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Helicity, linking, and writhe in a spherical geometry

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Abstract. Linking numbers, helicity integrals, twist, and writhe all describe the topology and geometry of curves and vector fields. The topology of the space the curves and fields live in, however, can affect the behaviour of these quantities. Here we examine curves and fields living in regions exterior to a sphere or in spherical shells. The winding of two curves need not be conserved because of the topology of a spherical shell. Avoiding the presence of magnetic monopoles inside the sphere is very important if magnetic helicity is to be a conserved quantity. Considerations of parallel transport are important in determining the transfer of helicity through the foot of a magnetic flux tube when it is in motion.

1. Introduction

In 1833 the German mathematician Carl Friedrich Gauss mentioned the idea of linking number as a brief note in his diary (Ricca & Nipoti, 2011). Gauss was considering the path of asteroids across the sky when he realised that an asteroid's path would be restricted to a range of latitudes unless its orbit linked the Earth's orbit. It is not clear how Gauss formalised this idea into the Gauss linking integral, but Ricca and Nipoti suggest that he might have used his knowledge of terrestrial magnetism for this purpose.

As Gauss noted (Epple, 1998), linking number is – most importantly – invariant to smooth deformation of the curves as long as they are not permitted to pass through each other: it tells you about the topology of a curve rather than its geometry. Linking number is an important and widely used tool in many areas of study including molecular biology (Fuller, 1978; White & Bauer, 1986), fluid dynamics (Ricca, 1998) and super-fluid dynamics (Barenghi *et al.*, 2001).

Many applications of topological and geometrical measures of linking, twisting, and writhing involve geometries that have a curved nature to them (Priest, 1982). For example, the magnetic coronal-loops in the sun's atmosphere share a spherical boundary (the photosphere). Also, biologists are interested in long molecules such as DNA, RNA, and proteins which can be connected to roughly spherical objects (e.g. cell nuclei Bates & Maxwell (2005)). The buckling and twisting of elastic rods also depends on the geometry of external restraints (Hannay, 1998; Neukirch & Starostin, 2008). The purpose of this work is to extend the existing concepts of linking number, twist and writhe, so that they work in a consistent manner in a spherical geometry.

Spherical geometry differs in several ways to flat space. First, the usual spherical coordinate system (ρ, θ, ϕ) contains points that cannot be uniquely described; if $\rho = 0$ or $\theta = 0$ or $\theta = \pi$, then one or more of the other components can be changed without altering a point's position. Another difference is that on a sphere, Euclid's fifth postulate breaks down: two parallel lines need not stay a fixed distance apart. As a consequence the idea of parallel vectors must be



carefully defined; one must first specify how to move a vector around – using a connection – before this is possible. A third difference is topological: the space outside of a sphere has non-trivial topology; it has a hole. The hole can play havoc with helicity integrals because magnetic monopoles can be hidden inside. Helicity in triply periodic geometries (i.e. a three-torus) can also be affected by the non-trivial topology (Berger, 1997).

We first review the expressions for self and mutual helicity of flux tubes. We then examine the helicity of two tubes in a spherical shell, taking care to avoid hidden monopoles. In section 4 we calculate the flux of helicity into the tubes due to boundary motions.

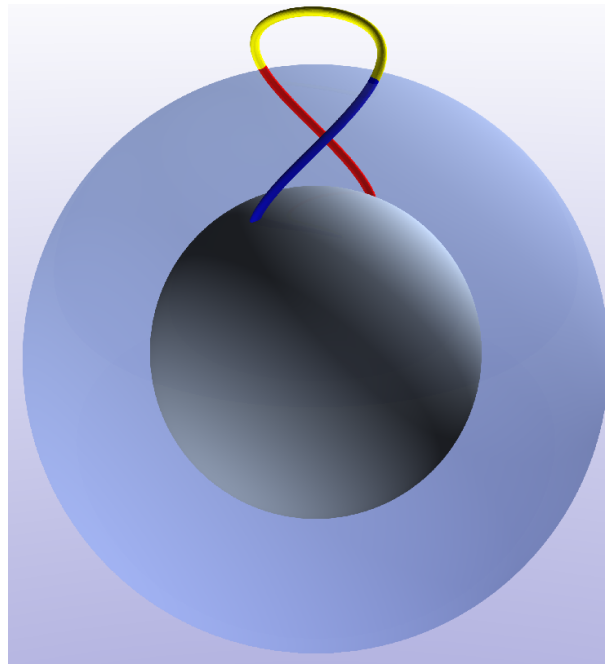


Figure 1. A loop extending out of a sphere. If we fix the intersection points with the inner and outer sphere, then the helicity in the shell between is conserved in the absence of resistivity.

2. The self and mutual helicity of two flux tubes

Magnetic helicity measures the net linking of pairs of fieldlines in a magnetic field (Moffatt, 1969). The total helicity of two tubes in a volume \mathcal{V} can be decomposed into a contribution from the self helicities of the individual tubes, plus a mutual helicity due to their winding number. Let \mathcal{T}_1 and \mathcal{T}_2 be the self helicities of tubes 1 and 2, and let w_{12} measure the winding of the tubes about each other. The self helicity of tube 1 arises from twisting T_{w1} of field lines inside the tube about its central axis, as well as the writhe W_{r1} of the axis itself:

$$\mathcal{T}_1 = T_{w1} + W_{r1} \quad (1)$$

(Berger & Prior, 2006). Let the magnetic fluxes of the two tubes be Φ_1 and Φ_2 . Then in total we have (Berger, 1986, 2009)

$$H = \mathcal{T}_1 \Phi_1^2 + \mathcal{T}_2 \Phi_2^2 + 2w_{12} \Phi_1 \Phi_2. \quad (2)$$

Next consider a volume with planar boundaries (see figure 2). Suppose two flux tubes stretch from bottom to top. The winding number between the two tubes w_{12} is given by $w_{12} = \alpha_{12}/(2\pi)$, where α_{12} is the net angle through which the tubes wind about each other.

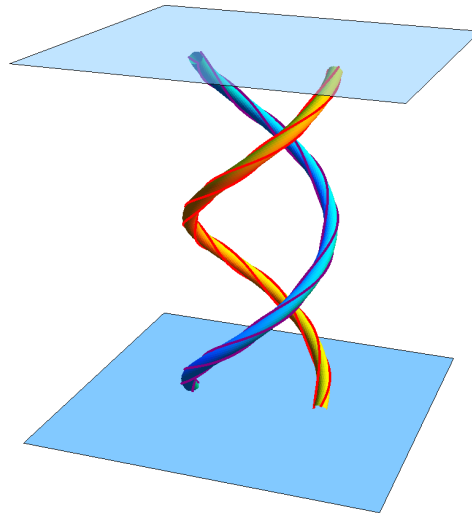


Figure 2. Two flux tubes extending between two parallel planes. The twist plus writhe of each individual tube is conserved, as well as the winding number between the two tubes.

If the boundaries are held fixed, then the helicity will be conserved if the tubes are distorted between the boundaries. The fluxes Φ_1 and Φ_2 will also be conserved. As the two fluxes can be chosen independently, we may conclude that \mathcal{T}_1 , \mathcal{T}_2 and w_{12} are separately conserved.

Next suppose that the footpoints of the tubes on the lower boundary move. Then the fluxes do not change, but the topological measures evolve. Suppose the angular rotation rate of footpoint 1 is $\dot{\alpha}_1$ and similarly for footpoint 2. The relative position vector between the centres of the two footpoints rotates with a rate $\dot{\alpha}_{12}$. Then

$$\frac{dH}{dt} = \frac{-1}{2\pi} (\dot{\alpha}_1 \Phi_1^2 + \dot{\alpha}_2 \Phi_2^2 + 2\dot{\alpha}_{12} \Phi_1 \Phi_2). \quad (3)$$

3. The helicity of two tubes in a spherical region

Suppose a loop of magnetic field rises out of a spherical boundary, for example, the surface of a star (see figure 1). We can build an external magnetic field to the sphere by summing over many loops, but here we will consider just one. We may wish to consider just the part of the loop inside a shell between two spherical boundaries, as in figure 1. The magnetic helicity of the entire loop will equal the helicity inside the shell between the two boundary spheres, plus the helicity above. Motions which vanish at the two spheres will separately conserve the helicities above and below.

The nature of spherical geometries leads to a strange effect. Can we deform the blue curve in figure 1 so as to reverse the sign of the crossing? It turns out you can – see figure 3. Here the end points of the curves have remained fixed, but the winding angle between the two curves has changed by minus one: $w_{12} \rightarrow w_{12} - 1$. In terms of the sign of the crossing between the tubes, going from the diagram on the left to the diagram on the right, a negative crossing has been replaced by a positive crossing.

It might seem that the helicity has changed; however, this is not the case: the helicity is still equal to Φ^2 . This will be easiest to see if we consider a cross-section extending along the tube in the shape of a ribbon (see figure 4). In the process of moving the middle part of the curve twist has been injected. As the curve is deformed winding is converted into twist. Figure 4 illustrates how a ribbon would change its twist by -2 under such a transformation. (This effect can be readily verified by experiment with a length of ribbon and a ball!) Thus from equation 2

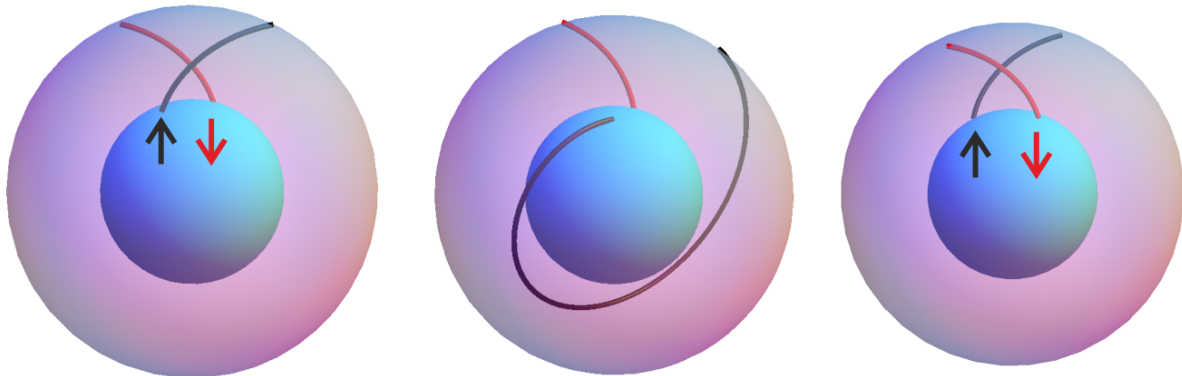


Figure 3. The foreground curve on the left curve smoothly deforms to sweep around the sphere, becoming the background curve in the right panel. In the left panel the curves wind about each other by $\alpha_{12} = \pi$ as they rise from the inner to the outer sphere; thus $w_{12} = 1/2$. On the other hand, as one tube has flux pointing inwards and one outwards, the crossing is negative giving a negative helicity contribution. In the right panel the winding is negative (alternatively, the crossing has become positive). Thus, despite the fixed endpoints, the winding of the two curves has changed from approximately $1/2$ to $-1/2$.

the helicity will be conserved, *provided the two fluxes are equal and opposite in magnitude*. Let the fluxes be $\Phi_1 = \Phi$, $\Phi_2 = -\Phi$. Then

$$H_{before} = \Phi^2 (\mathcal{T}_1 + \mathcal{T}_2 - 2w_{12}); \quad (4)$$

$$H_{after} = \Phi^2 ((\mathcal{T}_1 - 2) + \mathcal{T}_2 - 2(w_{12} - 1)). \quad (5)$$

Thus $H_{before} = H_{after}$.

Here an important distinction is apparent between the planar slice and spherical shell topologies. In the plane geometry of figure 2, the self and mutual terms are separately conserved even if the two fluxes differ. In the spherical case, however, the terms are not separately conserved; but total helicity conservation requires equal and opposite fluxes. Even though we do not directly enquire about what is happening inside the inner spherical boundary, we run into trouble if there is net magnetic flux coming out of it. Thus we must require that there are no monopoles inside the sphere (or at least equal numbers of positive and negative monopoles). For the planar case, net flux is allowed. We can always assume return fluxes at an infinite distance away from our tubes, so that there is no mutual helicity between our tubes and the return flux.

4. Change in helicity due to surface motions

Let the radius of the inner boundary be ρ_0 and consider two flux tubes crossing this boundary. We seek expressions for the change in helicity due to their two footpoints on the boundary moving about each other. A field with more flux tubes, or even a continuous field, can be analyzed by summing over many pairs of tubes.

For flux moving on a boundary due to motions parallel to the boundary with velocity \mathbf{V} , the helicity change can be written as a surface integral (Berger, 1986)

$$\frac{dH}{dt} = \int (\mathbf{A}_P \cdot \mathbf{V}) \mathbf{B} \cdot \hat{\mathbf{n}} \, d^2S, \quad (6)$$

where $\hat{\mathbf{n}}$ is the unit normal and \mathbf{A}_P is the unique Coulomb gauge (divergence-free) vector potential of a potential (vacuum) magnetic field with the same boundary flux.

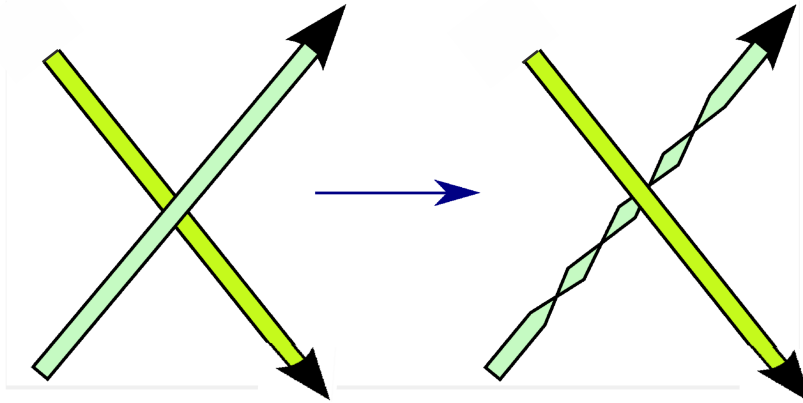


Figure 4. Before deforming: $\mathcal{T} = 0$ and $\mathcal{W} = -1$. Post deformation: $\mathcal{T} = -2$ and $\mathcal{W} = 1$. Hence in both cases $H = -\Phi^2$ as expected.

In some situations, Stoke’s theorem provides an easy method of determining \mathbf{A}_P . For a planar boundary, a circular footprint centred at the origin with normal flux Φ has

$$\mathbf{A}_P = \frac{\Phi}{2\pi r} \hat{\phi}, \quad (7)$$

where polar coordinates are (r, ϕ) and we assume r is greater than the radius of the footprint. We can then build up \mathbf{A}_P from linearly summing over footprints. But things are not so simple with spherical geometries!

In defining the problem we want to avoid the possibility of magnetic monopoles. If there is net flux coming through the surface then $\nabla \cdot \mathbf{B} \neq 0$ inside the sphere. The two flux tubes cutting our layer must have the same magnitude but oppositely directed radial fluxes to avoid this problem. The net radial flux would then be zero, so $\nabla \cdot \mathbf{B} = 0$ as required. We need to find the vector potential on the surface corresponding to these two tubes. Suppose footprint 1 is at the North pole and footprint 2 is at the location (θ_2, ϕ_2) . If we naively follow the procedure of equation 7, then we would say $\mathbf{A}_P = \mathbf{A}_1 + \mathbf{A}_2$ where

$$\mathbf{A}_1 = \frac{\Phi_1}{2\pi\rho_0 \sin \theta} \hat{\phi}, \quad (8)$$

and \mathbf{A}_2 is found by an appropriate rotation of this vector field (so that the \mathbf{A}_2 lines encircle spot 2). This approach fails however: the expression in equation 8 has a singularity at the South pole, which is not removed by adding \mathbf{A}_2 . It is better to only consider fields which explicitly have zero net flux before attempting to calculate the vector potential.

The problem with this approach is that in a sphere, it is no longer possible to study each tube individually. Instead, we will introduce a net return flux as in figure 5. We assume the areas \mathcal{A}_1 and \mathcal{A}_2 of the footprints are small compared to the area of the sphere. Then for each flux tube i

$$B_{i\rho} = \begin{cases} \frac{\Phi_i}{\mathcal{A}_1} - \frac{\Phi_i}{4\pi\rho_0^2} & \text{inside spot} \\ -\frac{\Phi_i}{4\pi\rho_0^2} & \text{outside spot} \end{cases} \quad (9)$$

If the two tubes have equal and opposite flux, then the return fluxes exactly cancel.

If footprint 1 is at the North pole, then we have (e.g. employing Stoke’s theorem) the vector potential due to its flux Φ_1 ,

$$\mathbf{A}_{P1}(\theta, \phi) = \left(\frac{1 + \cos \theta}{2} \right) \frac{\Phi_1}{2\pi\rho_0 \sin \theta} \hat{\phi}. \quad (10)$$

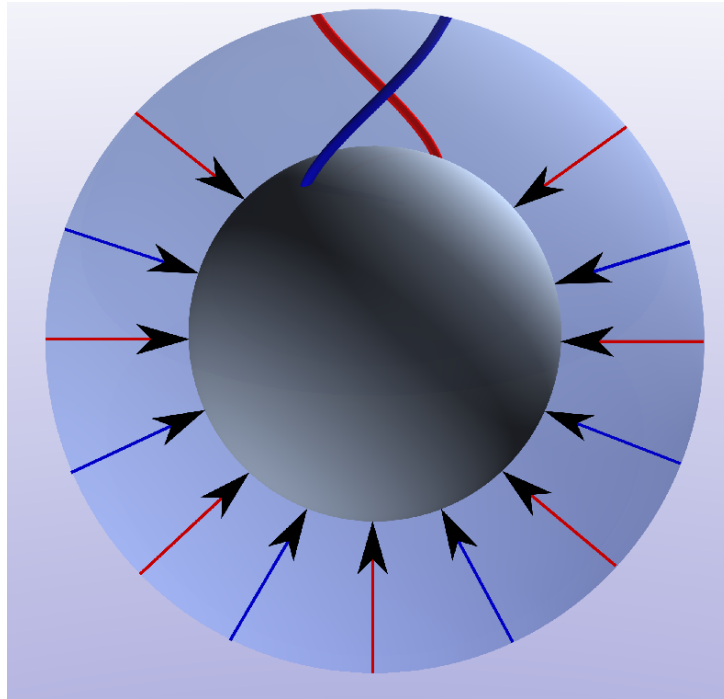


Figure 5. We introduce a net return flux to ensure that there are no magnetic monopoles. Each tube's B-field is shared out evenly and directed back into the surface.

Note that no singularity remains. The vector potential for \mathbf{A}_{P2} will have a similar form, but with Φ replaced by $-\Phi$ and with a rotation of the vector field to centre on footpoint 2.

Suppose that footpoint 1 on the lower sphere is located at the North pole and that the second footpoint is at co-latitude θ_0 . We wish to examine the change in helicity as footpoint 2 moves around on the bottom sphere. Let footpoint 2 move through an angle of $\delta\phi$ on the circle of co-latitude defined by θ_0 .

The change of helicity is:

$$\delta H = \Phi^2 (\delta\mathcal{T}_1 + \delta\mathcal{T}_2 + 2\delta w_{12}). \quad (11)$$

We will examine the linking term first. As tube 2 moves it will wind with tube 1 and also with the return flux. The amount of return flux it winds with will be proportional to the surface area of the polar cap that it circumscribes:

$$\delta w_{12} = \frac{\delta\phi}{2\pi} \left(1 - \frac{\text{area of polar cap at } \theta_0}{4\pi\rho_0^2} \right) \quad (12)$$

$$= \frac{\delta\phi}{2\pi} \left(\frac{1}{2} + \frac{\cos(\theta_0)}{2} \right). \quad (13)$$

Next we will need to work out the self-linking terms; these refer to the internal twist inside each flux tube. To measure the twist, we choose a field line on the axis of the tube and a field line on the boundary. The boundary field line will twist about the centre field line by T_w turns. The twist number T_w grows as the footpoint of the tube rotates - but what causes it to rotate? We will consider a vector which points from the centre of the footpoint to the chosen boundary line. We now ask how much this vector rotates.

Imagine walking around a circular path on a flat surface. If you always face the direction of travel, you will turn through an angle of 2π . Facing in the direction of travel corresponds to facing in the direction of the tangent vector $\hat{\mathbf{t}}$ to the path, which rotates by 2π . Similarly, a footpoint proceeding in this fashion will rotate by 2π . Suppose instead that you always face North, sometimes walking sideways, sometimes walking backwards. The vector $\hat{\mathbf{p}}$ representing the direction you are facing is the constant North vector, and there is no turning.

On a curved surface, the analogy of the non-turning constant vector $\hat{\mathbf{p}}$ is called the *parallel transport* vector. We will compare the forward-facing vector $\hat{\mathbf{t}}$ to the parallel transport vector $\hat{\mathbf{p}}$. After following a path, calculating the angle between these two vectors will give the net rotation of the footpoint. We will need to introduce some additional tools beforehand.

4.1. Parallel transport

Suppose we would like to transport a vector \mathbf{v} along a parametrised curve $\gamma(t)$, whose tangent vector is \mathbf{u} . Basis vectors are labelled $\hat{\mathbf{e}}_\beta$. Then standard differential geometry gives the covariant derivative of \mathbf{v} along the curve:

$$\frac{D\mathbf{v}}{Dt} = \left(\frac{dv^\beta}{dt} + u^\alpha v^\gamma \Gamma_{\gamma\alpha}^\beta \right) \hat{\mathbf{e}}_\beta. \quad (14)$$

Here the last term in this equation includes the Christoffel symbols $\Gamma_{\beta\alpha}^\gamma$, which describe how the basis vectors change.

This expression is known as the covariant derivative $\nabla_{\mathbf{u}}\mathbf{v}$. This derivative describes how a vector varies as it is transported along a curve. Suppose that we wish to transport \mathbf{v} without adding any extra rotation. More precisely, we require that for each infinitesimal amount \mathbf{v} is transported, it remains locally parallel to its previous direction. Then we say that \mathbf{v} is parallel transported along \mathbf{u} if

$$\nabla_{\mathbf{u}}\mathbf{v} = 0. \quad (15)$$

A geodesic is a curve whose tangent vector remains parallel when transported along the curve. For a sphere, these are curves that follow a great circle. Along other curves, there will be some acceleration or rotation of the tangent vector.

Suppose \mathbf{v} is a vector pointing from the centre of a footpoint to some point on its boundary. Keeping track of how this vector rotates tells us how much the footpoint rotates. So, when transporting the footpoint (and hence \mathbf{v}) along a curve which is not a great circle, the acceleration of the tangent vector will induce a corresponding rotation of the footpoint.

In particular, if the footpoint moves along a latitude line other than the equator, the rotation will inject helicity into the tube above. Of course, additional rotation can occur in addition to the transport contribution, for example if vorticity at the surface rotates the footpoint.

4.2. Twist calculation

We can now calculate how much \mathbf{v} rotates. The method here generalises from spherical geometries to other curved geometries. Expanding equation 15 gives:

$$0 = \left(\frac{dv^\beta}{dt} + u^\alpha v^\gamma \Gamma_{\gamma\alpha}^\beta \right), \quad (16)$$

where the metric connection gives

$$\Gamma_{\gamma\alpha}^\beta = g^{\beta\delta} \frac{1}{2} (g_{\delta\gamma,\alpha} + g_{\delta\alpha,\gamma} - g_{\gamma\alpha,\delta}). \quad (17)$$

There are only three non-zero Christoffel symbols:

$$\Gamma_{\phi\theta}^{\phi} = \Gamma_{\theta\phi}^{\phi} = \cot(\theta_0); \quad (18)$$

$$\Gamma_{\phi\phi}^{\theta} = -\sin(\theta_0)\cos(\theta_0). \quad (19)$$

For motion along a latitude line $u^{\phi} = 0$, we obtain

$$\partial_{\phi}v^{\theta} + v^{\phi}\Gamma_{\phi\phi}^{\theta} = 0; \quad (20)$$

$$\partial_{\phi}v^{\phi} + v^{\theta}\Gamma_{\theta\phi}^{\phi} = 0. \quad (21)$$

Differentiating the first equation, putting it into the second and doing the same the other way around we obtain

$$\frac{\partial^2 v^{\theta}}{\partial \phi^2} = -v^{\theta} \cos(\theta_0)^2; \quad (22)$$

$$\frac{\partial^2 v^{\phi}}{\partial \phi^2} = -v^{\phi} \cos(\theta_0)^2. \quad (23)$$

Letting $\Omega = \cos(\theta_0)$, these equations have solutions of the form:

$$v^{\theta}(\phi) = A \sin(\Omega\phi) + B \cos(\Omega\phi); \quad (24)$$

$$v^{\phi}(\phi) = C \sin(\Omega\phi) + D \cos(\Omega\phi). \quad (25)$$

We now consider a vector $\hat{\mathbf{p}}$ which initially points tangent to the latitude line at θ_0 . Thus it has the initial condition $\hat{\mathbf{p}}_0 = (p_0^{\theta}, p_0^{\phi}) = (0, 1/\sin\theta_0)$ (the factor of $1/\sin\theta_0$ is needed to make it a unit vector in the spherical coordinate system). We obtain for the parallel transport vector

$$\hat{p}^{\theta} = \sin(\Omega\phi); \quad (26)$$

$$\hat{p}^{\phi} = \frac{1}{\sin\theta_0} \cos(\Omega\phi). \quad (27)$$

In contrast, for motion along a latitude line the tangent vector at all times has $t^{\theta} = 0$. Suppose the motion proceeds along a net longitudinal angle $\delta\phi$. Relative to the parallel transport vector, the tangent vector will have rotated by an angle $\delta\phi \cos(\theta_0)$. This rotation corresponds to the *geodesic curvature* of the path.

Recall that our aim here was to calculate how much tube two rotates as it moves. If this tube is rotated positively, then the twist induced will be negative (as in equation 3). Thus if footpoint 1 stays at the North pole and footpoint 2 travels through an angle $\delta\phi$ at co-latitude θ_0 , then

$$\delta\mathcal{T}_1 = 0, \quad \delta\mathcal{T}_2 = -\frac{\delta\phi}{2\pi} \cos(\theta_0). \quad (28)$$

The net helicity change due to the motion of tube 2 (including winding) is

$$\delta\mathcal{H} = \Phi^2 (\delta\mathcal{T}_1 + \delta\mathcal{T}_2 + 2\delta H_{12}); \quad (29)$$

$$= \Phi^2 \frac{\delta\phi}{2\pi} \left[0 - \cos(\theta_0) + 2 \left(\frac{1}{2} + \frac{\cos(\theta_0)}{2} \right) \right]; \quad (30)$$

$$= \Phi^2 \frac{\delta\phi}{2\pi}. \quad (31)$$

This coincides with the change in helicity we would have if we were working in a flat geometry.

4.3. Uniform rotation

Next we will consider what would happen if footpoint 1 rotates as well, and both tubes move around 2π . This could be thought of as one complete solid rotation of the sphere. The tube at the north pole will rotate by 2π causing a negative twist of $\mathcal{T}_1 = -1$. The lower tube will also rotate about by $2\pi \cos(\theta_0)$, and so twist by $\mathcal{T}_2 = -2\pi \cos(\theta_0)$. The tubes will also wind with each other according to equation 13. The change in helicity is

$$\delta\mathcal{H} = \Phi^2 \left[-1 - \cos(\theta_0) + 2 \left(\frac{1}{2} + \frac{\cos(\theta_0)}{2} \right) \right]; \quad (32)$$

$$= 0, \quad (33)$$

which is what is expected as, relative to each other, there is no winding going on.

5. Conclusions

The helicity of two flux tubes can be thought of as a sum of self and mutual helicities. If the tubes are located in all space or are restricted to a half space, then the self and mutual helicities are separately conserved during ideal motion. However, for tubes in the space outside (or inside) a sphere, only the total helicity is necessarily conserved. We have investigated the helicity flux through a spherical boundary, and how it separates into individual and mutual terms, i.e. the spins of individual footpoints and the orbiting of footpoints about each other. Here measures of rotation must be carefully defined. The spin of an individual footpoint as it follows a path can be measured by the geodesic curvature, calculating the rotation of the tangent vector relative to the parallel transport vector. As a consistency check, a uniform rotation of the sphere should provide zero net helicity flow. This provides a constraint on how to calculate the orbit term as well. These methods will readily generalize to non-spherical geometries.

References

- BARENGHI, C., DONNELLY, R. & VINEN, W. 2001 **Quantized Vortex Dynamics and Superfluid Turbulence**. Springer Publishing, New York, USA.
- BATES, A. & MAXWELL, A. 2005 **DNA Topology**, 2nd edn. Oxford University Press, Oxford, UK.
- BERGER, M. 1986 Topological invariants of field lines rooted in planes. *Geophys. Astrophys. Fluid Dyn.* **34**, 265–281.
- BERGER, M. A. 1997 Magnetic helicity in a periodic domain. *J. Geophys Research* **102**, 2637–2644.
- BERGER, M. A. 2009 **Topological Magnetohydrodynamics and Astrophysics**. Springer, New York.
- BERGER, M. A. & PRIOR, C. 2006 The writhe of open and closed curves. *J. Phys. A* **39** (26), 8321–8348.
- EPPLE, M. 1998 Orbits of asteroids, a braid, and the first link invariant. *Math. Intelligencer* **20**, 45–52.
- FULLER, F. B. 1978 Decomposition of the linking number of a closed ribbon: a problem from molecular biology. *Proc. Nat. Acad. Sci. U.S.A.* **75** (8), 3557–3561.
- HANNAY, J. 1998 Cyclic rotations, contractibility, and gauss-bonnet. *J. Phys. A* **31**.
- MOFFATT, H. K. 1969 The degree of knottedness of tangled vortex lines. *Journal of Fluid Mechanics* **35**, 117–129.
- NEUKIRCH, S. & STAROSTIN, E. L. 2008 Writhe formulas and antipodal points in plectonemic dna configurations. *Phys. Rev. E* **78**, 041912.

- PRIEST, E. 1982 **Solar Magneto-Hydrodynamics**. D.Reidel Publishing Company, Dordrecht, Holland.
- RICCA, R. & NIPOTI, B. 2011 Gauss' linking number revisited. *Journal of Knot Theory and Its Ramifications* **20** (10), 1325–1343.
- RICCA, R. L. 1998 Applications of knot theory in fluid mechanics. In *Knot theory (Warsaw, 1995)*, *Banach Center Publ.*, vol. 42, pp. 321–346. Polish Acad. Sci., Warsaw.
- WHITE, J. & BAUER, W. R. 1986 Calculation of the twist and the writhe for representative models of dna. *J. Mol Biol* **189** (2), 329–341.