Dynamic Pricing of Network Goods with Boundedly Rational Consumers

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Abstract

We present a model of dynamic monopoly pricing for a good that displays network effects. In contrast with the standard notion of a rational-expectations equilibrium, we model consumers as boundedly rational, and unable either to pay immediate attention to each price change, or to make accurate forecasts of the adoption of the network good. Our analysis shows that the seller's optimal price trajectory has the following structure: the price is low when the user base is below a target level, is high when the user base is above the target, and is set to keep user base stationary once the target level has been attained. We show that this pricing policy is robust to a number of extensions, which include the product's user base evolving over time, and consumers basing their choices on a mixture of a myopic and a "stubborn" expectation ofadoption. Our results differ significantly from those that would be predicted by a model based on rational-expectations equilibrium, and are more consistent with the pricing of network goods observed in practice.

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1 Introduction

An important simplifying assumption made in economic models of network effects is that consumers are perfectly rational. This assumption is important in such models because the value to each consumer of a good or service that displays network effects (henceforth referred to as a "network good") is influenced by the consumption choices made by some or all other consumers. The solutions to such models are typically based on asserting the following "unboundedly" rational behavior: (a) consumers immediately react to each observable strategic decision made by a seller, and (b) having observed this decision, consumers form a common expectation of demand, and make their consumption choices unilaterally based on this expectation, which is then realized in equilibrium. This solution is commonly referred to as satisfying "fulfilled expectations", or as a "rational expectations equilibrium".

Some notion of perfect rationality is at the base of most current economic analysis, even though most researchers accept that agents are not in reality unboundedly rational. Such models continue to be used, perhaps because there is an implicit belief that the 'output' of analysis based on the approximation of unboundedly rationality agents is (reasonably) correct. In the specific case of network goods, however, many predictions of models based on unboundedly rational behavior do not appear to be a good description of reality. For example, it is believed that outcomes in markets for network goods are often path-dependent, and therefore, the dynamics of the adoption process and the path of choices made by a seller are important determinants of eventual outcomes. This would not be the case were consumers able to form rational expectations after each price change. It is therefore possible that managers, upon observing these aspects of real-world network markets, would hesitate to rely on the prescriptions of models that ascribe the extent of rationality, coordination and prediction accuracy that characterizes unboundedly rational behavior.

Our objective in this paper is to examine how the predictions of models of network effects change, if at all, under assumptions about consumer rationality that seem more realistic. We do so by presenting alternative models of demand for a network good, in which consumers are cognitively bounded, do not immediately react to every change in the seller's price, and furthermore, make their consumption choices based on a boundedly rational assessment of expected demand, which may depend on the current price, the current level of demand, and/or an exogenously specified "stubborn" expectation of equilibrium demand. We use these models of bounded rationality to study the dynamic pricing problem for a monopoly seller of a network good. The adoption choices of the consumers continuously influence the rate at which demand adjusts over time, and the monopoly seller therefore chooses the price *trajectory* that maximizes her discounted stream of profits. The rate at which demand adjusts over time is also affected by a parameter k that is proportional to the fraction of consumers in each period who "pay attention" to the current price, and will be called the *attention rate*.

Our first results show that when consumers are myopic in their expectations, the monopolist's optimal pricing trajectory is generated by a *target policy* with the following properties: when current demand is below the target, the price is low; when current demand is above the target, the price is high; and when current demand is at the target, the price is chosen to keep demand stationary. The target could be interpreted as the level of adoption below which the monopolist invests in building a user base, and above which the monopolist profits from exploiting her installed base. The result thus prescribes a form of penetration pricing that is not uncommon in markets for network goods. (See Sec.5 for examples.)

The theoretical results alluded to above are derived under the assumption that the cumulative distribution function (cdf) of the attentioon rate, k, is either a convex or concave funtion on a finite interval. The central case is the one in which the cdf is convex. The uniform cdf (purely linear in k) is a limiting case of the convex case, and in it the optimal pricing policy has the extreme form in which the price is zero when demand is below the target, and equals the maximum allowed by the model when demand is above the the target. The concave (but not linear) case presents two problematic features: (1) strictly speaking, the optimal policy does not exist, in the sense that when demand equals the target the optimal ”price” is really a probability distribution concentrated on the two points 0 and 1; (2) this probability distribution is the same for all strictly concave (i.e. not linear) cdfs. (For more on this point, see the end of Sec. 3.5.) The extension of the analysis to a more general class of cdfs remains a project for future research.

These results also show that the optimal demand target with myopic consumers is lower than the equilibrium level of demand predicted by a model with rational expectations. The difference between the target demand and the rational expectations equilibrium demand is a decreasing function of k, and tends to zero as k increases without bound, i.e., when all consumers react to price changes infinitely fast. (In fact, we show that, for a fairly general class of cdfs, the optimal policy for the rational expectations model cannot be the steady state of our model with myopic consumers.)

Our subsequent results extend the analysis in two directions, for the case of the uniform cdf. First, we examine how the monopolist's optimal price trajectory varies when the population of consumers evolves over time. That is, in each period, a constant fraction of consumers is (exognously) replaced by new ones, at a rate determined by a parameter c that is proportional to the fraction of consumers replaced in each period. The monopolist's optimal pricing trajectory continues to be generated by a target policy with the same qualitative properties as the one in the ”basic myopic case,” although with a strictly lower optimal demand target. Moreover, in this model, the price that keeps demand stationary at any desired level is progressively lower as c increases, and we discuss how this differentiates the effect of changes in the rate of replacement c, from corresponding changes in the rate of ”attention” k.

In a second extension we examine how the monopolist's optimal price trajectory varies when the expectation of demand formed by each consumer (who pays attention) is a weighted average of the myopic expectation and an exogenously specified ”stubborn” expectation. Our final theorem establishes that, again, the monopolist's optimal pricing trajectory continues to be generated by a target policy with the same qualitative properties as the one in the basic myopic case, but again with a lower target demand level. The target increases as consumers become less stubborn, eventually converging to the target demand level of the policy for purely myopic consumers.

We have organized the rest of this paper as follows. Section 2 describes our model of bounded rationality, our underlying discrete-time model, and the derivation of its continuous-time counterpart. Section 3 analyzes our ”base” model with myopic consumers. In it we derive the optimal price trajectory for this model, and contrast it with that predicted by the rational-expectations model. Section 4 extends the base model in the two ways described above. Section 5 summarizes and comments on our results, and sketches a program of future research. Section 6 contains some bibliographic remarks and the list of references. An Appendix (Sec. 7) contains those proofs not presented in the main body of the paper.

2 Overview of the Model

2.1 A Discrete-Time Model of Boundedly Rational Consumers

We introduce our continuous-time formulation of a monopolist's market for a network good as the limiting case of a discrete-time model. We emphasize that our discussion of the discrete-time model and the passage to the limit is only designed to motivate the continuous time formulation, and is therefore only "heuristic" in the sense that we leave out many of the mathematical details that would be required for a more rigorous discussion (see comments below).

In the discrete-time model, the length of each period is h, and time is divided into periods, $n = 0, 1, 2, ...$*ad inf.* Calendar time is denoted by t, a non-negative real number, so that period n begins at $t = nh$ and ends at $t = (n + 1)h$. A network good is provided by a monopolist, who sells the good in units of one period. (Think of the good as a service.) At the beginning of period n, the monopolist announces a price $p(t)$ (per unit time) for the time interval $nh \le t < (n + 1)h$. We assume that the price is constrained to be nonnegative, and is bounded above (more on this later). A continuum of consumers is indexed by a "type" parameter θ. We take the mass of the consumers to be unity, and the set of types to be the unit interval. If a consumer of type θ buys the service for one period, and the total mass of consumers who buys the service in that period is $q(t)$, then the consumer's (incremental) instantaneous utility in that period will be equal to

$$\theta q(t) - p(t). \tag{1}$$

Note that $p(t)$ and the decision of each consumer is constant during any one period. Let F be the cumulative distribution function of θ, i.e., the fraction of consumers with type less than or equal to θ is $F(\theta)$. For simplicity, we assume F to be absolutely continuous and strictly increasing on the the unit interval.

The first aspect of our model of bounded rationality is that of *bounded attention*. At the beginning each period (with two exceptions noted below), a "random" fraction kh of consumers of each type "pay attention to" the current price $p(t)$. Correspondingly, the remaining fraction $(1 - kh)$ of consumers of each type do not pay attention to the monopolist's price announcement, and their choice remains unchanged in period n from that in period $(n - 1)$. Notice that an equal fraction kh of consumers of each type "pay attention" in each period, and that the magnitude of this fraction depends on the length of the interval h. One might therefore interpret k as measuring a "rate of attention" of consumers to price changes, or a "rate of adjustment." Thus the average time between successive price checks by a consumer is $(1/k)$.

[Note: The mathematical problems of dealing with a continuum of random variables are well-known, and (as noted above) the preceding story is only a heuristic description of a model. For a precise description and analysis of the model see (Radner, 2003).]

The second aspect of our model of bounded rationality specifies how consumers who are paying attention form their *expectation of demand* for the coming period. Specifically, we assume that each consumer who notices the price $p(t)$ at the beginning of period n makes the same prediction, $q_E(t,h)$, of the total demand in period n. Therefore, a consumer of type θ who notices $p(t)$ will buy the good if and only if $\theta q_E(t,h) \geq p(t)$. In this paper we shall discuss several models expectation formation.

The first exception to our model of bounded attention occurs when $p(t) > q_E(t,h)$. In this case, we assume that $q(t) = 0$. The rationale is that if $p(t) > q_E(t,h)$ and every consumer expected the quantity demanded to be $q_E(t,h)$, then no consumer would want to buy the good in period t. In a sense, the occurrence of "$p(t) > q_E(t,h)$" is a "wake-up-call" for *all consumers*. Hence we shall assume directly that

$$0 \leq p(t) \leq q_E(t,h). \tag{2}$$

[Note: We should mention that some upper bound on price is desirable to assure that the model is mathematically well-behaved.]

The second exception to our model of bounded attention occurs when $q_E(t,h) = 0$. From the above constraint on $p(t)$, it would follow that $p(t) = 0$. For technical reasons, we shall postpone dealing with this case until Section 3.1.

To highlight how the dynamics of the system depend on the paramemter h, we change the notation slightly and let $q_h(t)$ denote the actual total demand at time $t = nh$.We shall now derive a difference equation for $q_h(t)$ in the case in which both $q_E(t,h)$ and $q_h(t$ are > 0. Let $w_h(\theta,t)$ denote the demand (per consumer) from consumers of type θ in period t. Thus

$$q_h(t) = \int_0^1 w_h(\theta,t)dF(\theta). \tag{3}$$

Recall that a "fraction" kh of consumers of type θ pay attention to $p(t)$, form a shared expectation of demand $q_E(t,h)$, and decide whether or not to adopt the product for period n. Therefore, if $\theta \geq [p(t)/q_E(t,h)]$, each consumer in this fraction kh adopts the product, and if $\theta < [p(t)/q_E(t,h)]$, then none of these consumers adopt the product. Since the remaining fraction $(1-kh)$ continue to do in period n what they were doing in period $n-1$, it follows that:

$$w_h(\theta,t) = \begin{cases} kh + (1-kh)w_h(\theta,t-h), & \theta \geq p(t)/q_E(t,h), \\ (1-kh)w_h(\theta,t-h), & \theta < p(t)/q_E(t,h). \end{cases} \tag{4}$$

The last two expressions imply

$$
\begin{aligned}
q_h(t) &= \int_0^1 (1-kh)w_h(\theta, t-h)dF(\theta) + \int_{p(t)/q_E(t,h),}^1 khdF(\theta) \\
&= (1-kh)q_h(t-h) + kh\left(1 - F\left[\frac{p(t)}{q_E(t,h)}\right]\right). \qquad (5)
\end{aligned}
$$

2.2 A Continuous Time Approximation

Our continuous-time model is obtained by letting the length h of the interval in the discrete-time model tend to zero. Recall that $q_E(t,h)$ is the predicted demand in period n, where $nh \leq t < (n+1)h$, and define

$$q_E(t) = \lim_{h \to 0} q_E(t,h).$$

Assume that $q_E(t)$ is well-defined, and depends at most on the current demand and price, $q(t)$ and $p(t)$, respectively. The resulting time-rate of change of demand is described in our first lemma.

Lemma 1 *If at time $t > 0$ the demand and price are $q(t)$ and $p(t)$, respectively, and*

$$
\begin{aligned}
0 &< q(t) < 1, \\
0 &< q_E(t) < 1, \\
0 &\leq p(t) \leq q_E(t),
\end{aligned}
$$

then the time-rate of change of demand is specified by:

$$q'(t) = k\left\{Q[q_E(t), p(t)] - q(t)\right\}, \qquad (6)$$

where

$$Q(x,y) \equiv 1 - F\left(\frac{y}{x}\right), \quad 0 < x \leq 1. \qquad (7)$$

Proof. From the law of motion at the end of the discrete-time model, it follows that

$$q_h(t) - q_h(t-h) = -khq_h(t-h) + kh\left(1 - F\left[\frac{p(t)}{q_E(t,h)}\right]\right).$$

Dividing both sides by h, and letting h tend to zero, gives the desired result. ■

Completing the specification of the law of motion at the boundary values is postponed to the following sections, where different specifications of the prediction function, q_E, are formulated. Also, it is assumed that the initial values, $q(0)$ and $q_E(0)$ are given.

The monopolist wants to choose a price trajectory $p(t)$ to maximize her profit. Assume, for simplicity, that her (marginal) cost of providing the service is zero; then her total discounted (variable) profit is

$$\int_0^\infty e^{-rt} p(t)q(t)dt, \qquad (8)$$

where $r > 0$ is her given discount rate, and $q(t)$ evolves according to the model. We shall analyze the maximization problem using both the the method of Pontryagin and the Bellman/Blackwell method of dynamic programming, in which the state variable is the current demand. The precise formulation of the maximization problem will depend on the particular cases we shall consider in the following sections.

3 Myopic Consumers

3.1 The Model with Myopic Consumers

This section describes the monopolist's optimal price trajectory for a class of models of bounded rationality in which consumers are myopic, i.e., their expected demand in the "immediate future" is equal to the current demand. In the model with continuous time, this reduces to

$$q_E(t) = q(t). \tag{9}$$

It will be convenient to take the monopolist's control variable, $\boldsymbol{a}(t)$, to be *the ratio of price to expected quantity demanded*, and hence

$$\boldsymbol{a}(t) = \frac{p(t)}{q(t)}, \tag{10}$$

with the assumption that $0 \leq \boldsymbol{a}(t) \leq 1$. [Note, to avoid any confusion, the entire trajectory of the control variable is denoted by $\boldsymbol{a}$, in boldface, and the control at time t is denoted by $\boldsymbol{a}(t)$.] For $0 < q(t) \leq 1$, the law of motion of the quantity demand given by Lemma 1 is

$$q'(t) = k\left[1 - F[\boldsymbol{a}(t)] - q(t)\right].$$

If $q(t) = 0$, then the constraint $0 \leq p(t) \leq q(t)$ implies that $p(t) = 0$, so that $\boldsymbol{a}(t) = 0/0$. We therefore modify the law of motion in Lemma 1 as follows: With a slight abuse of notation, define

$$m(q, a) = k\left[1 - F(a) - q\right], \quad 0 \leq q \leq 1, \quad 0 \leq a \leq 1, \tag{11}$$

and assume that the law of motion is

$$q'(t) = m[q(t), \boldsymbol{a}(t)], \quad 0 \leq q(t) \leq 1, \quad 0 \leq \boldsymbol{a}(t) \leq 1. \tag{12}$$

Note that the function m is continuous everywhere in the unit square. In particular, $m(0,1) = 0$, whereas $m(0,a) > 0$ for $a < 1$. Thus this modification of the law of motion removes the discontinuity in the version contained in Lemma 1.

Given the quantity $q(t)$ and the control $\boldsymbol{a}(t)$, the monopolist's profit at time t is $\boldsymbol{a}(t)q(t)^2$. Hence the monopolist's maximization problem is

$$\max_{\boldsymbol{a}} \int_0^\infty e^{-rt}\boldsymbol{a}(t)q(t)^2 dt, \tag{13}$$

subject to the law of motion (with its constraints), and given the initial quantity, $q(0)$.

In the following subsection, we give a precise definition of a Rational Expectations Equilibrium (REE), and show that, under suitable regularity conditions on the cumulative distribution function F, a REE cannot be a steady state of the system with myopic consumers and a profit-maximizing monopolist. In the subsequent three subsections we discuss the special cases in which the cumulative distribution of types is, respectively, (1) uniform, (2) convex, and (3) concave.

3.2 Rational Expectations Equilibrium is not Optimal

An alternative theory of consumer behavior is embodied in the concept of "rational-expectations equilibrium." Imagine that, when faced with a price p, each consumer *correctly predicts* the total demand at that price, and decides whether or not to subscribe on the basis of that prediction. Thus the total demand at that price must satisfy

$$q = 1 - F\left(\frac{p}{q}\right), \tag{14}$$

provided $q > 0$. Suppose now that the monopolist chooses the price, say p^{**}, to maximize pq subject to the last equation as a constraint; call the corresponding demand, q^{**}. Following standard terminology, we shall call such a pair (p^{**}, q^{**}) a *rational-expectations equilibrium* (REE). We shall show that, for a sufficiently "regular" cdf F, a REE cannot be a steady state for an optimal trajectory with boundedly rational myopic consumers.

In terms of the continuous-time model of the previous subsection, in which the control variable is $a = p/q$, the above constraint is replaced by

$$q = 1 - F(a), \quad 0 \leq q \leq 1, \quad 0 \leq a \leq 1. \tag{15}$$

Recall that F is absolutely continuous and strictly increasing, $F(0) = 0$, and $F(1) = 1$. Hence this last equation implicitly defines a mapping, say A, from q to a,

$$A(q) = F^{-1}(1 - q), \tag{16}$$

which is absolutely continuous and strictly decreasing, with $A(0) = 1, \quad A(1) = 0$, and the corresponding profit is

$$v(q) = A(q)q^2. \tag{17}$$

Define

$$\begin{aligned} q^{**} &= \arg\max_q v(q), & (18) \\ a^{**} &= A(q^{**}). & (19) \end{aligned}$$

Since $v(0) = 0$, it follows that $q^{**} > 0, \quad 0 < a^{**} < 1$. For the purpose of this subsection, we shall say that F is "regular" if q^{**} satisfies the usual first-order condition for a maximum of v,

$$v'(q^{**}) = 0. \tag{20}$$

Theorem 2 *If F is "regular," then the optimal REE cannot be a steady state of an optimal dynamic price trajectory with boundedly rational myopic consumers.*

Proof. Let (a^{**}, q^{**}) be a REE. Since $v\,(0) = 0$, the quantity q^{**} is strictly positive. We now demonstrate the existence of a price trajectory that, starting with the initial condition

$$q(0) = q^{**},$$

in the model of boundedly-rational myopic consumers, yields a profit that is strictly greater than $(1/r)v(q^{**})$. Let $u > 0$, and define the pricing trajectory $\boldsymbol{a}$ by

$$\boldsymbol{a}(t) = \begin{cases} 1, & 0 \leq t < u, \\ A[q(u)], & t \geq u. \end{cases} \tag{21}$$

The total discounted profit from $\boldsymbol{a}$ is

$$w(u) = \int_0^u e^{-rx} \left[q(x)\right]^2 dx + \frac{e^{-ru} v[q(u)]}{r}, \quad u \geq 0, \tag{22}$$

$$v[q(u)] = A[q(u)][q(u)]^2. \tag{23}$$

Differentiating w, one gets

$$w'(u) = e^{-ru} \left\{ \left[q(u)\right]^2 - v[q(u)] + \left(\frac{1}{r}\right) v'[q(u)]q'(u) \right\}. \tag{24}$$

Setting $u = 0$ one gets

$$\begin{aligned} w'(0) &= (q^{**})^2 - v(q^{**}) + \left(\frac{1}{r}\right) v'(q^{**})q'(0) && (25) \\ &= (q^{**})^2 - a^{**}(q^{**})^2 + 0 && (26) \\ &= (q^{**})^2(1 - a^{**}) && (27) \\ &> 0. && (28) \end{aligned}$$

But

$$w(0) = \left(\frac{1}{r}\right) v(q^{**}). \tag{29}$$

Hence, for some sufficiently small $\epsilon > 0$,

$$w(\epsilon) > w(0), \tag{30}$$

so that setting $u = \epsilon$ yields the monopolist a larger total discounted profit than charging the REE price forever from time zero, which completes the proof of the theorem.

In fact, as we shall see, in the cases discussed in this paper, the REE demand is larger, and the REE price is smaller, than the respective demand and price in a steady state of an optimal trajectory.

3.3 Uniformly Distributed ConsumerTypes

We now give a complete solution of the optimal dynamic price problem for the special case in which θ is distributed uniformly on the unit interval; thus

$$F(\theta) = \theta, \quad 0 \le \theta \le 1. \tag{31}$$

From (12) and (1) of Sec. 3.1, the the law of motion is

$$\begin{aligned} q'(t) &= m[q(t), \boldsymbol{a}(t)], && (32) \\ m(q,a) &= k\,[1 - a - q]\,, \quad 0 \le q \le 1, \quad 0 \le a \le 1, && (33) \end{aligned}$$

where $a = p/q$ is the control variable.

In our analysis we shall use the Bellman/Blackwell theory of dynamic programming. According to a theorem of Blackwell, in a problem of our form, an optmal trajectory can be generated by a *(stationary) policy*, which is a mapping α from the current demand ("state") to the current control variable, thus:

$$\boldsymbol{a}(t) = \alpha[q(t)]. \tag{34}$$

As a byproduct, the analysis gives the maximum total discounted profit, say $V(q)$, as a function of the initial demand, $q(0) = q$. This mapping is called the *value function for the optimal policy.*

We shall, in fact, show that the optimal policy has the following special form, which we shall call a *target policy.* For a given number, say s, with $0 < s \le 1$, *the target policy with target* s is defined by

$$\pi(q) = \begin{cases} 0, & q < s, \\ A(s), & q = s, \\ 1, & q > s, \end{cases} \tag{35}$$

where $A(q)$ is given by (**??** for the uniform distribution,

$$A(q) = 1 - q. \tag{36}$$

Recall that, if the current demand is q, and the monopolist chooses the control $A(q)$, then the demand will remain unchanged. i.e., $q'(t) = 0$.

For example, if the initial demand is less than the the target, the demand will increase, and the revenue will remain zero, until the demand reaches the target, after which the revenue will remain at $A(q)q^2$. Note that there is a tradeoff between reaching a higher target and getting there sooner. Let π^* be the optimal target policy. The policy, α, and several demand trajectories are shown in Figure 1.

We shall prove that π^* *is optimal among all policies.*

Theorem 3 *The optimal target policy* π^* *is optimal among all policies, and the optimal target is*

$$\sigma \equiv \frac{2k}{3k + r}.$$

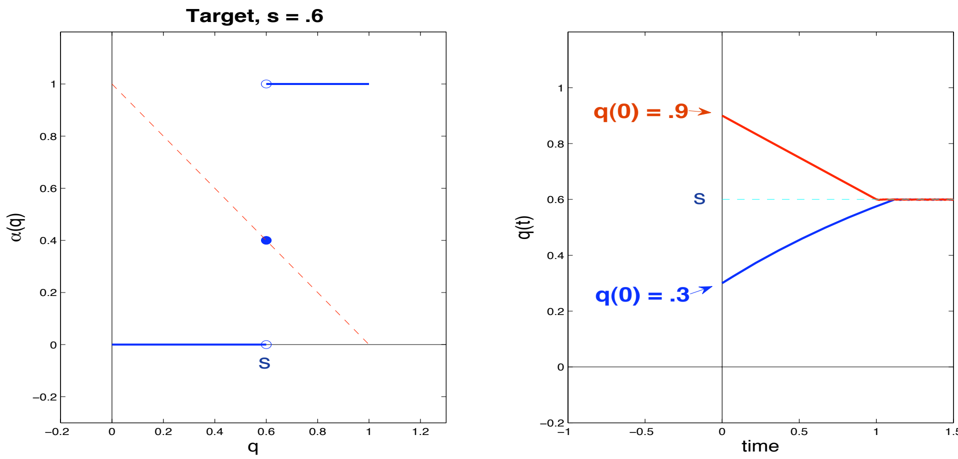


Figure 1: Left - The target policy, $\alpha(q)$, when the cdf is uniform. The target here is set equal to .6. Right - two trajectories using the target policy. The upper trajectory starts at $q(0) = .9$ and the lower trajectory starts at $q(0) = .3$. Both trajectories reach the target, $s = .6$, in finite time.

Furthermore, the value function for π^* is given by

$$V_{\pi^*}(q) = \frac{1}{r}\left[(1-q)^{-(r/k)}\right]\left[\sigma^2(1-\sigma)^{\left(1+\frac{r}{k}\right)}\right], \tag{37}$$

and $V_{\pi^*}(q)$ is increasing in k if the imtial demand q is less than the optimal target, σ.

The proof of this result is presented in Appendix 7.1.

It is straightforward to verify that the demand in the REE (Sec. 3.2) for the uniform case is $2/3$, which is larger than the the target in the optimal policy.

3.4 Convex Distributions of ConsumerTypes

In this section we consider the case where the cumulative distribution function of consumer types is strictly convex: $F''(a) > 0$, $0 < a < 1$. We will show that, in this case, there is a unique optimal trajectory and we will illustrate this result with the quadratic cdf $F(a) = a^2$. This optimal trajectory determines a stationary policy as a function of demand, q.

In this subsection, we use the notation $\dot{x}$ to denote the derivative with respect to time, whereas a 'prime' denotes the derivative with respect to the function's argument:

$$\dot{F}(\mathbf{a}(t)) = \frac{dF(\mathbf{a}(t))}{dt}, \qquad F'(a) = \frac{dF}{da}.$$

Recall from Equation (13) that the monopolist wants to solve the following

$$\begin{aligned}
\text{Maximize} \quad V(\mathbf{a}) &= \int_0^\infty e^{-rt}\mathbf{a}(t)q(t)^2\,dt = \int_0^\infty e^{-rt}h(t,q,\mathbf{a})\,dt \ , && (38)\\
\text{subject to} \ \dot{q}(t) &= k\,(\ 1 - F(\mathbf{a}(t)) - q(t)) = m(q(t),\mathbf{a}(t))\ , && (39)\\
q(0) &= q_0 \in (0,1), \qquad 0 \le a \le 1\,. && (40)
\end{aligned}$$

Our assumptions on F imply that $0 \le F \le 1$ for all a, $F'(a) \ge 0$ and $F''(a) > 0$ for all $a \in (0,1)$. Also note that, with these constraints on F, if $q_0 \in [0,1]$ then $q(t)$ will remain between 0 and 1.

We use an infinite-time Maximum Principle proved in (Weber, 2006), Proposition 2. While this problem doesn't satisfy all of the hypotheses of Proposition 2 in Weber, the conclusion still holds, and the proof - given in the Appendix 7.2 - closely follows the argument in that paper.

For the optimization problem (38), define the *current value Hamiltonian* function:

$$\begin{aligned}
H(t,q,\mathbf{a},\psi) &= h(t,q,\mathbf{a}) + \psi m(q,\mathbf{a}) && (41)\\
&= \mathbf{a}(t)q(t)^2 + k\psi(t)\,[1 - F(\mathbf{a}(t)) - q(t)]\ , && (42)
\end{aligned}$$

where $\psi(t)$ is the *adjoint*, or *co-state*, variable. We denote by $\mathcal{A}$ the set of *admissable* controls, in this case those that are bounded between 0 and 1.

Proposition 4 ***Pontryagin Maximum Principle:*** *Let $(q^*, \mathbf{a}^*)$ be a solution to the optimal control problem (38). Then there exists an absolutely continuous function $\psi : \mathbb{R}^+ \to \mathbb{R}$ such that the following optimality conditions are satisfied:*

1. *Adjoint Equation:*
$$\dot{\psi}(t) = r\psi(t) - \frac{\partial H}{\partial q}\,. \qquad (43)$$

2. *Maximality Condition*
$$H(t,q^*,\mathbf{a}^*,\psi) = \sup_{\mathbf{a}\in\mathcal{A}} H(t,q,\mathbf{a},\psi)\,. \qquad (44)$$

3. *Boundedness of the adjoint variable. The function $\psi(t)$ is positive and bounded on $\mathbb{R}^+$. In particular, there exists a $\bar{\psi}$ such that*
$$0 \le \psi \le \bar{\psi}$$
for all $t \in \mathbb{R}^+$.

Note that if $\mathbf{a}^*$ satisfies
$$\frac{\partial H}{\partial \mathbf{a}}\Big|_{\mathbf{a}^*} = 0, \qquad (45)$$
then it is a candidate solution for Equation (44).

In sections 3.4.1 and 3.4.2 we use Proposition 4 to find differential equations for the adjoint variable, $\psi(t)$, and for the optimal $\mathbf{a}^*(t)$, and we use the structure of the phase space of these variables and the boundedness of ψ to show that the optimal trajectories must converge to a unique equilibrium. Furthermore, using the geometry of the phase space, we show that, for every initial value of the demand, q_0, there is a unique optimal trajectory $(q(t), \mathbf{a}(t))$ through q_0.

3.4.1 Uniqueness of the optimal strategy

In this section, we analyze the system in the $q\psi$-plane to show that any optimal trajectory must lie on a curve. From this we conclude that, for each value of $q(t)$, there is a unique value $\psi(t)$ that corresponds to an optimal trajectory and hence, by Pontryagin's Maximum Principle, a unique optimal policy $\mathbf{a}(t) = \alpha(q(t))$.

Our strategy is to describe all possible long-term behaviors of the system by first finding all possible equilibria and their stability. This is done by showing that the nullclines of the system intersect each other exactly twice, once at the origin and once in the interior of the unit square. Thus, there are two equilibria of the system, one at $(0, 0)$ and one in the interior. Figure 2 shows an example of the phase portrait. A linear analysis shows that the interior equilibrium is a saddle, and therefore the long-term behavior of all trajectories that start in the unit square must fall into one of three categories:

1. The trajectory is unbounded.
2. The trajectory converges to the origin.
3. The trajectory converges to the interior saddle equilibrium along its stable manifold.

Since the first two categories cannot be optimal, the unique optimal strategy is the one that corresponds to a trajectory along the stable manifold of the saddle.

For clarity in the calculations, we let $u = F(a)$, so that $G(u) = F^{-1}(u) = a$. The equations of motion, Equations (39) and (43) become:

$$\dot{\psi} = (r + k)\psi - 2G(u)q\,, \tag{46}$$
$$\dot{q} = k(1 - u - q)\,. \tag{47}$$

In Appendix 7.3 we prove the following:

Theorem 5 *The system of differential equations given by Equations 46 and 47 has a unique equilibrium in the interior of the unit square. This interior equilibrium is a saddle.*

Because the interior equilibrium is a saddle, the only trajectories that converge to this equilibrium lie on its stable manifold. All other trajectories are either unbounded, or converge to $q = \psi = 0$. Since the latter is not optimal by the boundedness of the adjoint variable, (part 3 of Proposition 4) , all optimal

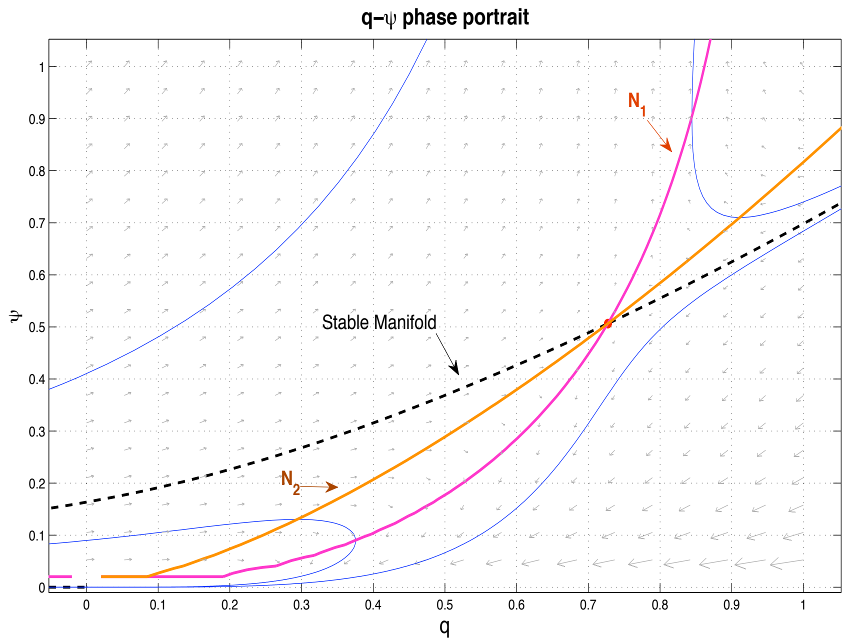


Figure 2: Phase portrait for the system in the $q\psi$-plane with $F(a) = a^2$. There is a unique interior equilibrium which is a saddle. All optimal trajectories lie on the stable manifold of this saddle, since other trajectories are either unbounded or remain at the equilibrium $q = \psi = 0$.

trajectories must lie on the stable manifold of the interior equilibrium. This stable manifold can be broken into two pieces: one with q-values below the interior equilibrium and one with q-values above the interior equilibrium. By construction, neither of these pieces can cross a nullcline, and therefore both q and ψ are either strictly increasing or strictly decreasing along each segment. Thus the stable manifold of the interior saddle equilibrium implicitly defines a one-one mapping from q to ψ.

A phase portrait for the case $F(a) = a^2$ is shown in Figure 2.

3.4.2 Properties of the optimal solutions

In this section, we analyze and summarize the properties of the optimal trajectories. The structure of the phase plane shows that an optimal strategy, $\mathbf{a}$ is increasing in q, and that the trajectory of $q(t)$ under this strategy moves monotonically towards the unique interior equilibrium demonstrated in the previous subsection. This equilibrium corresponds to the target in the case of a uniform distribution discussed in Section 3.3.

Equations (43), (45) and (39) give the following for the optimal triple $(q^*, \mathbf{a}^*, \psi^*)$:

$$
\begin{aligned}
\dot{\psi} &= (r+k)\psi - 2aq, && (48)\\
0 &= q^2 - k\psi F'(a), && (49)\\
\dot{q} &= k(1 - F(a) - q)\,. && (50)
\end{aligned}
$$

We use these to get an equation for the time evolution of the optimal $\mathbf{a}(t)$. The maximality condition, Equation (49), gives:

$$
F'(a) = \frac{q^2}{k\psi} \Rightarrow \psi = \frac{q^2}{kF'(a)}. \tag{51}
$$

Differentiation of Equation (51) with respect to time and substituting expressions for $\dot{q}$, ψ and $\dot{\psi}$ yields:

$$
\begin{aligned}
F''(a)\dot{a} &= \frac{2q\dot{q}\psi - q^2\dot{\psi}}{k\psi^2} && (52)\\
\Rightarrow \dot{a} &= \frac{F'(a)}{qF''(a)}\,[\,2k(1-F(a)-q) - (r+k)q + 2kaF'(a)]\,. && (53)
\end{aligned}
$$

Equations (39) and (53) give two differential equations for $q(t)$ and $\mathbf{a}(t)$ along an optimal trajectory. We show directly that this system has a unique equilibrium in the interior of the unit square.

Any equilibria of the system will lie on the intersection of the q and a nullclines, given by:

$$
\begin{aligned}
&\dot{q} = 0 \quad \Rightarrow \quad q = N_1(a) = 1 - F(a)\,, && (54)\\
&\dot{a} = 0 \quad \Rightarrow \quad F'(a) = 0, \text{ or } q = N_2(a) = \frac{2k\,[\,1 - F(a) + aF'(a)]}{r+3k}\,.
\end{aligned}
$$

We remark that the assumptions on F imply that $F'(a) > 0$ so that N_1 is strictly decreasing and concave for $a \in (0,1)$, and $N_1(0) = 1, N_1(1) = 0$.

For the other nullcline, we calculate that $N_2(0) = \dfrac{2k}{r+3k} < 1$, and the derivative of $N_2(a)$ is always positive:

$$
\frac{dN_2}{da} = \frac{2k}{(r+3k)}\,[\,-F'(a) + F'(a) + F''(a)] > 0
$$

since $F''(a) > 0$. This means that the graph of $N_2(a)$ starts below the graph of N_1 at $a = 0$, ends up above the graph of N_1 at $a = 1$, and the two graphs intersect exactly once, since N_2 is strictly increasing while N_1 is strictly decreasing. Therefore, there is *exactly one* equilibrium where the graphs of $N_1(a)$ and $N_2(a)$ intersect.

If $F'(0) = 0$, there is another equilibrium at $a = 0, q = 1$. Note that, if $F'(0) = 0$, trajectories in the qa-plane that converge to the equilibrium at $q = 1, a = 0$ correspond to trajectories in the ψq-plane with $\psi \to \infty$, since

$$a \to 0 \Rightarrow g(u) \to \frac{1}{F'(0)} \Rightarrow \psi \to \frac{1}{kF'(0)} \to \infty\,.$$

By the boundedness of ψ on optimal trajectories (Proposition 4, condition 3), trajectories in the qa-plane that converge to $(a, q) = (0, 1)$ are not optimal. Therefore, as in the previous section, optimal trajectories must lie on the stable manifold of the interior equilibrium.

The phase plane of the system, with nullclines depicted and arrows indicating the direction of the vector field is given in Figure 3. For this example, we have taken $F(a) = a^2$, so that we are in the case $F'(0) = 0$.

Let (q^{**}, a^{**}) denote the coordinates of this interior equilibrium, and denote by $\mathcal{W}_s$ the stable manifold of the saddle. Let (q_1, a_1) be a point on $\mathcal{W}_S$ with $q_1 < q^{**}$. Since this portion of $\mathcal{W}_S$ lies above N_2 and below N_1, the vector field at (q_1, a_1) is in the first quadrant, i.e. $\dot{q}(t) > 0$ and $\dot{\mathbf{a}}(t) > 0$, so that $q(t)$ and $\mathbf{a}(t)$ increase monotonically towards (q^{**}, a^{**}). In other words, $\mathcal{W}_S$ is strictly increasing in this region (as a function of q) so that, for every initial $q_1 < q^{**}$ there is a unique optimal strategy starting at a_1, and both $q(t)$ and $\mathbf{a}(t)$ will increase monotonically over time. A similar argument shows that for an initial $q_2 > q^{**}$ there is a unique optimal strategy starting at a_2 on $\mathcal{W}_S$, and both $q(t)$ and $\mathbf{a}(t)$ decrease monotonically towards (q^{**}, a^{**}) from this value.

By Pontryagin's Maximum Principle, Proposition 4, this phase portrait gives *all* candidates for optimal trajectories. To determine which of the orbits given by this phase portrait are optimal, we appeal to the analysis in subsection 3.4.1 to conclude that any optimal trajectory must lie on the stable manifold of the interior saddle equilibrium.

We can interpret this result as follows: $\mathcal{W}_s$ determines a one-to-one mapping from values of q to strategies which gives the *stationary policy*, $\alpha(q) = \mathbf{a}$ analogous to that described for the case of the uniform distribution in Section 3.3. Here we see that α is an *increasing* function of q. Also, as in the uniform case, $\alpha(q^{**}) = a^{**}$ is the policy such that $\dot{q} = 0$, which corresponds to the *target* in Section 3.3. The target is given analytically as the value (q^{**}, a^{**}) such that

$$\begin{aligned} q^{**} &= 1 - F(a^{**}) = \frac{2k}{r + 3k}\,[\,1 - F(a^{**}) + a^{**}F'(a^{**})] \\ \Rightarrow a^{**} &= \frac{r + k}{2kF'(a^{**})} \end{aligned} \tag{55}$$

Some typical strategies and demand trajectories are shown in the right two panels of Figure 3.

As mentioned in Section 3.2, the target demand is strictly less than the REE (Rational Expectations Equilibrium) if F is a strictly convex and "regular" cdf and if $r > 0$. This is the content of the following proposition, whose proof is in Appendix 7.4.

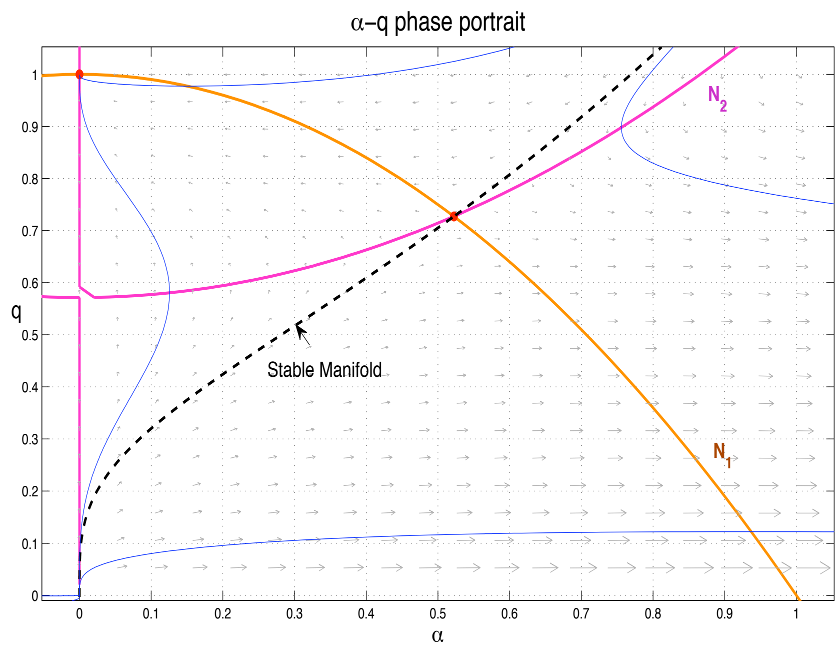


Figure 3: Phase portrait in the aq-plane showing the nullclines, $N_1 : \dot{q} = 0$ and $N_2 : \dot{a} = 0$. The interior equilibrium is a saddle, and the only optimal trajectories, where $0 < a < 1$ lie on the stable manifold of this equilibrium. In this example, $F(a) = a^2, k = 1, r = .5$.

Theorem 6 *If $F(a)$, the distribution of demand as a function of price, is regular and strictly convex, then the target demand, given by Equation (55) is less than or equal to the REE demand, given by Equation (17), with equality only in the case $r = 0$.*

3.4.3 Example: the quadratic case

In this section we work out the details in the case of a quadratic distribution function:

$$F(a) = \begin{cases} a^2 & 0 \leq a \leq 1 \\ 0 & a < 0 \\ 1 & a > 1 \end{cases} .$$

In the (q, a) variables, we have the following equations of motion:

$$\dot{a} = \frac{2a}{q}\left(2k(1-a^2-q) - (r+k)q + 4ka^2\right) \tag{56}$$

$$\dot{q} = k(1-a^2-q) \tag{57}$$

The interior equilibrium is at:

$$1 - a^2 = \frac{2k(1-a^2+2a^2)}{r+3k} \Rightarrow \frac{r+3k}{2k}(1-a^2) = 1 + a^2 \Rightarrow a^2 = \frac{2k}{r+5k}\left(\frac{r+k}{2k}\right)$$

$$\Rightarrow a = \sqrt{\frac{r+k}{r+5k}} \Rightarrow q = 1 - \frac{r+k}{r+5k} = \frac{4k}{r+5k} . \tag{58}$$

For an initial value $q(0) = q_0$, the optimal policy, $\mathbf{a}^*(q)$ will start at the corresponding point on the stable manifold of the interior equilibrium. This can be determined by explicitly using the formula for the cdf, F:

In terms of the adjoint variable, ψ, we first calculate $G(u) = F^{-1}(u) = \sqrt{u}$, $g(u) = G'(u) = \dfrac{1}{2\sqrt{u}}$, so that $g^{-1}(y) = \dfrac{1}{4y^2}$. Along optimal trajectories,

$$u = g^{-1}(k\psi/q^2) = \frac{1}{4(k\psi/q^2)^2} = \frac{q^4}{4k^2\psi^2} \Rightarrow G(u) = \frac{q^2}{2k\psi}.$$

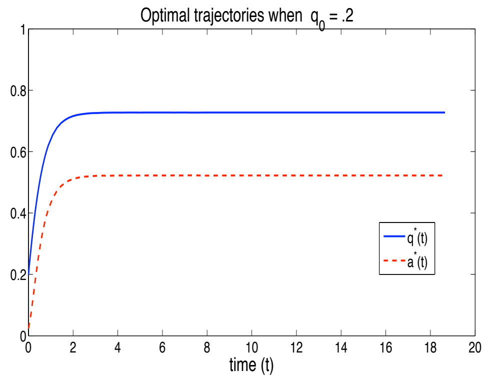
The equations of motion are, therefore:

$$\dot{q} = k\left(1 - \frac{q^4}{4k^2\psi^2} - q\right) , \tag{59}$$

$$\dot{\psi} = (r+k)\psi - \frac{q^3}{k\psi} . \tag{60}$$

The target equilibrium is given by the solution to

$$m\psi^2 = \frac{q^3}{r+k}, \quad q = 1 - \frac{q^4}{4k^2\psi^2} = 1 - \frac{q^4(r+k)}{4kq^3} = 1 - \frac{(r+k)q}{4k} ,$$



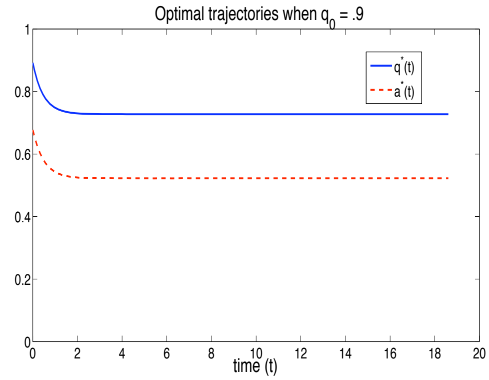


Figure 4: Sample optimal trajectories in the quadratic case: $F(a) = a^2, k = 1, r = .5$. Left: $q_0 = .2$, Right: $q_0 = .9$.

which is consistent with the previous calculation. In the $q\psi$-plane, the target equilibrium is:

$$q = \frac{4k}{r+5k}, \qquad \psi = \sqrt{\frac{q^3}{k(r+k)}} = \frac{8k}{r+5k}\sqrt{\frac{1}{(r+5k)(r+k)}}.$$

See Figure 4 for graphs of sample optimal trajectories when $k = 1, r = .5$.

3.5 Concave Distributions of ConsumerTypes

We now extend the example above for any cumulative distribution function F of consumer types that is *concave but not purely linear*. While there is no optimal stationary policy in the classical sense, we shall show that there is an optimal "measure-valued policy" which is independent of F, and whose structure is very similar to that of the optimal target policy in the case of a uniform cdf F. (See further remarks at the end of this subsection.)

We retain our basic assumption that F is absolutely continuous and strictly increasing on the unit interval, and in particular, that $F(0) = 0, F(1) = 1$. In addition, we assume that F is concave but not purely linear. We extend the set of permissible controls to include those that specify *a probability measure over the set of feasible controls at each time t*. Therefore, at any time t, define a *generalized control* as a probability measure $\psi(\cdot, t)$ on $[0, 1]$, and Ψ as the set of all such generalized controls. For a measure ψ on $[0, 1]$, define

$$\begin{aligned}
\bar{a}(\psi) &= \int a d\psi(a), \\
\bar{F}(\psi) &= \int F(a) d\psi(a), \\
\overline{m}(q, \psi) &= k\left[1 - \bar{F}(\psi) - q\right].
\end{aligned}$$

Then the immediate return at time t from $\psi(\cdot, t)$ is

$$\bar{a}(\psi[\cdot, t])\,[q(t)]^2\,, \tag{61}$$

and the law of motion is

$$q'(t) = \overline{m}[q(t), \psi(\cdot, t)]. \tag{62}$$

The monopolist's goal is to maximize the total discounted immediate return. Based on Gamkrelidze (1978), one can show that there is an optimal generalized control. As in the case of the uniform cdf we use the Bellman/Blackwell theory of dynamic programming to completely characterize the optimal generalized control. By a generalization of Blackwell's Theorem, the optimal generalized control can be represented by a policy that maps each current state into a probability measure on the unit interval. In fact, we shall show that this mapping takes a form that is analagous to the target policy in the uniform case. Let Φ be the set of measures on $[0, 1]$ such that the probability is concentrated on the endpoints of the interval, and let a *generalized target policy with target* s $(0 < s \leq 1)$ be a mapping φ_s from states q to Φ of the form:

$$\begin{aligned} \varphi_s(1, q) &= \begin{cases} 0, & q < s, \\ 1 - s, & q = s, \\ 1, & q > s. \end{cases} && (63) \\ \varphi_s(0, q) &= 1 - \varphi_s(1, q). && (64) \end{aligned}$$

Theorem 7 *If the cumulative distribution of consumer types F is concave and not purely linear, and satisfies all the other assumptions of this subsection, then the generalized target policy with target*

$$\sigma = \frac{2k}{3k + r}. \tag{65}$$

is optimal among all generalized policies

The proof of this result is presented in Appendix 7.5.

Note that the optimal generalized policy is *the same for all cdfs F in the class covered by the theorem.*

The interpretation of a generalized policy presents some difficulties. It is tempting to think of a generalized policy as a randomized strategy, although the rationale is different from that in the theory of games. An alternative interpretation is suggested by the example of "viscous demand" analyzed in (Radner and Richardson, 2003). There it is shown that the supremum of the monopolist's profit can be approximated arbitrarily closely by ordinary policies that ocscillate arbitrarily quickly. However, following that line of thought would take us beyond the scope of the present paper.

4 Extensions of the Model of Myopic Consumers

We now present two extensions of our model of myopic consumers. In each extension, we maintain the assumption of bounded attention and the uniform distribution of consumer types. Each extension establishes that the seller's optimal price trajectory retains the "target policy" structure derived in Theorem 3, although the level of the demand target and steady-state price vary in the two cases.

4.1 An Evolving Consumer Population

In the first extension we consider a situation in which, rather than being static, the population of potential customers for the network good evolves over time. We return to the discrete-time model of Section 2 to characterize this precisely, while maintaining each of its other assumptions. The length of each period is $h > 0$, and each period n starts at time $t = nh$. A fraction ch of consumers of each type (both adopters and non-adopters) is *replaced* in each period. That is, a fraction ch of existing consumers "leave" and an equal fraction ch of new consumers "arrive" and are added to the pool of potential customers. Each newly-arrived customer is assumed to be initially a non-adopter. The size of the total set of potential customers therefore remains constant, although it has a constant "rate of replacement", which is proportional to the parameter c.

Following the replacement at the beginning of each period, we assume that an equal fraction kh of consumers of each type pay attention to the new price in each period, and an equal fraction $(1-kh)$ do not. Here we face a modeling decision about how to model "myopic" expectations of the next period's demand. We make the simplest assumption, that the consumers who make their subscription decisions ignore the phenomenon of replacement, and so

$$q_E(t,h) = q(t-h). \tag{66}$$

Correspondingly, the control variable is

$$\boldsymbol{a}(t) = p(t)/q(t-h). \tag{67}$$

Proceeding to the continuuous-time model, as in Sections 2.2, 3.1, and 3.3, if at time t the demand and price are $q(t)$ and $p(t)$, respectively, then the time rate of change of demand (the law of motion) takes the form,

$$\begin{aligned} q'(t) &= m_c[q(t), \boldsymbol{a}(t)], & (68)\\ m_c(q,a) &= k\,(1-a) - (k+c)q, \quad 0 \le q \le 1, \quad 0 \le a \le 1. & (69) \end{aligned}$$

For example, for any price trajectory that causes an increase in demand, the positive rate of consumer replacement slows down the rate at which demand increases, in comparison with the case in which $c = 0$.

For any demand q, the control a that makes the time-derivative zero - the *stay-where-you-are control* - is

$$A_c(q) = 1 - \left(1 + \frac{c}{k}\right) q.$$

A stationary target policy π_c with target s $(0 < s < 1)$ is defined by

$$\pi_c(q) = \begin{cases} 0, & q < s, \\ A_c(s), & q = s, \\ 1, & q > s. \end{cases} \tag{70}$$

The next theorem states that the seller's optimal price trajectory is generated by a target policy, and the target is a strictly decreasing function of the replacement rate, c.

Theorem 8 *The seller's optimal price trajectory is generated by a target policy, and the optimal demand target is*

$$\sigma_c = \frac{2k}{3(k+c)+r}. \tag{71}$$

The price at the target is

$$P_c(\sigma_c) = A_c(\sigma_c)\sigma_c = \frac{2k[(k+c+r)]}{[3(k+c)+r]^2}. \tag{72}$$

The target and the price at the target are both decreasing in the replacement rate, c.

Note that the last statement imples that the steady state profit is also decreasing in c.

The proof of the theorem is presented in Appendix 7.6.

4.2 Myopic and "Stubborn" Consumers

Finally, we describe some properties of the monopolist's optimal price trajectory when consumers are something between being myopic and being "stubborn". Rather than basing their expectation of total demand in the next period on the current period's demand level, consumers who pay attention to the monopolist's price announcement partly base their prediction on a stubborn assessment, ω, of the total demand for the good. The extent to which they base their expectation on ω is determined by a parameter γ, where $0 \le \gamma \le 1$, and

$$q_E(t,h) = \gamma q(t-h) + (1-\gamma)\omega. \tag{73}$$

Proceeding as in Section 2, it follows that

$$q_E(t) = \gamma q(t) + (1-\gamma)\omega, \tag{74}$$

and the control variable is

$$\boldsymbol{a}(t) = \frac{p(t)}{\gamma q(t) + (1-\gamma)\omega}. \tag{75}$$

The law of motion is

$$q'(t) = m[q(t), \boldsymbol{a}(t)], \tag{76}$$

$$m(q,a) = k(1-a-q), \quad 0 \le q \le 1, \quad 0 \le a \le 1, \tag{77}$$

and the ”stay-where-you-are” control is

$$A(q) = (1-q) \tag{78}$$

(cf. 3.2). A stationary target policy π with target σ $(0 < \sigma < 1)$ is defined by

$$\pi(q) = \begin{cases} 0, & 0 < q < \sigma, \\ A(\sigma), & q = \sigma, \\ 1, & \sigma < q \le 1. \end{cases} . \tag{79}$$

For a given γ and ω, let π^* be the optimal target policy, and denote its target as $\sigma(\gamma, \omega)$. Our final theorem confirms that the structure of the pricing policy prescribed in Section 3.3 is qualitatively robust to this extension. However, the demand target is always lower.

Theorem 9 *(a) The monopolist's optimal price trajectory is generated by the target policy with target $\sigma(\gamma, \omega)$.*

(b) $\sigma(\gamma, w)$is strictly increasing in γ, and has the following values at its end points:

$$\sigma(0, \omega) = \frac{k}{2k+r}, \tag{80}$$

$$\sigma(1, \omega) = \frac{2k}{3k+r}. \tag{81}$$

(c) $\sigma(\gamma, \omega)$ is strictly decreasing in ω.

The proof of the theorem is presented in Appendix 7.7.

Notice that the demand target of Theorem 3 is a limiting case of the demand target above, when $\gamma = 1$. Parts (b) and (c) have a simple intuitive explanation. An increase in the installed base for a network good benefits the seller in two ways: through the direct increase in demand, and by increasing the willingness to pay of consumers. It is the latter property that increases the monopoly demand for the good beyond what a normal good would enjoy. Therefore, at any given stubborn expectation ω, a decrease in the weight γ placed on the current demand makes the good seem ”less like” a network good, and more like a normal good with an exogenously specified value that is proportionate to ω, thus reducing the steady-state user base that the seller finds optimal. Correspondingly, for any given γ, an increase in ω reduces the fraction of perceived user value that is influenced by actual current demand, and increases the corresponding fraction influenced by the ”stubborn” expectation. One might therefore expect outcomes that are qualitatively similar to those of Theorem 3 in the base model of Section 3.2 if, rather than being a pure network good as we have assumed, a fraction of the willingness-to-pay for the good is independent of the demand q.

5 Summary of Results and Future Research Directions

We have explored several variations of a model of optimal dynamic monopoly pricing of a network good with assumptions about consumer rationality that are more realistic than those embodied in the theory of rational expectations. First, we assume that consumers have bounded attention, which results in consumers varying in the delay with which they respond to changes in price announced by the monopolist. Second, we assume that consumers are unable to correctly forecast the demand at any time, but instead use some heuristic method for forecasting. In particular, if consumers are myopic in their demand forecasts, then it will be optimal for the monopolist to use a *target policy* with the following properties: when current demand is below the target, the price is low; when current demand is above the target, the price is high; and when current demand is at the target, the price is chosen to keep demand stationary. The target could be interpreted as the level of adoption below which the monopolist invests in building a user base, and above which the monopolist profits from exploiting her installed base.

This last outcome is different from what is obtained from a model in which the process of demand adjustment is based on the correct formation of expectations by rational consumers after each price announcement. Moreover, our prescribed price path seems to be similar to those often observed in practice. It is beyond the scope of this paper to survey the empirical literature on the pricing of network goods. Furthermore, our theoretical model is too simple to cover the complexities that one usually encounters in such markets. For one thing, pure monopolies are rare, although the occurrence of firms with considerable market power is not. For another, markets for network goods typically include several categories of customers, take the form of "'two-sided markets," etc. We confine ourselves here to describing two examples that illustrate these features, and yet give qualitative support to our general conclusions about pricing strategy.

An early example of a network good was telephone service, both local and long-distance (LD). Based on Alexander Graham Bell's 1876 patent of the telephone, AT&T (Bell System) was a legal monopolist until the patent's expiration in 1894. AT&T remained a monopolist in LD through acquisition of additional patents, as well as in local service in many areas. Thus, during the early years of the 20th century, AT&T was the dominant firm in local service in many areas, and the dominant firm in LD in the country. In this period, it gradually increased its dominance, attaining the status of an unregulated monopoly in many local jurisdictions (Gabel, 1994). This period ended in 1913 when AT&T settled an antitrust suit by the U.S. Attorney General with an out-of-court agreement (the "Kingsbury Commitment"), at which time its business began to be partially regulated (Temin, 1987, p. 10).

With respect to our model, the case of AT&T up to 1894 fits well. Although some competition after the end of the patent was to be expected, it would be hard to quantify and describe defensive strategies by AT&T during the patent period. The period between 1894 and 1913 presents several problems. First, AT&T was not a pure monopoly; there were competitors of varying importance in both local and LD service. Second, AT&T's local service was provided by local operating companies, which were not typically wholly-owned by AT&T, whereas LD service was provided by the parent AT&T company through interconnection with the local companies, not directly to the customers. Thirdly, customers were classified as "residence" or "business," with separate prices for each category. Fourth, before the Kingsbury Commitment, AT&T refused to interconnect with independent companies, thereby depriving their customers of high quality LD (Noll & Owen, p. 330).

Dealing adequately with these problems would require an elaboration of our model. Problems 2 and 3 would lead to a more complex optimization problem, but would probably not require a basic change in methodology. However, Problem 1would require a game-theoretic analysis, in which the players include both AT&T and its competitors. In particular, AT&T was accused of using "predatory pricing" to drive its competitors out of business, an illegal practice. There was some dispute among economists about whether AT&T's pricing was predatory. On the one hand,

"...at the start of the century AT&T's managers believed that residential service should be provided a rate that was less than the cost of direct service. This "loss" was more than made up by the higher charges that could then be set for business lines. This below-cost price is not an example of predation because the intent was to bring new customers onto the network and thereby raise the value to the existing customers." (Gabel, 1994, p. 545).

[Note the reference to the "network goods effect" in the last sentence above.] On the other hand, Gabel presents evidence of the "predatory" intent of AT&T's pricing, and of other anti-competitive behavior.]

A more recent example is PayPal. However, this is not a "pure example," either, since there are two classes of customers, individuals and businesses, and there is competition in the relevant market. PayPal provides a service that allows individuals to send money to other individuals and businesses via the Internet more cheaply and more conveniently than by check, bank transfer, or the usual credit-card transactions. Initially, PayPal offered a small rebate to individuals who signed up for their service, and charged nothing for payments made to other account holders. Also, the charges paid by businesses receiving money were significantly less than those charged by the standard credit-card companies. This strategy was followed for a few years, during which their customer base customer base grew rapidly. At this point PayPal started to make money by eliminating the rebates to new individual customers, narrowing the definition of "individual customer," and increasing the charges to business customers, although the latter were still less than standard credit-card charges, and "individual" customers were not charged for sending money. By the time it entered the profitable phase, PayPal had established a dominant position in its market. (For details, see Mendelson, 2002, for details.)

To sum up the preceding discussion, our model is too simple to fit all of the aspects of the examples, but its results provide a qualitative insight into pricing behavior that goes beyond that provided by standard economic theory using “unboundedly rational” consumers.

We note that our results are thus far based in part on certain assumptions about the distribution within the consumer population of the utility for the network good, and one topic for future research is to explore the implications of other assumptions.

We have shown that our main results are robust to a number of extensions. During our analysis of these extensions, we contrast how the demand target, and the price that keeps demand at that target, vary with changes in the rate at which myopic consumers pay attention to price changes, and the rate at which there is turnover in the potential consumer base. Similarly, we have examined how the target demand and the steady-state price vary with the extent to which a fraction of the consumers base their assessment of future demand on a ”stubborn” expectation expectation of future demand. While the extent to which each of these parameters characterize specific network goods and industries is an empirical question, managers who assess these parameters appropriately for their products can use our theory as a general basis on which to fine-tune the details of their pricing strategy.

A natural extensions of our theory would model myopic consumers who make imperfect observations of demand. Additionally, we model the rate of attention and the rate of replacement as exogenous variables, although sellers may in fact be able to influence these by making advertising and branding investments. An interesting direction for future research would be to extend our model to permit investments of this kind. This may also indicate how such investments should vary over time, since the impact of a change in either parameter on the seller’s profits depends on its timing.

We have noted that most markets for network goods are not ”’pure monopolies,” nor are they ”purely competitive. The analysis of oligopolies with unboundedly rational firms but boundedly rational consumers would naturally use the theory of dynamic strategic games. One of us has studied a duopoly model from this point of view, with consumers with bounded attention, but with no network effects (Radner, 2003). The results have the same flavor as those of the present paper, but there is a continuum of equilibria. The extension to models with network effects would appear to pose significant analytical challenges.

Finally, building on recent models of local network effects (Sundararajan, 2005b, Tucker, 2004), the rate at which consumers pay attention to products may not be constant across the population, but may be influenced by the adoption decisions of other consumers whom one is locally ”connected” to. This represents an interesting extension to our model of bounded rationality, one that is especially pertinent to network goods, and a direction of research we hope to pursue in the future.

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6 Bibliographic Notes and References

Our results add to a broad theoretical and applied literature on network effects[1]. The seminal papers, by Katz and Shapiro (1985, 1986) and by Farrell and Saloner (1985, 1986), and a large majority of the literature that followed, have focused more on analyses of oligopoly rather than monopoly pricing, and in contrast with our paper, almost always use the model of consumer behavior embodied in the concept of rational expectations. An exposition of the theory of rational expectations in economic analysis can be found in Radner (1982), and its use in defining "fulfilled-expectations" outcomes in the presence of network effects is described in Katz and Shapiro (1985).

The static model of network effects underlying our dynamic model is based on Rohlfs (1974), who provided the first model of monopoly pricing for a network good, and on the subsequent exposition by Economides (1996). The discussion of dynamic pricing closest to ours that we are aware of is by Dhebar and Oren (1985). Their formulation differs from ours in several respects, but in general it can be interpreted as incorporating a kind of bounded rationality on the part of the consumers. A special case of their model (Section 4 of their paper, with the parameter $\alpha = 0$) leads to a law of motion that is mathematically isomorphic to our benchmark case of purely myopic consumers with a uniform distribution of types (Section 3.3 above), and for that case they derive the optimal price trajectory. For their general model they discuss properties of the optimal price trajectory if the initial customer base is "small," and indicate that the monopolist will eventually price to keep the demand at a steady state (what we would call a "target"), but they do not obtain a complete characterization of the optimal policy. (For a related discussion, see Dhebar and Oren, 1986).

Cabral, Salant and Woroch (1999) study the dynamic pricing of a durable network good in a two-stage model with rational consumers, where they illustrate how the presence of network effects may overturn Coasian dynamics and lead to first period pricing that is lower than second-period pricing. Fudenberg and Tirole (2000) model dynamic pricing by a monopolist who sells a network good to overlapping generations of consumers who live for two periods, although they assume perfect rationality on the part of their consumers. Related papers that study single-period monopoly price discrimination based on a model of rational demand expectations include those by Oren, Smith and Wilson (1982) and by Sundararajan (2004, 2005a).

Shared information systems often display network effects, and our model may thus inform the literature on the adoption of such systems. For example,

[1] Comprehensive surveys of models of network effects can be found in Farrell and Klemperer (2004), and in Economides (1996).

Riggins, Kreibel and Mukhopadhyay (1994) model the two-stage adoption of an interorganizational system with positive and negative adoption externalities. While their model uses the standard notion of fulfilled expectations, they do discuss a case with myopic adopters. They show that subsidies are often necessary to induce adoption in the first stage, a result qualitatively similar to ours. Wang and Seidmann (1995) examine a related problem for the adoption of EDI in a two-sided network of buyers and suppliers, incorporating not just positive network effects from higher adoption, but negative (or "competitive") externalities imposed by a buyer (supplier) on other buyers (suppliers) by their adoption; a similar tradeoff is modeled by Westland (1992) as well. Nault and Dexter (2005) provide a general model of pricing by a monopoly provider of a "network alliance" service, wherein the number of adopters and the investments made by these adopters each influence the demand for every adopter's products. They show that, when combined with an exclusivity arrangement with participating members, the provider's optimal commission level restricts membership in the alliance.

The bounded rationality of agents in our model leads to a demand adjustment process that is "viscous", and is similar in this regard to the model of Radner and Richardson (2003) and of Radner (2003). These papers, however, model a good of constant value, and do not study network effects – rather, the rate at which demand adjusts to price announcements by sellers varies in proportion to the magnitude of the difference between the announced price and each consumer's willingness to pay. A model of network effects with boundedly rational consumers that is closely related to ours is by Arthur (1989), who studies adoption choices between two competing durable network goods. In his model, myopic consumers make their choices based on the current market share of each good. He shows that over time, the market share of one of the goods will tend to 100%, though one cannot predict ex-ante which of the two goods it would be, and outcomes are path-dependent. He does not model the choice of price, dynamic or otherwise, instead implicitly assuming that prices are fixed and exogenously specified. An extension of our model of bounded rationality to one with two competing sellers of network goods may provide insight into whether his results continue to hold when sellers can strategically alter their prices over time, although this extension remains unsolved at this time.

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7 Appendix

7.1 Uniform Distribution of Consumer Types

Consider an arbitrary target policy, π, and let s denote the corresponding target. Suppose that the initial state q is less than s. We shall first calculate the optimal target, s, starting from $q(0) = q$. Until it reaches s, $q(t)$ solves

$$q'(t) = k[1 - q(t)]. \tag{82}$$

The unique solution to the differential equation in (82) with the initial condition $q(0) = q$ is:

$$q(t) = 1 - e^{-kt}(1 - q). \tag{83}$$

As a consequence, if the initial state is $q < s$, the time T at which $q(T) = s$ solves

$$s = 1 - e^{-kT}(1 - q), \tag{84}$$

and therefore

$$T = \frac{1}{k} \ln \left(\frac{1 - q}{1 - s} \right). \tag{85}$$

Therefore, under the policy π, the value function is:

$$V_\pi(q) = sP(s) \left(\int_T^\infty e^{-rt} dt \right), \tag{86}$$

where $P(s) = qA(s)$, which simplifies to:

$$V_\pi(q) = \frac{1}{r} \left(\frac{1 - q}{1 - s} \right)^{-(r/k)} s^2(1 - s). \tag{87}$$

Equation (87) can be rewritten as:

$$V_\pi(q) = \frac{1}{r} \left[(1 - q)^{-(r/k)} \right] \left[s^2(1 - s)^{\left(1 + \frac{r}{k}\right)} \right]. \tag{88}$$

Differentiating (88) with respect to s yields:

$$\frac{dV_\pi(q)}{ds} = \frac{1}{r} \left[(1 - q)^{-(r/k)} \right] \left[2s(1 - s)^{\left(1 + \frac{r}{k}\right)} - \left[\left(1 + \frac{r}{k}\right) s^2(1 - s)^{\left(\frac{r}{k}\right)} \right] \right]. \tag{89}$$

For $0 < s < 1$, the right-hand side of (88) is strictly quasiconcave in s. Additionally, $\dfrac{dV_\pi(q)}{ds} = 0$ at $s = 0$ and $s = 1$, which are minima for which $V_\pi(q) = 0$ (In fact, both these statements are true for any function of the form $Kx^a(1-x)^b$ for $a, b \geq 1$).

As a consequence, the value $\sigma \in [0, 1]$ that maximizes $V_\pi(q)$ with respect to s solves

$$2(1 - \sigma) = \left(1 + \frac{r}{k}\right) \sigma, \tag{90}$$

which yields

$$\sigma = \frac{2k}{3k+r}. \tag{91}$$

The corresponding price is

$$\frac{2k(k+r)}{(3k+r)^2}. \tag{92}$$

(Note that when $r = 0$, this yields the rational expectations equilibrium quantity $q^* = 2/3$ and price $p^* = 2/9$.

Correspondingly, if the initial state is $q > s$, until it reaches s, $q(t)$ solves

$$q'(t) = -kq(t), \tag{93}$$

which corresponds uniquely to the demand trajectory:

$$q(t) = qe^{-kt}, \tag{94}$$

and a similar computation yields the value function:

$$V_\pi(q) = \frac{1}{2k+r}\left(q^2 + q^{-\frac{r}{k}} s^{(2+\frac{r}{k})}\left[\frac{2k(1-s) - rs}{r}\right]\right) \tag{95}$$

which is also maximized with respect to s by the value of σ in (91).

Denote by π^* the target policy with target σ. We shall now show that the target policy π^* is optimal. For any given policy α (not necessarily a target policy), if its value function is continuously differentiable, then the corresponding *Bellmanian Functional* is defined by

$$B_\alpha(q,a) = aq^2 - rV_\alpha(q) + V'_\alpha(q)\, m(q,a). \tag{96}$$

According to a well-known proposition, a policy α is optimal if it satisfies the Hamilton-Jacobi-Bellman condition:

$$B_\alpha(q,a) \leq 0 \text{ for all } q, a. \tag{97}$$

An alternative form for the last condition is

$$\alpha(q) = \arg\max_a B_\alpha(q,a). \tag{98}$$

This follows from the fact that, for all q,

$$B_\alpha[q, \alpha(q)] = 0,$$

which is readily verified. (In fact, this identity is true for any stationary policy whose value function is C^1.)

Hence, from the above,

$$B_{\pi^*}[q, \pi^*(q)] = \pi^*(q)q^2 - rV_{\pi^*}(q) + k\,(1 - q - \pi^*(q))\, V'_{\pi^*}(q) = 0. \tag{99}$$

It will be useful to define

$$G(q) \equiv q^2 - kV_\pi'(q). \tag{100}$$

and write $B_{\pi^*}(q,a)$ in the form,

$$\begin{aligned} B_{\pi^*}(q,a) &= aq^2 - rV_\pi(q) + k\left(1-q-a\right)V_\pi'(q) & (101) \\ &= -rV(q) + k(1-q)V'(q) + aG(q). & (102) \end{aligned}$$

Thus $B_{\pi^*}(q,a)$ is linear in a, and the coefficient of a is $G(q)$. Hence

$$\arg\max_a B_{\pi^*}(q,a) = \begin{cases} 0, & \text{if } G(q) < 0, \\ 1, & \text{if } G(q) > 0. \end{cases} \tag{103}$$

Recall that the *stay-where-you-are* policy is defined by

$$\mathbf{A}(q) = 1 - q. \tag{104}$$

With this policy, $q(t) = q(0)$ for all $t > 0$, and its value function is

$$V_{\mathbf{A}}(q) = \frac{\mathbb{A}(q)q^2}{r} = \frac{(1-q)q^2}{r}. \tag{105}$$

Case 1. $0 < q < \sigma$**:** In this case $\pi^*(q) = 0$, and

$$B_{\pi^*}[q, \pi^*(q)] = -rV_{\pi^*}(q) + k(1-q)V_{\pi^*}'(q) = 0, \tag{106}$$

so

$$\begin{aligned} kV_{\pi^*}'(q) &= \frac{rV_{\pi^*}(q)}{1-q}, & (107a) \\ G(q) &= q^2 - \frac{rV_{\pi^*}(q)}{1-q}, & (107b) \end{aligned}$$

and from (105), it follows that

$$G(q) < 0 \Leftrightarrow V_{\pi^*}(q) > V_{\mathbf{A}(q)}.. \tag{108}$$

Suppose that the monopolist uses the policy π such that $a = 0$ for $0 \le t < u$, and then switches to the "stay-where-you-are" policy $\mathbf{A}$ from then on. Since her price is zero for $0 \le t < u$, her resulting profit will be

$$g(u) \equiv e^{-ru}V_{\mathbf{A}}[q(u)] = \left(\frac{1}{r}\right) e^{-ru}[q(u)]^2[1-q(u)], \tag{109}$$

where $q(t)$ is determined by the differential equation $q'(t) = 1 - q(t)$ on the interval $[0, T)$, with $q(0) = q$. Note that

$$\begin{aligned} g(0) &= V_{\mathbf{A}}(q), & (110) \\ g(T) &= V_{\pi^*}(q), & (111) \end{aligned}$$

where, as before, T is the time at which $q(t)$ reaches the target σ under the policy π^*. Differentiating (109) with respect to u, and simplifying the resulting expression yields the derivative of g:

$$g'(u) = \left(\frac{1}{r}\right) e^{-ru}[q(u)][1-q(u)][2k-(3k+r)q(u)] > 0 \quad \text{for } 0 \le u < T, \tag{112}$$

since

$$q(u) < \sigma = \frac{2k}{3k+r} \text{ for } 0 \le u < T. \tag{113}$$

Hence, $g(u)$ is strictly increasing in u and so

$$V_{\pi^*}(q) = g(T) > g(0) = V_{\mathbf{A}}(q), \tag{114}$$

and $B_{\pi^*}(q, a)$ is maximized at $a = 0$.

Case 2. $q > \sigma$**:** In this case $\pi^*(q) = 1$. Using an analogous argument, we find that

$$kV'_{\pi^*}(q) = \frac{-rV_{\pi^*}(q) + q^2}{q}, \tag{115}$$

which leads to

$$G(q) > 0 \Leftrightarrow V_{\pi^*}(q) > V_{\mathbf{A}}(q). \tag{116}$$

The analogous expression for g is

$$\begin{aligned} g(u) &\equiv q\int_0^u e^{-rt}q(t)dt + e^{-ru}V_{\mathbf{A}}[q(u)] \qquad (117)\\ &= q\int_0^u e^{-rt}q(t)dt + \left(\frac{1}{r}\right) e^{-ru}[q(u)]^2[1-q(u)], \qquad (118)\end{aligned}$$

where $q(t)$ is defined by the differential equation

$$q'(t) = -kq(t), \quad q(0) = q \tag{119}$$

in $[0, T)$. Differentiating (118) with respect to u yields:

$$g'(u) = e^{-ru}\left([q(u)]^2 - [q(u)]^2[1-q(u)] + \frac{1}{r}\left(2q(u) - 3[q(u)]^2\right) q'(u)\right),$$

which simplifies to:

$$g'(u) = \frac{q(u)}{r} e^{-ru}\left[(3k+r)q(u) - 2k\right] q(u), \tag{120}$$

which is strictly positive, since

$$q(u) > \sigma = \frac{2k}{3k+r} \text{ for } 0 \le u < T. \tag{121}$$

Therefore, $g(u)$ is strictly increasing in u and so

$$V_{\pi^*}(q) = g(T) > g(0) = V_{\mathbf{A}}(q), \tag{122}$$

and therefore, $B_{\pi^*}(q, a)$ is maximized at $a = 1$.

Finally, note that, from (107a) and (115),

$$V'_{\pi^*}(\sigma^-) = V'_{\pi^*}(\sigma^+) = V'_{\pi^*}(\sigma) = \frac{\sigma^2}{k}, \tag{123}$$
$$G(\sigma) = 0, \tag{124}$$

so V_{π^*} is continuously differentiable for all q. Hence B_{π^*} satisfies the Bellman Optimality Condition, which completes the proof.

From the derivation of the optimal target policy above, we get the value function of the optimal policy:

$$V_{\pi^*}(q) = \frac{1}{r}\left[(1-q)^{-(r/k)}\right]\left[\sigma(1-\sigma)^{\left(1+\frac{r}{k}\right)}\right]. \tag{125}$$

A straightforward calculation shows that this expression is increasing in k if $q < \sigma$.

7.2 Proof of Proposition 4

Here we include an outline of the proof of the Maximum Principle, adapted from that given in (Weber, 2006) to the particular constraints of our problem. We sketch here the main arguments of the proof, with details only where modifications that are needed for our particular application.

We want to solve the infinite-horizon problem given in Equation 38

$$\begin{array}{lrcl} \text{(P)} \qquad \text{Maximize} & V(\mathbf{a}) & = & \displaystyle\int_0^\infty e^{-rt}\mathbf{a}(t)q(t)^2\,dt = \int_0^\infty e^{-rt}G(u)q^2\,dt\ , \\ \text{subject to} & \dot{q}(t) & = & k(1 - F(\mathbf{a}(t)) - q(t)) = k(1-u-q)\ , \\ & q(0) & = & q_0 \in [0,1], \qquad 0 \le a \le 1 \quad \text{or } 0 \le u \le 1. \end{array}$$

In what follows, it is more convenient to use $u = F^{-1}(a)$ as the control variable; we apologize for the clumsiness in notation.

Existence of a solution. The state space, Q, for the problem is the set of continuously differentiable real-valued functions $q : \mathbb{R} \mapsto [0,1]$, which is a nonempty, compact set. As remarked earlier, the assumptions on F imply that Q is invariant under the law of motion, and so admissible trajectories are uniformly bounded between 0 and 1. We also note that, for all $T > 0$,

$$\omega(T) = \int_T^\infty e^{-rt}G(u(t))q(t)^2\,dt \le \int_T^\infty e^{-rt}\,dt = \frac{e^{-rt}}{r}$$

defines a non-increasing positive function $\omega(T)$ with $\lim_{T\to\infty}\omega(T) = 0$.

The current value of the integrand in the value function: $h(u, q) = G(u)q^2$ satisfies

$$\frac{\partial^2 h}{\partial u^2} = q^2 G''(u) < 0$$

since $F = G^{-1}$ is convex, so h is strictly concave with respect to u. This concavity ensures the uniqueness of the optimal u^* in terms of q and the co-state variable, ψ. Theorem 3.6 in (Balder, 1983) can then be applied to ensure the existence of a solution to the infinite-horizon problem, (P).

Construction of the solution. As in (Weber, 2007) we define a sequence of finite time optimization problems, P_n, on time intervals $[0, T_n]$, where $T_n < T_{n+1}$ and $\lim_{n\to\infty} T_n \to \infty$. In our case, the control u is bounded by 1, so that $1 + |u| \leq 2$ for all controls. Denote by (q^*, u^*) an optimal pair for the infinite-horizon problem (P). We will require a sequence $v_n \in C^1(\mathbb{R}_+, \mathbb{R})$ and an increasing sequence of real constants $\epsilon_n \to \infty$ that satisfy:

$$\sup_{t\in\mathbb{R}_+} |v_n(t)| \leq 2, \quad \int_0^\infty e^{-rt} |v_n - u^*|^2 \, dt \leq \frac{1}{n}$$

and

$$\frac{1}{n(1+\epsilon_n)} \geq \omega(T_n)$$

Such sequences exist since $\{v_n\}$ can be obtained by successive approximations of u^* and $\{\epsilon_n\}$ by approximating the sequence $\dfrac{1}{n \cdot \omega(T_n)}$ which diverges. We can then define the sequence of optimization problems:

$$(P_n) \qquad \text{Maximize} \quad V(u) = \int_0^\infty e^{-rt} \left(G(u)q^2 - \frac{|v_n - u|^2}{1+\epsilon_n} \right) dt \ ,$$

subject to the same constraints as the problem (P). It can be shown (Weber, 2007) that there is an optimal u_n for each problem (P_n), and that, for any $T > 0$, these optimal solutions converge in L_2 to the optimal u^* for problem (P) as $n \to \infty$. As a consequence of this L_2 convergence, we can conclude that, for any $T > 0$ as $n \to \infty$:

$$u_n \to u^* \text{ in } L_2[0,T], \tag{126}$$
$$q_n \rightrightarrows q^* \text{ on } [0,T], \tag{127}$$
$$\dot{q}_n \to \dot{q}^* \text{ weakly in } L_1[0,T]. \tag{128}$$

See (Weber, 2007) for details and further references.

By the classical Pontryagin maximum principle (Pontryagin et al., 1962) there is an absolutely continuous adjoint variable, ψ_n for problem P_n, i.e. one that satisfies the Adjoint Equation, Equation 43, with

$$H_n(q_n, u_n, \psi_n) \overset{a.e.}{=} \sup_u \mathcal{H}_n(q_n, u, \psi_n) = \hat{H}_n(q_n, \psi_n)$$

where

$$\hat{H}_n = G(u_n)q_n^2 - \frac{|v_n - u_n|^2}{1+\epsilon_n} + \psi_n \left(-ku_n + k(1-q_n) \right)$$

Then we can show:

Claim 10 *$|\psi_n(0)|$ is bounded for large n.*

Proof: We drop the subscripts, n, for legibility. Using the Pontryagin Maximum principle for the finite horizon problem, the adjoint equation is:

$$\dot{\psi} = (r+k)\psi - 2G(u)q\,.$$

This has the solution:

$$\psi(T)e^{-(r+k)T} - \psi(0) = -2\int_0^T e^{-(r+k)t}G(u)q(t)\,dt\,. \tag{129}$$

For the finite time problem, the transversality condition implies that $\psi(T) = 0$ which implies that

$$\psi(0) = 2\int_0^T e^{-(r+k)t}G(u)q(t)\,dt\,.$$

Since $0 \leq G(u) \leq 1$ and $0 \leq q \leq 1$ we get the following bounds for $\psi(0)$:

$$0 \leq \psi(0) \leq 2\int_0^T e^{-(r+k)t}\,dt = \frac{2}{r+k}\,.$$

Since this will hold for all finite times, T, we have that $|\psi_n(0)|$ is bounded by $\dfrac{2}{r+k}$ independent of n, which proves the claim. ■

An identical argument to that given in (Weber, 2007) about the convergence properties of the solutions to the finite time problems as $T_n \to \infty$ shows that there is a sequence of adjoint solutions, ψ_n that converge uniformly on every interval $[0, T_n]$, to an adjoint solution, ψ, of the infinite-time problem. Furthermore $\dot{\psi}_n \to \dot{\psi}$ weakly in $L_1([0, T_n])$. We see that $\psi(t)$ is positive since $\psi_n > 0$ on $[0, T_n)$ for $n \to \infty$. Furthermore, the upper bound on $|\psi_n(0)|$ is also an upper bound of $|\psi_n(t)|$ for $t \in [0, T_n]$ for all n. This gives the bounds on the adjoint variable, ψ in Proposition 4.

7.3 Proof of Theorem 5

Along optimal trajectories, the maximality condition, Equation (44), implies that

$$g(u) = G'(u) = \frac{k\psi}{q^2} \Rightarrow u = g^{-1}\left(\frac{k\psi}{q^2}\right)\,. \tag{130}$$

To find the equilibria we calculate the nullclines by setting Equations (46) and (47) equal to 0. We can write u in terms of q and ψ using Equation (130) whence, after substitution into the nullcline equations, we get ψ in terms of q only. *Note the definitions of N_1 and N_2 differ from those in Section 3.4.2.*

$$\dot{q} = 0 \Rightarrow \quad \psi = N_1(q) \quad = \quad \frac{q^2 g(1-q)}{k}\,, \tag{131}$$

$$\dot{\psi} = 0 \Rightarrow \quad \psi = N_2(q) \quad = \quad \frac{q^2}{k}g\left(\lambda^{-1}\left(\frac{2k}{(r+k)q}\right)\right) \quad \text{for } q \neq 0\,. \tag{132}$$

where $L(u) = \ln G(u)$, and $\lambda(u) = L'(u) = \dfrac{g(u)}{G(u)}$. Since we assume that F is strictly increasing on $[0,1]$, we can define $G(1) = F^{-1}(1) = 1$. However the derivative of G is not well defined at $u(t) = 1$ or $u(t) = 0$, so the nullcline equations are only valid on the open interval $(0,1)$.

At an equilibrium where $q \neq 0$, we must have:

$$q = 1 - u = \left(\frac{2k}{r+k} \right) \frac{1}{\lambda(u)}.$$

Since $F'(a) > 0$ and increasing, we have $g(u) = G'(u) > 0$ and decreasing, so that $\lambda(u) = \dfrac{g(u)}{G(u)} > 0$ and decreasing as well, which means that $\dfrac{1}{\lambda(u)}$ is increasing, so that there is only one equilibrium at which $q \neq 0$.

We consider the case $q = 0$ separately: if $\dot{q}(t) = 0$ when $q(t) = 0$, we must have $u(t) = 1 \Rightarrow G(u(t)) = 1$, so that we could only have an equilibrium where $q = 0$ if $\psi = \dfrac{2q}{r+k} = 0$.

Note that $\lim_{q\to 1} N_1(q) = \lim_{u \to 0} \dfrac{g(u)}{k}$. On the other hand,

$$\lim_{q\to 1} N_2(q) = \lim_{q \to 1} \frac{1}{k} g\left(\lambda^{-1} \left(\frac{2k}{(r+k)q} \right) \right).$$

Thus, since g is a decreasing function, as long as $\lambda^{-1}\left(\dfrac{2k}{(r+k)q} \right) > 0$ (which holds if F is strictly convex), we have

$$\lim_{q\to 1} N_1(q) > \lim_{q\to 1} N_2(q).$$

On the other hand, if $g(u)$ is bounded as $u \to 1$, then, from the definition of λ, we know that $\lim_{u\to 0} \lambda(u) \to \infty$ so that we get

$$\lim_{q\to 0} N_1(q) = \lim_{q\to 0} N_2(q) = 0.$$

Therefore, the two graphs of the two nullclines can intersect at two equilibria: once in the interior of the unit square, and once at the origin. (See Figure 2 for the case $F(a) = a^2 \Rightarrow G(u) = \sqrt{u}$). The stability of the interior equilibrium can be calculated by looking at the linearized system, given by the Jacobian matrix:

$$J = \begin{bmatrix} r + k - 2qG'(u)\frac{du}{d\psi} & -2G(u) - 2qG'(u)\frac{du}{dq} \\ -k\frac{du}{d\psi} & -k\frac{du}{dq} - k \end{bmatrix}$$

The derivatives of u with respect to q and ψ can be evaluated using Equation (130):

$$\frac{du}{d\psi} = \frac{1}{g'(u)}\frac{k}{q^2} = \frac{g(u)}{g'(u)}\frac{1}{\psi}, \qquad \frac{du}{dq} = \frac{1}{g'(u)}\frac{-2k\psi}{q^3} = \frac{g(u)}{g'(u)}\frac{2}{q}. \tag{133}$$

The equilibrium is a saddle if the determinant of the Jacobian, evaluated at the equilibrium, is negative. Recall that $g = G'$.

$$\det(J) = -k\left(\left[(r+k) - 2qg(u)\frac{du}{d\psi} \right] \left(\frac{du}{dq} + 1 \right) + 2\left[G(u) + qg(u)\frac{du}{dq} \right] \frac{du}{d\psi} \right). \tag{134}$$

At the equilibrium, we know that $q = 1 - u = \dfrac{2k}{r+k}\dfrac{g(u)}{G(u)}$ and $(r+k)\psi = 2G(u)q$. Use these identities and Equations (133) to get:

$$\frac{du}{d\psi} = \frac{(r+k)^2}{4kg'(u)}, \qquad \frac{du}{dq} = -\frac{(r+k)G(u)}{kg'(u)},$$

which gives, after simplification:

$$\det(J) = -k\left[\frac{-(r+k)^2 G(u)}{2kg'(u)} + \frac{-(r+k)g(u)^2}{2G(u)g'(u)} + (r+k) \right]. \tag{135}$$

By our assumptions on F, $G(u)$ and $g(u)$ are positive, while $g'(u)$ is negative, so that all terms inside the brackets are positive, and therefore $\det(J)$ is negative, which implies that the interior equilibrium is a saddle.

7.4 The REE Demand is greater than the Target Demand in the case of a convex cdf.

Here is the proof of Theorem 6. From Equation (17):

$$v(q) = A(q)q^2 = F^{-1}(1-q)q^2.$$

For a regular cdf, F, the rational-expectations equilibrium (REE) is given by the value $\hat{q}$ that satisfies

$$\frac{dv}{dq}(q)|_{\hat{q}} = 0.$$

Letting $\hat{a} = F^{-1}(1-\hat{q})$ we get that the price at the REE must satisfy:

$$\begin{aligned} v'(\hat{q}) = 0 \quad &\Rightarrow \quad \frac{\hat{q}^2}{F'(F^{-1}(1-\hat{q}))(-1)} + 2F^{-1}(1-\hat{q})\hat{q} = 0, \\ &\Rightarrow \quad \hat{q} = 0 \quad \text{or } \hat{q} = 2F^{-1}(1-\hat{q})F'(F^{-1}(1-\hat{q})), \\ &\Rightarrow \quad 2\hat{a}F'(\hat{a}) = 1 - F(\hat{a}). \end{aligned}$$

In the case of the convex cdf, the target demand is given in Equation (55):

$$\begin{aligned} q^{**} \quad &= \quad \frac{2k}{r+3k}\left[\, 1 - F(a^{**}) + a^{**}F'(a^{**})\right] = 1 - F(a^{**}), \\ \Rightarrow \quad &2a^{**}F'(a^{**}) = \left[\, 1 - F(a^{**})\right]\left[\frac{r+k}{k} \right] = \left[\, 1 - F(a^{**})\right]\left[\, 1 + \frac{r}{k} \right]. \end{aligned}$$

We'd like to show that $a^{**} > \hat{a}$ so that $q^{**} < \hat{q}$, i.e. the target demand is less than the REE demand.

To show this, we note first that the function $aF'(a)$ is increasing in a, since its derivative: $F'(a) + aF''(a)$ is strictly positive for $a \in (0,1)$. Therefore, since

$$[\,1 - F(a)] < [\,1 - F(a)]\left(\,1 + \frac{r}{k}\right)$$

for all $r, k > 0$, we must have $\hat{a} < a^{**} \Rightarrow \hat{q} > q^{**}$, or **the target demand is strictly less than the REE demand** for $r > 0$, with equality when $r = 0$.

7.5 Concave Distributions of Consumer Types

Consider any generalized target policy with target s, and let $V_s(q)$ be the corresponding value function. For $q(t) = s$,

$$q'(t) = k\,[1 - s - (sF\,[0] + (1 - s)F\,[1])] = 0, \tag{136}$$

and the immediate return is $s^2(1 - s)$. Therefore,

$$V_s(s) = \int_0^\infty e^{-rt} s^2(1 - s)dt = \frac{s^2(1 - s)}{r}. \tag{137}$$

When $q < s$,

$$q'(t) = k[1 - q(t)], \tag{138}$$

with initial condition $q(0) = q$, yielding a solution

$$q(t) = 1 - e^{-kt}(1 - q). \tag{139}$$

The time T to get to s is therefore

$$T = \frac{1}{k}\log\left(\frac{1 - q}{1 - s}\right), \tag{140}$$

and the value function is:

$$V_s(q) = s^2(1 - s)\left(\int_T^\infty e^{-rt}dt\right), \tag{141}$$

or

$$V_s(q) = \frac{1}{r}\left[(1 - q)^{-(\rho)}\right]\left[s^2(1 - s)^{(1+\rho)}\right], \quad q < s. \tag{142}$$

Similarly, when $q > s$,

$$q'(t) = -kq(t),$$

and a similar sequence of steps yields the value function for $q > s$:

$$V_s(q) = \frac{1}{2k + r}\left(q^2 + q^{-\rho}s^{(2+\rho)}\left[\frac{2k(1 - s) - rs}{r}\right]\right), \quad q > s. \tag{143}$$

As in Section 3.2, that the value of s that maximizes $V_s(q)$ for both $q < s$ and $q > s$ is

$$\sigma = \frac{2k}{3k+r}. \tag{144}$$

After some elementary simplification, the corresponding value function $V_\sigma(q)$ is

$$V_\sigma(q) = \begin{cases} \frac{1}{r}\left[(1-q)^{-(\rho)}\right]\left[\sigma^2(1-\sigma)^{(1+\rho)}\right], & q<\sigma, \\ \frac{\sigma^2(1-\sigma)}{r}, & q=\sigma, \\ \frac{1}{2k+r}\left(q^2+\frac{k}{r}q^{-\rho}\sigma^{(3+\rho)}\right), & q>\sigma, \end{cases} \tag{145}$$

$$\text{where } \rho = r/k. \tag{146}$$

It is easily verified that $V_\sigma(q)$ is continuous at $q=\sigma$, and

$$V'_\sigma(\sigma^-) = V'_\sigma(\sigma^+) = \frac{\sigma^2}{k}, \tag{147}$$

which verifies that V_σ is continuously differentiable. One also verifies that $V'_\sigma(q) > 0$ for all q (see below).

The Bellmanian corresponding to $V_\sigma(q)$ is

$$\overline{B}(q,\psi) = q^2\bar{a}(\psi) - rV_\sigma(q) + kV'_\sigma\bar{F}(\psi), \tag{148}$$

where ψ denotes a generic measure in Ψ. Since F is strictly concave and $V'_\sigma(q) > 0$, for each q the Bellmanian is maximized in ψ by taking a $\psi \in \Phi(q)$. By a slight abuse of notation, denote a measure in $\Phi(q)$ by the corresponding probability $\phi(q)$. Thus for a measure $\phi \in \Phi(q)$, the Bellmanian equals

$$\overline{B}(q,\phi) = G(q)\phi - rV(q) + kV'(q)(1-q), \tag{149}$$

where

$$G(q) \equiv q^2 - kV'_\sigma(q). \tag{150}$$

The Bellmanian is therefore linear in ϕ, and consequently, is maximized by $\phi(q) = 0$ for $G(q) < 0$, and by $\phi(q) = 1$ for $G(q) > 0$.

Case 1. $q < \sigma$**:** In this case, one verifies that (after some simplification),

$$V'_\sigma(q) = \frac{\sigma^2}{k}\left(\frac{1-\sigma}{1-q}\right)^{(1+\rho)}, q<\sigma, \tag{151}$$

and therefore,

$$G(q) = \frac{q^2(1-q)^{(1+\rho)} - \sigma^2(1-\sigma)^{(1+\rho)}}{(1-q)^{(1+\rho)}} \text{ for } q<\sigma. \tag{152}$$

Since $q < \sigma$, it follows that $G(q) < 0$, and therefore, $\phi(q) = 0$.

Case 2. $q > \sigma$**:** In this case, one verifies that (again after some simplification),

$$V'_\sigma(q) = \frac{1}{2k+r}\left[2q - \left(\sigma^2\left(\frac{\sigma}{q}\right)^{(1+\rho)}\right)\right], q > \sigma, \tag{153}$$

and therefore,

$$G(q) = q^2 - \frac{k}{2k+r}\left[2q - \left(\sigma^2\left(\frac{\sigma}{q}\right)^{(1+\rho)}\right)\right] \text{ for } q > \sigma, \tag{154}$$

which simplifies to

$$G(q) = q^2\left(\left[1 - \left(\frac{\sigma}{q}\right)^{3+\rho}\right] - \left[\left(\frac{3+\rho}{2+\rho}\right)\left(\frac{\sigma}{q}\right)\left(1 - \left(\frac{\sigma}{q}\right)^{2+\rho}\right)\right]\right) \text{ for } q > \sigma. \tag{155}$$

This is of the form

$$K[\left(1 - x^{(y+1)}\right) - \frac{y+1}{y}x(1 - x^y)], \text{ with } x = \frac{\sigma}{q},\ y = 2 + \rho. \tag{156}$$

The identity

$$\frac{1 - x^{(y+1)}}{1 - x^y} > \frac{(y+1)}{y}x \text{ for } y > 0,\ 0 < x < 1 \tag{157}$$

establishes that $G(q) > 0$ and therefore, $\phi(q) = 1$.

Finally,

$$G(\sigma) = 0, \tag{158}$$

and therefore, $\overline{B}(\sigma, \phi)$ is constant in ϕ. The generalized target policy with target σ therefore satisfies the Hamilton-Jacobi-Bellman condition, which completes the proof.

7.6 An Evolving Consumer Population

The law of motion can be rewritten as

$$m(q, a) = k[1 - a - hq], \quad 0 < q \leq 1, 0 \leq a \leq 1, \tag{159}$$

where

$$h \equiv 1 + \frac{c}{k} \geq 1. \tag{160}$$

Now consider an arbitrary target policy with target s. Starting at $q < s$, until $q(t)$ reaches s, $q(t)$ satisfies the differential equation

$$q'(t) = k[1 - hq(t)],\ q(0) = q, \tag{161}$$

whose solution is

$$q(t) = \frac{1}{h}[1 - e^{-kht}(1 - hq)], \tag{162}$$

and proceeding as in the proof of Theorem 1 yields the value function

$$V(q) = \frac{1}{r}s^2[1-hs]^{1+\rho}(1-hq)^{-}\rho,\ q<s, \quad (163)$$

$$\text{where } \rho = \frac{r}{hk}. \quad (164)$$

The RHS is strictly quasiconcave for $0 < s \leq 1$, and it is maximized at

$$\sigma = \frac{2k}{3hk+r}, \quad (165)$$

which is our candidate optimal target. Correspondingly, starting at $q > s$, the value function solves to being

$$V(q) = \frac{1}{2hk+r}\left(q^2 + q^{-\rho}s^{(2+\rho)}\left[\frac{2k(1-hs)-rs}{r}\right]\right),\ q>s, \quad (166)$$

which is also maximized in s at σ.

Now, the Bellmanian functional for the target policy with target σ is

$$B(q,a) = aq^2 - rV(q) + m(q,a)V'(q), \quad (167)$$

which simplifies to

$$B(q,a = a\left[q^2 - kV'(q)\right] - [rV(q) + k - khV'(q)]. \quad (168)$$

Recalling the function $G(q)$ defined in Sec. 7.5

$$G(q) = q^2 - kV'(q), \quad (169)$$

it follows again that

$$\arg\max_a B(q,a) = \begin{cases} 0, & \text{if } G(q) < 0, \\ 1, & \text{if } G(q) > 0. \end{cases} \quad (170)$$

Differentiation the value function V yields, after some rearranging,

$$V'(q) = \frac{\sigma^2}{k}\left(\frac{1-h\sigma}{1-hq}\right)^{[1+\rho]}, \quad (171)$$

which in turn implies that,

$$G(q) = q^2\left(1 - \frac{f_0(\sigma)}{f_0(q)}\right), \quad q<\sigma, \quad (172)$$

where

$$f_0(x) \equiv x^2[1-hx]^{[1+\rho]}. \quad (173)$$

However, the function $f_0(x)$ is maximized at $x = \sigma$, which implies that $f_0(q) < f_0(\sigma)$ for $q < \sigma$, and therefore $G(q) < 0$ for $q < \sigma$. A similar computation, which is omitted, verifies that $G(q) > 0$ for $q > \sigma$. Finally, one can verify that $V_1(\sigma-,\sigma) = V_1(\sigma+,\sigma)$, and this completes the proof of the theorem.

7.7 Myopic and "Stubborn" Consumers

Consider an arbitrary target policy with target s. Proceeding as we did in Section 7.1 , we shall first characterize the optimal target, starting from $q(0) = q$. For the purpose of this subsection, it will be convenient to take the control variables to be price, $p(t)$, rather than $a(t)$.

We begin with the Case 1, in which $q < s$. Until $q(t)$ reaches s, the price is zero, and $q(t)$ satisfies the differential equation,

$$q'(t) = k[1 - q(t)]. \tag{174}$$

When $q(t)$ reaches the target, s, the price jumps to the "stay as you are" price,

$$P(s) = (1 - q)(\gamma s + (1 - \gamma)\omega)$$

As in Section 7.1, the value function for the target policy with target s is:

$$\begin{aligned}
V(q, s) &= \left(\frac{1-s}{1-q}\right)^{\rho} \frac{P(s)s}{r} && (175)\\
&= \frac{f(s)}{r(1-q)^{\rho}}, && (176)\\
\text{where } \rho &= \frac{r}{k}' \\
f(s) &= (1-s)^{\rho+1}\left[\gamma s^2 + (1-\gamma)\omega s\right]. && (177)
\end{aligned}$$

Hence the target that maximizes $V(q, s)$ is the value of s that maximizes $f(s)$. One verifies that

$$\begin{aligned}
f'(s) &= (1-s)^{\rho} G(s), \text{ where} && (178)\\
G(s) &= -\gamma(\rho+3)s^2 + [2\gamma - (\rho+2)(1-\gamma)\omega]s + (1-\gamma)\omega.
\end{aligned}$$

Note that $f'(s)$ and $G(s)$ have the same sign. Also, G is quadratic and concave, and $G(0) = (1 - \gamma)\omega > 0$. Hence f is maximized at the larger of the two roots of $G(s) = 0$. Call this root $\sigma(\gamma, \omega)$; it is the optimal target. Note that it is independent of the starting state, q.

We now show that, for $q < \sigma(\gamma, \omega)$, the target policy with target $\sigma(\gamma, \omega)$ is optimal among all policies. For the purpose of this proof, we define

$$\begin{aligned}
\hat{\sigma} &= \sigma(\gamma, \omega),\\
\hat{V}(q) &= V(q, \hat{\sigma}).
\end{aligned}$$

The Bellmanian functional for this policy is

$$B(q, p) = pq - r\hat{V}(q)) + k\left[1 - \frac{p}{\gamma q + (1-\gamma)\omega} - q\right]\hat{V}'(q). \tag{179}$$

Differentiating with respect to p, we have

$$B_2(q, p) = q - \frac{k\hat{V}'(q)}{\gamma q + (1-\gamma)\omega}. \tag{180}$$

Since

$$\hat{V}'(q) = \frac{f(\hat{\sigma})}{k(1-q)^{\rho+1}},$$

it follows that

$$B_2(q,p) = q - \frac{f(\hat{\sigma})}{(1-q)^{\rho+1}\left[\gamma q + (1-\gamma)\omega\right]}.$$

Hence $B_2(q,p) < 0$ if and only if

$$\begin{aligned} (1-q)^{\rho+1}\left[\gamma q^2 + (1-\gamma)\omega q\right] &< f(\hat{\sigma}), \text{ or} \\ f(q) &< f(\hat{\sigma}), \end{aligned}$$

which is true for $q < \hat{\sigma}$. This completes the proof of the optimality of the target policy with target $\hat{\sigma}$ in Case 1. The argument for Case 2, $q > \hat{\sigma}$, is analogous, and is omitted. Finally, one can verify that $V_1(\hat{\sigma}-,\hat{\sigma}) = V_1(\hat{\sigma}+,\hat{\sigma})$. This completes the proof of Part (a) of the theorem.

To prove Part (b), write $G(s)$ in (178) in the form

$$\begin{aligned} G(s,\gamma) &= \gamma g_b(s) + (1-\gamma) g_a(s), \text{ where} \\ g_a(s) &= -(\rho+3)s^2 + 2s, \\ g_b(s) &= -(\rho+2)\omega s + \omega. \end{aligned} \tag{181}$$

Recall that $\hat{\sigma} = \sigma(\gamma,\omega)$ is the larger root of

$$G[s,\gamma] = 0.$$

A standard "comparative statics" calculation yields

$$\sigma_1(\gamma,\omega) = -\frac{g_b[\sigma(\gamma,\omega)] - g_a[\sigma(\gamma,\omega)]}{\gamma g_b'[\sigma(\gamma,\omega)] + (1-\gamma) g_a'[\sigma(\gamma,\omega)]}. \tag{182}$$

Let σ_b be the positive root of $g_b(s) = 0$ (the other root is 0), and let σ_a be the root of $g_a(s) = 0$. Then

$$\sigma_b = \frac{2}{\rho+3}, \quad \sigma_a = \frac{1}{\rho+2}. \tag{183}$$

Note that

$$\frac{\sigma_b}{2} < \sigma_a < \sigma_b. \tag{184}$$

Note also that (1) $g_b(s)$ is decreasing and positive for $\frac{\sigma_b}{2} \leq s < \sigma_b$;(2) $g_a(s)$ is decreasing, and is negative for $\sigma_a < s \leq \sigma_b$, and (3) $\sigma_a < \sigma(\gamma,\omega) < \sigma_b$ for $0 < \gamma < 1$ (see Figure 5). Hence, by (182) and (184), $\sigma_1(\gamma,\omega) > 0$ for $0 < \gamma < 1$, which completes the proof of the theorem.

To prove part (c), first notice that, independent of the value of ω, part (b) establishes that for $0 < \gamma < 1$:

$$\frac{1}{2+\rho} < \sigma(\gamma,\omega) < \frac{2}{3+\rho}. \tag{185}$$

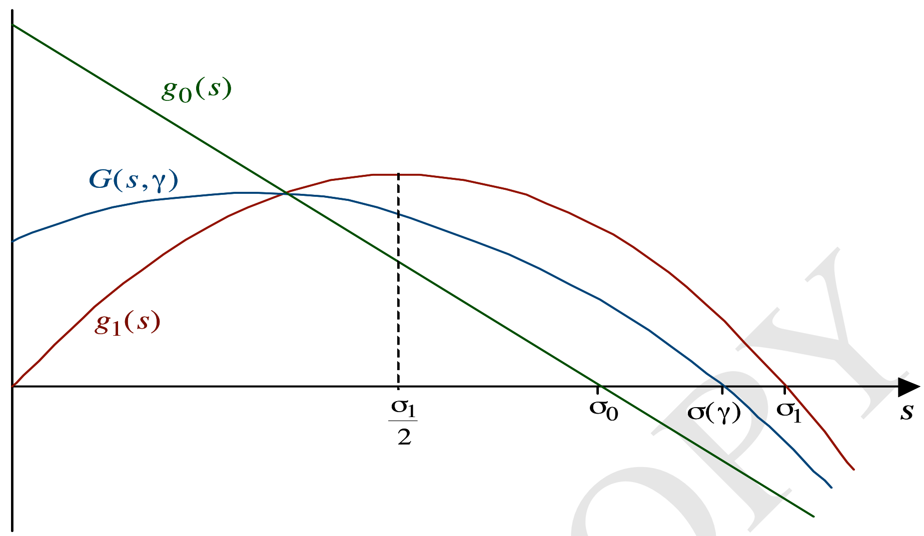

Figure 5: Illustrates the proof of part(b) of Theorem 9.

Also, from the second line of (178), σ is defined by

$$-\gamma(\rho+3)[\sigma(\gamma,\omega)]^2 + [2\gamma - (\rho+2)(1-\gamma)\omega]\sigma(\gamma,\omega) + (1-\gamma)\omega = 0. \quad (186)$$

Differentiating both sides of () with respect to ω and rearranging yields:

$$\sigma_2(\gamma,\omega) = -\left(\frac{(1-\gamma)[(2+\rho)\sigma(\gamma,\omega)-1)}{2\gamma[((3+\rho)\sigma(\gamma,\omega)-1)] + \omega[1-\gamma](2+\rho)}\right). \quad (187)$$

From (185), $(2+\rho)\sigma(\gamma,\omega) > 1$, and thus both the numerator and the denominator of the expression in parentheses on the LHS of (187) are strictly positive. This completes the proof of Part (c).