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Hyper generalized pseudo Q-symmetric semi-Riemannian manifolds

Adara M. Blaga[1]
Manoj Ray Bakshi[2]
Kanak Kanti Baishya[3]

[1] *Department of Mathematics, West University of Timişoara, Timişoara, România.*
adarablaga@yahoo.com

[2,3] *Department of Mathematics, Kurseong College, Kurseong, Darjeeling, India.*
raybakshimanoj@gmail.com; kanakkanti.kc@gmail.com

ABSTRACT

The object of the present paper is to study the properties of a hyper generalized pseudo Q-symmetric semi-Riemannian manifold, proving that under certain assumptions, it is a perfect fluid spacetime.

RESUMEN

El objetivo del presente artículo es estudiar las propiedades de una variedad semi-Riemanniana hiper generalizada pseudo Q-simétrica, probando que bajo ciertas condiciones, es un espacio-tiempo fluido perfecto.

Keywords and Phrases: Q-curvature tensor, perfect fluid spacetime.

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1 Introduction

Let R, S, L and r denote the curvature tensor, Ricci tensor, Ricci operator and the scalar curvature of a (semi)-Riemannian manifold, respectively. It is Mantica and Suh [5] who have introduced the notion of *Q-curvature tensor.* In an n-dimensional Riemannian or semi-Riemannian manifold (M^n, g) $(n > 2)$, the Q-curvature tensor is defined as

$$R(Y,U,V,W) = Q(Y,U,V,W) + \frac{\psi}{n-1}[g(Y,W)g(U,V) - g(Y,V)g(U,W)], \tag{1.1}$$

where Y, U, V, W are arbitrary vector fields on M^n and ψ is a scalar function. Semi-Riemannian manifolds with Ricci tensor S of the form

$$S(Y,V) = ag(Y,V) + bT(Y)T(V),$$

for any vector fields Y, V, are often termed as perfect fluid spacetimes, where a and b are scalars and the vector field ϱ, metrically equivalent to the 1-form T (that is, $g(Y,\varrho) = T(Y)$), is a unit time like vector field (that is, $g(\varrho,\varrho) = -1$).

An n-dimensional semi-Riemannian manifold is said to be *hyper generalized pseudo Q-symmetric* (which will be abbreviated hereafter as $(HGPQS)_n$) if it satisfies the equation

$$\begin{aligned} & (\nabla_X Q)(Y,U,V,W) \qquad (1.2)\\ = \; & 2A_1(X)Q(Y,U,V,W) + A_1(Y)Q(X,U,V,W) \\ & + A_1(U)Q(Y,X,V,W) + A_1(V)Q(Y,U,X,W) \\ & + A_1(W)Q(Y,U,V,X) + 2A_2(X)(g \wedge S)(Y,U,V,W) \\ & + A_2(Y)(g \wedge S)(X,U,V,W) + A_2(U)(g \wedge S)(Y,X,V,W) \\ & + A_2(V)(g \wedge S)(Y,U,X,W) + A_2(W)(g \wedge S)(Y,U,V,X), \end{aligned}$$

where

$$\begin{aligned} (g \wedge S)(Y,U,V,W) &= g(Y,W)S(U,V) + g(U,V)S(Y,W) \\ &\quad - g(Y,V)S(U,W) - g(U,W)S(Y,V) \end{aligned} \tag{1.3}$$

and A_1, A_2 are non-zero 1-forms whose g-dual vector fields will be denoted by θ_1 and θ_2, i.e. $A_1(X) = g(X,\theta_1)$ and $A_2(X) = g(X,\theta_2)$.

We organized our paper as follows: section 2 is concerned with preliminaries. After preliminaries, some curvature properties of $(HGPQS)_n$ manifolds are studied in section 3. It is obtained that the Q-curvature tensor in a $(HGPQS)_n$ manifold satisfies 2nd Bianchi's identity. It is further obtained that the scalar function ψ is always constant. In section 4 we investigate properties of divergence-free $(HGPQS)_n$ manifolds and we prove that a divergence-free $(HGPQS)_n$ manifold $(n > 2)$ under the assumption $A_1(Q(Y,U)V) = 0$ is a perfect fluid spacetime as well as the integral curves of the vector field ϱ are geodesics and the vector field ϱ is irrotational, if the associated vector fields ϱ and σ corresponding to the 1-forms T_1 and T_2 are related by $(r-1)\varrho + n\sigma = 0$.

2 Preliminaries

In this section, some relations useful to the study of $(HGPQS)_n$ manifolds are obtained. Let $\{e_i\}$ be an orthonormal basis of the tangent space at each point of the manifold, where $1 \le i \le n$.

From (1.1) we can easily verify that the tensor Q satisfies the following properties:

$$\begin{aligned} &(i) \quad Q(Y,U)V + Q(U,Y)V = 0, \\ &(ii) \quad Q(Y,U)V + Q(U,V)Y + Q(V,Y)U = 0, \end{aligned} \tag{2.1}$$

where $g(Q(X,Y)U,V) = Q(X,Y,U,V)$.

Also from (1.1) we have

$$\sum_{i=1}^{n} \epsilon_i Q(X,Y,e_i,e_i) = 0 = \sum_{i=1}^{n} \epsilon_i Q(e_i,e_i,W,U) \tag{2.2}$$

and

$$\begin{aligned} \sum_{i=1}^{n} \epsilon_i Q(e_i,Y,V,e_i) &= \sum_{i=1}^{n} \epsilon_i Q(Y,e_i,e_i,V) = S(Y,V) - \psi g(Y,V) \\ &=: Z(Y,V), \end{aligned} \tag{2.3}$$

where

$$\epsilon_i = g(e_i,e_i) = \pm 1, \ S(X,Y) = \sum_{i=1}^{n} \epsilon_i g(R(X,e_i)e_i,Y), \ r = \sum_{i=1}^{n} \epsilon_i S(e_i,e_i).$$

From (1.1) and (2.1) it follows that

$$\begin{aligned} &(i) \quad Q(X,Y,U,V) + Q(X,Y,V,U) = 0, \\ &(ii) \quad Q(X,Y,U,V) - Q(U,V,X,Y) = 0. \end{aligned} \tag{2.4}$$

3 Some curvature properties of $(HGPQS)_n$ manifolds

In this section we prove that in a $(HGPQS)_n$ manifold, the Q-curvature tensor satisfies 2nd Bianchi's identity, that is,

$$(\nabla_X Q)(Y,U,V,W) + (\nabla_Y Q)(U,X,V,W) + (\nabla_U Q)(X,Y,V,W) = 0. \tag{3.1}$$

In view of (1.1), (1.2) and (3.1) we get

$$\begin{aligned}
&(\nabla_X Q)(Y,U,V,W) + (\nabla_Y Q)(U,X,V,W) + (\nabla_U Q)(X,Y,V,W) \\
= \; & A_1(V)[Q(Y,U,X,W) + Q(U,X,Y,W) + Q(X,Y,U,W)] \\
&+A_1(W)[Q(Y,U,V,X) + Q(U,X,V,Y) + Q(X,Y,V,U)] \\
&+A_2(V)[(g \wedge S)(Y,U,X,W) + (g \wedge S)(U,X,Y,W) \\
&+(g \wedge S)(X,Y,U,W)] + A_2(W)[(g \wedge S)(Y,U,V,X) \\
&+(g \wedge S)(U,X,V,Y) + (g \wedge S)(X,Y,V,U)].
\end{aligned} \tag{3.2}$$

Using (1.3) and 1st Bianchi's identity for the Q-curvature tensor in (3.2) and then simplifying, we obtain (3.1).

Thus we can state the following:

Theorem 3.1. *The Q-curvature tensor in a $(HGPQS)_n$ manifold satisfies 2nd Bianchi's identity.*

Using (1.1) in (3.1), we have

$$\begin{aligned}
(\nabla_X R)(Y,U,V,W) \; &+ \; (\nabla_Y R)(U,X,V,W) + (\nabla_U R)(X,Y,V,W) \\
&- \; \frac{d\psi(X)}{(n-1)}[g(Y,W)g(U,V) - g(Y,V)g(U,W)] \\
&- \; \frac{d\psi(Y)}{(n-1)}[g(U,W)g(X,V) - g(U,V)g(X,W)] \\
&- \; \frac{d\psi(U)}{(n-1)}[g(X,W)g(Y,V) - g(X,V)g(Y,W)] = 0.
\end{aligned} \tag{3.3}$$

By virtue of 2nd Bianchi's identity for the Riemannian curvature tensor, (3.3) yields

$$\begin{aligned}
&\frac{d\psi(X)}{(n-1)}[g(Y,W)g(U,V) - g(Y,V)g(U,W)] \\
+ &\frac{d\psi(Y)}{(n-1)}[g(U,W)g(X,V) - g(U,V)g(X,W)] \\
+ &\frac{d\psi(U)}{(n-1)}[g(X,W)g(Y,V) - g(X,V)g(Y,W)] = 0.
\end{aligned} \tag{3.4}$$

Contracting U and V in (3.4), we have

$$(n-2)[d\psi(X)g(Y,W) - d\psi(Y)g(X,W)] = 0 \tag{3.5}$$

which yields after further contraction

$$(n-1)(n-2)d\psi(X) = 0.$$

This implies that $d\psi(X) = 0$, that is, ψ is constant since $n > 2$ and leads to the following:

Theorem 3.2. *In a $(HGPQS)_n$ manifold, the scalar function ψ is always constant.*

Consequently, one can easily bring out the following:

Theorem 3.3. *In a $(HGPQS)_n$ manifold, $(divQ)(X,Y)Z$ and $(divR)(X,Y)Z$ are equivalent.*

In view of (1.1), (1.2) and Theorem 3.2 we have

$$\begin{aligned} & (\nabla_X R)(Y,U,V,W) \\ = \; & 2A_1(X)Q(Y,U,V,W) + A_1(Y)Q(X,U,V,W) \\ & +A_1(U)Q(Y,X,V,W) + A_1(V)Q(Y,U,X,W) \\ & +A_1(W)Q(Y,U,V,X) + 2A_2(X)(g \wedge S)(Y,U,V,W) \\ & +A_2(Y)(g \wedge S)(X,U,V,W) + A_2(U)(g \wedge S)(Y,X,V,W) \\ & +A_2(V)(g \wedge S)(Y,U,X,W) + A_2(W)(g \wedge S)(Y,U,V,X) \end{aligned} \tag{3.6}$$

which yields

$$\begin{aligned} & (\nabla_X S)(U,V) \\ = \; & [F_1(X) + F_2(X)]S(U,V) + F_2(U)S(X,V) + F_2(V)S(U,X) \\ & +[F_3(X) + F_4(X)]g(U,V) + F_4(U)g(X,V) + F_4(V)g(U,X) \\ & +A_1(Q(X,U)V) - A_1(Q(V,X)U) \end{aligned} \tag{3.7}$$

after contraction over Y and W, where

$$\begin{aligned} F_1(X) &= A_1(X) + (n+1)A_2(X), \\ F_2(X) &= A_1(X) + (n-3)A_2(X), \\ F_3(X) &= rA_2(X) - \psi A_1(X) + 3A_2(LX), \\ F_4(X) &= rA_2(X) - \psi A_1(X) - A_2(LX), \end{aligned}$$

where L is the Ricci operator defined by $g(LX,Y) = S(X,Y)$.

Definition 3.4. *An n-dimensional semi-Riemannian manifold is called almost generalized pseudo Ricci symmetric if the non-flat Ricci curvature tensor satisfies the equation*

$$\begin{aligned} & (\nabla_X S)(U,V) \\ = \; & [A(X) + B(X)]S(U,V) + A(U)S(X,V) + A(V)S(U,X) \\ & +[C(X) + D(X)]g(U,V) + C(U)g(X,V) + C(V)g(U,X), \end{aligned}$$

where A, B, C and D are non-zero 1-forms whose g-dual vector fields will be denoted by γ_1, γ_2, δ_1 and δ_2, i.e. $A(X) = g(X,\gamma_1)$, $B(X) = g(X,\gamma_2)$, $C(X) = g(X,\delta_1)$ and $D(X) = g(X,\delta_2)$.

Thus we can state the following:

Theorem 3.5. *A $(HGPQS)_n$ manifold $(n > 2)$ under the assumption $A_1(Q(X,U)V) = A_1(Q(V,X)U)$ is necessarily almost generalized pseudo Ricci symmetric.*

Making use of (2.3) in (3.7), we get

$$\begin{aligned} & (\nabla_X Z)(U,V) \\ = \; & [F_1(X) + F_2(X)]Z(U,V) + F_2(U)Z(X,V) + F_2(V)Z(U,X) \\ & +[F_3(X) + \psi F_1(X) + F_4(X) + \psi F_2(X)]g(U,V) \\ & +[F_4(U) + \psi F_2(U)]g(X,V) + [F_4(V) + \psi F_2(V)]g(U,X), \end{aligned} \tag{3.8}$$

where $Z = S - \psi g$ is the tensor considered in ([4], [6], [7]). This leads to the following:

Theorem 3.6. *A $(HGPQS)_n$ manifold $(n > 2)$ under the assumption $A_1(Q(X,U)V) = A_1(Q(V,X)U)$ is necessarily almost generalized pseudo Z-symmetric.*

4 $(\mathbf{HGPQS})_n$ manifolds $(n > 2)$ with $div\mathbf{Q} = 0$

Let (M^n, g) be a semi-Riemannian manifold of dimension n and let $\{e_i\}$ be an orthonormal basis of the tangent space T_pM at any point $p \in M$ and $\epsilon_i = \pm 1$. Then the divergence of a vector field U is defined as

$$divU = \sum_{i=1}^{n} \epsilon_i g(\nabla_{e_i} U, e_i),$$

and the divergence of a tensor field of type $(1,3)$, which is a tensor field of type $(0,3)$, is defined as

$$(divK)(X,Y)Z = \sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} K)(X,Y)Z, e_i).$$

Now

$$\begin{aligned} (divQ)(Y,U)V \; &= \; \sum_{i=1}^{n} \epsilon_i g((\nabla_{e_i} Q)(Y,U)V, e_i) \\ &= \; \sum_{i=1}^{n} \epsilon_i [2A_1(e_i)Q(Y,U,V,e_i) + A_1(Y)Q(e_i,U,V,e_i) \\ & \quad +A_1(U)Q(Y,e_i,V,e_i) + A_1(V)Q(Y,U,e_i,e_i) \\ & \quad +A_1(e_i)Q(Y,U,V,e_i) + 2A_2(e_i)(g \wedge S)(Y,U,V,e_i) \\ & \quad +A_2(Y)(g \wedge S)(e_i,U,V,e_i) + A_2(U)(g \wedge S)(Y,e_i,V,e_i) \\ & \quad +A_2(V)(g \wedge S)(Y,U,e_i,e_i) + A_2(e_i)(g \wedge S)(Y,U,V,e_i)] \end{aligned}$$

$$\begin{aligned}
&= 3A_1(Q(Y,U)V) + A_1(Y)[S(U,V) - \psi g(U,V)] \\
&\quad -A_1(U)[S(Y,V) - \psi g(Y,V)] + 3A_2(Y)S(U,V) \\
&\quad +3A_2(LY)g(U,V) - 3A_2(LU)g(Y,V) - 3A_2(U)S(Y,V) \\
&\quad +A_2(Y)[(n-2)S(U,V) + rg(U,V)] \\
&\quad -A_2(U)[(n-2)S(Y,V) + rg(Y,V)] \\
&= 3A_1(Q(Y,U)V) + S(U,V)[A_1(Y) + (n+1)A_2(Y)] \\
&\quad -S(Y,V)[A_1(U) + (n+1)A_2(U)] \\
&\quad +g(U,V)[3A_2(LY) + rA_2(Y) - \psi A_1(Y)] \\
&\quad -g(Y,V)[3A_2(LU) + rA_2(U) - \psi A_1(U)] \\
&= 3A_1(Q(Y,U)V) + T_1(Y)S(U,V) - T_1(U)S(Y,V) \\
&\quad +T_2(Y)g(U,V) - T_2(U)g(Y,V),
\end{aligned}$$

hence

$$\begin{aligned}
(divQ)(Y,U)V &= 3A_1(Q(Y,U)V) + T_1(Y)S(U,V) - T_1(U)S(Y,V) \\
&\quad +T_2(Y)g(U,V) - T_2(U)g(Y,V),
\end{aligned} \tag{4.1}$$

where

$$\begin{aligned}
T_1(Y) &= A_1(Y) + (n+1)A_2(Y) =: g(Y,\varrho), \text{ for } \varrho = \theta_1 + (n+1)\theta_2, \\
T_2(Y) &= 3A_2(LY) + rA_2(Y) - \psi A_1(Y) =: g(Y,\sigma), \text{ for } \sigma = 3L\theta_2 + r\theta_2 - \psi\theta_1.
\end{aligned}$$

Assuming $(divQ)(Y,U)V = 0$ and $A_1(Q(Y,U)V) = 0$, we get from the above equation

$$T_1(Y)S(U,V) + T_2(Y)g(U,V) = T_1(U)S(Y,V) + T_2(U)g(Y,V). \tag{4.2}$$

Now contracting (4.2) over U and V we get

$$S(Y,\varrho) = rT_1(Y) + (n-1)T_2(Y). \tag{4.3}$$

Again putting $V = \varrho$ in (4.2) we get

$$(n-2)[T_1(Y)T_2(U) - T_1(U)T_2(Y)] = 0, \tag{4.4}$$

which under the assumption $n > 2$ implies $T_1(Y)T_2(U) = T_1(U)T_2(Y)$.

Now putting $U = \varrho$ in (4.2) and using (4.3) and (4.4) we get

$$T_1(\varrho)S(Y,V) + T_2(\varrho)g(Y,V) = T_1(Y)[rT_1(V) + nT_2(V)] \tag{4.5}$$

and we can state:

Theorem 4.1. *A divergence-free* $(HGPQS)_n$ *manifold* $(n > 2)$ *under the assumption* $A_1(Q(Y,U)V) = 0$ *is a perfect fluid spacetime with unit timelike vector field* ϱ*, provided the associated vector fields* ϱ *and* σ *corresponding to the 1-forms* T_1 *and* T_2 *are related by* $(r-1)\varrho + n\sigma = 0$.

In this case, (4.5) becomes

$$S(Y,V) = ag(Y,V) - T_1(Y)T_1(V), \tag{4.6}$$

where $a =: T_2(\varrho)$.

Again, $(divQ)(Y,U)V = 0$ gives

$$(\nabla_Y S)(U,V) - (\nabla_U S)(Y,V) = 0. \tag{4.7}$$

Now using (4.6) in (4.7) we find

$$\begin{aligned} & da(Y)g(U,V) - da(U)g(Y,V) \\ & -[T_1(V)(\nabla_Y T_1)(U) + T_1(U)(\nabla_Y T_1)(V)] \\ & +[T_1(V)(\nabla_U T_1)(Y) + T_1(Y)(\nabla_U T_1)(V)] = 0. \end{aligned} \tag{4.8}$$

Taking a frame field and contracting Y and V we get

$$(n-1)da(U) + [T_1(U)(\delta T_1) + (\nabla_\varrho T_1)(U)] = 0, \tag{4.9}$$

where

$$\delta T_1 = \sum_{i=1}^{n} \epsilon_i (\nabla_{e_i} T_1)(e_i).$$

Setting $V = Y = \varrho$ in (4.8) we find

$$(\nabla_\varrho T_1)(U) = -da(U) - da(\varrho)T_1(U). \tag{4.10}$$

Substituting (4.10) in (4.9) we get

$$(n-2)da(U) + T_1(U)(\delta T_1) - da(\varrho)T_1(U) = 0 \tag{4.11}$$

which yields

$$\delta T_1 = (n-1)da(\varrho) \tag{4.12}$$

for $U = \varrho$.

Using (4.12) in (4.11) we obtain

$$da(U) = -T_1(U)da(\varrho), \tag{4.13}$$

provided $n > 2$.

Putting $V = \varrho$ in (4.8) and using (4.13) we get

$$(\nabla_Y T_1)(U) - (\nabla_U T_1)(Y) = 0.$$

This means that the 1-form T_1 is closed, that is,

$$dT_1(Y, U) = 0.$$

Hence

$$g(\nabla_U \varrho, Y) = g(\nabla_Y \varrho, U) \text{ for all } U, Y, \tag{4.14}$$

which yields

$$g(\nabla_\varrho \varrho, Y) = g(\nabla_Y \varrho, \varrho), \tag{4.15}$$

for $U = \varrho$. Since $g(\nabla_Y \varrho, \varrho) = 0$, from (4.15) it follows that $g(\nabla_\varrho \varrho, Y) = 0$ for all Y. Hence $\nabla_\varrho \varrho = 0$. This implies that the integral curves of the vector field ϱ are geodesics. Therefore we can state the following:

Theorem 4.2. *In a divergence-free $(HGPQS)_n$ manifold $(n > 2)$ under the assumption $A_1(Q(Y,U)V) = 0$, the integral curves of the unit timelike vector field ϱ are geodesics, provided the associated vector fields ϱ and σ corresponding to the 1-forms T_1 and T_2 are related by $(r-1)\varrho + n\sigma = 0$.*

Taking into account that the divergence of the conformal curvature tensor of a Riemannian manifold (M^n, g) is ([3], [6]):

$$\begin{aligned}(divC)(X,Y)Z &= \frac{n-3}{n-2}[(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z)] \\ &= \frac{n-3}{n-2}(divQ)(X,Y)Z,\end{aligned} \tag{4.16}$$

for any vector fields X, Y, Z on M^n, from the Lemma 2.1 of [2] we infer

Theorem 4.3. *Let (M, g) be a $(HGPQS)_n$ perfect fluid spacetime $(n > 2)$. If $(divQ)(X,Y)Z = 0$, for any vector fields X, Y, Z on M, then the unit timelike vector field ϱ is irrotational.*

Also, in [2] was proved the following result:

Theorem 4.4. [2] *Let (M, g) be a $(HGPQS)_n$ perfect fluid spacetime $(n > 2)$. If $(divQ)(X,Y)Z = 0$, for any vector fields X, Y, Z on M, then (M, g) is a GRW spacetime whose fiber is Einstein.*

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