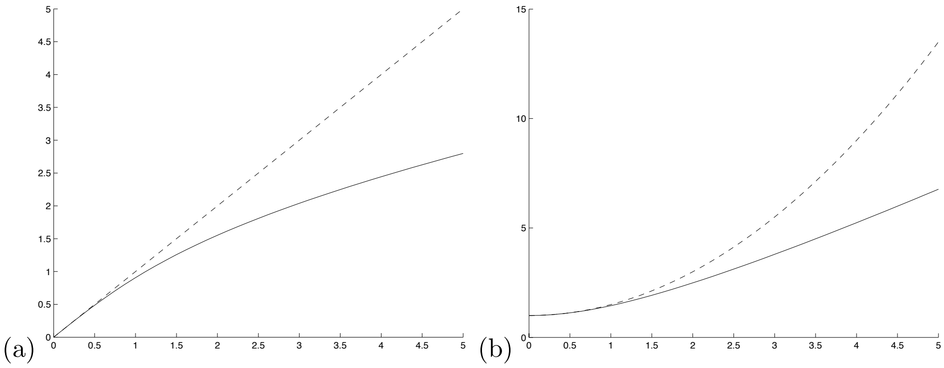

Fig. 1 Solution for steady rotating flow in dimensionless variables. In (a) the azimuthal velocity v is shown and in (b) the height h. Solid curves show the results for SW2, obtained by solving (4.26) with (4.27). For comparison the dashed curves show the correct solution (4.17) which is consistent with SW3.

4.4 *SW3 as a truncation of the three-dimensional Navier–Stokes approximation*

We have argued for SW3 as the appropriate dissipative term for the shallow water equations but not yet determined a possible value for ς in (4.8). Here we will consider how SW3 arises as a truncation of the full Navier–Stokes equation in which the flow field and pressure field are constrained to take the form

$$\bar{\boldsymbol{u}} = (u(x,y,t), v(x,y,t), z\lambda(x,y,t)), \tag{4.29}$$

$$\bar{p} = \rho g(h(x,y,t) - z), \tag{4.30}$$

in Cartesian coordinates and components. Our aim is not to discuss which physical effects (for example rotation or shallowness of the fluid layer) constrain the fluid motion, but simply what stresses arise if this form is assumed. We will substitute this into the full three-dimensional equations and integrate the stress in such a way as to preserve the form in (4.29), (4.30). As such we are discussing a truncation rather than an asymptotic approximation. For discussion of the inviscid shallow water equations as an approximation and its limitations see (**26**) and references therein, and with viscosity see (**17**, **18**).

We will use a bar to denote quantities and derivatives in three dimensions, where the notation duplicates that we have used so far for two dimensions. We take $\boldsymbol{u} = (u, v)$, $\bar{\boldsymbol{u}}$ as in (4.29), $\nabla = (\partial_x, \partial_y)$ and $\bar{\nabla} = (\partial_x, \partial_y, \partial_z)$, for example. The three dimensional fluid is taken to be incompressible, so that

$$\bar{\nabla} \cdot \bar{\boldsymbol{u}} = \partial_x u + \partial_y v + \lambda = 0. \tag{4.31}$$

Note that the shallow water mass conservation equation (4.1) becomes

$$D_t h = h\lambda \qquad (D_t \equiv \partial_t + \boldsymbol{u} \cdot \nabla). \tag{4.32}$$

We take the usual three dimensional stress tensor for the full incompressible flow (4.29) given by

$$\bar{\boldsymbol{t}} = \bar{\nabla}\bar{\boldsymbol{u}} + (\bar{\nabla}\bar{\boldsymbol{u}})^T = \begin{pmatrix} 2\partial_x u & \partial_y u + \partial_x v & z\partial_x\lambda \\ \partial_y u + \partial_x v & 2\partial_y v & z\partial_y\lambda \\ z\partial_x\lambda & z\partial_y\lambda & 2\lambda \end{pmatrix}. \tag{4.33}$$

Now consider a curve C bounding an area S in the (x, y)-plane, and let V be the three-dimensional cylinder of fluid with base S, extending to the free surface $z = h(x, y)$. The unit vector field $\boldsymbol{n}$ is normal to the curve C in the (x, y) plane, while the unit vector field $\bar{\boldsymbol{n}}$ is normal to all of V in three dimensions: the two coincide on the vertical surfaces of V.

We calculate the integrated force on the volume V. Dealing with pressure is standard: the horizontal component of the total force from pressure is

$$\boldsymbol{f}_V = -\int_{\partial V} \bar{p}\boldsymbol{n}\, dS. \tag{4.34}$$

This arises only from the vertical sides of the surface S bounding V (the pressure is zero on top and at the base it has only a vertical component) and so we may integrate p in (4.30) over the vertical coordinate z from zero to h to give

$$\boldsymbol{f}_V = -\rho g \int_C \tfrac{1}{2}h^2 \boldsymbol{n}\, ds = -\rho g \int_S \nabla \tfrac{1}{2}h^2\, dS. \tag{4.35}$$

The surface integral over S that results corresponds precisely to the gradient term in (4.2).

We now turn to the total viscous stress on the volume V, which is

$$\bar{\boldsymbol{g}}_V = \mu \int_{\partial V} \bar{\boldsymbol{n}} \cdot \bar{\boldsymbol{t}}\, dS = \mu \int_V \bar{\nabla} \cdot \bar{\boldsymbol{t}}\, dV \tag{4.36}$$

by the divergence theorem. Using the explicit form (4.33) and (4.31) to calculate $\bar{\nabla} \cdot \bar{\boldsymbol{t}}$ we obtain the horizontal components of the force as

$$\boldsymbol{g}_V = \mu \int_V \nabla^2 \boldsymbol{u}\, dV, \tag{4.37}$$

which on performing the integral from zero to h gives

$$\boldsymbol{g}_V = \mu \int_S h \nabla^2 \boldsymbol{u}\, dS. \tag{4.38}$$

This is precisely the integrated form of SW1 (4.6), as it would appear on the right hand side of (4.2). However we have seen that SW1 is energetically inconsistent: how has this arisen from such a simple physical argument? The problem appears to be that the stress tensor $\bar{\boldsymbol{t}}$ does not vanish on the free surface $z = h(x, y, t)$ given its form (4.33). What we are observing is a contribution from unphysical stresses on the free surface. There is a contradiction between the dependence (4.29) and the imposition of a stress free boundary condition on the free surface. In reality, at large Reynolds number, this would be resolved by a boundary layer where the flow (4.29) adjusts to a stress free condition; in this boundary layer the form (4.29) must break down.

Although such layers could be treated in an asymptotic model, bringing in additional physics, within the truncated system, it is natural to attempt to fix this problem by simply taking the integral of the stress over the vertical sides of the cylinder V, omitting any contributions from the top or base, which could be absorbed into thin, unresolved boundary layers. In this case we calculate the horizontal component of the stress force $\boldsymbol{n} \cdot \bar{\boldsymbol{t}}$ integrated over the vertical cylinder surface. This involves only the 2×2 subblock of $\bar{\boldsymbol{t}}$ given by the two-dimensional stress tensor

$$\boldsymbol{t} = \nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T. \tag{4.39}$$

Integrating also over z from zero to h gives the horizontal components of the force as

$$\boldsymbol{g}_V = \mu \int_C h\boldsymbol{n} \cdot [\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T]\, ds, \tag{4.40}$$

and finally using the two-dimensional divergence theorem yields

$$\boldsymbol{g}_V = \mu \int_S \nabla \cdot [h(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^T)]\, dS. \tag{4.41}$$

This is the integrated form of (4.2) with SW3 (4.8) and $\varsigma = 0$, and this form of shallow water dissipation has properties suitable for idealised models.

Interestingly, the value $\varsigma = -2$ also has merits in building in more of the dissipation arising from the vertical flow $z\lambda(x, y, t)$. In a shallow water scaling argument based on a thin layer depth, Marche (**18**) retains the $t_{33} = 2\lambda = -2(\partial_x u + \partial_y v)$ term from (4.33) in the equation for the vertical component of velocity, where it contributes to a pressure of the form

$$\bar{p} = \rho g(h(x, y, t) - z) - 2\mu(\partial_x u + \partial_y v). \tag{4.42}$$

Integrating this additional component as in (4.35) gives a contribution to the horizontal momentum balance corresponding to $\varsigma = -2$ in (4.8).

It is worth noting the relation of the value of ς with the underlying physics. Consider a fluid flow $\boldsymbol{u} = (ax, -ay)$ with h constant, to be treated as a local approximation or as maintained somehow by external body forces. Then energy is dissipated by viscosity according to (4.16), regardless of the value of ς, as one would expect because of straining of fluid elements in a horizontal plane. However for the flow $\boldsymbol{u} = (ax, ay)$, $h = h_0 e^{-2at}$ there is no viscous loss of energy for $\varsigma = 1$ despite similar straining of individual elements, now taking place in a vertical plane. For $\varsigma = 0$ or -2, energy is dissipated viscously in both cases. In fact in the more general case of a strain flow $\boldsymbol{u} = (ax, by)$ in the plane, corresponding to $\bar{\boldsymbol{u}} = (ax, by, -(a + b)z)$ in three dimensions, the true dissipation is proportional to $2a^2 + 2b^2 + 2(a + b)^2$, which is also obtained for $\varsigma = -2$. For $\varsigma = 0$ we have $2a^2 + 2b^2$. For either value, $\varsigma = 0$ or -2, both forms of dissipation only vanish if $a = b = 0$ which is physically sensible.

5. Concluding remarks

We have considered the form of the viscous, dissipative term in the Navier–Stokes equation for flow on a two-dimensional surface, and flow in a plane governed by the shallow water equations. Our point of view has been to obtain a system with the best conservation properties, in particular giving conservation of angular momentum and dissipation of energy. In each case what we would consider the best formulation involves obtaining the stress vector from the argument based on the underlying physics, of taking the divergence of a symmetric stress tensor, CS3 in (3.11) and SW3 in (4.8) with $\varsigma = 0$ or -2 (**18**). In the former case, of flow on a surface it is notable that many papers in the literature instead make use of the Laplace–de Rham operator (CS1) or the Laplacian based on the metric (CS2). In the context of the shallow water equations there has been much discussion from the point of view of atmospheric and oceanic modelling, and the approach of using a symmetric stress tensor is generally accepted; less clear is the best form.

There are important areas we have not discussed as they involve the modelling of transport by waves and turbulence. In this case the effective dissipative term may be anisotropic, with differing horizontal and vertical components (in a three-dimensional model). We refer the reader to papers such as (**10**, **11**, **16**, **22**) for further discussion and note that in these cases it is still desirable to approximate the stress tensor to capture the appropriate phenomena, while keeping its symmetry for angular momentum conservation (**9**), rather than to approximate the equations after the divergence of the stress tensor is taken. We have also not concerned ourselves with hyperviscosity but note that that higher powers of the Laplacian generally lead to a loss of energy and angular momentum conservation (**10**, **16**).

To finish, we write down the equations for shallow water flow on a general curved surface S, based on the discussion above. In intrinsic coordinates we require

$$\rho[\partial_t U^i + U^j \nabla_j U^i) + \rho g \nabla^i h = \mu D^i, \tag{5.1}$$

$$\partial_t h + \nabla_i (h U^i) = 0 \tag{5.2}$$

with the equivalent of SW3 (4.8) for $\varsigma = 0$ (say)

$$D^j = h^{-1} \nabla_i (h T^{ij}) = \nabla_i T^{ij} + h^{-1} T^{ij} \nabla_i h, \quad T^{ij} = \nabla^i U^j + \nabla^j U^i. \tag{5.3}$$

So, in physical components, on a sphere, we obtain

$$\rho(\partial_t u + u\partial_\theta u + s^{-1} v \partial_\phi u - s^{-1} c\, vv) + \rho g \partial_\theta h = \mu d_\theta, \tag{5.4}$$

$$\rho(\partial_t v + u\partial_\theta v + s^{-1} v \partial_\phi v + s^{-1} c\, uv) + \rho g s^{-1} \partial_\phi h = \mu d_\phi, \tag{5.5}$$

$$\partial_t h + s^{-1} \partial_\theta (shu) + s^{-1} \partial_\phi (hv) = 0. \tag{5.6}$$

To calculate the dissipative terms we use $T_{ij} = 2E_{ij}$ given in (B.13–B.14) below (and not T_{ij} in (B.15–B.16)) to obtain (reinstating a general value of ς),

$$\begin{aligned} d_\theta &= (\Delta_{\mathrm{T}} \boldsymbol{u})_\theta + (1-\varsigma)\partial_\theta (\nabla \cdot \boldsymbol{u}) \\ &\quad + h^{-1}[(\partial_\theta h) 2(\partial_\theta u) + s^{-1}(\partial_\phi h)(\partial_\theta v + s^{-1}\partial_\phi u - s^{-1} cv)], \end{aligned} \tag{5.7}$$

$$\begin{aligned} d_\phi &= (\Delta_{\mathrm{T}} \boldsymbol{u})_\phi + (1-\varsigma) s^{-1} \partial_\phi (\nabla \cdot \boldsymbol{u}) \\ &\quad + h^{-1}[(\partial_\theta h)(\partial_\theta v + s^{-1}\partial_\phi u - s^{-1} cv) + s^{-1}(\partial_\phi h) 2(s^{-1}\partial_\phi v + s^{-1} cu)]. \end{aligned} \tag{5.8}$$

Here the terms in $\Delta_{\mathrm{T}}\boldsymbol{u}$ are listed in (3.26), (3.27), and $\nabla \cdot \boldsymbol{u} = T^i_i$ is given in (3.14) above. For a steady solid body axisymmetric motion $u = 0$, $v = as$, $h = h(\theta)$ the tensor T_{ij} vanishes and thus so do all the dissipative terms, leaving $h = h_0 + \frac{1}{2}a^2 s^2/g$ determined by (5.4), analogously to (4.17). Note that an equation for viscous shallow water flow on a sphere is written down in (**15**) but using a traceless symmetric stress tensor, (4.8) with $\varsigma = 1$; we would argue that it is preferable to take $\varsigma = 0$ or $\varsigma = -2$ (**18**).

APPENDIX A

Here we summarise the analysis of Il'in (**4**) concerning how the operators such as $\mathrm{curl}_{\mathrm{v}}$ and $\mathrm{curl}_{\mathrm{s}}$ are defined. We use $\boldsymbol{x} = (x_1, x_2, x_3) = (x, y, z)$ as Cartesian coordinates in $\mathbb{R}^3$. Given a surface S embedded in $\mathbb{R}^3$ with unit normal vector field $\boldsymbol{n}$, then locally (in a coordinate patch) we can parameterise points on S by coordinates X_1, X_2, say as

$$\boldsymbol{x} = \boldsymbol{f}(X_1, X_2), \tag{A.1}$$

defined in such a way that the coordinate lines on S are orthogonal in $\mathbb{R}^3$. Thus the unit vector fields pointing along the coordinate lines, given by

$$\boldsymbol{e}_1 = \left|\frac{\partial \boldsymbol{x}}{\partial X_1}\right|^{-1} \frac{\partial \boldsymbol{x}}{\partial X_1}, \quad \boldsymbol{e}_2 = \left|\frac{\partial \boldsymbol{x}}{\partial X_2}\right|^{-1} \frac{\partial \boldsymbol{x}}{\partial X_2} \tag{A.2}$$

on S are everywhere perpendicular. With a suitable choice of orientation, the normal vector field $\boldsymbol{n} \equiv \boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$ completes an orthogonal basis at each point on S.

Now a coordinate X_3 is introduced to parameterise points off S via:

$$\boldsymbol{x} = \boldsymbol{f}(X_1, X_2) + X_3\, \boldsymbol{n}(X_1, X_2). \tag{A.3}$$

In other words one simply follows the normal vector $\boldsymbol{n}$ at $\boldsymbol{x} = \boldsymbol{f}(X_1, X_2)$ a distance X_3 from the point $\boldsymbol{x}$ on S. This will only generally be possible for a finite range of X_3 in a given coordinate patch (before the normal lines start to intersect) but this is not a problem. With this definition coordinate lines of constant X_1 or X_2 are simply translated a distance X_3 along the field of unit normals $\boldsymbol{n}$. Similarly the orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 = \boldsymbol{n}\}$ at each point $(X_1, X_2, 0)$ on S is simply translated along the normal $\boldsymbol{n}$ at that point. In this way an orthogonal coordinate system (X_1, X_2, X_3) and orthonormal basis $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3 = \boldsymbol{n}\}$ is defined at points off, but near to, S.

Through this definition $\boldsymbol{n} = \nabla X_3$ and so this field is curl-free as required. Note that this is a stronger condition than the Frobenius integrability condition

$$\boldsymbol{n} \cdot \nabla \times \boldsymbol{n} = 0 \tag{A.4}$$

for a vector field normal to a surface, and results from the careful definition of the coordinate system. For an illustrative example, if the surface S is an elliptical cylinder $x^2/a^2 + y^2/b^2 = 1$, it is temping to take the gradient of this function and set

$$\boldsymbol{n} = (x/a^2, y/b^2, 0)(x^2/a^4 + y^2/b^4)^{-1/2} \tag{A.5}$$

to define $\boldsymbol{n}$ on and off S. However this satisfies only the Frobenius integrability condition, and $\nabla \times \boldsymbol{n}$ is non-zero unless $a = b$. Instead the correct way to proceed is to parameterise the surface S via, say

$$\boldsymbol{x} = (a \cos X_1, b \sin X_1, X_2), \tag{A.6}$$

so that for points on S,

$$\boldsymbol{e}_1 = (-a \sin X_1, b \cos X_1, 0)\gamma, \ \boldsymbol{e}_2 = (0, 0, 1), \ \boldsymbol{n} \equiv \boldsymbol{e}_3 = (b \cos X_1, a \sin X_1, 0)\gamma, \tag{A.7}$$

with $\gamma(X_1) = (a^2 \sin^2 X_1 + b^2 \cos^2 X_1)^{-1/2}$. We extend the coordinate system off S via (A.3),

$$\boldsymbol{x} = (a \cos X_1 + b\gamma X_3 \cos X_1, b \sin X_1 + a\gamma X_3 \sin X_1, X_2). \tag{A.8}$$

It may easily be checked that the corresponding basis (see (A.2)) is orthogonal at each point. Although everything is written in an inconvenient, parameterised form, the vector field $\boldsymbol{n} = \nabla X_3$ is by construction curl-free.

With this orthogonal coordinate system in $\mathbb{R}^3$, the usual formulae for vector calculus operators hold. In $\mathbb{R}^3$ the metric takes the form

$$g_{ij} = \mathrm{diag}(h_1^2, h_2^2, h_3^2), \quad h_3 \equiv 1, \tag{A.9}$$

the latter by construction. Here h_1 and h_2 are generally functions of (X_1, X_2, X_3). In $\mathbb{R}^3$ we have then

$$\mathrm{grad}\, \chi = \boldsymbol{e}_1 h_1^{-1} \frac{\partial \chi}{\partial X_1} + \boldsymbol{e}_2 h_2^{-1} \frac{\partial \chi}{\partial X_2} + \boldsymbol{n} h_3^{-1} \frac{\partial \chi}{\partial X_3}\,. \tag{A.10}$$

A scalar field $\chi(X_1, X_2)$ that is at the outset defined only on S may be extended off S with (A.8) to be constant in the X_3 direction; its gradient has no $\boldsymbol{n} = \boldsymbol{e}_3$ component, and is tangential to S (cf. (3.6)).

We have in $\mathbb{R}^3$, witing $\boldsymbol{u} = u_1\boldsymbol{e}_1 + u_2\boldsymbol{e}_2 + u_3\boldsymbol{e}_3$, for the curl,

$$\mathrm{curl}\, \boldsymbol{u} = \frac{\boldsymbol{e}_1}{h_2 h_3}\left(\frac{\partial(h_3 u_3)}{\partial X_2} - \frac{\partial(h_2 u_2)}{\partial X_3}\right) + \frac{\boldsymbol{e}_2}{h_3 h_1}\left(\frac{\partial(h_1 u_1)}{\partial X_3} - \frac{\partial(h_3 u_3)}{\partial X_1}\right) + \frac{\boldsymbol{n}}{h_1 h_2}\left(\frac{\partial(h_2 u_2)}{\partial X_1} - \frac{\partial(h_1 u_1)}{\partial X_2}\right). \tag{A.11}$$

From this (with $h_3 = 1$) it is evident that

$$\operatorname{curl}_s \chi = \frac{\boldsymbol{e}_1}{h_2}\frac{\partial \chi}{\partial X_2} - \frac{\boldsymbol{e}_2}{h_1}\frac{\partial \chi}{\partial X_1}, \tag{A.12}$$

$$\operatorname{curl}_v \boldsymbol{u} = \frac{1}{h_1 h_2}\left(\frac{\partial(h_2 u_2)}{\partial X_1} - \frac{\partial(h_1 u_1)}{\partial X_2}\right), \tag{A.13}$$

from (3.3). The values obtained at points on S are clearly independent of the extension of the fields χ and $\boldsymbol{u}$ in the neighbourhood of S.

For the divergence of a general field $\boldsymbol{u}$ we have in $\mathbb{R}^3$,

$$\operatorname{div} \boldsymbol{u} = (h_1 h_2 h_3)^{-1}\left(\frac{\partial(h_2 h_3 u_1)}{\partial X_1} + \frac{\partial(h_3 h_1 u_2)}{\partial X_2} + \frac{\partial(h_1 h_2 u_3)}{\partial X_3}\right). \tag{A.14}$$

For a flow tangential to the surface S, we have $u_3 = 0$ on S. We specify that the field $\boldsymbol{u}$ is extended off S using (A.8) in such a way that $u_3 = 0$ in a neighbourhood of S. Then with $h_3 = 1$ the divergence of the extended field $\boldsymbol{u}$ in $\mathbb{R}^3$ amounts to the two-dimensional divergence of the original flow $\boldsymbol{u}$ on the surface itself, independent of the extension.

In terms of the intrinsic geometry, this machinery now starts with the two-dimensional metric induced by (A.9)

$$g_{ij} = \operatorname{diag}(h_1^2, h_2^2), \tag{A.15}$$

(using the same symbol) and may be followed without any reference to an embedding in $\mathbb{R}^3$. There is a correspondence between definitions and derivations in either framework, for example in (3.42)

$$(\eta_{ij}) = \begin{pmatrix} 0 & h_1 h_2 \\ -h_1 h_2 & 0 \end{pmatrix}, \quad (\eta^{ij}) = \begin{pmatrix} 0 & h_1^{-1} h_2^{-1} \\ -h_1^{-1} h_2^{-1} & 0 \end{pmatrix}, \tag{A.16}$$

and so with reference to the physical components of a vector $u_1 = h_1 U^1 = h_1^{-1} U_1$, $u_2 = h_2 U^2 = h_2^{-1} U_2$,

$$\eta^{ij} \nabla_j \chi = h_1^{-1} h_2^{-1} (\partial_2 \chi, -\partial_1 \chi)^i = (\operatorname{curl}_s \chi)^i \tag{A.17}$$

corresponding to (A.12). A similar result may be obtained for $\operatorname{curl}_v \boldsymbol{u}$, these together establishing (3.43).

APPENDIX B

In this appendix we summarise standard results concerning the differential geometry of a unit sphere. The metric is given by

$$g_{ij} = \operatorname{diag}(1, s^2), \quad g^{ij} = \operatorname{diag}(1, s^{-2}). \tag{B.1}$$

Here we abbreviate $s = \sin\theta$, $c = \cos\theta$ in terms of spherical polar coordinates (θ, ϕ) on the surface of the unit sphere. The only non-vanishing Christoffel symbols are:

$$\Gamma^1{}_{22} = -sc, \quad \Gamma^2{}_{12} = \Gamma^2{}_{21} = c/s. \tag{B.2}$$

with the covariant derivative given by :

$$\nabla_i U^j = \partial_i U^j + \Gamma^j{}_{ik} U^k, \quad \nabla_i U_j = \partial_i U_j - \Gamma^k{}_{ij} U_k, \quad \nabla_i T^{jk} = \partial_i T^{jk} + \Gamma^j{}_{il} T^{lk} + \Gamma^k{}_{il} T^{jl}, \tag{B.3}$$

etc. The non-vanishing Riemann tensor components are given by

$$R_{1212} = -R_{1221} = -R_{2112} = R_{2121} = s^2, \tag{B.4}$$

or

$$R^1{}_{212} = -R^1{}_{221} = s^2, \quad R^2{}_{121} = -R^2{}_{112} = 1, \tag{B.5}$$

while the Ricci tensor $R_{jk} = R^i{}_{jik}$ is the identity matrix

$$R_{ij} = g_{ij}, \quad R^i{}_j = \delta^i{}_j. \tag{B.6}$$

Note that the distinction between physical components of a vector (referred to an orthogonal basis) and the contravariant and covariant components, stressed in section 3.2 carries across to tensors. So the stress tensor $\boldsymbol{t}$ may be written in lower-case, physical components, referred to the usual orthonormal basis, as

$$\boldsymbol{t} = t_{11}\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\theta}} + t_{12}\hat{\boldsymbol{\theta}} \otimes \hat{\boldsymbol{\phi}} + t_{21}\hat{\boldsymbol{\phi}} \otimes \hat{\boldsymbol{\theta}} + t_{22}\hat{\boldsymbol{\phi}} \otimes \hat{\boldsymbol{\phi}} \tag{B.7}$$

or may be written with contravariant components as

$$\boldsymbol{t} = T_{11} d\theta \otimes d\theta + T_{12} d\theta \otimes d\phi + T_{21} d\phi \otimes d\theta + T_{22} d\phi \otimes d\phi \tag{B.8}$$

or in covariant as

$$\boldsymbol{t} = T^{11}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\theta + T^{12}\boldsymbol{e}_\theta \otimes \boldsymbol{e}_\phi + T^{21}\boldsymbol{e}_\phi \otimes \boldsymbol{e}_\theta + T^{22}\boldsymbol{e}_\phi \otimes \boldsymbol{e}_\phi \tag{B.9}$$

(where $\boldsymbol{e}_\theta$, $\boldsymbol{e}_\phi$ give the dual basis to $d\theta$, $d\phi$). These are then related via the metric with

$$T^{11} = T_{11} = t_{11}, \quad T^{12} = s^{-2}T_{12} = s^{-1}t_{12}, \quad T^{21} = s^{-2}T_{21} = s^{-1}t_{21}, \quad T^{22} = s^{-4}T_{22} = s^{-2}t_{22}. \tag{B.10}$$

We also note that the geometrical nature of the stress tensor, in the framework of differential geometry, is discussed in (**3**, **27**).

Returning to calculations, on the sphere we have

$$\nabla_1 U_1 = \partial_1 U_1, \quad \nabla_1 U_2 = \partial_1 U_2 - s^{-1}cU_2, \tag{B.11}$$

$$\nabla_2 U_1 = \partial_2 U_1 - s^{-1}cU_2, \quad \nabla_2 U_2 = \partial_2 U_2 + scU_1, \tag{B.12}$$

and so

$$E_{11} = \partial_1 U_1, \quad E_{22} = \partial_2 U_2 + scU_1, \tag{B.13}$$

$$E_{12} = E_{21} = \tfrac{1}{2}(\partial_1 U_2 + \partial_2 U_1 - 2s^{-1}cU_2). \tag{B.14}$$

We define the stress tensor in an explicitly divergence-free form by subtracting off the (zero) divergence of $\boldsymbol{u}$, namely $E^k_k = \partial_1 U_1 + s^{-2}\partial_2 U_2 + s^{-1}cU_1$. In other words we set $T_{ij} = 2E_{ij} - g_{ij}E^k_k$, giving

$$T_{11} = -s^{-2}T_{22} = \partial_1 U_1 - s^{-2}\partial_2 U_2 - s^{-1}cU_1, \tag{B.15}$$

$$T_{12} = T_{21} = \partial_1 U_2 + \partial_2 U_1 - 2s^{-1}cU_2. \tag{B.16}$$

Now we need

$$D^1 = \nabla_i T^{i1} = \partial_1 T^{11} + \partial_2 T^{21} + s^{-1}cT^{11} - scT^{22}, \tag{B.17}$$

$$D^2 = \nabla_i T^{i2} = \partial_1 T^{12} + \partial_2 T^{22} + 2s^{-1}cT^{12} + s^{-1}cT^{21}, \tag{B.18}$$

which yields for CS3

$$D^1 = (\partial_1^2 U_1 + s^{-1}c\partial_1 U_1 + s^{-2}\partial_2^2 U_1) - 2s^{-3}c\partial_2 U_2 + s^{-2}(s^2 - c^2)U_1, \tag{B.19}$$

$$D^2 = s^{-2}(\partial_1^2 U_2 - s^{-1}c\partial_1 U_2 + s^{-2}\partial_2^2 U_2) + s^{-3}c\partial_2 U_1 + 2s^{-2}U_2, \tag{B.20}$$

and becomes (3.26,3.27) when translated into physical components. For CS2 a similar calculation, that is taking the divergence of $\nabla^i U^j$, gives

$$D^1 = (\partial_1^2 U_1 + s^{-1} c \partial_1 U_1 + s^{-2} \partial_2^2 U_1) - 2s^{-3} c \partial_2 U_2 - s^{-2} c^2 U_1 \tag{B.21}$$

$$D^2 = s^{-2}(\partial_1^2 U_2 - s^{-1} c \partial_1 U_2 + s^{-2} \partial_2^2 U_2) + 2s^{-3} c \partial_2 U_1 + s^{-2} U_2, \tag{B.22}$$

and becomes (3.24,3.25).

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(Ignore the 'Appendix C' above which is a result of qjmam.cls (style file) that I cannot easily remove.)

APPENDIX C

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