NET Institute*

www.NETinst.org

Working Paper #11-07

October 2011

Optimal Consumer Network Structure Formation under Network Effects: Seeds Controllability and Visibility

Yifan Dou
Tsinghua University

Marius F. Niculescu
Georgia Inst. of Technology

D. J. Wu
Georgia Inst. of Technology

* The Networks, Electronic Commerce, and Telecommunications ("NET") Institute, http://www.NETinst.org, is a non-profit institution devoted to research on network industries, electronic commerce, telecommunications, the Internet, "virtual networks" comprised of computers that share the same technical standard or operating system, and on network issues in general.

Optimal Consumer Network Structure Formation under Network Effects: Seeds Controllability and Visibility[1]

Yifan Dou

Tsinghua University, Haidian District, Beijing, 100086, China, douyf.03@sem.tsinghua.edu.cn

Marius F. Niculescu, D. J. Wu

Georgia Institute of Technology, 800 West Peachtree Street, N.W., Atlanta, GA 30308-1149, USA

{marius.niculescu, dj.wu}@mgt.gatech.edu

Abstract

Understanding the process of software adoption is of paramount importance to software start-ups. We study a monopolistic seller's optimal consumer network structure formation (seeding, segmentation, sequencing, and pricing strategies) under network effects. We demonstrate the importance of adoption sequencing as well as controllability over the seeding process to seller's profit, consumer surplus, and social welfare. Under multi-pricing, full information, and full control over the seeding process, with both multiplicative and additive forms of network effects, we show that all segments contain only paying customers except the first one, which contains both seeded and paying customers; and segments are opened in order of the customer valuation. Further, the seller's optimal strategy is socially optimal. Under single-pricing and limited seeding control, worst case seeding (where all seeds go to the high-valuation customers) leads to higher social welfare and consumer surplus than uniform seeding, as the former covers a larger portion of the market while charging a lower price. In the case of random seeding with limited control, we identify an optimal strategy and conditions under which the optimal price is not affected by the randomness of seeding.

Key words: network effects; software; seeding; adoption sequencing; price discrimination

JEL Classification: D85, L12, L86, M15, M31

---

[1]We thank the NET Institute (www.NETinst.org) and the Center for International Business Education and Research (CIBER) at Georgia Institute of Technology for financial support. Yifan Dou thanks additional financial support from the National Natural Science Foundation of China under grant 70890082, and Tsinghua University Scientific Research Initiative Program grant 2010 1081741. An earlier and abbreviated version of this paper will appear in Proceedings of the 32nd International Conference on Information Systems (ICIS), under the title of "Software Adoption under Network Effects: Optimal Seeding, Sequencing, and Pricing".

1. Introduction

In order to jumpstart the process of new software adoption, many software vendors are embracing *seeding* strategies by giving away products for free to a fraction of the potential customer base to take advantage of the network effects (Aral et al. 2011; Galeotti and Goyal 2009; Niculescu and Wu 2011). For example, Microsoft offers fully functional software to both early-stage startups (via its BizSpark program[2]) and high-school/college students (via its DreamSpark program[3]) for free on a global scale. IBM recently followed suit by offering free software to startups (via its new Global Entrepreneur program[4]), and free software to faculty and college students (via IBM Academic Initiatives). In 2009, Autodesk seeded 100 clean-tech startups with software bundles each worth \$150,000[5]. Salesforce offers its CRM enterprise edition software for free to more than 9, 000 non-profit organizations[6].

Despite there being a rich and growing literature on the economics of network effects (Economides 1996; Farrell and Saloner 1986; Katz and Shapiro 1994), little is known about the *structure formation* of a software consumer network (Niculescu et al. 2011), and its impact as measured by seller's profit, consumer surplus, and social welfare. Our study attempts to fill this gap by exploring the role of adoption sequencing in *controlling* the process of creating a consumer network, which goes beyond the extant literature that mostly focuses on installed base growth.

This paper aims to shed light on relevant managerial questions for software entrepreneurs. First, what percentage of the market should be seeded and how should the seeds be allocated? Second, how should the seller segment the market and price each segment? Third, what would be the associated adoption sequence and consumer network formation process?

We investigate a monopolistic seller's optimal seeding, market segment sequencing, and pricing strategies. Our analysis contributes to the literature on price discrimination by considering the impacts adoption sequencing in the presence of network effects. We stress the importance of adoption sequence as well as seller's degree of control over the seeding process to seller profit, consumer surplus and social welfare. Under a multiple-pricing scheme, full information and full control over the seeding process, we show that it is optimal to seed only inside the segment with lowest valuation. The optimal sequence of opening the segments follows an ascending order of customer valuation. Furthermore, seller's optimal strategy is socially optimal. Under a single-pricing scheme and limited seeding control, worst case seeding (where all seeds go to highest-valuation customers) leads to a higher social welfare and consumer surplus than uniform seeding; as the former covers a larger market while charging a lower price. In the case of random seeding with limited control, we identify an optimal strategy and conditions under which the optimal price is not affected by the randomness of seeding.

---

[2]http://www.microsoft.com/bizspark/
[3]http://www.webpronews.com/microsoft-dreamspark-ignites-interest-2008-02.
[4]http://www-304.ibm.com/isv/startup
[5]http://nvcatoday.nvca.org/index.php/autodesk-offering-cleantech-entrepreneurs-free-software-grants.html
[6]Private communication, Robert Pickeral, Senior VP, Global Technology Services, salesforce.com.

The rest of the paper is organized as follows. We first provide a brief review of the literature in Section 2. Section 3 studies the optimal seeding when the seller has full control over the allocation outcome of the seeding process. In Section 4, we extend the analysis to the case where the seller has only limited control over the allocation outcome of the seeding process. Section 5 compares seller profit, consumer surplus and social welfare under different models. Section 6 concludes.

2. Literature Review

We use Table 1 to summarize related literature on seeding and sampling. Our review is by no means exhaustive. Rather we intend to use Table 1 to highlight some gaps in the literature and justify our intended contributions.

As shown in Table 1, there is a small but growing body of literature on seeding. A software firm may give away the fully functional new product to a fraction of the potential customer base in order to simply catalyze the adoption process (Jiang and Sarkar 2009). Under dynamic pricing, seeding is optimal only for limited situations but does not appear to be optimal when there is a premature downturn/stalling in sales (Lehmann and Esteban-Bravo 2006). Most of the literature follows the classic Bass diffusion model (Bass 1969), which is at the macro level instead of the individual level as we do. In a different context (social networks), the role of seeding has recently been explored via simulation (Aral et al. 2011; Galeotti and Goyal 2009). In a two-period setting under word-of-mouth effects, Niculescu and Wu (2011) find that uniform seeding is always dominated by either time-limited freemium models (give away the product to the entire customer base in the first period and charge afterwards) or conventional for-fee models.

Product sampling and free demonstration are also widely used as methods to boost adoption. Product sampling involves giving away samples to customers in order to update their priors on the product value. Jain et al. (1995) study the optimal number of free samples based on the classic Bass model. Heiman and Muller (1996) extend this study by controlling the sampling time. Bawa and Shoemaker (2004) empirically test impacts of sampling and document that free samples can produce measurable long-term effects on sales. Cheng and Tang (2010) compare the free-trial and versioning strategies and find that the strength of network effects plays a critical role. Cheng and Liu (2010) extend the model to include time-sensitive network effects and derive conditions under which a time-limited model may be preferred.

To the best of our knowledge, the extant literature has not explored in depth how the seller should control the adoption process under various seeding scenarios. In this study, we attempt to fill this gap by focusing on the path of adoption, taking into account the individual-level consumer adoption decision as well as the seller's degree of control over the outcome of the seeding process.

3. Seeding with Full Control

In this section, we present our general model. We assume the seller has full information about the consumers. Initially, we assume the seller has full control over the allocation of the seeds by being able to pick and choose which customer to seed. Essentially, we consider price discrimination in the presence of network effects. We begin

Table 1. Position of this Paper to the relevant literature on seeding and sampling

	Seller Side				Consumer Side			
	Full Seeding Control	Limited Seeding Control	Price Discrimination	Decision Variables	Heterogeneity/ social interaction level	Network Externality	Structure of Installed Base	Key Research Question
Jain et. al (1995)	√	√	×	Seeding ratio	Innovators and imitators	Word-of-mouth	×	Numerically find the optimal sample size
Lehmman and Esteban-Bravo (2006)	√	×	×	Seeding ratio	Innovators and imitators	innovators are influenced only by innovators but imitators are influenced by both innovators and imitators	×	Numerically characterize when it is optimal to seed imitators
Jiang and Sarkar (2009)	√	×	×	Seeding ratio	Two-types (high & low)	Word-of-mouth	×	Optimal seeding strategy
Galeotti and Goyal (2009)	√	√	×	Seeding ratio	Degree of social connections are on [0,1]	Peers-driven	√	Optimal Seeding strategy depends on the content of social interaction
Aral et al. (2011)	√	√	×	Seeding ratio	Position in network	Peers-driven	√	Empirically show that seeding over 0.2% is wasteful
Niculescu and Wu (2011)	×	√	×	Seeding ratio & price	Customer types are uniformly distributed on [0,1]	Word-of-mouth	×	Uniform seeding is dominated by freemium and for-fee models
Heiman and Muller (1996)	×	√	×	Time span of free trial	N/A	Word-of-mouth	×	Empirically study optimal time span of free trial
Bawa and Shoemaker (2004)	√	×	×	Whether to give away samples	3 segments	N/A	×	Empirically shows sampling brings long term impact
Cheng and Tang (2010)	×	√	×	Prices of free trial and versions	Customer types are uniformly distributed on [-(a-1), 1]	Additive form	×	Free trial outperforms versioning in the case of strong network effects
Cheng and Liu (2011)	×	√	×	Price and time span of free trial	Customer types are uniformly distributed on [-(a-1), 1]	Additive form	×	Find the optimal time span and the optimal price
This paper	√	√	√	**Seeding ratio, sequence & Price**	**Customer types are uniformly distributed on [0,1]**	**Multiplicative form**	√	**Optimal seeding, sequence, and pricing strategies**

with several very simple examples to highlight key ideas in our model. Next, we then characterize seller's optimal strategies. We then extend our analysis to consider general utility function, general distribution function, and additive form of network effects.

3. 1. The Model Setup

We consider a software market with a monopolistic seller and a mass of potential customers. The total number of the customers is K. Following the standard literature, customer type θ is assumed to be uniformly distributed on [0,1]. Denote $\delta \in [0,1]$ as the *current* fraction of customers who have already adopted the product, then customer θ's *willingness-to-pay* (WTP) is $u(\delta,\theta) = \delta K\theta$.

The above multiplicative setting captures network effects in that early adopters would influence the WTP of late adopters. Later on, we extend our analysis to consider general utility function, distribution function and additive form of network effects. Customers are assumed not to be able to collude and arbitrage cannot occur among customers. We further assume that they are myopic in the sense that a potential customer adopts as soon as her updated WTP based on current installed base exceeds the price. If more than one customer's WTP exceeds the price at the same time, we break the tie by assuming that the customer with the higher type adopts first.

The seller wishes to divide the entire market into n disjoint segments $(\theta_{i-1},\theta_i]$ $(i = 1,2,...,n)$ where $\theta_0 = 0$ and $\theta_n = 1$. Within each segment, the seller can choose to seed some customers and charge the rest a unique price $p_i > 0$. Denote $(\theta_{i-1},\theta_i]$ as interval i. throughout the rest of the paper, we use "segment" and "interval" interchangeably. We assume throughout this section that the seller has full information over customer type and their WTP. We consider several scenarios corresponding to seller's degree of control over the seeding process (full, limited, or no control) and the capability to observe the seeding outcome.

We assume the seller can also control the sequence of releasing the product by segments. Let σ be the sequence of opening the segments, where $\sigma(j) = m$ means that interval j is the m^{th} opened segment in the sequence.

Denote Λ as the seeding strategy (identifying customers who receive the product for free) and $\mathbf{P} = (p_1, p_2, ..., p_n)$ as the price vector. The seller problem becomes:

$$\max_{\sigma,\mathbf{P},\Lambda} \Pi(\sigma,\mathbf{P},\Lambda). \tag{1}$$

For each customer, network effects manifest within and across intervals as each potential customer's WTP is driven by the total number of existing adopters. Following the literature on software adoption (e.g., Niculescu and Wu 2011), we assume that the software development cost is sunk and marginal reproduction and distribution costs are negligible. We are interested in the optimal structure formation (seeding, adoption sequencing, and pricing) of a software consumer network, and its impact on seller profit, consumer surplus, and social welfare.

3.2. Discrete Examples

For simplicity, we will use a few discrete customer type examples to illustrate our key ideas and intended insights. Our purpose is to show that seeding and sequencing matter in consumer network formation and business performance. These examples are based on a popular MBA class network game originally developed by Prof. Abraham Seidmann of University of Rochester (Seidmann 2009). It is straightforward to see that our continuous customer type model is a generalization of the discrete customer case, and all our major results apply to the discrete case as well.

Consider a set of six customers indexed by i ={A, B, C, D, E, F}, corresponding to a set of customer types {1, 2, 3, 4, 5, 6}. If there are M existing adopters, then customer i 's WTP is $M \times i$ due to network effects. When there is no adopter, customer i 's WTP is equal to her type.

First let us consider the case that no customer is seeded and only one price can be charged for all customers. We show that the optimal pricing is to $p^* = 5$, as other pricing strategies either leave money on the table or stall the adoption process prematurely (thus suboptimal). The adoption sequence associated with optimal pricing follows the sequence of F, E, D, C, B, and finally to the last paying customer, A. We depict this adoption sequence in Figure 1, in which each block/node represents a customer identified in the middle cell. The top cell records the price charged to customer i and the bottom cell is her current WTP which increases as the number of adopters grows. Solid lines connect adopters that form the customer network, while dotted lines indicate network effects of existing adopters to potential new adopters. Note that consumers B to F all enjoy net surplus at the end of this adoption process due to the continuous growth of the installed base post their adoption. Such benefit may not be present at the time of their adoptions, case of point is customer E, whose net surplus is zero (5 – 5 = 0) when she first adopted.

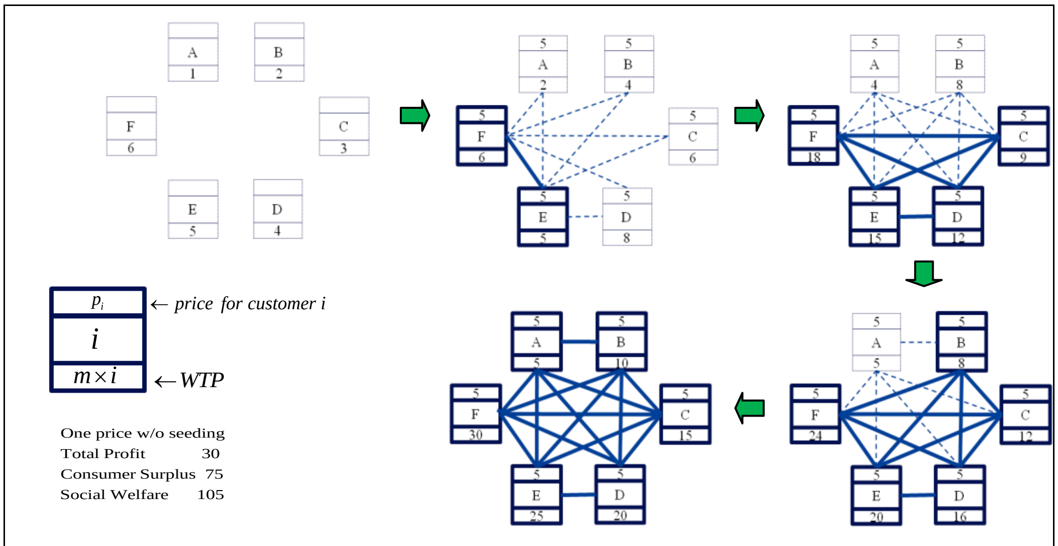


Figure 1. No Seeding, One Price, Full Information

Next, we consider the case that no customer is seeded but the seller can conduct perfect price discrimination. It can be shown that the optimal strategy is penetration pricing by charging p^* (A, B, C, D, E, F) = (1, 2, 6, 12, 20, 30) and the associated adoption sequence follows A, B, C, D, E and finally F. Note that this optimal sequence is in the reverse order of that under the no seeding and one price.

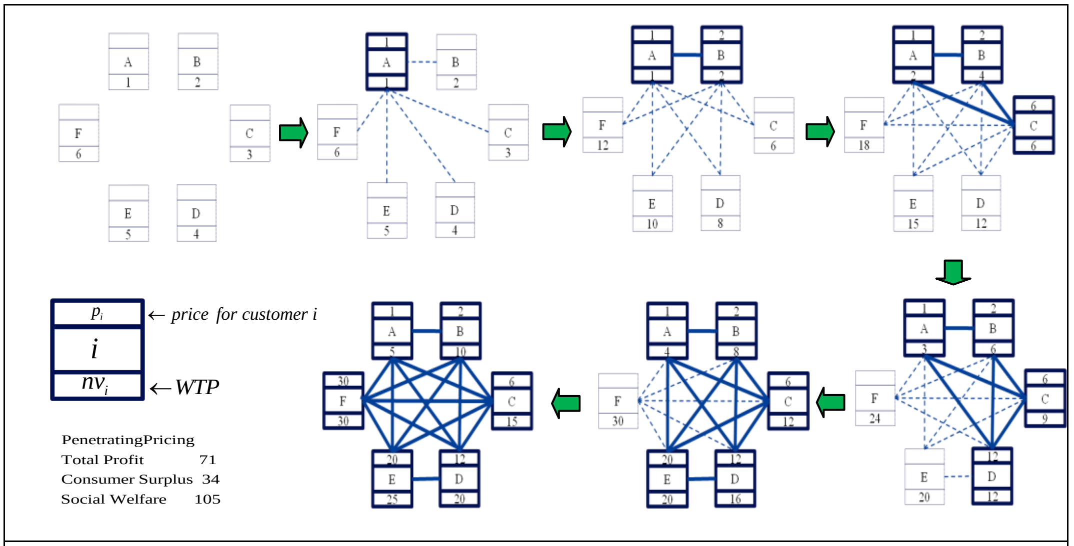


Figure 2. No Seeding, Perfect Price Discrimination, Full Information

In the third example, we allow seeding and assume customers A, B, and C are seeded. Then customer F's WTP is updated from the initial value of 6 to 18 = 3*6 due to three existing adopters (seeded customers A, B, and C). If the price is 18, F adopts as this price equals F's current WTP. E follows suit after F adopts, as E's WTP is updated from the initial value of 5 to be 4*5=20, which exceeds the price. Finally D adopts as 5*4 = 20 > 18. The seller makes a profit of 18*3 = 54. At the end of the adoption process, F enjoys a surplus of 12 (= 30 – 18) because her final WTP is 30 and the price F paid is 18. Total surplus of all 6 customers is 51, leading to a social welfare of 105.

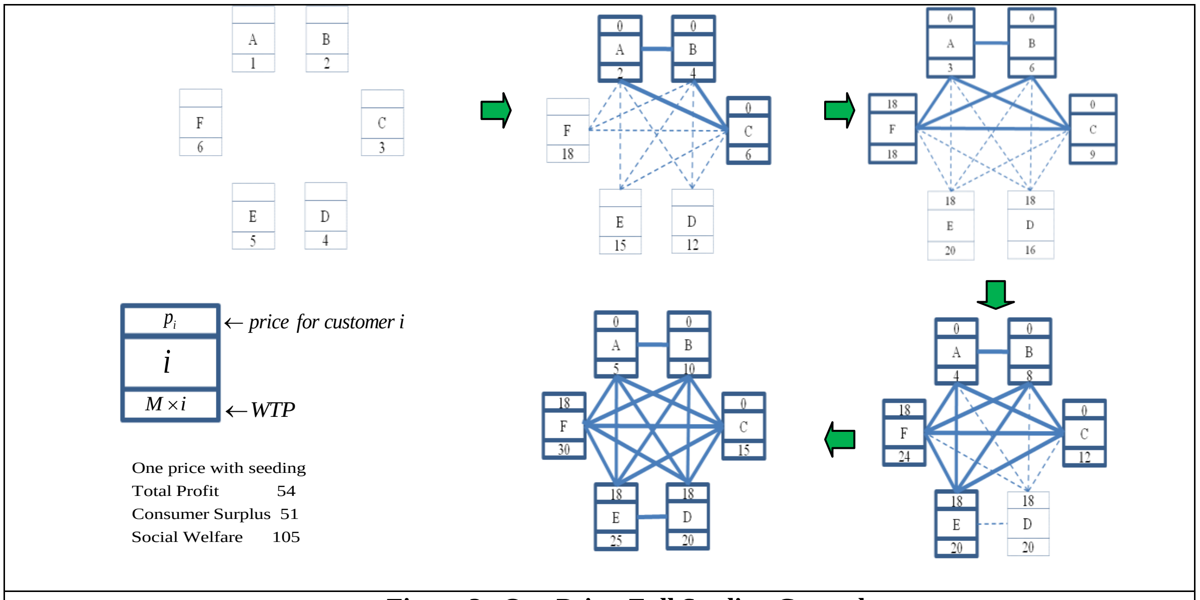


Figure 3. One Price, Full Seeding Control

As proved in the next section, such a seeding and sequencing strategy is optimal if the seller wishes to charge only one positive price. Comparing with Figure 1 and 2, we can see that, when only one price can be charged, seeding significantly improves the profit under no seeding. However, the profit under seeding does not exceed the profit with price discrimination.

Figure 4 visualizes the consumer adoption network formation process if the seller wishes to charge two positive prices. We will show that the optimal seeding and sequencing strategy is as follows. The seller should balance to the extent possible the size of the three segments (free, low price, and high price).

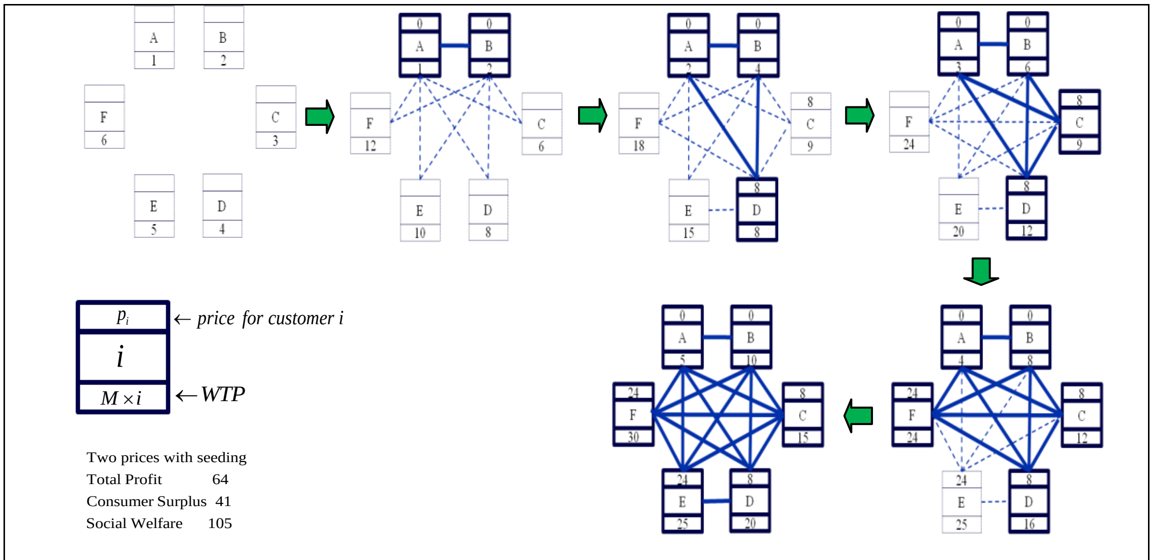


Figure 4. Two Prices, Full Seeding Control

In particular, in this example, each segment should have two customers. The seeds should go to the lower end of the market $\{A, B\}$. The seller then opens the low price segment {C, D} by charging a price that is equal to D's WTP, which is 2*4 = 8. Within the segment, C's WTP becomes 3*3 = 9 immediately after D adopts, so C would also adopt. Finally, the seller opens the high price segment {E, F}, by charging a price that is equal to the WTP of the highest type customer (F), which is 4*6 = 24. E adopts after F, as 5*5 = 25 > 24. Notice that this strategy remains socially optimal. However, by adding one more segment, the seller is able to profit more and give away fewer.

What if, for some reasons, the seller is unable to fully control the seeding process? Assume the seller wishes to seed three customers and charge one price to the rest. Suppose that a random draw allocates the seeds to customers $\{A, C, E\}$ and the seller observes this outcome. It is straightforward to see that the optimal price is 16, yielding adoption of F, followed by D but B would not adopt (as 5*2 = 10 < 16). As depicted in Figure 3, and compared with Figure 1, seller profit, consumer surplus and social welfare all decrease. This example illustrates the potential market inefficiency if the seller is unable to fully control the seeding process.

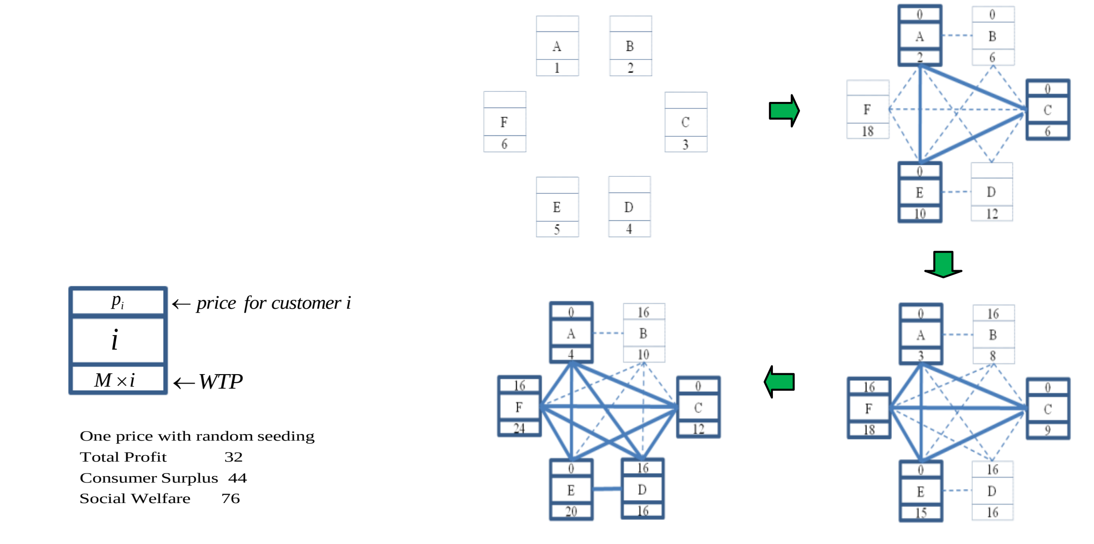


Figure 5. One Price, Limited Seeding Control

3.3. Optimal Solution to Basic Model with Full Seeding Control

We first solve the seller's problem with the assumption that the seller can fully control the seeding allocation process. Lemma 1 characterizes the necessary conditions for optimality to the seller’s problem (as defined earlier).

Lemma 1 (Necessary conditions for optimality). *The seller’s optimal strategies must satisfy the following necessary conditions:*

(a) *The market is fully covered; (ii) all segments contain only paying customers except the first one, which contains both seeded and paying customers;*

(b) *Segments are opened in order of the customer valuation; a higher valuation segment will not be opened until the immediately lower valuation segment has completed the adoption process.*

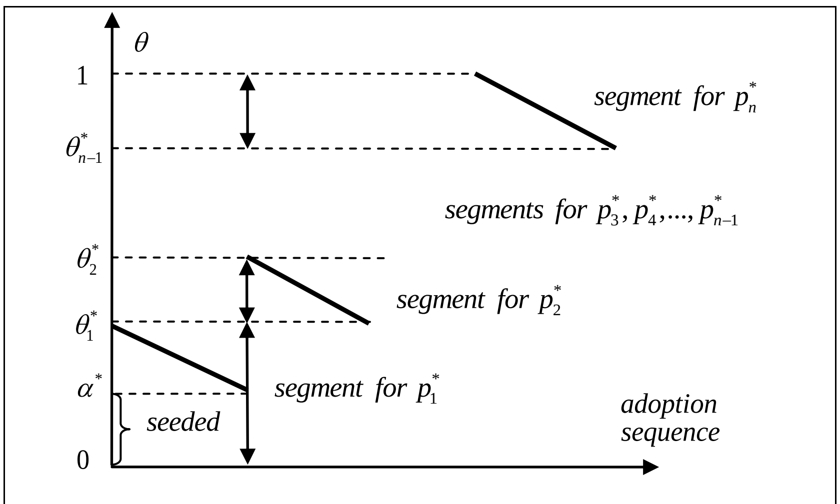


Figure 6. The Optimal Adoption Sequence with *n* Prices

It follows immediately from Lemma 1 that finding the optimal seeding strategy Λ^* is equivalent to finding the optimal seeding mass α^* at the beginning of the lowest segment. Figure 6 illustrates the optimal adoption sequence. The horizontal axis depicts the adoption sequence and y-axis refers to customer's initial WTP. Segment $[0,\theta_1^*]$ is opened first by first seeding $[0,\alpha^*]$ and then charging a price $p_1^*\left(\alpha^*\right)$ to the rest $(\alpha^*,\theta_1^*]$, followed in order by segments $(\theta_1^*,\theta_2^*]$ with price p_2^*. Within each segment, the adoption goes from high to low customer type. After everyone in the segment has adopted, next segment will be opened, so on and so forth. Given Lemma 1, Proposition 1 gives the optimal strategy.

Proposition 1 (n-price optimal strategies). *The seller's optimal strategies are the following:*

(a) *(Optimal Seeding Λ^*) Seed exactly half of the lowest segment* $[0,\alpha^*]$ *with* $\alpha^* = \theta_1^* K/2.$

(b) *(Optimal Segmentation) Size equally each segment except the first one, with the following cutoff points* θ_i^* *where* $\theta_i^* = \dfrac{i+1}{n+1}, i = 1,2,...n.$

(c) *(Optimal Pricing) Optimal price for each segment is*

$$p_i^* = \theta_{i-1}^* K \theta_i^* = \frac{i(i+1)K}{(n+1)^2},\ i = 2,...,n.$$

(d) *(Optimal Profit) the seller's optimal profit is*

$$\pi_n^* = \frac{n(n+2)K^2}{3(n+1)^2}. \tag{2}$$

Proposition 1 provides several insights on the structure of the seller's problem. First, as expected, seller's profit increases as the number of segments increases, and is bounded by $\dfrac{K^2}{3}$. Second, when the seller can charge only one single price, percentage of seeded market is the largest. In this case, optimal seeding ratio, price, and profit are given by $\alpha^* = \dfrac{1}{2}$, $p^* = \dfrac{K}{2}$, $\pi^* = \dfrac{K^2}{4}$, respectively.

3.4. Extension: General Utility Function

In this section, we consider a more general form of utility function $u(\delta,\theta) = \delta K w\left(\theta\right)$, where $w(\theta)$ is continuous, increasing, and second order differentiable. Assume further $w(0) = 0$, $w(1) = 1$. To focus on adoption sequencing, we restrict our analysis to the case when the seller charges every paying customer a single price (i.e., $n = 1$). It is straightforward to verify that Lemma 1 holds in this generalized utility case. Lemma 2 below characterizes the optimal seeding and pricing strategy:

Lemma 2. *The optimal strategy* (α, p) *must satisfy the following constraint:*

$$p \le \min_{\alpha < x \le 1} (\alpha + 1 - x) w(x) K.$$

Given any seeding strategy, the above constraint embodies three necessary conditions for optimal pricing and sequencing. First, for adoption to start, price cannot exceed the WTP of the highest type (i.e., $p \le \alpha K$). Second, for adoption to cover the entire non-seeded market, price cannot exceed the WTP of the last paying customer (i.e., $p \le w(\alpha)K$); and adoption should not stall in between, which requires $p \le \min_{\alpha < x < 1} (\alpha + 1 - x) w(x) K$. Note that under uniform distribution assumption of consumer types, it is straightforward to see that the first two conditions are identical. The seller's problem is now defined as:

$$\max_{p,\alpha} p(1-\alpha)K$$

$$s.t.\ \ p \le \min_{\alpha < x \le 1} (\alpha + 1 - x) w(x) K$$

$$p > 0,\ \ \alpha \in (0,1).$$

Solving the seller's problem, we have the following Proposition 2.

Proposition 2. *The following hold true:*

(a). When $w(x)$ *is concave, then the optimal strategy is* $\alpha^* = \frac{1}{2}$ *and* $p^* = \frac{K}{2}$*;*

(b).When $w(x)$ *is convex, then the optimal strategy is of the form* $p^* = w(\alpha^*)$ *where* $\alpha^* \ge \frac{1}{2}$ *and satisfies equation* $\frac{w(\alpha^*)}{w'(\alpha^*)} = 1 - \alpha^*$ *which has a unique solution when* $\frac{w'(x)}{w''(x)} \ge \frac{1-x}{2}$ *for* $x \in [0,1]$.

Proposition 2 extends our findings under the basic model where the utility function is linear to the more general non-linear case. Under a concave utility function, the optimal strategy remains identical as in the basic model. This is due to the fact that optimal pricing remains the same and concavity ensures the remaining two constraints are satisfied, because $(\alpha + 1 - \alpha) w(\alpha) K = w(\alpha) K \ge \alpha K = p^*$.

Note that the optimal seeding ratio is larger than $\frac{1}{2}$, in which case the seller gives away free products to the majority of the market in order to capture the amplified WTP of a few high type customers through network effects. For example, if $w(\alpha) = \alpha^{10}$, it can be shown that the optimal seeding ratio is $\alpha^* = \frac{10}{11}$, meaning the seller seeds 10 customers in order to harvest one paying customer. This free/fee ratio is referred in practice as the freemium rate, and some software entrepreneurs are advocating a "good" freemium rate is around 10, but our model suggests that the optimal freemium rate depends on the form of the consumer's utility function.

3.5. Extension: General Distribution Function

So far we have assumed that customers are uniformly distributed. Now we consider a general customer type cumulative distribution function (CDF) $F(\theta)$ with density function $f(\theta)$. We assume the WTP function is in linear form such that $w(\theta)=\theta.$ We restrict to the case of no price discrimination, i.e., $n=1$. It can be verified that (a) of Lemma 1 holds for this case of general distribution function. We have the following Lemma 3 as the necessary condition for the optimal strategy.

Lemma 3. *The optimal strategy* (α, p) *must satisfy the following constraint:*

$$p \le \min_{\alpha < x \le 1} \left[F(\alpha)+1-F(x)\right]xK.$$

The proof of Lemma 3 is similar to the proof of Lemma 2. First, for the adoption to start, price cannot exceed the WTP of the highest type (i.e., $p \le F(\alpha)K$). Second, for the adoption to complete, pricing cannot exceed the WTP of the last paying customer (i.e., $p \le \alpha K$). Finally, to ensure the continuous adoption flows, we must have all customers in between would also adopt, which requires $p \le \min_{\alpha < x < 1} \left[F(\alpha)+1-F(x)\right]xK.$ The seller's problem is now defined as:

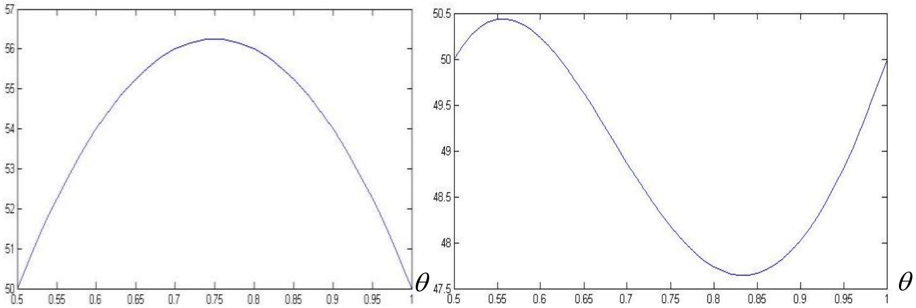
$$\max_{p,\alpha} p\left[1-F(\alpha)\right]K$$

$$s.t.\ \ p \le \min_{\alpha < x \le 1} \left[F(\alpha)+1-F(x)\right]xK.$$

$$p>0,\ \ \alpha \in (0,1).$$

The optimal solution to the seller's problem is subject to the customer type distribution function. We use Figure 7 below to show numerically how customer distribution affects WTP functions, and, implicitly, the pricing strategies. For each type, the figure depictes the WTP function precisely at the adoption time when all the seeds have been allocated and higher type customers have adopted. Customers in Panel (a) follow a uniform distribution and customers in Panel (b) follow a truncated normal distribution over $[0,1]$. From panel (a), it is straightforward to see that the optimal price is 50 (as we proved in Proposition 1).

However, under the truncated normal distribution, as Panel (b) indicates, the WTP function does not reach the minimum at boundaries. If the price is set at 50, the adoption sequence would have stalled after the first paying customer adopts. In this case, the solution given by Proposition 1 is no longer optimal. We need to use Lemma 3 to find the optimal pricing strategy. In Panel (b) of Figure 5, the minimum is approximately 47.645<50, which is reached at customer type around 0.83. Hence, compared with the optimal strategy under the uniform distribution, the seller needs to either decrease the price or increase the number of seeds. Computed numerically, the optimal solution is $\alpha^* \approx 0.59$ and $p^* \approx 64.72$ under the truncated normal distribution.



(a) *WTP under* $F(\theta) = \theta$ (b) *WTP under* $F(\theta) \sim N(0.5, 0.25)$

Figure 7. The WTP functions at adoption time under different distributions (K=100, seeding interval $[0, 0.5]$**)**

This example illustrates how customer distribution affects the optimal strategies. Essentially, when the minimum WTP (at adoption time) is not reached at the boundaries, Lemma 3 is needed to find the optimal price. Otherwise, there would be customers leftover without the product, which is suboptimal under full visibility due to Lemma 1.

3.6. Extension: Additive Form of Network Effects

So far, we have been using the multiplicative form to characterize the network effects. In this subsection, we extend our model to an additive form of network effects (e.g. Conner 1995; Jing 2007; Cheng and Tang 2010) following the standard linear form:

$$u(\delta, \theta) = \theta + \gamma \delta K,$$

where γ characterizes the strength of the network effects and u represents the current WTP . Consider the case where the type distribution is uniform. It can be verified that Lemma 1 still holds (see proof of Proposition 3 in the appendix). The following result characterizes the optimal firm strategies in this setting.

Proposition 3 (*n*-price optimal strategies). The seller's optimal strategies are the following:

(a) (Optimal Seeding Λ^*) The optimal seeding strategy is

$$\alpha^* = \begin{cases} (1-\gamma K) / \left[1-\gamma K + (1+\gamma K)n\right], \gamma K < 1 \\ 0, \gamma K = 1 \\ (\gamma K - 1) / \left[\gamma K - 1 + (1+\gamma K)n\right], \gamma K > 1. \end{cases}$$

(b) (Optimal Segmentation) The optimal cutoff points θ_i^* are

$$\theta_i^* = \begin{cases} \left[1-\gamma K + (1+\gamma K)i\right] / \left[1-\gamma K + (1+\gamma K)n\right], \gamma K < 1 \\ i / n, \gamma K = 1 \\ \left[\gamma K - 1 + (1+\gamma K)i\right] / \left[\gamma K - 1 + (1+\gamma K)n\right], \gamma K > 1. \end{cases}$$

(c) (Optimal Pricing) Optimal price for each segment is

$$p_i^* = \begin{cases} (1+\gamma K)\left[(1+\gamma K)i-\gamma K\right]/\left[1-\gamma K+(1+\gamma K)n\right],\ \gamma K<1 \\ (2i-1)/n,\ \gamma K=1 \\ (1+\gamma K)\left[(1+\gamma K)i-1\right]/\left[\gamma K-1+(1+\gamma K)n\right],\ \gamma K>1. \end{cases}$$

Comparing Propositions 1 and 3, we can see that the optimal market segmentation structure is similar under our two forms of network effects. The only difference is the optimal seeding ratio. Under the multiplicative setting, the market is equally segmented and the mass of seeded customers is equal to the mass of paying customers in intervals 2 to n. In contrast, under additive setting, the mass of seeded customers is smaller than the mass of customers in any other segment. Specifically, in the case of no price discrimination when $n=1$, the ratio of seeded customers is

$$\alpha^* = \begin{cases} (1-\gamma K)/2,\ \gamma K<1 \\ 0,\ \gamma K=1 \\ (\gamma K-1)/2\gamma K,\ \gamma K>1, \end{cases}$$

which is always smaller than $1/2$, the optimal seeding ratio under the multiplicative setting. This can be explained by the structure of network effects. Under the additive form of network effects, seeded customers contribute the same amount of network effects to other potential buyers. Under the multiplicative form of network effects, however, the network effects are larger for the potential buyers with higher types. In the latter case, the seller should seed more customers to profit more from the high-end customers.

We also observe that the seeding fraction α^* is decreasing in the number of segments (n). An interesting case is when $\gamma K=1$, *in which case* seeding is not necessary. For illustration purposes, we explain the intuition behind this result in the simple case when $n=1$. If a fraction α of customers are seeded, the WTP of customer x is $u(\alpha,x)=x+\beta(\alpha+1-x)=1+\alpha$. Hence the optimal price is $1+\alpha$. Profit is $(1-\alpha)(1+\alpha)K=(1-\alpha^2)K$ which is maximized at $\alpha=0$. This indicates that the additional value created by seeding is smaller than the profit the seller can obtain by charging the seeded customers.

Because the optimal market structure is similar under both forms of network effects, we will use the multiplicative form in the rest of the paper.

4. Seeding with Limited Control

So far we have assumed that the seller has full controllability over the seeding process. We now extend the analysis to the case where the seller cannot fully control the seeding outcome. We consider two sub-cases: (1). although the seller cannot fully control seeding (limited controllability), he is able to observe the seeding outcome (full visibility); (2) the seller can neither fully control seeding (limited controllability) nor observe the seeding outcome (limited visibility). We will discuss these two sub-cases respectively. In the case of limited controllability but with full visibility, we consider two potential scenarios: the worst seeding case (when all the seeds go to the highest valuation customers) and the uniform case (when the seeds are uniformly distributed among all customers).

In the case of limited controllability and limited visibility, we identify conditions for an optimal pricing strategy that is independent of the seeding outcome.

For simplicity, we restrict our analysis to the case when WTP is linear such that $w(\theta)=\theta,$ the consumer type distribution is uniform, and there is no price discrimination ($n=1$).

4.1. Worst Seeding Case

The worst case occurs when all the seeds go to the high end of the market. In this case, let us denote the seller's optimal strategy by (α^w, p^w), where all the seeds go to the interval $[1-\alpha^w, 1]$. To ignite the adoption process, the seller is constrained to charge $p^w \le (1-\alpha^w)\alpha^w K$. Since customer types follow the uniform distribution, given any p^w, there must exist a customer θ^w who is the marginal customer satisfying $p^w=(1-\theta^w)\theta^w K$. For any customer $x \in (\theta^w, \alpha^w)$, at the moment of adoption, the WTP is $x(1-x)K$, which is concave with minimum value attained at one of the boundaries. Therefore

$$p^w = (1-\theta^w)\theta^w K \le (1-\alpha^w)\alpha^w K.$$

The mass of paying customers is $(1-\theta^w-\alpha^w)K.$ The seller's optimization problem becomes

$$\max_{\alpha^w,\theta^w}(1-\alpha^w-\theta^w)(1-\theta^w)\theta^w K^2$$

$$s.t.\ \ (1-\theta^w)\theta^w \le (1-\alpha^w)\alpha^w$$

$$(1-\alpha^w-\theta^w)K>0$$

$$\alpha^w,\theta^w \in (0,1).$$

We prove in the appendix that the optimal strategy is $\alpha^w=\theta^w=\frac{3-\sqrt{3}}{6},\ p^w=\frac{K}{6}, \pi^u=\frac{\sqrt{3}K^2}{18}.$ Note that, in this worst case scenario, the market is *not* fully covered: customers in interval $\left[0,\frac{3-\sqrt{3}}{6}\right]$ end up not adopting the software, thus contributing no value to the seller or other adopters.

4.2. Uniform Seeding Case

The second limited control scenario we consider is uniform seeding where the seeds are uniformly distributed among all customers. Assume the seeding ratio is α^u, such that there are $\alpha^u\theta K$ seeded customers in interval $[0,\theta]$, for any $\theta \in [0,1]$.

Given price p^u, for adoption to start, we need $p^u \le \alpha^u K$. Denote by θ^u the last paying customer. For any paying customer $x \in [\theta^u, 1]$, at the moment of adoption, the WTP is $(1-x+\alpha^u x)xK$ which is also concave with the minimum value achieved on the boundaries. Therefore we have $p^u=\theta^u(1-\theta^u+\alpha\theta^u)K$. Note that the number of paying customers is $(1-\alpha^u)(1-\theta^u)K$ which is always non-negative for any $\alpha^u,\theta^u \in (0,1)$. The seller's problem becomes

$$\max_{\alpha^u,\theta^u}(1-\theta^u+\alpha^u\theta^u)\theta^u(1-\alpha^u)(1-\theta^u)K^2$$

$$s.t.\ (1-\theta^u+\alpha^u\theta^u)\theta^u \le \alpha^u$$

$$\alpha^u,\theta^u \in (0,1).$$

We show in the appendix the seller's optimal strategies are $\alpha^u=\frac{1}{4},\theta^u=\frac{1}{3},p^u=\frac{K}{4},\pi^u=\frac{K^2}{8}.$

Note that the seller's profit decreases from full control, to uniform seeding, and to worst seeding, which highlights the importance of controllability of the seeding process, from the seller's perspective.

4.3. Random Seeding with Limited Control

In this section, we discuss the case where the seller can neither fully control seeding nor observe the seeding outcome.

Instead of jointly optimizing on both the number of seeds and price, we introduce an optimal pricing strategy which is subject to the seller's controllability over the allocation of the seeds. We will show that our optimal pricing strategy is not affected by the seeding outcome. This "trick" could dramatically simplify the seller's decision making process when he faces randomness in seeding.

When seeds are not controllable, we assume that all the seeds go to countably many intervals on $[0,1]$. Thus we can formalize any possible seeding outcome in the following way: For $\theta\in[0,1]$, assume $g(\theta)$ represents the number of seeded customers with a lower initial WTP than customer θ. If the total number of seeds is αK, we require $g(\theta)$ satisfy: $g(0)=0;\ g(1)=\alpha K;\ g(\theta)\le\theta K;\ g'(\theta)\ge 0.$

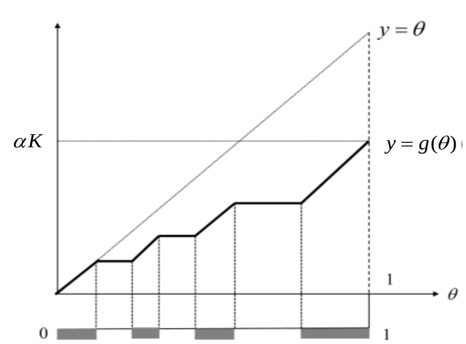


Figure 8. An example of general random seeding

Since all the seeds go to countably many seeding intervals, $g(\theta)$ is differentiable almost everywhere (a.e.). Figure 8 shows an example of $g(\theta)$ where the dark area on the horizontal line is seeded, corresponding to an increase in $g(\theta)$ where $g(\theta)=1>0$. The slope of the curve is 1 (same for the curve $y=\theta$) at the seeded customers and equal to 0 otherwise (except at a countable number of points). Hence $g''(\theta)=0$ holds almost everywhere.

Denote $\alpha^r K$ as the *maximum mass of seeds the seller can seed with limited controllability*, where we define limited controllability as the seller's capability to seed without full visibility the low-end half of the market (i.e., customers in $[0,\frac{1}{2}]$). In other words, the seller can seed $\alpha^r K$ customers in $[0,\frac{1}{2}]$ but cannot control the seeding outcome nor see precisely the types of those seeded customers. For example, if $\alpha^r=\frac{1}{2}$, it means that the seller can seed everyone in interval $[0,\frac{1}{2}]$, which would lead to the maximum profit as computed in Proposition 1. However, if $\alpha^r=\frac{1}{6}$, it means that the seller can seed one third of the customers in $[0,\frac{1}{2}]$ but it could either be $[0,\frac{1}{6}]$, $[\frac{1}{3},\frac{1}{2}]$ or any other seeding outcome (e.g., discontinuous). Thus, α^r indicates a certain degree to which the seller can identify that some customers belong to the low-end. Given α^r, Proposition 4 provides an optimal pricing strategy which is not affected by the randomness of the seeding process.

Proposition 4. Given α^r, *if the seller is restricted/committed to seeding $\alpha^r K$ and can do so with limited controllability (defined as above), when $\alpha^r \le \frac{\sqrt{2}-1}{2}$ the optimal pricing strategy is $p^*=\alpha^r K$.*

Proposition 4 provides an optimal pricing strategy for the seller who can only seed the customers with limited controllability and limited visibility. It suggests that if α^r is not too big in that $\alpha^r \le \frac{\sqrt{2}-1}{2}$, charging $p=\alpha^r K$ always yields the maximum profit regardless of the randomness of the seeding process. This result could be of great help to small software startups who have limited visibility/information of their target customers. It will be much easier for them to identify the low-end half of the market and distribute the seeds since the optimal price does not depend on the seeding outcome but on the number of seeds.

4.4. Information Acquisition

When the seller cannot observe the seeding outcome, an alternative is to pay for the information on customers' WTP (e.g., hire a target advertising company or purchase customer data from a third party). We consider this case in this section.

Without loss of generality, we assume the cost of acquiring information on the customers in interval $[0,\alpha]$ is $c\alpha^2K^2$. This form implies that the marginal cost of acquiring customer information increases as the size of the targeted customer group increases. Recall that in this section we have assumed a uniformly-distributed customer type. After spending $c\alpha^2K^2$ to obtain user information, the seller can perfectly seed the customers in $[0,\alpha]$. The seller's problem is

$$\max_{p,\alpha} p(1-\alpha)K - c\alpha^2K^2$$

$$s.t.\ \ p \le \alpha K$$

$$p > 0,\ \ \alpha \in (0,1).$$

The optimal solution to this problem is $p^* = \alpha K = \dfrac{K}{2(1+c)}$. As c increases, the optimal number of seeds is decreasing since the cost of information acquisition increases. For a seller with limited seeding controllability, the following result provides cost thresholds that can assist the seller in choosing between (1) optimal seeding by paying for information acquisition and (2) random seeding with no information acquisition.

Proposition 5 . *If the seller commits to seeding* $\alpha^r K$ *and can do so with limited controllability, then*

(a) *When information acquisition cost exceeds a threshold such that* $c > \dfrac{1}{2\alpha^r} - 1$, *random seeding with limited control is optimal;*

(b) *When information acquisition cost is below a threshold such that* $c < \dfrac{1}{4\alpha^r(1-\alpha^r)} - 1,$ *acquiring customer information is optimal;*

(c) *Otherwise when* $c \in \left[\dfrac{1}{4\alpha^r(1-\alpha^r)} - 1, \dfrac{1}{2\alpha^r} - 1\right]$, *either strategy can be optimal depending on the realization of the random seeding.*

Proposition 5 provides useful guidelines that can help the seller choose between different strategies. It formalizes the intuition that information acquisition outperforms random seeding when the information acquisition cost is small. However, when this cost is moderate (see case c in Proposition 5), either one can be optimal depending on the randomness in the seeding outcome.

5. Social Welfare Analysis

In this section, we compare the social welfare implications of four models: (1) *n*-price with optimal seeding, (2) one price with optimal seeding, (3) one price with uniform seeding, and (4) one price with worst seeding. Table 2 summarizes the seller's profit, consumers' surplus, and social welfare corresponding to each model. Note that consumer surplus and social welfare are both computed after the entire adoption process has been completed, as we show earlier on in our discrete case examples.

Table 2. Social Welfare Analysis				
	Model	Seller's Profit	Consumer Surplus	Social Welfare
Multiple-price	n-Price with optimal seeding	$\frac{n(n+2)K^2}{3(n+1)^2}$	$\left[\frac{1}{6}+\frac{1}{3(n+1)^2}\right]K^2$	$\frac{K^2}{2}$
One price	Optimal Seeding	$\frac{K^2}{4}$	$\frac{K^2}{4}$	$\frac{K^2}{2}$
	Uniform Seeding	$\frac{K^2}{8}$	$\frac{11K^2}{48}$	$\frac{17K^2}{48}$
	Worst Seeding	$\frac{\sqrt{3}K^2}{18}$	$\left[\frac{5+\sqrt{3}}{24}\right]K^2$	$\left[\frac{15+7\sqrt{3}}{72}\right]K^2$

From Table 2, optimal seeding leads to social welfare maximization regardless of how the market is segmented (however, the seller makes more profit by increasing segments). The social welfare decreases when the seller loses control over the seeding process. Interestingly, the case of worst seeding has a higher social welfare than the case of uniform seeding, because the former is associated with a lower optimal price (than the latter), thus attracting more paying customers. In other words, more customers are left unserved in the case of uniform seeding.

Consumer surplus is the other side of the same coin. The case of worst seeding offers consumers a higher surplus, due to the increase in the social welfare and decrease in seller's profit. Compared with the case of uniform seeding, worse seeding covers a larger portion of the market at a lower price.

Finally, it is worth noting where the consumer surplus is coming from. Under our setting, consumers are myopic and they adopt the software as soon as their WTP exceeds the price charged to them. However, they ultimately enjoy additional surplus as the installed base grows over time.

6. Conclusion

Understanding the process of software adoption is of paramount importance to software start-ups, who are increasingly embracing the seeding strategy to jumpstart adoption and boost the willingness-to-pay of potential paying customers via network effects. We study the seller's optimal seeding, sequencing, and pricing strategies in the presence of network effects. We demonstrate the importance of sequencing as well as controllability over the seeding process to seller's profit, consumer surplus, and social welfare.

With both multiplicative and additive forms of network effects, we find that under multiple pricing and full control of the seeding process, it is optimal to seed only the lower half of the lowest valuation segment and then charge non-zero prices to every other customer. The optimal sequence of opening the segments follows an ascending order of customer types, while within each segment, paying customers adopt in descending order of their types. Social welfare is maximized under optimal seeding but decreases when the seller loses control over the seeding process. Under single pricing and limited seeding control, worst case seeding has a higher social welfare and consumer surplus than uniform seeding, because the former covers a larger market at a lower price. In the case of random seeding with limited control, we identify an optimal strategy and conditions under which the optimal price is not affected by the randomness of seeding. The model in this paper provides a new perspective to study the adoption path and to enhance our understanding of software adoption dynamics. Specifically, we focus on how to shape the software adoption process via seeding in a consumer network. Our framework and findings may help software vendors to efficiently and effectively design their marketing strategies.

References

Aral, S., L. Muchnik, A. Sundararajan. 2011. Engineering social contagions: Optimal network seeding and incentive strategies. Working Paper. New York University.

Bass, F. M. 1969. A new product growth for model consumer durables. *Management Science* **15**(9) 215-227.

Bawa, K., R. Shoemaker. 2004. The effects of free sample promotions on incremental brand sales. *Marketing Science* **23**(3) 345-363.

Cheng, H. K., Y. Liu. 2011. Software free trial: How long is optimal? *Information Systems Research*. Forthcoming.

Cheng, H. K., Q. C. Tang. 2010. Free trial or no free trial: Optimal software product design with network effects. *European Journal of Operational Research* **205**(2) 437-447.

Conner, K. R. 1995. Obtaining strategic advantage from being imitated: When can encouraging 'clones' pay? *Management Science* **41**(2) 209-225.

Dou, Y. F., M. F. Niculescu, D. J. Wu. 2011. Software adoption under network effects: Optimal seeding, sequencing, and pricing. *Proceedings of the 32nd International Conference on Information Systems* (ICIS), December 4-7.

Economides, N. 1996. The economics of networks. *International Journal of Industrial Organization* **14**(6) 673-699.

Farrell, J., G. Saloner. 1986. Installed base and compatibility: Innovation, product preannouncements and predation. *The American Economic Review* **76**(5) 940-955.

Galeotti, A., S. Goyal. 2009. Influencing the influencers: A theory of strategic diffusion. *The RAND Journal of Economics* **40**(3) 509-532.

Heiman, A., E. Muller. 1996. Using demonstration to increase new product acceptance: Controlling demonstration time. *Journal of Marketing Research* **33**(4) 422-430.

Jain, D., V. Mahajan, E. Muller. 1995. An approach for determining optimal product sampling for the diffusion of a new product. *Journal of Product Innovation Management* **12**(2) 124-135.

Jiang, Z., S. Sarkar. 2009. Speed matters: The role of free software offer in software diffusion. *Journal of Management Information Systems* **26**(3) 207-240.

Katz, M. L., C. Shapiro. 1994. Systems competition and network effects. *The Journal of Economic Perspectives* **8**(2) 93-115.

Lehmann, D.R., M. Esteban-Bravo. 2006. When giving some away makes sense to jump-start the diffusion process. *Marketing Letters* **17**(4) 243-254.

Niculescu, M.F., D.J. Wu. 2011. When should software firms commercialize new products via freemium business models? Working Paper. Georgia Institute of Technology. http://papers.ssrn.com/ sol3/papers.cfm? abstract_id=1853603 (earlier version presented at WISE 2010, St. Louis, Dec 11-12)

Niculescu, M.F., H. Shin, S. Whang. 2011. Underlying consumer heterogeneity in markets for subscription-based IT services with network effects. Working Paper. Georgia Institute of Technology. Northwestern University. Stanford University. http://papers.ssrn.com/ sol3/papers.cfm? abstract_id=1869168.

Shapiro, C., H.R. Varian. 1999. *Information Rules*. Harvard Business School Press, Boston.

Seidmann, A. 2009. An innovative network externality game. IT Teaching Workshop, University of Minnesota, Carlson School of Management, May 22nd.

Appendix

Proof of Lemma 1.

(a). (i) We show that under optimality, the market is fully covered. We do so in two steps.

First, we show that in each segment, the seller should complete seeding before start selling. Suppose otherwise: after adoption by a set of paying customers, the seller seeds a set of non-buying customers. Then seeding the same set of non-buying customers before selling strictly increases the WTP of each and every customer in the same set of paying customers. The seller thus can make a higher profit by charging a higher price without shrinking the installed base of paying customers.

Second, we prove by contradiction that, the market is fully covered under optimality. Suppose otherwise: the market is not fully covered. There exists at least one segment, say, $(\theta_{k-1}, \theta_k]$ that is not fully covered. Given any $p_k > 0$, since by assumption adoption occurs from high to low type within each segment, there exists a marginal customer $\varepsilon_k > \theta_{k-1}$ such that customers in $(\theta_{k-1}, \varepsilon_k)$ would not pay (all customers above ε_k can afford to pay for the product). Thus, any uncovered customers must be in this interval $(\theta_{k-1}, \varepsilon_k)$. However, seeding these non-buying customers before charging will only increase WTP of the rest customers in this segment and all customers in subsequent segments. Thus the seller can raise the price by $\theta_{k-1}(\varepsilon_k - \theta_{k-1})$ for this segment without shrinking the installed base of paying customers, thus making a strictly higher profit. This contradicts the optimality of the original strategy.

(ii). We show that under optimality, all segments contain only paying customers except for the first one, which contains both seeded and paying customers.

We first show that in each segment, seeding the low end is optimal (where the mass of seeding could be zero, i.e., no seeding is a special case of seeding). Suppose otherwise: under optimality, there exists at least one segment, say, $(\theta_{k-1}, \theta_k]$ that contains at least one interval $(\varepsilon_k, \varepsilon_{k+1}]$ of seeded customers who are adjacent to a set of paying customers $(\varepsilon_{k-1}, \varepsilon_k]$. For any customer x in $(\varepsilon_{k-1}, \varepsilon_k]$, at the time of adoption, the size of installed base is $(I + \varepsilon_k - x)K$, where I is the mass of the installed base prior to the adoption by customer ε_k. Hence the WTP of customer x is $(I + \varepsilon_k - x)xK$. Now let us consider a candidate strategy such that customers in $(\varepsilon_{k-1}, \varepsilon_k^c]$ are seeded (see Figure 7 below), where $\varepsilon_k^c = \varepsilon_{k-1} + \varepsilon_{k+1} - \varepsilon_k$ and customers in $(\varepsilon_k^c, \varepsilon_{k+1}]$ is charged, and all the other things remaining unchanged. We map one-on-one each customer $x \in (\varepsilon_{k-1}, \varepsilon_k]$ to a customer y in $(\varepsilon_k^c, \varepsilon_{k+1}]$ by shifting, $y = x + \varepsilon_{k+1} - \varepsilon_k$. Note that the mass of the installed base prior to ε_{k+1} remains I, such that the WTP of customer y is $(I + \varepsilon_{k+1} - y)yK$. Note that, since $\varepsilon_{k+1} - y = \varepsilon_k - x$ due to one-on-one mapping, we have

$$(I + \varepsilon_{k+1} - y)yK - (I + \varepsilon_k - x)xK = (I + \varepsilon_k - x)yK - (I + \varepsilon_k - x)xK = (I + \varepsilon_k - x)(y - x)K > 0.$$

This means that WTP of all paying customers in $(\varepsilon_{k-1}, \varepsilon_{k+1}]$ is increasing under the candidate strategy. Thus the seller makes a higher profit using the candidate strategy. This contradicts the optimality of the original strategy.

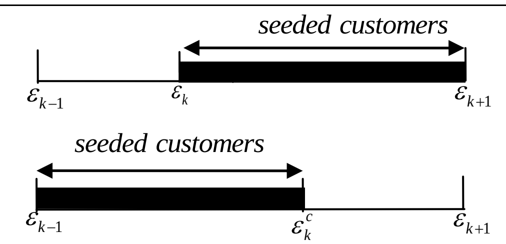


Figure 7. The Market Segmentation of Optimal Strategy

Given this property and (i), under the optimal strategy $(\sigma^*, \mathbf{P}^*, \Lambda^*)$, the structure of the market can be visualized using Figure 8, where black intervals represent seeded customers in each segment. Denote the number of seeded customers in each segment as $\alpha_i \geq 0$ ($i = 1, 2, ..., n$).

Suppose segment 1 is the k^{th} to be opened in the optimal sequence, i.e., $\sigma(1) = k$ and $1 \leq k \leq n$. We denote $A = \{j_1, j_2, j_3, ..., j_{k-1}\}$ as the set of $(k-1)$ segments opened prior to interval 1 that has a total joint mass of U paying customers with the lowest paying type as $\underline{\theta}$. Note that $\underline{\theta} \geq \theta_1$.

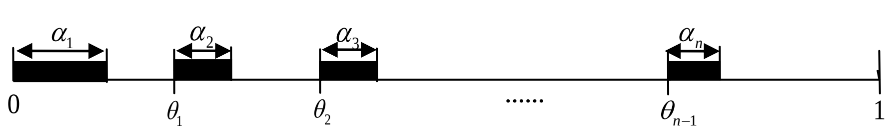


Figure 8. The Market Segmentation of Optimal Strategy

We denote the optimal price for each segment as p_i . Note that at any moment along the adoption process, the current fraction of adopters from the intervals in A is no greater than U. Hence $p_1 \le (U + \sum_{i=1}^{k} \alpha_{j_i})\theta_1 K$ and total profit of interval 1 satisfies $\pi_1 \le (U + \sum_{i=1}^{k} \alpha_{j_i})\theta_1(\theta_1 - \alpha_1)K^2$. Denote the profit from interval i as π_i and the total seller profit as Π . Now consider a candidate strategy:

$$\theta_i^c = \theta_i + \sum_{j=i+1}^{n} \alpha_j \;,\; \sigma^c(i) = \begin{cases} 1, i = 1 \\ \sigma(i) + 1, i \in A \\ \sigma(i), \textit{otherwise} \end{cases} \;,\; \alpha_i^c = \begin{cases} \sum_{j=1}^{n} \alpha_j, i = 1 \\ 0, \textit{otherwise} \end{cases} \text{, and } p_i^c = \begin{cases} \theta_1^c \sum_{j=1}^{n} \alpha_j K, i = 1 \\ \\ p_i + \theta_1 \underline{\theta}, K\ i \in A \\ p_i, \textit{otherwise.} \end{cases}$$

Under the candidate strategy, customers in $[0, \sum_{j=1}^{n} \alpha_j]$ are seeded. The number of paying customer in each segment remains the same. Figure 9 visualizes the market structure.

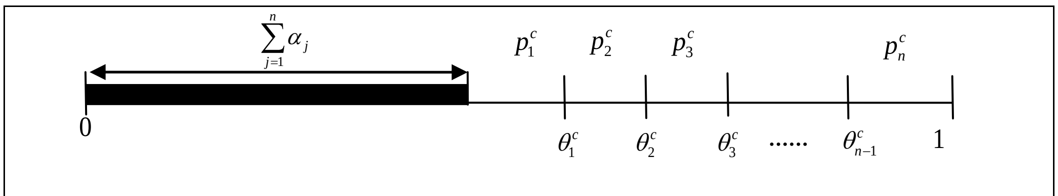


Figure 9. The Market Segmentation of Candidate Strategy

Let us take a look at how profit changes for each interval under the candidate strategy. For each segment in A , at the time it is opened, the user base is increased by at least θ_1 and customers are replaced by higher type customers. Therefore, p_i^c is acceptable for all paying customers of intervals in A. For each segment not in A or equal to 1, under the new segmentation, the paying customers assigned to each segment has a higher type, compared with the optimal strategy. Therefore the original price is still acceptable. Hence only the profit of interval 1 may decrease while the profit in all intervals in A increases. Denote the profit from interval i as π_i^c and the total profit as Π^c . Similarly, π_i^c is a function of σ_i^c, p_i^c, α_i^c and θ_i^c . The improvement in profit is

$$\begin{aligned}
\Pi^c - \Pi &= \pi_1^c \text{-} \pi_1 + \sum_{i \in A} [\pi_i^c - \pi_i] + \sum_{\substack{i \notin A \\ i \neq 1}} [\pi_i^c - \pi_i] \\
&= \pi_1^c \text{-} \pi_1 + \sum_{i \in A} [\pi_i^c - \pi_i] \\
&= \theta_1^c \sum_{i=1}^{n} \alpha_i (\theta_1 - \alpha_1) K^2 - (U + \sum_{i=1}^{k} \alpha_{j_i}) \theta_1 (\theta_1 - \alpha_1) K^2 + U \theta_1 \underline{\theta} K^2 \\
&= (\theta_1 + \sum_{i=2}^{n} \alpha_i) \sum_{i=1}^{n} \alpha_i (\theta_1 - \alpha_1) K^2 - \theta_1 \sum_{i=1}^{k} \alpha_{j_i} (\theta_1 - \alpha_1) K^2 + U \theta_1 (\underline{\theta} - \theta_1 + \alpha_1) K^2 \\
&\geq 0,
\end{aligned}$$

which implies that the total profit under the candidate strategy is non-decreasing. Therefore under the optimal strategy, $\alpha_j = 0$ for $j = 2, 3, ..., n$ and all seeds go to interval 1. This completes the proof of (a).

(b). We prove part (b) by induction.

Step (i): From (a), we know that segment 1 is opened first, i.e., $\sigma(1) = 1$.

Step (ii): Assume $\sigma(i) = i$ for any $i = 1, 2..., T$ where $1 \leq T \leq n-1$. We shall prove that $\sigma^*(T+1) = T+1$ is optimal. Suppose in optimal sequence, $\sigma^*(T+1) = Q$ and $Q > T+1$. For $j_k = (\sigma^*)^{-1}(k)$, we denote by $B = \{j_{T+1}, j_{T+2}, j_{T+3}, ..., j_{Q-1}\}$ the set consisting of all the intervals opened between interval T and interval $T+1$. For all the intervals in B, we assume that paying customers account for a fraction of G in total and the lowest type of paying customers in B is $\underline{\theta}$. Note that $\underline{\theta} > \theta_{T+1}$.

Given a sequence σ, we denote the profit from interval i as $\pi_i(\sigma)$ and the total profit as $\Pi(\sigma)$.

$$\Pi(\sigma^*) = \sum_{i \leq T} \pi_i(\sigma^*) + \sum_{i \in B} \pi_i(\sigma^*) + \pi_{T+1}(\sigma^*) + \sum_{\substack{i > T+1 \\ i \notin B}} \pi_i(\sigma^*).$$

For interval $T+1$ to start, it must be the case that $p_{T+1}^* \leq (\theta_T + G)\theta_{T+1} K$, which indicates the total profit of $T+1$ is bounded by $(\theta_T + G)\theta_{T+1}(\theta_{T+1} \text{-} \theta_T)K$. Now let us consider a candidate strategy with sequence σ^c and price p_i^c

$$\sigma^c(i) = \begin{cases} i, i \leq T+1 \\ \sigma^*(i) + 1, i \in B \\ \sigma^*(i), \textit{otherwise}, \end{cases} \qquad p_i^c = \begin{cases} \theta_{T+1}\theta_T K, i = T+1 \\ p_i^* + \underline{\theta}(\theta_{T+1} - \theta_T)K, i \in B \\ p_i^*, \textit{otherwise}. \end{cases}$$

The number of seeds and the market segmentation remain the same as those in the old strategy. Under this candidate strategy, the size of installed base prior to the adoption in interval $T+1$ is θ_T. Therefore, $p_{T+1}^c = \theta_T \theta_{T+1} K$ which can be accepted by all the customers in interval $T+1$ because the WTP of customer x is $(\theta_T + \theta_{T+1} - x)xK \geq \theta_T \theta_{T+1} K$ for any $x \in (\theta_T, \theta_{T+1}]$. Hence $\pi_{T+1}(\sigma^c) = \theta_{T+1}\theta_T(\theta_{T+1} - \theta_T)K^2$. For customers in interval

i with $i \in B$, the prices can be increased because the installed base is increased by $(\theta_{T+1} - \theta_T)K$. The total profit under the candidate strategy is

$$\Pi(\sigma^c) = \sum_{i \le T+1} \pi_i(\sigma^c) + \sum_{i \in B} \pi_i(\sigma^c) + \sum_{\substack{i > T+1 \\ i \notin B}} \pi_i(\sigma^c).$$

Therefore

$$\begin{aligned}
\Pi(\sigma^c) - \Pi(\sigma^*) &= \sum_{i \in B} [\pi_i(\sigma^c) - \pi_i(\sigma^*)] + \pi_{T+1}(\sigma^c) - \pi_{T+1}(\sigma^*) \\
&\ge \sum_{i \in B} [\pi_i(\sigma^c) - \pi_i(\sigma^*)] + \theta_T \theta_{T+1}(\theta_{T+1} \text{-} \theta_T) K^2 \text{-} (\theta_T + G)\theta_{T+1}(\theta_{T+1} \text{-} \theta_T) K^2 \\
&= \sum_{i \in B} [\pi_i(\sigma^c) - \pi_i(\sigma^*)] - G\theta_{T+1}(\theta_{T+1} \text{-} \theta_T) K^2 \\
&\ge G\underline{\theta}(\theta_{T+1} \text{-} \theta_T) K^2 - G\theta_{T+1}(\theta_{T+1} \text{-} \theta_T) K^2 \\
&> 0,
\end{aligned}$$

which suggests that the profit can be improved under the candidate strategy. This contradicts the optimality of the original strategy. Given the induction hypothesis, we know interval $T+1$ can be fully covered by charging $p_{T+1}^c = \theta_T \theta_{T+1} K$. If any new interval is opened before the last customer θ_T adopts, p_{T+1}^c cannot be increased because otherwise the adoption of interval $T+1$ will not start, therefore it only decreases the user base of the new interval. Hence opening a new interval before all customers in interval $T+1$ adopt is never profit-improving. This proves part (b) of Lemma 1. Q.E.D.

Proof of Proposition 1.

Given Lemma 1, we know that under the optimal strategy, everyone in interval i can afford the maximum price which can start the adoption of interval i . Therefore the seller's problem becomes:

$$\max_{\alpha,\, \theta_i,\, i=1,2,\ldots,n-1} \underbrace{\alpha\theta_1 K}_{p_1}(\theta_1 - \alpha) + \sum_{i=2}^{n-1} \underbrace{\theta_i \theta_{i-1} K}_{p_i}(\theta_i - \theta_{i-1}) + \underbrace{\theta_{n-1} K}_{p_n}(1 - \theta_{n-1}) \tag{3}$$

$$s.t. \quad 0 \le \theta_1 \le \theta_2 \le \ldots \le \theta_{n-1} \le 1.$$

First order condition yields the following system of equations:

$$\begin{cases}
\alpha^* = \dfrac{\theta_1^*}{2}, \\
\theta_1^* = \dfrac{\alpha^* + \theta_2^*}{2}, \\
\ldots \\
\theta_i^* = \dfrac{\theta_{i-1}^* + \theta_{i+1}^*}{2}, \; i = 2, 3, \ldots, n-1,
\end{cases}$$

which gives the relationship of $\theta_{i-1} = \dfrac{i}{i+1}\theta_i$. Then it immediately follows that

$$\theta_i^* = \frac{i+1}{n+1}.$$

The optimal profit can be obtained by plugging this back to the objective function in (3). Q.E.D.

Proof for Lemma 2.

The constraint in Lemma 2 provides the condition for p to be sufficiently low in order for the entire market to be covered. By a similar argument as in the proof of (a) in Lemma 1, under the optimal strategy, seeds should be given away to the low type customers in the beginning and all the other customers will pay for the product.

Suppose that under the optimal strategy (α^*, p^*), the customers in $[0,\alpha^*]$ are seeded and all customers in $(\alpha^*,1]$ would pay for the product. For any customer $x \in (\alpha^*,1]$, at the moment of adoption, the current size of the installed base is $(\alpha^* + 1 - x)K$. Hence the WTP of customer x is $(\alpha^* + 1 - x)w(x)K$. Since everyone x must eventually purchase, the following inequality must hold

$$p^* \le \min_{\alpha < x \le 1} (\alpha + 1 - x)w(x)K.$$

Proof of Proposition 2.

(a). If $w(x)$ is concave, our imposed boundary conditions imply that $w(x) > x$ for $x \in (0,1)$. After seeding the customers in $[0,\alpha]$, for any paying customer $x \in (\alpha,1]$, her WTP function at the adoption time is $(\alpha + 1 - x)w(x)K$ which can be shown to be concave and reaches minimum at $x = 1$. Hence WTP of customer $x \in (\alpha,1]$ is no smaller than α. Therefore $\min_{\alpha < x \le 1} (\alpha + 1 - x)w(x) = \alpha$. The maximization problem becomes

$$\max_{p,\alpha} p(1-\alpha)K$$

$$s.t. \quad p \le \alpha.$$

Since the objective function is increasing in p, the constraint is binding. The maximization problem becomes $\max_{\alpha} \alpha(1-\alpha)K^2$. Solving the seller's optimization problem results in $\alpha^* = \frac{1}{2}$.

(b). If $w(x)$ is convex, with the imposed boundary conditions, we have $w(x) < x$ for $x \in (0,1)$. After seeding the customers on $[0,\alpha]$, for any customer x in $[\alpha,1]$, the WTP function is $(\alpha + 1 - x)w(x)K$. The value of this function at α is equal to $w(\alpha)$. The convexity implies that $\frac{w(x)}{x} \ge \frac{w(\alpha)}{\alpha}$ under our setting. When $1 \ge x \ge \alpha \ge 0$, $w(\alpha) < \alpha,\ w(x) < x$, we have

$$\frac{w(x)}{x} \ge \frac{w(\alpha)}{\alpha} \Rightarrow \frac{w(x) - w(\alpha)}{x - \alpha} \ge \frac{w(x)}{x} \Rightarrow x[w(x) - w(\alpha)] \ge w(x)(x - \alpha),$$

therefore, we have $w(x)-w(\alpha)\geq x[w(x)-w(\alpha)]\geq w(x)(x-\alpha)$ since $x\leq 1$. Therefore we have $(\alpha+1-x)w(x)\geq w(\alpha)$. This ensures that the constraint in Lemma 2 is always binding at $p=w(\alpha)K$. The seller's problem reduces to

$$\max_{\alpha} w(\alpha)(1-\alpha)K^2.$$

Note that $w(0)(1-0)=w(1)(1-1)=0$, the objective function is positive inside the interval. Further, the object function is continuous and bounded. Therefore the optimal α^* must satisfy the first order condition, $w'(\alpha^*)(1-\alpha^*)=w(\alpha^*)$.

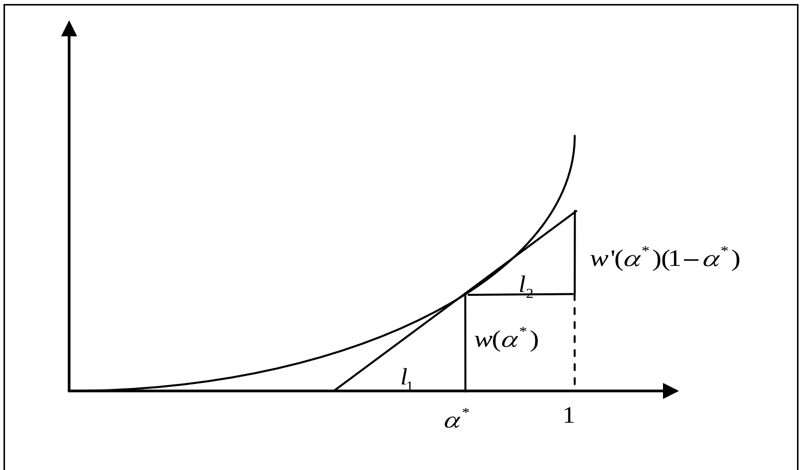


Figure 10. The Optimal Seeding Ratio

In Figure 10, α^* is the optimal seeding ratio, suggesting that all the customers on $[0,\alpha^*]$ are seeded. We have shown that α^* must satisfy $w'(\alpha^*)(1-\alpha^*)=w(\alpha^*)$. Hence in Figure 10, we have $l_1=l_2=1-\alpha^*$. Since $l_1+l_2\leq 1$, we have $2(1-\alpha^*)\leq 1$ which is equivalent to $\alpha^*\geq\frac{1}{2}$.

We now show that the existence of solution to above equation. Define $\phi(\alpha)=w'(\alpha)(1-\alpha)-w(\alpha)$. Since $\phi(\alpha)$ is continuous, $\phi(0)=w'(0)>0$, $\phi(1)=-w(1)=-1<0$. There must exists an α^* such that $\phi(\alpha^*)=0$.

When $\frac{w'(x)}{w''(x)}\geq\frac{1-x}{2}$, the objective function is concave in α, which leads to a unique interior solution. Q.E.D.

Proof of Proposition 3.

We first show that Lemma 1 holds for the additive form of network effects.

It is straightforward to show that in each segment, the seller should complete seeding before start selling and the market is fully covered under optimality. Next we show that in each segment, seeding the low end is optimal. Consider the same example as in Figure 7. For any customer x in $(\varepsilon_{k-1},\varepsilon_k]$, at the time of adoption, the size of the installed base is $I+\varepsilon_{k+1}-x$, the WTP of customer x is $x+\gamma K(I+\varepsilon_{k+1}-x)$. After mapping one-on-one each customer $x\in(\varepsilon_{k-1},\varepsilon_k]$ to a customer y in $(\varepsilon_k^c,\varepsilon_{k+1}]$, the WTP of customer y is

$$y+\gamma K(I+\varepsilon_{k+1}-\varepsilon_k+\varepsilon_{k+1}-y)=y+\gamma K(I+\varepsilon_{k+1}-x).$$

Hence

$$y+\gamma K(I+\varepsilon_{k+1}-x)-x-\gamma K(I+\varepsilon_{k+1}-x)=y-x>0,$$

which implies that the seller can charge a higher price if the seeds go to the low end of each interval. Thus the structure of market under the optimal strategy can be shown as in Figure 8. With the same setting in proof of Lemma 1, we discuss three cases: (*i*) $\gamma K<1$; (*ii*) $\gamma K=1$, and (*iii*) $\gamma K>1$.

case (i): $\gamma K<1$. In interval 1, the customer with the smallest WTP is customer type α_1. Hence the optimal price for interval 1 must satisfy $p_1\le\alpha_1+\gamma K(U+\sum_{i=1}^{k}\alpha_{j_i}+\theta_1-\alpha_1)$. The total profit of interval 1 must satisfy $\pi_1=p_1\alpha_1\le\left[\alpha_1+\gamma K(U+\sum_{i=1}^{k}\alpha_{j_i}+\theta_1-\alpha_1)\right](\theta_1-\alpha_1)K$. Now let us consider the following candidate strategy

$$\theta_i^a=\theta_i+\sum_{j=i+1}^{n}\alpha_j\ ,\ \sigma^a(i)=\begin{cases}1,\ i=1\\ \sigma(i)+1,\ i\in A\\ \sigma(i),\ otherwise\end{cases}\ ,\ \alpha_i^a=\begin{cases}\sum_{j=1}^{n}\alpha_j,\ i=1\\ 0,\ otherwise\end{cases}\text{, and }\ p_i^a=\begin{cases}\sum_{j=1}^{n}\alpha_j+\gamma K(\sum_{j=1}^{n}\alpha_j+\theta_1-\alpha_1),\ i=1\\ \\ p_i+\gamma K\theta_1\ i\in A\\ p_i,\ otherwise.\end{cases}$$

Under this candidate strategy, customers in $[0,\sum_{j=1}^{n}\alpha_j]$ are seeded. The number of paying customer in each segment remains the same. The market structure under this candidate strategy is the same as illustrated in Figure 9. Denote the profit from interval i as π_i^a and the total profit as Π^a. Similarly, π_i^a is a function of σ_i^a, p_i^a, α_i^a, and θ_i^a. The improvement in profit is

$$\begin{aligned}\Pi^a-\Pi&=\pi_1^a-\pi_1+\sum_{i\in A}[\pi_i^a-\pi_i]+\sum_{\substack{i\notin A\\ i\ne 1}}[\pi_i^a-\pi_i]\\
&=\pi_1^a-\pi_1+\sum_{i\in A}[\pi_i^a-\pi_i]\\
&\ge\left[\sum_{j=1}^{n}\alpha_j+\gamma K(\sum_{j=1}^{n}\alpha_j+\theta_1-\alpha_1)\right](\theta_1-\alpha_1)K-\left[\alpha_1+\gamma K(U+\sum_{i=1}^{k}\alpha_{j_i}+\theta_1-\alpha_1)\right](\theta_1-\alpha_1)K+U\gamma K\theta_1K\\
&\ge(\sum_{j=1}^{n}\alpha_j-\gamma KU)(\theta_1-\alpha_1)K+U\gamma\theta_1K^2\\
&\ge 0,\end{aligned}$$

which implies that the total profit under the candidate strategy is non-decreasing but all seeds go to interval 1.

case (ii): $\gamma K = 1$. In interval 1, the WTP is a constant. Hence the optimal price for interval 1 must satisfy $p_1 = \theta_1 + U + \sum_{i=1}^{k} \alpha_{j_i}$. The total profit of interval 1 must satisfy $\pi_1 = \left(\theta_1 + U + \sum_{i=1}^{k} \alpha_{j_i} \right)(\theta_1 - \alpha_1) K$. Now let us consider the following candidate strategy

$$\theta_i^a = \theta_i + \sum_{j=i+1}^{n} \alpha_j \ , \ \sigma^a(i) = \begin{cases} 1, i = 1 \\ \sigma(i) + 1, i \in A \\ \sigma(i), otherwise \end{cases} \ , \ \alpha_i^a = \begin{cases} \sum_{j=1}^{n} \alpha_j, i = 1 \\ 0, otherwise \end{cases} \text{, and } p_i^a = \begin{cases} \sum_{j=1}^{n} \alpha_j + \sum_{j=2}^{n} \alpha_j + \theta_1, i = 1 \\ \\ p_i + \theta_1 \ i \in A \\ p_i, otherwise. \end{cases}$$

This candidate strategy results in the same market structure as illustrated in Figure 9. The improvement in profit is equal to

$$\begin{aligned}
\Pi^a - \Pi &= \pi_1^a \text{-} \pi_1 + \sum_{i \in A} [\pi_i^a - \pi_i] + \sum_{\substack{i \notin A \\ i \neq 1}} [\pi_i^a - \pi_i] \\
&= \pi_1^a \text{-} \pi_1 + \sum_{i \in A} [\pi_i^a - \pi_i] \\
&= \left(\sum_{j=1}^{n} \alpha_j + \sum_{j=2}^{n} \alpha_j + \theta_1 \right)(\theta_1 - \alpha_1) K - \left(\theta_1 + U + \sum_{i=1}^{k} \alpha_{j_i} \right)(\theta_1 - \alpha_1) K + U\theta_1 K \\
&\geq (\sum_{j=2}^{n} \alpha_j - U)(\theta_1 - \alpha_1) K + U\theta_1 K \\
&\geq 0,
\end{aligned}$$

which implies that the total profit under this candidate strategy is non-decreasing but all seeds go to interval 1.

Case (iii): $\gamma K > 1$. In interval 1, the customer with the smallest WTP is θ_1. Hence the optimal price for interval 1 must satisfy $p_1 \leq \theta_1 + \gamma K \left(U + \sum_{i=1}^{k} \alpha_{j_i} \right)$. The total profit of interval 1 must satisfy $\pi_1 \leq \left[\theta_1 + \gamma K \left(U + \sum_{i=1}^{k} \alpha_{j_i} \right) \right] (\theta_1 - \alpha_1) K$. Now let us consider the following candidate strategy

$$\theta_i^a = \theta_i + \sum_{j=i+1}^{n} \alpha_j \ , \ \sigma^a(i) = \begin{cases} 1, i = 1 \\ \sigma(i) + 1, i \in A \\ \sigma(i), otherwise \end{cases} \ , \ \alpha_i^a = \begin{cases} \sum_{j=1}^{n} \alpha_j, i = 1 \\ 0, otherwise \end{cases} \text{, and } p_i^a = \begin{cases} \theta_1 + \sum_{j=2}^{n} \alpha_j + \gamma K \sum_{j=1}^{n} \alpha_j, i = 1 \\ \\ p_i + \gamma K \theta_1 \ i \in A \\ p_i, otherwise. \end{cases}$$

This candidate strategy results in the same market structure as illustrated in Figure 7. The improvement in profit is equal to

$$
\begin{aligned}
\Pi^a - \Pi &= \pi_1^a \text{-} \pi_1 + \sum_{i\in \mathrm{A}} [\pi_i^a - \pi_i] + \sum_{\substack{i\notin \mathrm{A}\\ i\neq 1}} [\pi_i^a - \pi_i] \\
&= \pi_1^a \text{-} \pi_1 + \sum_{i\in \mathrm{A}} [\pi_i^a - \pi_i] \\
&= \left(\theta_1 + \sum_{j=2}^{n} \alpha_j + \gamma K \sum_{j=1}^{n} \alpha_j\right)(\theta_1 - \alpha_1)K - \left(\theta_1 + \gamma K\left(U + \sum_{i=1}^{k} \alpha_{j_i}\right)\right)(\theta_1 - \alpha_1)K + U\gamma K\theta_1 K \\
&\geq (\sum_{j=2}^{n} \alpha_j - \gamma K U)(\theta_1 - \alpha_1)K + U\gamma\theta_1 K^2 \\
&\geq 0,
\end{aligned}
$$

which implies that the total profit under this candidate strategy is non-decreasing but all seeds go to interval 1.

Next we will show that (b) of Lemma 1 holds for the additive form of network effects. We consider three cases:

Case (i): $\gamma K < 1$. It must be the case that $p_{T+1}^* \leq \theta_T + \gamma K(G + \theta_{T+1})$. The profit of interval $T+1$ is bounded by $\left[\theta_T + \gamma K(G + \theta_{T+1})\right](\theta_{T+1} - \theta_T)$. Consider the following candidate strategy

$$
\sigma^a(i) = \begin{cases} i,\ i \leq T+1 \\ \sigma^*(i)+1,\ i \in B \\ \sigma^*(i),\ otherwise, \end{cases} \qquad \text{and} \qquad p_i^a = \begin{cases} \theta_T + \gamma K \theta_{T+1},\ i = T+1 \\ p_i^* + \gamma K(\theta_{T+1} - \theta_T),\ i \in B \\ p_i^*,\ otherwise. \end{cases}
$$

Under this candidate pricing strategy, it can be verified that all customers can still afford to buy.

$$
\begin{aligned}
\Pi(\sigma^a) - \Pi(\sigma^*) &= \sum_{i\in B} [\pi_i(\sigma^a) - \pi_i(\sigma^*)] + \pi_{T+1}(\sigma^a) - \pi_{T+1}(\sigma^*) \\
&\geq \sum_{i\in B} [\pi_i(\sigma^a) - \pi_i(\sigma^*)] + (\theta_T + \gamma K\theta_{T+1})(\theta_{T+1} \text{-} \theta_T)K \text{-} \left[\theta_T + \gamma K(G + \theta_{T+1})\right](\theta_{T+1} \text{-} \theta_T)K \\
&= \sum_{i\in B} [\pi_i(\sigma^a) - \pi_i(\sigma^*)] - \gamma G(\theta_{T+1} \text{-} \theta_T)K^2 \\
&\geq G\gamma(\theta_{T+1} \text{-} \theta_T)K^2 - \gamma G(\theta_{T+1} \text{-} \theta_T)K^2 \\
&= 0,
\end{aligned}
$$

which suggests that the profit can be improved under this candidate strategy. This contradicts to the optimality of the original strategy. Hence $\sigma(T+1) = T+1$.

Case (ii): $\gamma K = 1$. It must be the case that $p_{T+1}^* \leq \theta_T + G + \theta_{T+1}$. The profit of interval $T+1$ is bounded by $\left[\theta_T + G + \theta_{T+1}\right](\theta_{T+1} - \theta_T)$. Consider the following candidate strategy

$$
\sigma^a(i) = \begin{cases} i,\ i \leq T+1 \\ \sigma^*(i)+1,\ i \in B \\ \sigma^*(i),\ otherwise, \end{cases} \qquad p_i^a = \begin{cases} \theta_T + \theta_{T+1},\ i = T+1 \\ p_i^* + \theta_{T+1} - \theta_T,\ i \in B \\ p_i^*,\ otherwise. \end{cases}
$$

Under this candidate pricing strategy, it can be verified that all customers can still afford to buy.

$$\begin{aligned}\Pi(\sigma^a)-\Pi(\sigma^*)&=\sum_{i\in B}[\pi_i(\sigma^a)-\pi_i(\sigma^*)]+\pi_{T+1}(\sigma^a)-\pi_{T+1}(\sigma^*)\\&\geq\sum_{i\in B}[\pi_i(\sigma^a)-\pi_i(\sigma^*)]+\left(\theta_T+\theta_{T+1}\right)(\theta_{T+1}\text{-}\theta_T)K\text{-}\left[\theta_T+G+\theta_{T+1}\right](\theta_{T+1}\text{-}\theta_T)K\\&=\sum_{i\in B}[\pi_i(\sigma^a)-\pi_i(\sigma^*)]-G(\theta_{T+1}\text{-}\theta_T)K\\&\geq G(\theta_{T+1}\text{-}\theta_T)K-G(\theta_{T+1}\text{-}\theta_T)K\\&=0,\end{aligned}$$

which suggests that the profit can be improved under this candidate strategy. This contradicts to the optimality of the original strategy. Hence $\sigma(T+1)=T+1$.

Case (iii): $\gamma K>1$. It must be the case that $p^*_{T+1}\leq\theta_{T+1}+\gamma K(G+\theta_T)$. The profit of interval $T+1$ is bounded by $\left[\theta_{T+1}+\gamma K(G+\theta_T)\right]\left(\theta_{T+1}-\theta_T\right)$. Consider the following candidate strategy

$$\sigma^a(i)=\begin{cases}i,\, i\leq T+1\\\sigma^*(i)+1,\, i\in B\\\sigma^*(i),\, otherwise,\end{cases}\qquad\text{and}\qquad p_i^a=\begin{cases}\theta_{T+1}+\gamma K\theta_T,\, i=T+1\\p_i^*+\gamma K(\theta_{T+1}-\theta_T),\, i\in B\\p_i^*,\, otherwise.\end{cases}$$

Under this candidate pricing strategy, it can be verified that all customers can still afford to buy.

$$\begin{aligned}\Pi(\sigma^a)-\Pi(\sigma^*)&=\sum_{i\in B}[\pi_i(\sigma^a)-\pi_i(\sigma^*)]+\pi_{T+1}(\sigma^a)-\pi_{T+1}(\sigma^*)\\&\geq\sum_{i\in B}[\pi_i(\sigma^a)-\pi_i(\sigma^*)]+\left(\theta_{T+1}+\gamma K\theta_T\right)(\theta_{T+1}\text{-}\theta_T)K\text{-}\left[\theta_{T+1}+\gamma K(G+\theta_T)\right](\theta_{T+1}\text{-}\theta_T)K\\&=\sum_{i\in B}[\pi_i(\sigma^a)-\pi_i(\sigma^*)]-\gamma G(\theta_{T+1}\text{-}\theta_T)K^2\\&\geq\gamma G(\theta_{T+1}\text{-}\theta_T)K^2-\gamma G(\theta_{T+1}\text{-}\theta_T)K^2\\&=0,\end{aligned}$$

which suggests that the profit can be improved under this candidate strategy. This contradicts to the optimality of the original strategy. Hence $\sigma(T+1)=T+1$.

In all three cases above, given the candidate pricing strategy, interval $T+1$ can be fully covered. Hence it is not optimal if any new interval is opened before the last customer θ_T adopts. This verifies (b) of Lemma 1 under the additive form of network effects. The optimal market structure remains the same, as illustrated in Figure 4.

Given the optimal market structure, next we now solve the seller's problem in each of the three cases.

Case (i): $\gamma K<1$. $p_i^*=\theta_{i-1}+\gamma K\theta_i$. The seller's problem is

$$\max_{\alpha,\,\theta_i,i=1,2,..,n-1}\underbrace{(\alpha+\gamma K\theta_1)}_{p_1}(\theta_1-\alpha)K+\sum_{i=2}^{n-1}\underbrace{(\theta_{i-1}+\gamma K\theta_i)}_{p_i}(\theta_i-\theta_{i-1})K+\underbrace{(\theta_{n-1}+\gamma K)}_{p_n}(1-\theta_{n-1})K$$

$$s.t. \quad 0 \le \theta_1 \le \theta_2 \le ... \le \theta_{n-1} \le 1.$$

Analogues to the solution approach of Proposition 1, we obtain:

$$\alpha^* = \frac{1-\gamma K}{1-\gamma K+(1+\gamma K)n}, \theta_i^* = \frac{1-\gamma K + i(1+\gamma K)}{1-\gamma K+(1+\gamma K)n}, p_i^* = \frac{[i(1+\gamma K)-\gamma K](1+\gamma K)K}{1-\gamma K+(1+\gamma K)n}.$$

Case (ii): $\gamma K = 1.$ $\quad p_i^* = \theta_{i-1} + \theta_i.$ The seller's problem is

$$\max_{\alpha,\, \theta_i, i=1,2,..,n-1} \underbrace{(\alpha+\theta_1)}_{p_1}(\theta_1-\alpha)K + \sum_{i=2}^{n-1}\underbrace{(\theta_{i-1}+\theta_i)}_{p_i}(\theta_i - \theta_{i-1})K + \underbrace{(\theta_{n-1}+1)}_{p_n}(1-\theta_{n-1})K$$

$$s.t. \quad 0 \le \theta_1 \le \theta_2 \le ... \le \theta_{n-1} \le 1.$$

Solving, gives us

$$\alpha^* = 0, \theta_i^* = \frac{i}{n}, p_i^* = \frac{(2i-1)K}{2n}.$$

Case (iii): $\gamma K > 1.$ $\quad p_i^* = \theta_i + \gamma K \theta_{i-1}.$ The seller's problem is

$$\max_{\alpha,\, \theta_i, i=1,2,..,n-1} \underbrace{(\theta_1+\gamma K\alpha)}_{p_1}(\theta_1-\alpha)K + \sum_{i=2}^{n-1}\underbrace{(\theta_i+\gamma K\theta_{i-1})}_{p_i}(\theta_i - \theta_{i-1})K + \underbrace{(1+\gamma K\theta_{n-1})}_{p_n}(1-\theta_{n-1})K$$

$$s.t. \quad 0 \le \theta_1 \le \theta_2 \le ... \le \theta_{n-1} \le 1.$$

Solving, gives us

$$\alpha^* = \frac{\gamma K-1}{\gamma K-1+(1+\gamma K)n}, \theta_i^* = \frac{\gamma K-1+i(1+\gamma K)}{\gamma K-1+(1+\gamma K)n}, p_i^* = \frac{[i(1+\gamma K)-1](1+\gamma K)K}{\gamma K-1+(1+\gamma K)n}.$$ Q.E.D.

Derivation of the Optimal Strategy in the Worst Seeding Case.

The seller's optimization problem is

$$\begin{aligned} &\max_{\alpha,\theta}(1-\alpha-\theta)(1-\theta)\theta K^2 \\ &s.t.\ (1-\theta)\theta \le (1-\alpha)\alpha \\ &\quad 1-\alpha-\theta > 0 \\ &\quad \alpha,\theta \in (0,1). \end{aligned}$$

The first constraint is $(1-\theta)\theta \le (1-\alpha)\alpha \Rightarrow \begin{cases} \text{(i).}\ \ \theta = \alpha \\ \text{(ii).}\ \ \theta+\alpha \ge 1 \text{ if } \theta > \alpha \\ \text{(iii).}\ \theta+\alpha \le 1 \text{ if } \theta < \alpha. \end{cases}$

Case (ii) is dropped since we need $1-\theta-\alpha > 0$ because of the second constraint. For any θ, since the objective function is always decreasing in α, we need the minimum value of α, which is $\alpha = \theta$ under both (i) or (iii) cases. Note that any θ satisfying case (iii) will be smaller than $\frac{1}{2}$. Thus, the maximization problem becomes

$$\max_{\theta}(1-2\theta)(1-\theta)\theta K^2,$$

which gives $\theta^* = \frac{3-\sqrt{3}}{6}$.

Derivation of the Optimal Strategy in the Uniform Seeding Case.

The constraint maximization problem is

$$\begin{aligned}&\max_{\alpha,\theta}(1-\theta+\alpha\theta)\theta(1-\alpha)(1-\theta)K^2\\&s.t.\ \ (1-\theta+\alpha\theta)\theta \le \alpha\\&\qquad \alpha,\theta\in(0,1).\end{aligned}$$

The constraint is equivalent to $\alpha \ge \frac{\theta}{1+\theta}$. Therefore, we have

$$\begin{aligned}&\max_{\alpha,\theta}\left[1-(1-\alpha)\theta\right](1-\alpha)(1-\theta)\theta K^2\\&s.t.\ \ \alpha \ge \frac{\theta}{1+\theta}.\end{aligned}$$

For any θ, the optimal α for unconstrained problem is $\alpha^* = 1-\frac{1}{2\theta} \le 1-\frac{1}{1+\theta} = \frac{\theta}{1+\theta}$, which implies that the constraint is always binding. Therefore the maximization becomes

$$\max_{\theta}\frac{\theta(1-\theta)K^2}{(1+\theta)^2}.$$

It can be shown that $\theta^* = \frac{1}{3}$.

Proof of Proposition 4.

Given α^r, the optimal price p must satisfy $p \le \alpha^r K.$ For each customer θ, at the time of adoption, the WTP is $(g(\theta)+1-\theta)\theta K$ which is a function of θ. Hence we simplify the notation of WTP function $u(\delta,\theta)$ to $u(\theta)$. For customers in $(\frac{1}{2},1]$, the WTP function is

$$u(\theta) = (1-\theta+\alpha^r)\theta K, \quad \theta\in(\frac{1}{2},1]$$

which is deterministic and satisfies $u(\theta) \ge \alpha^r K$ for all $\theta\in(\frac{1}{2},1]$. This indicates that the last paying customer x must be on the low-end half. Given any price p, we will have a last paying customer x and the number of paying customer is $N^r = \left(1-x-[\alpha^r - g(x)]\right)K.$ The profit maximization problem becomes

$$\begin{aligned}&\max_{x}\ x\left[1-x+g(x)\right]\left[1-x+g(x)-\alpha^r\right]K^2\\&s.t.\ x\left[1-x+g(x)\right] \le \alpha^r\end{aligned}$$

Given interval $[0,\frac{1}{2}]$, the WTP function $u(x)=x[1-x+g(x)]K$ satisfies

$$\frac{du}{dx}=[1-2x+g(x)+g'(x)x]K\geq 0 \;\; a.e.$$

This suggests that the WTP function is non-decreasing (a.e.) on the low-end half. Since the WTP function is continuous, we know that there will be at least one $x\in[0,\frac{1}{2}]$ satisfying $u(x)=\alpha^r K$. Now our task is equivalent to show that $\pi(y)\leq\pi(x)$ for all $y\leq x$. For any $y\leq x$, there are two possible cases:

Case (i): Customer y is seeded. This is equal to say that $g'(y)=1$. Without loss of generality, we assume $g(y)=y-c$. Therefore, the profit function near the neighborhood of y is

$$y(1-y+y-c)(1-y+y-c-\alpha^r)K^2=y(1-c)(1-c-\alpha^r)K^2$$

which is always increasing in y. This indicates that the neighborhood of marginal customer should not be seeded because the seller can always get higher profit by raising the price;

Case (ii): y is not seeded. Let us assume in a small neighborhood $[y-\rho, y+\rho]$, none customers are seeded. Therefore, the profit function is differentiable in this neighborhood. What we need to show is $\frac{d\pi(y)}{dy}\geq 0$ for $y\leq x$. Without loss of generality, since y is not seeded, we can assume $g(y)=s$ in this case.

The profit function is $\pi(y)=y(1-y+s)(1-y+s-\alpha^r)K^2$. Hence we have

$$\frac{d\pi(y)}{dy}=\left[3y^2-2(2+2s-\alpha^r)y+(1+s)(1+s-\alpha^r)\right]K^2. \tag{4}$$

The symmetry axis of this quadratic function is $\frac{2+2s-\alpha^r}{3}\geq\frac{2-\alpha^r}{3}\geq\frac{1}{2}$ for $\alpha^r\leq 0.5$. This indicates that all possible y is to the left of symmetry axis. Since $y\leq x$, our task is equivalent to show $\frac{d\pi(y)}{dy}|_{y=x}\geq 0$.

Note that s is actually a function of x. Therefore, s in the neighborhood of x is not always the same to s in the neighborhood of y. Let us assume the profit function in the neighborhood of x is π_1 and that of neighborhood of y is π_2. Fixing y, we can see that the profit function $y(1-y+s)(1-y+s-\alpha^r)K^2$ is always increasing in s, therefore, π_1 dominates π_2.

The solutions to the cubic function $y(1-y+s)(1-y+s-\alpha^r)K^2=0$ include $y=0$, $y=1+s-\alpha^r$, $y=1+s$. This indicates that in interval $[0,\frac{1}{2}]$, the profit functions are unimodal. Since $y\leq x$, as long as $\frac{d\pi(y)}{dy}|_{y=x}\geq 0$, we will have $\pi_1(x)>\pi_2(y)$ (see Figure 11 below for illustration). It is possible that x is seeded. In that case, the WTP curve at the neighborhood of x intersects with $\pi_1(x)$ at x and does not change the value at x, therefore we can skip that case.

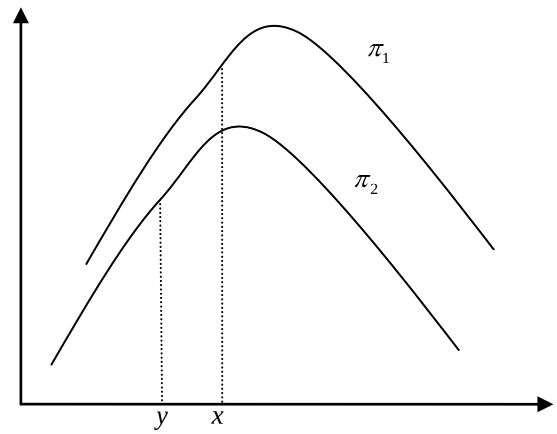


Figure 11. The profit functions at neighborhoods of *y* and *x*

At x, we have $x\left[1-x+g(x)\right]=\alpha^r$, therefore, $1+g(x)=\frac{\alpha^r}{x}+x$, plug this back to (4)

$$\frac{(\alpha^r)^2}{x}(\frac{1}{x}-1)-\alpha^r(2-x)$$

Our task is equivalent to show

$$\alpha^r \geq \frac{x(2-x)}{\frac{1}{x}-1}=x^2(1+\frac{1}{1-x}) \tag{5}$$

To ensure this always hold true for any x, we need to find the biggest possible value of the right hand side. Note that the right hand side is increasing in x, therefore, for each α^r, we need to show what is the maximum possible x. Since $x(1-x+s)=\alpha^r$, we have $x=\frac{1+s-\sqrt{(1+s)^2-4\alpha^r}}{2}$,

which is decreasing in s. The smallest possible s only occurs in the case shown in Figure 12. The horizontal line presents the customers from 0 to 1. All customers in the dark interval are seeded. If all seeds go to the low end half, the seeding outcome shown in following figure gives the smallest number of $g(x)$ for any $x\in[0,\frac{1}{2}]$

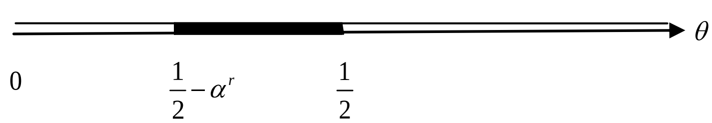


Figure 12. The case where g(x) is smallest for all $x\in[0,\frac{1}{2}]$

In Figure 12, at the point $\frac{1}{2}-\alpha^r$, the WTP is $(\frac{1}{2}-\alpha^r)(\frac{1}{2}+\alpha^r)=\frac{1}{4}-(\alpha^r)^2$. We discuss two cases in turn.

Case (i): if $\alpha^r < \frac{\sqrt{2}-1}{2}$, $\frac{1}{4}-(\alpha^r)^2 > \alpha^r$, which means $g(\max x)=0$ and $\max x = \frac{1-\sqrt{1-4\alpha^r}}{2}$. Plugging this back into (5) , we have

$$\alpha^r > (\frac{1-\sqrt{1-4\alpha^r}}{2})^2(1+\frac{2}{1+\sqrt{1-4\alpha^r}}),$$

which is true for all $\alpha^r < \frac{\sqrt{2}-1}{2}$.

Case (ii): if $\alpha^r \geq \frac{\sqrt{2}-1}{2}$, $\frac{1}{4}-(\alpha^r)^2 \leq \alpha^r$, the WTP function at this point is less than α^r, which means $g(\max x)=\max x-\frac{1}{2}+\alpha^r$, plug this back into $x(1-x+g(x))=\alpha$, we have $\max x=\frac{2\alpha^r}{1+2\alpha^r}$. Hence

$$\alpha^r \geq (\frac{2\alpha^r}{1+2\alpha^r})^2(1+1+2\alpha^r)$$

which is equivalent to

$$1-4\alpha^r-4(\alpha^r)^2 \geq 0$$

which requires $\alpha^r \leq \frac{\sqrt{2}-1}{2}$. This only holds for $\alpha^r = \frac{\sqrt{2}-1}{2}$.

Combine Case (i) and (ii), we have when $\alpha^r \leq \frac{\sqrt{2}-1}{2}$, so (5) always holds and $\frac{d\pi(y)}{dy}|_{y=x} \geq 0$. Q.E.D.

Proof of Proposition 5.

Under random seeding with limited control, we have shown in the proof of Proposition 4 that all the customer in $(\frac{1}{2},1]$ will purchase if the price is set to $\alpha^r K$. Therefore the profit from the high-end half is $\frac{\alpha^r K^2}{2}$, which gives lower bound. Now we consider optimal seeding with an information acquisition cost c. The optimal profit is $\frac{K^2}{4(1+c)}$. We discuss three cases in turn.

Case (i): When $c > \frac{1}{2\alpha^r}-1$, it is equivalent to say the optimal profit is no greater than $\frac{\alpha^r K^2}{2}$. Hence we have (a) of Proposition 5.

Case (ii): When $c < \frac{1}{4\alpha^r(1-\alpha^r)} - 1$, the profit of optimal seeding with acquisition cost is no less than $\alpha^r(1-\alpha^r)K^2$ which is equal to the profit under the best scenario of random seeding with $\alpha^r K$ seeds. Hence it is optimal for the seller to buy the information.

Case (iii): $c \in \left[\frac{1}{4\alpha^r(1-\alpha^r)} - 1, \frac{1}{2\alpha^r} - 1\right]$, The profit from optimal seeding with cost is $\frac{K^2}{4(1+c)} \in \left[\frac{\alpha^r K^2}{2}, \alpha^r(1-\alpha^r)K^2\right]$. The profit from random seeding is smaller than $\alpha^r(1-\alpha^r)K^2$. Therefore, either could be the dominating strategy, which is subject to the realization of the random seeding. Q. E. D