Extended domain for fifth convergence order schemes

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ABSTRACT

We provide a local as well as a semi-local analysis of a fifth convergence order scheme involving operators valued on Banach space for solving nonlinear equations. The convergence domain is extended resulting a finer convergence analysis for both types. This is achieved by locating a smaller domain included in the older domain leading this way to tighter Lipschitz type functions. These extensions are obtained without additional hypotheses. Numerical examples are used to test the convergence criteria and also to show the superiority for our results over earlier ones. Our idea can be utilized to extend other schemes using inverses in a similar way.

RESUMEN

Entregamos un análisis local y uno semi-local de un esquema de quinto orden de convergencia que involucra operadores con valores en un espacio de Banach para resolver ecuaciones no-lineales. El dominio de convergencia es extendido resultando en un análisis de convergencia más fino para ambos tipos. Esto se logra ubicando un dominio más pequeño incluido en el dominio antiguo, entregando funciones de tipo Lipschitz más ajustadas. Estas extensiones se obtienen sin hipótesis adicionales. Se usan ejemplos numéricos para verificar los criterios de convergencia y también para mostrar que nuestros resultados son superiores a otros anteriores. Nuestra idea se puede utilizar para extender otros esquemas usando inversos de manera similar.

1 Introduction

In this article, B_1, B_2 are standing for Banach spaces, $D \subset B_1$ is denoting a convex and open set, and $F : D \longrightarrow B_2$ is considered differentiable according to the Fréchet notion. One of the most important tasks is the location of a solution x_* of nonlinear equation

$$F(x) = 0. \tag{1.1}$$

Solving equation $F(x) = 0$ is useful because using modeling (Mathematical) problems from many areas can be formulated as (1.1). The explosion of technology requires the development of higher convergence schemes. Starting from the quadratically convergent Newton's method higher order schemes develop all the time [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19].

Recently, Singh et al. [13] provided a semi-local convergence for efficient fifth order scheme under Lipschitz continuity on F'' defined as follows

$$
\begin{aligned}
y_n &= x_n - F'(x_n)^{-1} F(x_n) \\
z_n &= y_n - F'(x_n)^{-1} F(y_n) \\
x_{n+1} &= z_n - F'(y_n)^{-1} F(z_n).
\end{aligned}
\tag{1.2}
$$

Later in [14] the applicability of scheme (1.2) was extended using w- continuity conditions. In general, the convergence domain is small. That is why we develop a technique where a tighter domain than before is obtained containing the iterates. This way the new w-functions are tighter leading to a finer semi-local convergence analysis. It is worth noticing that these extensions do not involve new hypotheses because the new w-functions are specializations of the old one. Hence, we extend the applicability of the method. It turns out that the local convergence analysis can be extended too.

For example: Let $B_1 = B_2 = \mathbb{R}$, $\Omega = [-\frac{1}{2}, \frac{3}{2}]$. Define G on Ω by

$$
G(x) = \begin{cases} x^3 \log x^2 + x^5 - x^4, & x \neq 0 \\ 0, & x = 0. \end{cases}
$$

Then, we get $x_* = 1$, and

$$G'(x) = 3x^2 \log x^2 + 5x^4 - 4x^3 + 2x^2,$$

$$G''(x) = 6x \log x^2 + 20x^3 - 12x^2 + 10x,$$

$$G'''(x) = 6 \log x^2 + 60x^2 - 24x + 22.$$

Obviously $G'''(x)$ is not bounded on Ω. So, the convergence of scheme (1.2) is not guaranteed by the analysis in [13, 14]. In this study we use only assumptions on the first derivative to prove our results. Relevant studies can be found in [6, 19].

The structure of the rest of the article involves local and semi-local convergence analysis in Section 2 and Section 3, respectively. The numerical experiments appear in Section 4.

2   Local convergence

It is easier for the local convergence of method (1.2), if we develop some real functions. We start with a function ω_0 defined on the interval $I = [0, \infty)$ with values in I satisfying $\omega_0(0) = 0$. Assume equation

$$\omega_0(t) = 1 \tag{2.1}$$

has a least positive solution called ρ_0. Assume the existence of function ω, continuous increasing defined on $I_0 = [0, \rho_0)$ with values in I satisfying $\omega(0) = 0$. Define functions λ_1 and μ_1 on I_0 as follows

$$\lambda_1(t) = \frac{\int_0^1 \omega((1-\theta)t)d\theta}{1 - \omega_0(t)}$$

and

$$\mu_1(t) = \lambda_1(t) - 1.$$

These definitions lead to $\mu_1(0) = -1$ and $\mu_1(t) \longrightarrow \infty$ as $t \longrightarrow \rho_0^-$. Then, the theorem on intermediate value assure the existence of solutions for the equation $\mu_1(t) = 0$ in $(0, \rho_0)$. Set R_1 to be the least such solution. Assume equation

$$\omega_0(\lambda_1(t)t) = 1 \tag{2.2}$$

has a least positive solution called ρ_1. Set $I_1 = [0, \rho_2)$, $\rho_2 = \min\{\rho_0, \rho_1\}$. Define functions λ_2 and μ_2 on I_1 as follows

$$\lambda_2(t) = \frac{\int_0^1 \omega((1-\theta)\lambda_1(t)t)d\theta\lambda_1(t)}{1 - \omega_0(\lambda_1(t)t)}$$

and

$$\mu_2(t) = \lambda_2(t) - 1.$$

This time we also have $\lambda_2(0) = -1$ and $\lambda_2(t) \longrightarrow \infty$ as $t \longrightarrow \rho_2^-$. Call R_2 the smallest solution of equation $\lambda_2(t) = 0$ in $(0, \rho_2)$. Assume equation

$$\omega_0(\lambda_2(t)t) = 1 \tag{2.3}$$

has a least positive solution called ρ_3. Set $I_2 = [0, \rho_4)$, $\rho_4 = \min\{\rho_2, \rho_3\}$. Consider functions λ_3 and μ_3 on I_2 as follows

$$\lambda_3(t) = \left[\frac{\int_0^1 \omega((1-\theta)\lambda_2(t)t)d\theta}{1 - \omega_0(\lambda_2(t)t)} + \frac{(\omega_0(\lambda_2(t)t) + \omega_0(\lambda_1(t)t)\int_0^1 v(\theta\lambda_2(t)t)d\theta}{(1 - \omega_0(\lambda_2(t)t))(1 - \omega_0(\lambda_1(t)t))} \right] \lambda_2(t)$$

and

$$\mu_3(t) = \lambda_3(t) - 1,$$

where $v : I_2 \longrightarrow I$ is an increasing and continuous function. By these functions, we obtain $\mu_3(0) = -1$ and $\mu_3(t) \longrightarrow \infty$ as $t \longrightarrow \rho_4^-$. Let R_3 stand for the smallest solution of equation $\mu_3(t) = 0$ in $(0, \rho_4)$. A radius of convergence can be given as follows

$$R = \min\{R_i\}, \ i = 1, 2, 3. \tag{2.4}$$

Then, for all $t \in [0, R)$.

$$0 \ \leq \ \omega_0(t) < 1 \tag{2.5}$$

$$0 \ \leq \ \omega_0(\lambda_1(t)t) < 1 \tag{2.6}$$

$$0 \ \leq \ \omega_0(\lambda_1(t)t) < 1 \tag{2.7}$$

$$0 \ \leq \ \omega_0(\lambda_2(t)t) < 1 \tag{2.8}$$

and

$$0 \leq \lambda_i(t) < 1. \tag{2.9}$$

Denote by $U(x_*, \gamma)$ a ball of center x_* and with a radius $\gamma > 0$. Then, $\bar{U}(x_*, \gamma)$ stands for the closure of $U(x_*, \gamma)$.

We base the local convergence on this notation and the conditions (C).

(c1) $F : D \longrightarrow B_2$ is differentiable according to Fréchet, and $x_* \in D$ with $F(x_*) = 0$ is a simple solution.

(c2) There exists an increasing and continuous real function ω_0 on I satisfying $\omega_0(0) = 0$ and such that for all $x \in D$

$$\|F'(x_*)^{-1}(F'(x) - F'(x_*))\| \leq \omega_0(\|x - x_*\|).$$

Set $U_0 = D \cap U(x_*, \rho_0)$.

(c3) There exists a function ω on I_0 continuous and increasing satisfying $\omega(0) = 0$ such that for all $x, y \in U_0$

$$\|F'(x_*)^{-1}(F'(y) - F'(x))\| \leq \omega(\|y - x\|).$$

Set $U_1 = D \cap U(x_*, \rho_4)$.

(c4) There exists a function v on I_2 continuous and increasing, such that for all $x \in U_1$

$$\|F'(x_*)^{-1}F'(x)\| \leq v(\|x - x_*\|).$$

(c5) $\bar{U}(x_*, R) \subseteq D$.

(c6) There exists $R_1 \geq R$ such that

$$\int_0^1 \omega_0(\theta R_1)d\theta < 1.$$

Set $U_2 = D \cap \bar{U}(x_*, R_1)$.

Theorem 2.1. *Assume hypotheses (C) hold and starting point* $x_0 \in U(x^*, R) - \{x^*\}$. *Then the following assertions are valid, sequence* $\{x_n\}$ *belongs in* $U(x_*, R) - \{x_*\}$ *and converges to* $x_* \in U(x_*, R)$ *so that this limit point uniquely solves equation* $F(x) = 0$ *in the set* U_2.

Proof. Let $z \in U(x^*, R) - \{x_*\}$ and utilize (c2), (2.4) and (2.5) to obtain

$$\|F'(x_*)^{-1}(F'(z) - F'(x_*))\| \leq \omega_0(\|z - x_*\|) \leq \omega_0(R) < 1,$$

which together with a result by Banach [12] for linear operators whose inverse exists imply

$$\|F'(z)^{-1}F'(x_*)\| \leq \frac{1}{1 - \omega_0(\|z - x^*\|)}. \tag{2.10}$$

In particular, by scheme (1.2) y_0, z_0 are well defined since if we set $z = x_0 \in U(x_*, R) - \{x_*\}$, and $F'(x_0)$ is invertible. Then, by (2.4), (2.8) (for $k = 1$), (c1), (c3) and (2.10) (for $z = x_0$), we have

$$
\begin{aligned}
\|y_0 - x_*\| &= \|x_0 - x_* - F'(x_0)^{-1}F(x_0)\| \\
&\leq \|F'(x_0)^{-1}F'(x_*)\| \left[\int_0^1 \|F'(x_*)^{-1}[F'(x_0 + \theta(x_0 - x_*)) - F'(x_0)](x_0 - x_*)d\theta\| \right] \\
&\leq \frac{\int_0^1 \omega((1-\theta)\|x_0 - x_*\|)d\theta}{1 - \omega_0(\|x_0 - x_*\|)}\|x_0 - x_*\| \\
&\leq \lambda_1(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\| < R. \tag{2.11}
\end{aligned}
$$

Hence, $y_0 \in U(x_*, R)$. Using the second substep of method (1.2) and replacing x_0, y_0, by y_0, z_0, respectively as in (2.10) and (2.11), we get

$$
\begin{aligned}
\|z_0 - x_*\| &\leq \frac{\int_0^1 \omega((1-\theta)\|y_0 - x_*\|)d\theta}{1 - \omega_0(\|y_0 - x_*\|)}\|y_0 - x_*\| \\
&\leq \frac{\int_0^1 \omega((1-\theta)\lambda_1(\|x_0 - x_*\|)\|x_0 - x_*\|)d\theta\lambda_1(\|x_0 - x_*\|)\|x_0 - x_*\|}{1 - \omega_0(\lambda_1(\|x_0 - x_*\|)\|x_0 - x_*\|} \\
&\leq \lambda_2(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|. \tag{2.12}
\end{aligned}
$$

That is $z_0 \in U(x_*, R)$ and also x_1 exists (for $y_0 = z$, in (2.10)). Notice that (c1), (c4), (2.12) and

$$F(z_0) = F(z_0) - F(x_*) = \int_0^1 F'(x_* + \theta(z_0 - x_*))d\theta(z_0 - x_*),$$

we obtain that

$$
\begin{aligned}
&\|F'(x_*)^{-1}F'(z_0)\| \\
&\leq \int_0^1 v(\theta\|z_0 - x_*\|)d\theta\|z_0 - x_*\| \\
&\leq \int_0^1 v(\theta\lambda_2(\|x_0 - x_*\|)\|x_0 - x_*\|d\theta\lambda_2(\|x_0 - x_*\|)\|x_0 - x_*\|. \tag{2.13}
\end{aligned}
$$

Moreover, by the last substep of method (1.2), (2.4), (2.5), (2.8) (for $k = 3$), (2.10), (2.13) (for

$z = x_0, y_0$), (2.11) and (2.12), we have in turn that

$$
\begin{aligned}
\|x_1 - x_*\| &\leq \|z_0 - x_* - F'(z_0)^{-1}F(z_0)\| &(2.14)\\
&\quad + \|F'(z_0)^{-1}[(F'(y_0) - F'(x_*)) + (F'(x_*) - F'(z_0))]F'(y_0)^{-1}F(z_0)\|\\
&\leq \Biggl[\frac{\int_0^1 \omega((1-\theta)\|z_0 - x_*\|)d\theta}{1 - \omega_0(\|z_0 - x_*\|)}\\
&\quad + \frac{(\omega_0(\|z_0 - x_*\|) + \omega_0(\|y_0 - x_*\|))\int_0^1 v(\theta\|z_0 - x_*\|)d\theta}{(1 - \omega_0(\|z_0 - x_*\|))(1 - \omega_0(\|y_0 - x_*\|))} \Biggr] \|z_0 - x_*\|\\
&\leq \lambda_3(\|x_0 - x_*\|)\|x_0 - x_*\| \leq \|x_0 - x_*\|, &(2.15)
\end{aligned}
$$

so $x_1 \in U(x_*, R)$. Replacing x_0, y_0, z_0, x_1 by x_k, y_k, z_k, x_{k+1}, in the previous computations we obtain

$$
\|y_k - x_*\| \leq \lambda_1(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\| < R, \tag{2.16}
$$

$$
\|z_k - x_*\| \leq \lambda_2(\|x_k - x_*\|)\|x_k - x_*\| \leq \|x_k - x_*\| \tag{2.17}
$$

and

$$
\|x_{k+1} - x^*\| \leq \lambda_3(\|x_k - x^*\|)\|x_k - x_*\| \leq \|x_k - x_*\|, \tag{2.18}
$$

so y_k, z_k, x_{k+1} stay in $U(x_*, R)$ and $\lim_{k \to \infty} x_k = x_*$. Furthermore, let $x_*^1 \in U_2$ with $F(x_*^1) = 0$. In view of (c2) and (c6) we obtain

$$
\begin{aligned}
\left\| F'(x_*)^{-1}\left(\int_0^1 F'(x_* + \theta(x_*^1 - x_*))d\theta - F'(x_*) \right) \right\| &\leq \int_0^1 \omega_0(\theta\|x_*^1 - x_*\|)d\theta\\
&\leq \int_0^1 \omega_0(\theta R_1)d\theta < 1,
\end{aligned}
$$

so $x_*^1 = x_*$, since $T = \int_0^1 F'(x_* + \theta(x_*^1 - x_*))d\theta$ is invertible and

$$
0 = F(x_*^1) - F(x_*) = T(x_*^1 - x_*).
$$

$\square$

Remark 2.2. *1. In view of (2.10) and the estimate*

$$
\begin{aligned}
\|F'(x^*)^{-1}F'(x)\| &= \|F'(x^*)^{-1}(F'(x) - F'(x^*)) + I\|\\
&\leq 1 + \|F'(x^*)^{-1}(F'(x) - F'(x^*))\| \leq 1 + L_0\|x - x^*\|
\end{aligned}
$$

condition (2.13) can be dropped and M can be replaced by

$$
M(t) = 1 + L_0 t
$$

or

$$
M(t) = M = 2,
$$

since $t \in [0, \frac{1}{L_0})$.

2. *The results obtained here can be used for operators F satisfying autonomous differential equations [2] of the form*

$$F'(x) = P(F(x))$$

where P is a continuous operator. Then, since $F'(x^*) = P(F(x^*)) = P(0)$, we can apply the results without actually knowing x^*. For example, let $F(x) = e^x - 1$. Then, we can choose: $P(x) = x + 1$.

3. *Let $\omega_0(t) = L_0 t$, and $\omega(t) = Lt$. In [2, 3] we showed that $r_A = \frac{2}{2L_0 + L}$ is the convergence radius of Newton's method:*

$$x_{n+1} = x_n - F'(x_n)^{-1} F(x_n) \text{ for each } n = 0, 1, 2, \cdots \tag{2.19}$$

under the conditions (2.11) and (2.12). It follows from the definition of R in (2.4) that the convergence radius R of the method (1.2) cannot be larger than the convergence radius r_A of the second order Newton's method (2.19). As already noted in [2, 3] r_A is at least as large as the convergence radius given by Rheinboldt [12]

$$r_R = \frac{2}{3L}, \tag{2.20}$$

where L_1 is the Lipschitz constant on D. The same value for r_R was given by Traub [15]. In particular, for $L_0 < L_1$ we have that

$$r_R < r_A$$

and

$$\frac{r_R}{r_A} \to \frac{1}{3} \text{ as } \frac{L_0}{L_1} \to 0.$$

That is the radius of convergence r_A is at most three times larger than Rheinboldt's.

4. *It is worth noticing that method (1.2) is not changing when we use the conditions of Theorem 2.1 instead of the stronger conditions used in [13, 14]. Moreover, we can compute the computational order of convergence (COC) defined by*

$$\xi = \ln \left(\frac{\|x_{n+1} - x^*\|}{\|x_n - x^*\|} \right) / \ln \left(\frac{\|x_n - x^*\|}{\|x_{n-1} - x^*\|} \right)$$

or the approximate computational order of convergence

$$\xi_1 = \ln \left(\frac{\|x_{n+1} - x_n\|}{\|x_n - x_{n-1}\|} \right) / \ln \left(\frac{\|x_n - x_{n-1}\|}{\|x_{n-1} - x_{n-2}\|} \right).$$

This way we obtain in practice the order of convergence in a way that avoids the bounds involving estimates using estimates higher than the first Fréchet derivative of operator F. Note also that the computation of ξ_1 does not require the usage of the solution x^*.

3   Semi-local convergence analysis

Let $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(B_2, B_1)$ exists at $x_0 \in D$, where $\mathcal{L}(B_2, B_1)$ denotes the set of bounded linear operators from B_2, B_1 and the following conditions hold.

(1) $\|\Gamma_0\| \leq \beta_0$.

(2) $\|\Gamma_0 F(x_0)\| \leq \eta_0$.

(3)' $\|F'(x) - F'(x_0)\| \leq M_0\|x - x_0\|$ for all $x \in D$. Set $D_0 = D \cap U\left(x_0, \frac{1}{\beta_0 M_0}\right)$.

(3) $\|F''(x)\| \leq M$ for all $x \in D_0$.

(4) $\|F''(x) - F''(y)\| \leq \omega(\|x - y\|)$ for all $x, y \in D_0$ for a continuous nondecreasing function ω, $\omega(0) \geq 0$ such that $\omega(tx) \leq t^p\omega(x)$ for $t \in [0, 1], x \in (0, \infty)$ and $p \in [0, 1]$.

Then, as in [13, 14], let $r_0 = M\beta_0\eta_0$, $s_0 = \beta_0\eta_0\omega(\eta_0)$ and define sequences $\{r_k\}, \{s_k\}$ and $\{\eta_k\}$ for $k = 0, 1, 2, \ldots$, by

$$r_{k+1} = r_k\varphi(r_k)^2\psi(r_k, s_k), \tag{3.1}$$

$$s_{k+1} = s_k\varphi(r_k)^{2+p}\psi(r_k, s_k)^{1+p}, \tag{3.2}$$

$$\eta_{k+1} = \eta_k\varphi(r_k)\psi(r_k, s_k), \tag{3.3}$$

where

$$\varphi(t) = \frac{1}{1 - tg(t)} \tag{3.4}$$

$$g(t) = \left(1 + \frac{t}{2} + \frac{t^2}{2(1-t)}\left(1 + \frac{t}{4}\right)\right) \tag{3.5}$$

and

$$\psi(t, s) = \frac{t^2}{2(1-t)}\left(1 + \frac{t}{4}\right)\left[\frac{s}{1+p}\left(\frac{t^{1+p}}{2^{1+p}} + \frac{1}{2+p}\left(\frac{t^2}{2(1-t)}\left(1 + \frac{t}{4}\right)\right)^{1+p}\right)\right.$$
$$\left. + \frac{t}{2}\left(t + \frac{t^2}{2(1-t)}\left(1 + \frac{t}{4}\right)\right)\right]. \tag{3.6}$$

Remark 3.1. *In [14] the following conditions were used instead of (3), (4), respectively*

(3)' $\|F''(x)\| \leq M_1$ for all $x \in D$

(4)' $\|F''(x) - F''(y)\| \leq \omega_1(\|x - y\|)$ for all $x, y \in D$ and ω_1 as ω.

    But, we have

$$D_0 \subseteq D,$$

    so

$$M_0 \leq M_1$$

$$M \leq M_1$$

and

$$\omega(\theta) \leq \omega_1(\theta).$$

Examples where the preceding items are strict can be found in [1, 2, 3, 4, 5, 6]. Notice that (3)' is used to determine D_0 leading to $M = M(D_0, x_)$. Hence, the results in [13, 14] can be rewritten with M replacing M_1. So, if $M < M_1$ the new semi-local convergence analysis is finer. This is also done under the same computational effort because in practice finding ω_1, M_1 requires finding ω, M_0, M as special cases. This technique can be used to extend the applicability of other schemes involving inverses in an analogous fashion. Hence, the proof of the following semi-local convergence result for scheme (1.2) is omitted.

Theorem 3.2. *Let $r_0 = M\beta_0\eta_0 < \nu, s_0 = \beta_0\eta_0\omega(\eta_0)$ and assumptions (1)-(4) hold. Then, for $\bar{U}(x_0, R\eta_0) \subseteq D$, where $R = \frac{g(r_0)}{1-\delta\gamma}$, the sequence $\{x_k\}$ generated by (1.2) converges to the solution x_* of $F(x) = 0$. Moreover, $y_k, z_k, x_{k+1}, x_* \in \bar{U}(x_0, R\eta_0)$ and x_* is the unique solution in $U\left(x_0, \frac{2}{M_0\beta_0} - R\eta_0\right) \cap D$. Furthermore, we have*

$$\|x_k - x_*\| \leq g(r_0)\delta^k \frac{\gamma^{\frac{(4+q)^k-1}{3+q}}}{1 - \delta\gamma^{(4+q)^k}} \eta_0.$$

4 Numerical Examples

Example 4.1. *Let us consider a system of differential equations governing the motion of an object and given by*

$$F_1'(x) = e^x, \ F_2'(y) = (e-1)y + 1, \ F_3'(z) = 1$$

with initial conditions $F_1(0) = F_2(0) = F_3(0) = 0$. Let $F = (F_1, F_2, F_3)$. Let $\mathcal{B}_1 = \mathcal{B}_2 = \mathbb{R}^3, D = \bar{U}(0,1), p = (0,0,0)^T$. Define function F on D for $w = (x, y, z)^T$ by

$$F(w) = \left(e^x - 1, \frac{e-1}{2}y^2 + y, z\right)^T.$$

The Fréchet-derivative is defined by

$$F'(v) = \begin{bmatrix} e^x & 0 & 0 \\ 0 & (e-1)y+1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that using the $(\mathcal{A})$ conditions, we get for $\alpha = 1, w_0(t) = (e-1)t, w(t) = e^{\frac{1}{e-1}}t, v(t) = e^{\frac{1}{e-1}}$. The radii are

$$R_1 = 0.3826919122323857447298678380 3208, R_2 = 0.33841523581069998805048726353562,$$

$$R_3 = 0.32249343047238987480795913143083 \text{ and } R = R_3.$$

Example 4.2. *Let* $\mathcal{B}_1 = \mathcal{B}_2 = C[0,1]$, *the space of continuous functions defined on* $[0,1]$ *be equipped with the max norm. Let* $D = \overline{U}(0,1)$. *Define function* F *on* D *by*

$$F(\varphi)(x) = \varphi(x) - 5 \int_0^1 x\theta\varphi(\theta)^3 d\theta. \tag{4.1}$$

We have that

$$F'(\varphi(\xi))(x) = \xi(x) - 15 \int_0^1 x\theta\varphi(\theta)^2\xi(\theta)d\theta, \ \text{ for each } \ \xi \in D.$$

Then, we get that $x^* = 0$, *so* $w_0(t) = 7.5t, w(t) = 15t$ *and* $v(t) = 2$. *Then the radii are*

$$R_1 = 0.06666666666666666666666666666667, R_2 = 0.059338915721683857529278327547217,$$

$$R_3 = 0.047722035514509826559237382070933 \text{ and } R = R_3.$$

Example 4.3. *Returning back to the motivational example at the introduction of this study, we have* $w_0(t) = w(t) = 96.6629073t$ *and* $v_1(t) = 2$. *The parameters for method (1.2) are*

$$R_1 = 0.0068968199414654552878434223828208, R_2 = 0.0061008926455964288676492301988219,$$

$$R_3 = 0.0044632430213268044563723613293386 \text{ and } R = R_3.$$

5   Conclusion

In general, the convergence domain of iterative schemes is small limiting their applications. Hence, any attempt to increase it is very important. This is achieved here by finding smaller $\omega-$ functions than before which are also specialization of the previous ones. Hence, the extensions are obtained under the same computational cost. Our idea can be used to extend the usage of other schemes in a similar way. Numerical experiments further demonstrate the superiority of our findings.

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