Risk Choice Under High-Water Marks [*]

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Abstract

I provide a closed-form solution to the optimal dynamic risk choice of a fund manager who is compensated under a high-water mark contract. The manager's optimal risk choice varies with the distance between the fund's asset value and its high-water mark. Negative returns increase the manager's effective risk aversion ('de-leveraging') when the value of his outside option is low, termination is 'strict', or management fees are high, and decrease his effective risk aversion ('gambling') otherwise. I show that in the absence of limits on risk taking, it is *never* optimal for a manager to walk away. When there are risk limits, walk-away can be optimal following losses.

1 Introduction

High-water mark (HWM) contracts are the predominant performance-based incentive scheme used to compensate managers in the hedge fund industry. The contract pays a fund manager a fixed fraction of the fund's return in excess of its historical maximum cumulative return– the high-water mark. An important question is, what is the optimal risk choice of a manager under this simple compensation mechanism? This question is particularly poignant in the case of hedge funds since they are amongst the most unconstrained and sophisticated investors. Moreover, understanding how the HWM impacts hedge funds' risk tolerance may be important in understanding price dynamics in markets where they are an influential investor or liquidity provider. HWM-style incentives are also implicit in other types of performance-based compensation (e.g. executive compensation), so understanding the optimal risk taking behavior they induce is relevant in settings beyond hedge funds.

Despite the great value in understanding the answer to this question, there are only a few results in the literature on the optimal risk choice of a manager facing HWM incentives. In this paper, I greatly expand the set of known results by providing a closed-form solution to this problem. The solution shows that the manager's optimal risk choice varies as a function of the distance between the fund's asset value and its high-water mark. Two main types of risk-choice dynamics are possible: (i) 'loss-driven de-leveraging', whereby the manager reduces fund leverage as the fund's value falls further below the HWM,[1] and (ii) 'gambling for resurrection', where the manager instead increases risk-taking as the fund's value falls further below the HWM. I show how the specific dynamic that arises is determined jointly by several key characteristics of the manager's environment: the value of his outside option, the 'termination policy' or distance the fund can drop below the HWM before triggering the manager's termination, and the discounted value of management fees (the manager's 'inside payoff'). I further solve for the manager's optimal walk-away policy. I show that walk-away is *never* optimal in the absence of limits on the manager's risk-taking and why risk limits can make walk-away optimal following negative returns.

The main antecedent to this paper is Panageas and Westerfield (2009) (henceforth PW), who are the first to provide closed-form results for the HWM risk-choice problem. Similar to PW, I solve for the risk-choice of an indefinitely-tenured, risk-neutral manager facing a HWM. PW show that such a manager optimally acts like a CRRA investor with a (fixed)

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[1] In the model the manager controls risk-choice by changing the weight of the risky and riskless asset in the fund's portfolio, i.e. by changing fund leverage.

risk-aversion of less than one. As they emphasize, this result shows that despite the (call) option-like nature of the HWM payout, even a *risk-neutral* manager does *not* put unbounded weight on risky assets. Their result provides an important underpinning for the results in this paper. However, the framework in PW does not incorporate a number of important considerations that fundamentally impact the nature of the manager's optimal risk-taking behavior. Incorporating these considerations reveals a rich class of possible risk-taking dynamics on the part of the manager. The constant risk choice found by PW then appears as a unique special case of the general solution. While incorporating these factors into the manager's problem raises substantially the complexity of the problem, I am still able to find closed-form solutions that illustrate how the different factors impact the manager's risk choice policy.

Two important considerations that I incorporate into the manager's problem are: (1) the manager can be terminated when bad performance reduces the fund value to a given percentage of the HWM, and (2) the manager receives an outside payoff (i.e., has an outside opportunity) in case of termination or walk-away.[2] This outside payoff is proportional to the scale of the fund, so the manager of a larger fund will have a larger outside payoff. Both considerations are highly relevant to managers in practice. For hedge fund managers in particular, a common view is that the outside option has an important influence on risk-taking. As this paper shows, the nature of the manager's optimal risk choice depends greatly on both considerations. In contrast, in PW the manager is never terminated–he continues to run the fund so long as fund assets are non-zero. Moreover, his outside payoff is implicitly fixed at zero.

The risk of termination enters prominently into the manager's dynamic risk choice. When losses drop the fund value below the HWM, bringing it closer to the termination boundary, the manager must weigh two offsetting considerations. On the one hand, the greater distance from the HWM means the manager must wait longer until he receives any more performance fees. This pushes him in the direction of taking more risk in order to reduce the expected discounted time to payment. On the other hand, the closer proximity to the termination point reduces the manager's margin for error. This makes the manager effectively more 'risk-averse', and pushes him towards reducing risk.[3] As I show, the solution weighs these

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[2]The outside payoff is the value to the manager in utils or equivalently dollars (since he is risk neutral), of his opportunity cost. I do not take a stand on the specific source of this, but a natural candidate is the value of starting a new fund or the value to the manager of his leisure.

[3]As stated above, the manager is assumed to be risk-neutral. However, as shown, his indirect utility function has curvature, so he behaves as if he is risk-averse.

two competing considerations against each other. Which consideration dominates depends on the termination policy, the manager's outside payoff, and the discounted value of non-performance payouts, such as management fees, received by the manager while he is inside the fund (the 'inside payoff').

If the outside payoff is not much higher than the inside payoff, or the termination point is relatively close to the HWM, then avoiding termination dominates the desire to reduce the time to the HWM. In this case, the manager will reduce risk as the fund falls further below the HWM. Indeed, the solution to the manager's problem shows that the manager's effective 'risk-aversion' increases sharply as a function of the distance below the HWM, causing him to drastically reduce fund leverage. Both the overall level of risk taken by the manager and the rate of deleveraging are determined jointly by the outside payoff, termination point, and inside payoff. Unlike the result in PW, the set of potential (endogenous) manager 'risk-aversions' expands to include all positive values, as opposed to just values less than one.

An additional finding is that the (endogenous) functional form for the manager's effective risk-aversion that arises here bears a very close analytical resemblance to the (exogenously specified) external habits preferences of Campbell and Cochrane (1999), with the surplus ratio corresponding to the distance between the fund's wealth and the high-water mark. However, in contrast with external habits, variation in the manager's effective risk aversion is driven directly by realized returns rather than by changes in consumption.

If the outside payoff is high relative to the inside payoff, or the termination point is far below the HWM, then the desire to reduce the time to the HWM will dominate. In this case, the manager's effective risk-aversion decreases and he takes more risk as the fund falls below the HWM. The dynamic is again nonlinear, with risk-taking rising increasingly rapidly as termination is approached. This increased risk-taking can be viewed as a form of 'gambling for resurrection' by the manager, since it is triggered by losses. However, unlike 'pure' gambling, the manager's position in the risky asset remains finite and well-defined.

I further extend the manager's problem to incorporate the impact of a continuous rate of fund withdrawals by investors and to include a management fee. A higher rate of fund withdrawal makes the fund manager effectively *less* risk-averse, causing him to take more risk. This reflects the fact that in the face of a greater rate of withdrawal, it becomes more important to quickly increase the fund's value (and reach the HWM). This results in the manager discounting his continuation value more strongly and therefore decreases the level of risk aversion induced by the HWM. In contrast, a higher management fee decreases

risk taking by raising the inside payoff to the manager. This is because the value of the perpetuity represented by the management fees makes managing the fund more attractive for the manager relative to his outside payoff, which increases the importance of avoiding termination.

I then consider the manager's optimal walk-away decision. If the manager's outside payoff is positive, then the manager may consider voluntarily walking away from the fund, and this walk-away option may have an important influence on the manager's risk choices. I therefore allow for walk-away and solve jointly for the manager's risk choice and optimal walk-away decision. Remarkably, the solution shows that it is actually *never* optimal for the manager to exercise his walk-away option. Only when the fund value is zero will the manager trivially 'walk away'. To understand this, consider a fund manager who walks away when the fund value drops to some fraction of the HWM. This fund manager is better off pushing down this walk-away point and simultaneously raising his risk-taking *globally*, as this combination results in a greater expected payout for him at any level of fund wealth. This suggests, however, that manager walk-away may be optimal if the manager is constrained in his ability to increase risk/leverage. This constraint might arise in the form of margin constraints, position limits, or it may be exogenously imposed by investors. To investigate this intuition, I solve the manager's problem with position limits. I find that when the position constraint binds, then it can be optimal for the manager to exercise his walk-away option. Moreover, the manager will exercise his walk-away option earlier when his outside payoff is bigger and when the position limit is lower, and thus binds more tightly.

Finally, I consider an extension of the model where the rate of investors' withdrawals depends on the fund's performance. I assume that if fund assets falls below some percentage of the HWM then this triggers an increase in the rate of investor withdrawals, reflecting some loss of faith in the manager by investors. In response, the manager locally increases risk-taking despite the increased distance from the HWM and the closer proximity to termination. This response is consistent with the idea of 'gambling for resurrection'; increased losses by the fund manager actually induce him to become more aggressive in his risk taking. A corresponding global effect is that the manager becomes more cautious in the region close to the HWM, when the rate of withdrawals is still low, since he takes into account the risk of greater future withdrawals in the solution to his dynamic problem.

Relation to the Literature

This paper is relate to the literatures on incentives frictions in delegated asset man-

agement, option-based compensation, and the impact of financial intermediation on asset prices. As indicated above, this paper builds on results in Panageas and Westerfield (2009). PW itself is closely related to Goetzmann, Ingersoll and Ross (2003) (henceforth GIR). GIR provide closed-form solutions for the present value of the hedge fund manager's fees and investors' claim under a high-water mark contract when the risk-choice of the manager is *exogenously fixed* at a constant level. The framework in their paper is quite general and incorporates important considerations such as termination, which is triggered when fund wealth falls to a given percentage of the HWM, investor withdrawals, and management fees. This paper follows that framework in incorporating these features. Moreover, the solution for the manager's value function in this paper is equivalent to the discounted expected value of the manager's payoffs. In contrast to GIR, however, this paper solves for the manager's optimal risk-choice, which is in general very different from the fixed level assumed in their paper. Indeed, understanding the manager's optimal dynamic risk-taking is the central objective of this paper. This also accounts for why aspects of the manager's problem, such as his outside payoff and walk-away option, are very important to this paper but do not appear in GIR. On the technical side, this paper also has similarities to Browne (1997, 2000), who solves a set of control problems where the goal is to maximize an expected discounted reward from attaining a goal.

Carpenter (2000) solves for the risk choice of a risk-averse manager facing a finite horizon, who is compensated with a single call option. The HWM contract is akin to compensation with a series of call options, where an option that is exercised is replaced with one with a higher strike price (the new HWM). Moreover, with a HWM the horizon is indefinite. This leads to important differences in the predicted risk choice of the manager. In Carpenter's model, the risk choice of the manager goes to infinity as the asset value decreases to zero, while in Panageas and Westerfield (2009), the risk choice of the manager remains constant. Panageas and Westerfield (2009) show that the difference between the finite horizon in Carpenter (2000) and the indefinite horizon of the HWM model is a key to this difference since with an indefinite horizon the continuation value of the manager has an important role in attenuating risk taking. This papers shows that with a HWM and an indefinite horizon, it is in fact possible for risk taking to decrease *or* increase as losses increase the distance between fund wealth and the HWM. PW is a special case that is right at the boundary, where risk taking remains constant. In contemporaneous work, Lan, Wang, and Yang (2011) solve numerically the problem of a HWM-compensated hedge-fund manager whose alpha-generating strategy suffers from decreasing returns to scale.

Although the high-water mark contract is closely associated with the hedge fund industry, similar incentives arrangements are often implicit in other contexts. An example is a manager of a corporation whose compensation is in part based on firm performance reaching new highs (e.g., record earnings or stock price). This compensation sensitivity might result from bonuses that are performance-sensitive, or from the granting over time of a series of executive stock options. If increasing firm risk can improve measured performance, then the manager faces a problem very similar to the one modeled here.

There is also a growing literature on the impact of financial intermediation and incentives-related frictions on asset price dynamics. For instance, He and Krishnamurthy (2010) develop a model where the risk-bearing capacity of a risk-tolerant financial intermediary sector is a central driver of risk premiums and hence asset prices. In particular, the behavior of intermediaries is potentially an important factor in explaining the dynamics of asset prices during the recent financial crisis. By solving for the dynamic risk choices of a prominent group of intermediaries (i.e. hedge funds), this paper helps to understand mechanisms that may have contributed to the crisis's dynamics. For example, an apparently important feature of the financial crisis was the de-leveraging of the normally risk-tolerant hedge fund sector. For hedge funds facing the problem modeled here, this paper demonstrates that losses would induce de-leveraging. Moreover, the effect is non-linear so that large losses would induce an increasingly sharp rate of de-leveraging. This means that a large macroeconomic shock that leads to losses across many hedge funds simultaneously would cause a coordinated and sharp de-leveraging in the hedge fund sector.[4]

Finally, there is an expanding empirical literature that focuses especially on hedge funds. Early works include Fung and Hsieh (1997), Ackermann, McEnally, and Ravenscraft (1999), Brown, Goetzmann, and Ibbotson (1999), and Brown, Goetzmann, and Park (2001). Recent work includes Agrawal, Daniel, and Naik (2009), Fung, Hsieh, Naik, and Ramadorai (2008), Aragon and Nanda (2010), Ang, Gorovyy, and van Inwegen (2011), and Ray (2011).

The paper proceeds as follows. Section 2 lays out the model describing the manager's optimal risk choice problem. Section 3 derives its solution, analyzes comparative statics, and develops the intuition behind the solution's functional forms. Section 4 analyzes the walk-away decision and shows that voluntary walk-away is never optimal for an unconstrained manager. It then extends the model to include risk limits, solves it, and shows that in this case walk-away can be optimal. Section 5 considers an extension where losses trigger an

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[4]Such an effect would naturally augment other potential drivers of de-leveraging, such as increases in margin requirements.

increase in the rate of investor withdrawals and demonstrates how this leads to a dynamic increase in risk taking. Section 6 concludes. Proofs not appearing in the main text are left to the Appendix.

2   Model

A risk-neutral manager allocates a portfolio between one risky asset and a money market account. The price of the money market account evolves according to

$$\frac{dP_{0,t}}{P_{0,t}} = rdt$$

where r is the fixed interest rate earned by the money market account. The manager can go long or short the money market account. The price of the risky security evolves according to

$$\frac{dP_{1,t}}{P_{1,t}} = \mu dt + \sigma dB_t$$

where $\mu > r$ and $\sigma > 0$ are constant and B_t is a one-dimensional Brownian motion. Like PW, I do not take a stand on the source of the risky investment's expected return since I study how the manager responds to this investment opportunity and not why it exists. The risky asset expected return could just reflect an equilibrium risk premium or even the manager's subjective expectation of returns. However, a reasonable assumption is that μ represents risk-adjusted excess return ('alpha') that the manager is able to generate through some proprietary skill. Moreover, the manager may be benchmarked so that he is not compensated for systematic risks and remains 'market-neutral'. This is equivalent to considering risky-asset dynamics that are risk-adjusted, i.e., taking the risky-asset dynamics above as being given under the *risk-neutral* measure of the fund's representative investor.

The manager chooses the fraction π_t of fund wealth W_t to invest in the risky asset at time t. The remaining $1 - \pi_t$ then goes into the money market account. In the baseline model there are no limits on the position that the manager can take in the stock or money market account, except that the portfolio strategy is admissable (in $\mathcal{L}^2$) and the transversality condition holds.

As in GIR and PW, I model the fund as being invested in by a group of investors with the same HWM. This eliminates the need to keep track of a different high-water marks for each investor. In practice, this will arise if all investors currently in the fund began investing

in the fund at the same time, or if inflows into the fund only occur when it is at the HWM. This assumption does *not* place any restriction on withdrawals, which I model as occurring at a continuous rate ϕ_t. I do not model inflows. In the baseline model, ϕ_t is set constant, whereas in an extension I consider a case where ϕ_t varies.

The HWM is denoted by H_t, which is the highest level that the net assets of the fund have reached, subject to some adjustments. As in GIR, the HWM is adjusted down for withdrawals and is adjusted up at a contractually stated rate, which I set equal to r. Adjusting the HWM upwards at r implies that the manager does not earn performance fees for earning the risk-free rate on funds (it is a form of benchmarking). A final adjustment to the HWM accounts for the management fee, which is denoted by m. As in GIR, and in practice, the three adjustments to H_t (for ϕ_t, r, and m) are proportional; i.e., post-adjustment, the ratio W_t/H_t remains unchanged. When $W_t < H_t$ and the fund is not reaching a new high, these are the only adjustments made to H_t. Hence, in the region $W_t < H_t$, H_t evolves deterministically as follows

$$dH_t = (r - \phi_t - m)H_t dt$$

The other region of importance is when the fund is at the high-water mark, $W_t = H_t$. When the fund's wealth increases from $W_t = H_t$ to $W_t = H_t + \varepsilon$, a performance fee of $k\varepsilon$ is paid, fund wealth is reduced by $k\varepsilon$, and the HWM is reset to $H + \varepsilon$. It will be convenient to have a notation for just these 'ε' increases. Therefore, let these increases be denoted by dH_t^ε, so that $dH_t = dH_t^\varepsilon + (r - \phi_t - m)H_t dt$.

Termination of the manager may occur in two ways. The first is if fund wealth drops to some 'low' proportion C of the high-water mark, at which point investors lose confidence in the manager. Following GIR and PW, I also allow for an exogenous random termination of the fund that is assumed to be Poisson with intensity λ. This may represent, for example, the possibility of a liquidity shock for investors that induces liquidation of the fund. Upon termination, the manager receives an outside payoff that has value $\underline{V}_t$. One natural interpretation of this is that it is the value to the manager of starting or managing a new fund. With this in mind, I assume that the outside payoff of the manager is proportional to the current magnitude of the fund. I capture magnitude by the current HWM.[5] This assumption is important for tractability, since it keeps the problem homogenous. Moreover, it is very

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[5]Of course this also means that the outside payoff is proportional to the size of the fund at the termination (lower) boundary, $W_t = CH_t$, which is typically the point at which the outside payoff is realized.

plausible; the manager of a large fund should have a more valuable outside opportunity than the manager of a small fund. I therefore let $\underline{V}_t = g_0\beta_1 H_t$. The constant chosen here is g_0, while β_1 is a normalization factor that arises below. The value of g_0 is freely specifiable, subject to some economic restrictions derived below.

We can now formally represent the manager's objective. Define τ to be a stopping time that is equal to the termination time of the manager, or ∞ if he is never terminated. Since the manager is risk-neutral, his objective is to maximize the expected discounted value of the fees he receives up until termination, plus the discounted value of the outside payoff, which is received upon termination,

$$V_t = \max_{\pi_s} E\left[\int_t^\tau e^{-\rho(s-t)}(mW_s ds + kdH_s^\varepsilon) + e^{-\rho(\tau-t)}\underline{V}_\tau\right] \tag{1}$$

where ρ is the manager's time-discount factor. Note that since $\mu - r > 0$ is adjusted for systematic risk (i.e., it is pure 'alpha'), this objective is unchanged if the expectation is taken under the representative investor's risk-neutral measure.[6,7]

3   Solution

I start by deriving the Hamilton-Jacobi-Bellman (HJB) equation satisfied by the solution to the manager's problem. I then demonstrate a solution that satisfies these equations and also look at some special cases of the solution that help to understand its dependence on parts of the model.

Since the manager's problem is homogenous in H_t, it will be beneficial to use as state

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[6]If the manager is not risk-neutral, then his personal pricing measure will in general depend on his outside wealth, income stream, and the size of the fund relative to these. If fund payouts are an important source of consumption for the manager, then in general this will induce the manager to be more cautious in his portfolio choice in order to decrease the volatility of the fund's performance and his income/consumption stream. I conjecture, however, that the results derived here will not change qualitatively, but leave the solution of this problem to future work.

[7]In PW it is futher assumed that the manager is excluded from trading in private accounts, rendering the market incomplete from his vantage point. Their motivation for this requirement is that in a complete market equilibrium there is no risk-adjusted excess return on the risky asset. This implies (by their solution) that the manager will take unbounded positions. This assumption is not necessary here because $\mu - r$ is assumed to be a (positive) *risk-adjusted* return. Moreover, assuming no-arbitrage, it is implicit that the market is incomplete, otherwise the positive-alpha asset would be an arbitrage.

variables both H_t and a new variable

$$X_t = \frac{W_t}{H_t}$$

that measures the proportional distance of current fund wealth from the HWM. Taking differentials gives $dX_t = \frac{dW_t}{H_t} - \frac{W_t}{H_t^2} dH_t$. Substituting in the dynamics of W_t and H_t then shows that in the region $\{C < X_t < 1\}$ the dynamics of X_t are given by:

$$dX_t = X_t \pi_t (\mu - r) dt + X_t \pi_t \sigma dB_t \tag{2}$$

On the boundary $\{X_t = 1\}$ there is the payout of the performance fee and the reset of the HWM: (i) $dH_t = dH_t^\varepsilon$, (ii) $dW_t = -k dH_t^\varepsilon$, so that $dX_t = -(1+k) dH_t^\varepsilon / H_t$.

Now let $V = V(X_t, H_t)$ denote the value function of the manager, given by the solution to (1). Then, for $t < \tau$, the process $\int_0^t e^{-\rho s} (mW_s ds + k dH_s^\varepsilon) + e^{-\rho \tau} \underline{V}_\tau$ is a martingale under the maximizing choice of π, and satisfies the following HJB equation:

$$0 = -(\rho + \lambda)V + mW_t + \lambda \underline{V}_t + \sup_\pi \{V_X X \pi (\mu - r) + \frac{1}{2} V_{XX} X^2 \pi^2 \sigma^2\} \tag{3}$$

$$+ V_H H (r - \phi_t - m) + k dH^\varepsilon - V_X (1+k) \frac{dH^\varepsilon}{H} + V_H dH^\varepsilon$$

where the last three terms are non-zero only at the boundary $\{X_t = 1\}$.

3.1 Model Solution

For the baseline model, I make two changes to (3) before solving it. First, I assume that ϕ_t is a constant ϕ. In an extension I relax this assumption and allow ϕ to take on a low and high value depending on the fund's performance. The second change relates to the management fee payout and is necessary for analytical tractability. While I continue to assume that funds are withdrawn from fund wealth at the rate m, and H_t continues to be appropriately adjusted (downwards) at this rate, I assume that the payout received by the manager is a constant proportion of H_t, given by $m_H H_t$. This serves as an approximation. For much of the analysis, I set management fees to zero ($m = m_H = 0$) so this has no effect. Where management fees are non-zero, a natural choice is to set m_H to be a fraction of m in order to get a similar average level of payout.[8] It is possible to relax the assumption of constant

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[8]A 'conservative' choice is $m_H = Cm$ since then $m_H H < mW_t$.

m_H, and get closer to a constant proportion of W_t, by having m_H decrease piecewise with the value of W_t. However, the given assumption provides a high degree of tractability and will allow much clarity regarding how the management fee impacts the solution and manager behavior.

Assuming that $V_X \geq 0$ and $V_{XX} < 0$,[9] we can use the first-order condition to find the optimizing value of π in (3),

$$\pi^* = \frac{-(\mu - r)}{\sigma^2} \frac{V_X}{X V_{XX}} \tag{4}$$

Substituting π^*, the expression for $\underline{V}_t$, and $m_t W_t = m_H H_t$ into (3) and simplifying the resulting expression gives

$$0 = -(\rho + \lambda)V + m_H H_t + \lambda g_0 \beta_1 H_t - \omega \frac{V_X^2}{V_{XX}} + V_H H(r - \phi_t - m) \tag{5}$$

$$+ k dH^\varepsilon - V_X (1+k) \frac{dH^\varepsilon}{H} + V_H dH^\varepsilon$$

where ω is defined as

$$\omega = \frac{1}{2} SR^2 = \frac{1}{2} \frac{(\mu - r_f)^2}{\sigma^2}$$

Conjecture that

$$V(X_t, H_t) = \beta_1 H_t G(X_t) \quad . \tag{6}$$

The homogeneity of V in H_t is a consequence of the scale invariance of the problem. The coefficient β_1, which was introduced earlier, is a normalization constant whose value is chosen in order that $G(1) = 1$.

The first line of (5) holds throughout, while the second line applies only at $X = 1$ and thus serves as the boundary condition on V. The solution for V must therefore satisfy the two parts separately. Consider first the second line. Substituting in the conjecture shows that the following condition must hold at $X = 1$ for the three terms in the second line to sum to zero

$$k - \beta_1 G_X(1)(1+k) + \beta_1 G(1) = 0 \quad .$$

Solving for β_1 and applying the normalization $G(1) = 1$ gives

$$\beta_1 = \frac{k}{G_X(1)(1+k) - 1} \quad . \tag{7}$$

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[9]This assumption is verified by the solution, as shown below.

The solution for the ordinary differential equation (ODE) represented by the top line of (5) is given by

$$G(X_t) = \left(\frac{X_t - D_0}{D_1}\right)^{\eta} + D_2 \tag{8}$$

Substituting this back into the ODE gives four equations that must be jointly satisfied by η, D_0, D_1 and D_2. The solution for D_2 is

$$D_2 = \frac{m_H \beta_1^{-1} + \lambda g_0}{\rho + \lambda - r + \phi + m}$$

I assume that the denominator in D_2, which as I explain below has a natural interpretation as a present-value factor for a perpetuity, is positive, i.e., $\rho + \lambda - r + \phi + m > 0$. The solution for η is given by

$$\eta = \frac{\rho + \lambda - r + \phi + m}{\omega + \rho + \lambda - r + \phi + m} \tag{9}$$

It is important to note that since $\mu - r > 0$ (i.e., positive 'alpha') then $\omega > 0$ and hence $0 < \eta < 1$. This means that $V_{XX} < 0$, and hence V is concave, consistent with the assumption made in deriving π^* in (4). D_0 and D_1 depend on the boundary conditions for $G(X_t)$, which follow directly from the boundary conditions for $V(X_t, H_t)$. They are: $G(C) = g_0$ and $G(1) = 1$. Solving for D_0 and D_1 gives

$$D_0 = C - \frac{1 - C}{(1 - D_2)^{1/\eta} - (g_0 - D_2)^{1/\eta}}(g_0 - D_2)^{1/\eta}$$

$$D_1 = \frac{(1 - C)}{(1 - D_2)^{1/\eta} - (g_0 - D_2)^{1/\eta}}$$

3.1.1 The 'Inside Payoff'

Consider the quantity represented by D_2. The term $\beta_1 H_t D_2$ is the value of an indefinite-horizon payout stream that is discounted at the rate $\rho + \lambda - r + \phi + m$. The rate of payout is H_t times $m_H + \lambda g_0 \beta_1$, which captures the management fee $(m_H H_t)$ and the expected outside payoff from stochastic termination $(\lambda g_0 \beta_1 H_t)$. The 'discount rate' captures four factors that discount or reduce the value of these future payments: the manager's time-discount factor ρ, the stochastic termination intensity (λ), the rate of outflows (ϕ), and the management fee itself (m), plus one factor, the contractual growth rate of the HWM (r), which increases future payments.

Note that the discounted expected value of the payout stream captured by D_2 does not include the expected value of performance fees. It also does not include the expected value of the outside payoff realized at the termination point C. It therefore captures the expected value of payouts, excluding performance fees, earned by the manager while he is managing 'inside' the fund. I therefore refer to the value of these payouts as the 'inside ex-performance payout', or more concisely, as the 'inside payout'. Note that the inside payout does not depend on X_t and therefore represents a lower-bound on the expected discounted payout (i.e. value function) of the manager. Therefore, $V(X_t, H_t) \geq \beta_1 H_t D_2$. This then implies that the manager's value function at the termination point must be greater than or equal to the value of the inside payout: $V(C, H_t) \geq \beta_1 H_t D_2$. This shows that we need a different boundary condition in the case that the outside option is less than the inside option, i.e., in case $g_0 < D_2$. In this case, we replace the condition $G(C) = g_0$ with

$$\pi_t^*(C) = 0 \quad .$$

That is, when the inside option is greater than the outside option, the manager avoids termination by optimally reducing risk taking to zero as X_t approaches the termination point C. As will be clear below from the expression for π^*, this condition implies that

$$D_0 = C$$
$$D_1 = \frac{(1 - C)}{(1 - D_2)^{1/\eta}}$$

It follows immediately that the condition $\pi_t^*(C) = 0$ is equivalent to $G(C) = D_2$.[10] In other words, the manager's value function at the termination boundary is in general equal to the maximum of the outside and inside payoffs.

3.1.2 Parameter Restrictions

An important restriction is that the manager's value function at the HWM should be greater than his outside payoff. Otherwise, the manager's problem is vacuous–he should immediately leave the fund to obtain the outside payoff. This consideration gives the restriction $g_0 < 1$.

An additional restriction on the solution, which also appears in Panageas and Westerfield (2009), is that $\beta_1 > 0$. Intuitively, β_1 is a multiplier of H_t that captures the accumulated

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[10]Note that the expressions for D_0 and D_1 in this case can be obtained by substituting in $g_0 = D_2$ in their counterparts above.

value of future increases of H_t on the manager's value function. The requirement $\beta_1 > 0$ ensures that the discounted sum of expected future payments, given by the manager's value function, is finite. From (7), this condition is equivalent to $G_X(1)(1 + k) > 1$.

When these parameter restrictions hold, a standard martingale verification argument confirms that value function V_t is indeed given by (6) and (8) and the optimal value of π_t by (4) (see e.g., Browne (1997), Oksendal (2003), and Panageas and Westerfield (2009)).

3.2 Optimal Risk-Choice

Substituting the solution for the manager's value function (equations (6) and (8)) into (4) and simplifying, we obtain the following expression for the manager's optimal risk-choice, i.e., his investment in the risky asset,

$$\pi_t^* = \frac{1}{(1 - \eta)(\frac{X_t}{X_t - D_0})} \frac{\mu - r}{\sigma^2} \tag{10}$$

As the solution shows, unless $D_0 = 0$, the risk-choice of the manager depends dynamically on X_t. In fact, the proportion invested in the risky asset by the manager is the same as would be chosen by a risk-averse investor whose relative risk aversion equals $(1 - \eta)\frac{X_t}{X_t - D_0}$. This expression reveals that there are three cases to consider for the relationship between X_t and the manager's effective risk-aversion/risk-choice : (i) $D_0 > 0$, (ii) $D_0 < 0$, and (iii) $D_0 = 0$.

Consider first the case where $D_0 > 0$. Note what happens to the effective 'risk-aversion' of the manager as X_t decreases from the maximum value of 1 (when the fund is at the HWM). In this case, the effective 'risk-aversion' of the manager *increases* and he becomes more cautious. He therefore chooses to *de-lever* the fund (reduce π_t^*). Figure 1 illustrates this relationship between the distance of the fund from the HWM (captured by X_t), the manager's effective risk-aversion, and the manager's risk-choice. Note that the impact of losses on leverage and risk-aversion can be strongly non-linear. Indeed, a simple calculation shows that the elasticity of the manager's effective risk-aversion with respect to X_t is given by $\frac{-D_0}{X_t(X_t - D_0)}$. Hence, when $D_0 > 0$, risk-aversion increases and leverage decreases more quickly as losses push the manager further below the HWM. Furthermore, the maximum effective risk-aversion in this case, occurring at $X_t = C$, is unbounded (corresponding to the case $D_0 = C$). Thus, the range of possible effective risk-aversions in this case includes the whole positive interval. It is also quite interesting to note that the dynamics of 'risk-aversion'

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in this case strongly resemble an external habits process. In the external habit process of Campbell and Cochrane (1999), the local coefficient of risk-aversion is (in their notation) $\gamma \frac{C_t}{C_t - H_t}$. Hence, variation in C_t acts like variation in X_t in driving the changes in risk-aversion. However, unlike the habits model, here the effective risk-aversion and its variation arise endogenously from the risk-neutral manager's problem rather than being assumed into the manager's preferences. Another important difference is that returns themselves rather than consumption shocks are the source of variation driving risk-aversion variation. Still, the two processes have very similar nonlinear dynamics that are both 'counter-cyclical' in the sense that negative shocks raise risk-aversion.

When $D_0 < 0$, the risk-choice of the manager actually *increases* as he falls further below the HWM. This is illustrated in Figure 2, which plots a comparison of π_t^* for the three cases of D_0. Note how the dash-dot line, which illustrates the case of $D_0 < 0$, rises as X_t falls, while the opposite holds for the case of $D_0 > 0$, illustrated by the dashed line. What explains this difference and the dependence on D_0. As discussed above, and detailed in the next section, D_0 captures the joint impact on the manager's risk-choice of factors such as the outside payoff of the manager g_0, and the termination point, C. For example, a *higher* outside payoff corresponds to a lower value of D_0. Hence, an important factor contributing to $D_0 < 0$ is a sufficiently 'high' outside payoff. Now consider how this interacts with the manager's distance from the HWM. As the manager drops further below the HWM, two considerations influencing the manager's risk-choice are: (1) the expected time until he is paid any further performance fees increases and (2) the likelihood increases that he will be terminated. For a manager with a sufficiently high outside payoff, as captured by $D_0 < 0$, risk-choice will depend more strongly on the first consideration. Such a manager will increase risk-choice because he puts more weight on shortening the time until payment than on the increased likelihood of termination that will result from this policy. On the other hand, a manager with outside payoff sufficiently low, so $D_0 > 0$, is more concerned about the second consideration. Hence, he decreases risk taking to reduce the probability of fund wealth falling to the point of termination. Finally, for the case of $D_0 = 0$, the manager has a constant effective risk-aversion and risk-choice that does not vary with X_t. This is shown as the solid line in Figure 2. This case includes the result in PW as a special case, as discussed below.

3.2.1 Comparative Statics

I assume throughout this section that $m = m_H = 0$, which allows simple analytical character-izations of all of the comparative statics. In a subsequent section I analyze $m > 0$, $m_H > 0$. The forces at work in the solution are the same with $m > 0$, $m_H > 0$ but in that case finding the value of D_2 requires solving a non-linear equation (see the Appendix) that does not have an explicit solution and does not lend itself to such simple analytical expressions.[11]

Lemma 1. *Let $m = m_H = 0$. The following obtain:*

1. $\dfrac{d\pi_t^*}{dC} < 0$

2. $\dfrac{d\pi_t^*}{dg_0} > 0$

3. $\lim\limits_{\omega \to 0} \pi_t^* \to \infty \quad and \quad \lim\limits_{\omega \to \infty} \pi_t^* \to \infty$

   Moreover, $\dfrac{d\pi_t^*}{d\omega} < 0 \;\; if \;\; \omega < \rho + \lambda + \phi - r$

4. $\dfrac{d\pi_t^*}{d\phi} > 0$

As equation (10) shows, π_t^* depends on two endogenously-determined constants, D_0 and η. D_0 captures the impact on the manager's optimal risk-choice of his termination point and the excess of his outside payoff over his inside payoff. A higher value for C increases D_0 and from (10) it is clear that this reduces π_t^* at all values of X_t, including when the fund is at the HWM. Hence, a 'stricter' termination policy induces the manager to be globally more risk-averse. A *decrease* in the value of the outside payoff g_0 also increases D_0, and therefore increases the manager's effective 'risk-aversion' and reduces his risk-choice. It is interesting to note that, for a fixed value of C, the maximum value of D_0 is C, corresponding to the case $g_0 = D_2$. That is, manager risk-aversion is maximized when the outside payoff has no excess value over the inside payout. In this case, it is clear from (10) that as $X_t \to C$, risk-aversion goes to infinite and $\pi_t^* \to 0$. It is also interesting to take a look at the shape of the manager's expected discounted payoff (i.e. value function) as a function of X_t and of g_0. This is illustrated in Figure 3 , which plots $V(X_t)$ corresponding to three cases of g_0. The solid line corresponds to the largest value of g_0, while the dash-dot line, which has the most curvature, corresponds to $g_0 = D_2 = 0$.

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[11]This arises because the value of D_2 depends on β_1 which itself depends on D_2 via D_0 and D_1.

The impact of ω, the (squared) Sharpe Ratio of the risky asset (and hence also of the manager's portfolio), is non-monotonic. Varying ω changes two quantities in (10): the quantity $(\mu - r)/\sigma^2$, *and* the endogenously determined constant η. The interaction of these two changes produces the non-monotonicity. As the Appendix shows, π_t^* can be rewritten as

$$\pi_t^* = \frac{\sqrt{2}}{\sigma} \left(\sqrt{\omega} + \frac{\rho + \lambda - r + \phi}{\sqrt{\omega}} \right) \frac{X_t - D_0}{X_t}$$

which shows that π_t^* gets large for *both* large and small ω. The result that π_t^* gets large for large omega is consistent with the intuition that an improved investment opportunity increases investment in the risky-asset. The surprising result that π_t^* also gets larger for smaller ω was also shown in *PW* and continues to hold true here. As they point out, effective 'risk aversion' is endogenously determined and depends on the importance of the manager's continuation value. When ω is sufficiently small, the decrease in effective 'risk-aversion' dominates the decreased attractiveness of the investment opportunity, so that π_t^* actually decreases.

By decreasing the rate of fund growth and the future size of the fund, an increase in the withdrawal rate ϕ diminishes the importance of the future or continuation value in the manager's problem. Since this continuation value acts to attenuate the manager's incentive for risk taking, its decrease implies greater risk taking by the manager. Figure 4 illustrates a comparison of $\pi^*(X_t)$ for three values of ϕ, with the solid line representing the lowest rate of outflow and the dash-dot line the highest.

Finally, I examine the impact of the incentive fee, k, on the value function of the manager, maintaining the assumption $m = m_H = 0$. In PW, increasing k for the risk-neutral manager unambiguously *decreases* his value function because the resulting decrease in the growth rate of fund assets outweighs the benefit of the immediate payout to manager. Interestingly, it turns out that this result no longer holds here in general. Differentiating V with respect to k gives

$$V_k(X_t, H_t) = \frac{G_X(1) - 1}{[G_X(1)(1 + k) - 1]^2} H_t G(X_t)$$

Hence, the sign of the derivative depends on $G_X(1) \gtrless 1$. $G_X(1)$ acts like a discount factor with a value that is closely tied to the level of the manager's effective 'risk-aversion'. Hence, when the manager's overall level of 'risk-aversion' is high, increases in k increase his value function. In this case, the value of immediate payout is now greater than the loss in fund growth rate, and the manager is better off. For example, this happens when C is high or g_0

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is low. On the other hand, when 'risk-aversion' is low, the opposite is true, as in the special case in PW.

3.3 Special Cases

In order to gain more intuition about the general solution, I look at some special cases. Consider first the model in PW. Since PW do not explicitly consider an outside payoff in their paper, they are implicitly setting $g_0 = 0$. In addition, there is no termination, so implicitly, $C = 0$. Finally, $D_2 = 0$. Substituting these values in gives $D_0 = 0$, $D_1 = 1$, and

$$G(X_t) = X_t^\eta \qquad \pi_t^* = \frac{1}{1-\eta}\frac{\mu - r}{\sigma^2} \qquad \beta_1 = \frac{k}{\eta(1+k)-1}$$

which are the same as in PW. In this case, 'risk-aversion' is just $1 - \eta$, which has a value less than one, and is independent of X_t. Hence, the manager behaves exactly as would a CRRA investor. Note that we can actually attain the same value function and optimal risk-choice for the manager when $C > 0$, by simultaneously increasing the outside payoff to $g_0 = C^\eta$. As Lemma 1 shows, decreasing (increasing) the outside payoff from this value decreases (increases) the manager's risk-choice. When g_0 is decreased to 0, then $D_0 = C$, $D_1 = 1 - C$, and

$$G(X_t) = \left(\frac{X_t - C}{1-C}\right)^\eta \qquad \pi_t^* = \frac{1}{(1-\eta)(\frac{X_t}{X_t-C})}\frac{\mu - r_f}{\sigma^2} \qquad \beta_1 = \frac{k}{\eta(1-C)^{-1}(1+k)-1}$$

Notice that in this case effective risk-aversion increases without bound and $\pi_t^* \to 0$ as $X_t \to C$. The manager is least 'risk-averse' at the HWM, where his effective risk-aversion is $(1-\eta)\frac{1}{1-C}$. This adjustment by $(1-C)^{-1}$, which reflects the rise in 'risk-aversion' relative to PW, also appears in the coefficient β_1. Indeed, for the same value of k and all other parameters, it increases the discounting inherent in β_1 and therefore helps to ensure that the restriction $\beta_1 > 0$ holds. We can further see from this term how the result in PW that $V_k < 0$ can cease to hold. In particular, $G_X(1) = \eta(1-C)^{-1}$ and hence, $G_X(1) > 1$ when $C > 1 - \eta$. Thus, if termination is sufficiently strict then $V_k \geq 0$. If not, then $V_k < 0$, which is the case for PW, where it is implicit that $C = 0$.

3.4 Management Fee

An increase in the management fee has two conflicting effects on the manager's problem. The first, call it the "withdrawal effect", is that, like a rate of withdrawal, the management fee reduces the growth rate of the fund. This effect is captured in the expression for η and in the denominator of D_2, where the impact of m is the same as the impact of ϕ. Like an increase in ϕ, an increase in m discounts more strongly the continuation value of the fund in the manager's problem and therefore leads to an increase in risk-choice. Taken in isolation, this effect also reduces the value function of the manager. Of course, unlike a withdrawal, the management fee is paid to the manager. Hence, the second effect, call it the "payout effect", is that an increase in m_H increases the stream of payments to the manager. This effect is captured by m_H in the numerator for D_2, which increases the value function of the manager. Since an increase in D_2 reduces the manager's risk-choice, this payout effect pulls the manager's risk choice in the opposite direction from the withdrawal effect.

We can investigate this analytically by considering a dm increase in both m and m_H at the point $m = m_H = 0$. First, it is clear by inspection that $\frac{d\eta}{dm} > 0$, which implies greater risk taking by the manager. Second, differentiating D_2 gives

$$\frac{d\,D_2}{d\,m}\bigg|_{m\,=\,m_H\,=\,0} = \frac{\beta_1^{-1}(\rho + \lambda - r + \phi) - \lambda g_0}{(\rho + \lambda - r + \phi)^2} \quad .$$

This shows that the sign of $\frac{d\,D_2}{dm}$ at $m = m_H = 0$ depends on the sign of the numerator, which determines whether the payout or withdrawal effect dominates on D_2. If the numerator is negative, then the withdrawal effect dominates and D_2 is reduced, which reinforces the increase in risk-choice coming from η. In this case the impact of an increase in management fee is unambiguously to increase risk-choice. On the other hand, if the numerator is positive, then the payout effect dominates, D_2 increases, and this acts to reduce the manager's risk taking. There are then two opposite forces on risk-choice.

The net effect of these two forces depends on the specific parameter values and may further depend on the value of X_t. That is, an increase in m and m_H may increase risk-choice for a region of values of X_t but decrease risk-choice over another region. Figure 5 plots an example of this. The top panel shows the manager's expected discounted payoff (value function) and the bottom panel his risk-choice, for $m = m_H = 0$, $m = m_H = 0.02$ and $m = m_H = 0.04$. The top panel shows that for this parametrization, the increase in management fee increases the manager's expected payoff for all X_t. The impact on risk-

choice is non-monotonic, as the bottom panel shows. For high values of X_t, the higher management fee is associated with a higher risk-choice. In this region the withdrawal effect dominates. However, as X_t approaches the termination point, the risk-choices converge and then reverse ordering. For X_t close to the termination point, the payout effect dominates and risk-choice is lower for the higher management fee.

The impact on risk-choice of increasing C, g_0, or ϕ when $m > 0$, $m_H > 0$ are qualitatively similar to when $m = m_H = 0$. If β_1 were held constant, then analyzing the comparative statics would remain analytically simple. However, β_1 does vary with these parameters if $m_H > 0$, which complicates the analytical expressions. For C and g_0, the change in β_1 reinforces the forces which drive the comparative static under $m = m_H = 0$. For example, as shown for $m = m_H = 0$, an increase in C or decrease in g_0 reduces risk-choice and decreases the value function of the manager. This implies a decrease in β_1, which further increases D_2 when $m_H > 0$. This increase in D_2 reflects the increased relative importance in these cases of the inside payout. Since this makes the manager more 'risk-averse', it reinforces the decrease in risk-choice. Figure 6 plots a comparative static for C with $m_H = 2\%$. The plot shows how risk-choice decreases globally as the value of C is increased. For ϕ, the decrease in β_1 actually counteracts the increase in risk-choice induced by an increase in ϕ, so the net effect is more complicated. Figure 7 plots a comparative static for ϕ with $m = m_H = 2\%$. For the given parameterization, the plot shows that an increase in ϕ increases risk taking. However, the difference in risk taking across ϕ values narrows and becomes negligible as X_t decreases towards the termination point.

4 Optimal Walk-Away and Position Limits

I now consider, in addition to termination, the potential that the manager voluntarily leaves the fund ('walk-away') to receive his outside payoff. Naturally, this walk-away decision depends on the value of X_t. Let C_w be the value of X_t at which the manager walks away from the fund. Suppose that $C_w \leq C$. Then walk-away has no impact on the manager's problem because termination will always come into effect before the manager walks away. The manager's problem is then unaffected. On the other hand, if $C_w > C$, then the manager will always walk away before he reaches the deterministic termination point. Note that in this case, his problem simply becomes equivalent to one where $C = C_w$. The form of the solution remains the same, except C is replaced with C_w. However, since C_w is a choice of the manager, we can consider the manager's *optimal* choice of C_w. This is given by the

following proposition.

Proposition 1. *The manager will always choose C_w to be less than or equal to C. In other words, it is **never** optimal for the manager to walk away.*

To prove this, I solve for the optimal walk-away point by requiring that the *smooth-pasting* condition must hold at C_w^*,

$$V_X(X_t, H_t)\big|_{X_t=C_w} = \beta_1 H_t \eta \frac{1}{D_1} \left(\frac{C_w - D_0}{D_1}\right)^{\eta-1} = 0 \tag{11}$$

This condition must hold if C_w^* is an internal optimum (see e.g, Dixit and Pindyck (1994)). However, it is straightforward to see that this condition *cannot* hold, because $D_1 > 0$ and $0 < \eta$ imply that $V_{X|X=C_w} > 0$. Another way to see this is to note that since $V_{XX} < 0$, $V_{X|X=C_w} = 0$ implies $V_{X|X>C_w} < 0$, contradicting $V_X \geq 0$. Furthermore, $V_{X|X=C_w} > 0$ implies that V is *decreasing* in C_w.[12] Hence, the manager is always better off pushing his walk-away point towads zero. In other words, it is *never* optimal for him to walk-away.

The Appendix provides another proof of Proposition 1. It shows that $V_C(X_t, H_t) < 0$ for all X_t. Therefore, reducing C increases V everywhere. It is therefore optimal to set C_w to the lowest possible value, i.e., walk-away is never optimal for the manager. Figure 8 provides a useful graphical view of this result by plotting $G(X_t)$ (which is a bit more illustrative than plotting $V(X_t)$ due to the inherent normalization) for different values of C_w. As the top panel in the Figure shows, decreasing C_w raises G everywhere, making it more elongated and decreasing its curvature.

When a fund incurs large losses and is pulled far away from the HWM, it may take a lot longer for it to reach the HWM again and for the manager to once again be paid performance fees. It may seem quite counterintuitive then that it is never optimal for the manager to walk away from the fund. Why is this the case? To understand this, it will be useful to consider the manager's optimal risk-choice as a function of C_w (when $C_w > C$ and there is walk-away). By Lemma 1, π_t^* increases globally as C_w is decreased. This is plotted in the bottom panel of Figure 8. With a decrease in the walk-away point, the manager becomes in effect less 'risk-averse' and takes on more risk for all values of X_t. Hence, rather than walk away when $X_t = C_w^1$, a manager can do better for himself by walking away only when $X_t = C_w^2 < C_w^1$ *and* increasing his risk-choice everywhere. The optimal risk-choice increases

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[12]This follows from the argument in Dixit and Pindyck (1994) on the necessity of the smooth-pasting condition.

the rate at which he will reach the HWM from any value of X_t, so that he can extract a higher payoff and is better off than choosing to walk away at C_w^1.

Another useful, if more technical, way to understand this is as follows. Consider the value function of the manager given by the dash-dot curve in Figure 8 at the walk-away point, given by the intersection of the curve with the x-axis. It might appear that, since the value function is downward sloping ($V_X < 0$), it would be better for the manager to just walk away at this point, maintaining a flat value function for lower values of X_t, then to continue running the fund. This corresponds to the common intuition alluded to above that the manager may walk away from the fund when fund wealth has dropped sufficiently. This would be true *if* the manager were required to hold to the same risk-choice policy. However, this is not the case, as can be seen from the bottom panel of Figure 8. When the walk-away point is moved down, the manager responds to this by changing his policy function to increase risk-choice everywhere, which improves his expected discounted payoff everywhere.

To further understand this result and the value of the option to walk away, I consider a manager who faces a constraint on the weight invested in the risky asset. I then solve his problem, including the optimal walk-away decision. The solution shows that when the constraint is binding, walk-away can be optimal for the manager in some scenarios.

4.1   With Binding Position Limits

To constrain the risk taking of the manager, I add the following position-limit to the manager's problem

$$\pi_t \leq \overline{\pi} \quad . \tag{12}$$

Such a limit could naturally occur for a number of reasons. For example, it may be exogenously imposed by investors who want to constrain the risk taking of the manager. Alternatively, it may arise simply from margin constraints. Finally, the manager may self-impose this limit because he believes that crossing some risk threshold raises a red flag for current or future investors, perhaps because it signals he is 'gambling'.

Consider first the case where the *un*constrained manager has $D_0 \geq 0$. Since $\pi^*(X_t)$ is increasing in X_t in this case, a reasonable conjecture in the presence of the constraint is that the solution will have $\pi^*(X_t)$ increasing and unconstrained on a region $[C, \mathbb{X}]$, and then hitting the constraint on $(\mathbb{X}, 1]$ (this includes the potential that the constraint never actually

binds, i.e., $\mathbb{X} = 1$). The solution below verifies this conjecture. From the proof of Proposition 1, it follows that it will never be optimal for the manager to walk away if C_w is in an interval of X where the manager is unconstrained. Thus, in this case there can only be optimal walk-away if the manager is constrained on the whole interval $[C_w, 1]$. I first solve this case and then consider the other possible outcomes arising from the general problem.

Proposition 2. *Walk-away can be optimal for the manager at C_w, when the position-limit (12) binds on all of $[C_w, 1]$ and C_w solves the equation*

$$\frac{\gamma_2 C_w^{-\gamma_1} - \gamma_1 C_w^{-\gamma_2}}{\gamma_2 - \gamma_1} = \frac{1}{g_0} \tag{13}$$

where γ_1, γ_2 are the solutions to the following quadratic equation

$$\frac{1}{2}\gamma_i^2 \sigma^2 \bar{\pi}^2 + \gamma_i \left(-\frac{1}{2}\sigma^2 \bar{\pi}^2 + \bar{\pi}(\mu - r) \right) - (\rho - r + \phi) = 0$$

For simplicity and to reduce notational clutter, the proposition is stated with $m = m_H = 0$ and $\lambda = 0$. However, with adjustment to the equations, Proposition 2 would also hold more generally if these were non-zero.

Assuming (12) binds on all of $[C_w, 1]$, we can proceed from the HJB equation (3) by substituting in $\bar{\pi}$ for π as the maximizing risky-asset weight. Once we solve for V, we can go back to check that the assumption of a binding constraint holds. The conjecture for the solution to the HJB equation remains the same as (6), as does (7). However, because the solution $\pi_t^* = \bar{\pi}$ is not an internal optimum, the equation satisfied by $G(X)$ is given by

$$0 = -(\rho - r + \phi)G + G_x X \bar{\pi}(\mu - r) + \frac{1}{2}G_{XX}\sigma^2 \bar{\pi}^2 X^2 \quad .$$

The solution to this equation is

$$G(X_t) = A_1 X^{\gamma_1} + A_2 X^{\gamma_2} \tag{14}$$

where γ_1, γ_2 solve the quadratic equation given in Proposition 2. I let γ_1 be the negative root and γ_2 the positive root. The boundary conditions, $G(1) = 1$ and $G(C_w) = g_0$, imply the following solutions for the A_i:

$$A_1 = \frac{C_w^{\gamma_2} - g_0}{C_w^{\gamma_2} - C_w^{\gamma_1}} \qquad\qquad A_2 = 1 - A_1$$

It remains to determine C_w such that walk-away is indeed optimal. An optimal C_w must satisfy the *smooth-pasting* condition

$$G_X(C_w) = 0 \tag{15}$$

Straightforward but tedious substitution of G and the A coefficients into this condition gives equation (13) that determines the unique optimal C_w. If also $C_w \geq C$ (since termination is still permitted), then indeed walk-away is optimal. If it is instead the case that $C_w < C$, then the manager would be terminated prior to his walk-away and this does not represent the solution. We are then in one of the different cases of the general solution, which are solved below.

The following lemma follows from the result of Proposition 2.

Lemma 2. *The optimal walk-away point is **increasing** in the outside payoff,*

$$\frac{d C_w}{d g_0} > 0.$$

This is proved by showing that the left-hand term in (13) is decreasing in C_w. This follows by taking the derivative of this left-hand term in C_w and showing that it is negative using the fact that $\gamma_1 < 0$, $\gamma_2 > 0$ and $C_w < 1$. Since the right-hand term is decreasing in g_0, it follows that $\frac{d C_w}{d g_0} > 0$.

The intuition behind this result is that a greater outside payoff makes it less worthwhile for the manager to continue managing the fund at low X_t since his ability to generate value is limited by the constraint on risk. As the constraint is loosened, the manager can generate more value by increasing risk so the benefit of leaving decreases and the optimal walk-away point decreases.

We can now also verify that (12) does indeed bind on $[C_w, 1]$. We can check this by verifying that the Lagrange multiplier on the constraint is positive in the maximization of the manager's objective. Denote this Lagrange multiplier by ψ. It is given by

$$\psi = V_X X(\mu - r) + \bar{\pi} X^2 \sigma^2 V_{XX}$$

The top panel of Figure 9 plots $V(X_t)$ for examples of an unconstrained, constrained, and occasionally constrained (derived below) manager. It is also helpful to compare the corresponding $G(X_t)$, and these are plotted in the bottom panel. Note that the *constrained*

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manager's value function need *not* be concave, in contrast to the *unconstrained* manager's value function. Moreover, when there is optimal walk-away (i.e., the optimal C_w is greater than C), then the manager's value function *must* be convex at the walk-away point. To see this, note that $\psi > 0$ and $V_X(C_w) = 0$ implies that $V_{XX}(C_w) > 0$.

If it is instead the case that $C_w < C$, then walk-away is *not* optimal. There are then two other cases to consider. The first is that the constraint remains globally binding even though walk-away is not optimal. The solution to this takes the same form as (14), but now the boundary condition is $G(C) = g_0$ and (15) is no longer imposed. Using the resulting solution for G, we must then check that the constraint does indeed bind everywhere, consistent with our assumption. If this is the case, then this is the solution for the value function. Otherwise, the constraint only binds occasionally or the manager is unconstrained. The case of an occasionally binding constraint is solved below.

Now consider the case where the unconstrained manager has $D_0 < 0$. Since the unconstrained $\pi(X_t)^*$ is decreasing in X_t, in the presence of the constraint the solution to the problem will have $\pi^*(X_t)$ *constrained* on a region $[C, \mathbb{X}]$ and unconstrained on $(\mathbb{X}, 1]$. If $\mathbb{X} = 1$, then the manager is constrained on the whole interval $[C, 1]$ ($[C_w, 1]$ if there is optimal walk-away) and this is the case discussed above. On the other hand, if $\mathbb{X} < 1$, then this is the case of an occasionally binding constraint, which is solved below.

Finally, for completeness, note that the expression for β_1 remains (7). It is clearly the case that the constrained $V(X_t, H_t)$ will be less than its unconstrained counterpart and that when the unconstrained V is finite, the constrained V will be as well. Since β_1 is just the value function evaluated at $X_t = 1$ and $H_t = 1$, this further implies that the constrained β_1 value will be less than the unconstrained β_1.

4.2   Occasionally Binding Limits

If the constraint does *not* bind everywhere, then there are two cases, corresponding to whether $D_0 \geq 0$ or $D_0 \leq 0$ for the unconstrained manager. If $D_0 \geq 0$, then the constraint binds on $(\mathbb{X}, 1]$ where $C < \mathbb{X} < 1$, or the manager is unconstrained. Assume that $C < \mathbb{X} < 1$, so that the manager is constrained some of the time. Since the manager is unconstrained on the lower region, walk-away is *not* optimal, as shown above. The form of V remains the same as in (6) but now the solution for G is split into two regions. Let $\underline{G}(X_t)$ be the solution on the region $[C, \mathbb{X}]$ and $\overline{G}(X_t)$ be the solution on $(\mathbb{X}, 1]$. Then $\underline{G}(X_t)$ has the form (8) since the HJB equation on this region corresponds to an unconstrained manager. Correspondingly,

$\overline{G}(X_t)$ takes the form (14) as the HJB equation on this region corresponds to a constrained manager. In addition to $\underline{G}(C) = g_0$ and $\overline{G}(1) = 1$, there are now three other boundary conditions,

1. $\underline{G}(\mathbb{X}) = \overline{G}(\mathbb{X})$

2. $\underline{G}_X(\mathbb{X}) = \overline{G}_X(\mathbb{X})$

3. $0 = V_X \mathbb{X}(\mu - r) + \overline{\pi} \mathbb{X}^2 \sigma^2 V_{XX}$

The first two conditions match the value and first derivative of the two parts of G across the change of regions. The third condition says that at the point $\mathbb{X}$, where the constraint begins to bind, π_t^* equals $\overline{\pi}$. This means the risk-choice of the manager is continuous. The five boundary conditions jointly pin down the values of the five constants, D_0, D_1, A_1, A_2, and $\mathbb{X}$. It then remains to check that (i) $C < \mathbb{X} < 1$ and (ii) the constraint (12) does indeed bind on $(\mathbb{X}, 1]$.

Figure 9 plots $V(X_t)$ (top panel) and $G(X_t)$ (bottom panel) for a comparison of an occasionally constrained manager alongside the corresponding completely constrained (with and without walk-away) and unconstrained counterparts. To produce the plot, $\overline{\pi}$ is increased starting from a low value, where the manager is globally constrained and there is walk-away. As $\overline{\pi}$ is increased, it becomes no longer optimal to walk-away. As $\overline{\pi}$ is increased further, the manager becomes only occasionally constrained and $\mathbb{X}$ increases from a low value. When $\overline{\pi}$ has increased sufficiently that $\mathbb{X}$ increases to 1, the manager becomes unconstrained. The top plot shows how $V(X_t)$ increases as the risk-limit $\overline{\pi}$ is raised, while the bottom plot shows how $G(X_t)$ becomes increasingly concave. It is clear in the bottom plot how the slope of G is zero at the walk-away point in the unconstrained case with walk-away (solid line), while it is positive in the other cases. Figure 10 plots $\pi^*(X_t)$ corresponding to the cases in Figure 9. It is interesting to note that for low X_t, the risk-choice of the globally constrained manager (dashed line) is actually greater than than for the unconstrained and occasionally-constrained manager.

It remains to take care of the case corresponding to $D_0 \leq 0$. In this case the constraint binds on $[C, \mathbb{X}]$, with $C < \mathbb{X} < 1$. Again, the form of V remains (6) and the solution for G is split into two regions. Let $\underline{G}(X_t)$ be the solution on the region $[C, \mathbb{X}]$ and $\overline{G}(X_t)$ be the solution on $(\mathbb{X}, 1]$. Now, it is $\underline{G}(X_t)$ that has the form (14), since the manager is constrained on the lower region, while $\overline{G}(X_t)$ takes the form (8) since the manager is unconstrained on the upper region. The boundary conditions include the same five as above: $\underline{G}(C) = g_0$,

$\overline{G}(1) = 1$, $\underline{G}(\mathbb{X}) = \overline{G}(\mathbb{X})$, $\underline{G}_X(\mathbb{X}) = \overline{G}_X(\mathbb{X})$, and $0 = V_X \mathbb{X}(\mu - r) + \overline{\pi} \mathbb{X}^2 \sigma^2 V_{XX}$. However, because the constraint binds on the lower region, it *is* now possible that there will be optimal walk-away, $C_w > C$. If this is the case, C is replaced with C_w above, and C_w satisfies the smooth-pasting condition, $\underline{G}_X(C_w) = 0$. This possibility for optimal walk-away corresponds to the common intuition that a fund manager who experiences negative performance may want to walk-away. In this case, the manager also appears to be 'gambling for resurrection' prior to walking away, since he increases risk taking in reaction to the negative performance. If he incurs continued negative returns and X_t declines further, then eventually the walk-away decision is taken.

5   Withdrawals and Aggressive Risk Choice

Under the baseline model, the rate of withdrawal from the fund is constant and does not respond to the fund's performance. However, it is quite plausible that investors' rate of withdrawal would increase when the fund is sufficiently far below the HWM. Such a plausible pattern of withdrawals could have an important effect on the manager's incentives and the dynamics of risk-choice. To that end, consider two rates of withdrawal, ϕ_1, ϕ_2, with $\phi_1 < \phi_2$, and assume that investor's rate of withdrawal is ϕ_1 when $X_t \in (\mathcal{X}, 1]$ and increases to ϕ_2 when $X_t \in [C, \mathcal{X}]$. For simplicity I restrict the problem to two rates of withdrawal, but adding further rates is straightforward.

On each region, the HJB equation and value function are the same as in (5) and (6). Let $G_1(X_t)$ and $G_2(X_t)$ denote the G function corresponding to ϕ_1 and ϕ_2 respectively, and define analagously the coefficients η_i, $D_{0,i}$ and $D_{1,i}$ for each G_i function, $i = 1, 2$. Two of the boundary conditions are $G_2(C) = g_0$ and $G_1(1) = 1$. The two additional boundary conditions are (i) $G_1(\mathcal{X}) = G_2(\mathcal{X})$ and (ii) $\frac{dG_1}{dX}(\mathcal{X}) = \frac{dG_2}{dX}(\mathcal{X})$, i.e., the values and first derivatives match across the change of regions.

Equation (9) for η shows that $\phi_1 < \phi_2$ implies that $\eta_1 < \eta_2$. The impact of η_i on effective 'risk-aversion' in π_t^* suggests that an increase in the outflow rate should in some way induce greater risk taking by the manager. However, we must also incorporate the solutions for $D_{0,i}$ and $D_{1,i}$ into the optimal risk-choice to see if such an effect indeed holds. It turns out that this is indeed the case, as formalized by the following Proposition,

Proposition 3. *Let the outflow rate be given by ϕ_1 for $X_t \in (\mathcal{X}, 1]$ and by ϕ_2 for $X_t \in [C, \mathcal{X}]$,*

with $\phi_1 < \phi_2$. Then

$$\frac{\lim_{X_t \downarrow \mathcal{X}} \pi_t^*}{\lim_{X_t \uparrow \mathcal{X}} \pi_t^*} = \frac{1 - \eta_2}{\eta_2} \frac{\eta_1}{1 - \eta_1} < 1$$

The Proposition shows that the increase in the rate of withdrawals induces a jump up in the manager's risk-choice. The extent of this increase depends on the relative magnitudes of η_1 and η_2. An increase in outflows increases the rate at which the size of the fund erodes and therefore diminishes the importance of the change in continuation value on the manager's problem. Since the continuation value serves to attenuate or 'discipline' the manager's risk taking, its decreased importance results in the increased risk taking.

To prove the proposition, use the boundary condition equating the first derivatives of G_1 and G_2 at $\mathcal{X}$ and substitute in the expression for G_i to get,

$$\eta_1 G_1(\mathcal{X}) \frac{1}{\mathcal{X} - D_{0,1}} = \eta_2 G_2(\mathcal{X}) \frac{1}{\mathcal{X} - D_{0,2}}$$

By the value-matching boundary condition we have $G_1(\mathcal{X}) = G_2(\mathcal{X})$, so that

$$\frac{\mathcal{X} - D_{0,1}}{\mathcal{X} - D_{0,2}} = \frac{\eta_1}{\eta_2}$$

Substituting this result into the expression for π_t^* gives the equation in the Proposition, while the inequality follows by the observation that $\phi_1 < \phi_2$ implies $\eta_1 < \eta_2$. The Appendix contains details on the full solution of the problem.

Figure 11 plots $\pi^*(X_t)$ (top panel) and $G(X_t)$ (bottom panel) for an example where the withdrawal rate increases when X_t falls (solid line) compared to the case where the withdrawal rate is constant at ϕ_1 (dashed-line). For this example, $D_{0,i} > 0$, so losses induce the manager to reduce risk taking. Hence, within reach ϕ_i region, risk-choice monotonically decreases as X_t decreases. However, as the top panel shows, for the case where the decrease in X_t triggers an increase in the rate of withdrawal, there is a jump up in the manager's risk taking at the boundary between the regions. The top panel also shows that the manager chooses a lower level of risk taking in the upper region relative to the constant withdrawal case, even though the withdrawal rate at that point is the same. That is, the potential of a future increase in the withdrawal rate makes him more 'risk-averse' even far away from the actual switch in withdrawal rates as he seeks to lower the chance of entering the lower region. The bottom panel shows that $G(X)$ is lower everywhere for the manager facing the increase in the withdrawal rate, including at the point (marked by the circle), where the switch in

withdrawals takes place. The same relation holds true for the value function $V(X_t)$.

This result on the implications of performance-based withdrawal rates provides another structural mechanism consistent with the idea that a fund manager with HWM-based incentives may choose to 'gamble for resurrection'. The mechanism here is different than the one above since here $D_0 > 0$ can be the case, and absent the loss-induced increase in the withdrawal rate, the manager would reduce risk taking (as shown by the dashed line in Figure 11). The result shows, however, that if losses induce an increase in withdrawals by investors, then the manager may respond by becoming more aggressive and *increasing* risk taking in an attempt to 'resurrect' the fund's fortunes and stem the rate of withdrawals from the fund.

6   Conclusion

Explicit high-water mark contracts are widespread in the money management industry, in particular for hedge fund managers. Their main feature is also implicit in other kinds of performance-based management compensation. This paper seeks to understand the optimal dynamic risk taking of a manager who is compensated under a high-water mark contract. By providing closed-form solutions to this problem, it extends greatly the set of known results.

This paper demonstrates that a manager operating under a higher-water mark contract may display two general types of risk-taking dynamics. The particular dynamic that arises depends jointly on several characteristics of the manager and his environment: his outside payoff, the termination policy, and his 'inside payoff'. In particular, the termination policy and outside payoff of the manager combine to powerfully impact the risk-taking dynamics of the manager. When the termination policy is strict or the manager's outside payoff is low, the risk-neutral manager's effective 'risk-aversion' is high and negative fund returns induce an increase in the manager's effective risk-aversion. This causes the manager to de-lever the fund at an increasingly rapid rate as the fund's value drops further below the high-water mark. On the other hand, if the termination policy is 'loose' and the manager's outside payoff is high, the manager's effective 'risk-aversion' will be low and the impact of negative returns on effective risk aversion and leverage will go in the opposite direction. The manager will then appear to engage in 'gambling' by increasing risk taking as the fund's value falls further below the high-water mark. At the boundary of these two cases, the impact of the termination policy and outside payoff offset and the manager's effective risk aversion remains

constant in the distance of the fund's value from the high-water.

This paper further examines the option of the fund manager to walk away from the fund. A common intuition is that a fund manager who is far below the high-water mark will opt to just walk away from the fund. This paper demonstrates that this is *never* optimal so long as the manager faces no exogenous risk limits. Regardless of his outside payoff, the manager will always prefer delaying walk-away and increasing his risk-taking globally over an earlier walk-away. *However*, when there are risk limits, this is no longer generally true. When risk limits bind following negative returns, then it may become optimal for the manager to exercise his walk away option.

Finally, this paper shows that if losses trigger an increase in the rate of investor withdrawals, the manager will *increase* risk taking at the point that this occurs. To guard against the risk of future withdrawals, the manager will also *reduce* risk taking when the fund is closer to the high-water mark. Hence, the impact of loss-triggered increases in investor withdrawals on risk taking is non-monotonic, as is the induced optimal risk taking dynamic. Since the manager in this situation may increase risk taking following losses, his behavior may also be described as 'gambling for resurrection'.

The results of this paper suggest that variation in financial intermediaries' effective risk aversion due to HWM-style incentives may be an important factor in these intermediaries' impact on asset prices. Hedge funds in particular are an important subgroup of financial intermediaries since they tend to be risk-tolerant and levered, and are often thought of as a very sophisticated segment of investors. The results presented here suggest that a negative return shock common to a broad swathe of hedge funds could cause an across-the-board reduction in their risk tolerance and induce de-leveraging. Moreover, the nonlinear nature of the de-leveraging implies that it should be particularly noticeable and significant when the shock is large. An interesting direction for future research is to analyze the impact that intermediaries facing high-water mark style incentives have on asset prices in equilibrium. This may be particularly interesting for understanding price variation and even contagion effects in 'sophisticated' market segments, such as derivatives markets, where hedge funds are prominent players.

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Figure 1: Distance from the HWM, Risk-Choice, and Effective Risk-Aversion
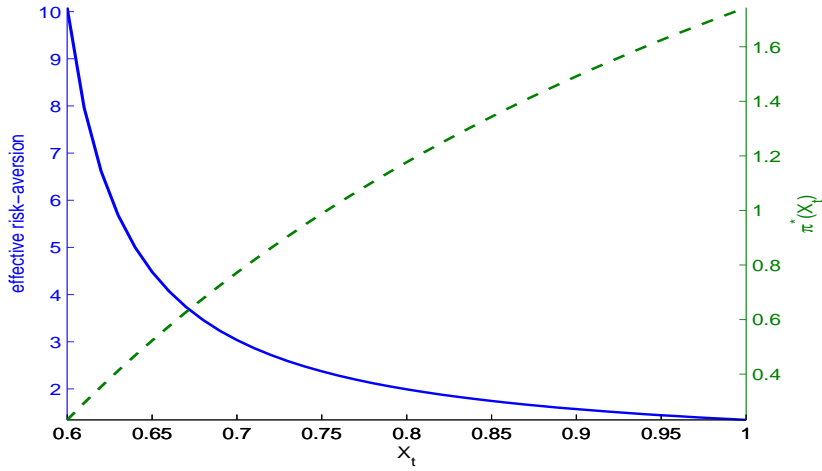
Figure 1 plots the manager's effective risk-aversion $(1 - \eta)\frac{X_t}{X_t - D_0}$ (solid line, left axis), and $\pi_t^*(X_t)$ (dashed line, right axis) against X_t (ratio of fund wealth to the hwm). The parameters for the plot are: $C = 0.6$, $g_0 = 0.35$, $\phi = 0.03$, $\rho = 0.03$, $\lambda = 0$, $m = m_H = 0$, $k = 0.2$, $\mu = 0.07$, $r = 0.01$, and $\sigma = 0.16$.

Figure 2: Outside Payoff and Risk-Choice
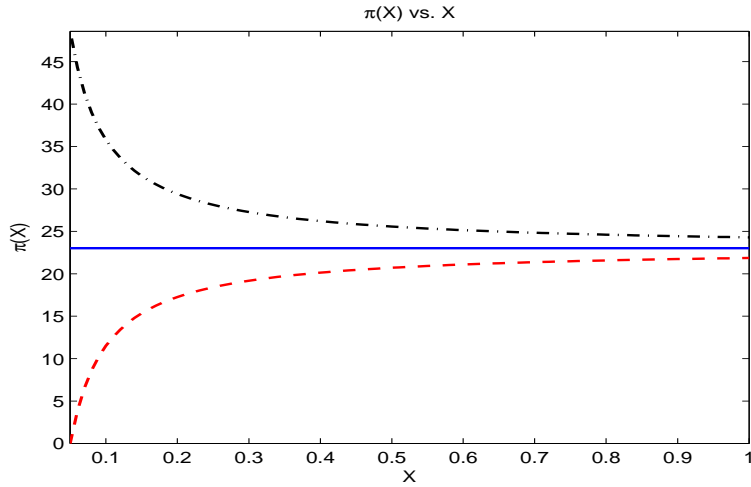
Figure 2 plots $\pi^*(X_t)$ for three cases that illustrate the possible relationships between π_t^* and X_t: (i) $D_0 = 0$ (solid line), (ii) $D_0 > 0$ (dashed line), and (iii) $D_0 < 0$ (dash-dot line).

Figure 3: Comparative Statics For Outside Payoff

V(X) vs. X
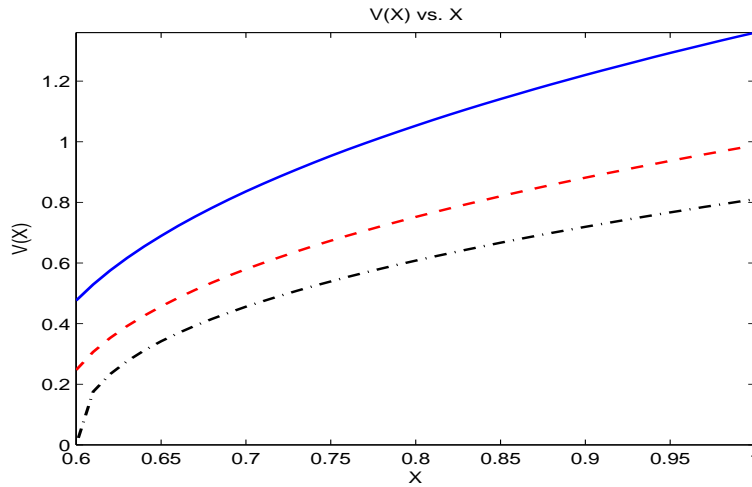
Figure 3 plots a comparison of $V(X_t)$ for $g_0 = 0.35$ (solid line), $g_0 = 0.25$ (dashed line), and $g_0 = 0$ (dash-dot line). The remaining parameters are $C = 0.6$, $\phi = 0.03$, $\rho = 0.03$, $\lambda = 0$, $m = m_H = 0$, $k = 0.2$, $\mu = 0.07$, $r = 0.01$, and $\sigma = 0.16$.
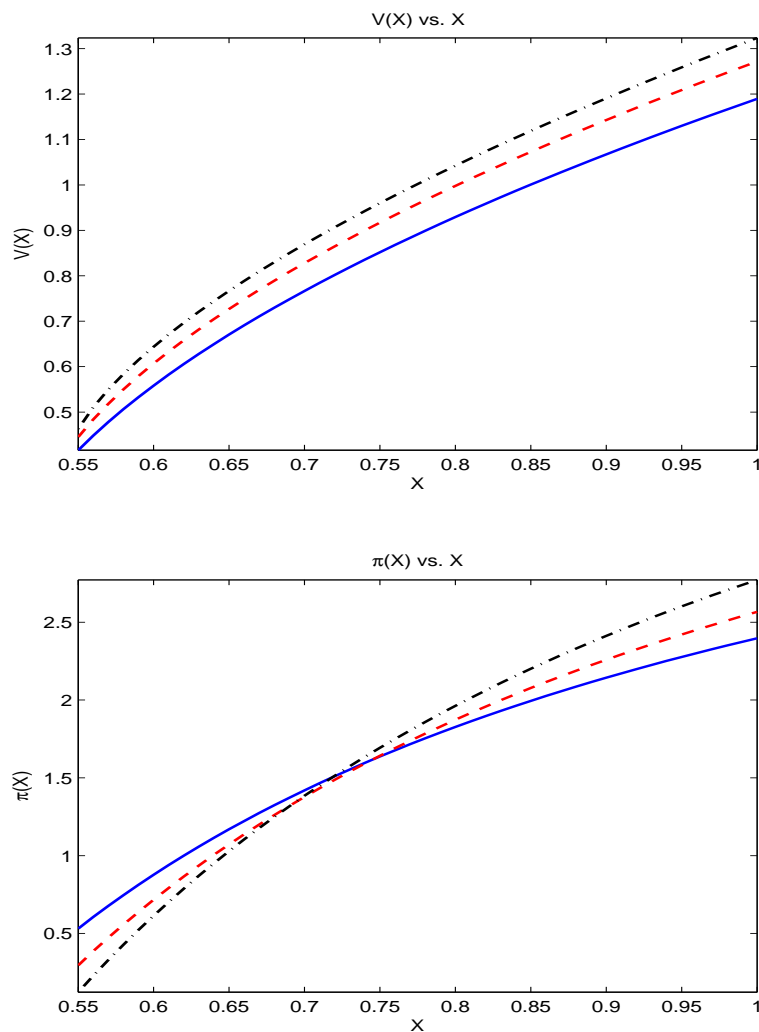
Figure 4: Comparative Statics For Withdrawal Rate

$\pi(X)$ vs. X
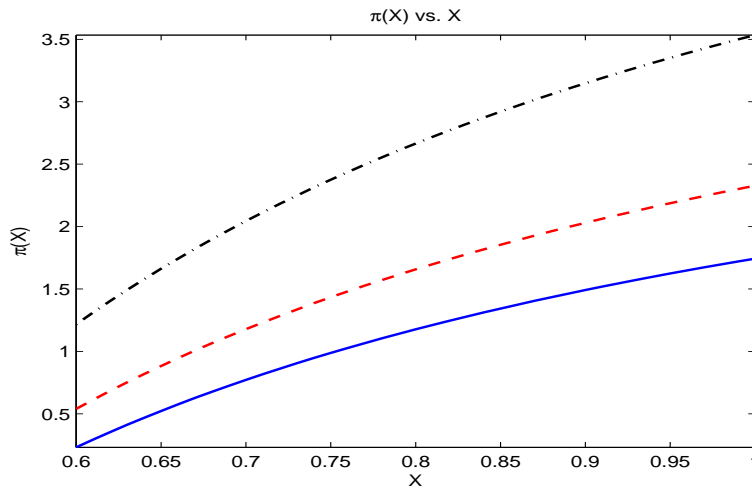
Figure 4 plots a comparison of $\pi^*(X_t)$ for $\phi = 0.03$ (solid line), $\phi = 0.06$ (dashed-line), and $\phi = 0.12$ (dash-dot line). The remaining parameters are $C = 0.6$, $g_0 = 0.35$, $\rho = 0.03$, $\lambda = 0$, $m = m_H = 0$, $k = 0.2$, $\mu = 0.07$, $r = 0.01$, $\sigma = 0.16$.

Figure 5: Management Fee Comparative Static

Figure 5 plots a comparison of $V(X_t)$ (top panel) and $\pi^*(X_t)$ (bottom panel) for $m = m_H = 0$ (solid line), $m = m_H = 0.02$ (dashed line) and $m = m_H = 0.04$ (dash-dot line). The remaining parameters are $k = 0.20$, $C = 0.55$, $g_0 = 0.35$, $\lambda = 0$, $\phi = 0.05$, $\rho = 0.03$, $r = 0.01$, $\mu = 0.07$, and $\sigma = 0.16$.

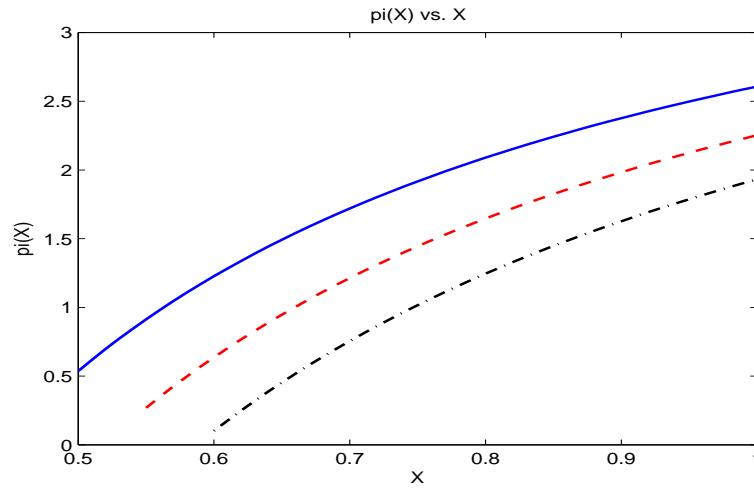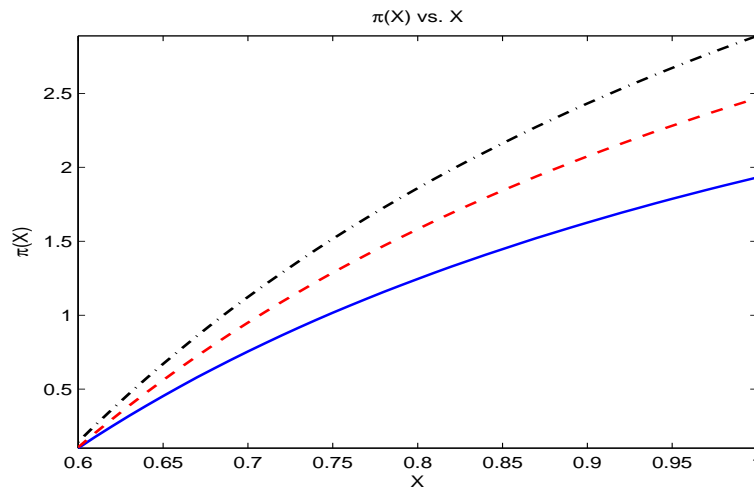Figure 6: Termination Point Comparative Static With $m = 2\%$

pi(X) vs. X

Figure 6 plots $\pi^*(X_t)$ for $C = 0.5$ (sold line), $C = 0.55$ (dashed line), and $C = 0.6$ (dash-dot line) for $m = m_H = 0.02$, $k = 0.20$, $g_0 = 0.35$, $\lambda = 0$, $\phi = 0.03$, $\rho = 0.03$, $r = 0.01$, $\mu = 0.07$, and $\sigma = 0.16$.

Figure 7: Withdrawal Rate Comparative Static With $m = 2\%$

$\pi(X)$ vs. X

Figure 7 plots $\pi^*(X_t)$ for $\phi = 0.03$ (sold line), $\phi = 0.07$ (dashed line), and $\phi = 0.1$ (dash-dot line) for $m = m_H = 0.02$, $k = 0.20$, $g_0 = 0.35$, $\lambda = 0$, $\phi = 0.03$, $\rho = 0.03$, $r = 0.01$, $\mu = 0.07$, and $\sigma = 0.16$.
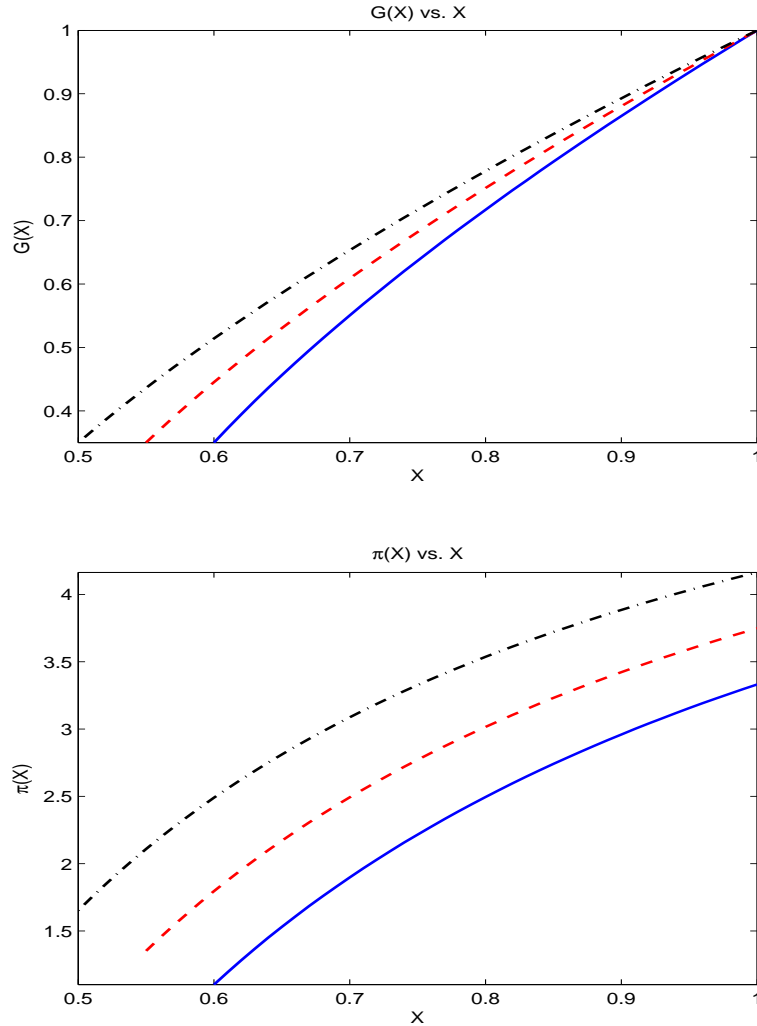
Figure 8: Walk-Away is <u>Not</u> Optimal

Figure 8 plots a comparison of $G(X_t)$ (top panel) and $\pi^*(X_t)$ (bottom panel) for successively decreasing values of C_w (or equivalently C), with the solid line corresponding to the highest value, and the dash-dot line the lowest value of C_w. The panels show that decreasing C_w increases $G(X_t)$ and $\pi^*(X_t)$ everywhere. The parameters are $C_w = 0.6$ (solid line), $C_w = 0.55$ (dashed-line), $C_w = 0.50$ (dash-dot line), and $g_0 = 0.35$, $\rho = 0.03$, $\phi = 0.11$, $\lambda = 0$, $m = m_H = 0$, $k = 0.2$, $\mu = 0.07$, $r = 0.01$, and $\sigma = 0.16$.

Figure 9: Fully and Occasionally Binding Constraints

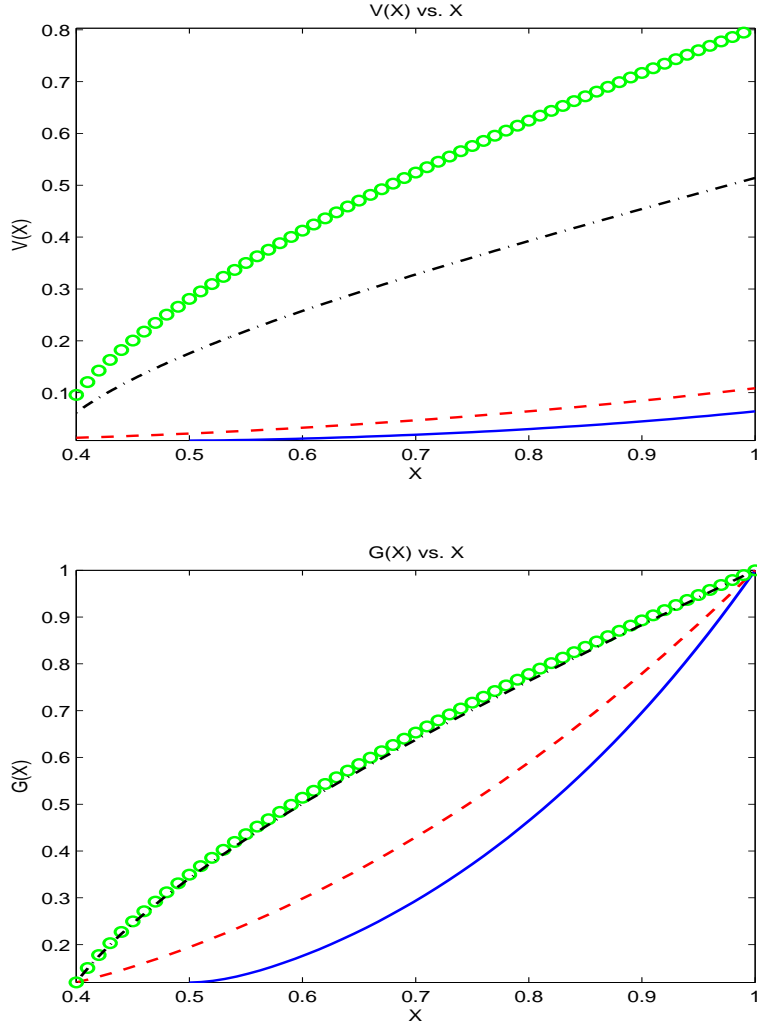Figure 9 plots $V(X_t)$ (top panel) and $G(X_t)$ bottom panel for a manager who is (i) completely constrained with optimal walk-away (solid line, $\bar{\pi} = 0.50$), (ii) completely constrained with no walk-away (dashed line, $\bar{\pi} = 0.75$), (ii) occasionally binding constraint (dash-dot line, $\bar{\pi} = 2.0$) and unconstrained (circles). The (optimal) walk-away point in (i) is $C_w = 0.5$. The termination point is $C = 0.40$. The values of the other parameters are $\phi = 0.11$, $\rho = 0.03$, $\lambda = 0$, $g_0 = 0.35$, $m = m_H = 0$, $k = 0.2$, $\mu = 0.07$, $r = 0.01$, and $\sigma = 0.16$

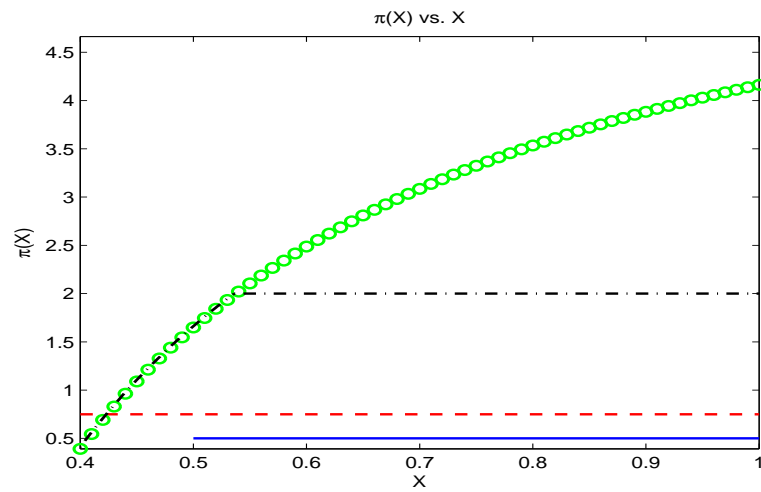Figure 10: Fully and Occasionally Binding Constraints

Figure 10 plots $\pi^*(X_t)$ corresponding to the four cases of fully and partially binding constraints shown in Figure 9.

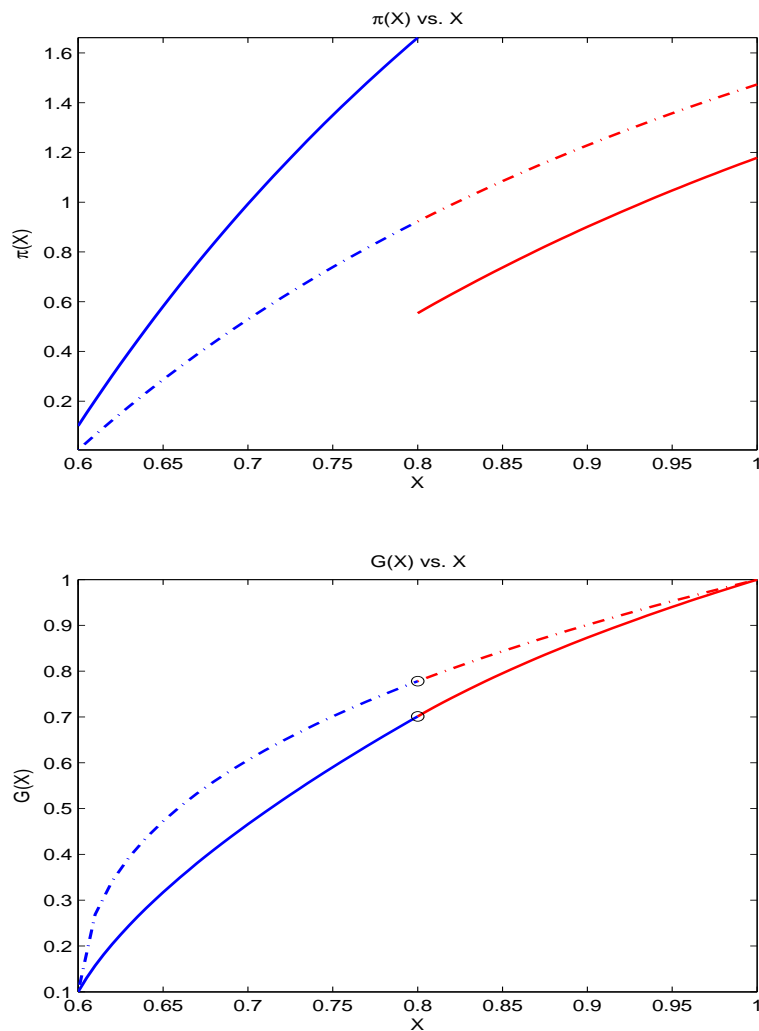Figure 11: Outflows Increase When The Fund Falls Far Below the HWM

Figure 11 plots $\pi^*(X_t)$ (top panel) and $G(X_t)$ (bottom panel) for a manager facing outflow rates of ϕ_1 on $(\mathcal{X}, 1]$ and ϕ_2 on $[C, \mathcal{X}]$. The solid line represents the case where $\phi_1 = 0.02 < \phi_2 = 0.1$, while for the dash-dot line $\phi_1 = \phi_2 = 0.02$. The black circle indicates the point $\mathcal{X} = 0.80$. The parameters are $C = 0.6$, $g_0 = 0.1$, $\rho = 0.03$, $\lambda = 0$, $m = m_H = 0$, $k = 0.2$, $\mu = 0.07$, $r = 0.01$, and $\sigma = 0.16$.

Appendix

A   Derivations and Proofs

A.1   Comparative Statics for Risk-Choice

I calculate comparative statics for π^* of the parameters C, g_0, ω and ϕ. Throughout I set $m_H = 0$.

Comparative Static for C.   I show that $\frac{d\,\pi_t^*}{d\,C} < 0$. Since the sum of the terms that multiply C in D_0 is positive, it is clear that D_0 is increasing in C. From the expression for π_t^*, it follows that an increase in D_0 reduces π_t^* for any value of X_t.

It is also useful to see that term

$$\frac{1-C}{(1-D_2)^{1/\eta} - (g_0 - D_2)^{1/\eta}}(g_0 - D_2)^{1/\eta}$$

that appears in D_0 is positive because $0 < g_0 - D_2 < 1 - D_2$. Hence, $D_0 \le C$ and D_0 equals C when $g_0 = D_2$.

Comparative Static for g_0.   I show that $\frac{d\,\pi_t^*}{d\,g_0} > 0$. To show this note that

$$g_0 - D_2 = \frac{\rho - r + \phi}{\rho + \lambda - r + \phi}g_0$$

The assumption $g_0 > D_2$ implies that $\rho - r + \phi > 0$, so that $g_0 - D_2$ is increasing in g_0. Next, note that

$$\frac{g_0 - D_2}{1 - D_2} = \frac{(\rho - r + \phi)g_0}{\rho + \lambda - r + \phi - \lambda g_0}$$

which is also increasing in g_0. Taken together, these two observations show that the second term in D_0 is increasing in g_0 and (since this term is subtracted) $\frac{d\,D_0}{d\,g_0} < 0$. From the expression for π_t^*, it then follows that $\frac{d\,\pi_t^*}{d\,g_0} > 0$.

Comparative Static for ω.   A change in ω has both a direct effect on π_t^* via the term $(\mu - r)/\sigma$ and an indirect via a change in the value of η. To see the net effect, note that

$$\frac{1}{1-\eta}\frac{\mu - r}{\sigma^2} = \frac{\sqrt{2}}{\sigma}\left(\sqrt{\omega} + \frac{\rho + \lambda - r + \phi}{\sqrt{\omega}}\right) \quad .$$

Hence,

$$\pi_t^* = \frac{\sqrt{2}}{\sigma}\left(\sqrt{\omega} + \frac{\rho + \lambda - r + \phi}{\sqrt{\omega}}\right)\frac{X_t - D_0}{X_t} \tag{A.1.1}$$

Below it is shown that as ω is varied, the term $(X_t - D_0)/X_t$ is bounded from below and above for $C < X_t \leq 1$. On the other hand, it is clear that the first term becomes unboundedly large as $\omega \to \infty$ *and* as $\omega \to 0$. Therefore,

$$\lim_{\omega \to 0} \pi_t^* \to \infty$$

$$\lim_{\omega \to \infty} \pi_t^* \to \infty$$

Taking the derivative of this first term with respect to ω gives

$$-\frac{1}{\sqrt{2}\sigma}\left(\frac{\rho + \lambda + \phi - r - \omega}{\omega^{3/2}}\right) = \begin{cases} < 0 & \text{if} \quad \omega < \rho + \lambda + \phi - r \\ > 0 & \text{if} \quad \omega > \rho + \lambda + \phi - r \end{cases}$$

Next, note that $\dfrac{d}{d\omega}\left(\dfrac{X_t - D_0}{X_t}\right) = -\dfrac{1}{X_t}\dfrac{d D_0}{d\eta}\dfrac{d\eta}{d\omega}$ and it is clear by inspection that $\dfrac{d\eta}{d\omega} < 0$. Moreover,

$$\text{sgn}\left(\frac{d D_0}{d\eta}\right) = \text{sgn}\left[\frac{d}{d\eta}\left(\frac{1 - D_2}{g_0 - D_2}\right)^{\frac{1}{\eta}}\right] < 0$$

since $1 - D_2 > g_0 - D_2 > 0$. Therefore,

$$\frac{d}{d\omega}\left(\frac{X_t - D_0}{X_t}\right) < 0$$

Therefore, $\dfrac{d\pi_t^*}{d\omega} < 0$ for 'small' ω, including for all $\omega < \rho + \lambda + \phi - r$.

Comparative Static for ϕ The impact of ϕ on π_t^* works through the change in η and the change in D_0,

$$\frac{d\pi_t^*}{d\phi} \propto \frac{1}{(1-\eta)^2}\frac{d\eta}{d\phi}\left(\frac{X_t - D_0}{X_t}\right) - \frac{1}{1-\eta}\frac{1}{X_t}\left(\frac{d D_0}{d\eta}\frac{d\eta}{d\phi} + \frac{d D_0}{d D_2}\frac{d D_2}{d\phi}\right)$$

It is easy to see that $\dfrac{d\eta}{d\phi} > 0$. Also, as shown above, $\dfrac{d D_0}{d\eta} < 0$. Finally, a straightforward differentiation shows that $\dfrac{d D_0}{d D_2} > 0$ and $\dfrac{d D_2}{d\phi} < 0$. Hence, all the terms complement each

other, so that $\dfrac{d\pi_t^*}{d\phi} > 0$.

A.2 Optimal Walk-Away

Assume $m = 0$ for simplicity. I show here that $\dfrac{dV(X_t, H_t)}{dC} < 0$ for any X_t, H_t. This shows that *reducing* C increases V everywhere. Since voluntarily walking away from the fund is equivalent to the manager increasing C beyond the point at which investors terminate him, this result demonstrates that walk-away is *never* optimal for the fund manager.

Recall that

$$V(X_t, H_t) = \beta_1 H_t \left(\left(\frac{X_t - D_0}{D_1} \right)^\eta + D_2 \right)$$

β_1, D_0 and D_1 are functions of C but η is not and when $m = 0$, neither is D_2. Therefore, it suffices to show that $\dfrac{d\beta_1}{dC} < 0$ and $\dfrac{d}{dC}\left(\dfrac{X - D_0}{D_1} \right) < 0$. The denominator of β_1 is

$$\eta \left(\frac{1}{1 - D_0} \right) \left(\frac{1 - D_0}{D_1} \right)^\eta (1 + k) - 1$$

The boundary condition $G(1) = 1$ implies that $\left(\frac{1 - D_0}{D_1} \right)^\eta = 1 - D_2$ and is therefore not a function of C. Therefore,

$$sgn\left(\frac{d\beta_1}{dC} \right) = -sgn\left\{ \frac{d}{dC}\left(\frac{1}{1 - D_0} \right) \right\} = -sgn\left(\frac{dD_0}{dC} \right)$$

and $\frac{dD_0}{dC}$ is positive by inspection of the expression for D_0. Next, note that $\frac{X - D_0}{D_1} = \frac{X - 1}{D_1} + \frac{1 - D_0}{D_1}$. Since $\frac{1 - D_0}{D_1} = (1 - D_2)^{1/\eta}$, it is not a function of C. Hence,

$$sgn\left\{ \frac{d}{dC}\left(\frac{X - 1}{D_1} \right) \right\} = -sgn\left(\frac{dD_1^{-1}}{dC} \right) = sgn\left(\frac{dD_1}{dC} \right)$$

which is negative by inspection.

A.3 Solution with Change in Withdrawal Rate

As explained in the main text, the solution for $G(X_t)$ is divided into two pieces $G_1(X_t)$ and $G_2(X_t)$ corresponding to the outflow rates ϕ_1 and ϕ_2, respectively. The solution for $G_i(X_t)$ maintains the same form as in (8). In addition we have the boundary conditions specified in the main text. For simplicity, I solve here for the case where $m = 0$ and $\lambda = 0$, so that

$D_{2,i} = 0$, but it is completely straightforward to incorporate $D_{2,i} > 0$. The solutions for $D_{0,i}$ and $D_{1,i}$ are as follows:

$$D_{0,1} = \frac{\mathcal{X} - G_1(\mathcal{X})^{\frac{1}{\eta_1}}}{1 - G_1(\mathcal{X})^{\frac{1}{\eta_1}}} \qquad\qquad D_{1,1} = \frac{1 - \mathcal{X}}{1 - G_1(\mathcal{X})^{\frac{1}{\eta_1}}}$$

$$D_{0,2} = \frac{CG_2(\mathcal{X})^{\frac{1}{\eta_2}} - \mathcal{X}g_0^{\frac{1}{\eta_2}}}{G_2(\mathcal{X})^{\frac{1}{\eta_2}} - g_0^{\frac{1}{\eta_2}}} \qquad\qquad D_{1,2} = \frac{\mathcal{X} - C}{G_2(\mathcal{X})^{\frac{1}{\eta_2}} - g_0^{\frac{1}{\eta_2}}}$$

The value matching condition gives that $G_1(\mathcal{X})^{\frac{1}{\eta_1}} = G_2(\mathcal{X})^{\frac{1}{\eta_2}}$. Denote this common value by $\mathcal{G}$. Substituting the solution into the derivative-matching condition and using the value-matching condition gives

$$\frac{\mathcal{X} - D_{0,1}}{\mathcal{X} - D_{0,2}} = \frac{\eta_1}{\eta_2}$$

Substituting in for the $D_{0,i}$ and rearranging gives the following equation which determines the value $\mathcal{G}$

$$\frac{\left(\frac{1}{\mathcal{G}}\right)^{\frac{1}{\eta_1}} - 1}{\frac{1}{\mathcal{X}} - 1} - \frac{\eta_2}{\eta_1} \frac{1 - \left(\frac{g_0}{\mathcal{G}}\right)^{\frac{1}{\eta_2}}}{1 - \frac{C}{\mathcal{X}}} = 0 \quad .$$