

SEVERAL NEW QUADRATURE FORMULAS FOR POLYNOMIAL INTEGRATION IN THE TRIANGLE

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Abstract. We present several new quadrature formulas in the triangle for exact integration of polynomials. The points were computed numerically with a cardinal function algorithm which imposes that the number of quadrature points N be equal to the dimension of a lower dimensional polynomial space. Quadrature formulas are presented for up to degree $d = 25$, all which have positive weights and contain no points outside the triangle. Seven of these quadrature formulas improve on previously known results.

Key words. multivariate integration, quadrature, cubature, fekete points, triangle, polynomial approximation

AMS subject classifications. 65D32 65D30 65M60 65M70

1. Introduction. We consider a set of N points $\{z_1, z_2, \dots, z_N\}$ and associated weights $\{w_1, w_2, \dots, w_N\}$ to be a quadrature formula of strength d if the quadrature approximation for a domain Ω ,

$$\int_{\Omega} g \simeq \sum_{j=1}^N w_j g(z_j),$$

is exact for all polynomials g up to degree d . Among all quadrature formulas of strength d , the optimal ones are those with the fewest possible points N . The quadrature problem has been extensively studied and has a long history of both theoretical and numerical development. For a recent review, see [3, 10, 6, 4]. An on-line database containing the many of the best known quadrature formula is described in [5]. Much of these results are also collected and distributed on CD-ROM in the book [12].

One successful approach for numerically finding quadrature formulas dates to [11]. A generalized version was used recently in [15]. Newton's method is used to solve the nonlinear system of algebraic equations for the quadrature weights and locations of the points. Symmetry is used to reduce the complexity of the problem. If the quadrature points are invariant under the action of a group G , then the number of equations can be reduced to the dimension of the polynomial subspace invariant under G .

Recently, a cardinal function algorithm has been developed which can provide additional reduction in the complexity of the quadrature problem [13]. It is motivated by a similar cardinal function Fekete point algorithm [14]. The key idea is to look for quadrature formula that have the same number of points as the dimension of a lower dimensional polynomial space. One can then construct a cardinal function basis for this lower dimensional space, make use of a multi-variate generalization of the Newton-Cotes quadrature weights and derive a remarkable expression analytically relating the variation in the quadrature weights to the variation of the quadrature points. The net result is a significant reduction in the number of equations and unknowns, while still retaining analytic expressions for the gradients necessary to apply steepest descent or Newton iterations.

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Symmetry can still be used to further reduce the complexity of the problem if needed. However here we have been able to find optimal quadrature sets of strength 9 through 25, subject only to the cardinal function constraint without imposing any symmetry constraints on the solutions.

2. Notation. Let $\xi = (\xi_1, \xi_2)$ be an arbitrary point in \mathfrak{R}^2 . We will work in the right triangle, $\xi_1 \geq -1$, $\xi_2 \geq -1$ and $\xi_1 + \xi_2 \leq 0$. Let \mathcal{P}_d be the finite dimensional vector space of polynomials of at most degree d ,

$$\mathcal{P}_d = \text{span}\{\xi_1^n \xi_2^m, m + n \leq d\}.$$

We also define

$$N = \dim \mathcal{P}_d = \frac{1}{2}(d+1)(d+2).$$

The monomials $\xi_1^n \xi_2^m$ are notoriously ill-conditioned, so it is necessary to describe \mathcal{P}_d with a more reasonable basis. For this we use the orthogonal Kornwinder-Dubiner polynomials $\{g_{m,n}\}$ [1, 9, 7],

$$g_{m,n}(\xi) = P_m^{0,0} \left(\frac{\xi_1}{1-\xi_2} \right) (1-\xi_2)^m P_n^{2m+1,0}(\xi_2)$$

where $P_n^{\alpha,\beta}$ are the Jacobi Polynomials with weight (α, β) and degree n . In this basis, $\mathcal{P}_d = \{g_{m,n}, m+n \leq d\}$. Suitable recurrence relations for these polynomials are given in [8].

3. Quadrature formula with N points. We note that given a set of N non-degenerate points in the triangle $\{z_j\}$, we can obtain the generalized Newton-Cotes weights by solving the $N \times N$ system:

$$\sum_{j=1}^N w_j g_{m,n}(z_j) = \int g_{m,n} d\xi \quad \forall g_{m,n} \in \mathcal{P}_d \quad (3.1)$$

By construction, the Newton-Cotes weights and the points $\{z_i\}$ give a quadrature formula which exactly integrates our N basis functions, and thus

$$\sum_{j=1}^N w_j g(z_j) = \int g d\xi \quad \forall g \in \mathcal{P}_d.$$

Because any set of N quadrature points of strength d or greater must satisfy Eq. 3.1, the weights for all such quadrature formulas must be the Newton-Cotes weights.

To obtain quadrature points of strength greater than d , one must optimize the location of the points $\{z_j\}$. Here we present results using the algorithm from [13] to perform this optimization. The goal is to find points $\{z_j\}$ which exactly integrate all of \mathcal{P}_{d+e} for the largest possible e .

4. Degrees of Freedom bound. There is one degree of freedom for each coordinate of each point, for a total of $2 \dim \mathcal{P}_d$. Since we are using Newton-Cotes weights, we automatically integrate exactly all of \mathcal{P}_d . The number of additional equations that must be satisfied to integrate exactly all of \mathcal{P}_{d+e} is thus $\dim \mathcal{P}_{d+e} - \dim \mathcal{P}_d$. If we require that the degrees-of-freedom in the location of the quadrature points is at least as large as the number of equations that must be satisfied, we arrive at a lower bound on N given by $\dim \mathcal{P}_{d+e} \leq 3 \dim \mathcal{P}_d = 3N$.

5. Results. Our results for the triangle are summarized in Table 5. Except for quadrature formulas associated with $d = 3$ and $d = 4$, we were able to obtain the optimal solution (fewest number of points) subject to the cardinal function constraint on the number of points and the degrees-of-freedom lower bound:

$$N = \dim \mathcal{P}_d, \quad (5.1)$$

$$\dim \mathcal{P}_{d+e} \leq 3N. \quad (5.2)$$

All the quadrature points have positive weights and no points lie outside the triangle, although neither of these properties is in any way guaranteed by the cardinal function algorithm. The errors presented in the table is the max norm of the quadrature error over all the ortho-normal basis functions:

$$\max_{g_{m,n} \in \mathcal{P}_{d+e}} \left| \sum_i w_i g_{m,n}(z_i) - \int g_{m,n} d\xi \right|$$

with normalization $\int g_{m,n}^2 d\xi = 2$ (the area of the right triangle). Many of the quadrature sets are invariant under the symmetry group of rotations and reflections of the triangle, D_3 . The solutions which do not have this symmetry are denoted with *asym* in the table.

Degree of cardinal functions (d)	Number of Points (N)	Degree of Exact Integration (d+e)	Error	Notes
1	3	2	4.4×10^{-16}	
2	6	4	9.7×10^{-16}	
3	10	5	1.7×10^{-14}	
4	15	7	2.1×10^{-14}	
5	21	9	2.8×10^{-14}	
6	28	11	4.7×10^{-15}	asym
7	36	13	2.2×10^{-14}	asym,new
8	45	14	1.8×10^{-15}	
9	55	16	8.6×10^{-15}	asym,new
10	66	18	3.3×10^{-14}	asym,new
11	78	20	2.8×10^{-14}	asym,new
12	91	21	2.9×10^{-14}	new
13	105	23	3.3×10^{-14}	new
14	120	25	4.3×10^{-14}	asym,new

TABLE 5.1

Quadrature points computed with the cardinal function algorithm. In all cases, the quadrature weights are positive and the points are not outside the triangle. Solutions which are not D_3 symmetric are denoted by asym. Solutions which improve upon previously published results are denoted by new.

Quadrature formulas denoted by *new* in the table represent formulas which improve upon the best previously published results, as taken from the extensive database described in [5] as well as the quadrature points presented in [15] (which are not included in the database as of this writing). The new solutions for integration degree $d + e$ from 16 to 25 have fewer points than the previously published results. For $d + e = 13$, the previous result with the fewest number of quadrature points [2] has $N = 36$. The points in [2] are symmetric, but some are outside the triangle and not

all weights are positive. The result presented here also has 36 points, all of which are inside the triangle, the weights are positive, but the points are asymmetric.

Plots for all of the quadrature points are given in the figures. For the plots, the right triangle has been mapped lineally to the equilateral triangle in order to make the asymmetry in the points more visible. The coordinates of the quadrature points are given in Appendix A. They are available electronically by downloading the \TeX source of this paper from the arXiv.

6. Summary. We have presented results from a cardinal function algorithm for computing multi-variate quadrature points. The algorithm was applied to the triangle, where optimal (in the sense of Equations 5.1 and 5.2) formulas were constructed for integrating polynomials up to degree 25. Seven of these quadrature formulas improve on previously known results.

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Appendix A. Plots of quadrature points.

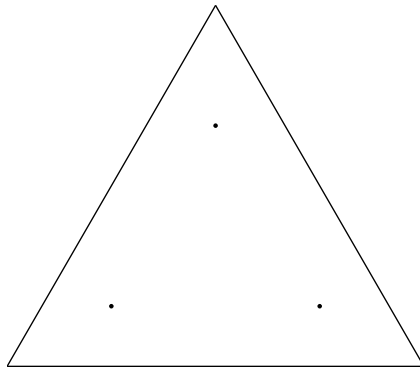


FIG. A.1. *Quadrature points for the triangle which exactly integrate polynomials of degree 2.*

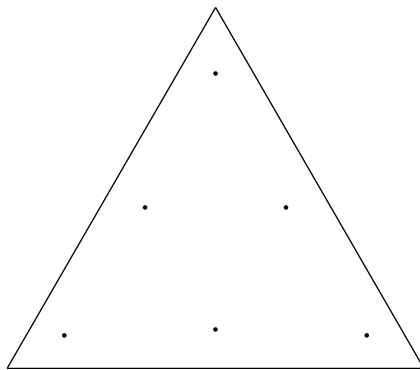


FIG. A.2. *Quadrature points for the triangle which exactly integrate polynomials of degree 4.*

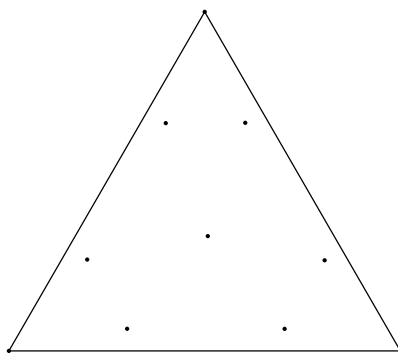


FIG. A.3. *Quadrature points for the triangle which exactly integrate polynomials of degree 5.*

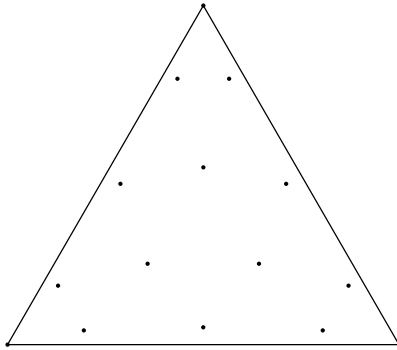


FIG. A.4. *Quadrature points for the triangle which exactly integrate polynomials of degree 7.*

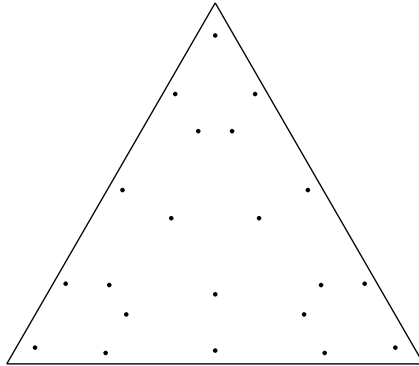


FIG. A.5. *Quadrature points for the triangle which exactly integrate polynomials of degree 9.*

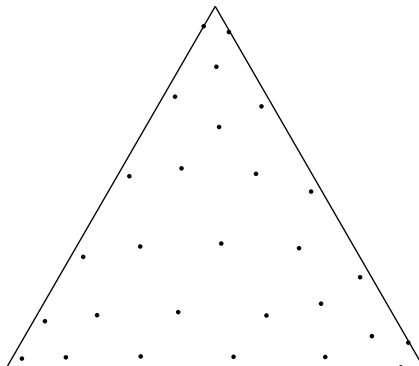


FIG. A.6. *Quadrature points for the triangle which exactly integrate polynomials of degree 11.*

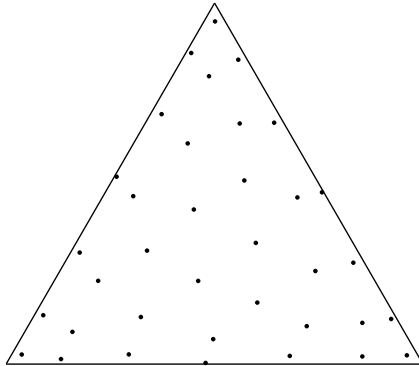


FIG. A.7. *Quadrature points for the triangle which exactly integrate polynomials of degree 13.*

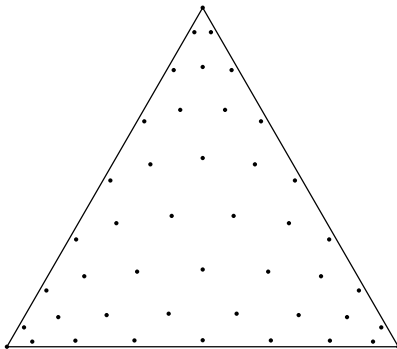


FIG. A.8. *Quadrature points for the triangle which exactly integrate polynomials of degree 14.*

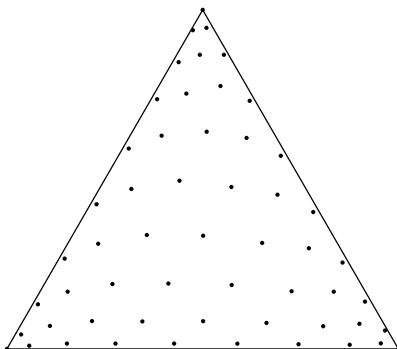


FIG. A.9. *Quadrature points for the triangle which exactly integrate polynomials of degree 16.*

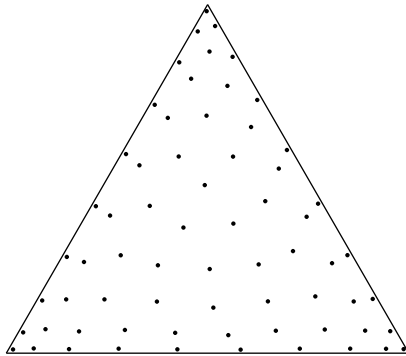


FIG. A.10. *Quadrature points for the triangle which exactly integrate polynomials of degree 18.*

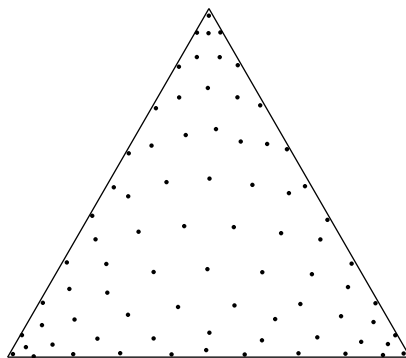


FIG. A.11. *Quadrature points for the triangle which exactly integrate polynomials of degree 20.*

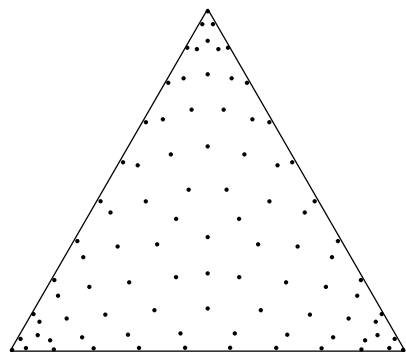


FIG. A.12. *Quadrature points for the triangle which exactly integrate polynomials of degree 21.*

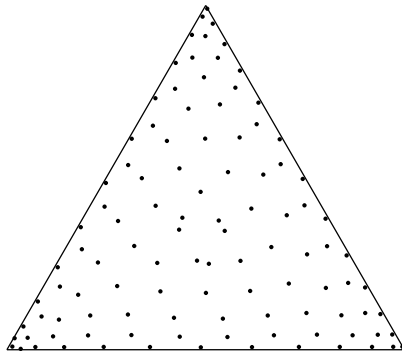


FIG. A.13. *Quadrature points for the triangle which exactly integrate polynomials of degree 23.*

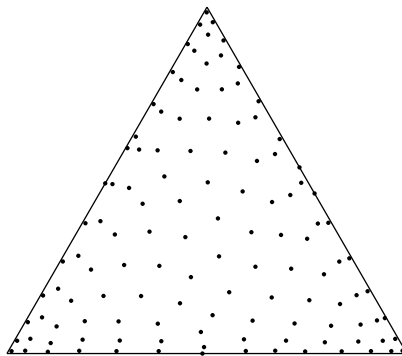


FIG. A.14. *Quadrature points for the triangle which exactly integrate polynomials of degree 25.*

Appendix B. Tables of quadrature points. We now list the coordinates of the quadrature points described in Table 5. For each line, we give the first two barycentric coordinates of each point (equivalent to the x and y coordinates after an equilateral triangle is linearly mapped to the unit right triangle $x \geq 0$, $y \geq 0$ and $x + y \leq 1$) followed by the associated quadrature weight. The third barycentric coordinate is defined such that the sum of all three coordinates is one. These points are available electronically by downloading the \TeX source of this paper from the arXiv.

Tables of Points					
integration degree=2 N=3:			0.1927920403641 0.4036039798179 0.1881601469167		
0.1666666666667	0.6666666666667	0.6666666666667	integration degree=11 N=28:		
0.6666666666667	0.1666666666667	0.6666666666667	0.0000000000000	0.9451704450174	0.0114082494033
0.1666666666667	0.1666666666667	0.6666666666667	0.9451704450173	0.0000000000000	0.0114082494033
integration degree=4 N=6:			0.9289002405719	0.0685505797224	0.0132691285720
0.0915762135098	0.0915762135098	0.2199034873106	0.0685505797224	0.9289002405717	0.0132691285720
0.8168475729805	0.0915762135098	0.2199034873106	0.0243268355615	0.0243268355616	0.0155865773350
0.0915762135098	0.8168475729805	0.2199034873106	0.1279662835335	0.0277838749488	0.0408274780428
0.1081030181681	0.4459484909160	0.4467631793560	0.0277838749488	0.1279662835337	0.0408274780429
0.4459484909160	0.1081030181681	0.4467631793560	0.0287083428360	0.7498347588657	0.0579849665116
0.4459484909160	0.4459484909160	0.4467631793560	0.7498347588656	0.0287083428360	0.0579849665116
integration degree=5 N=10:			0.7228007909707	0.2497602062385	0.0601385247663
0.0000000000000	1.0000000000000	0.0262712099504	0.2497602062386	0.7228007909707	0.0601385247663
1.0000000000000	0.0000000000000	0.0262716612068	0.0865562992839	0.8325513856997	0.0625273888433
0.0000000000000	0.0000000000000	0.0274163947600	0.8325513856998	0.0865562992839	0.0625273888433
0.2673273531185	0.6728199218710	0.2348383865823	0.3061619157672	0.0303526617491	0.0639684321504
0.6728175529461	0.2673288599482	0.2348412238268	0.0303526617491	0.3061619157675	0.0639684321504
0.0649236350054	0.6716530111494	0.2480251793114	0.4868610595047	0.4868610595047	0.0661325872161
0.6716498539042	0.0649251690029	0.2480304922521	0.6657904293017	0.1765456154219	0.0668503236820
0.0654032456800	0.2693789366453	0.2518604605529	0.1765456154221	0.6657904293016	0.0668503236821
0.2693767069140	0.0654054874919	0.2518660533658	0.0293121007360	0.5295657488669	0.0686904305977
0.3386738503896	0.3386799893027	0.4505789381914	0.5295657488667	0.0293121007360	0.0686904305977
integration degree=7 N=15:			0.1444673824391	0.1444673824391	0.1002717543859
1.0000000000000	0.0000000000000	0.0102558174092	0.3299740111411	0.5361815729050	0.1143136784099
0.0000000000000	0.0000000000000	0.0102558174092	0.5361815729052	0.3299740111409	0.1143136784099
0.0000000000000	0.0000000000000	0.0102558174092	0.5511507516862	0.1437790861923	0.1223648146752
0.7839656651012	0.0421382841642	0.1116047046647	0.1437790861923	0.5511507516862	0.1223648146752
0.1738960507345	0.7839656651012	0.1116047046647	0.3348066587327	0.1529619437161	0.1394422334178
0.1738960507345	0.0421382841642	0.1116047046647	0.1529619437161	0.3348066587327	0.1394422334178
0.0421382841642	0.1738960507345	0.1116047046647	0.3430183498147	0.3430183498147	0.1744377829182
0.7839656651012	0.1738960507345	0.1116047046647	integration degree=13 N=36:		
0.0421382841642	0.7839656651012	0.1116047046647	0.0242935351590	0.9493059293846	0.0166240998757
0.4743880861752	0.4743880861752	0.1679775595335	0.0265193427722	0.0242695130640	0.0166811699778
0.4743880861752	0.0512238276497	0.1679775595335	0.9492126023551	0.0265067966437	0.0166830569067
0.0512238276497	0.4743880861752	0.1679775595335	0.0033775763749	0.4767316412363	0.0175680870083
0.2385615300181	0.5228769399639	0.2652238803946	0.4757672298101	0.5198921829102	0.0184474661845
0.5228769399639	0.2385615300181	0.2652238803946	0.5190783193471	0.0055912706202	0.0197942410188
0.2385615300181	0.2385615300181	0.2652238803946	0.8616839745321	0.0133996048618	0.0203540395855
integration degree=9 N=21:			0.1249209759926	0.8613054321334	0.0206852863940
0.0451890097844	0.0451890097844	0.0519871420646	0.0138565453861	0.1247733717358	0.0208271366086
0.0451890097844	0.9096219804312	0.0519871420646	0.0211887064222	0.8438438351223	0.0317819778279
0.9096219804312	0.0451890097844	0.0519871420646	0.8432296787219	0.1354563645830	0.0320472035241
0.7475124727339	0.0304243617288	0.0707034101784	0.1354231797865	0.0213482820656	0.0320607681146
0.2220631655373	0.0304243617288	0.0707034101784	0.3088853510679	0.0221919663014	0.0430765959183
0.7475124727339	0.2220631655373	0.0707034101784	0.6685057595169	0.3089012879389	0.0438473415339
0.2220631655373	0.7475124727339	0.0707034101784	0.0226545012557	0.6691709943321	0.0439209672733
0.0304243617288	0.7475124727339	0.0707034101784	0.2808515408772	0.6924718155106	0.0479951923691
0.0304243617288	0.2220631655373	0.0707034101784	0.6922446749051	0.0268723345026	0.0483806260733
0.1369912012649	0.2182900709714	0.0909390760952	0.0268617447119	0.2810093973222	0.0484867423375
0.6447187277637	0.2182900709714	0.0909390760952	0.1141778485470	0.7973581413586	0.0556964488024
0.1369912012649	0.6447187277637	0.0909390760952	0.7974807922061	0.0879806508791	0.0561026364356
0.2182900709714	0.1369912012649	0.0909390760952	0.0892807293894	0.1145020561128	0.0565190123693
0.2182900709714	0.2182900709714	0.0909390760952	0.1052487892455	0.6686904119922	0.0689289890670
0.6447187277637	0.1369912012649	0.0909390760952	0.6663022280740	0.2275051631832	0.0717213336089
0.0369603304334	0.4815198347833	0.1032344051380	0.2307803737547	0.1054572561221	0.0727453920976
0.4815198347833	0.0369603304334	0.1032344051380	0.1705059157540	0.5174064398658	0.0788807336737
0.4815198347833	0.4815198347833	0.1032344051380	0.5086593973043	0.3170523855209	0.0810114345512
0.4036039798179	0.1927920403641	0.1881601469167	0.3141823862281	0.1810706361659	0.0825725299055
0.4036039798179	0.4036039798179	0.1881601469167	0.4617460817864	0.4678594539804	0.0842044567330
			0.0693087496081	0.4622856042085	0.0843585533305

0.4651955259268 0.0724357805669 0.0851969868488 0.2662302843647 0.0144783956308 0.0224186693682
0.2578625857893 0.6131395039177 0.0902845328052 0.8673504065214 0.0591679410400 0.0230122616993
0.6112627766779 0.1300360834609 0.0914283143485 0.0741493666957 0.8634782575061 0.0236813902500
0.1305182135934 0.2581713828884 0.0916279065409 0.0159285948360 0.4191238955238 0.0257464643368
0.4281437991828 0.2362005969817 0.1025573374896 0.0156061028068 0.5809222921146 0.0257956801608
0.3356995783730 0.4311026308588 0.1033159661413 0.5910094817484 0.0159251452651 0.0258072327610
0.2305424298836 0.3456013949376 0.1035854367193 0.4034771496889 0.5806700368104 0.0260343232059
integration degree=14 N=45:
0.0000000000000 1.0000000000000 0.0010616711990 0.0678493700650 0.0761218678591 0.0265784761831
1.0000000000000 0.0000000000000 0.0010616711990 0.4265968590272 0.0157509692312 0.0267532329238
0.0000000000000 0.0000000000000 0.0010616711990 0.0670982507890 0.7741898312421 0.0375787806641
0.0573330873026 0.0151382269814 0.0131460236101 0.7528310231480 0.0819119495639 0.0383065894195
0.0573330873026 0.0151382269814 0.0131460236101 0.7753727783557 0.1577128457292 0.0412364778098
0.9275286857160 0.0573330873026 0.0131460236101 0.1689073157787 0.7503943099742 0.0389619825852
0.0151382269814 0.0573330873026 0.0131460236101 0.1687335832919 0.0708311507268 0.0394604111547
0.9275286857160 0.0151382269814 0.0131460236101 0.0821244708436 0.1762996626771 0.0518230042269
0.0151382269814 0.9275286857160 0.0131460236101 0.6288705363345 0.0807744953317 0.0512872438483
0.8159625040711 0.1659719969565 0.0242881926949 0.0811413015266 0.3054373589776 0.0516405641935
0.8159625040711 0.0180654989724 0.0242881926949 0.2969112065080 0.6227485988871 0.0518230042269
0.1659719969565 0.8159625040711 0.0242881926949 0.0767542314171 0.6247247149546 0.0528527988181
0.0180654989724 0.8159625040711 0.0242881926949 0.6223022338845 0.3011485821166 0.0538505573027
0.1659719969565 0.0180654989724 0.0242881926949 0.3103786288051 0.0779098365079 0.0541895329319
0.0180654989724 0.1659719969565 0.0242881926949 0.0819218215187 0.4603633038351 0.0584737146444
0.3165475556378 0.0186886898773 0.0316799866332 0.4717022665013 0.0821554006797 0.0592863168363
0.6647637544849 0.0186886898773 0.0316799866332 0.4546603415250 0.4637565033890 0.0594358276749
0.0186886898773 0.6647637544849 0.0316799866332 0.1701091339237 0.6422277808188 0.0631800255863
0.0186886898773 0.3165475556378 0.0316799866332 0.6406004329487 0.1898293537256 0.0632926845153
0.3165475556378 0.6647637544849 0.0316799866332 0.1912267583717 0.1739955685343 0.064070361772
0.6647637544849 0.3165475556378 0.0316799866332 0.1885315767070 0.4798914070406 0.0812040595918
0.0192662192492 0.4903668903754 0.0349317947036 0.4772929957691 0.3348356598119 0.0814437513530
0.4903668903754 0.0192662192492 0.0349317947036 0.3126974621760 0.4957972197259 0.0814679201241
0.4903668903754 0.4903668903754 0.0349317947036 0.4961225945946 0.1927553668904 0.0815050548084
0.0875134669581 0.8249730660837 0.0383664533945 0.1928805312867 0.3161015807261 0.0815164664939
0.0875134669581 0.0875134669581 0.0383664533945 0.3360041453816 0.1894892801290 0.0816931059623
0.8249730660837 0.0875134669581 0.0383664533945 0.3337280550848 0.3343571021811 0.0923218334531
integration degree=18 N=66:
0.0935526036219 0.2079865423167 0.0578369491210 0.0116731059668 0.9812565951289 0.0025165756986
0.935526036219 0.6984608540613 0.0578369491210 0.9810030858388 0.0071462504863 0.0025273452007
0.2079865423167 0.0935526036219 0.0578369491210 0.0106966317092 0.0115153933376 0.0033269295333
0.6984608540613 0.0935526036219 0.0578369491210 0.6984608540613 0.0126627518417 0.0370123620615 0.0086135525742
0.2079865423167 0.6984608540613 0.0578369491210 0.0974892983467 0.5380088595149 0.0725821687394
0.3645018421383 0.0974892983467 0.0725821687394 0.3645018421383 0.0974892983467 0.0725821687394
0.5380088595149 0.0974892983467 0.0725821687394 0.5380088595149 0.3645018421383 0.0725821687394
0.3645018421383 0.5380088595149 0.0725821687394 0.3645018421383 0.0974892983467 0.0725821687394
0.0974892983467 0.3645018421383 0.0725821687394 0.0974892983467 0.5565708210253 0.0897856524107
0.5565708210253 0.2217145894873 0.0897856524107 0.2217145894873 0.2217145894873 0.0897856524107
0.2217145894873 0.2217145894873 0.0897856524107 0.3860471669296 0.2279056661408 0.1034544533617
0.3860471669296 0.2279056661408 0.1034544533617 0.3860471669296 0.3860471669296 0.1034544533617
integration degree=16 N=55:
1.0000000000000 0.0000000000000 0.0006202599851 0.2763025250863 0.0121741311386 0.0170953520675
0.0000000000000 1.0000000000000 0.0006315174712 0.0109658368561 0.4194306712466 0.0173888854559
0.0000000000000 0.0000000000000 0.0007086601559 0.4289110517884 0.5599616067469 0.0174543962439
0.9398863583577 0.0049848744634 0.0055163716168 0.4215420555115 0.0116475994785 0.0178406757287
0.0543806683058 0.9386405618617 0.0062692407656 0.5711258590444 0.0118218313989 0.0178446863879
0.0093940049164 0.0526424462697 0.0078531408826 0.5826868270511 0.4057889581177 0.0179046337552
0.0164345086362 0.9469035517351 0.0094551483864 0.0130567806713 0.2725023750868 0.0181259756201
0.9469487269862 0.0363373677167 0.0097824511271 0.0130760400964 0.7224712523233 0.0184784838882
0.0426604005768 0.0151224541799 0.0099861643489 0.7263437062407 0.2602984019251 0.0185793564371
0.0122269495439 0.8693773510664 0.0137553818816 0.0687230068637 0.0631417272721 0.0203217151777
0.8673696521047 0.1204917285774 0.0140979178040 0.8652302101529 0.0720611837338 0.0213771661809
0.8456744021389 0.0157763967870 0.0149646864337 0.0648599071037 0.8590433543910 0.0231916854098
0.1395759632103 0.8448120870375 0.0156097503612 0.1483494943362 0.7888788352240 0.0274426710859
0.1317821743231 0.0135009605584 0.0157683693348 0.0624359898396 0.1493935499354 0.0290301922340
0.0157955126300 0.1455274938536 0.0175794546383 0.7871369011735 0.0656382042757 0.0294522738505
0.7365462884436 0.0155697540908 0.0204113840270 0.0519104921610 0.5255635695605 0.0299436251629
0.0139688430330 0.7379836894450 0.0209562878616 0.1543129927444 0.0716383926917 0.0307026948119
0.2547895186039 0.7297615689771 0.0210713412998 0.2617842745603 0.0621479485288 0.0325263365863
0.7316386522555 0.2543076683315 0.0217646760202 0.767257872813 0.1658211554831 0.0327884208506
0.0157253728951 0.2696239795791 0.0222288408699 0.2582103676627 0.6800119766139 0.0331234675192

0.0679065925147 0.7571515437782 0.0346167526875 0.0567625702001 0.7409121894959 0.0305668796074
0.5293578274804 0.4121503841107 0.0347081373976 0.0483837933475 0.6075135660978 0.0306067413002
0.0666036150484 0.2612513087886 0.0347372049404 0.1080612809760 0.1122081510437 0.0309330068201
0.0585675461899 0.3902236114535 0.0348528762454 0.6185605900991 0.2698753703035 0.0309773820835
0.0644535360411 0.6373626559761 0.0348601561186 0.7721296013497 0.1114117395333 0.0313146250545
0.6748138429151 0.0637583342061 0.0355471569975 0.6115734801133 0.3389367677931 0.0313573493392
0.3914602310369 0.5503238090563 0.0360182996383 0.3381326103376 0.0494693938787 0.0314320469287
0.6487701492307 0.2836728360263 0.0362926285843 0.1173084128254 0.7696451309795 0.0315182143894
0.3946498220408 0.0605175522554 0.0381897702083 0.2674551260596 0.1115718808154 0.0324248137985
0.5390137151933 0.0611990176936 0.0392252800118 0.6542100160026 0.1906548314700 0.0347512152386
0.1627895082785 0.6861322141035 0.0482710125888 0.0538297481158 0.3358616826849 0.0350393454927
0.6812436322641 0.1567968345899 0.0489912121566 0.1848840324117 0.1551831523851 0.0350717420310
0.1542832287802 0.1667512624020 0.0497220833872 0.3376267104744 0.6081402596294 0.0352129215334
0.2522727750445 0.2504803933395 0.0507065736986 0.6067102034499 0.0542632795598 0.0352615504981
0.2547981532407 0.4994090649043 0.0509771994043 0.4612614085496 0.0688176670722 0.0366403220343
0.1485580549194 0.5756023096087 0.0521360063667 0.1525465365671 0.6510240845749 0.0367733107670
0.2930239606436 0.5656897354162 0.0523460874925 0.0700582543543 0.4661904392742 0.0371675662937
0.2808991272310 0.1437921574248 0.0524440683552 0.4704201379032 0.4634826455353 0.0373371571606
0.4820989592971 0.2518557535865 0.0527459644823 0.1216461693746 0.2381494875516 0.0403973346588
0.5641878245444 0.1462966743153 0.0529449063728 0.6371404052702 0.1238399384513 0.0413580040638
0.1307699644344 0.4489577586117 0.0542395594501 0.2379904515119 0.6370216452326 0.0421957791870
0.1479692221948 0.3001174386829 0.0543470203419 0.1483929857177 0.4894188577780 0.0495451004037
0.5638684222946 0.2813772089298 0.0547100548639 0.3598069571550 0.1452880866253 0.0500419261141
0.4361157428790 0.4252053446420 0.0557288345913 0.4941441055095 0.3610216383818 0.0505794587115
0.3603263935285 0.2599190004889 0.0577734264233 0.1440360687981 0.3513508341887 0.0520037210188
0.4224188334674 0.1453238443303 0.0585393781623 0.5019764440004 0.1435491663293 0.0521533567886
0.3719001833052 0.3780122703567 0.0609039250680 0.3555423834298 0.5016491599502 0.0524899152358
0.2413645006928 0.3847563284940 0.0637273964449 0.2443439540771 0.2406052129104 0.05499159762516
integration degree=20 N=78:
0.0089411337112 0.0086983293702 0.0021744545399 0.2437064989342 0.5109017277055 0.0599609997426
0.9792622629807 0.0102644133744 0.0028987135265 0.5122200807321 0.2452737973543 0.0599915272129
0.0105475382112 0.9785514202515 0.0030846029337 0.2526038315178 0.3700319555094 0.06341333449
0.0023777061947 0.0636551098604 0.0034401633104 0.3759895652851 0.2505406611631 0.0635311861108
0.0630425115795 0.0041506347509 0.0041898472012 0.3729077987144 0.3753750277549 0.0637206605672
0.9308422496730 0.0048053482263 0.0044738051498 integration degree=21 N=91:
0.0629076555490 0.9316790069481 0.0047054420814 0.0035524391922 0.0035524391922 0.0006704436439
0.93159624478305 0.062624881801 0.0048867935750 0.0035524391922 0.9928951216156 0.0006704436439
0.0061951689415 0.9293587058564 0.0051927643369 0.9553548273730 0.0087898929093 0.0045472608074
0.0287125819237 0.0310202122997 0.0074073058981 0.0358552797177 0.0087898929093 0.0045472608074
0.9293844478305 0.0342152968219 0.0079755410301 0.9553548273730 0.0358552797177 0.0045472608074
0.0375457566621 0.9257868884669 0.0083550522910 0.0087898929093 0.0358552797177 0.0045472608074
0.0086895739064 0.1584971251510 0.0096166660864 0.0087898929093 0.9553548273730 0.0045472608074
0.1547597053965 0.8363606657688 0.0096318257850 0.0358552797177 0.9553548273730 0.0045472608074
0.8331025294185 0.0089257244824 0.0098577460758 0.8865264879047 0.1082329745017 0.0052077585320
0.8374231073526 0.1529167304078 0.0102657880301 0.8865264879047 0.0052405375935 0.0052077585320
0.1559362505234 0.0094966240058 0.0103188103111 0.0052405375935 0.1082329745017 0.0052077585320
0.0098599642095 0.8342211493596 0.0106291001630 0.0052405375935 0.8865264879047 0.0052077585320
0.4055873733289 0.0074389302008 0.0106881306895 0.1082329745017 0.8865264879047 0.0052077585320
0.5964727898618 0.3956330809311 0.0106969021010 0.1082329745017 0.0052405375935 0.0052077585320
0.0080747800416 0.4031319425903 0.0109026461714 0.0466397432150 0.9067205135700 0.0065435432887
0.0075073977721 0.5851609594681 0.0109899783575 0.0466397432150 0.0466397432150 0.0065435432887
0.3936764519237 0.5974896592899 0.0113423055229 0.9067205135700 0.0466397432150 0.0065435432887
0.5846530726212 0.0087250464968 0.0120535642930 0.2075720456946 0.0082759241284 0.0092737841533
0.4870804112120 0.0202129229912 0.0139619193821 0.2075720456946 0.7841520301770 0.0092737841533
0.2683512811785 0.7202340088668 0.0141147991536 0.7841520301770 0.2075720456946 0.0092737841533
0.7223956288748 0.2662399366456 0.0141930347046 0.0082759241284 0.7841520301770 0.0092737841533
0.2716826742357 0.0112882698808 0.0144212676268 0.0082759241284 0.2075720456946 0.0092737841533
0.0112580842046 0.7169695963325 0.0144704346855 0.7841520301770 0.0082759241284 0.0092737841533
0.0115034734370 0.2740067110166 0.0144949769872 0.0858119489725 0.0314836947701 0.0095937782623
0.7140525900564 0.0113511560497 0.0145386775694 0.8827043562574 0.0314836947701 0.0095937782623
0.4902871053112 0.4936491841468 0.0145964190926 0.8827043562574 0.0858119489725 0.0095937782623
0.0201423425209 0.4832573459601 0.0147314578466 0.8827043562574 0.8827043562574 0.0095937782623
0.0361107464859 0.0935679501582 0.0167463963304 0.8827043562574 0.0858119489725 0.0095937782623
0.8607998819851 0.0397379067075 0.0168955500458 0.0314836947701 0.8827043562574 0.0095937782623
0.1005891526001 0.8586343419352 0.0169422662884 0.6688778233826 0.0095150760625 0.0114247809167
0.0918740717058 0.0395513001973 0.0173070172095 0.0095150760625 0.3216071005550 0.0114247809167
0.860488296191 0.0966224057079 0.0174524546493 0.0095150760625 0.6688778233826 0.0114247809167
0.0439842178673 0.8561886349107 0.0177217222159 0.6688778233826 0.3216071005550 0.0114247809167
0.2011017606735 0.7449115835626 0.0282824024023 0.3216071005550 0.6688778233826 0.0114247809167
0.7449993726263 0.0536865638166 0.0284996712488 0.3216071005550 0.0095150760625 0.0114247809167
0.0532186641310 0.1963754275935 0.0285005646539 0.4379999543113 0.0099859785681 0.0117216964174
0.7453984647401 0.1982065805550 0.0300647223478 0.0099859785681 0.5520140671206 0.0117216964174
0.1957289932876 0.0555713833156 0.0302031277082 0.4379999543113 0.5520140671206 0.0117216964174
0.1092532057988 0.6100036182413 0.0303987136077 0.0099859785681 0.4379999543113 0.0117216964174

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0.7974931072148 0.0405093994119 0.0188197155232 0.2740287679608 0.7173981847948 0.0087963441074
0.0405093994119 0.1619974933734 0.0188197155232 0.7173982224778 0.2740287304386 0.0087963448112
0.0405093994119 0.7974931072148 0.0188197155232 0.2394976858234 0.0081859182262 0.0091304195716
0.1619974933734 0.7974931072148 0.0188197155232 0.0081859185845 0.2394975566677 0.0091304213611
0.7974931072148 0.1619974933734 0.0188197155232 0.0068836152075 0.4843740892687 0.0092821748751
0.1619974933734 0.0405093994119 0.0188197155232 0.4843741485699 0.0068836232949 0.0092821815662
0.3864215551955 0.3864215551955 0.0235260980271 0.4960767772741 0.4960767529507 0.0094499806178
0.3864215551955 0.2271568896090 0.0235260980271 0.6112936776245 0.3804323691239 0.0094627468484
0.2271568896090 0.3864215551955 0.0235260980271 0.3804323980345 0.6112936466533 0.0094627485294
0.8090129379329 0.0954935310336 0.0235571466151 0.7303890713524 0.0083987179701 0.0095555772285
0.0954935310336 0.8090129379329 0.0235571466151 0.0083987168639 0.7303890895407 0.0095555792843
0.0954935310336 0.0954935310336 0.0235571466151 0.6128525675612 0.0075475979695 0.0096138842488
0.2745425238718 0.0479840480721 0.0268246207430 0.0075475961037 0.6128525484582 0.0096138846826
0.0479840480721 0.6774734280561 0.0268246207430 0.0079525316513 0.3559773826721 0.0099991524212
0.6774734280561 0.0479840480721 0.0268246207430 0.3559774870460 0.0079525358502 0.0099991551850
0.6774734280561 0.2745425238718 0.0268246207430 0.9110236977966 0.0437233665345 0.0100301319277
0.2745425238718 0.6774734280561 0.0268246207430 0.0437233605166 0.9110236807446 0.0100301346636
0.0479840480721 0.2745425238718 0.0268246207430 0.0388480061835 0.0967030908282 0.0124936676185
0.4053472446667 0.5429849622344 0.0314289776779 0.0967032117936 0.0388479942386 0.0124936726125
0.0516677930989 0.4053472446667 0.0314289776779 0.0873226911312 0.0873226620391 0.0140197309137
0.4053472446667 0.0516677930989 0.0314289776779 0.0421445202084 0.8485617789108 0.0143336216896
0.5429849622344 0.0516677930989 0.0314289776779 0.8485617974961 0.0421445420915 0.0143336272125
0.0516677930989 0.5429849622344 0.0314289776779 0.8477921333864 0.1067435942472 0.0153604142740
0.5429849622344 0.4053472446667 0.0314289776779 0.1067435889398 0.8477921328146 0.0153604183425
0.1877738615539 0.1068148267588 0.0337196192159 0.1833966521991 0.0416340521608 0.0184523825614
0.7054113116872 0.1877738615539 0.0337196192159 0.0416340541167 0.1833965196930 0.0184523863146
0.7054113116872 0.1068148267588 0.0337196192159 0.7611632251560 0.1941599202852 0.0195833983573
0.1068148267588 0.7054113116872 0.0337196192159 0.1941599254144 0.7611632153938 0.0195834019994
0.1877738615539 0.7054113116872 0.0337196192159 0.7579378747173 0.0439826608586 0.0197632751342
0.1068148267588 0.1877738615539 0.0337196192159 0.0439826512395 0.7579378242308 0.0197632766677
0.1195059712009 0.3057122990643 0.0427745294213 0.0369760535918 0.5363186076436 0.0198806391019
0.1195059712009 0.5747817297348 0.0427745294213 0.5363187134342 0.0369760780935 0.0198806485776
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