Forecasting and Information Sharing in Supply Chains Under Quasi-ARMA Demand

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Abstract

In this paper, we revisit the problem of demand propagation in a multi-stage supply chain in which the retailer observes ARMA demand. In contrast to previous work, we show how each player constructs the order based upon its *best* linear forecast of leadtime demand given its available information. In order to characterize how demand propagates through the supply chain we construct a new process which we call quasi-ARMA or QUARMA. QUARMA is a generalization of the ARMA model. We show that the typical player observes QUARMA demand and places orders that are also QUARMA. Thus, the demand propagation model is QUARMA-in-QUARMA-out. We study the value of information sharing between adjacent players in the supply chain. We demonstrate that under certain conditions information sharing can have unbounded benefits. Our analysis hence reverses and sharpens several previous results in the literature involving information sharing and also opens up many questions for future research.

KEYWORDS: Supply Chain Management, Information Sharing, Time Series, ARMA, Invertibility.

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1 Introduction

We study demand propagation in a multi-stage supply chain in which the retailer observes ARMA demand with Gaussian white noise (shocks). Similar to previous research, we assume each supply chain player constructs a linear forecast for the leadtime demand and uses it to determine the order quantity via a periodic review myopic order-up-to policy. The new feature in our paper is the derivation of the *best* linear forecast of leadtime demand for each supply chain player given the particular information that is available to that player. This is important since under the assumption of Gaussian white noise the best linear forecast is the best forecast. Therefore, in the absence of the best linear forecast, the order quantity determined by a player would be sub-optimal, see for example Johnson and Thompson (1975).

In order to construct its best linear forecast, we describe how a supply chain player must first, via a traditional time-series methodology, characterize the information available to the player. We assume that information sharing occurs only between contiguous players. We show that under certain assumptions there arise only four mutually exclusive and exhaustive information sets when contiguous players either share or do not share their information (shocks). This permits us to exactly construct a recursive process that characterizes how demand propagates from player to player when each player uses his or her best linear forecast. Under each of the four information sets, we show that the demand propagation model is QUARMA-in-QUARMA-out where a QUARMA model is defined in (3) and is a more general time series model than an ARMA model. This result generalizes the ARMA-in-ARMA-out property proposed by Zhang (2004). Furthermore, we show that under certain conditions information sharing can have tremendous benefits. Our analysis hence reverses and sharpens several of the previous results in the literature involving information sharing and also opens up many questions for future research.

Our paper is an extension of the work of Lee, So and Tang (2000) and Raghunathan (2001), Zhang (2004) and Gaur, Giloni, and Seshadri (GGS, 2005), who study the value of information sharing in supply chains. Zhang and GGS extend the original work of Lee, So and Tang (2000) and Raghunathan (2001) by studying the value of information sharing in supply chains where the retailer serves an ARMA(p, q) demand as opposed to AR(1) demand. In each of these papers, the retailer places orders with a supplier using a periodic review order-up-to policy. Both the supplier and the retailer know the parameters of the demand process, however, the retailer may or may not choose to share information about the actual realizations of demand with the supplier. Zhang (2004) studies how the order process propagates upstream in a supply chain under the assumption that ARMA demand to the retailer is invertible. Zhang does not consider the case when demand becomes non-invertible at any stage of the chain, a phenomenon that GGS point out can happen even though the retailer's demand is invertible.

The concept of invertibility is central to an understanding of the value of information sharing, and also to an understanding of how demand propagates through a supply chain. In any ARMA(p, q) model for demand at a given stage of the supply chain, a linear combination of the present observation and p past demand observations is set equal to a constant plus a linear combination of the present and q past values of a particular white noise (shock) series. The given player in the supply chain does not directly observe the white noise series, but only the present and the past values of his or her own demand. To facilitate the construction of the optimal forecast and the evaluation of the corresponding mean squared forecast errors, it is useful to represent a given demand series as a constant plus a (potentially infinite) linear combination of present and past shocks. Zhang assumes that the current shock can be obtained from present and past demand observations, and uses these shocks to construct forecasts for leadtime demand (equation (11) in Zhang). Unfortunately, as pointed out by GGS, this is not always possible due to lack of invertibility. Informally, invertibility of a demand series with respect to a particular shock series is the property under which it is possible to obtain the current shock by linear operations on the present and past demand observations. In Section 2, we provide a precise definition of invertibility, as well as the necessary and sufficient conditions for invertibility of an ARMA(p, q) model with respect to the shocks in its defining equation, in terms of the q moving average parameters in that equation.

GGS show that the supplier's demand may not be invertible with respect to the retailer's shocks, even when the retailer's demand is invertible with respect to its own shocks. Therefore, in this case, Zhang's order process cannot be utilized since it would require information that is not available to the supplier, while the order process proposed by GGS can be used. The GGS order process for this case utilizes a sub-optimal forecast as opposed to the best linear forecast of leadtime demand, and hence, results in larger inventory related costs compared to those under the best linear forecast.

In this paper, we provide an order process for the retailer as well as (under some restrictions) for each upstream supply chain player based on its best linear forecast of leadtime demand where its demand is non-invertible with respect to the retailer's shocks, and information is not shared between the retailer and the supplier. We also derive the resulting optimal mean squared forecast error. In this case, the demand observed by the supplier is less informative than the retailer's demand, because the supplier can recover neither the retailer's shocks nor the retailer's demand based on the supplier's own demand. Here, there is a benefit to the supplier when the retailer shares its demand information with the supplier, because then the supplier can base the forecast of leadtime demand on the retailer's shocks. We show that in some circumstances the improvement from demand sharing can be unbounded, as measured by the ratio of the mean squared errors of the resulting forecasts.

The implications of our research, both for two-stage and multi-stage supply chains, are in contrast to the managerial implications suggested by both Zhang and GGS. For example, Zhang concludes that information sharing is generally not beneficial to the supplier, nor to any other upstream player, since he states that the player in question will be able to infer the previous player's demand and ultimately the retailer's shocks. GGS concludes that under an additional condition, once the supplier's demand is not invertible with respect to the retailer's shocks, information sharing would be required by all upstream players in order to construct their order processes.

In contrast to Zhang, we show that in some circumstances there can be great benefit from information sharing to a supply chain player when the demand observed by the player is not invertible with respect to the previous player's *full information shocks*, see Definition 2. *Informally, a player's full information shocks are those shocks that convey the full information that is available to the player. This information is based upon either the player's own current and historical demand or a sharing arrangement with the previous player. Thus, even though a certain set of shocks might be driving the order process of the previous player, the current player might not have access to those shocks unless the previous player shares them.* Furthermore, we show that when a supply chain player is able to obtain the previous player's full information shocks, its order process is similar to that suggested by Zhang. If a supply chain player is not able to obtain or construct the previous player's full information shocks, we show that the forecast proposed by Zhang cannot be used.

For the majority of our paper, we assume the supply chain is such that at each individual stage, either the equivalent of full information shocks are shared or there is no sharing. In contrast to GGS, we show that under such assumptions, even if a supply chain player's demand is not invertible with respect to the previous player's shocks and the previous player does not share information it is possible to construct its best linear forecast of leadtime demand. In other words, information sharing is never needed to determine a player's order process and hence the propagation of demand can be described precisely with or without information sharing at any stage of the chain.

Even though we allow only for the possible sharing of information between contiguous players in the chain, the patterns of the effects of information sharing on mean squared forecasting error can be quite complex. As we show in Section 6, there can be a difference between sharing of demand and sharing of shocks between players $k-1$ and k when $k \geq 3$. Until now, researchers have studied demand sharing, whereas we show that shock sharing at times provides greater information. Further, if one were to relax the assumption that information is shared only between contiguous players, there exists future work on how to characterize the information sets for each player in the supply chain.

Our research is valuable to managers because it even pertains to the simple case where a retailer observes AR(1) demand, since even in this case, the retailer's order process can be a non-invertible ARMA$(1,1)$ process with respect to the retailer's shocks. As noted by other researchers, another reason why our research is valuable to managers is that actual demand patterns often follow higher order autoregressive processes due to the presence of seasonality and business cycles. For example, the monthly demand for a seasonal item in its simplest form is approximately an AR(12) process. More general ARMA(p, q) processes are found to fit demand for long life cycle goods such as fuel, food products, machine tools, etc. as observed in Chopra and Meindl (2001) and Nahmias (1993). Aviv (2003) proposes an adaptive inventory replenishment policy that utilizes the Kalman filter technique in which the types of demand considered are ARMA models as well as more general demand processes.

The structure of our paper is as follows. In Section 2 we discuss the model setup with regard to our assumptions on all supply chain players, define the concept of invertibility and continue our discussion of its importance in supply chain management under ARMA demand. In Section 3 we discuss the retailer's order process along with the mean square forecast error of aggregated demand over the lead time. In Section 4, we demonstrate how a player observing demand that is not invertible with respect to the previous player's full information shocks can represent its demand as an invertible ARMA process with respect to a new set of shocks. In Section 5, we show how demand propagates upstream in a supply chain under the assumption that all players either shared the equivalent of their full information shocks or did not share any information. We also discuss the value of information sharing for the various scenarios discussed previously. In Section 6, we demonstrate that the value to player k of receiving demand information shared by player $k-1$ may be different than the value to player k of receiving shocks shared by player $k-1$. In Section 7, we provide a summary of our results and include some suggestions for future research.

2 The Model; Invertibility

We consider a K-stage supply chain where at discrete equally-spaced time periods, the retailer (assumed to be at stage 1) faces external demand $\{D_{1,t}\}$, for a single item. Let $\{D_{1,t}\}$ follow a covariance stationary $ARMA\,(p, q_1)$ process with $p \geq 0$, $q_1 \geq 0$,

$$D_{1,t} = d + \phi_1 D_{1,t-1} + \phi_2 D_{1,t-2} + \cdots + \phi_p D_{1,t-p} + \epsilon_{1,t} - \theta_{1,1}\epsilon_{1,t-1} - \theta_{1,2}\epsilon_{1,t-2} - \cdots - \theta_{1,q_1}\epsilon_{1,t-q_1}, \quad (1)$$

where $\{\epsilon_{1,t}\}$ is a sequence of uncorrelated random variables with mean zero and variance σ_1^2, and $\phi_1, \ldots, \phi_p$ and $\theta_{1,1}, \ldots, \theta_{1,q_1}$ are known constants such that $\phi_p \neq 0$ and all roots of the polynomial $1 - \phi_1 z - \cdots - \phi_p z^p$ are outside the unit circle. We also assume that all roots of the polynomial $1 - \theta_{1,1} z - \cdots - \theta_{1,q_1} z^{q_1}$ are on or outside the unit circle. This ensures that the retailer's demand is invertible with respect to its full information shocks. (See the discussion below.) If $\theta_{1,1}, \ldots, \theta_{1,q_1}$ are all equal to zero, then $\{D_{1,t}\}$ is an AR(p) process. It is often useful to express (1) in terms of the backshift operator, B, where $B^s(\epsilon_{1,t}) = \epsilon_{1,t-s}$. In order to do so, let $\phi(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$ and $\theta_1(B) = 1 - \theta_{1,1} B - \theta_{1,2} B^2 - \cdots - \theta_{1,q_1} B^{q_1}$. Then $\{D_{1,t}\}$ in (1) can be expressed as

$$\phi(B) D_{1,t} = d + \theta_1(B)\epsilon_{1,t}. \quad (2)$$

Since $\{D_{1,t}\}$ is covariance stationary, $E[D_{1,t}]$ exists and is constant for all t, $Var[D_{1,t}]$ is a finite constant, and the covariance between $D_{1,t}$ and $D_{1,t+h}$ depends on h but not on t.

Following Lee, So and Tang (2000) and Zhang (2004), we assume that the shocks $\{\epsilon_{1,t}\}$ are Gaussian white noise. We note that this implies that full information shocks (when they exist) at all subsequent stages are also Gaussian due to the manner in which demand and full information shocks propagate upstream a supply chain (see Theorem 1). Let the replenishment leadtime from the retailer's supplier to the retailer be ℓ_1 periods. Similarly, let the replenishment leadtime from the player at stage $k+1$ to stage k be ℓ_k periods. We assume that all supply chain players use a myopic order-up-to inventory policy where negative order quantities are allowed, and d is sufficiently large so that the probability of negative demand or negative orders is negligible. In time period t, after demand $D_{1,t}$ has been realized, the retailer observes the inventory position and places order $D_{2,t}$ with its supplier. The retailer receives the shipment of this order at the beginning of period $t + \ell_1$, where $\ell_1 \geq 1$. The sequence of events at all supply chain players is similar. We assume that the ℓ_k period lead time guarantee holds, i.e., if the player at stage $k+1$ does not have enough stock to fill an order from the player at stage k, then the player at stage $k+1$ will meet the shortfall from an alternative source, possibly by incurring an additional cost. See, for example, Lee So and Tang or Chen (2003) for a discussion of this assumption. Excess demand at the retailer is backlogged.

With respect to the information structure, we assume, as did GGS and others, that the form and parameters of the model generating player $k-1$'s demand are known by player k, but player $k-1$'s demand realizations and/or full information shocks may be private knowledge. When there is no information sharing, the retailer's supplier receives an order of $D_{2,t}$ in time period t from the retailer. On the other hand, when there is information sharing, the retailer's supplier receives the order $D_{2,t}$ as well as information about $D_{1,t}$ and/or $\epsilon_{1,t}$ in time period t from the retailer. As we show later in the paper, if the retailer's supplier receives information about $D_{1,t}$ in time period t from the retailer, he will be able to construct the shock series $\{\epsilon_{1,t}\}$ as described in (1). Unfortunately, as we show in Section 6, this is not necessarily the case for upstream player k who receives information about $D_{k-1,t}$ in time period t from player $k-1$ (that is, it might not be possible for player k to construct the shock series $\{\epsilon_{k-1,t}\}$). Therefore, the potential information sharing arrangements between upstream supply chain players at contiguous stages can be quite complex.

Central to our study of information sharing and how demand $\{D_{1,t}\}$ propagates upstream in a supply chain is the concept of invertibility, for which we provide a definition below. First, we define the *quasi-ARMA model*. In Theorem 1, we show that under certain assumptions, ARMA demand to the retailer propagates to *quasi-ARMA* or *QUARMA* demand for upstream players even when a given player's demand is non-invertible with respect to the previous player's shocks. Hence, the QUARMA model is central to our study of the propagation of demand in a supply chain. One manner in which a given player's demand is non-invertible with respect to the previous player's shocks is when the given player's demand, represented in terms of the previous player's shocks, has a coefficient of zero on the current shock. This case is important, as it is here that the potential for the improvement from shock sharing is unbounded. In other words, when presented with such demand, if the previous player shares its shocks, the given player may have a perfect forecast of leadtime demand, whereas without information sharing the forecast will not be perfect.

Consider a demand series $\{D_t\}$ expressed in terms of the shocks $\{\epsilon_t\}$,

$$D_t = d + \phi_1 D_{t-1} + \phi_2 D_{t-2} + \cdots + \phi_p D_{t-p} + \epsilon_{t-J} - \theta_1 \epsilon_{t-J-1} - \theta_2 \epsilon_{t-J-2} - \cdots - \theta_q \epsilon_{t-J-q}, \quad (3)$$

where $J \geq 0$ entails a lagging of the shocks by J time units. We show that the scenario $J > 0$ in (3) can arise at any stage beyond the retailer's, even though the retailer's demand has $J = 0$, without loss of generality. We denote the model (3) with $J \geq 0$ by QUARMA(p, q, J). By convention, we say that a constant demand series (see Remark 1) is QUARMA with $J = \infty$. In terms of the backshift operator, the QUARMA(p, q, J) model may be expressed as $\phi(B)D_t = d + B^J\theta(B)\epsilon_t$, where $\theta(B) = 1 - \sum_{j=1}^{q} \theta_j B^j$ if $J < \infty$ and $\theta(B) = 0$ if $J = \infty$. When $J = 0$, $\{D_t\}$ is ARMA(p, q) with respect to $\{\epsilon_t\}$, so that the QUARMA$(p, q, 0)$ model reduces to an ARMA(p, q) model. The AR and MA polynomials of the QUARMA(p, q, J) model in (3) may have common roots. A QUARMA(p, q, J) model in which the AR and MA polynomials do not share any common roots is said to be in *minimal form*.

Definition 1 *$\{D_t\}$ is invertible with respect to shocks $\{\epsilon_t\}$ in (3) with $J < \infty$ if the current and previous shocks can be recovered by a constant plus a linear combination of present and past values of the demand process $\{D_t\}$ or as a constant plus the limit of a sequence of linear combinations of present and past values of the demand process $\{D_t\}$.*

Questions of invertibility are not relevant in (3) with $J = \infty$ (since $\{D_t\}$ does not depend on the shocks $\{\epsilon_t\}$), nor are questions of information sharing since if a player observes constant demand, there is no need for information sharing. Thus, henceforth when we discuss invertibility or information sharing, we implicitly assume (unless stated otherwise) that the upstream player's demand is not constant. It is clear from the definition that if $0 < J < \infty$, $\{D_t\}$ cannot be invertible with respect to $\{\epsilon_t\}$. It is well-known that for the ARMA case $J = 0$ (see Brockwell and Davis 1991, pp. 127-129), $\{D_t\}$ is invertible with respect to shocks $\{\epsilon_t\}$ in (3) if and only if all the roots of the equation

$$1 - \theta_1 z - \theta_2 z^2 - \cdots - \theta_q z^q = 0 \tag{4}$$

lie outside or on the unit circle. It follows that for the QUARMA(p, q, J) model with any $0 \leq J < \infty$, $\{D_t\}$ is invertible with respect to the lagged shocks $\{B^J \epsilon_t\}$ if and only if all roots of (4) lie on or outside the unit circle.

In all of the QUARMA representations of demand $\{D_t\}$ with respect to shocks $\{\epsilon_t\}$ considered in this paper, the AR polynomial of upstream players is the same as the AR polynomial of the retailer. Since it was assumed that the retailer's AR polynomial has all of its roots outside the unit circle, this is also the case for upstream players. Although it is further assumed that the retailer's demand is in minimal form, it is possible for an upstream player to observe QUARMA demand whose AR and MA polynomials share at least one common root. Such a common root must lie outside the unit circle since all the roots of the AR polynomials in this paper are outside the unit circle. Therefore, in the event that $\{D_t\}$ is QUARMA(p, q, J) with respect to $\{\epsilon_t\}$ where $\phi(z)$ and $\theta(z)$ share at least one common root, consider the associated QUARMA representation of $\{D_t\}$ with respect to $\{\epsilon_t\}$ in minimal form with AR polynomial $\phi^m(z)$ and MA polynomial $\theta^m(z)$. Based upon our assumptions on $\phi(B)$, it follows that $\{D_t\}$ is invertible with respect to $\{\epsilon_t\}$ if and only if all roots of the original MA polynomial $\theta(z)$ lie on or outside the unit circle. This is because although invertibility is based upon the roots of $\theta^m(z)$, the roots of $\theta^m(z)$ lie on or outside the unit circle if and only if the roots of the original MA polynomial $\theta(z)$ lie on or outside the unit circle.

Under our definition of invertibility, the demand series is invertible with respect to shocks $\{\epsilon_t\}$ if and only if the present and past demand observations generate the same linear space (up to an additive constant) as the present and past shocks, so that from the point of view of linear operations the shock series contains no more information than the demand series. We note that our definition of invertibility coincides with the extended definition considered by Brockwell and Davis (1991, p. 128). The extended definition includes the case where the shocks $\{\epsilon_t\}$ can only be recovered as a constant plus the limit of a sequence of linear combinations of present and past values of the demand process $\{D_t\}$ as opposed to a constant plus a linear combination of present and past values of $\{D_t\}$. This corresponds to the case where at least one root of (4) lies on the unit circle (see Brockwell and Davis 1991, Problem 3.8, p. 111) and can result in the next player's demand being constant (see Remark 1).

As noted in the introduction, invertibility of player k's demand with respect to $\{\epsilon_{k-1,t}\}$, the full information shocks of player $k-1$, is central to our discussion of the value of information sharing in a supply chain. If $\{D_{k,t}\}$ is invertible with respect to $\{\epsilon_{k-1,t}\}$, then player k can utilize present and past values of $\{\epsilon_{k-1,t}\}$ in its forecast of its leadtime demand. On the other hand, if $\{D_{k,t}\}$ is not invertible with respect to $\{\epsilon_{k-1,t}\}$, then player k can utilize $\epsilon_{k-1,t}$ in its forecast of its leadtime demand if player $k-1$ shares information equivalent to its full information shocks. If $\{D_{k,t}\}$ has a QUARMA representation with respect to $\{\epsilon_{k-1,t}\}$ where $J > 0$ then $\{D_{k,t}\}$ is not invertible with respect to $\{\epsilon_{k-1,t}\}$ since the current shock $\epsilon_{k-1,t}$ cannot be observed by player k. In the next section, we discuss the retailer's order process and thus provide a starting point for understanding how demand propagates upstream in a supply chain.

3 Determining the Retailer's Order

As mentioned in the previous section, we assume that the retailer observes ARMA(p, q) demand as given by (1). We assume that the ARMA(p, q) model of the retailer's demand is in minimal form, i.e., the values of p and q are such that there are no common roots of the AR and MA polynomials. We further assume, as in Lee, So, and Tang, Raghunathan, Zhang, and GGS, that the retailer utilizes a periodic review system. Let $S_{1,t}$ be the order up to level that minimizes the retailer's total expected holding and shortage costs in period $t + \ell_1$. Let $\mathcal{M}_t^{D_1} = \overline{sp}\{1, D_{1,t}, D_{1,t-1}, \ldots\}$, i.e., the Hilbert space generated by $\{1, D_{1,t}, D_{1,t-1}, \ldots\}$ with inner product given by the covariance. We refer to $\mathcal{M}_t^{D_1}$ as the linear past of $\{D_{1,t}\}$. Similarly, let $\mathcal{M}_t^{\epsilon_1} = \overline{sp}\{1, \epsilon_{1,t}, \epsilon_{1,t-1}, \ldots\}$. For the linear past of two time series, for example, $\{D_{1,t}\}$ and $\{D_{2,t}\}$, we write $\mathcal{M}_t^{D_1,D_2} = \overline{sp}\{1, D_{1,t}, D_{2,t}, D_{1,t-1}, D_{2,t-1}, \ldots\}$.

In this paper, we only consider linear forecasting since it follows from our assumptions that all random variables appearing in the paper are Gaussian, in which case the best possible forecast is a linear one. In this context, a given player's information set can always be represented as the linear past of some collection of one or more time series. We denote the full information set available to player 1 as $\mathcal{M}_t^1$. We assumed in Section 2 that the retailer's demand $\{D_{1,t}\}$ is invertible with respect to the shocks $\{\epsilon_{1,t}\}$. This entails no loss of generality, since the only shocks observable to the retailer are those obtained from an invertible ARMA(p, q) representation of its demand. Therefore, $\mathcal{M}_t^1 = \mathcal{M}_t^{D_1} = \mathcal{M}_t^{\epsilon_1}$. This is the retailer's information set available at time t after $D_{1,t}$ has been observed. We refer to $\{\epsilon_{1,t}\}$ as the retailer's full information shocks. Let σ_1 be the standard deviation of $\epsilon_{1,t}$.

Therefore, the retailer's order-up-to level at time t is given by $S_{1,t} = m_{1,t} + k\sigma_1\sqrt{v_1}$ where $m_{1,t}$ is the best linear forecast of $\sum_{i=1}^{\ell_1} D_{1,t+i}$ in the space $\mathcal{M}_t^1$ and v_1 is the associated mean squared forecast error. Thus, the retailer's order to its supplier is $D_{2,t} = D_{1,t} + S_{1,t} - S_{1,t-1} = D_{1,t} + m_{1,t} - m_{1,t-1}$.

Since it is assumed that $\{D_{1,t}\}$ is invertible with respect to $\{\epsilon_{1,t}\}$, we have $\epsilon_{1,t} \in \mathcal{M}_t^1$. It follows that $\epsilon_{1,t}$ is observable to the retailer, as are $\epsilon_{1,t-1}, \epsilon_{1,t-2}, \ldots$. Furthermore, since $\{D_{1,t}\}$ is ARMA with respect to $\{\epsilon_{1,t}\}$ with all roots of $\phi(z)$ outside the unit circle, it has a one-sided $MA(\infty)$ representation with respect to these shocks.

Proposition 1 *(Zhang, 2004). The retailer's demand and its best linear forecast of lead-time demand can be represented as MA(∞) processes with respect to the retailer's full information shocks $\{\epsilon_{1,t}\}$. Its order $\{D_{2,t}\}$ to its supplier is given by (5) with respect to the retailer's full information shocks $\{\epsilon_{1,t}\}$. In particular with $\mu_d = d/(1 - \phi_1 - \cdots - \phi_p)$, the retailer's demand is $D_{1,t} = \sum_{i=0}^{\infty} \psi_{1,i}\epsilon_{1,t-i} + \mu_d$, where the $\{\psi_{1,i}\}$ are given in (19). Furthermore, the retailer's order is*

$$D_{2,t} = (\sum_{j=0}^{\ell_1} \psi_{1,j})\epsilon_{1,t} + \sum_{i=1}^{\infty} \psi_{1,i+\ell_1}\epsilon_{1,t-i} + \mu_d. \quad (5)$$

See the Appendix for a version of the proof that lays the groundwork for our development.

Equation (5) provides a representation of the retailer's order as a constant plus a linear combination of present and past values of $\{\epsilon_{1,t}\}$, that are in the retailer's information set. Although the retailer's order is equivalent to its supplier's demand, $\{D_{2,t}\}$ can be expressed in terms of various sets of shocks. As will be shown in Theorem 1, $\{D_{2,t}\}$ has a QUARMA representation with respect to $\{\epsilon_{1,t}\}$. However, it is possible that $\{\epsilon_{1,t}\}$ may not be in the supplier's information set in the absence of information sharing. This occurs when $\{D_{2,t}\}$ is not invertible with respect to $\{\epsilon_{1,t}\}$ and hence the supplier will not be able to obtain the retailer's current shock from the supplier's present and past demand observations.

Thus, in order to study how ARMA demand propagates upstream in a supply chain, it is necessary for us to understand the potential information sets faced by a given supply chain player. In the above case of the retailer, its information set is straightforward. However, the information set that is available to supply chain player k with $k \geq 2$ can be quite complex. We discuss some of these information sets in Section 5. But, before doing so, we first describe how a player observing demand that is not invertible with respect to the previous player's full information shocks can represent its demand as an invertible ARMA process with respect to a new set of shocks.

Remark 1 *It is possible that* $\{D_{2,t}\}$ *in (5) reduces to a constant,* $D_{2,t} = \mu_d$ *for all* t*. This case arises when* $1 + \sum_{j=1}^{\ell_1} \psi_{1,j} = 0$ *and* $\psi_{1,i+\ell_1} = 0$ *for* $i \geq 1$*. It can be easily seen that these conditions are satisfied when* $\{D_{1,t}\}$ *is an MA(*q_1*) process with respect to* $\{\epsilon_{1,t}\}$ *with* $q_1 \leq \ell_1$ *and the MA polynomial has a root at* 1*. In Theorem 1, we provide conditions when* $\{D_{k,t}\}$ *reduces to a constant when player* $k-1$*'s demand is QUARMA with respect to* $\{\epsilon_{k-1,t}\}$*.*

4 Representation of Noninvertible ARMA Demand

Consider an ARMA(p, q) model for a demand series $\{D_t\}$ with respect to a shock series $\{\epsilon_t\}$, that is, $\phi(B)D_t = d + \theta(B)\epsilon_t$ where $\theta(B) = 1 - \theta_1 B - \cdots - \theta_q B^q$. As discussed in Section 2, the series $\{D_t\}$ is non-invertible with respect to $\{\epsilon_t\}$ if and only if at least one root of $\theta(z)$ lies inside the unit circle, where z is a complex variable. This situation may be faced by a given supply chain player with respect to the previous player's full information shocks. If the previous player $(k-1)$ does not share any information with the given player (k), then in order to forecast its leadtime demand, under the assumptions in Section 5 along with the assumption that player k's demand is not constant, player k first represents its demand as ARMA(p, q) with respect to a new set of shocks that are observable in the sense that they generate the same linear past as the given player's demand series, such that the new ARMA(p, q) model yields the same variance and autocorrelations as the original one. We now explain how this is done for the ARMA series $\{D_t\}$ originally expressed with respect to $\{\epsilon_t\}$.

Since $\{D_t\}$ is ARMA(p, q) with respect to shocks ϵ_t,

$$\prod_{s=1}^{p}(1 - a_s^{-1}B)D_t = d + \prod_{s=1}^{q}(1 - b_s^{-1}B)\epsilon_t, \tag{6}$$

where $a_s, 1 \leq s \leq p$ are the roots of the polynomial equation

$$1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p,$$

$b_s, 1 \leq s \leq q$ are the roots of the polynomial equation

$$1 - \theta_1 z - \theta_2 z^2 - \cdots - \theta_q z^q.$$

Since $\{D_t\}$ is not invertible with respect to shocks $\{\epsilon_t\}$, there exists at least one $1 \le r \le q$ such that $|b_r| < 1$. Without restriction of generality, we assume that $|b_s| > 1, 1 \le s \le h$, and $|b_s| < 1, h < s \le q$. It follows (see for example Brockwell and Davis (1991, pp. 125-127)) that $\{D_t\}$ can be represented as an *invertible* ARMA(p, q) with respect to a new set of shocks $\{\gamma_t\}$ with variance $\sigma_\gamma^2 = (\prod_{h<s\le q} |b_s|)^{-2})\sigma_\epsilon^2$. Since $|b_s| < 1$, for $h < s \le q$, $\prod_{s=h+1}^{q} |b_s|^{-2} > 1$ and hence $\sigma_\gamma^2 > \sigma_\epsilon^2$. This invertible representation has form

$$\prod_{s=1}^{p}(1 - a_s^{-1}B)D_t = d + \prod_{s=1}^{h}(1 - b_s^{-1}B)\prod_{s=h+1}^{q}(1 - \bar{b}_s B)\gamma_t, \tag{7}$$

where $\bar{b}_s$ is the complex conjugate of b_s. Thus

$$\phi(B)D_t = d + \theta(B)\epsilon_t = d + \theta^\dagger(B)\gamma_t \tag{8}$$

where

$$\theta^\dagger(B) = \prod_{s=1}^{h}(1 - b_s^{-1}B)\prod_{s=h+1}^{q}(1 - \bar{b}_s B) \tag{9}$$

has all of its roots outside the unit circle. Thus, the ARMA(p,q) representation of $\{D_t\}$ in (8) is invertible with respect to $\{\gamma_t\}$ and hence $\gamma_t, \gamma_{t-1}, \ldots$ are observable to the player observing demand $\{D_t\}$, i.e., $\mathcal{M}_t^\gamma = \mathcal{M}_t^D$. Furthermore, in such a case, $\{\gamma_t\}$ is the only set of shocks with the following properties: (i) $\{D_t\}$ is invertible with respect to $\{\gamma_t\}$, (ii) the roots of $\phi(B)$ are all outside the unit circle, and (iii) the coefficient in front of γ_t is one in the ARMA representation of $\{D_t\}$.

The above methodology is widely available in the time series literature, but to the best of our knowledge has not been noted in the OM literature. This methodology is central to our paper as it demonstrates how a supply chain player can construct shocks that include as much information as is available to the player without information sharing. In contrast, GGS did not use this methodology when considering a supplier's demand that is not invertible with respect to the retailer's shocks and hence its proposed forecast is sub-optimal from the supplier's perspective. Furthermore, the fact that $\sigma_\gamma^2 > \sigma_\epsilon^2$ provides the potential for value of information sharing.

5 The Propagation of Demand Under a Restriction on Information Sharing

Player k must forecast its leadtime demand, using all information available to it. A variety of information sets can arise here, depending on sharing arrangements and on considerations of invertibility. We denote the full information set available to player k as $\mathcal{M}_t^k$. This entails $\mathcal{M}_t^{D_k}$ possibly together with information obtained through sharing with player $k-1$ when $k \geq 2$.

Definition 2 *We say that $\{\epsilon_{k,t}\}$ are player k's full information shocks if there exists a white noise sequence, $\{\epsilon_{k,t}\}$, such that $\mathcal{M}_t^k = \mathcal{M}_t^{\epsilon_k}$.*

In this section, we show how demand propagates upstream (and how player k can construct its full information shocks) in a supply chain under the assumption that all players either shared the equivalent of their full information shocks or did not share any information. In order to accomplish this task, we need to understand the various information sets that a supply chain player may face. In Theorem 1 below, we show that there are four such sets, which are mutually exclusive and exhaustive. The theorem thus needs to establish the existence and construction of each player's full information shocks which itself is contingent upon demonstrating those information sets that are available to a supply chain player. Furthermore, it is only possible to determine which information sets are available to a supply chain player when one models how demand propagates upstream the supply chain. Hence, Theorem 1 necessarily includes these three parts along with a fourth that although appears to only be a special case would result in the theorem being incomplete if left out.

In other words, in order to determine its order to player k, player $k-1$ must first represent its demand, and then its best linear forecast of its leadtime demand, each as a constant plus a linear combination of its present and past full information shocks, $\{\epsilon_{k-1,t}\}$. Finally, using the order-up-to-policy, player $k-1$ constructs its order $\{D_{k,t}\}$ to player k again as a constant plus a linear combination of its present and past full information shocks, $\{\epsilon_{k-1,t}\}$. The forecasting and order construction is performed using these representations. Theorem 1 establishes the existence of such representations by constructing player k's full information shocks based on its information set. There are several properties which propagate from one stage to the next, as described in the theorem. These are proved by induction. One of these properties is the QUARMA structure of the demand at each stage (with respect to both the given player's and the previous player's full information shocks), leading to a QUARMA-in-QUARMA-out property. This structure is used to determine whether or not a given demand series is invertible with respect to its own full information shocks, the previous player's full information shocks, or to lagged versions of either set of shocks. The QUARMA structure also allows a given player to represent its observed demand as a constant plus a linear combination of its own present and past full information shocks.

The four information sets that supply chain player k may face are the following, with $\subset$ denoting strict subspace here and throughout the paper.

$$
\begin{aligned}
&(i)\mathcal{M}_t^k = \mathcal{M}_t^{k-1} \ and\ \mathcal{M}_t^{D_k} \neq \mathbb{R} \\
&(ii)\mathcal{M}_t^k = \mathcal{M}_t^{D_k} = \mathcal{M}_{t-J}^{\epsilon_{k-1}} \ where\ 1 \leq J < \infty,\ and\ \mathcal{M}_t^{D_k} \neq \mathbb{R} \\
&(iii)\mathcal{M}_t^k = \mathcal{M}_t^{D_k} \ where\ \mathcal{M}_t^{D_k} \subset \mathcal{M}_t^{k-1} \ and\ \mathcal{M}_t^{D_k} \neq \mathcal{M}_{t-J}^{\epsilon_{k-1}},\ for\ J \geq 0,\ and\ \mathcal{M}_t^{D_k} \neq \mathbb{R} \\
&(iv)\mathcal{M}_t^{D_k} = \mathbb{R}. \qquad (10)
\end{aligned}
$$

In case (i), player k has access to the full information shocks of player $k-1$. This can occur either because it is possible to represent $\{D_{k,t}\}$ as an invertible ARMA process with respect to $\{\epsilon_{k-1,t}\}$ or becase $\{D_{k,t}\}$ is not invertible with respect to $\{\epsilon_{k-1,t}\}$ but player $k-1$ shared the equivalent of its full information shocks with player k.

In case (ii), player k's information set is equivalent to the linear past of its demand which in turn is equivalent to the linear past of a lagged set of player $k-1$'s full information shocks. This occurs when $\beta_{k-1} = 0$ in Equation (12) below, player $k-1$ has not shared its full information shocks with player k, but player k's demand is invertible with respect to the lagged set of player

$k-1$'s shocks.

In case (iii), player k's information set is equivalent to the linear past of its demand which in turn is a strict subset of player $k-1$'s information set. This occurs when player k's demand is not invertible with respect to player $k-1$'s full information shocks nor with respect to any lagged version of player $k-1$'s full information shocks. The forecasting in this case is not done based on the original shocks. Instead a more general methodology as described in Section 4 must be used.

In case (iv), player k's demand is constant.

Some thought reveals that intuitively these four sets are mutually exclusive and collectively exhaustive for the information sharing assumptions made in this section. This is proven in Theorem 1 below, our main result on the propagation of demand. To initialize this propagation, we think of the player previous to the retailer as the retailer itself. That is, we define $\{D_{0,t}\} = \{D_{1,t}\}$, $\{\epsilon_{0,t}\} = \{\epsilon_{1,t}\}$, and $\ell_0 = \ell_1$. We also note that if $\{D_{k-1,t}\}$ is QUARMA(p, q_{k-1}, J_{k-1}) with respect to $\{\epsilon_{k-1,t}\}$ with $J_{k-1} < \infty$ and MA parameters $\theta_{k-1,1}, \cdots, \theta_{k-1,q_{k-1}}$, then $\{D_{k-1,t}\}$ can be represented as a constant plus a linear combination of its present and past full information shocks, i.e., $D_{k-1,t} = \mu_d + \sum_{i=0}^{\infty} \psi_{k-1,i}\epsilon_{k-1,t-i}$ where

$$\psi_{k-1,i} = \begin{cases} 0 & i < J_{k-1} \\ 1 & i = J_{k-1} \\ \sum_{j=1}^{p} \phi_j \psi_{k-1,i-j} - \theta_{k-1,i-J_{k-1}} & i > J_{k-1} \end{cases} \tag{11}$$

and $\theta_{k-1,i} = 0$ if $i > q_{k-1} + J_{k-1}$. If $\{D_{k-1,t}\}$ is QUARMA$(p, 0, \infty)$ with respect to $\{\epsilon_{k-1,t}\}$, then $\psi_{k-1,i} = 0$ for all i and $\{D_{k-1,t}\}$ reduces to a constant. From (11), it can be seen that J_{k-1} is an integer equal to the index of the first $\psi_{k-i,i}$ that is nonzero. $\tilde{J}_k$ and J_k have similar interpretations where $\tilde{J}_k$ refers to the QUARMA representation of $\{D_{k,t}\}$ with respect to $\epsilon_{k-1,t}$ and J_k refers to the QUARMA representation of $\{D_{k,t}\}$ with respect to $\epsilon_{k,t}$. An important quantity for the computation of player $k-1$'s order to player k, as well as the values of $\tilde{J}_k$ and J_k is

$$\beta_{k-1} = \sum_{i=0}^{\ell_{k-1}} \psi_{k-1,i}. \tag{12}$$

Theorem 1 *Let $k \geq 1$. Assume that player $k-1$ shares with player k either nothing or the equivalent of player $k-1$'s full information shocks, and that similar arrangements hold between all previous players. In the case where a player's demand is constant, information sharing to or from the player is irrelevant. Then*

(I) Player $k-1$'s order to player k with respect to player $k-1$'s full information shocks: $\{D_{k,t}\}$ has a $QUARMA(p, q_k, \tilde{J}_k)$ representation with respect to $\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$ of form $\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$, where $\{\lambda_{k,i}\}$ are defined in (32) and (33), $\tilde{J}_0 = \tilde{J}_1 = 0$, $\tilde{J}_k \geq 0$ is formally defined in (31) and $q_k = \max\{\max(p, q_{k-1} - \ell_{k-1} + J_{k-1}) - \tilde{J}_k, 0\}$. If $\tilde{J}_k < \infty$, then $\tilde{\theta}_k(B)$ is a polynomial in B of order q_k with leading coefficient 1 and $\tilde{\theta}_{k,j} = -\frac{\lambda_{k,\tilde{J}_k+j}}{\lambda_{k,\tilde{J}_k}}$, $j = 1, \cdots, q_k$. If $\tilde{J}_k = \infty$, then $\tilde{\theta}_k(B) = 0$.

(II) The four information sets: The four sets in (10) are the mutually exclusive and exhaustive possibilities for player k's information set. These determine player k's full information shocks $\{\epsilon_{k,t}\}$ as follows: In case (i), $\epsilon_{k,t} = \lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$; in case (ii), $\epsilon_{k,t} = \lambda_{k,\tilde{J}_k}\epsilon_{k-1,t-\tilde{J}_k}$; in case (iii), $\epsilon_{k,t} = \lambda_{k,\tilde{J}_k}(\tilde{\theta}_k^{\dagger})^{-1}(B)[\phi(B)D_{k,t} - d]$; in case (iv), $\epsilon_{k,t} = 0$.

(III) Player k's demand with respect to its full information shcks: $\{D_{k,t}\}$ has a $QUARMA(p, q_k, J_k)$ representation with respect to $\{\epsilon_{k,t}\}$ of form $\phi(B)D_{k,t} = d + B^{J_k}\theta_k(B)\epsilon_{k,t}$. If $\tilde{J}_k < \infty$, $\theta_k(B)$ is a polynomial in B of order q_k with leading coefficient 1. In case (i), $\theta_k(B) = \tilde{\theta}_k(B)$, $J_k = \tilde{J}_k < \infty$; in case (ii), $\theta_k(B) = \tilde{\theta}_k(B)$, $J_k = 0$ and $0 < \tilde{J}_k < \infty$; in case (iii), $\theta_k(B) = \tilde{\theta}_k^{\dagger}(B)$, $J_k = 0$ and $0 \leq \tilde{J}_k < \infty$; in case (iv), $\theta_k(B) = \tilde{\theta}_k B = 0$ and $J_k = \tilde{J}_k = \infty$.

(IV) Constant demand: For $k \geq 2$, if $\{D_{k-1,t}\}$ is $QUARMA(0, q^m_{k-1}, J_{k-1})$ in minimal form with respect to $\{\epsilon_{k-1,t}\}$ with an MA polynomial that has a root at 1 and $\ell_{k-1} \geq J_{k-1} + q^m_{k-1}$, then $\tilde{J}_k = \infty$ and $\{D_{k,t}\}$ is constant.

The full information shocks in part (II) of Theorem 1 have a specific multiplicative constant, chosen in order to ensure that the coefficient of the first shock with a nonzero coefficient (if there exists such a shock) in the QUARMA representation in Part (III) of the theorem is one. This value of $\lambda_{k,\tilde{J}_k}$ in (32) and (33), and the variance σ_k^2 of player k's full information shocks $\epsilon_{k,t}$, depends upon sharing arrangements between player $k-1$ and player k as well as considerations of invertibility of player k's demand with respect to player $k-1$'s full information shocks. Each distinct scenario (except for the case of constant demand) corresponds to a particular subcase of (10) as described in Corollary 1 below, which follows from the proof of Theorem 1 and from Lemma 2.

The QUARMA models in parts (I) and (III) of theorem 1 can have MA and AR polynomials with common roots even though this is not the case at the retailer. This is important because in such a case a QUARMA representation of $\{D_{k-1,t}\}$ with respect to $\{\epsilon_{k-1,t}\}$ in minimal form will have $p^m_{k-1} < p$. Furthermore, the existence of common roots may result in $p^m_{k-1} = 0$ which together with the other conditions in part (IV) of Theorem 1 will result in $\{D_{k,t}\}$ being constant.

In order to interpret the theorem and ease the discussion of the value of information sharing, Corollary 1 is included as it maps the results of Theorem 1 to the possible situations that a supply chain player may face. Once a player's full information shocks and their variance are determined, we show below in Section 5.1 how a player's best linear forecast and hence MSFE can be constructed. Finally, the MSFE of the no information sharing case can be compared to that of the information sharing case in order to determine the value of information sharing which we show below in Section 5.2.

Corollary 1 *Under the conditions of Theorem 1,*

- *if* $\{D_{k,t}\}$ *is invertible with respect to* $\{\epsilon_{k-1,t}\}$*, then case (i) holds,* $\beta_{k-1} \neq 0$*,* $\lambda_{k,\tilde{J}_k} = \beta_{k-1}$ *and* $\sigma^2_k = \beta^2_{k-1}\sigma^2_{k-1}$*;*
- *if* $\{D_{k,t}\}$ *is not invertible with respect to* $\{\epsilon_{k-1,t}\}$*,* $\beta_{k-1} \neq 0$*, and player* $k-1$ *shared the equivalent of its full information shocks with player* k*, then case (i) holds,* $\lambda_{k,\tilde{J}_k} = \beta_{k-1}$ *and* $\sigma^2_k = \beta^2_{k-1}\sigma^2_{k-1}$*;*
- *if* $\{D_{k,t}\}$ *is not invertible with respect to* $\{\epsilon_{k-1,t}\}$*,* $\beta_{k-1} = 0$*, and player* $k-1$ *shared the equivalent of its shocks with player* k*, then case (i) holds,* $\lambda_{k,\tilde{J}_k} = \psi_{k-1,\ell_{k-1}+\tilde{J}_k}$ *and* $\sigma^2_k = \psi^2_{k-1,\ell_{k-1}+\tilde{J}_k}\sigma^2_{k-1}$*;*
- *if* $\{D_{k,t}\}$ *is not invertible with respect to* $\{\epsilon_{k-1,t}\}$*,* $\beta_{k-1} = 0$*, player* $k-1$ *did not share with player* k*, and* $\tilde{\theta}_k(B)$ *has all roots outside the unit circle, then case (ii) holds,* $\lambda_{k,\tilde{J}_k} =$

$\psi_{k-1,\ell_{k-1}+\tilde{J}_k}$ *and* $\sigma_k^2 = \psi^2_{k-1,\ell_{k-1}+\tilde{J}_k}\sigma^2_{k-1}$;

- *if* $\{D_{k,t}\}$ *is not invertible with respect to* $\{\epsilon_{k-1,t}\}$, $\beta_{k-1} \neq 0$, *and player* $k-1$ *did not share with player* k, *then case (iii) holds,* $\lambda_{k,\tilde{J}_k} = \beta_{k-1}$ *and* $\sigma_k^2 = \beta_{k-1}^2(\prod_{h<s\leq q_k}|b_s|)^{-2})\sigma_{k-1}^2$;
- *if* $\{D_{k,t}\}$ *is not invertible with respect to* $\{\epsilon_{k-1,t}\}$, $\beta_{k-1} = 0$, *player* $k-1$ *did not share with player* k, *and* $\tilde{\theta}_k(B)$ *has at least one root inside the unit circle, then* $\lambda_{k,\tilde{J}_k} = \psi_{k-1,\ell_{k-1}+\tilde{J}_k}$ *and* $\sigma_k^2 = \psi^2_{k-1,\ell_{k-1}+\tilde{J}_k}(\prod_{h<s\leq q_k}|b_s|)^{-2})\sigma_{k-1}^2$;

5.1 The Best Linear Forecast

We now turn to a discussion of forecasting leadtime demand from the point of view of player k, and its associated mean squared forecast error. We state a lemma on forecasting of a general demand series $\{D_t\}$ that can be represented as a constant plus a linear combination of present and past values of a white noise sequence $\{\epsilon_t\}$ with $\mathcal{M}_t^\epsilon$ assumed known. This lemma and its proof were utilized in the proof of Theorem 1.

Lemma 1 *If demand* $\{D_t\}$ *can be represented as a constant plus a linear combination of present and past values of a white noise series* $\{\epsilon_t\}$ *with* $\mathcal{M}_t^\epsilon$ *known of form* $D_t = \mu_d + \sum_{j=0}^\infty \psi_j\epsilon_{t-j}$, *the best linear forecast of* ℓ*-step leadtime demand (the projection into* $\mathcal{M}_t^\epsilon$*), is given by* $m_t = \sum_{i=0}^\infty \omega_{i+\ell}\epsilon_{t-i} + \ell\mu_d$ *and the associated mean squared forecast error is given by* $\sigma_\epsilon^2\sum_{i=0}^{\ell-1}\omega_i^2$, *where* ω_i *are defined in (24) and* $\sigma_\epsilon^2 = var(\epsilon_t)$.

We now discuss how to use this lemma in conjunction with Corollary 1. First, in all situations considered in this section, player $k-1$'s order to player k, $\{D_{k,t}\}$, can be represented as a constant plus a linear combination of present and past $\{\epsilon_{k-1,t}\}$. However, player k will only be able to use this representation to forecast its leadtime demand if $\mathcal{M}_t^k = \mathcal{M}_t^{\epsilon_{k-1}}$, that is, in case (i) of (10). Zhang (2004) implicitly assumes that present and past shocks $\epsilon_{k-1,t}$, $\epsilon_{k-1,t-1}, \cdots$ are *always* observable to player k. Unfortunately, this assumption does not hold in cases (*ii*) and (*iii*) of (10).

Second, by Theorem 1 part (III), player k has a QUARMA representation of its demand with respect to its full information shocks $\{\epsilon_{k,t}\}$. However, player k's full information shocks and hence its QUARMA representation will differ depending upon which scenario listed in Corollary 1 player k may face. The scenario faced by player k depends upon invertibility of $\{D_{k,t}\}$ with respect to $\{\epsilon_{k-1,t}\}$ and whether or not there exists a sharing arrangement between player $k-1$ and player k. Lemma 1 provides player k's mean squared error of its forecast of leadtime demand in terms of the variance of its full information shocks, $\sigma_k^2 = var(\epsilon_{k,t})$. Specifically, we let $\omega_{k,i} = 0$ for $i < 0$, $\omega_{k,0} = \psi_{k,0}$, $\omega_{k,i} = \omega_{k,i-1} + \psi_{k,i}$ for $0 < i < \ell_k$ and $\omega_{k,i} = \omega_{k,i-1} + \psi_{k,i} - \psi_{k,i-\ell_k}$ for $i \geq \ell_k$, where $\psi_{k,i}$ follows from (11) with $k-1$ replaced by k. Player k's best linear forecast of leadtime demand and associated mean squared forecast error are given by $m_{k,t}$ and $\sigma_k^2 \sum_{i=0}^{\ell_k - 1} \omega_{k,i}^2$ respectively, where

$$m_{k,t} = \sum_{i=0}^{\infty} \omega_{k,i+\ell_k} \epsilon_{k,t-i} + \ell_k \mu_d. \tag{13}$$

At this stage, we show how it is possible to recursively construct the propagation of demand through the supply chain. Player $k-1$ places an order to player k that is QUARMA with respect to its full information shocks as per part (I) of Theorem 1. Next, one needs to determine if player k's demand is invertible with respect to $\{\epsilon_{k-1,t}\}$. In the event that player k's demand is not invertible with respect to $\{\epsilon_{k-1,t}\}$, one needs to consider if there is a sharing arrangement between player $k-1$ and player k. Based upon these two determinations, one can construct player k's full information shocks and can determine which specific case of Corollary 1 player k faces. Then, player k's QUARMA representation of its demand can be determined as in part (III) of Theorem 1 along with its best linear forecast of leadtime demand as in (13). Player k's order to player $k+1$ follows the myopic order-up-to policy and the propagation of demand continues upstream similarly.

5.2 The Value of Information Sharing

Whether or not there is an information sharing agreement between players $k-1$ and k, and whether or not player k's demand is invertible with respect to player $k-1$'s full information shocks, player k is able to use Theorem 1 in conjunction with Corollary 1 as well as the proof of Lemma 2 to construct its best linear forecast of leadtime demand given the information available to it. Below, we determine the value of information sharing, by comparing the mean square forecast error of each no information sharing scenario to the associated information sharing scenario. In other words, player k's mean squared forecast error depends on the variance of its full information shocks as well as on the values of $\omega_{k,i}$. However, both of these quantities depend on which scenario from Corollary 1 player k faces.

We propose to measure the value of information sharing by the ratio of player k's mean square forecast error where player $k-1$ did not share its shocks to where player $k-1$ did share its shocks. In other words, $V = \frac{MSFE_k^{NS}}{MSFE_k^{S}}$. It is clear that $V \geq 1$ and when $V = 1$ there is no value to information sharing. Below, we discuss the value of information sharing by considering those cases from Corollary 1.

If $\{D_{k,t}\}$ is invertible with respect to $\{\epsilon_{k-1,t}\}$, then there is no value of information sharing between player $k-1$ and player k since the information available to player k from the linear past of its demand is equivalent to player $k-1$'s full information set, i.e., $V = 1$. If $\{D_{k,t}\}$ is not invertible with respect to $\{\epsilon_{k-1,t}\}$, then there generally is value of information sharing between player $k-1$ and player k. See Proposition 2 below showing the possibility of unbounded gain from information sharing. We note that in those cases where $\mathcal{M}_t^{D_k} \subset \mathcal{M}_t^{k-1}$, it is possible that there may be no value to information sharing (see Pierce, 1975). Typically, though, if $\mathcal{M}_t^{D_k} \subset \mathcal{M}_t^{k-1}$, then if player $k-1$ does not share its full information shocks with player k, there is information loss and player k's MSFE of its aggregated demand over the lead time is larger than if full information shocks had been shared.

We numerically demonstrate the value of information sharing and the potential for a simple AR(1) demand to result in benefit of information sharing for an upstream player. In all figures in this section, we assume that the retailer observes ARMA(1,1) demand with leadtime $\ell_1 = 1$. In Figure 1, the enclosed region shows those values of ϕ and θ_1 of the retailer's demand model such that the supplier's demand will not be invertible with respect to the retailer's shocks. We point out that the enclosed region consists of one quarter of all possible ARMA(1,1) demand configurations for the retailer if the leadtime is 1. In Figure 2, we study the same situation as mentioned above except here we measure the value of information sharing for the supplier in terms of $ln(V)$ where $V = \frac{MSFE^{NS}}{MSFE^{S}}$ and we assume that the retailer observes ARMA(1,1) demand with $\phi = -.7$. As θ_1 approaches .3, the value of information sharing approaches infinity since $\beta_1 = 1 + \phi - \theta_1$ approaches 0 (See Proposition 2). From Figure 1, it can be seen that even when the retailer observes AR(1) demand, i.e,. when $\theta_1 = 0$, the supplier's demand will not be invertible with respect to the retailer's shocks if $\phi < -.5$. Hence our paper is crucial to understanding how a simple AR(1) demand propagates along a supply chain and the value of information sharing within it. For example, if $\phi = -.6$ and $\theta_1 = 0$, if $\ell_1 = 1$, the retailer's order to the supplier is given by $D_{2,t} = .6D_{2,t-1} + d + \epsilon_{1,t} - 1.5\epsilon_{1,t-1}$. Hence $\{D_{2,t}\}$ is not invertible with respect to $\{\epsilon_{1,t}\}$. In the event the retailer does not share with the supplier and $\ell_2 = 1$, then the supplier's MSFE is $.36\sigma_1^2$ whereas if the supplier would sub-optimally use the GGS forecast, the MSFE would be $.52\sigma_1^2$, i.e. $\frac{MSFE^{GGS}}{MSFE} = \frac{13}{9}$.

Proposition 2 *If $\ell_k \leq \tilde{J}_k$, then there exists unbounded gain to player k from player $k-1$ sharing its full information shocks.*

Proof. Since $\ell_k \leq \tilde{J}_k$, it follows that if player $k-1$ shares its full information shocks with player k, then $\omega_{k,i} = 0$ for $0 \leq i \leq \ell_k - 1$ and hence the mean square forecast error of leadtime demand, $\sigma_k^2 \sum_{i=0}^{\ell_k-1} \omega_{k,i}^2$, under information sharing is 0. On the other hand, if player $k-1$ does not share with player k, then by Theorem 1 (III) , $J_k = 0$, $\omega_{k,0} = 1$ and hence $MSFE_k^{NS} \geq \sigma_k^2 > 0$.

□

Remark 2 *Proposition 2 demonstrates where there exists unbounded gain to player k from player $k-1$ sharing its full information shocks. However, as can be seen from Figure 2, there can be great value to information sharing even in a neighborhood around such a configuration where there exists unbounded value of information sharing.*

In the next section, we relax the assumption that every supply chain player shares either its full information shocks or shares nothing. There, we demonstrate that it is possible for player k's demand not to be invertible with respect to player $k-1$'s shocks and yet there is no value to player k for player $k-1$ to share its demand. Besides being mathematically interesting, this also is in contrast to results of GGS.

6 Shock Sharing versus Demand Sharing

In this section, we assume that the retailer through player $k-2$ shared either nothing or the equivalent of their full information shocks. We show that under these assumptions the value to player k of receiving demand information shared by player $k-1$ may be different than the value to player k of receiving shocks shared by player $k-1$. However, we leave the study of the structure of the propagation of demand when sharing of demand information occurs as well as research on the related topic of sharing between non-contiguous players to future work.

Here, we discuss specific scenarios where $\{D_{k,t}\}$ is not invertible with respect to $\{\epsilon_{k-1,t}\}$ under which there are various possible relationships among $\mathcal{M}_t^{D_{k-1}}$, $\mathcal{M}_t^{D_k}$, $\mathcal{M}_t^{D_{k-1},D_k}$, and $\mathcal{M}_t^{\epsilon_{k-1}}$. We show that the value to player k of receiving demand information shared by player $k-1$ may be : nonexistent; positive, but less than that obtained from player $k-1$ sharing its shocks $\{\epsilon_{k-1,t}\}$; or positive and equivalent to that obtained from player $k-1$ sharing its shocks $\{\epsilon_{k-1,t}\}$.

The first scenario we consider is where $k=2$ and $\{D_{2,t}\}$ is not invertible with respect to $\{\epsilon_{1,t}\}$. Here, it is equivalent for the retailer to share its demand or its full information shocks with the supplier since $\{D_{1,t}\}$ is always invertible with respect to $\{\epsilon_{1,t}\}$ and hence $\mathcal{M}_t^{D_2} \subset \mathcal{M}_t^{D_1} = \mathcal{M}_t^{\epsilon_1}$.

For the rest of this section, we consider the scenario where $k=3$ and $\{D_{2,t}\}$ is not invertible with respect to $\{\epsilon_{1,t}\}$ but the retailer shared the equivalent of its full information shocks with the supplier (player 2). We further assume that $\{D_{3,t}\}$ is not invertible with respect to $\{\epsilon_{2,t}\}$, $\tilde{J}_2 = \tilde{J}_3 = 0$ and both $\tilde{\theta}_2(B)$ as well as $\tilde{\theta}_3(B)$ have no roots on the unit circle. Since the retailer shared its shocks with the supplier, $\epsilon_{2,t} = \beta_1 \epsilon_{1,t}$,

$$\begin{aligned} \phi(B) D_{2,t} &= d + \tilde{\theta}_2(B)(\beta_1 \epsilon_{1,t}) \\ \phi(B) D_{3,t} &= d + \tilde{\theta}_3(B)(\beta_1 \beta_2 \epsilon_{1,t}). \end{aligned} \tag{14}$$

Since $\{D_{2,t}\}$ is not invertible with respect to $\{\epsilon_{1,t}\}$ and $\tilde{J}_2 = 0$, at least one root of $\tilde{\theta}_2(z)$ lies within the unit circle, but not at 0. Similarly, at least one root of $\tilde{\theta}_3(z)$ lies within the unit circle, but not at 0. From these facts along with (14),

$$\epsilon_{1,t} = \frac{(\phi(B) D_{2,t} - d)\tilde{\theta}_2^{-1}(B)}{\beta_1} = \frac{(\phi(B) D_{3,t} - d)\tilde{\theta}_3^{-1}(B))}{\beta_1 \beta_2}. \tag{15}$$

Multiplying both sides of (15) by $\tilde{\theta}_2(B)\tilde{\theta}_3(B)\phi^{-1}(B)$, we obtain

$$\beta_2 \tilde{\theta}_3(B) D_{2,t} = \tilde{\theta}_2(B) D_{3,t} + \frac{d}{\phi(1)} \left[\beta_2 \tilde{\theta}_3(1) - \tilde{\theta}_2(1) \right]. \tag{16}$$

It is possible that $\tilde{\theta}_3(z)$ and $\tilde{\theta}_2(z)$ have common roots. Denote $\tilde{\theta}_3^m(z)$ and $\tilde{\theta}_2^m(z)$ as the polynomials associated with (16) after the cancelation of common factors. After canceling common factors, it follows from (16) that

$$\beta_2 \tilde{\theta}_3^m(B) D_{2,t} = \tilde{\theta}_2^m(B) D_{3,t} + \frac{d}{\phi(1)} \left[\beta_2 \tilde{\theta}_3^m(1) - \tilde{\theta}_2^m(1) \right]. \tag{17}$$

Therefore, $\{D_{2,t}\}$ can be represented as a constant plus a linear combination of *only* present and past values of $\{D_{3,t}\}$ if and only if $\tilde{\theta}_3^m(z)$ has all of its roots outside the unit circle. Similarly, $\{D_{3,t}\}$ can be represented as a constant plus a linear combination of *only* present and past values of $\{D_{2,t}\}$ if and only if $\tilde{\theta}_2^m(z)$ has all of its roots outside the unit circle. There are thus four potential cases regarding the relationship among $\mathcal{M}_t^{D_2}$, $\mathcal{M}_t^{D_3}$ and $\mathcal{M}_t^{D_2,D_3}$. In each of the four cases below, if the supplier shared its shocks or nothing with player 3, then player 3 can determine its best linear forecast of leadtime demand as described in Section 5. We now focus on the potential benefit to player 3 of receiving demand shared by the supplier.

- Case 1. Both $\tilde{\theta}_3^m(z)$ and $\tilde{\theta}_2^m(z)$ have all roots outside the unit circle. Here, $\mathcal{M}_t^{D_2} = \mathcal{M}_t^{D_3} = \mathcal{M}_t^{D_2,D_3} \subset \mathcal{M}_t^{\epsilon_2}$ and there is no value to player 3 receiving the supplier's demand information, but there may be value to player 3 receiving the supplier's shocks.

- Case 2. $\tilde{\theta}_3^m(z)$ has at least one root inside the unit circle but $\tilde{\theta}_2^m(z)$ has all roots outside the unit circle. Here, by similar arguments as those mentioned in Section 2, $\mathcal{M}_t^{D_3} \subset \mathcal{M}_t^{D_2} = \mathcal{M}_t^{D_2,D_3} \subset \mathcal{M}_t^{\epsilon_2}$ and there may be value to player 3 of receiving the supplier's demand information but more value to player 3 of receiving the supplier's shocks. If the supplier shared its demand information with player 3, player 3 can determine shocks $\gamma_{2,t} = \beta_1 \tilde{\theta}_2^{\dagger^{-1}}(B)[\phi(B)D_{2,t} - d]$ from the knowledge of $\{D_{2,t}\}$. Substituting $D_{2,t} = (d + \tilde{\theta}_2^{\dagger}(B)\gamma_{2,t})\phi^{-1}(B)$ in the left hand side of (17), then multiplying both sides by $\phi(B)$, and using the fact that $\tilde{\theta}_2^m(z)$ has all of its roots outside the unit circle, we find that $\{D_{3,t}\}$ is QUARMA with respect to full information shocks $\epsilon_{3,t} = \beta_2\gamma_{2,t}$. Thus, $\{D_{3,t}\}$ can be represented as a one-sided MA(∞) with respect to full information shocks $\epsilon_{3,t}$ which can then be used by player 3 to forecast its leadtime demand.

- Case 3. $\tilde{\theta}_2^m(z)$ has at least one root inside the unit circle but $\tilde{\theta}_3^m(z)$ has all roots outside the unit circle. Here, $\mathcal{M}_t^{D_2} \subset \mathcal{M}_t^{D_3} = \mathcal{M}_t^{D_2,D_3} \subset \mathcal{M}_t^{\epsilon_2}$ and there is no value to player 3 of receiving the supplier's demand information, but there may be value to player 3 receiving the supplier's shocks.

- Case 4. Both $\tilde{\theta}_3^m(z)$ and $\tilde{\theta}_2^m(z)$ have at least one root inside the unit circle. Here, $\mathcal{M}_t^{D_2} \not\subseteq \mathcal{M}_t^{D_3}$ (so there may be value to player 3 of receiving the supplier's demand information), $\mathcal{M}_t^{D_3} \not\subseteq \mathcal{M}_t^{D_2}$, and $\mathcal{M}_t^{D_2,D_3} \subseteq \mathcal{M}_t^{\epsilon_2}$. If the supplier shared its demand information with player 3, player 3 can determine its best linear forecast of leadtime demand by applying the procedure proposed by Brockwell and Dahlhaus (2004), the application of which we leave to future research.

We leave the details of the framework discussed in the above mentioned cases for future research. Nevertheless, we provide numerical examples that demonstrate that the value of demand sharing can be less than the value of full information shock sharing. Although we enumerated four cases above, we have not yet found specific configurations where either case 2 or case 3 could occur. On the other hand, we were able to easily find many examples of both Case 1 and Case 4. The coefficients were found through optimization and are reported here to four significant figures.

An example of a Case 1 situation is $D_{1,t} = d - 0.7373D_{1,t-1} + \epsilon_{1,t} - .11\epsilon_{1,t-1} + .06\epsilon_{1,t-2} - .22\epsilon_{1,t-3}$ where $\ell_1 = 1$, $\ell_2 = 2$, and $\ell_3 = 1$. It follows that if the retailer shares its shocks with the supplier, $D_{2,t} = d - 0.7373D_{2,t-1} + \epsilon_{2,t} + 5.2207\epsilon_{2,t-1} - 1.4406\epsilon_{2,t-2}$. If the supplier shares its shocks with player 3, then $D_{3,t} = d - 0.7373D_{3,t-1} + \epsilon_{3,t} + 5.4834\epsilon_{3,t-1}$ and $MSFE^S = 0.01268\sigma_1^2$. We note that -0.1824 is a root of the polynomial $\theta_2(z)$ as well as $\theta_3(z)$. On the other hand, if either the supplier shares its demand or nothing with player 3, then $MSFE^{NS} = 0.3812\sigma_1^2$. The fact that Case 1 examples can occur is important since it is such situations where the value of demand sharing is nonexistent but there is value to shock sharing.

In all of the case 4 examples that we found, the value of demand sharing was equal to the value of shock sharing up to rounding errors since the MSFE in case 4 was calculated numerically via the framework suggested by Brockwell and Dahlhaus (2004). For example, consider $D_{1,t} = d - .75D_{1,t-1} + \epsilon_{1,t}$ where $\ell_1 = 1$, $\ell_2 = 2$, and $\ell_3 = 1$. It follows that if the retailer shares its shocks with the supplier, $D_{2,t} = d - .75D_{2,t-1} + \epsilon_{2,t} + 3\epsilon_{2,t-1}$ where $\epsilon_{2,t} = .25\epsilon_{1,t}$. If the supplier shares its shocks with player 3, then $D_{3,t} = d - .75D_{3,t-1} + \epsilon_{3,t} + 1.56\epsilon_{3,t-1}$ and $MSFE^S = [(.25)(1.5625)\sigma_1]^2$. On the other hand, if the supplier shares nothing with player 3, then $D_{3,t} = d - .75D_{3,t-1} + \epsilon_{3,t} + \frac{1}{1.56}\epsilon_{3,t-1}$ and $MSFE^{NS} = [(.25)(1.5625)(1.56)\sigma_1]^2$. The MSFE when the supplier shares its demand with player 3 was calculated to be the same as in the shock sharing case.

We have not yet managed to construct any examples where the value of demand sharing is positive but less than the value of shock sharing. If indeed, it is impossible for the value of demand sharing to be between the two extremes (for all possible configurations with respect to sharing arrangements between players and invertibility), then it would follow that Theorem 1 holds regardless of whether supply chain players share nothing, their demand, or their shocks. We leave further analysis on this issue for future research.

7 Directions For Future Research

In this paper, we showed that in a supply chain where the retailer observes ARMA demand and where each supply chain player either shares its full information shocks or shares nothing, the propagation of demand follows a QUARMA-in-QUARMA-out property. Under the same conditions, we provided a methodology where a supply chain player can forecast its leadtime demand via the best linear forecast given its available information and we characterized where there may be value to information sharing.

This paper may facilitate research on several related problems. First is the continuation of the study as to whether the value of demand sharing can be restricted to comparison of no sharing and shock sharing. Second is the somewhat related issue of sharing between non-contiguous supply chain players. Third, is the case where at some levels of the chain, there exist more than one player, each of who observes QUARMA demand. Last is the generalization to the study of the propagation of demand and the value of information sharing when the retailer observes demand that can be expressed as an MA(∞) but is not ARMA such as in long-memory time series models.

Appendix

Proof of Proposition 1:

Let $d_{1,t} = D_{1,t} - \mu_d$ where $\mu_d = \frac{d}{1-\phi_1-\phi_2-\cdots-\phi_p}$. Since the roots of the polynomial $1 - \phi_1 z - \cdots - \phi_p z^p$ are all outside the unit circle, $\{d_{1,t}\}$ has the one-sided $MA(\infty)$ representation,

$$d_{1,t} = \sum_{i=0}^{\infty} \psi_{1,i}\epsilon_{1,t-i}, \tag{18}$$

where $\{\psi_{1,i}\}$ is a square-summable sequence. Following standard time series methodology (c.f. Brockwell and Davis, 1991, pp. 91-92),

$$\psi_{1,i} = \begin{cases} 0 & i < 0, \\ 1 & i = 0, \\ \sum_{j=1}^{p} \phi_j \psi_{1,i-j} - \theta_{1,i} & i \geq 1 \end{cases} \tag{19}$$

where by convention, $\theta_{1,i} = 0$ if $i > q_1$. Thus by Lemma 1,

$$m_{1,t} = \sum_{i=\ell_1}^{\infty} \omega_{1,i}\epsilon_{1,t+\ell_1-i} + \ell_1\mu_d = \sum_{i=0}^{\infty} \omega_{1,i+\ell_1}\epsilon_{1,t-i} + \ell_1\mu_d, \tag{20}$$

and the mean square forecast error is

$$\sigma_1^2 \sum_{i=0}^{\ell_1-1} \omega_{1,i}^2. \tag{21}$$

Thus, from the fact that $D_{1,t} = \mu_d + \sum_{j=0}^{\infty} \psi_{1,j}\epsilon_{1,t-j}$, (23), (24) and (25) with appropriate subscripts, we observe that the retailer's order to its supplier follows

$$\begin{aligned} D_{2,t} = D_{1,t} + m_{1,t} - m_{1,t-1} &= \sum_{i=0}^{\infty} \psi_{1,i}\epsilon_{1,t-i} + \sum_{i=0}^{\infty} \omega_{1,i+\ell_1}\epsilon_{1,t-i} - \sum_{i=0}^{\infty} \omega_{1,i+\ell_1}\epsilon_{1,t-1-i} + \mu_d \\ &= (1 + \omega_{1,\ell_1})\epsilon_{1,t} + \sum_{i=1}^{\infty} \psi_{1,i+\ell_1}\epsilon_{1,t-i} + \mu_d \\ &= (1 + \sum_{j=1}^{\ell_1} \psi_{1,j})\epsilon_{1,t} + \sum_{i=1}^{\infty} \psi_{1,i+\ell_1}\epsilon_{1,t-i} + \mu_d. \end{aligned} \tag{22}$$

Proof of Lemma 1:

Since $D_t = \mu_d + \sum_{j=0}^{\infty} \psi_j \epsilon_{t-j}$ and $\sum_{i=1}^{\ell} D_{1,t} = \ell\mu_d + \sum_{n=1}^{\ell} \sum_{j=0}^{\infty} \psi_j \epsilon_{t+n-j}$, by rearranging terms, it can be shown that

$$\sum_{i=1}^{\ell} D_{t+i} = \sum_{i=0}^{\infty} \omega_i \epsilon_{t+\ell_1-i} + \ell\mu_d, \tag{23}$$

where

$$\omega_i = \begin{cases} 0 & i < 0 \\ \psi_i & i = 0 \\ \omega_{i-1} + \psi_i & 0 < i < \ell \\ \omega_{i-1} + \psi_i - \psi_{i-\ell} & i \geq \ell. \end{cases} \tag{24}$$

Hence, the best linear forecast of demand over the lead-time is

$$m_t = \sum_{i=\ell}^{\infty} \omega_i \epsilon_{t+\ell-i} + \ell\mu_d = \sum_{i=0}^{\infty} \omega_{i+\ell} \epsilon_{t-i} + \ell\mu_d, \tag{25}$$

where $m_t \in \mathcal{M}_t^{\epsilon}$, the forecast error

$$\sum_{i=0}^{\ell-1} \omega_i \epsilon_{t+\ell-i} \tag{26}$$

is in the orthogonal complement of $\mathcal{M}_t^{\epsilon}$, and the mean square forecast error is

$$\sigma_\epsilon^2 \sum_{i=0}^{\ell-1} \omega_i^2. \tag{27}$$

Proof of Theorem 1:

This is a proof by induction. First, we show (I), (II), and (III), are satisfied by the retailer (part (IV) of the theorem does not apply to the retailer). It is assumed that $\{D_{0,t}\} = \{D_{1,t}\}$ and $\{\epsilon_{0,t}\} = \{\epsilon_{1,t}\}$. This along with the fact that $\{D_{1,t}\}$ is invertible with respect to $\{\epsilon_{1,t}\}$ implies that $\{D_{1,t}\}$ has an invertible $QUARMA(p, q_1, 0)$ representation with respect to both $\{\epsilon_{0,t}\}$ as well as with respect to $\{\epsilon_{1,t}\}$. It follows that $\mathcal{M}_t^1 = \mathcal{M}_t^{D_1} = \mathcal{M}_t^0$. Hence, for the retailer, case (i) of the four cases in (10) always holds. Thus, we have shown the theorem holds for the retailer.

Next, we show that (IV) is satisfied when $k = 2$. It was already assumed that the retailer's QUARMA demand with respect to $\{\epsilon_{1,t}\}$ is in minimal form. The condition in part (IV) will occur if the retailer's demand, $\{D_{1,t}\}$, is QUARMA$(0, q_1, 0)$ with respect to $\{\epsilon_{1,t}\}$, $q_1 \leq \ell_1$ and $\theta_1(z)$ has a root at 1. By the argument described in Remark 1, $\tilde{J}_2 = \infty$ and $D_{2,t}$ is constant.

Next, we assume that Theorem 1 holds for player $k-1$, $k \geq 2$. In particular, we assume that

$$\phi(B)D_{k-1,t} = d + \theta_{k-1}(B)\epsilon_{k-1,t-J_{k-1}}. \tag{28}$$

where $\{\epsilon_{k-1,t}\}$ are player $k-1$'s full information shocks. Furthermore, we assume that if $\{D_{k-2,t}\}$ is QUARMA$(0, q^m_{k-2}, J^m_{k-2})$ in minimal form with an MA polynomial that has a root at 1, then $\tilde{J}_{k-1} = \infty$ and $\{D_{k-1,t}\}$ is constant. We show that these assumptions imply that (I), (II),(III) and (IV) hold for player k.

From (28) and Lemma 2, it follows that $\{D_{k,t}\}$ is QUARMA$(p, q_k, \tilde{J}_k)$ with respect to shocks $\lambda_{k,\tilde{J}_k}\{\epsilon_{k-1,t}\}$ where q_k is defined in (34), $\tilde{J}_k$ is defined in (31), and $\lambda_{k,\tilde{J}_k}$ is defined in (33).

To prove II and III, we now proceed from the point of view of player k, which has knowledge of $\tilde{\theta}_k(B)$. There are four mutually exclusive and exhaustive situations that player k may face, and these correspond precisely to cases (i), (ii), (iii), and (iv) of (10). (See the discussion just below (10).)

The first situation is where player $k-1$ shares its full information shocks with player k or the following two conditions hold: $\tilde{\theta}_k(z)$ has all roots outside the unit circle and $\tilde{J}_k = 0$. In this situation, player k is in case (i). In such a case, its information set is $\mathcal{M}_t^{\epsilon_{k-1}}$. Thus, $\epsilon_{k,t} = \lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$ and the QUARMA representation for $\{D_{k,t}\}$ with respect to $\{\epsilon_{k,t}\}$ has the same parameters as the QUARMA representation for $\{D_{k,t}\}$ with respect to $\lambda_{k,\tilde{J}_k}\{\epsilon_{k-1,t}\}$, that is, $\theta_k(B) = \tilde{\theta}_k(B)$, $J_k = \tilde{J}_k$.

The second situation is where player $k-1$ shares no information with player k, $\tilde{\theta}_k(z)$ has all roots outside the unit circle and $0 < \tilde{J}_k < \infty$. Then player k is in case (ii) and its information set is $\mathcal{M}^{\epsilon_{k-1}}_{t-\tilde{J}_k}$. Thus, $\epsilon_{k,t} = \lambda_{k,\tilde{J}_k}\epsilon_{k-1,t-\tilde{J}_k}$, and the QUARMA representation for $\{D_{k,t}\}$ with respect to $\{\epsilon_{k-1,t}\}$ is the same as the QUARMA representation for $\{D_{k,t}\}$ with respect to $\lambda_{k,\tilde{J}_k}\{\epsilon_{k-1,t}\}$ except that the $\epsilon_{k,t}$ are not shifted, that is, $\theta_k(B) = \tilde{\theta}_k(B)$, $J_k = 0$.

The third situation is where player $k-1$ shares no information with player k, and $\tilde{\theta}_k(z)$ has at least one root inside the unit circle. Then player k is in case (iii). In such a case, its full information shocks are those obtained from constructing an ARMA representation of $\{D_{k,t}\}$ with respect to the shocks $\{\epsilon_{k,t}\}$ such that $\mathcal{M}_t^{\epsilon_k} = \mathcal{M}_t^{D_k}$. Following the methodology described in Section 4, starting from the fact that by (I), $\{D_{k,t}\}$ has an ARMA(p, q_k) representation with respect to a white noise sequence with AR polynomial $\phi(B)$ and moving average polynomial $\tilde{\theta}_k(B)$, we obtain $\epsilon_{k,t} = \lambda_{k,\tilde{J}_k}(\tilde{\theta}_k^{\dagger})^{-1}(B)[\phi(B)D_{k,t} - d]$. Thus, we have $\theta_k(B) = \tilde{\theta}_k^{\dagger}(B)$, $J_k = 0$, so that $\{D_{k,t}\}$ has a stationary, invertible ARMA(p, q_k) representation with respect to $\{\epsilon_{k,t}\}$.

The fourth situation is where either player $k-1$'s demand is constant or it is QUARMA with respect to $\{\epsilon_{k-1,t}\}$ such that $\tilde{J}_k = \infty$ as discussed in part (I) of the theorem. It follows that player k's demand is constant. In such a case, its full information shocks are $\{\epsilon_{k,t}\} = 0$.

It follows from Lemma 2 and Lemma 3 that (IV) holds.

□

Lemma 2 *If $\{D_{k-1,t}\}$ is QUARMA(p, q_{k-1}, J_{k-1}) with respect to $\{\epsilon_{k-1,t}\}$ of form $\phi(B)D_{k-1,t} = d + B^{J_{k-1}}\theta_{k-1}(B)\epsilon_{k-1,t}$, where $\theta_{k-1}(B)$ is a polynomial in B of order q_{k-1} with leading coefficient 1, if $0 \le J_{k-1} < \infty$ and $\theta(B) = 0$ if $J_{k-1} = \infty$, then $\{D_{k,t}\}$ has a QUARMA$(p, q_k, \tilde{J}_k)$ representation with respect to $\{\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}\}$ of form $\phi(B)D_{k,t} = d + B^{\tilde{J}_k}\tilde{\theta}_k(B)\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t}$, where $\{\lambda_{k,\tilde{J}_k}\}$ is defined in (33), $\tilde{\theta}_k(B)$ is a polynomial in B of order q_k with leading coefficient 1 if $0 \le \tilde{J}_k < \infty$, $\tilde{\theta}_k(B) = 0$ if $\tilde{J}_k = \infty$ and $q_k = \max\{\max(p, q_{k-1} - \ell_{k-1} + J_{k-1}) - \tilde{J}_k, 0\}$.*

Proof. Since by definition $\{\epsilon_{k-1,t}\}$ are player $k-1$'s full information shocks, it follows from Lemma 1 with $\psi_i = \psi_{k-1,i}$ that

$$m_{k-1,t} = \sum_{i=0}^{\infty} \omega_{k-1,i+\ell_{k-1}}\epsilon_{k-1,t-i} + \ell_{k-1}\mu_d, \tag{29}$$

where $\omega_{k-1,i}$ are given by (24) with $\psi_{k-1,i}$ replacing ψ_i. Similar to the way in which the retailer determined its order via the order-up-to policy, player $k-1$'s order to player k follows $D_{k,t} = D_{k-1,t} + m_{k-1,t} - m_{k-1,t-1}$. It follows from (22) with $k-1$ replacing 1 and recognizing that due to the QUARMA demand for player $k-1$, $\psi_{k-1,0}$ may be 0, that

$$D_{k,t} = \beta_{k-1}\epsilon_{k-1,t} + \sum_{i=1}^{\infty} \psi_{k-1,i+\ell_{k-1}}\epsilon_{k-1,t-i} + \mu_d. \tag{30}$$

Further, let

$$\tilde{J}_k = I(\beta_{k-1} = 0) \inf\{i > 0 | \psi_{k-1,i+\ell_{k-1}} \neq 0\}. \tag{31}$$

Next, we write

$$D_{k,t} - \phi_1 D_{k,t-1} - \cdots - \phi_p D_{k,t-p} = d + \sum_{i=0}^{\infty} \lambda_{k,i}\epsilon_{k-1,t-i}, \tag{32}$$

where $\lambda_{k,i}$ are constants to be determined. It follows from (30) and (32) that

$$\lambda_{k,i} = \begin{cases} \beta_{k-1} & i = 0 \\ \psi_{k-1,\ell_{k-1}+i} - \phi_i\beta_{k-1} & i = 1 \\ \psi_{k-1,\ell_{k-1}+i} - \sum_{j=1}^{i-1} \phi_j \psi_{k-1,\ell_{k-1}+i-j} - \phi_i\beta_{k-1} & 1 < i \leq p \\ \psi_{k-1,\ell_{k-1}+i} - \sum_{j=1}^{p} \phi_j \psi_{k-1,\ell_{k-1}+i-j} & i > p \end{cases} \tag{33}$$

Note that if $p > q_{k-1} - \ell_{k-1} + J_{k-1}$, then from (33) along with (11), $\lambda_{k,i} = 0$ for all $i > p$. Furthermore, if $p \leq q_{k-1} - \ell_{k-1} + J_{k-1}$, then from (33) along with (11), $\lambda_{k,i} = 0$ for all $i > q_{k-1} - \ell_{k-1} + J_{k-1}$. From the definition of $\tilde{J}_k$ in (31), $\lambda_{k,i} = 0$ for $i < \tilde{J}_k$. Thus $\lambda_{k,i}$ can be nonzero only for $\tilde{J}_k \leq i \leq \max\{p, q_{k-1} - \ell_{k-1} + J_{k-1}\}$. If $\tilde{J}_k > \max\{p, q_{k-1} - \ell_{k-1} + J_{k-1}\}$, then $\tilde{J}_k = \infty$ and $D_{k,t}$ is constant.

We define

$$q_k = \max\{\max\{p, q_{k-1} - \ell_{k-1} + J_{k-1}\} - \tilde{J}_k, 0\}. \tag{34}$$

If $\tilde{J}_k < \infty$, then let $\tilde{\theta}_{k,j} = -\frac{\lambda_{k,\tilde{J}_k+j}}{\lambda_{k,\tilde{J}_k}}$ for $1 \leq j \leq q_k$. Hence, if $\tilde{J}_k < \infty$,

$$D_{k,t} - \phi_1 D_{k,t-1} - \cdots - \phi_p D_{k,t-p} = d + \lambda_{k,\tilde{J}_k}\epsilon_{k-1,t-\tilde{J}_k} - \tilde{\theta}_{k,1}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t-1-\tilde{J}_k} - \cdots - \tilde{\theta}_{k,q_k}\lambda_{k,\tilde{J}_k}\epsilon_{k-1,t-q_k-\tilde{J}_k}. \tag{35}$$

If $\tilde{J}_k = \infty$, then $q_k = 0$, $\tilde{\theta}_k(B) = 0$ and $D_{k,t} = \mu_d$.

□

Lemma 3 *For $k \geq 2$, if $\{D_{k-1,t}\}$ is $QUARMA(0, q^m_{k-1}, J_{k-1})$ in minimal form with respect to $\{\epsilon_{k-1,t}\}$ where $\theta^m_{k-1}(z)$ has a root at 1, and $\ell_{k-1} \geq J_{k-1} + q^m_{k-1}$, then $\tilde{J}_k$ defined in (31) is equal to ∞.*

Proof. Assume that for $k \geq 2$, $\{D_{k-1,t}\}$ is QUARMA$(0, q^m_{k-1}, J_{k-1})$ in minimal form with respect to $\{\epsilon_{k-1,t}\}$ where $\theta^m_{k-1}(z)$ has a root at 1, and $\ell_{k-1} \geq J_{k-1} + q^m_{k-1}$. Since $\theta^m_{k-1}(z)$ has a root at 1, $1 - \sum_{i=1}^{q^m_{k-1}} \theta_i = 0$. Since $\{D_t\}$ is QUARMA$(0, q^m_{k-1}, J_{k-1})$, $\psi_i = 0$ for $i < J_{k-1}$, and for $i > J_{k-1} + q^m_{k-1}$. Furthermore, $\psi_{j+J_{k-1}} = -\theta_j$ for $0 \leq j \leq q^m_{k-1}$. Since $\ell_{k-1} \geq J_{k-1} + q^m_{k-1}$, $\beta_{k-1} = \sum_{j=0}^{\ell_{k-1}} \psi_{k-1,j} = 0$. It follows that $\lambda_{k,i}$ defined in (33) are equal to 0 for all i.

□

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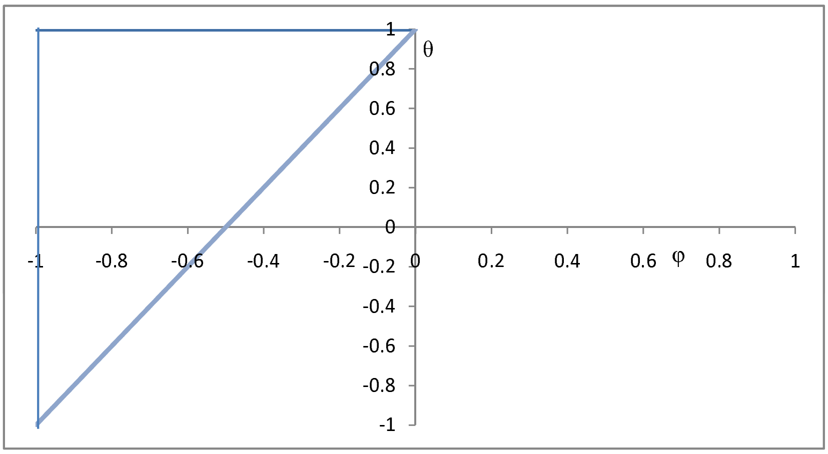

Figure 1: In this figure, the retailer observes an ARMA(1,1) demand with ϕ given on the X axis and θ given on the Y axis. The enclosed triangle represents those configurations of the retailer's demand where the supplier's demand will not be invertible with respect to the retailer's shocks.

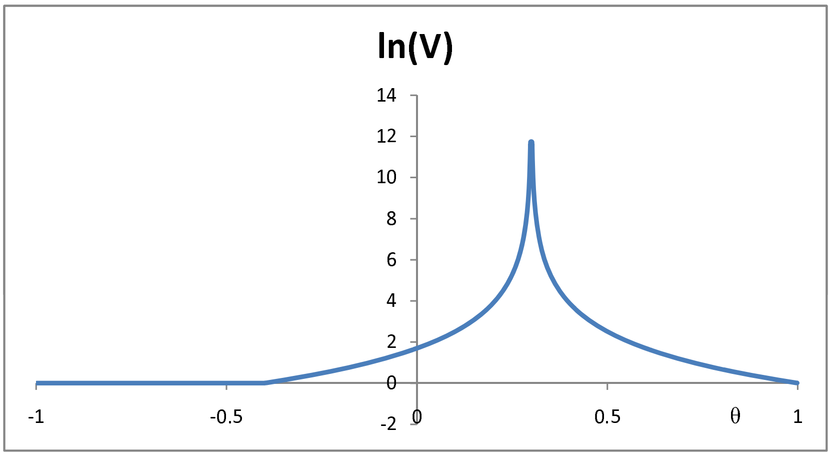

Figure 2: In this figure, the retailer observes an ARMA(1,1) demand with $\phi = -.7$. The X axis represents θ and the Y axis the natural logarithm of the value of information sharing.