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Bundling and Competition for Slots[*]

Doh-Shin Jeon[†] and Domenico Menicucci[‡]

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Abstract

We study competition among upstream firms when each of them sells a portfolio of distinct products and the downstream has a limited number of slots (or shelf space). In this situation, we study how bundling affects competition for slots. When the downstream has k number of slots, social efficiency requires that it purchases the best k products among all upstream firms' products. We find that under bundling, the outcome is always socially efficient but under individual sale, the outcome is not necessarily efficient. Under individual sale, each upstream firm faces a trade-off between quantity and rent extraction due to the coexistence of the internal competition (i.e. competition among its own products) and the external competition (i.e. competition from other firms' products), which can create inefficiency. On the contrary, bundling removes the internal competition and the external competition among bundles makes it optimal for each upstream firm to sell only the products belonging to the best k. This unambiguous welfare-enhancing effect of bundling is novel.

Key words: Bundling, Competition among Portfolios, Limited Slots (or Shelf Space)

JEL Code: D4, K21, L13, L41, L82

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1 Introduction

In vertical relations, very often each upstream firm sells a portfolio of distinct products which compete for limited slots (or shelf space) of downstream firms. In this situation, upstream firms may employ bundling as a strategy to win over the competition for the limited slots. The practice of bundling has been a major antitrust issue and a subject of intensive research in the past. However, to the best of our knowledge, the theoretical Industrial Organization literature seems to have neglected competition among portfolios of distinct products and, in particular, no paper has studied how bundling affects competition for limited slots in this framework. In this paper, we attempt to provide a new perspective on bundling by analyzing how it affects competition among portfolios of distinct products for limited slots, and social welfare.

Examples of the situation we described above are abundant. For instance, in the movie industry, each movie distributor has a portfolio of distinct movies and buyers (either movie theaters or TV stations) have limited slots. More precisely, the number of movies that can be projected in a season (or in a year) by a theater is constrained by time and the number of rooms. Likewise, the number of movies that a TV station can show at prime time during a year is also limited. Actually, allocation of slots in movie theaters has been one of the main issues of the last presidential election in France regarding the movie industry[1]. Furthermore, bundling in this industry (known as block booking[2]) was declared illegal in two supreme court decisions in U.S.: *Paramount Pictures*, where blocks of films were rented for theatrical exhibition, and *Loew's*, where blocks of films were rented for television exhibition. In addition, recently in *MCA Television Ltd. v. Public Interest Corp.* (11th Circuit, April 1999), the court of appeals reaffirmed the *per se* illegal status of block booking.

Another example we have in mind is that of manufacturers' competition for a retailer's shelf space. Typically, each manufacturer produces a range of different products (for instance, think about all the products sold under the brand name of Nestle) and manufacturers compete for a retailer's limited shelf space. In this context, manufacturers having a large portfolio of products may practice bundling (often called full-line forcing) for their

[1] Cahiers du Cinema (April, 2007) proposes to limits the number copies per film since certain movies by saturating screens limits other films' access to screens and asks the presidential candidates' opinions about the issue.

[2] Block booking refers to "the practice of licensing, or offering for license, one feature or group of features on the condition that the exhibitor will also license another feature or group of features released by distributors during a given period" (Unites States v. Paramount Pictures, Inc., 334 U.S. 131, 156 (1948)).

advantage and there has been antitrust cases related to this practice: Procter & Gamble / Gillette[3] and Société des Caves de Roquefort.[4]

Moreover, since we consider a general model in which a firm can bundle any number of products, our setup can be applied to bundling a large number of information goods, a common practice in the Digital era (for instance, bundling of electronic academic journals).

In our model, we will assume away any asymmetric information or any uncertainty about values of products. This will allow us to depart from the existing literature on block booking or bundling (see the review of the related literature later on in this section) and to identify what seems to us a first-order effect of bundling by focusing on the downstream firm's slot constraint. More precisely, we consider competition between two upstream firms selling to a downstream firm who has $k(>0)$ number of slots. In our setting, a product needs to occupy a slot to generate some value (i.e. a profit) to the downstream firm. The upstream firms' products are heterogenous in terms of the value that each of them generates to the downstream firm. Therefore, social efficiency requires the downstream firm to purchase and use the best k products among all products owned by the two upstream firms. We focus on studying how bundling affects the set of the products that are purchased and consumed by the downstream firm.

As the main result, we find that under bundling the outcome of competition is always socially efficient, while this is not necessarily the case under individual sale. Under individual sale, each upstream firm faces a trade-off between quantity and rent extraction due to the coexistence of the internal competition (i.e. competition among its own products) and the external competition (i.e. competition from other firms' products): as a firm increases the number of products it induces the downstream firm to buy, it should abandon more rent for each product it sells. This trade-off can make the outcome inefficient. On the contrary, bundling removes the internal competition and the external competition among bundles makes it optimal for each upstream firm to sell only his own products which belong to the best k.

We think that this unambiguous welfare-enhancing effect of bundling is pretty novel. Furthermore, we show that the efficiency property of bundling is very general in that it holds regardless of whether we consider a sequential or simultaneous game or whether or not we allow firms to contract directly on exclusive use of slots. The existing literature analyzing bundling in a second-degree price discrimination framework often justifies the rule of reason standard regarding bundling. Our result has strong policy implications that go beyond the rule of reason.

[3] DG Competition case COMP/M.3732

[4] Conseil de la Concurrence, Decision 04-D-13, 8th April 2004.

There are only a few papers on block booking. The leverage theory, on which the Supreme Court's decisions were based, that block booking allows a distributor to extend its monopoly power in a desirable movie to an undesirable one was criticized by Stigler since the distributor is better off by selling only the desirable movie at a higher price. Instead of the leverage theory, Stigler (1968) proposed a theory based on second-degree price discrimination. However, Kenney and Klein (1983) point out that simple price discrimination explanation since block booking is inconsistent with the facts of *Paramount* and *Loew's* for the prices of the blocks varied a great deal across markets and argue that block booking mainly prevents exhibitors from oversearching, i.e. from rejecting films revealed ex post to be of below-average value from an ex ante average-valued package. Their hypothesis is empirically tested in a recent paper by Hanssen (2000) but the author finds little support for the hypothesis[5] and proposes that block booking was primarily intended to cheaply provide films in quantity.

Most papers on bundling study bundling of two (physical) goods in the context of second-degree price discrimination and focus on either surplus extraction (Schmalensee, 1984, McAfee et al. 1989, Salinger 1995 and Armstrong 1996, 1999) in a monopoly setting or entry deterrence (Whinston 1990 and Nalebuff 2004) in a duopoly setting. Bakos and Brynjolfsson (1999, 2000)'s papers are an exception, in that they study bundling of a large number of information goods, but they maintain the second-degree price discrimination framework. Their first paper shows that bundling allows a monopolist to extract more surplus (since it reduces the variance of average valuations by the law of large numbers) and thereby unambiguously increases social welfare;[6] the second paper applies this insight to entry deterrence (we do not address the entry deterrence issue). Since our novelty is that we assume complete information and hence full surplus extraction is possible under the monopoly setting, the rent extraction issue does not arise in our environment and there is no use in applying the law of large number.

In Jeon-Menicucci (2006), we took a framework similar to the one in this paper to study bundling electronic academic journals. More precisely, publishers owning portfolios of distinct journals compete to sell them to a library who has a fixed budget to allocate between journals and books. Publishers are assumed to have complete information about the value that the library obtains from a journal and about the budget. We found that bundling is a profitable strategy both in terms of surplus extraction and entry deterrence. Conventional wisdom says that bundling has no effect in such a setting and this is true without the budget constraint. However, when the budget constraint binds, we found that

[5]But Kenny and Klein (2000) do not agree with Hanseen's analysis.

[6]See also Armstrong (1999).

each firm has a strict incentive to adopt bundling but bundling reduces social welfare by reducing the library's consumption of journals and monographs. In this paper, instead of focusing on the budget constraint of the buyer, we focus on his slot constraint. Another difference is that Jeon-Menicucci (2006) focus on products (journals) of homogeneous value while in this paper we consider products of heterogenous value. In spite of similarities of the frameworks, the result we obtain here is contrary to the one in the previous paper since we find that the allocation under bundling is socially efficient.

Finally, to our knowledge, Shaffer (1991) is the only paper that explicitly models the downstream firm's limited shelf space.[7] He considers an upstream monopolist selling two substitutable products with variable quantity. He finds that brand specific two-part tariffs alone do not allow the monopolist to capture the maximum rent from the downstream firm but full-line forcing (equivalent to bundling) does. We consider products of independent values and hence the rent extraction issue Shaffer considers does not arise. In a general setup of competition in which seller i has n_i number of products and the buyer has $k(< \sum_i n_i)$ number slots, we study how bundling affects the set of the products that occupy the slots. Although products have independent values, competition arises because of the slot constraint as long as there are at least two sellers.

In what follows, section 2 presents the model. Section 3 (4) characterizes the equilibrium under individual sale (bundling). Section 5 compares individual sale with bundling in terms of social welfare. Section 6 shows that when bundling is allowed, firms have an incentive to bundle their products. Section 7 shows that the efficiency property of bundling is very general. Section 8 concludes. Most proofs are gathered in the Appendix.

2 The setting

2.1 Model

There are two upstream firms, denoted by $i = A, B$, and a downstream firm, denoted by D. Each firm i has a portfolio of $n_i(\geq 1)$ products for $i = A, B$, and all products are distinct; let $n \equiv n_A + n_B$. Firm D has a limited number of slots (or shelf space) to distribute the upstream firms' products: the number of slots is given by $k(\geq 1)$. Given (n_A, n_B, k), we assume for simplicity that the cost of producing each product is zero for $i = A, B$ and the cost of distributing each product is zero for D.

D's distribution of a product requires one unit of slot. Therefore, D can distribute at most k number of products and we assume $n > k$. In this setup, we consider products of

[7] See also Vergé (2001) who performs the social welfare analysis in the setup of Shaffer (1991).

heterogenous value and study how bundling affects the set of the products occupying the limited slots. More precisely, we are interested in knowing when D distributes the best k number of products. Let u_i^j denote the gross profit that D obtains from distributing the j-th best product of i; thus $u_i^1 \geq u_i^2 \geq ... \geq u_i^{n_i} > 0$ for $i = A, B$. Let u^j denote the gross profit (or surplus) that D obtains from the j-th best product among all products owned by both upstream firms, thus $u^1 \geq u^2 \geq ... \geq u^n$. We assume $u^k > u^{k+1}$. Let m_i^* denote the number of i's products in the set of the k best products among all products owned by both upstream firms: by definition, $m_A^* + m_B^* = k$. We assume $m_i^* \geq 1$ for $i = A, B$.[8] In this paper we study the case of $n_A \geq k \geq n_B$, but we can extend easily the analysis to the case of $n_A < k$ or $k < n_B$. Then, without loss of generality, we suppose $n_A = k$.

Under individual sale, firm i chooses $p_i^j > 0$ for its product with value u_i^j, and we define $w_i^j \equiv u_i^j - p_i^j$ as the net profit that D obtains from buying this product. Let $\mathbf{p}_i \equiv (p_i^1, p_i^2, ..., p_i^{n_i})$ and $\mathbf{w}_i \equiv (w_i^1, w_i^2, ..., w_i^{n_i})$ denote the vectors of prices and of net profits for the products of firm i, respectively. It is clear that there is a one-to-one correspondence between $\mathbf{p}_i$ and $\mathbf{w}_i$, and therefore we can equivalently express firm i's decision problem in terms of either $\mathbf{p}_i$ or $\mathbf{w}_i$. However, when we use $\mathbf{w}_i$ we need to recall that $w_i^j < u_i^j$ for $i = A, B$ and $j = 1, ..., n_i$. In particular, we will sometimes refer to the condition

$$w_A^1 < u_A^1, \quad w_A^2 < u_A^2, \quad ..., \quad w_A^k < u_A^k \tag{1}$$

for firm A.

Under bundling, firm i sells a bundle B_i at a price $P_i (> 0)$.

2.2 Nonexistence of equilibrium in a simultaneous game

Before presenting the timing of the sequential game that we study, we below illustrate that equilibrium often does not exist in a simultaneous game under individual sale. Suppose that A and B choose simultaneously $\mathbf{p}_A$ and $\mathbf{p}_B$. As usual, we adopt as a tie-breaking rule that if indifferent among different products, D buys the product which generates the highest gross profit.

Suppose that A (B) has two (one) products and $k = 2$. Assume $(u_A^1, u_A^2, u_B^1) = (3, 1, 2)$. Without loss of generality, we can assume that A chooses $\mathbf{p}_A$ such that $3 - p_A^1 \geq \max\{0, 1 - p_A^2\}$: the net profit that D makes from buying A's best product is positive and larger than the one it makes from buying A's second best product. Given $\mathbf{p}_A$ satisfying $3 - p_A^1 \geq \max\{0, 1 - p_A^2\}$, B's best response is to choose p_B^1 such that $2 - p_B^1 =$

[8] The analysis of individual sale applies to $m_i^* = 0$. $m_i^* \geq 1$ simplifies the exposition of the analysis of bundling.

$\max\{0, 1 - p_A^2\}$: B can find such a price since $p_A^2 > 0$ and hence $\max\{0, 1 - p_A^2\}$ cannot be larger than 1. Consider first the case in which $1 \geq p_A^2 > 0$. In this case, B's best response is $p_B^1 = 1 + p_A^2$. However, then A can deviate by charging $p_A^{2\prime} = p_A^2 - \varepsilon$ for $\varepsilon(> 0)$ small enough and sell both products. Consider now the case in which $p_A^2 > 1$. In this case, B's best response is $p_B^1 = 2$. However, then A can deviate by charging $p_A^{2\prime} = 1 - \varepsilon$ for $\varepsilon(> 0)$ small enough and sell both products. Therefore, there is no equilibrium (in pure strategy).[9]

The above example illustrates well the commitment issue that A faces. On the one hand, if A can commit not to sell its second best product, then A and B can sell their best products at the prices that extract D's whole surplus. However, this outcome cannot be an equilibrium since A has an incentive to deviate by undercutting B's price (i.e. charging $p_A^{2\prime} = 1 - \varepsilon$) to sell both products. On the other hand, there cannot be an equilibrium in which D buys both products of A and does not buy B's product. Therefore, in what follows, we will consider a sequential game in which firm A can commit to its prices before B chooses its prices.

2.3 Timing and tie-breaking rules

We consider the following sequential timing. When bundling is prohibited (i.e. under individual sales),

Stage 1. A chooses $\mathbf{p}_A$;

Stage 2. after observing $\mathbf{p}_A$, B chooses $\mathbf{p}_B$;

Stage 3. D makes its purchase decision.

When bundling is allowed, at stage 1 (stage 2), A (B) decides whether or not to bundle his products and $\mathbf{p}_A$ or P_A ($\mathbf{p}_B$ or P_B) accordingly: in addition, if firm i decides to bundle his products, it also decides which products to include into the bundle.

In what follows we use the concept of subgame perfect Nash equilibrium (SPNE) to determine the outcome of this game. Thus we start with D's purchases at stage three. In the case of individual sales, D chooses the k products yielding the highest non-negative net profits. However, it is necessary to specify how D deals with ties, i.e. with products which have the same net profit. Therefore we introduce the following tie-breaking rules.

T1: If D is indifferent between buying a product from A and a product from B (both with non-negative net profits), and cannot buy both of them, then D buys B's product.

[9] Even if we allow firms to charge zero price, the equilibrium does not exist. Suppose now $p_A^2 = 0$: this together with $3 - p_A^1 \geq \max\{0, 1 - p_A^2\}$ implies $p_A^1 \leq 2$. In this case, B's best response is $p_B^1 = 1$. However, then A can deviate by charging $(p_A^{1\prime}, p_A^{2\prime}) = (3, 2)$ for instance. Then, A sells only its best product but at a higher price.

T1 is motivated by the fact that in our sequential game, given the price of A's product in question, B, as the follower, can always lower by $\varepsilon > 0$ its price to break D's indifference. Formally, in some cases B has no best reply without this assumption.

T2: If D is indifferent among different products offered by the same firm, and cannot buy both of them, D buys the product that generates the highest gross value.

T2 is a standard tie-breaking rule.

Finally, we introduce the following tie-breaking rule for the upstream firms.

T3: Under bundling, if including a product into the bundle that i sells does not strictly increase i's profit, i prefers not to include the product into the bundle.

T3 makes a perfect sense when there is an infinitesimal cost of production. Although we do not model any production cost for simplicity, T3 captures the essential effect that would result from a positive production cost.

3 Individual sale

3.1 A preliminary result

Recall that we have set $w_i^j = u_i^j - p_i^j$ for $i = A, B$ and $j = 1, ..., n_i$, and $\mathbf{w}_i = (w_i^1, w_i^2, ..., w_i^{n_i})$ for $i = A, B$. In $\hat{\mathbf{w}}_i \equiv (w_i^{(1)}, w_i^{(2)}, ..., w_i^{(n_i)})$ we order instead the net profits in a decreasing way, which means that $w_i^{(1)} \geq w_i^{(2)} \geq ... \geq w_i^{(n_i)}$. We now prove a simple and intuitive result: there is no loss of generality in assuming that $w_i^{(j)} = w_i^j$ for all j.

Lemma 1 *Without loss of generality, we can restrict our attention to the case in which $w_i^1 \geq w_i^2 \geq ... \geq w_i^{n_i}$ (i.e., $\mathbf{w}_i = \hat{\mathbf{w}}_i$) for i=A,B.*

In particular, lemma 1 implies the following monotonicity condition for firm A, which we will use repeatedly in the remaining of the paper:

$$w_A^1 \geq w_A^2 \geq ... \geq w_A^k \tag{2}$$

The lemma also implies that when firm i is selling m_i number of products, i is actually selling its products with the m_i highest gross values.

3.2 Stage two

Now we apply backwards induction to firm B, by examining his decision at stage two. Precisely, we take $\mathbf{w}_A$ as given and consider the following questions: given $m \in \{1, ..., n_B\}$, is it feasible for B to sell m units? If so, what is the highest profit B can make by selling m units?

Lemma 2 *Given $\mathbf{w}_A$ and $m \in \{1, ..., n_B\}$, it is feasible for B to sell m units if and only if $u_B^m > w_A^{k-m+1}$. In this case, the highest profit B can earn by selling m products is $u_B^1 + ... + u_B^m - m \max\left\{w_A^{k-m+1}, 0\right\}$.*

The basic idea of the lemma is that D buys m units of B if and only if m products of B are among the k products with the highest net profits. For instance, consider the case of $m = 1$. If $w_A^k \geq u_B^1$, then B cannot sell any product because the inequality $w_A^k > w_B^1$ necessarily holds and therefore D will buy k products from A and none from B. If instead $w_A^k < u_B^1$, B succeeds in selling his best product by charging a sufficiently low price p_B^1 such that $w_A^k < u_B^1 - p_B^1$ and p_B^j large enough for $j \geq 2$. Precisely, from T1, the highest price which induces D to buy B's best product is $p_B^1 = u_B^1 - \max\left\{w_A^k, 0\right\}$. In words, B can sell his best product only if the k-th best product of A gives D a net profit that is smaller than the gross profit of the best product of B. In short, it must be possible for B to push out the k-th best product of A by pricing aggressively enough his own best product.

For an arbitrary value of m in $\{1, ..., n_B\}$, the same argument shows that the inequality $w_A^{k-m+1} < u_B^m$ is necessary, i.e. it must be possible for B to block out the $(k - m + 1)$-th best product of A by pricing suitably his own m best products. Otherwise, $w_A^{k-m+1} > w_B^m$ and therefore D will buy at least $k - m + 1$ units from 1, and at most $k - (k - m + 1) = m - 1$ from B. When $w_A^{k-m+1} < u_B^m$, B succeeds in selling m products by charging prices $p_B^1, ..., p_B^m$ such that $w_B^1 = ... = w_B^m = \max\left\{w_A^{k-m+1}, 0\right\}$ (again, recall T1), or equivalently $p_B^j = u_B^j - \max\left\{w_A^{k-m+1}, 0\right\}$ for $j = 1, ..., m$ and p_B^j large for $j = m+1, ..., n^B$; the resulting profit for B is $u_B^1 + ... + u_B^m - m \max\left\{w_A^{k-m+1}, 0\right\}$.

In view of lemma 2 we define as follows the profit B can make by selling m units, for $m \in \{1, ..., n_B\}$:[10]

$$\pi_B(m) \equiv \begin{cases} u_B^1 + ... + u_B^m - m \max\left\{w_A^{k-m+1}, 0\right\} & \text{if } u_B^m > w_A^{k-m+1}; \\ 0, & \text{otherwise.} \end{cases}$$

In order to examine how π_B depends on m, we begin by noticing that the higher is m, the more restrictive is the inequality $u_B^m > w_A^{k-m+1}$. Thus, if B is unable to sell m units because $u_B^m \leq w_A^{k-m+1}$, he is *a fortiori* unable to sell $\tilde{m} > m$ units.

[10] This profit depends also on $\mathbf{w}_A$, even though we do not emphasize this fact in the notation.

Now we consider a case in which $u_B^{m+1} > w_A^{k-m} > 0$, so that B is able to sell $m+1$ products (and also fewer than $m+1$) and we below examine how increasing his sale by one more product affects B's profit. When B sells m units, we have seen that he earns a profit of $u_B^1 + ... + u_B^m - mw_A^{k-m+1}$ by charging prices $p_B^j = u_B^j - w_A^{k-m+1}$ for $j = 1, ..., m$; these prices are determined by the fact that B needs to block out the $(k-m+1)$-th best product of A. If instead he sells $m+1$ units, B needs to push out the $(k-m)$-th best product of A, which is more valuable than the $(k-m+1)$-th. Prices are then $\hat{p}_B^j = u_B^j - w_A^{k-m}$ for $j = 1, ..., m+1$, and $\hat{p}_B^j < p_B^j$ for $j = 1, ..., m$. This generates a loss for B, on his m best units, equal to $m(w_A^{k-m} - w_A^{k-m+1})$. However, now B gains $\hat{p}_B^{m+1} = u_B^{m+1} - w_A^{k-m} > 0$ from the sale of the $(m+1)$-th unit. Whether B prefers selling $m+1$ units to m units depends on the comparison between the loss $m(w_A^{k-m} - w_A^{k-m+1})$ and the gain $u_B^{m+1} - w_A^{k-m}$. In other words, (2) makes B face *a trade-off between quantity and (per unit) rent extraction*: as B increases the number of products he sells, he must leave more surplus per unit to D.

3.3 Stage one

We first study the optimal pricing conditional on that A sells $k-m$ units. And then, we study the optimal m that maximizes A's profit.

3.3.1 A's profit when he sells $k-m$ units

Now we consider the first stage of the game in order to determine the profit A can make as a function of the number of products he sells. Hence, suppose that A wants to sell $k-m$ units for $m \in \{0, 1, ..., n_B\}$. Then, we inquire whether (i) there exists $\mathbf{w}_A$ such that, taking into account the best response by B, induces D to buy $k-m$ units from A; (ii) within the set of $\mathbf{w}_A$ which allow A to sell $k-m$ units, we identify the vector that maximizes A's profit.

Formally, the conditions that allow A to sell $k-m$ products can be stated by using the following incentive constraints:

$$(\text{IC}_{m,m'}) \qquad \pi_B(m) \geq \pi_B(m') \quad \text{for} \quad \text{any} \quad m' \neq m \quad \text{and} \quad m' \in \{1, ..., n_B\} \tag{3}$$

Condition (3) means that B prefers to sell m units rather than $m' \neq m$. In particular, (3) implies that B is not going to push out the $(k-m)$-th best unit of A (nor any better product of A),[11] and therefore D will buy $k-m$ number of products from A. Then A's

[11] Actually, it suffices to satisfy the constraints $(\text{IC}_{m,m'})$ for all $m' > m$. However, it turns out that it is costless for A to satisfy also the constraints $(\text{IC}_{m,m'})$ for $m' < m$ (see the proof of Proposition 1).

profit is given by:

$$\pi_A(k-m) \equiv \sum_{j=1}^{k-m} (u_A^j - w_A^j)\mathbf{1}_{[w_A^j \geq 0]}$$

which, we note, is not affected by $(w_A^{k-m+1}, ..., w_A^k)$. We investigate below whether there is a set of $\mathbf{w}_A$ which satisfy (3) and, if so, we maximize $\pi_A(k-m)$ in this set.

We start by observing that it is certainly possible for A to sell $k - n_B$ units, and that he can do so without leaving any surplus to D on these products. In order to show the details, suppose that A chooses $p_A^j = u_A^j$ for $j = 1, ..., k - n_B$ and p_A^j high enough for $j = k - n_B + 1, ..., k$. In this way, A's n_B worst products are not competing with B's products while A's best $k - n_B$ products give D zero surplus. Then, B will reply by charging $p_B^j = u_B^j$ for $j = 1, ..., n_B$, and D will buy $k - n_B$ products from A and n_B products from B, earning no profit.

When A's objective is to induce B to sell only $m(< n_B)$ products, as it will become clear later on, B has two strategies: *accommodation* or *fight*. "Accommodation" means that B contents himself with occupying m slots. "Fight" means that B tries to occupy more than m slots by blocking out some extra units of A. Obviously, to achieve his goal, A must choose prices such that B prefers accommodation to fight, which is equivalent to the property that ($\mathrm{IC}_{m,m'}$) is satisfied for all $m' > m$. What makes the case of $m = n_B$ straightforward is that B sells *all* his n_B units by accommodating, and thus he will not fight.

The next proposition characterizes the condition under which A is able to sell $k - m$ units and the profit maximizing vector $\mathbf{w}_A$ (hence, the optimal prices) conditional on selling $k - m$ units. For expositional facility, we introduce the following notation. Given $m \in \{0, 1, ..., n_B - 1\}$, let

$$\mu_m^{k+1-m''} \equiv \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''}) \qquad \text{for} \quad m'' = m+1, ..., n_B \tag{4}$$

Proposition 1 *For a given $m \in \{0, 1, ..., n_B - 1\}$,*

(i) a. A can find $\mathbf{w}_A$ that induces D to buy $k - m$ units from A if and only if

$$u_A^{k+1-m''} > \mu_m^{k+1-m''} \qquad \textit{for} \quad m'' = m+1, ..., n_B \tag{5}$$

b. Let $\hat{m} \in \{0, 1, ..., n_B - 1\}$ denote the smallest m for which (5) is satisfied [we set $\hat{m} = n_B$ if (5) fails to hold for any $m \in \{0, 1, ..., n_B - 1\}$]. Then, (5) is satisfied also for $m = \hat{m} + 1, ..., n_B$.

(ii) If $m \geq \hat{m}$, the profit maximizing $\mathbf{w}_A$ for A is as follows:

a. when $m = 0$, $w_A^1 = ... = w_A^k = u_B^1$;

b. when $m \in \{1, ..., n_B - 1\}$,

$$\begin{aligned} w_A^{k-m+1} &= w_A^{k-m+2} = ... = w_A^k = 0 & (6)\\ w_A^{k-m''+1} &= \max\{w_A^{k-m''+2}, \mu_m^{k-m''+1}\} \quad \textit{for} \quad m'' = m+1, ..., n_B & (7)\\ w_A^1 &= w_A^2 = ... = w_A^{k-n_B} = w_A^{k-n_B+1} & (8)\end{aligned}$$

We below give the intuition of the results in Proposition 1; we focus on explaining the profit maximizing $\mathbf{w}_A$ conditional on selling $k-m$ units for $m \geq 1$, described in Proposition 1(ii)b.[12] Given A's objective to sell $k-m$ units, A should structure his prices for the best $k-m$ products (the ones to sell) very differently from the prices for the m worst units (the ones not to sell). On the one hand, regarding the m worst products, it is optimal to charge very high prices (higher than their values) so that B does not face any competition from them; precisely, (6) reveals that choosing $w_A^{k-m+1} = w_A^{k-m+2} = ... = w_A^k = 0$ is optimal. The reason is that this pricing maximizes B's profit from accommodation and hence reduces B's temptation to fight. In fact, the pricing allows B to extract full surplus $u_B^1 + ... + u_B^{m'}$ from his best m' products if he wants to sell only $m' \leq m$ products. Then, obviously, B strictly prefers selling m units to selling less than m, and hence downward incentive constraints (i.e. ($\mathrm{IC}_{m,m'}$) for $m' < m$) are trivially satisfied. On the other hand, regarding the best $k-m$ units to sell, the prices should be competitive enough to make it unprofitable for B to sell more than m units. In particular, A cannot extract full surplus from these products since if he attempts to do that, B can sell all of his n_B products by leaving no surplus per product to D, given T1.

To explain the optimal pricing of the best $k-m$ units, suppose that B wants to sell $m+1$ units instead of m units. Lemma 2 shows that B can sell $m+1$ products only if $w_A^{k-m} < u_B^{m+1}$. In this case, B makes a profit equal to $\pi_B(m+1) = u_B^1 + ... + u_B^{m+1} - (m+1)w_A^{k-m}$ and we have

$$\pi_B(m+1) - \pi_B(m) = u_B^{m+1} - (m+1)w_A^{k-m}$$

As we discussed after Lemma 2, $u_B^{m+1} - (m+1)w_A^{k-m}$ is composed of the loss $-mw_A^{k-m}$ on B's best m units (with respect to selling them at full prices) plus the gain $u_B^{m+1} - w_A^{k-m}$ from selling the $(m+1)$-th unit. Therefore, $w_A^{k-m} \geq \frac{u_B^{m+1}}{m+1} = \mu_m^{k-m}$ allows to satisfy $\pi_B(m) \geq \pi_B(m+1)$: note that it is less restrictive than $w_A^{k-m} \geq u_B^{m+1}$. Hence, the smallest value of w_A^{k-m} satisfying ($\mathrm{IC}_{m,m+1}$) is $w_A^{k-m} = \mu_m^{k-m}$, as described in (7). In order to deter B from selling $m+2$ units, we can argue as before. A sufficient condition is $w_A^{k-m-1} \geq u_B^{m+2}$, but

[12] Proposition 1(ii)a is straightforward, as the best way for A to sell k products is to set $w_A^1 = ... = w_A^k$ equal to the value of B's best product, u_B^1, provided that $u_A^k > u_B^1$.

when $w_A^{k-m-1} < u_B^{m+2}$ we must have:

$$\pi(m+2) - \pi(m) = u_B^{m+1} + u_B^{m+2} - (m+2)w_A^{k-m-1} \leq 0,$$

which is equivalent to $w_A^{k-m-1} \geq \mu_m^{k-m-1} = \frac{1}{m+2}(u_B^{m+1} + u_B^{m+2})$. Therefore, ($\text{IC}_{m,m+2}$) is satisfied if $w_A^{k-m-1} \geq \min\{\mu_m^{k-m-1}, u_B^{m+2}\}$. However, w_A^{k-m-1} should also satisfy the monotonicity condition (2) (in particular, $w_A^{k-m-1} \geq w_A^{k-m}$). From $w_A^{k-m-1} \geq \min\{\mu_m^{k-m-1}, u_B^{m+2}\}$ and $w_A^{k-m-1} \geq w_A^{k-m}$, we find that the smallest value of w_A^{k-m-1} satisfying ($\text{IC}_{m,m+2}$) is $w_A^{k-m-1} = \max\{w_A^{k-m}, \mu_m^{k-m-1}\}$, as described in (7).[13] By iterating the argument we obtain the smallest values of w_A^{k-m}, $w_A^{k-m-1}, ..., w_A^{k-n_B+1}$ which satisfy (3), as described in (7). This explains the pricing of the worst $n_B - m$ units of A among the $k - m$ units to sell. Finally, regarding the pricing of the best $k - n_B$ units to sell, we observe that the variables in $(w_A^1, ..., w_A^{k-n_B})$ do not affect (3) and thus each of them can be set equal to $w_A^{k-n_B+1}$ to satisfy the monotonicity condition (2), as described in (8). In this way we have found the smallest values of $w_A^1, ..., w_A^k$ which satisfy (2) and (3).

As we mentioned in section 2, the values in $\mathbf{w}_A$ are feasible only if they satisfy (1) since otherwise there exist no prices $p_A^1 > 0, ..., p_A^k > 0$ such that $w_A^j = u_A^j - p_A^j$ for $j = 1, ..., k$. Hence, u_A^j must be larger than the profit-maximizing w_A^j characterized in Proposition 1(ii). This is why (5) is necessary and sufficient for A to be able to sell $k - m$ units. Notice that Proposition 1(i)b implies that there is an $\hat{m}$ between 0 and n_B such that A is able to sell any number of units between 0 and $k - \hat{m}$, but out arguments above imply that A will always sell at least $k - n_B$ units, if $k > n_B$.

3.3.2 Maximizing A's profit with respect to m

Since Proposition 1 allows to compute $\pi_A(k-m)$ for any $m \geq \hat{m}$, the profit-maximizing m can be found by comparing $\pi_A(k-n_B)$, $\pi_A(k-n_B+1)$, ..., $\pi_A(k-\hat{m})$. Before seeing a few examples and a useful property of π_A, we can improve our understanding of the problem of A by comparing $\pi_A(k-m)$ with $\pi_A(k-m+1)$, in order to examine the incentives of A to increase his supply. Let us use here $w_A^1(m), ..., w_A^{k-m}(m)$ to denote D's net profits from buying A's products, as determined by (7)-(8), when A sells $k - m$ products.

Then we find

$$\begin{aligned} w_A^{k-m}(m) &= \mu_m^{k-m}, \quad w_A^{k-m-1}(m) = \max\{\mu_m^{k-m}, \mu_m^{k-m-1}\}, ..., \\ w_A^{k-n_B+1}(m) &= \max\{\mu_m^{k-m}, \mu_m^{k-m-1}, ..., \mu_m^{k-n_B+1}\} = w_A^1(m) = ... = w_A^{k-n_B}(m). \end{aligned}$$

[13]Actually, w_A^{k-m-1} must be equal to the highest between w_A^{k-m} and $\min\{\mu_m^{k-m-1}, u_B^{m+2}\}$, but (i) when $\min\{\mu_m^{k-m-1}, u_B^{m+2}\} = \mu_m^{k-m-1}$, we can write $w_A^{k-m-1} = \max\{\mu_m^{k-m}, \mu_m^{k-m-1}\}$; (ii) when $\min\{\mu_m^{k-m-1}, u_B^{m+2}\} = u_B^{m+2}$, $w_A^{k-m-1} = \max\{\mu_m^{k-m}, \mu_m^{k-m-1}\}$ still holds because u_B^{m+2} is much smaller than u_B^{m+1} and it turns out that this implies that $w_A^{k-m} = \mu_m^{k-m}$ is larger than both u_B^{m+2} and μ_m^{k-m-1}.

When instead A sells $k-m+1$ products, we have:

$$\begin{aligned} w_A^{k-m+1}(m-1) &= \mu_{m-1}^{k-m+1}, \quad w_A^{k-m}(m-1) = \max\{\mu_{m-1}^{k-m+1}, \mu_{m-1}^{k-m}\}, ..., \\ w_A^{k-n_B+1}(m-1) &= \max\{\mu_{m-1}^{k-m+1}, \mu_{m-1}^{k-m}, ..., \mu_{m-1}^{k-n_B+1}\} = w_A^1(m-1) = ... = w_A^{k-n_B}(m-1). \end{aligned}$$

It is straightforward to see from (4) that $\mu_{m-1}^{k+1-m''} > \mu_m^{k+1-m''}$ for any $m'' \in \{m+1, ..., n_B\}$, thus we have $w_A^{k+1-m''}(m-1) > w_A^{k+1-m''}(m)$ for any $m'' \in \{m+1, ..., k\}$.

The latter inequality is very intuitive: in order to sell one extra unit, (i.e. $k-m+1$ rather than $k-m$ units). A must increase the rent it abandons to D for all the $k-m$ initial units. Thus, when we compare $\pi_A(k-m+1) = \sum_{j=1}^{k-m+1}(u_A^j - w_A^j(m-1))$ with $\pi_A(k-m) = \sum_{j=1}^{k-m}(u_A^j - w_A^j(m))$, we see that $\pi_A(k-m+1)$ contains the additional term $u_A^{k-m+1} - w_A^{k-m+1}(m-1) > 0$, which is A's profit on the $(k-m+1)$-th unit sold, but A's profit on each of his first $k-m$ units is reduced from $u_A^j - w_A^j(m)$ to $u_A^j - w_A^j(m-1)$, as we just proved that $w_A^j(m-1) > w_A^j(m)$ for $j \in \{1, ..., k-m\}$. In words, as it is the case with B, A also faces a *trade off between quantity and rent extraction*: as A sells more units, it should leave more surplus per unit to D. Precisely, as A increases its sales from $k-m$ to $k-m+1$, inducing B to accommodate becomes more difficult for two reasons. First, B's ability to fight is now stronger since he can use his m-th best unit, with value u_B^m, which was previously sold. Second, B has now less to lose by trying to push out a product of A, since selling $m-1$ products makes the profit from accommodation (described just after Lemma 2) smaller than when selling m. Therefore, when A sells one extra unit, in order to induce B not to fight, A should make his units more competitive by leaving D a higher surplus for each unit.

We now present a result which simplifies the task of finding the optimal m. Precisely, we prove a concavity-like property of π_A which states that the marginal profit for A from selling one extra unit is decreasing: the profit increase from selling $k-m+2$ products instead of $k-m+1$ is smaller than the profit increase from selling $k-m+1$ products instead of $k-m$.

Proposition 2 *(i) Suppose that it is feasible for A to sell $k-m+2$ units (i.e. $m-2 \geq \hat{m}$). Then $\pi_A(k-m+2) - \pi_A(k-m+1) \leq \pi_A(k-m+1) - \pi_A(k-m)$.*
(ii) The optimal m for A, denoted by m_A^{**}, is characterized as follows: $\pi_A(m_A^{**}) \geq \max\{\pi_A(m_A^{**}-1), \pi_A(m_A^{**}+1)\}$ if $k-n_B+1 \leq m_A^{**} \leq k-\hat{m}-1$, $\pi_A(m_A^{**}) \geq \pi_A(m_A^{**}-1)$ if $m_A^{**} = k-\hat{m}$, $\pi_A(m_A^{**}) \geq \pi_A(m_A^{**}+1)$ if $m_A^{**} = k-n_B$.

Notice that the concavity-like property of π_A described in Proposition 2(i) implies immediately Proposition 2(ii): in order to test the optimality of m_A^{**}, it suffices to compare the profit as the number of products to sell for A is decreased by one unit or increased by one unit. In what follows, to give further insight, we study some specific settings.

3.3.3 When only the local incentive constraint ($\mathrm{IC}_{m,m+1}$) matters

Let us present first the simple case in which only the local incentive constraint ($\mathrm{IC}_{m,m+1}$) matters. We saw that when A wants to sell $k-m$ units, downward incentive constraint are trivially satisfied but satisfying upward constraints requires A to abandon some surplus to D. We below present a special case in which satisfying only ($\mathrm{IC}_{m,m+1}$) is sufficient to satisfy (3), and this makes it straightforward to derive $\pi_A(k-m)$.

Corollary 1 *Given m such that $\hat{m} \leq m \leq n_B - 2$, if $u_B^{m+2} \leq \frac{1}{m+1}u_B^{m+1}$ then (5) is equivalent to $u_A^{k-m} > \frac{1}{m+1}u_B^{m+1}$. When this condition is satisfied, (6)-(8) imply $w_A^1 = ... = w_A^{k-m} = \frac{1}{m+1}u_B^{m+1} > 0 = w_A^{k-m+1} = ... = w_A^k$; thus $\pi_A(k-m) = u_A^1 + ... + u_A^{k-m} - \frac{k-m}{m+1}u_B^{m+1}$.*

Precisely, if u_B^{m+2} is sufficiently smaller than u_B^{m+1}, it turns out that $\mu_m^{k-m} \geq \mu_m^{k-m-1} \geq ... \geq \mu_m^{k-n_B+1}$ and then (5) is satisfied if and only if $u_A^{k+1-m''} > \mu_m^{k+1-m''}$ holds for $m'' = m+1$, or equivalently $u_A^{k-m} > \frac{1}{m+1}u_B^{m+1}$. If this condition is satisfied, then the optimal prices for A are such that the products he wants to sell give a constant net profit to D equal to $\frac{1}{m+1}u_B^{m+1}$, the profit satisfying ($\mathrm{IC}_{m,m+1}$) with equality. If the condition $u_B^{m+2} \leq \frac{1}{m+1}u_B^{m+1}$ holds *for every* $m \in \{\hat{m}, ..., n_B - 2\}$, then we have

$$\pi_A(k-m+1) - \pi_A(k-m) = u_A^{k-m+1} - \frac{1}{m}u_B^m - (k-m)(\frac{1}{m}u_B^m - \frac{1}{m+1}u_B^{m+1}).$$

Note however that the conditions $\frac{1}{\hat{m}+1}u_B^{\hat{m}+1} \geq u_B^{\hat{m}+2}$, $\frac{1}{\hat{m}+2}u_B^{\hat{m}+2} \geq u_B^{\hat{m}+3}$, ..., $\frac{1}{n_B-1}u_B^{n_B-1} \geq u_B^{n_B}$ are somewhat restrictive, since they imply that the values of B's products decrease quite quickly. This also suggests that in general more than one upward incentive constraints matter, as in the examples below.

3.3.4 Example 1: When $n_B = 3$

Suppose that $n_B = 3$. In order to sell $k-3$ units, A sets

$$p_A^1 = u_A^1, \quad p_A^2 = u_A^2, \quad ..., \quad p_A^{k-3} = u_A^{k-3}, \quad p_A^{k-2} \geq u_A^{k-2}, \quad p_A^{k-1} \geq u_A^{k-1}, \quad p_A^k \geq u_A^k.$$

and then B chooses $p_B^1 = u_B^1$, $p_B^2 = u_B^2$, $p_B^3 = u_B^3$. Hence, $\pi_A(k-3) = u_A^1 + u_A^2 + ... + u_A^{k-3}$. In order to find $\pi_A(k-2)$ we have to consider ($\mathrm{IC}_{2,3}$), which is given by

$$(\mathrm{IC}_{2,3}) \quad w_A^{k-2} \geq \frac{1}{3}u_B^3.$$

Therefore, A chooses

$$p_A^1 = u_A^1 - \frac{1}{3}u_B^3, \quad p_A^2 = u_A^2 - \frac{1}{3}u_B^3, \quad ... \quad , p_A^{k-2} = u_A^{k-2} - \frac{1}{3}u_B^3, \quad p_A^{k-1} \geq u_A^{k-1}, \quad p_A^k \geq u_A^k.$$

which is feasible only if $u_A^{k-2} > \frac{1}{3}u_B^3$. Then, B plays $p_B^1 = u_B^1$, $p_B^2 = u_B^2$. Hence, $\pi_A(k-2) = u_A^1 + u_A^2 + ... + u_A^{k-2} - \frac{k-2}{3}u_B^3$.
In order to find $\pi_A(k-1)$ we need to consider both (IC$_{1,2}$) and (IC$_{1,3}$), which are given by:

$$(\text{IC}_{1,2}) \quad w_A^{k-1} \geq \frac{1}{2}u_B^2.$$

$$(\text{IC}_{1,3}) \quad w_A^{k-2} \geq \max\{\frac{1}{2}u_B^2, \frac{1}{3}(u_B^2 + u_B^3)\}.$$

Hence, satisfying the incentive constraints is feasible if $u_A^{k-1} > \frac{1}{2}u_B^2$ and $u_A^{k-2} > \max\{\frac{1}{2}u_B^2, \frac{1}{3}(u_B^2 + u_B^3)\}$. Then, A chooses

$$\begin{aligned} p_A^j &= u_A^j - \max\{\frac{1}{2}u_B^2, \frac{1}{3}(u_B^2 + u_B^3)\} \text{ for } j = 1, ..., k-2; \\ p_A^{k-1} &= u_A^{k-1} - \frac{1}{2}u_B^2, \quad p_A^k \geq u_A^k. \end{aligned}$$

Then $\pi_A(k-1) = u_A^1 + u_A^2 + ... + u_A^{k-2} + u_A^{k-1} - (k-2)\max\{\frac{1}{2}u_B^2, \frac{1}{3}(u_B^2 + u_B^3)\} - \frac{1}{2}u_B^2$.

Finally, A is able to sell k units if and only if $u_A^k > ku_B^1$, and then $\pi_A(k) = u_A^1 + u_A^2 + ... + u_A^{k-2} + u_A^{k-1} + u_A^k - ku_B^1$.

In order to fix the ideas, suppose that $u_B^2 > 2u_B^3$, so that $\max\{\frac{1}{2}u_B^2, \frac{1}{3}(u_B^2 + u_B^3)\} = \frac{1}{2}u_B^2$. Then, from Proposition 2(ii), we see for instance that it is optimal for A to sell $k-2$ products if $\pi_A(k-2) \geq \max\{\pi_A(k-1), \pi_A(k-3)\}$, which is equivalent to $u_A^{k-2} \geq \frac{k-2}{3}u_B^3$ and $u_A^{k-1} \leq \frac{k-2}{2}(u_B^2 - u_B^3) + \frac{1}{2}u_B^2$. The first inequality implies that the gain on the $(k-2)$-th sold by A, $u_A^{k-2} - \frac{1}{3}u_B^3$, is larger than his loss on the $k-3$ units, $\frac{k-3}{3}u_B^3$, with respect to selling them at full prices. The second inequality means that selling the $(k-1)$-th unit yields a profit of $u_A^{k-1} - \frac{1}{2}u_B^2$ but results in a loss of $\frac{k-2}{2}(u_B^2 - u_B^3)$, which is larger than $u_A^{k-1} - \frac{1}{2}u_B^2$.

3.3.5 Example 2: When all B's products have the same value

Suppose that $u_B^1 = u_B^2 = ... = u_B^{n_B} \equiv u_B > 0$. In this case, for $m(= 1, ..., n_B - 1)$ and $m''(= m+1, ..., n_B)$, we find that $\mu_m^{k+1-m''} = \frac{m''-m}{m''}u_B$. Thus $\mu_m^{k+1-m''}$ is increasing in m''. Given m, the profit-maximizing $w_A^1, ..., w_A^{k-m}$, determined by (7)-(8), are

$$\begin{aligned} w_A^{k-m} &= \frac{1}{m+1}u_B, w_A^{k-m-1} = \frac{2}{m+2}u_B, ..., w_m^{k-n_B+2} = \frac{n_B - m - 1}{n_B - 1}u_B, \\ w_m^{k-n_B+1} &= \frac{n_B - m}{n_B}u_B = w_A^1 = ... = w_A^{n_B}. \end{aligned}$$

If $m \geq \hat{m}$, we have that $\pi_A(k-m) = u_A^1 + ... + u_A^{k-m} - [\frac{1}{m+1} + \frac{2}{m+2} + ... + \frac{n_B-m-1}{n_B-1} + \frac{n_B-m}{n_B}(k - n_B + 1)]u_B$.

In order to find the optimal m, we exploit lemma 2. Thus, $m = n_B$ is optimal if $\pi_A(k - n_B) \geq \pi_A(k - n_B + 1)$, i.e. if $\frac{u_A^{k-n_B+1}}{u_B} \leq \frac{k-n_B+1}{n_B}$. Finally, for m between 1 and $n_B - 1$, m is optimal if $\pi(k - m) - \pi(k - m - 1) \geq 0$ and $\pi(k - m + 1) \leq \pi(k - m)$, i.e.

$$\frac{1}{m} + \frac{1}{m+1} + ... + \frac{1}{n_B - 1} + \frac{k - n_B + 1}{n_B} \geq \frac{u_A^{k-m+1}}{u_B} \qquad \text{and}$$

$$\frac{u_A^{k-m}}{u_B} \geq \frac{1}{m+1} + \frac{1}{m+2} + ... + \frac{1}{n_B - 1} + \frac{k - n_B + 1}{n_B}$$

4 Bundling

In this section we initially assume that each firm practices bundling. Precisely, at stage one firm A chooses q_A of his products to include into his bundle B_A, and a price P_A for B_A.

At stage two, after observing the move of A, B chooses q_B of his products to include into his bundle B_B and a price P_B for B_B. In order to specify the value of a bundle for D, it is not enough to specify the number of units it contains, but it is necessary to know the precise products in the bundle. However, it is intuitive that if i inserts q_i of his products in M_i, it is optimal for him to choose the best q_i products among the ones he can sell. For $i = A, B$, let $U_i(q_i)$ denote the gross value of M_i for D if it includes the q_i best products of i: $U_i(q_i) = \sum_{j=1}^{q_i} u_i^j$. Let $U_{AB}(q_{A,}q_B)$ denote D's gross profit from buying both bundles, taking into account the capacity constraint of D; thus $U_{AB}(q_A, q_B) \leq U_A(q_A) + U_B(q_B)$, with equality if and only if $q_A + q_B \leq k$. Therefore, the net profit of D from buying only M_i is $U_i(q_i) - P_i$, while D's profit from buying B_A and B_B is $U_{AB}(q_A, q_B) - P_A - P_B$.

As we did for the game with individual sales, we apply backwards induction starting with stage three. Clearly, D determines his purchase by maximizing his own payoff. About tie-breaking, we assume that D buys both bundles if $U_{AB}(q_A, q_B) - P_A - P_B \geq \max\{U_A(q_A) - P_A, U_B(q_B) - P_B\}$, while he buys B_B if $U_B(q_B) - P_B = U_A(q_A) - P_A > U_{AB}(q_A, q_B) - P_A - P_B$ (this is consistent with T1).

At stage two, given (q_A, P_A), B wants to choose q_B and (the maximal) P_B such that D decides to buy B_B. In order to achieve this objective, B can choose between two strategies as under individual sale: accommodation and fight. B can try to induce D to buy both bundles or try to induce D to buy only B_B (and block out B_A). Recall from section 2 that m_i^* is the number of firm i's products of among the k best products overall, thus $m_A^* + m_B^* = k$. Before starting the analysis, it is useful to introduce the function

$$\bar{q}(q_A) = \begin{cases} \min\{n_B, k - q_A\} & \text{if } q_A < m_A^* \\ m_B^* & \text{if } q_A \geq m_A^* \end{cases}$$

The interpretation of $\bar{q}(q_A)$ is as follows: if D has purchased B_A which includes A's best q_A units, then $\bar{q}(q_A)$ is the maximal number of products of B which D would effectively distribute under the slot constraint in case D buys B_B as well. The next lemma characterizes B's optimal strategy at stage 2.

Lemma 3 *At stage two, given a pair* (q_A, P_A)*,*
(i) B fights by choosing $q_B = n_B$ *and* $P_B = U_B(n_B) - U_A(q_A) + P_A$ *if* $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$*;*
(ii) B accommodates by choosing $q_B = \bar{q}(q_A)$ *and* $P_B = U_{AB}[q_A, \bar{q}(q_A)] - U_A(q_A)$ *if* $P_A \leq U_{AB}(q_A, n_B) - U_B(n_B)$*.*

Not surprisingly, B ends up fighting (accommodating) when P_A is large (small). Precisely, in order to fight, B includes all its products into B_B since this decreases the relative value to D of buying both bundles against buying only B_B, and at the same times maximizes the value of B_B. Then it is feasible for B to block B_A out when D's profit from buying only B_A is smaller than D's profit from buying only B_B, which is equivalent to $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$.

Suppose now that B accommodates B_A. Notice that for any q_A, B finds it optimal to induce D to buy and distribute at least his m_B^* best units since $\bar{q}(q_A) \geq m_B^*$. Furthermore, if $q_A < m_A^*$, it is optimal for B to sell more than m_B^* units (if $n_B > m_B^*$) since his profit is equal to $U_{AB}(q_A, q_B) - U_A(q_A)$. However, if $q_A \geq m_A^*$, it is optimal for B to sell only m_B^* units since including more than m_B^* units does not affect his profit. Notice also that as long as $q_A \geq m_A^*$ and B accommodates B_A, D buys and distributes at least m_A^* best products of A.

Corollary 2 *Suppose that B accommodates* B_A*. Then,*

(i) For any q_A*, B can induce D to buy and distribute at least his* m_B^* *best units; hence A can never induce D to buy and distribute more than his* m_A^* *best units.*

(ii) Suppose $q_A \geq m_A^*$*. D always buys and distributes at least the* m_A^* *best units of A; hence B can never induce D to buy and distribute more than his* m_B^* *best units.*

The next proposition describes the equilibrium under bundling and shows that each firm i chooses $q_i = m_i^*$ along the equilibrium path.

Proposition 3 *When both firms practice bundling, there exists a unique SPNE and equilibrium strategies are as follows:*
(i) A chooses $q_A = m_A^*$ *and* $P_A = U_{AB}(m_A^*, n_B) - U_B(n_B)$*;*

(ii) B plays as described by Lemma 3, and along the equilibrium path chooses $q_B = m_B^*$ and $P_B = U_{AB}(m_A^*, m_B^*) - U_A(m_A^*) = U_B(m_B^*)$.
(iii) D buys both bundles and hence consumes the k best among both firms' products.

Proof. At stage one, firm A will choose (q_A, P_A) in a way which induces B to accommodate (otherwise A makes no profit), hence $P_A = U_{AB}(q_A, n_B) - U_B(n_B)$ and q_A is selected in order to maximize $U_{AB}(q_A, n_B)$. Then it follows that $q_A = m_A^*$ because $q_A < m_A^*$ implies $U_{AB}(q_A, n_B) < U_{AB}(m_A^*, n_B)$, while $q_A > m_A^*$ implies that some units are included in B_B but do not increase the profit of A. Given $q_A = m_A^*$ and $P_A = U_{AB}(m_A^*, n_B) - U_B(n_B)$, B will choose as described by Lemma 3(ii); thus $q_B = \bar{q}(m_A^*) = m_B^*$ and $P_B = U_{AB}(m_A^*, m_B^*) - U_A(m_A^*) = U_B(m_B^*)$. ■

Given B's best response described in Lemma 3, A chooses the largest P_A which induces B to accommodate (i.e. $P_A = U_{AB}(q_A, n_B) - U_B(n_B)$) and sets $q_A = m_A^*$, since $U_{AB}(q_A, n_B)$ increases with q_A up to $q_A = m_A^*$ and then becomes constant. Then, the highest P_B at which B induces D to buy both bundles is $P_B = U_{AB}(m_A^*, q_B) - U_A(m_A^*)$, from Lemma 3(ii), and this is maximized at $q_B = m_B^*$ since $U_{AB}(m_A^*, q_B)$ increases with q_B up to $q_B = m_B^*$ and then becomes constant. Therefore, D ends up consuming the k best among both firms' products.[14]

5 Social welfare comparison

In this section, we compare the outcome under individual sale and the one under bundling in terms of the social welfare.

In our model, social welfare is equivalent to the gross profit of D and thus it is maximized if D consumes the k best products among both firms' products. In the previous section we found that in the unique SPNE under bundling, D consumes precisely these products. Therefore, under bundling, we always have the socially efficient outcome.

On the contrary, under individual sale, there is no particular reason for competition to lead to the efficient outcome. The analysis in section 3 shows that each firm faces a trade-off between quantity and rent extraction. In our sequential game, the profile of products effectively consumed by D is determined by the first mover, firm A. However, there is no reason that the trade-off for firm A induces him to sell m_A^* number of products. Although we completely characterized each firm's strategy in equilibrium, having a general characterization of the condition under which competition under individual sale leads to the efficient outcome is messy since it depends on the vectors $(u_A^1, ..., u_A^{n_A})$ and $(u_B^1, ..., u_B^{n_B})$.

[14] A higher P_B would make D buy only M_A.

In order to provide the intuition for why the competition under individual sale does not necessarily leads to the efficient allocation, we below provide two simple examples. Let m_i^{**} denote the number of products sold by firm i under individual sale. Obviously, we have $m_A^{**} + m_B^{**} = k$. In what follows, we give an example of $m_A^{**} < m_A^*$ and another example of $m_A^{**} > m_A^*$.

Example of $m_A^{**} < m_A^*$

Suppose $n_B = 1$ and $u_A^k > u_B^1$, so that $m_B^* = 0$. We have

$$\pi_A(k-1) = U_A(k-1);$$

$$\pi_A(k) = U_A(k-1) + u_A^k - ku_B^1.$$

Therefore,

$$\pi_A(k-1) - \pi_A(k) > 0 \quad \text{if and only if} \quad u_A^k < ku_B^1.$$

Hence, if $u_A^k < ku_B^1$, we have $m_A^{**} = k-1 < m_A^* = k$. The intuition for this result is simple. If A sells only $k-1 = k - n_B$ products, he can extract full surplus from his $k-1$ best products since B will not fight. However, if A chooses to sell all k products, it has to leave D a net surplus equal to u_B^1 for each of his product in order to block out B's product. This trade-off between quantity and rent extraction makes A sell only his $k-1$ best products when u_B^1 is not too smaller than u_A^k.

Example of $m_A^{**} > m_A^*$

Consider the setting of example 2 in section 3: we have $u_A^1 > ... > u_A^{m_A^*} > u_B > u_A^{m_A^*+1} > ... > u_A^k$, $u_B^1 = ... = u_B^{m_B^*} = u_B$. Suppose $m_B^* = n_B$. We have:

$$\pi_A(k - m_B^*) = U_A(k - m_B^*);$$

$$\pi_A(k - m_B^* + 1) = U_A(k - m_B^*) + u_A^{m_A^*+1} - (k - m_B^* + 1)\frac{u_B}{m_B^* + 1}.$$

Therefore,

$$\pi_A(k - m_B^* + 1) - \pi_A(k - m_B^*) > 0 \quad \text{iff} \quad u_A^{m_A^*+1} > \frac{(k - m_B^* + 1)}{m_B^* + 1}u_B.$$

For instance, if $u_A^{m_A^*+1}$ is close to u_B^1, we have $\pi_A(k-m_B^*+1) - \pi_A(k-m_B^*) > 0$ if $m_B^* > k/2$.

To sharpen the intuition, suppose $m_A^* = 0$, $m_B^* = k \geq 2$, $u_A^1 \simeq u_B$ Then, $\pi_A(k-m_B^*) = 0$. Hence, A has to sell at least one product to generate a profit. Suppose that A charges $p_A^1 = \varepsilon(> 0)$ very small and very high prices on the other products. If B accommodates A's product, B's profit is $(k-1)u_B$. Instead, if B blocks A's product out, B's profit is

$ku_B - k(u_A^1 - \varepsilon) \simeq k\varepsilon$. Therefore, B prefers accommodation and hence A can sell his inferior product. This example is symmetric to the previous example: A takes advantage of B's trade-off between quantity and rent extraction in order to sell his inferior product.

The next proposition summarizes the main finding in terms of social welfare comparison:

Proposition 4 *(social welfare) (i) Under bundling, the outcome is socially efficient: D always consumes the k best products among both firms' products.*

(ii) Under individual sale, it is not necessarily the case: D can consume some products which are not the k best either from A or from B.

(iii) Therefore, social welfare is higher under bundling than under individual sale.

The intuition for the result in Proposition 4 is that under individual sale, firm i faces not only competition from firm $j(\neq i)$'s products but also competition among its own products while under bundling there is no competition among its own products. This implies that the trade-off between quantity and rent extraction which creates the inefficiency under individual sale does not exist under bundling. Actually, the price that firm i can command for its bundle weakly increases as it includes more products. Furthermore, this price strictly increases as long as the additional product that is included into the bundle belongs to the best k products among all products included in both firms' bundles. This is why firm i includes exactly his m_B^* best products into the bundle and D consumes the k best products by purchasing both bundles.

6 Incentives to bundle

We have examined above the two different regimes of no bundling and bundling. Now we inquire which regime will endogenously emerge when each seller can choose between bundling and no bundling. In short, we find that bundling is weakly dominant for firm B and, given that B bundles, also for A it is weakly dominant to practice bundling.

Proposition 5 *(i) If firm B can make a profit by pricing his products independently, then B can make at least the same profit by bundling.*
(ii) Given that B chooses to bundle, if firm A can make a profit by pricing his products independently, then A can make at least the same profit by bundling.

While Proposition 5(i) suggests that B never loses from bundling Proposition 5(ii) establishes the same result given that B bundles, as established by Proposition 5(i).

Thus, bundling emerges endogenously when it is not forbidden. In order to improve our understanding of this fact, it might be useful to examine the benefits of B from bundling when A uses individual sales. In the case in which B also uses individual sales and wants to sell m products (this objective is attainable if and only if $u_B^m > w_A^{k-m+1}$) we know from Lemma 2 that his profit is $u_B^1 + ... + u_B^m - mw_A^{k-m+1}$. On the other hand, we show in the proof of Proposition 5 that he can make $U_B(m) - (w_A^{k-m+1} + w_A^{k-m+2} + ... + w_A^k)$ by bundling. Therefore, with individual sales he leaves D a net profit equal to w_A^{k-m+1} on each unit of the m units he sells,[15] for a combined value of mw_A^{k-m+1}. With bundling, instead, he needs to leave to D the net value of the m worst products of A, $w_A^{k-m+1} + w_A^{k-m+2} + ... + w_A^k$, which is (weakly) smaller than mw_A^{k-m+1}. The reason for the result is that with individual sales, D can replace each single product of B with the $k - m + 1$-th product of A if the product of B yields D a profit smaller than w_A^{k-m+1}. On the other hand, D has less flexibility when B bundles as he and can substitute B_B only with the m worst units of A. This gives an edge to B and allows him to extract a (weakly) higher price from D.

7 Robustness

In this section, we show that the efficiency property of bundling is robust in various settings.

7.1 Exclusive contracts

In all previous sections, after buying a number of products, D has the freedom to choose the products to occupy the slots. In this subsection, we allow firms to sign exclusive contracts on the use of slots such that if D buys q_i number of products from firm i, i =A,B, D should allocate exclusively q_i number of slots on i's products. Introducing exclusive contracts does not affect the analysis of individual sale since under individual sale, D buys only the products that he will effectively distribute. However, introducing exclusive contracts might affect the analysis of bundling. For instance, we know from corollary 2 that without exclusive contracts, firm i can never induce D to distribute more than m_i^* units. However, if firms can sign exclusive contracts, for instance, firm A can force D to distribute more than m_A^* units. Then, the question is to know whether it is profitable for A to do that.

The next proposition shows that the equilibrium outcome under bundling is the same regardless of whether firms use exclusive contracts or not.[16]

Proposition 6 *Under exclusive contracts, there exists a unique SPNE and equilibrium strategies are as follows, in which* $\hat{q}_B(q_A) \equiv \min\{k - q_A, n_B\}$*:*

[15] w_A^{k-m+1} is the value for D of the best product of A he does not purchase.

[16] Notice however that B's best response is different from the one described in lemma 3.

(i) A chooses $q_A = m_A^*$ and $P_A = U_{AB}(m_A^*, n_B) - U_B(n_B)$;
(ii) Given a pair (q_A, P_A), B blocks B_A out by playing $q_B = n_B$ and $P_B = U_B(n_B) - U_A(q_A) + P_A$ when $P_A > U_A(q_A) + U_B[\hat{q}_B(q_A)] - U_B(n_B)$; conversely, B accommodates by playing $q_B = \hat{q}_B(q_A)$ and $P_B = U_{AB}[q_A, \hat{q}_B(q_A)] - U_A(q_A) = U_B[\hat{q}_B(q_A)]$ if $P_A \leq U_A(q_A) + U_B[\hat{q}_B(q_A)] - U_B(n_B)$.
(iii) Along the equilibrium path, (q_A, P_A, q_B, P_B) are like in the SPNE described in Proposition 3, and thus D still buys both bundles and consumes the k best products among both firms' products.

In order to provide an intuition for our result, we consider what happens if A includes $m_A^* + 1$ units instead of m_A^* units into his bundle. Remember that without exclusive contracts, this does not affect the set of the products that will be effectively distributed by D, which implies that (i) this does affect B's response and (ii) the price that A charges for the bundle remains the same as well. Hence, under T3, A prefers including only m_A^* units into his bundle.

Consider now exclusive deals. Then, if A includes one more unit into his bundle, this affects B's choice between accommodation and flight since if B accommodates B_A, B can sell only $m_B^* - 1$ units and obtains profit equal to $U_B(m_B^* - 1)$, which is smaller than $U_B(m_B^*)$. If B blocks B_A out, he chooses $q_B = n_B$ and P_B such that

$$U_B(n_B) - P_B \geq U_A(m_A^* + 1) - P_A.$$

This implies that in order to induce B to accommodate B_A, A must choose P_A such that

$$U_B(m_B^* - 1) \geq U_B(n_B) - U_A(m_A^* + 1) + P_A.$$

Hence, A's profit is $U_B(m_B^* - 1) + U_A(m_A^* + 1) - U_B(n_B)$, which is smaller than his profit when A sells only m_A^* units $(U_B(m_B^*) + U_A(m_A^*) - U_B(n_B))$. It is interesting to notice that the difference in A's profits is exactly equal to

$$u_A^{m_A^*+1} - u_B^{m_B^*} < 0.$$

If A sells $m_A^* + 1$ units through exclusive contracts, it induces D to replace the m_B^*-th best product of B with the $m_A^* + 1$-th best product of A, which is inferior to the former. Hence, A should compensate D for the reduction in D's surplus by reducing its price by $u_B^{m_B^*} - u_A^{m_A^*+1}$. Therefore, A finds optimal to sell only m_A^* units.

7.2 Simultaneous moves

We can prove that, under bundling, the outcome described by proposition 3 is unaltered if the firms play simultaneously rather than sequentially. This makes more robust our result.

Proposition 7 *Under bundling, if A and B choose* (q_A, P_A) *and* (q_B, P_B) *simultaneously then the unique Nash equilibrium of the game is such that*
(i) A plays $q_A = m_A^*$ *and* $P_A = U_{AB}(m_A^*, n_B) - U_B(n_B)$*, B plays* $q_B = m_B^*$ *and* $P_B = U_{AB}(n_A, m_B^*) - U_A(n_A)$.
(ii) D buys both bundles and consumes the k *best among both firms' products.*

Proof. In order for $(q_A, P_A), (q_B, P_B)$ to be an equilibrium it is necessary that both bundles are purchased by D, otherwise a firm i which makes no profit could would deviate by choosing $q_i = m_i^*$ and $P_i > 0$ but close to zero. Hence, in view of Lemma 3(ii), it is necessary that $P_A \leq U_{AB}(q_A, n_B) - U_B(n_B)$ and $P_B \leq U_{AB}(n_A, q_B) - U_A(n_A)$, otherwise one firm can block the other firm's bundle out. Then, in equilibrium the inequalities will hold as equalities, and $q_i = m_i^*$ for $i = A, B$ since $U_{AB}(q_A, n_B) - U_B(n_B)$ is maximized with respect to q_A at $q_A = m_A^*$ and $U_{AB}(n_A, q_B) - U_A(n_A)$ is maximized with respect to q_B at $q_B = m_B^*$. ■

It is interesting that in the game with simultaneous moves, A makes the same profit as when moves are sequential, while B makes a lower profit. The reason is that, with simultaneous moves, B's bundle runs the risk of being pushed out by A if he chooses a too high price, something which does not occur when B moves second.

We also notice that the outcome of the game with bundling remains the same as the one described by propositions 3 and 7 also if we consider a sequential structure in which B moves first and A moves second (in this case A's profit is higher). In order to see why this result obtains, it suffices to write Lemma 3 after swapping indexes and then repeating the proof of Proposition 3.

8 Conclusion

We studied how bundling affects competition for limited slots in a general setting in which each upstream firm owns a portfolio of distinct products. We found that the outcome under bundling is socially efficient in that only the best products occupy the limited slots. We also proved that this results is quite robust; the results holds regardless of whether we consider a simultaneous or a sequential game, regardless of the order of the players if we consider a sequential game, regardless of whether or not we allow firms to sign exclusive contracts on the use of slots.

On the contrary, we showed that under individual sale, the outcome is not necessarily efficient. Under simultaneous game, there is no equilibrium in pure strategy. Under sequential game, the number of products that each upstream firm sells is determined by a trade-off between quantity and rent extraction such that there is no particular reason to expect that this number coincides with the socially efficient one.

This unambiguous welfare-enhancing result under competing portfolios is quite novel and has strong policy implications which go beyond the rule of reason standard based on the existing literature on bundling.

9 Appendix

9.1 Proof of Lemma 1

What matters for D's purchases (hence for A's and B's profits) are the vectors $\hat{\mathbf{w}}_A$ and $\hat{\mathbf{w}}_B$. Given $(\hat{\mathbf{w}}_A, \hat{\mathbf{w}}_B)$, suppose that $\mathbf{w}_B \neq \hat{\mathbf{w}}_B$ and let m_B denote the number of products which D purchases from B; this means that D buys from B the products with net profits $w_B^{(1)}, w_B^{(2)}, ..., w_B^{(m_B)}$. Let $u_B^{(j)}$ represent D's gross profit of the product with the net profit $w_B^{(j)}$. Then, B's profit is given by

$$\pi_B = \sum_{j=1}^{m_B} \left[u_B^{(j)} - w_B^{(j)} \right].$$

Now suppose that B choose prices $\tilde{p}_B^j = u_B^j - w_B^{(j)}$ for $j = 1, ..., n_B$, and denote by $\tilde{w}_B^j$ the resulting net profits for D. Then the same vector $\hat{\mathbf{w}}_B$ as before is obtained and $\tilde{w}_B^1 = w_B^{(1)} \geq \tilde{w}_B^2 = w_B^{(2)} \geq ... \geq \tilde{w}_B^{n_B} = w_B^{(n_B)}$. Thus, T1 and T2 imply that D will still purchase m_B number of products from B, and now B's profit is

$$\tilde{\pi}_B = \sum_{j=1}^{m_B} (u_B^j - \tilde{w}_B^j)$$

By definition of u_B^j, $\tilde{\pi}_B$ is at least as large as π_B and, in particular, $\tilde{\pi}_B > \pi_B$ if $\sum_{j=1}^{m_B} u_B^j > \sum_{j=1}^{m_B} u_B^{(j)}$, that is if the products sold initially by B are different from B's m_B products with the highest net profits.

The above argument applies to firm B since it chooses $\mathbf{p}_B$ after observing $\mathbf{p}_A$, and thus can take $\mathbf{w}_A$ as given. Conversely, firm A cannot take $\mathbf{w}_B$ as given and the argument must be slightly augmented as follows. If, given $\mathbf{w}_A$, it is optimal for B to choose prices such that a certain $\mathbf{w}_B$ is obtained, any $\mathbf{p}_A$ which leaves unaltered $\mathbf{w}_A$ leaves unaffected the incentives for firm B, and also his best reply prices. This allows to argue as above for B: in case that $\mathbf{w}_A \neq \hat{\mathbf{w}}_A$, let A choose $\tilde{p}_A^j = u_A^j - w_A^{(j)}$ for $j = 1, ..., k$ so that $\tilde{w}_B^j = w_B^{(j)}$ for $j = 1, ..., k$ and the same vector $\hat{\mathbf{w}}_A$ as before is obtained. Then, with respect to the initial situation, (i) B will not change his reply; (ii) D will still buy m_A products of A; (iii) A's profit will not decrease.

9.2 Proof of Proposition 1

Proof of (i)a There exists $\mathbf{w}_A$ such that D will buy $k-m$ units from A if and only if there exists $\mathbf{w}_A$ which satisfies (1), (2) and (3). Thus, since it is more likely that (1) is satisfied the smaller are $w_A^1, ..., w_A^k$, in order to prove (i)a we first find the smallest values of $w_A^1, ..., w_A^k$ which satisfy (2) and (3), and then we show that these values satisfy (1) if and only if (5) holds.

By Lemma 2, there exists $\mathbf{p}_B$ such that D buys m' units from B if and only if $u_B^{m'} > w_A^{k-m'+1}$. In particular, it is feasible for B to sell $m \in \{1, ..., n_B - 1\}$ units if and only if $u_B^m > w_A^{k-m+1}$. If firm A chooses w_A^{k-m+1} such that $w_A^{k-m+1} \geq u_B^m$, then it would actually sell at least $k-m+1$ units; thus it must be the case that $u_B^m > w_A^{k-m+1}$. This inequality implies $u_B^{m'} > w_A^{k-m'+1}$ for $m' = 1, ..., m-1$. Therefore, for $m' < m$, ($\text{IC}_{m,m'}$) is equivalent to

$$\pi_B(m) - \pi_B(m') = u_B^{m'+1} + ... + u_B^m - m \max\left\{w_A^{k-m+1}, 0\right\} + m' \max\left\{w_A^{k-m'+1}, 0\right\} \geq 0. \quad (9)$$

For $m'' > m$, instead, $u_B^m > w_A^{k-m+1}$ does not imply $u_B^{m''} > w_A^{k-m''+1}$. In case that $u_B^{m''} \leq w_A^{k-m''+1}$, we have $\pi_B(m'') = 0$ and then ($\text{IC}_{m,m''}$) is trivially satisfied. In case that $u_B^{m''} > w_A^{k-m''+1}$, then ($\text{IC}_{m,m''}$) is equivalent to

$$\pi_B(m'') - \pi_B(m) = u_B^{m+1} + ... + u_B^{m''} - m'' \max\left\{w_A^{k-m''+1}, 0\right\} + m \max\left\{w_A^{k-m+1}, 0\right\} \leq 0. \quad (10)$$

Therefore, (3) reduces to (9) for $m' = 1, ..., m-1$, and to $u_B^{m''} \leq w_A^{k-m''+1}$ and/or (10) for $m'' = m+1, ..., n_B$.

We first prove that it is convenient to choose $w_A^{k-m+1} = w_A^{k-m+2} = ... = w_A^k = 0$. For $m'' = m+1, ..., n_B$, the value of w_A^{k-m+1} which most relaxes (10) is $w_A^{k-m+1} = 0$, and this [together with (2)], implies $w_A^{k-m+2} = ... = w_A^k = 0$; these values of $(w_A^{k-m+2}, ..., w_A^k)$ satisfy (9) for any $m' \in \{1, ..., m-1\}$ and do not affect ($\text{IC}_{m,m''}$) for $m'' > m$. Thus, with $w_A^{k-m+1} = w_A^{k-m+2} = ... = w_A^k = 0$ we have taken care of (9). We now turn our attention to (10).

Given $w_A^{k-m+1} = 0$, (10) is equivalent to $w_A^{k-m''+1} \geq \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$. In particular, for $m'' = m+1$ we find

$$w_A^{k-m} \geq \frac{1}{m+1} u_B^{m+1} \quad (11)$$

This condition is less restrictive than $w_A^{k-m} \geq u_B^{m+1}$, the other way to satisfy ($\text{IC}_{m,m+1}$), and therefore ($\text{IC}_{m,m+1}$) is satisfied if and only if (11) holds – notice that the right hand side of (11) is μ_m^{k-m}. For $m'' = m+2$, ($\text{IC}_{m,m+2}$) is satisfied if and only if

$$w_A^{k-m-1} \geq \min\left\{\frac{1}{m+2}(u_B^{m+1} + u_B^{m+2}), u_B^{m+2}\right\} \quad (12)$$

and since $u_B^{m+1} \geq u_B^{m+2}$, either one can be the minimum in the right hand side of (12). Likewise, for $m'' = m+3, ..., n_B$, ($\text{IC}_{m,m''}$) is satisfied if and only if

$$w_A^{k-m''+1} \geq \min\left\{\frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''}), u_B^{m''}\right\} = \min\{\mu_m^{k-m''+1}, u_B^{m''}\}$$

In general, however, we cannot set $w_A^{k-m''+1} = \min\{\mu_m^{k-m''+1}, u_B^{m''}\}$ for $m'' = m+1, ..., n_B$ because (2) may be violated. The lowest values for $w_A^{k-m}, w_A^{k-m-1}, ..., w_A^{k-n_B+1}$ which satisfy ($\text{IC}_{m,m''}$) and (2) are given by $w_A^{k-m''+1} = \max\left\{w_A^{k-m''+2}, \min\{\mu_m^{k-m''+1}, u_B^{m''}\}\right\}$ for $m'' = m+1, ..., n_B$, but we can actually prove that this is equivalent to setting $w_A^{k-m''+1} = \max\{w_A^{k-m''+2}, \mu_m^{k-m''+1}\}$ for $m'' = m+1, ..., n_B$, or equivalently $w_A^{k-m''+1} = \max\{\mu_m^{k-m}, ..., \mu_m^{k-m''+1}\}$ for $m'' = m+1, ..., n_B$. Precisely, if $\min\{\mu_m^{k-m''+1}, u_B^{m''}\} = u_B^{m''}$ then $\max\left\{w_A^{k-m''+2}, \min\{\mu_m^{k-m''+1}, u_B^{m''}\}\right\} = w_A^{k-m''+2} = \max\{w_A^{k-m''+2}, \mu_m^{k-m''+1}\}$. In order to see this fact, suppose that $\min\{\mu_m^{k-m''+1}, u_B^{m''}\} = u_B^{m''}$ for some $m'' \in \{m+2, m+3, ..., n_B\}$, and that this is the smallest m'' with this property. Then $u_B^{m''} \leq \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$, or equivalently $u_B^{m''} \leq \frac{1}{m''-1}(u_B^{m+1} + ... + u_B^{m''-1}) = \mu_m^{k-m''+2}$. On the other hand, $\min\{\mu_m^{k-m''+2}, u_B^{m''-1}\} = \mu_m^{k-m''+2}$ by definition of m'', thus $w_A^{k-m''+2} \geq \mu_m^{k-m''+2}$ and $w_A^{k-m''+1} = \max\{w_A^{k-m''+2}, u_B^{m''}\} = w_A^{k-m''+2}$. Furthermore $\mu_m^{k-m''+2} \geq \mu_m^{k-m''+1}$ is true because it is equivalent to $u_B^{m''} \leq \mu_m^{k-m''+2}$, which we know to be true. Thus, $w_A^{k-m''+1}$ can be written as $\max\{w_A^{k-m''+2}, \mu_m^{k-m''+1}\}$, both when $\mu_m^{k+1-m''} < u_B^{m''}$ (this is obvious) and when $\mu_m^{k+1-m''} \geq u_B^{m''}$ (as we just proved).

Finally, we observe that no incentive constraint imposes any restriction on $w_A^1, w_A^2, ..., w_A^{k-n_B}$; thus we can pick $w_A^1 = w_A^2 = ... = w_A^{k-n_B} = w_A^{k-n_B+1}$ to satisfy (2).

In this way we have identified the lowest values of $w_A^1, ..., w_A^k$ which satisfy (2) and (3), and they are described by (6)-(8). However, these values are feasible if and only if they satisfy (1). Clearly, the conditions $w_A^j < u_A^j$ for $j \in \{m+1, ..., k\}$ are satisfied given (6). For $j \in \{k-n_B+1, ..., k-m\}$ we have $w_A^j = \max\{w_A^{j+1}, \mu_m^j\}$, and thus $w_A^j < u_A^j$ for $j \in \{k-n_B+1, ..., k-m\}$ if and only if (5) is satisfied. Finally, from $u_A^{k-n_B+1} > w_A^{k-n_B+1}$ it follows that $u_A^j > w_A^j = w_A^{k-n_B+1}$ for $j = 1, ..., k-n_B$. This establishes that A is able to sell $k-m$ units if and only if (5) is satisfied.

Proof of (i)b Now we suppose that (5) is satisfied for a certain $m^* \in \{0, 1, ..., n_B - 2\}$, and show that (5) is satisfied also for $m = m^* + 1$. If A wants to sell $k - m^* - 1$ units, (5) reduces to $u_A^{k+1-m''} > \mu_{m+1}^{k+1-m''}$ for $m'' = m^* + 2, ..., n_B$. This condition holds, as long as (5) is satisfied, because it involves a subset of the inequalities which appear in (5) and $\mu_{m+1}^{k+1-m''} < \mu_m^{k+1-m''}$ for $m'' = m^* + 2, ..., n_B$.

Proof of (ii) If we assume that (5) is satisfied for a certain m, then it is straightforward to see that the values of $w_A^1, ..., w_A^k$ determined by (6)-(8) maximize the profit of A. Indeed, (6)-(8) identify the smallest values of $w_A^1, ..., w_A^k$ which satisfy (2) and (3), and $\pi_A(k-m)$

is decreasing in $w_A^1, ..., w_A^k$.

9.3 Proof of Proposition 2

Since $\pi_A(k-m) = \sum_{j=1}^{k-m}[u_A^j - w_A^j(m)]$, we find

$$\begin{aligned}
\pi_A(k-m+1) - \pi_A(k-m) &= u_A^{k-m+1} - w_A^{k-m+1}(m-1) - \sum_{j=1}^{k-m}[w^j_{A(m-1)} - w_A^j(m)] \\
&= u_A^{k-m+1} - w_A^{k-m+1}(m-1) - [w_A^{k-m}(m-1) - w_A^{k-m}(m)] \\
&\quad -[w_A^{k-m-1}(m-1) - w_A^{k-m-1}(m)] - ... - [w_A^1(m-1) - w_A^1(m)]
\end{aligned}$$

and

$$\begin{aligned}
\pi_A(k-m+2) - \pi_A(k-m+1) &= u_A^{k-m+2} - w_A^{k-m+2}(m-2) - \sum_{j=1}^{k-m+1}[w_A^j(m-2) - w_A^j(m-1)] \\
&= u_A^{k-m+2} - w_A^{k-m+2}(m-2) - [w_A^{k-m+1}(m-2) - w_A^{k-m+1}(m-1)] \\
&\quad -[w_A^{k-m}(m-2) - w_A^{k-m}(m-1)] \\
&\quad -[w_A^{k-m-1}(m-2) - w_A^{k-m-1}(m-1)] - ... \\
&\quad -[w_A^1(m-2) - w_A^1(m-1)]
\end{aligned}$$

In order to prove that $\pi_A(k-m+2) - \pi_A(k-m+1) \leq \pi_A(k-m+1) - \pi_A(k-m)$ it suffices to show that

$$w_A^{k-m+1}(m-1)+[w_A^{k-m}(m-1)-w_A^{k-m}(m)]+[w_A^{k-m-1}(m-1)-w_A^{k-m-1}(m)]+...+[w_A^1(m-1)-w_A^1(m)]$$

is smaller (or equal) than

$$\begin{aligned}
&w_A^{k-m+2}(m-2) + [w_A^{k-m+1}(m-2) - w_A^{k-m+1}(m-1)] + [w_A^{k-m}(m-2) - w_A^{k-m}(m-1)] \\
&+[w_A^{k-m-1}(m-2) - w_A^{k-m-1}(m-1)] + ... + [w_A^1(m-2) - w_A^1(m-1)]
\end{aligned}$$

since $u_A^{k-m+2} \leq u_A^{k-m+1}$. In order to accomplish this task, we first prove that

$$w_A^{k-m+1}(m-1) \leq w_A^{k-m+2}(m-2) + w_A^{k-m+1}(m-2) - w_A^{k-m+1}(m-1) \tag{13}$$

and then we show that

$$w_A^{k+1-m''}(m-1) - w_A^{k+1-m''}(m) \leq w_A^{k+1-m''}(m-2) - w_A^{k+1-m''}(m-1) \tag{14}$$

for $m'' = m+1, ..., k$.

We find from (4) and (7) that $w_A^{k+1-m}(m-1) = \frac{1}{m}u_B^m$, $w_A^{k+2-m}(m-2) = \frac{1}{m-1}u_B^{m-1}$ and $w_A^{k+1-m}(m-2) = \max\{\frac{1}{m-1}u_B^{m-1}, \frac{1}{m}(u_B^{m-1} + u_B^m)\}$. Thus (13) is equivalent to $\frac{2}{m}u_B^m \leq$

$\frac{1}{m-1}u_B^{m-1} + \max\{\frac{1}{m-1}u_B^{m-1}, \frac{1}{m}(u_B^{m-1}+u_B^m)\}$, and it is easy to see that this inequality holds for either value of $\max\{\frac{1}{m-1}u_B^{m-1}, \frac{1}{m}(u_B^{m-1}+u_B^m)\}$.

About (14), we start by observing that if the inequalities $\mu_m^{k-m} \leq \mu_m^{k-m-1} \leq ... \leq \mu_m^{k-n_B+1}$ hold, then $w_A^{k+1-m''}(m) = \mu_m^{k+1-m''}$ for $m'' = m+1, ..., n_B$. In the opposite case, $\mu_m^{k+1-m''} > \mu_m^{k+1-(m''+1)}$ for some m'' between $m+1$ and $n_B - 1$ and we use $m''(m)$ to denote the smallest m'' for which this inequality holds;[17] notice that by using (4) we find that $\mu_m^{k+1-m''} > \mu_m^{k+1-(m''+1)}$ is equivalent to $\mu_m^{k+1-m''} = \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''}) > u_B^{m''+1}$. Then it turns out that $\mu_m^{k+1-m''} > \mu_m^{k+1-(m''+1)}$ for $m'' = m''(m)+1, ..., n_B - 1$,[18] and thus $w_A^{k+1-m''}(m) = \mu_m^{k+1-m''}$ for $m'' = m+1, ..., m''(m)$, and $w_A^{k+1-m''}(m)$ is constantly equal to $\mu_m^{k+1-m''(m)}$ for $m'' = m''(m)+1, ..., n_B$.
Likewise, $\mu_{m-1}^{k+1-m''} > \mu_{m-1}^{k+1-m''-1}$ if and only if $\mu_{m-1}^{k+1-m''} = \frac{1}{m''}(u_B^m + u_B^{m+1} + ... + u_B^{m''}) > u_B^{m''+1}$, and we let $m''(m-1)$ denote the smallest m'' between m and $n_B - 1$ for which this inequality holds. Notice that $m''(m-1) \leq m''(m)$ because $\mu_{m-1}^{k+1-m''} - \mu_m^{k+1-m''} = \frac{1}{m''}u_B^m > 0$. Finally, $\mu_{m-2}^{k+1-m''} > \mu_{m-2}^{k+1-m''-1}$ if and only if $\mu_{m-2}^{k+1-m''} = \frac{1}{m''}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''}) > u_B^{m''+1}$, and we let $m''(m-2)$ denote the smallest m'' between $m-1$ and n_B for which this inequality is satisfied; we have $m''(m-2) \leq m''(m-1)$ because $\mu_{m-2}^{k+1-m''} - \mu_{m-1}^{k+1-m''} = \frac{1}{m''}u_B^{m-1} > 0$. Thus, as m'' goes from $m+1$ to n_B, $w_A^{k+1-m''}(m-2)$ may become constant at some point, but not later than $w_A^{k+1-m''}(m-1)$, which in turn will not become constant (if it will) later than $w_A^{k+1-m''}(m)$.
Now we prove that (14), or equivalently

$$2w_A^{k+1-m''}(m-1) \leq w_A^{k+1-m''}(m-2) + w_A^{k+1-m''}(m), \tag{15}$$

is satisfied for $m'' = m+1, ..., k$.
Step 1 The case of $m+1 \leq m'' < m''(m-2)$. Then $w_A^{k+1-m''}(m-2) = \frac{1}{m''}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''})$, $w_A^{k+1-m''}(m-1) = \frac{1}{m''}(u_B^m + u_B^{m+1} + ... + u_B^{m''})$ and $w_A^{k+1-m''}(m) = \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$. As a consequence, (15) reduces to $2u_B^m \leq u_B^{m-1} + u_B^m$, which is satisfied.
Step 2 The case of $m''(m-2) \leq m'' < m''(m-1)$. Then $w_A^{k+1-m''}(m-2) = \frac{1}{m''(m-2)}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''(m-2)}) > \frac{1}{m''}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''})$, $w_A^{k+1-m''}(m-1) = \frac{1}{m''}(u_B^m + u_B^{m+1} + ... + u_B^{m''})$ and $w_A^{k+1-m''}(m) = \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$. We know that (15) would hold if $w_A^{k+1-m''}(m-2)$ were equal to $\frac{1}{m''}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''})$, thus (15) *a fortiori* holds since $w_A^{k+1-m''}(m-2) > \frac{1}{m''}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''})$.

[17] If $\mu_m^{k-m} \leq \mu_m^{k-m-1} \leq ... \leq \mu_m^{k-n_B+1}$, then we set $m''(m) = n_B$. A similar remark applies to $m''(m-1)$ and $m''(m-2)$ defined below.

[18] We know that $\mu_m^{k+1-m''} > \mu_m^{k+1-(m''+1)}$ is equivalent to $u_B^{m+1} + ... + u_B^{m''} > m''u_B^{m''+1}$, and when this inequality is satisfied at $m'' = m''(m)$ we find that it is satisfied also at $m'' = m''(m)+1$ since $u_B^{m''+1} \geq u_B^{m''+2}$.

Step 3 The case of $m''(m-1) \leq m'' < m''(m)$. Then $w_A^{k+1-m''}(m-2) = \frac{1}{m''(m-2)}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''(m-2)})$, $w_A^{k+1-m''}(m-1) = \frac{1}{m''(m-1)}(u_B^m + u_B^{m+1} + ... + u_B^{m''(m-1)}) > \frac{1}{m''}(u_B^m + u_B^{m+1} + ... + u_B^{m''})$ and $w_A^{k+1-m''}(m) = \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$. We know from step 2 that (15) holds at $m'' = m''(m-1) - 1$. As m'' increases to $m''(m-1)$, and beyond, $w_A^{k+1-m''}(m-1)$ and $w_A^{k+1-m''}(m-2)$ remain constant while $w_A^{k+1-m''}(m)$ increases. Thus (15) is still satisfied.

Step 4 The case of $m''(m) \leq m'' \leq n_B$. Then $w_A^{k+1-m''}(m-2) = \frac{1}{m''(m-2)}(u_B^{m-1} + u_B^m + u_B^{m+1} + ... + u_B^{m''(m-2)})$, $w_A^{k+1-m''}(m-1) = \frac{1}{m''(m-1)}(u_B^m + u_B^{m+1} + ... + u_B^{m''(m-1)})$ and $w_A^{k+1-m''}(m) = \frac{1}{m''(m)}(u_B^{m+1} + ... + u_B^{m''(m)})$. We know from step 3 that (15) holds at $m'' = m''(m) - 1$. As m'' increases to $m''(m)$, and beyond, we have that $w_A^{k+1-m''}(m-1)$, $w_A^{k+1-m''}(m-2)$ and $w_A^{k+1-m''}(m)$ all remain constant; thus (15) still holds.[19]

Step 5 The case of $m'' = n_B + 1, ..., k$. From (8) we see that in this case (15) is reduced to $2w_A^{k+1-n_B}(m-1) \leq w_A^{k+1-n_B}(m-2) + w_A^{k+1-n_B}(m)$, and we have proved in step 4 that this inequality is satisfied.

9.4 Proof of Corollary 1

We know from Proposition 1 that $w_A^{k-m} = \frac{1}{m+1}u_B^{m+1}$ and that $w_A^{k+1-m''} = \max\{\frac{1}{m+1}u_B^{m+1}, \frac{1}{m+2}(u_B^{m+1} + u_B^{m+2}), ..., \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})\}$ (recall that $\mu_m^{k+1-m''} = \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})$) for $m'' = m+2, ..., n_B$. Given that $\frac{1}{m+1}u_B^{m+1} \geq u_B^{m+2}$, we infer that $\frac{1}{m+1}u_B^{m+1} \geq u_B^{m+3} \geq ... \geq u_B^{m''}$. This implies that $\frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''}) \leq \frac{1}{m''}[u_B^{m+1} + \frac{1}{m+1}u_B^{m+1}(m''-m-1)] = \frac{1}{m+1}u_B^{m+1}$. Thus, $\max\{\frac{1}{m+1}u_B^{m+1}, \frac{1}{m+2}(u_B^{m+1} + u_B^{m+2}), ..., \frac{1}{m''}(u_B^{m+1} + ... + u_B^{m''})\} = \frac{1}{m+1}u_B^{m+1}$ and $w_A^{k+1-m''} = \frac{1}{m+1}u_B^{m+1}$ for $m'' = m+2, ..., n_B$.

9.5 Proof of Lemma 3

For B, it is possible to block B_A out if and only if the following inequalities are satisfied:

$$U_B(q_B) - P_B > U_{AB}(q_A, q_B) - P_A - P_B \quad (16)$$

$$U_B(q_B) - P_B \geq U_A(q_A) - P_A \quad (17)$$

Given (q_A, P_A), in order to relax (16) it is the best for B to choose $q_B = n_B$, as the left hand side of (16) increases (weakly) more quickly with respect to q_B than the right hand side; at $q_B = n_B$, (16) reduces to $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$. Furthermore, the highest P_B

[19] By invoking very similar argument to the ones used in steps 1-4 we can deal with the case in which $m''(m-2) = m''(m-1)$, or $m''(m-1) = m''(m)$, or $m''(m-2) = m''(m-1) = m''(m)$. We skip the details for the sake of brevity.

consistent with (17) is $U_B(q_B) - U_A(q_A) + P_A$, and this is maximized at $q_B = n_B$. Therefore, blocking out B_A is feasible if and only if $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$, and in such a case it yields B a payoff of $U_B(n_B) - U_A(q_A) + P_A$.
Conversely, for B to accommodate B_A, it is necessary and sufficient that (q_B, P_B) satisfies $U_{AB}(q_A, q_B) - P_A - P_B \geq \max\{U_A(q_A) - P_A, U_B(q_B) - P_B\}$, or equivalently

$$P_B \leq U_{AB}(q_A, q_B) - U_A(q_A) \tag{18}$$

$$P_A \leq U_{AB}(q_A, q_B) - U_B(q_B) \tag{19}$$

Hence, by accommodating B_A, B can realize a profit of $P_B = U_{AB}(q_A, q_B) - U_A(q_A)$
Suppose now that $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$, so that it is feasible for B to block B_A out. Then it is easy to see that $U_B(n_B) - U_A(q_A) + P_A > U_{AB}(q_A, q_B) - U_A(q_A)$, which means that B gets a strictly larger profit by pushing B_A out than by accommodating. Therefore, B will block M_A whenever this it feasible, i.e. when $P_A > U_{AB}(q_A, n_B) - U_B(n_B)$; B will instead accommodate when $P_A \leq U_{AB}(q_A, n_B) - U_B(n_B)$. In the latter case, it is profitable for B to choose $q_B = \bar{q}(q_A)$ because this maximizes $U_{AB}(q_A, q_B) - U_A(q_A)$. We have thus identified B's strategy in any SPNE.

9.6 Proof of Proposition 5

(i) Suppose that B does not bundle, and let S_B the set of products D buys from B, with profit $\sum_{j \in S_B} p_B^j$. Then, let B offer the bundle composed of the products in S_B, at the price $\sum_{j \in S_B} p_B^j$. With respect to the previous setting, D has now less flexibility in his purchases since he cannot buy only a few products in S_B. However, he can still buy the same products he was buying previously, and at the same aggregate price. Thus D will buy the same products of A as before, and the bundle of B. This means that, by bundling suitably a set of products, B can make at least the same profit as with individual sales.
(ii) Suppose that A does not practice bundling. Then, at stage two, B chooses q_B and P_B after observing $w_A^1, ..., w_A^k$. We start by finding B's best reply. First notice that q_B needs to satisfy $u_B^{q_B} \geq w_A^{k-q_B+1}$, otherwise D will not distribute all products included in B_B. Then, in order to determine the optimal P_B, we have that D's payoff is $U_B(q_B) - (w_A^{k-q_B+1} + w_A^{k-q_B+2} + ... + w_A^k)$ if he buys B_B, while it is $w_A^1 + ... + w_A^k$ otherwise. Thus, the optimal P_B is $U_B(q_B) - (w_A^{k-q_B+1} + w_A^{k-q_B+2} + ... + w_A^k)$, and the optimal q_B is the largest value which satisfies such that $u^{q_B} \geq w_A^{k-q_B+1}$.

In the case that A wants to sell $k-m$ units, he needs to choose $w_A^1 = ... = w_A^{k-m} = u_B^{m+1}$ (that requires $u_A^{k-m} > u_B^{m+1}$, or equivalently $k-m \leq m_A^*$) and, for instance, $w_A^{k-m+1} = ... = w_A^k = 0$. The profit A can make by selling $k-m$ products with no bundling is therefore

$u_A^1 + ... + u_A^{k-m} - (k-m)u_B^{m+1}$. In the case that A chooses to bundle, and still wants to sell $k - m \leq m_A^*$ units, we know from Lemma 3(ii) that the highest profit he can make is $U_{AB}(k-m, n_B) - U_B(n_B)$, which is equal to $u_A^1 + ... + u_A^{k-m} - (u_B^{m+1} + ... + u_B^{n_B})$, the value of the $k-m$ best units of A minus the value of the $n_B - m$ worst units of B. In order to conclude the proof, we show that $u_A^1 + ... + u_A^{k-m} - (k-m)u_B^{m+1} \leq u_A^1 + ... + u_A^{k-m} - (u_B^{m+1} + ... + u_B^{n_B})$. This inequality is equivalent to $u_B^{m+1} + ... + u_B^{n_B} \leq (k-m)u_B^{m+1}$, which is satisfied as (i) the number of terms on the left hand side is $n_B - m \leq k - m$; (ii) each of the terms on the left hand side is not larger than u_B^{m+1}. In particular, equality holds if and only if $k = n_B$ and $u_B^{m+1} = u_B^{n_B}$.

9.7 Proof of Proposition 6

As we know from the proof of Lemma 3, for B it is possible to push B_A out if and only if (16) and (17) are satisfied. Again, the highest P_B which satisfies (17) is $P_B = U_B(q_B) - U_A(q_A) + P_A$, and it is maximized with respect to q_B at $q_B = n_B$. About (16), it is still true that it is most relaxed when $q_B = n_B$, but it is important to notice that D cannot buy both B_A and B_B when $q_A + q_B > k$, since he cannot distribute all objects in both bundles. Thus, (16) reduces to $P_A > U_{AB}(q_A, n_B) - U_B(n_B) = U_A(q_A)$ when $q_A \leq k - n_B$ and becomes irrelevant when $q_A > k - n_B$.[20] In the latter case, B can block B_A out if and only if $P_A > U_A(q_A) - U_B(n_B)$, otherwise $P_B = U_B(n_B) - U_A(q_A) + P_A \leq 0$.
If instead B wants to accommodate, then (q_B, P_B) needs to satisfy (18)-(19) and $q_B \leq k - q_A$, otherwise D cannot buy $B_A \& B_B$; thus the right hand side of (19) is equal to $U_A(q_A)$. In order to maximize the right hand side of (18), B chooses $\hat{q}_B(q_A) = \min\{k - q_A, n_B\}$ and earns profit $U_B(\hat{q}_B)$, as long as $P_A \leq U_A(q_A)$. Therefore, in the case of $q_A > k - n_B$ and $U_A(q_A) - U_B(n_B) < P_A \leq U_A(q_A)$, B can choose between accommodating and pushing out B_A. By comparing the respective profits $U_B(k - q_A)$ and $U_B(n_B) - U_A(q_A) + P_A$, we see that B pushes B_A out when $P_A > U_A(q_A) + U_B(k - q_A) - U_B(n_B)$.
By using these results, at stage one A infers that he can earn $U_A(q_A)$ by choosing $q_A \leq k - n_B$, while he can make $U_A(q_A) + U_B(k - q_A) - U_B(n_B)$ if $q_A > k - n_B$. Since $k \geq n_B$, we have that $m_A^* \geq k - n_B$ and thus 1's profit is maximized at $q_A = m_A^*$.

[20] In other words, when $q_A > k - n_B$, by choosing $q_B = n_B$ firm B reduces the possible choices of D to buying only M_A or buying only M_B.

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