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Modeling Volatility in Prediction Markets

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Nowadays, there is a significant experimental evidence of excellent ex-post predictive accuracy in certain types of prediction markets, such as markets for elections. This evidence shows that prediction markets are efficient mechanisms for aggregating information and are more accurate in forecasting events than traditional forecasting methods, such as polls. Interpretation of prediction market prices as probabilities has been extensively studied in the literature, however little attention so far has been given to understanding volatility of prediction market prices. In this paper, we present a model of a prediction market with a binary payoff on a competitive event involving two parties. In our model, each party has some underlying "ability" process that describes its ability to win and evolves as an Ito diffusion. We show that if the prediction market for this event is efficient and accurate, the price of the corresponding contract will also follow a diffusion and its instantaneous volatility is a particular function of the current claim price and its time to expiration. We generalize our results to competitive events involving more than two parties and show that volatilities of prediction market contracts for such events are again functions of the current claim prices and the time to expiration, as well as of several additional parameters (ternary correlations of the underlying Brownian motions). In the experimental section, we validate our model on a set of InTrade prediction markets and show that it is consistent with observed volatilities of contract returns and outperforms the well-known GARCH model in predicting future contract volatility from historical price data. To demonstrate the practical value of our model, we apply it to pricing options on prediction market contracts, such as those recently introduced by InTrade. Other potential applications of this model include detection of significant market moves and improving forecast standard errors.

Key words: prediction market; stochastic model applications; volatility

1. Introduction

Nowadays, there is a significant evidence of excellent efficiency and ex-post predictive accuracy in certain types of prediction markets, such as markets for presidential elections (Wolfers and Zitzewitz 2004a). Berg et al. (2003) show that Iowa Electronic Markets significantly outperform polls in predicting the results of national elections. Moreover, they found "*no obvious biases in the market forecasts and, on average, considerable accuracy, especially for large, U.S. election markets*". Leigh and Wolfers (2006) provide statistical evidence that Australian betting markets for 2004 Australian elections were at least weakly efficient[1] and

[1] future stock price cannot be predicted from historical prices

responded very quickly to major campaign news. Luckner et al. (2008) report that prediction markets for the FIFA World Cup outperform predictions based on the FIFA world ranking. According to official press releases, Hollywood Stock Exchange prediction market consistently shows 80% accuracy for predicting Oscar nominations (HSX 2008).

One should also acknowledge that prediction markets are not perfect information aggregation mechanisms and may suffer from certain types of behavioral biases, such as the "favorite-longshot bias" (Thaler and Ziemba 1988). Reports on behavioral biases and other inefficiencies in prediction markets are not uniform even for different types of prediction markets on a single exchange. For example, Tetlock (2004) found that sports wagering markets on TradeSports.com overreact to news and exhibit "reverse favorite-longshot bias", however the same inefficiencies are not observed in financial markets *on the same exchange* even though both types of markets have similar structure, liquidity and trading volume. Amazingly, prediction markets with high liquidity can sometimes be less efficient than low-liquidity prediction markets and the "forecasting resolution of market prices actually worsens with increases in liquidity" (Tetlock 2008).

Notwithstanding the biases mentioned above, the success of public prediction markets as information aggregation mechanisms led to internal corporate applications of prediction markets for forecasting purposes and as decision support systems (Berg and Rietz 2003). Chen and Plott (2002) show that prediction markets on sales forecasting inside Hewlett-Packard Corporation performed significantly better than traditional corporate forecasting methods in most of the cases. In another example, Google has launched internal prediction markets in April 2005. Cowgill et al. (2008) report that Google's experiment with prediction markets revealed a number of biases such as optimism and overpricing of favorites, however "*as market participants gained experience over the course of our sample period, the biases become less pronounced*". It is even hypothesized that prediction markets can be used for analysis and evaluation of governmental policies (Wolfers and Zitzewitz 2004b).

Despite significant experimental evidence that prediction market prices are good estimates of actual probabilities of the events happening, some researchers have argued that there is little existing theory supporting this practice (Manski 2006). Several papers published in the last few years attempted to provide such theoretical background. For example, Wolfers and Zitzewitz (2006) describe sufficient conditions under which prediction market prices correspond to mean population beliefs.[2] They also show that, for a broad class of models, prediction market prices must be close to the mean beliefs of traders, even when these conditions are invalid.

We can see that interpretation of prediction market prices as probabilities has been extensively studied in the literature. Nevertheless, little attention so far has been paid to understanding *volatility* of prediction market prices, a surprising fact, given that volatility is one of the most crucial concepts in the analysis of markets. Volatility has intrinsic interest to prediction market researchers, not only as a measure of market dynamics, but also for its numerous practical applications. Even a simple task of distinguishing 'normal' market moves from major events can significantly benefit from having a volatility model.

So, our research question emerges naturally: "*If the price of a claim*[3] *in a prediction market is expectation of the actual probability of the event happening, what can we say about volatility of the claim?*" To answer this question we need a model of evolution of the underlying event. Consider a contingent claim that pays \$1 at time T, if and only if some event A happens. A simple model may say that whether the event will happen is predetermined at time $t < T$ and this information is known to *informed* traders. A model like that was used to study information dissemination in financial markets and informational role of prices in disclosing information held by insiders. In a seminal paper, Kyle (1985) considers a model of a betting market with three kinds of traders: one risk-neutral informed trader (insider) that knows the real value of the bet, a mass of random noise traders, and a competitive risk-neutral market maker. Kyle shows that, in this model, there is a unique linear equilibrium in which price follows a Brownian motion and variance of the market uncertainty about value of the bet decreases linearly until the end of the trading, when all insider's information is revealed to the market. Back et al. (2000) extended the analysis to the market with several informed traders having different but potentially correlated signals. Both papers, however, present striking examples of market inefficiency: the information is revealed gradually and full revelation happens only at T.

[2] A sufficient but not necessary condition for this is for beliefs and wealth to be uncorrelated.

[3] In this paper we adopt a popular financial convention of referring to a prediction market contract as a contingent claim or just a claim.

If in these models we enforce the efficiency assumption (i.e., that available information is disseminated to the market instantly), then the price of a contingent claim will not fluctuate even in the presence of noisy traders.

So, why would the price of a claim fluctuate in an *efficient* prediction market? Departing from the framework of Kyle (1985) that relies on presence of informational asymmetries to explain price fluctuations, we propose the explanation that fluctuation happens because the event A is *not* predetermined at time $t < T$, even if all information available at time t is revealed to all market players. To model this uncertainty, we introduce the notion of "*abilities*" for the event participants. These abilities are not constant, but are evolving over time, and so we model them as latent[4] stochastic processes. At expiration, the state of the "ability" processes defines what is the outcome of the event: the party with the highest "ability" wins. Since the "ability" processes evolve stochastically over time, the current state of the claim reveals only partial information about the future, and the larger the time to expiration, the less certain we are about the final state of the process. For our modeling purposes, we will further assume that abilities evolve as Ito diffusions, a particular type of stochastic process that is convenient to work with, as we can apply well-developed tools from stochastic calculus. Adopting a diffusion-like approach makes our model bear certain similarity to a recently published result showing that a simple diffusion model provides a good fit of the evolution of the winning probability in the 2004 Presidential elections market at InTrade.com (Chen et al. 2008). However, Chen et al. (2008) consider only a very restrictive parametric specification[5] and do not analyze the volatility of prices.

As the main theoretical contribution of this paper, we show that the parameters of the underlying stochastic processes affect the claim price *but* do not affect its volatility. In fact, one of the simplest, yet most amazing formulas in the paper shows that if we adopt the diffusion model and the underlying "ability" processes are homoscedastic, the instantaneous volatility of the claim, for an event involving two parties, is fully defined by its current price and the time to expiration.[6] The results generalize to events with more than two parties, however the volatility expression becomes more complex, has no closed form (to the best of our knowledge) and depends on additional parameters of "ability processes" such as ternary correlations.

[4] Latent to researchers however observed by traders.

[5] No drift, constant volatility

[6] If the underlying "ability" processes are heteroscedastic, in practice, one can combine our model with any conditional heteroscedasticity model (say, GARCH) as we show in Section 4.

The rest of the paper is organized as follows. Section 2 gives a short overview of the volatility concept and positions our approach within the stream of research on volatility modeling. Section 3 presents our main theoretical results for pricing of bets in "ideal" prediction markets where the underlying event can be described by a pair of diffusion process. An extension of the theoretical framework to events involving more than two competing parties is quite technical and is presented in Appendix A. Section 4 presents our experimental results obtained for a collection of InTrade prediction markets that show that the volatility pattern derived from our diffusion model is consistent with empirically observed return volatilities. Section 5 discusses the experimental results and outlines research questions and directions for further research on this topic. Section 6 presents an application of our model to pricing options on prediction market claims. Finally, Section 7 concludes the paper with a short summary of the theoretical and empirical results and their managerial implications.

2. Volatility in Financial Markets

What is volatility? Wikipedia[7] defines volatility as "the standard deviation of the continuously compounded returns of a financial instrument with a specific time horizon." Volatility is a natural measure of risk in financial markets as it describes the level of uncertainty about future asset returns. Empirical studies of volatility can be traced as far as Mandelbrot (1963) who observed that large absolute changes in the price of an asset are often followed by other large absolute changes (not necessarily of the same sign), and small absolute changes are often followed by small absolute changes. This famous fact is usually referred to as "volatility clustering" or "volatility persistence" and is nowadays a "must have" requirement for any volatility model in the financial literature. It is was not until two decades later that the first successful model of volatility forecasting was developed by Engle (1982). The insight of Engle's ARCH[8] model was that, in order to capture "volatility clustering", one should model volatility conditional on previous returns: if the square of the previous return is large one would expect the square of the current return to be large as well. That gave rise to the famous pair of equations:

$$r_t = h_t \varepsilon_t$$

[7] http://en.wikipedia.org/wiki/Volatility_(finance)

[8] AutoRegressive Conditional Heteroscedasticity

$$h_t^2 = \alpha + \beta r_{t-1}^2$$

where r_t represent return, h_t volatility and ε_t are i.i.d. residuals. Engle's model was later generalized by Bollerslev (1986) as GARCH[9] to allow for lagged volatility in the volatility equation and the topic exploded by producing a multitude of different volatility models in the next two decades (Bera and Higgins 1993). The research on volatility modeling was primarily guided by the observed properties of the empirical distribution of returns in financial markets, so understanding these properties helps understanding the taxonomy and evolution of volatility models (Engle and Patton 2001). According to Engle and Patton (2001) a good volatility model must be able to generate return distribution possessing the following properties:

1. **Volatility clustering**: large absolute returns must be followed by large absolute returns.

2. **Mean reversion**: long run volatility forecast must be equal to its unconditional volatility i.e., volatility "comes and goes."

3. **Asymmetry (or leverage) effect**: negative shocks generally have higher impact on volatility than positive shocks.[10]

4. **Heavy tails**: financial returns generally have much heavier tails than the normal distribution.

One can now see why it is probably not a good idea to just take an "off-the-shelf" volatility model from financial literature and apply it to prediction markets:

1. To the best of our knowledge, there is no empirical evidence that prediction market returns show significant "volatility clustering." In the experimental part of this paper, we study a collection of InTrade contracts and indeed provide evidence that volatility of daily returns has some persistency, however experimental results show that our model outperforms GARCH in volatility forecasting for InTrade contracts.

2. We have strong reasons to suggest that prediction market volatilities are not mean-reverting. The rationale for this follows from our theoretical model: prediction market volatility is not stationary as it explicitly depends on the remaining time until contract expiration.

[9] Generalized AutoRegressive Conditional Heteroscedasticity

[10] The effect is usually present in stock markets but might be absent in some types of financial markets where there are no absolutely "negative/positive" news, for example, currency markets.

3. Asymmetry is not meaningful within the context of prediction markets as well. Consider, for example, a Presidential elections contract on win of the Republican nominee vs. similar contract on win of the Democratic nominee. An asymmetric model will suggest that if the price of the first contract drops, its volatility must go up more than if it rises by the same amount. Same logic can be applied to the second contract and we obtain a contradiction by noting that gain of the first contract is loss of the second contract and, as prices of both contracts sum to a constant, both of them must have the same volatility.

The major property of the empirical distribution of financial returns that we can also see replicated in the prediction market returns is the presence of "heavy tails". We will discuss this property and its modeling applications more in the Section 5 of this paper.

One may now ask that if "off-the-shelf" volatility models are not likely to work for prediction markets, whether there are some results from financial literature that we can borrow? The answer is "yes", if we adopt a model of latent "ability" processes, we can use modeling techniques employed for option pricing.[11] Indeed, assuming the existence of a latent "ability" process evolving as an Ito diffusion, a prediction market contract can be priced in the same way as a binary option on a stock whose risk-neutral price evolution is the same as the evolution of our latent "ability" process.

The next section develops this idea by presenting the binary scenario which is the simplest to solve and generates the most elegant results.

3. Volatility Model for Prediction Markets on Events With Two Competing Parties

A diffusion model can be most naturally introduced for bets on competitive events such as presidential elections or the Super Bowl finals. Consider a prediction market for a simple event in which two parties (McCain vs. Obama or Patriots vs. Giants) compete with each other. Assume that each party has some strength value like potential to win. Denote strength of the first party as S_1, and strength of the second party as S_2. We consider S_1 and S_2 to be stochastic processes and refer to them as *ability* processes. What is the price of a contingent claim sold at time t that will pay \$1 if the first party wins at the final moment of time T? Assuming

[11] Be careful, however, that we borrow the "tricks" from risk-neutral pricing but not the arguments in defense of risk-neutral pricing such as Delta hedging. Those do not work for prediction markets as discussed in the Section 5.

that the prediction market is efficient and unbiased it must be that $\pi(t) = \mathbf{E}\left[P_w(S_1(T), S_2(T))\,|\mathscr{F}_t\right]$, where P_w is the instantaneous winning probability of the first party as a function of both party's abilities, and $\mathscr{F}_t$ represents the information set at time t.[12][13] Note that in our "ideal" market all market players have the same information set ($\mathscr{F}_t$) which includes all information on evolution of the ability processes until the current moment of time.

In order to complete the model we need to make two additional assumptions:

1. We should describe stochastic processes S_1 and S_2 driving the underlying abilities of each team.

2. We should describe instantaneous winning probability P_w of the first party as a function of both party's abilities.

We model S_1 and S_2 as Ito processes:

$$\begin{aligned} dS_1 &= a_1(S_1, t)dt + b_1(S_1, t)dW_1 \\ dS_2 &= a_2(S_2, t)dt + b_2(S_2, t)dW_2 \end{aligned} \tag{1}$$

where $a_1(s,t)$, $a_2(s,t)$ are drifts and $b_1(s,t)$, $b_2(s,t)$ are volatilities, potentially different for each process S_1, S_2. The processes are driven by two standard Brownian motions W_i that can be correlated with $\mathrm{corr}(W_1, W_2) = \rho_{12}$ ($|\rho_{12}| < 1$). As for P_w, we can either assume that the strongest party always wins or allow for additional randomness by including smoothing using either logit or probit function. In the following, for simplicity, we will assume no smoothing and use $P_w(S_1, S_2) = \mathbb{I}_+(S_1 - S_2)$, where $\mathbb{I}_+$ is the indicator function for positive numbers, i.e. the strongest party wins[14]. We discuss first a relatively simple case, where the drifts and volatilities of the "ability" processes remain constant over time.

3.1. Constant coefficients

Consider a constant coefficients model: $a_i(S_i,\ t) \equiv \mu_i,\ b_i(S_i, t) \equiv \sigma_i$. In other words, S_1 and S_2 are assumed to be Brownian motions with drifts:

$$\begin{aligned} dS_1 &= \mu_1 dt + \sigma_1 dW_1 \\ dS_2 &= \mu_2 dt + \sigma_2 dW_2 \end{aligned}$$

[12] Formally, it is a σ-algebra generated by $(S_1(s),\ S_2(s)),\ 0 \le s \le t$.

[13] In this paper we take risk-free rate to be zero though theoretical results can be easily extended to any constant risk-free rate. A zero rate approximation is sufficient for all practical purposes considering the speculative nature of prediction market contracts.

[14] Who wins in case of equal abilities is not important as this is a zero-probability event.

Note that the difference process ($S = S_1 - S_2$) can be written as

$$dS = \mu dt + \sigma dW,$$

where $\mu = \mu_1 - \mu_2$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$ and $W = \frac{1}{\sigma}(\sigma_1 W_1 - \sigma_2 W_2)$ is a standard Brownian motion. Now, by Markovity of our stochastic processes, the price $\pi(t)$ of a prediction market bid can be written simply as:

$$\pi(t) = P\{S(T) > 0 | S(t)\},$$

We can further expand $\pi(t)$ as:

$$\pi(t) = P\left\{W(T) - W(t) > -\frac{S(t) + \mu(T-t)}{\sigma}\right\} \tag{2}$$

As W is a Brownian motion, $W(T) - W(t)$ is a normal random variable with mean zero and volatility $\sigma\sqrt{T-t}$, so:

$$\pi(t) = \mathscr{N}\left(\frac{S(t) + \mu(T-t)}{\sigma\sqrt{T-t}}\right), \tag{3}$$

where $\mathscr{N}$ is the cumulative density function of the standard normal distribution.

While this is a closed-form result, it depends on the unobserved value $S(t)$[15] and therefore is not directly useful. We can obtain deeper insight by analyzing the *evolution* of the price process. By applying Ito's formula (Oksendal 2005) to $\pi(t)$, we get:

$$d\pi(t) = \frac{\partial \pi}{\partial t}dt + \frac{\partial \pi}{\partial S}dS + \frac{1}{2}\frac{\partial^2 \pi}{\partial S^2}(dS)^2. \tag{4}$$

Now, from Equation 3

$$\begin{aligned}
\frac{\partial \pi}{\partial t} &= \frac{1}{2\sigma\left(\sqrt{T-t}\right)^3}\phi\left(\frac{S(t)+\mu(T-t)}{\sigma\sqrt{T-t}}\right)(S(t) - \mu(T-t)) \\
&= \frac{1}{2\sigma(T-t)}\phi\left(\mathscr{N}^{-1}(\pi(t))\right)\left(\mathscr{N}^{-1}(\pi(t)) - 2\mu\sqrt{T-t}\right), \\
\frac{\partial \pi}{\partial S} &= \frac{1}{\sigma\sqrt{T-t}}\phi\left(\frac{S(t)+\mu(T-t)}{\sigma\sqrt{T-t}}\right) = \frac{1}{\sigma\sqrt{T-t}}\phi\left(\mathscr{N}^{-1}(\pi(t))\right),
\end{aligned}$$

[15] Our derivation is based on the assumption that $\sigma(S(t)) \subset \mathscr{F}_t$ i.e. abilities of each party are public information at time t. Thought it is reasonable to assume that this information is well-known to market players, it might not be available to researchers estimating the model (or might be perceived as subjective).

$$\begin{aligned}\frac{\partial^2 \pi}{\partial S^2} &= -\frac{1}{(\sigma\sqrt{T-t})^3}\phi\left(\frac{S(t)+\mu(T-t)}{\sigma\sqrt{T-t}}\right)(S(t)+\mu(T-t)) \\ &= -\frac{1}{\sigma^3(T-t)}\phi\left(\mathscr{N}^{-1}(\pi(t))\right)\mathscr{N}^{-1}(\pi(t)),\end{aligned}$$

where ϕ stands for probability density function of the standard normal distribution. We can substitute these expressions to the diffusion Equation 4 together with $dS = \mu dt + \sigma dW$ and $(dS)^2 = \sigma^2 dt$. The terms for the drift cancel[16] and we get an equation:

$$d\pi(t) = 0\ dt + \frac{1}{\sqrt{T-t}}\phi\left(\mathscr{N}^{-1}(\pi(t))\right)dW.$$

We can see that instantaneous volatility of $\pi(t)$ (call it $\Sigma(t)$) is given by the expression:

$$\Sigma(t) = \frac{1}{\sqrt{T-t}}\phi\left(\mathscr{N}^{-1}(\pi(t))\right), \tag{5}$$

which depends *only* on the current price $\pi(t)$ and the time to expiration $T-t$ and does *not* depend on any parameters of the underlying latent stochastic processes.

3.2. General Case For Binary Claims

The result we obtained in the previous example for the constant coefficient case can be generalized with some restrictions on parameters of the underlying diffusions as stated by the next Theorem. In particular, we relax the assumption that drifts or volatilities of "ability" processes are constant, though we need to assume that the coefficients are non-stochastic and that, if the dependency on the current process state is present, it is linear. Note that this theorem covers both the case of a standard Brownian Motion with drift as well as the case of a log-normal Brownian Motion with drift as the latter can be written as $dS = \mu S dt + \sigma S dW$. It also implicitly covers the case of a "barrier" bet that pays one dollar if $S_1(T) > K$ where K is a fixed constant - just take $\mu_2 \equiv \sigma_2 \equiv 0$.

THEOREM 1. (Contingent claim pricing in a simple prediction market with two competing parties) *Consider a complete probability space* $(\Omega, \mathscr{F}, P)$ *on which we have a two-dimensional Brownian motion* $(W_1(t),\ W_2(t);\ \mathscr{F}_t);\ 0 \le t \le T$, *where* $\mathscr{F}_t$ *is a natural filtration for* $W_{1,2}$ *i.e.* $\mathscr{F}_t =$

[16] As they should because of the law of iterated expectations.

$\sigma\left((W_1(s),\, W_2(s));\, 0 \le s \le t\right)$. *Assume that each* $(W_i(t);\ \mathscr{F}_t)$ *is a standard Brownian motion and* $\operatorname{corr}(W_1(t),\ W_2(t)) = \rho_{12},\ |\rho_{12}| < 1$. *Consider a prediction market for a competitive event such that underlying ability processes* S_1 *and* S_2 *are* $\mathscr{F}_t$*-measurable and satisfy the diffusion equation:*

$$dS_i = \mu_i(t)(\alpha S_i + \beta)dt + \sigma_i(t)(\alpha S_i + \beta)dW_i,$$

where $\mu_i(t)$ *is a continuous function,* $\sigma_i(t)$ *is a continuous non-negative function,* $\alpha \ge 0$ *and* $P\{\alpha S_i + \beta > 0\} = 1$. *Define the contingent claim price process* $\pi(t)$ *as*

$$\pi(t) = \mathbf{E}\left[\mathbb{I}\left(S_1(T) > S_2(T)\right) | \mathscr{F}_t\right].$$

Under conditions above, $\pi(t)$ *is an Ito's diffusion with zero drift and instantaneous volatility given by the following expression:*

$$\Sigma(t) = \frac{\sqrt{\sigma_1^2(t) + \sigma_2^2(t) - 2\rho_{12}\sigma_1(t)\sigma_2(t)}}{\sqrt{\int_t^T \left(\sigma_1^2(u) + \sigma_2^2(u) - 2\rho_{12}\sigma_1(u)\sigma_2(u)\right) du}} \phi\left(\mathscr{N}^{-1}\left(\pi(t)\right)\right).$$

In other words, $d\pi(t) = \Sigma(t)dW$ *where* W *is some standard Brownian motion with respect to filtration* $\mathscr{F}_t$.

Proof: See Appendix B.

COROLLARY 1. *In case of constant volatility of the underlying ability processes(*$\sigma_i(t) \equiv \sigma_i$*), the volatility of the contingent claim is just*

$$\Sigma(t) = \frac{1}{\sqrt{T-t}} \phi\left(\mathscr{N}^{-1}\left(\pi(t)\right)\right)$$

The Theorem 1 extends our previous result that given the current price of the contingent claim and its time to expiration we can determine its instantaneous volatility without knowing *any* parameters of the underlying ability processes such as their drifts or volatilities *unless* there are "seasonal" effects in the volatility of the "ability" processes in which case our constant coefficient estimate (Equation 5) needs to be scaled by the ratio of the current volatility of the "ability" processes to the future average volatility of the "ability" processes until the expiration. For example, if the current claim price is 0.5, the time to expiration is 10 time units and

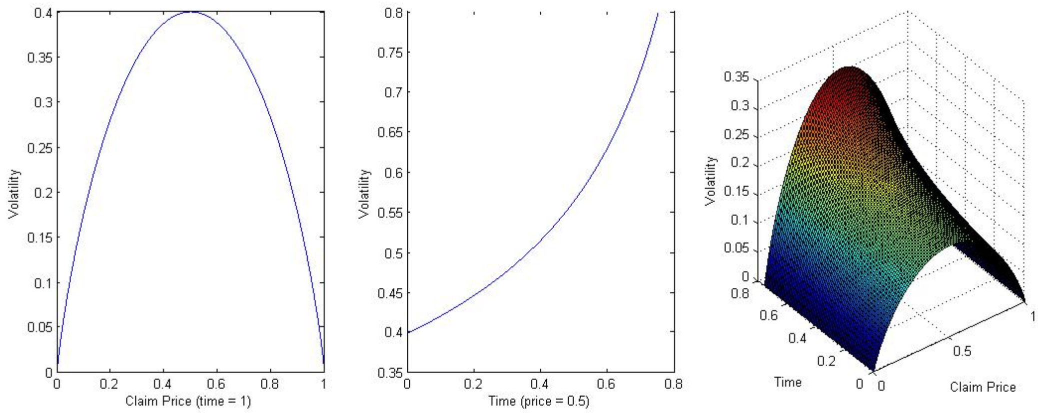

Figure 1 **Claim Volatility As Function of Its Current Price and Time (claim expires at $T = 1$, current time $t = 0$)**

we assume no "seasonal" effects, then its instantaneous volatility (with respect to the same time units) should be $\frac{1}{\sqrt{10}} \frac{1}{\sqrt{2\pi}} \approx 0.126$.

Note that our formula predicts that:

1. Having the claim price fixed, volatility of the claim is proportional to inverse square root of the time until expiration.

2. Having the time to expiration fixed, volatility of the claim is a strictly decreasing function of the distance between the claim price and 0.5 and it goes to zero as the claim price approaches 0.0 or 1.0.

The dependency of the claim volatility on the claim price and the time to expiration is shown in Figure 1.

One, however, might be more interested in the behavior of claim volatility if we don't force its price to be constant but let the claim evolve "naturally". Two interesting questions that might be asked based on our model are:

1. What is the expected instantaneous claim volatility at some future moment of time r?

2. What is the expected average volatility of the claim from the current moment of time until the future moment of time r?

The first of these questions is answered by Theorem 2 which says that our best forecast of the instantaneous volatility in the future is the current claim volatility weighted by "seasonal" effects if necessary.

THEOREM 2. (Instantaneous volatility is a martingale) *In the setting of the Theorem 1, $\frac{\Sigma(t)}{\sigma(t)}$ is a martingale i.e.,*

$$\forall\, r \in [t,T] \;\; \mathbf{E}\left[\frac{\Sigma(r)}{\sigma(r)}\,|\mathscr{F}_t\right] = \frac{\Sigma(t)}{\sigma(t)}.$$

In particular, if we assume that $\sigma_i(t) \equiv \sigma_i$, then $\Sigma(t)$ is a martingale.

Proof: See Appendix B.

The second question is answered by the Theorem 3.

THEOREM 3. (Average expected claim volatility) *In the setting of the Theorem 1, take $r \in (t,T)$ and define*

$$\Lambda = \frac{\int_t^r \sigma^2(u)du}{\int_t^T \sigma^2(u)du}$$

i.e., Λ is the volatility-weighted ratio of the time elapsed to the total contract duration (in particular, if we assume that $\sigma_i(t) \equiv \sigma_i$, then $\Lambda = \frac{r-t}{T-t}$). Then,

$$\mathbf{E}\left[(\pi(r)-\pi(t))^2\,|\mathscr{F}_t\right] = \int_0^\Lambda \frac{1}{\sqrt{1-\lambda^2}}\phi^2\left(\frac{1}{\sqrt{1+\lambda}}\mathscr{N}^{-1}(\pi(t))\right)d\lambda \tag{6}$$

Proof: See Appendix B.

COROLLARY 2. ("Ex-post expected" price trajectory) *In the setting of Theorem 3,*

$$\mathbf{E}\left[\pi(r)\,|\mathscr{F}_t,\ \pi(T)\right] = \pi(t) + \frac{1}{\pi(t)}\int_0^\Lambda \frac{1}{\sqrt{1-\lambda^2}}\phi^2\left(\frac{1}{\sqrt{1+\lambda}}\mathscr{N}^{-1}(\pi(t))\right)d\lambda.$$

Proof: Direct application of the Theorem 1 from Pennock et al. (2002), which says that, if the market price in a prediction market is an unbiased estimate of the probability of the actual event E, then

$$\mathbf{E}\left[\pi(t)\,|\pi(t-1),\ E\right] = \pi(t-1) + \frac{\mathrm{Var}\{\pi(t)|\pi(t-1)\}}{\pi(t-1)}.$$

Q.E.D.

The corollary 2 deserves some clarification. Our model was built under assumption that the claim price is a martingale i.e. $\mathbf{E}\left[\pi(r)\,|\mathscr{F}_t\right] = \pi(t)$. It follows that an "average" of *all* price trajectories of a claim from point $(t,\pi(t))$ is just a horizontal line. Imagine now that an observer (but not a trader) has access to an oracle

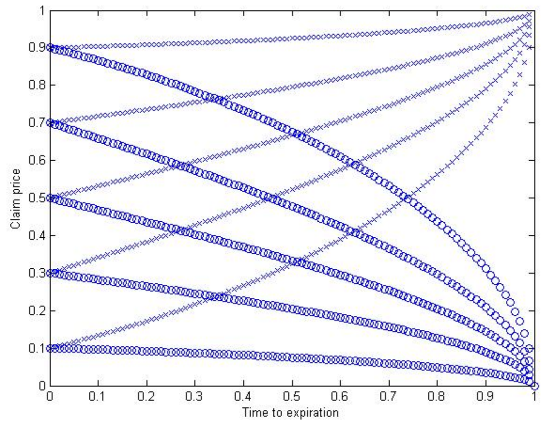

Figure 2 **Expected claim price conditional on event happening ('x') or not ('o') (claim expires at $T = 1$, current time $t = 0$)**

that can say whether the event will actually happen or not. Naturally, if the oracle says "yes", it eliminates all price trajectories converging to zero. The corollary 2 tells us what one would obtain by averaging all remaining trajectories that converge to one. We show the expected price trajectories for several initial price values in the Figure 2. Note that all price trajectories converge either to 1.0 (event happens) or to 0.0 (event does not happen).

In conclusion of this section, we would like to note that our model can be extended to handle claims with three or more competing parties. The extension is based on similar ideas, but the results and the derivations become quite technical, so we list the details in Appendix A.

4. Experimental Evaluation

In this section we present the experimental evaluation of our model on a set of InTrade prediction market contracts.

4.1. Data

Our data set contained daily observations for a collection of InTrade prediction market contracts. InTrade is an online Dublin-based trading exchange web site founded in 2001 and acquired by TradeSports.com in 2003. The trading unit on InTrade is a contract with a typical settlement value of $10, which is measured on a 100 points scale. To be consistent with our convention that the winning contract pays $1, we have normalized all price data to be in $[0, 1]$ range, so, for example, a 50 points price of InTrade contract is represented by 0.5

Variable	*Min*	*Max*	*Mean*	*Median*	*Standard Deviation*
Observations per contract	1	1427	375.76	237	334.52
Time to expiration	2	1933	417.68	347	306.35
Daily deviations	-0.93	0.942	-1.5923e-005	0	0.018869
Returns	-0.9975	449	0.019172	0	1.4059
Price	0.001	0.995	0.244	0.075	0.32092
Volume	0	9	0.46122	0	1.4053

Table 1 Descriptive statistics for 901 InTrade contracts (338,563 observations)

in our data set. The full data set we analyzed included daily closing price and volume data for a collection of 901 InTrade contracts obtained by periodic crawling of InTrade's web site.[17] Table 1 provides the descriptive statistics for our sample. Note that we use term "price difference" or "absolute return" to represent price changes between two consecutive days $a_t = p_t - p_{t-1}$, while just "return" refers to $r_t = \frac{p_t - p_{t-1}}{p_{t-1}}$.

A closer examination of the data reveals several interesting facts that must be taken into account in our empirical application. At first, $288,119$ of $338,563$ observations that we have (i.e. $> 85\%$ of the whole data sets) had zero price change since the previous day. Moreover, $287,430$ of these observations had zero daily trading volume, what means that most of the time price did not change because of the absence of any trading activity. We presume that absence of activity in many of InTrade prediction markets can be attributed partially to low liquidity of most of the markets in our sample and partially to high transaction costs for contracts executed on the exchange.[18] Even if we exclude observations with $a_t = 0$ from our data set, returns exhibit the following "round number" bias: absolute returns divisible by 5 ticks[19] occur much more frequently than absolute returns of similar magnitude that are not divisible by 5 ticks. For example, 5 ticks price difference has occurred $2,228$ times in our data set as opposed to 624 times for 4 tick difference and 331 times for 6 tick difference. In fact, more than half of all non-zero price differences in our data set are divisible by 5 ticks. The bias can be easily seen in Figure 3 that plots absolute return sizes against corresponding frequencies on regular and log scales. Similar effect can be observed for prices, as shown in Figure 4.

[17] InTrade keeps historical data for each contract on the web site, however expired contracts disappear from the web site after certain amount of time. This is why periodic crawling was necessary.

[18] InTrade charges a 5 points commission from each such contract.

[19] One InTrade tick is equal to 0.1 InTrade point or 0.1 cent in our normalization.

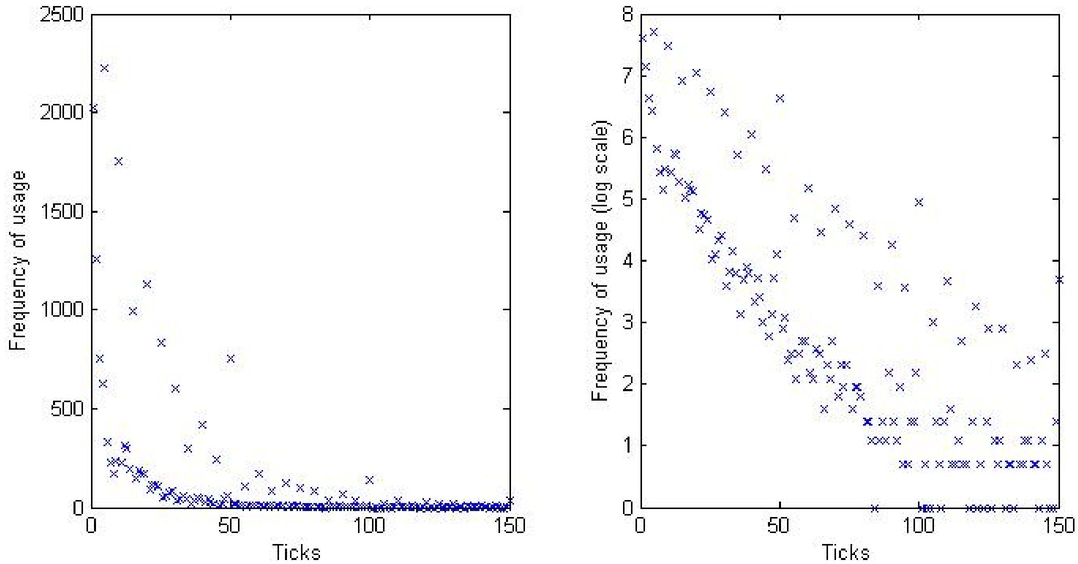

Figure 3 **Number of times a deviation of a certain size has occurred (tick = 0.1 InTrade point; zero deviations excluded)**

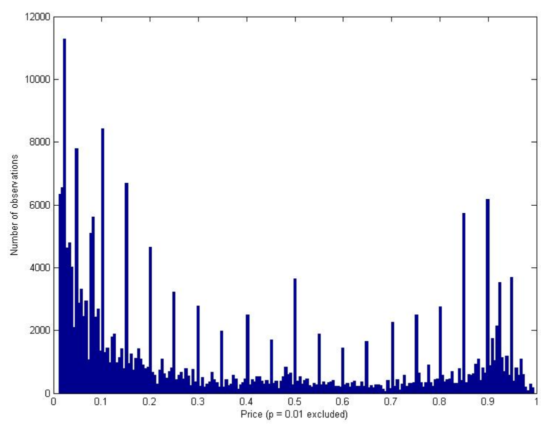

Figure 4 **Histogram of prices**

4.2. Model Test

Our theoretical model predicts that, for each observation, *conditional on price history*:

$$a_t = h_t \varepsilon_t, \tag{7}$$

$$h_t = \sqrt{\int_0^{\Lambda} \frac{1}{\sqrt{1-\lambda^2}} \phi^2 \left(\frac{1}{\sqrt{1+\lambda}} \mathcal{N}^{-1}(p_t) \right) d\lambda} \tag{8}$$

where h_t is the conditional volatility of absolute returns, $\Lambda = \frac{1}{T-t}$[20][21] and ε_t are independent identically distributed standard[22] random variables[23]. While most of InTrade contracts in our sample have more than two possible outcomes, we believe the qualitative behavior of contract prices can be well captured by our simple result for binary model. To test our statement we propose taking logs of the absolute value of Equation 7:[24]

$$\log(|a_t|) = \mathbf{E}\left[\log|\varepsilon_t|\right] + \log(h_t) + \log|\varepsilon_t| - \mathbf{E}\left[\log|\varepsilon_t|\right].$$

Now, let $\gamma = \mathbf{E}\left[\log|\varepsilon_t|\right]$ and $\mu_t = \log|\varepsilon_t| - \mathbf{E}\left[\log|\varepsilon_t|\right]$. Note that $\mathbf{E}\left[\mu_t\,|\mathscr{F}_t\right] = 0$ and

$$\log(|a_t|) = \gamma + \log(h_t) + \mu_t,$$

so we're in a regression-like setting even though the residuals are not normally distributed.

We suggest two different tests of our model, both of which require taking absolute values of price differences and regressing them on a constant and $\log(h_t)$. The first test checks for *presence* of heteroscedasticity effects of time and price as predicted by our theory. The null hypothesis here is that coefficient on $\log(h_t)$ is equal to zero and we want to reject it. To the contrary, the second test checks that the *magnitude* of heteroscedasticity effects is consistent with our theory. The null hypothesis here is that coefficient on $\log(h_t)$ is equal to one and we do not want to reject it.

Results of this regression are given in the Table 2, standard errors are corrected for heteroscedasticity[25]. Note that we included zero deviations (but not zero volume trading days), but to avoid taking logs of zero we added a small smoothing factor (10^{-4}) under the $\log(|a_t|)$. In Table 2 we report the results of *three* regressions. The first regression included all observations, the second regression included all observations with price in $(0.05, 0.95)$ range and the last regression included all observations with price in $(0.1, 0.9)$

[20] As we take daily returns, time must be measured in days.

[21] In this section we assume that the underlying "ability" process is homoscedastic i.e. $\sigma_i(t) \equiv \sigma_i$.

[22] mean zero, variance one

[23] We do not know what is the distribution of ε_t, however for small values of Λ it must be close to normal.

[24] We take logs instead of squares of returns as logs are more robust to outliers in the data (which are plenty because of the "fat tails" of the return distribution, as will be discussed later). Similar results can be obtained with squares of returns.

[25] As one can see from the Figure 5, residuals are indeed heteroscedastic as when volatility is high so is the residual.

Variable	*Coefficient*	*Robust Std. Err.*	*95% Conf. Lower Bound*	*95% Conf. Upper Bound*
constant	-3.38	0.031	-3.4437	-3.323
$\log(h_t)$	0.630	0.005	0.6204	0.639
constant	-2.46	0.086	-2.6338	-2.298
$\log(h_t)$	0.846	0.020	0.8078	0.886
constant	-2.10	0.115	-2.3266	-1.876
$\log(h_t)$	0.946	0.028	0.8900	1.001

Table 2 Three pooled regressions of logs of absolute returns (R^2 = 0.109, 0.027, 0.022)

range. We report results with excluded marginal observations because such observations are most seriously affected by the bias depicted in Figure 3. Moreover, there is substantial evidence from the prior research suggesting that people tend to overvalue small probabilities and undervalue near certainties, i.e., the so called "favorite-longshot bias" (Thaler and Ziemba 1988). This effect may be especially strong in our sample as it happened to be skewed towards low price contracts (see Figure 4). As we see from Table 2, the null hypothesis for the first test is strongly rejected in all three cases, meaning that there are indeed strong heteroscedasticity effects of the current contract price and the time until contract expiration. For the second test, while in the first two cases the null hypothesis is rejected at 95% confidence level, the coefficient on $\log(h_t)$ improves (i.e. approaches the predicted value of 1) as we exclude the marginal observations.

At that point we may get concerned that our results might be affected by potential heterogeneity of contracts. While the basic theory suggests that ε_t are i.i.d. residuals, in practice we may expect correlations in volatility levels for the same contract, for example, due to different liquidity levels of contracts in the sample (some contracts, like those for democratic primaries are very liquid but many others are not, and may display lower or higher volatility because of liquidity issues). In order to alleviate these concerns we also included contract-specific effects to our regression. In the Table 3 we report results of panel data regressions with fixed effects for contracts. Note that the Breusch and Pagan LM test rejects absence of effects ($\chi^2(1) = 5448.76$, Prob $> \chi^2 = 0.0000$) and the Hausman test does not reject the random effects model[26] ($\chi^2(1) = 0.97$, Prob $> \chi^2 = 0.3258$).

We also noted that because of potential auto-correlation in absolute returns we may underestimate actual

[26] We still report results of the fixed effects model as the number of samples that we have makes efficiency not a concern. Results of the random effects model are quantitatively similar.

Variable	*Coefficient*	*Robust Std. Err.*	*95% Conf. Lower Bound*	*95% Conf. Upper Bound*
$\log(h_t)$	0.6438	0.013112	0.61810	0.66951
$\log(h_t)$	1.0370	0.039503	0.95957	1.11443
$\log(h_t)$	1.1910	0.049832	1.09341	1.28876

Table 3 Three fixed effect regressions of logs of absolute returns (within R^2 = 0.0456, 0.0244, 0.0268)

standard errors; we tried a random-effects regression with AR(1) disturbances (GLS estimator). Indeed, we found significant auto-correlation in residuals ($\rho = 0.526$); correcting for autocorrelation makes standard errors slightly larger but does not affect the results significantly.

Furthermore, we have replicated all results using the instantaneous volatility from Equation 5 instead of the average daily volatility from Equation 8. As most of the observations in our sample are relatively far from the expiration date, instantaneous volatilities were close to daily averages and the regression results were not significantly different.

As instantaneous volatility expression provides particularly nice separation of time and price effects, we also suggest running a simple visual test in addition to regression based testing. As we have plenty of observations, we can use data to estimate sample means of the $\log(|a_t|)$ conditional on fixed price or fixed time to expiration. We can then plot the means against predictions of our model and see if the qualitative behavior is similar. Results that we obtained are presented in Figure 5. Solid lines shown on the picture are plots of $\log(\phi(\mathcal{N}^{-1}(p_{t-1})))$ and $-0.5\log(T-t)$ shifted by a constant so that they match the data means.

4.3. Predicting future volatility and comparison with GARCH model

So far, in our experimental results we were ignoring potential heteroscedasticity of the “ability” processes and, therefore, potential heteroscedasticity of the contract prices. Our main concern is that, if “volatility clustering” indeed has significant presence in prediction market prices, one might be able to forecast volatility better by using historical data than by using our theoretical results. In general, one cannot prove or disprove such a statement as different prediction markets and even different contracts can exhibit different levels of “volatility clustering”, and so the answer can only be given on a per-contract basis. In this paper, we report our test for a sample of 51 InTrade contracts where each contract represents the Democratic Party Nominee winning Electoral College Votes of a particular state in 2008 Presidential Election.

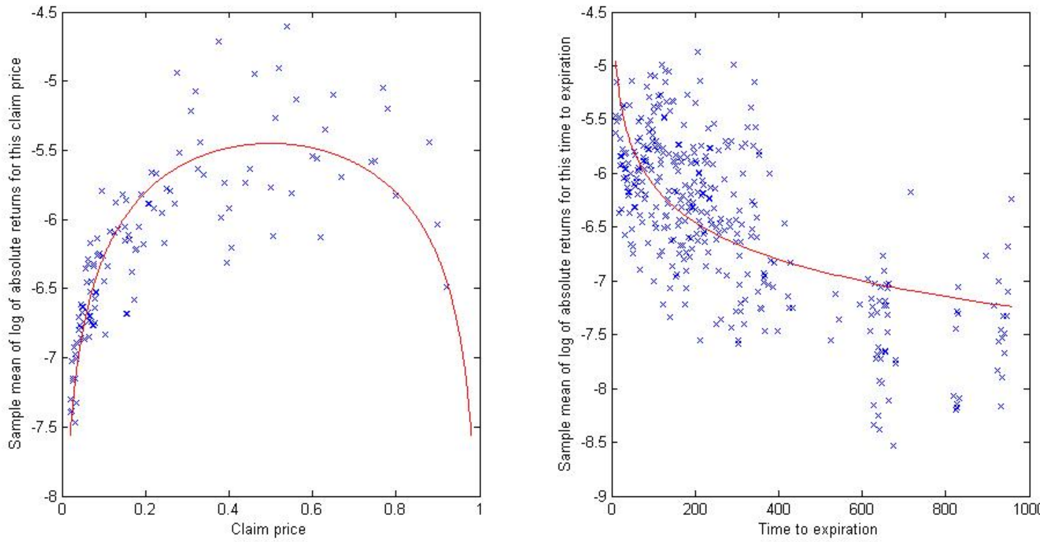

Figure 5 Sample means of log of absolute returns conditional on fixed price or time to expiration against predictions of our model.

In our test we compared three models, the GARCH(1,1) model (Bollerslev 1986), our model assuming homoscedasticity of the "ability" processes, and the GARCH(1,1) model applied to standardized residuals from our model[27]. Note that our model is non-parametric while other two models require estimating parameters of the GARCH process from historical data. As we are more interested in forecasting volatility than in explaining it, we separated observations for each contract into two equal parts - the first part was used to learn parameters of the GARCH processes (on a per contract basis) and the second part was used for evaluation of forecasting accuracy. We have compared all three models in terms of total log-likelihood on the testing part of the sample as this is a natural fit criterion for GARCH models. Results are given in Table 4. In 26 out of 51 cases the model 3 (GARCH on standardized residuals) provided the best forecasts and in 17 out of the rest 25 cases our model outperformed regular GARCH. Overall, the results support the hypothesis that there is a significant "volatility clustering" in the data, however our model is generally better in forecasting volatility than the results based on pure historical data. Moreover, our model seems to capture effects of time and price on claim volatility that are orthogonal to the heteroscedasticity captured by GARCH. The orthogonal nature of these two different sources of heteroscedasticity implies that one can significantly

[27] Standardized residual is the return r_t normalized by our prediction of the return volatility which is $\frac{h_t}{p_t}$ where h_t is given by Equation 8

improve forecasts of future contract volatility by running GARCH on standardized residuals from our model rather than running it directly on the return data.

5. Discussion and Future Work

Overall, our experiments show that realized volatility of prediction markets is consistent with what our diffusion-based theory would predict. The model provides good predictions of market volatility without requiring one to know *anything* about the market except for the current price and time to expiration. However, there are still areas for improving the model and providing even better fit. We outline three directions for improvement in future work:

1. Our model ignores market micro-structure as well as possible behavioral biases such as "favorite-longshot" and rounding biases. No market (especially prediction market) is ideally liquid - the bid-ask queue has always finite depth and there are usually transaction costs. As well, traders are not fully rational. It is an interesting research question to examine whether the volatility model can be extended to capture the effects of the structure of the bid-ask queue and/or some standard behavioral biases.

2. Our idea of two underlying "ability" processes being driven by Brownian motions might not ideally represent the real stochastic process driving the event probability. As we already mentioned before, most of the contracts in our sample involve more than two competing entities while in the experimental part we analyzed them as if they were binary cases. While the Appendix provides full derivation of the volatility expression in case of more than two competing parties, the result depends on unobserved correlations between Brownian motions driving the data and therefore cannot be directly applied to our data set.[28] Even if using the binary scenario is fine, we put significant restrictions on the claim price dynamics by assuming that they are driven by Brownian motions. Brownian motions result in sample paths that are continuous and short-term price changes that are *almost* normally distributed. However in our sample we frequently observe price jumps that are completely improbable if we assume normal distribution of returns. For example, the contract for Michigan to hold new democratic primaries in 2008 had a 59 cents price drop on March 19th, a move that, according to our estimates, constitutes almost 15 (!) standard deviations. One might want to extend

[28] Another interesting extension of this work would be to show how the correlations can be uncovered from the historical price data.

Contract	*Model 1* log L	*Model 2* log L	*Model 3* log L
ALABAMA.DEM	-431.09	**92.341**	59.567
ALASKA.DEM	4.8287	**69.42**	33.529
ARIZONA.DEM	67.601	114.71	**140.06**
MONTANA.DEM	-152.22	**122.64**	-98.139
NEBRASKA.DEM	-381.61	-29.548	**1.445**
ARKANSAS.DEM	65.974	138.61	**156.86**
NEVADA.DEM	180.46	240.81	**256.33**
CALIFORNIA.DEM	-4850.1	-10412	**-1047.2**
NEWHAMPSHIRE.DEM	228.38	260.14	**267.53**
COLORADO.DEM	322.91	294.21	**326.68**
CONNECTICUT.DEM	-198.42	**425.92**	289.61
DELAWARE.DEM	330.8	**433.69**	354.24
GEORGIA.DEM	127.85	121.84	**130.57**
NEWJERSEY.DEM	378.27	**398.59**	388.51
HAWAII.DEM	553.4	567.01	**588.64**
IDAHO.DEM	**158.06**	112.52	141.09
NEWMEXICO.DEM	310.15	302.49	**319.12**
NEWYORK.DEM	223.36	**447.15**	405.75
ILLINOIS.DEM	589.39	553.41	**613.2**
NTH.CAROLINA.DEM	-206.83	**155.25**	42.05
NTH.DAKOTA.DEM	75.26	**90.081**	83.445
INDIANA.DEM	112.67	140.14	**146.4**
IOWA.DEM	**329.78**	329.35	329.39
OHIO.DEM	270.74	269.71	**302.36**
OKLAHOMA.DEM	-8.6928	99.777	**119.05**
KANSAS.DEM	27.434	108.34	**123.38**
KENTUCKY.DEM	131.22	110.77	**134.07**
LOUISIANA.DEM	102.25	119.51	**134.51**
MAINE.DEM	**366.05**	330.65	336.14
MARYLAND.DEM	440.08	469.23	**487.67**
OREGON.DEM	**358.88**	356.9	348.78
MASSACHUSETTS.DEM	**518.96**	503.46	510.21
PENNSYLVANIA.DEM	324.09	325.93	**335.2**
MICHIGAN.DEM	271.28	**300.39**	292.54
MINNESOTA.DEM	324.2	323.13	**324.94**
RHODEISLAND.DEM	378.95	**494.51**	445.77
STH.CAROLINA.DEM	95.084	**101.23**	83.047
STH.DAKOTA.DEM	113.23	112.25	**121.73**
MISSISSIPPI.DEM	**133.45**	128.62	111.81
TENNESSEE.DEM	-559.44	**80.524**	61.8
MISSOURI.DEM	182.58	202.71	**207.44**
DISTRICTOFCOLUMBIA.DEM	Not converged	**598.3**	Not converged
TEXAS.DEM	141.63	117.9	**144.61**
UTAH.DEM	Not converged	**109.58**	Not converged
VERMONT.DEM	431.11	429.22	**452.49**
VIRGINIA.DEM	240.58	242.67	**262.89**
WASHINGTON.DEM	**436.91**	424.04	434.15
WESTVIRGINIA.DEM	-27.236	94.784	**96.654**
WISCONSIN.DEM	266.56	**301.05**	282.31
WYOMING.DEM	-1285.7	**82.772**	-66.499
FLORIDA.DEM	**226.92**	209.12	225.51

Table 4 **Model Comparison in terms of log-likelihood outside of training sample. Model 1: GARCH(1,1); Model 2: our model; Model 3:- GARCH(1,1) on standardized residuals from Model 2**

our model to cover cases when the underlying "ability" processes are driven by a jump-diffusion i.e. have a Poisson jump component in addition to a regular Brownian motion. Using jump-diffusion will allow for modeling of more interesting dynamics such as sudden arrival of new significant information to the market.

3. Finally, in our work, we price claims as if traders are risk-neutral. Indeed there is no "pricing kernel" or "stochastic discount factor" in our pricing formula. As prediction market bets are somewhat similar to binary options, it is tempting to say that we just borrow a risk-neutral pricing approach from the option pricing literature. Unfortunately, the traditional argument to support risk-neutral pricing of options, i.e. traders can replicate the option by trading in the underlying stock(s) (Delta hedging) does **not** work for prediction markets as the underlying either does not physically exist or cannot be traded. Nevertheless, we can suggest at least three alternative arguments in defense of risk-neutral pricing:

(a) In some prediction markets traders may indeed behave as risk-neutral either because the market uses "play money" instead of real ones (like Hollywood Stock Exchange which uses "Hollywood dollars") or because trader's participation is limited (Iowa Prediction Markets limit traders' positions by 500$).

(b) Wolfers and Zitzewitz (2006) show that under certain conditions prediction market prices maybe close to the mean population beliefs even if the traders are not risk-neutral, so risk-neutral pricing might be a valid approach even in the presence of certain risk-aversion.

(c) As we already described in the beginning of the paper, there is significant experimental evidence that prediction market prices are unbiased estimates of the actual event probability.

Note that the last argument was our main justification in this paper. Without trying to answer the question of "why is it so?", we just adopted risk-neutral pricing to see where the theory leads us. The results we have obtained in the experimental part of this paper, might be seen as a joint test of market efficiency and non-bias assumptions, as well as our diffusion model.

6. Application: Pricing options on prediction market contracts

This section presents application of our model to pricing options on prediction market contracts. Options are popular financial instruments with numerous applications such as risk hedging or speculation on volatility. A classic "vanilla" call option on a security provides the right but not the obligation to buy a specified quantity of the security at a set strike price at a certain expiration date.[29] A major breakthrough in option pricing was achieved by Fischer Black and Myron Scholes who were first to discover the PDE.[30] approach to option pricing and obtain a closed form solution for "vanilla" call price now known as the Black-Scholes formula (Black and Scholes 1973, Merton 1973).

In this section, we consider binary options on prediction market claims, like those recently introduced by InTrade. Such binary option will pay $1 if on option's expiration date (T') the underlying contract's price is larger than the strike price (K). Note that option's expiration date (T') is different from the expiration date of the underlying contract (T).[31] For example, InTrade's option X.15OCT.OBAMA.>74.0 will pay 100 points ($1 in our normalization) if on $T' =$ 15 OCT 2008, the price of the 2008.PRES.OBAMA contract is at least 74 points (K). The expiration date for the underlying contract is $T =$ 09 NOV 2008.

As we already assume risk-neutrality, it is easy to define the option price:

$$c_t = P\{\pi(T') > K|\pi(t)\}. \tag{9}$$

We know the evolution process for the underlying contract:

$$d\pi(t) = 0\ dt + \frac{\sqrt{\sigma_1^2(t) + \sigma_2^2(t) - 2\rho_{12}\sigma_1(t)\sigma_2(t)}}{\sqrt{\int_t^T \left(\sigma_1^2(u) + \sigma_2^2(u) - 2\rho_{12}\sigma_1(u)\sigma_2(u)\right) du}} \phi\left(\mathcal{N}^{-1}\left(\pi(t)\right)\right) dW,$$

so it only remains to calculate the expectation. The result is stated by the following theorem.

THEOREM 4. (Pricing options on prediction market claims in a simple prediction market with two competing parties) *In the setting of the Theorem 1, take $T' \in (t,T)$ and consider a binary option on the contract $\pi(s)$ with payoff*

$$c_t = P\{\pi(T') > K|\pi(t)\}.$$

Define $\lambda = \frac{\int_t^{T'} \sigma^2(u)du}{\int_t^T \sigma^2(u)du}$ i.e. λ is the volatility-weighted ratio of the time to the option expiration to the time to

[29] In this section we consider European style options only. An American style option will give owner the right to exercise it on or before the expiration date.

[30] Partial Differential Equation

[31] Naturally, it must be that $T' < T$.

the contract expiration (in particular, if we assume that $\sigma_i(t) \equiv \sigma_i$, then $\lambda = \frac{T'-t}{T-t}$). Then the option price is given by the equation

$$c_t = \mathcal{N}\left(\sqrt{\frac{1}{\lambda}}\mathcal{N}^{-1}(\pi(t)) - \sqrt{\frac{1}{\lambda} - 1}\mathcal{N}^{-1}(K)\right). \tag{10}$$

Proof: See Appendix B.

Several interesting observations can be made from this formula:

1. The option price is strictly increasing in the current contract price and strictly decreasing in the strike price.

2. As time to the option expiration converges to time to the contract expiration ($\lambda \to 1$), the option price converges to the current contract price ($\pi(t)$).

3. If the current contract price is 0.5 and the option expires halfway to the contract expiration,[32] then the option price is equal to the strike price $c_t = K$.

To give the reader better intuition of how the option price behaves with varying parameters λ and K we include a plot of option prices for a fixed value of the current price ($\pi(t) = 0.5$) in the Figure 6.

One can also take derivative of the option price with respect to the strike price to retrieve the risk-neutral density of the claim price at time T'. The result is

$$f(p) = \frac{\phi\left(\sqrt{\frac{1}{\lambda}}\mathcal{N}^{-1}(\pi(t)) - \sqrt{\frac{1}{\lambda} - 1}\mathcal{N}^{-1}(p)\right)}{\phi\left(\mathcal{N}^{-1}(p)\right)}\sqrt{\frac{1}{\lambda} - 1}. \tag{11}$$

We have plotted it for different values of the current price and λ in the Figure 7.

7. Managerial Implications and Conclusions

Although volatility is one of the most widely studied concepts in financial markets, little attention, so far, has been given to understanding volatility of betting or prediction markets. This is unfortunate as being able to predict market volatility in prediction markets can be as important as in financial markets. Practical applications may include detection of significant market moves, improving forecast standard errors in

[32] as weighted by volatilities of the underlying "ability" processes

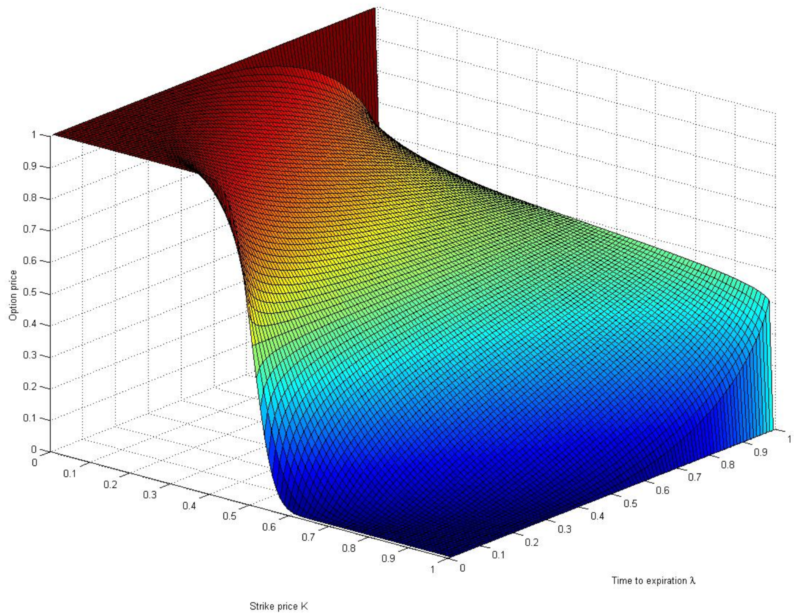

Figure 6 Option prices for prediction market contracts as a function of the time to option expiration and the strike price. The current contract price is 0.5.

prediction markets, pricing claims in conditional prediction markets and pricing options on prediction market claims.

This paper is the first attempt to provide a theoretical model of prediction market volatility. In doing so, we assume "ideal", i.e., unbiased and efficient prediction market and assume that the event being predicted is driven by a set of latent diffusion processes. Combination of both assumptions results in some interesting theoretical results, such as the fact that instantaneous volatility of a claim on a binary event is a particular function of the current claim price and its time to expiration. The result does not hold for events with more than two competing parties as we show that claim prices for such events will also depend on correlations between latent processes.

Volatility results we obtained bear certain similarity to the family of autoregressive conditional heteroscedasticity models started by the seminal paper of Engle (1982). The main difference is that ARCH models represent conditional volatility of returns in a stock market as a function of *previous absolute returns*, a well-known effect of volatility clustering, while our model suggests that conditional volatility in a prediction market is a function of the current *price* and the *time to expiration*. The second difference is that while ARCH

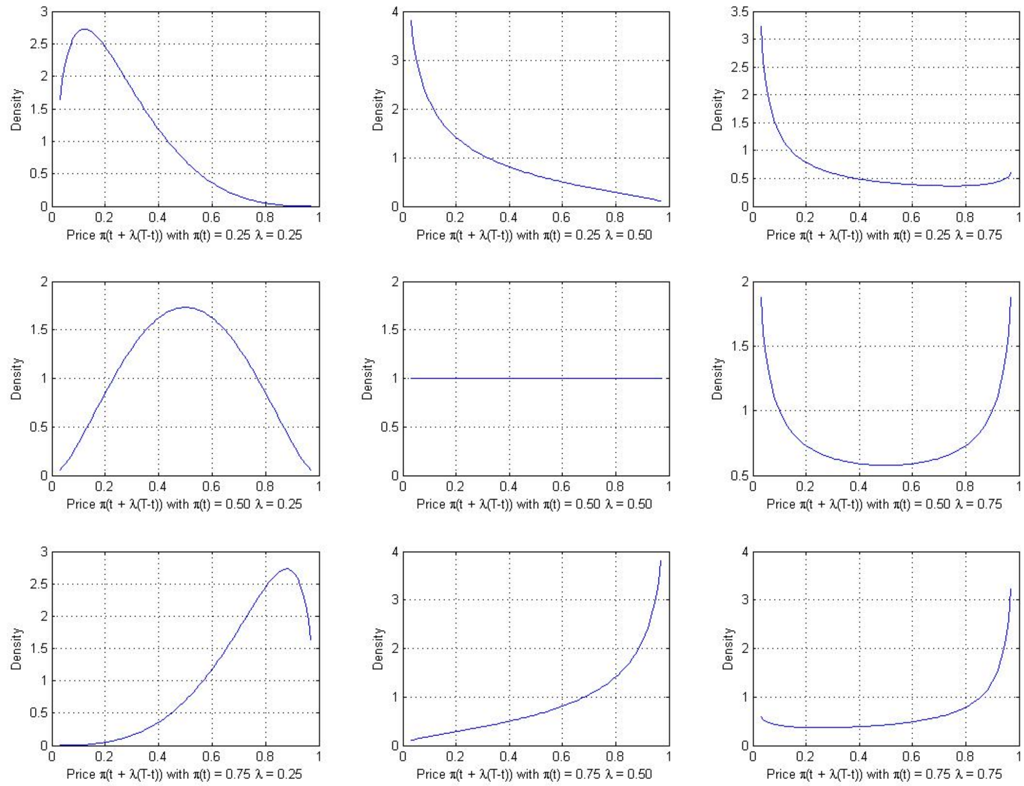

Figure 7 Risk-neutral densities for future contract prices. Rows correspond to $\pi(t) = 0.25,\ 0.5,\ 0.75$ **and columns correspond to** $\lambda = 0.25,\ 0.5,\ 0.75$**.**

models are usually experimental, our model of conditional heteroscedasticity can be derived theoretically from a stochastic model of "ability" processes.

While our theory is based on a model of "ideal" market, our experimental results for a collection of real InTrade prediction markets show that volatility patterns of real prediction markets are consistent with what our model predicts, especially if we exclude marginal (very high or very low priced) observations from the data set. Our results for a sample 51 of InTrade contracts on 2008 Presidential Elections demonstrate that our model is better in forecasting volatility than GARCH applied to historical price data, although the best performance is obtained by combining both models.

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References

Back, Kerry, C. Henry Cao, Gregory A. Willard. 2000. Imperfect competition among informed traders. *The Journal of Finance* **55**(5) 2117–2155.

Bera, Anil, Matthew Higgins. 1993. ARCH models: Properties, estimation, and testing. *Journal of Economic Surveys* **7**(4) 305–366.

Berg, Joyce, Robert Forsythe, Forrest Nelson, Thomas Rietz. 2003. Results from a dozen years of election futures markets research. Working Draft, The Handbook of Experimental Economics Results.

Berg, Joyce, Thomas Rietz. 2003. Prediction markets as decision support systems. *Information Systems Frontiers* **5**(1) 79–93.

Black, Fischer, Myron Scholes. 1973. The pricing of options and corporate liabilities. *Journal of Political Economy* **81**(3) 637–654.

Bollerslev, Tim. 1986. Generalized autoregressive conditional heteroscedasticity. *Journal of Econometrics* **31** 307–327.

Chen, Kay-Yut, Charles Plott. 2002. Information aggregation mechanisms: Concept, design and field implementation for a sales forecasting problem.

Chen, Melvin Keith, Jonathan Edwards Ingersoll, Jr., Edward H. Kaplan. 2008. Modeling a presidential prediction market. *Management Science* **54**(8) 1381–1394.

Cowgill, Bo, Justin Wolfers, Eric Zitzewitz. 2008. Using prediction markets to track information flows: Evidence from Google. Working Paper.

Engle, Robert F. 1982. Autoregressive conditional heteroscedasticity with estimates of the variance of united kingdom inflation. *Econometrica* **50**(4) 987–1008.

Engle, Robert F., Andrew J. Patton. 2001. What good is a volatility model? *Quantitative Finance* **1**(2) 237–245.

HSX. 2008. Hollywood stock exchange traders hit 80% of Oscar nominations for the 80th annual academy awards. Available at `http://www.hsx.com/about/press/080123.htm`.

Johnson, Herb. 1987. Options on the maximum or the minimum of several assets. *Journal of Financial and Quantiative Analysis* **22**(3) 277–283.

Kyle, Albert S. 1985. Continous auctions and insider trading. *Econometrica* **53**(6) 1315–1336.

Leigh, Andrew, Justin Wolfers. 2006. Competing approaches to forecasting elections: Economic models, opinion polling and predictionmarkets. *The Economic Record* **82**(258) 325–340.

Luckner, Stefan, Jan Schroder, Christian Slamka. 2008. On the forecast accuracy of sports prediction markets. *Negotiation, Auctions, and Market Engineering*, vol. 2. Springer, 227–234.

Mandelbrot, Benoit. 1963. The variation of certain speculative prices. *Journal of Business* **36**(4) 394–419.

Manski, Charles F. 2006. Interpreting the predictions of prediction markets. *Economic Letters* **91**(3) 425–429.

Merton, Robert. 1973. Theory of rational option pricing. *Bell Journal of Economics and Management Science* **4**(1) 141–183.

Oksendal, Bernt. 2005. *Stochastic Differential Equations: An Introduction with Applications*. Springer.

Pennock, David M., Sandip Debnath, Eric J. Glover, C. Lee Giles. 2002. Modeling information incorporation in markets with application to detecting and explaining events. *18th Conference on Uncertainty in Artificial Intelligence (UAI 2002)*. 405–413.

Tetlock, Paul. 2004. How efficient are information markets? Evidence from an online exchange. Working Paper.

Tetlock, Paul. 2008. Liquidity and prediction market efficiency. Working Paper.

Thaler, Richard, William Ziemba. 1988. Anomalies: Parimutuel betting markets: Racetracks and lotteries. *Journal of Economic Perspectives* **2**(2) 161–174.

Wikipedia. 2008. Multivariate normal distribution. Available at `http://en.wikipedia.org/wiki/Multivariate_normal_distribution`.

Wolfers, Justin, Eric Zitzewitz. 2004a. Prediction markets. Working Paper.

Wolfers, Justin, Eric Zitzewitz. 2004b. Using markets to evaluate policy: The case of the Iraq war. Working Paper.

Wolfers, Justin, Eric Zitzewitz. 2006. Interpreting prediction market prices as probabilities. Working Paper.

On-line companion follows

Appendix A: Extending our Model to N parties

One of the limitations of the model we studied so far is that we consider only bets with at most two competing parties. What if we have $N > 2$ parties (for example, Clinton, McCain and Obama) competing? Can the results be generalized to the case when each of the parties have a separate "abilities" process? The answer is "partially, yes". We start from a ternary ($N = 3$) example and later extend it to the case of arbitrary number of assets; the technique we use is borrowed from (Johnson 1987). To simplify notation we will also assume constant coeffients of the underlying diffusions, though it is not essential for derivation.

In the ternary case we have 3 ability processes

$$\begin{cases} dS_1 = \mu_1 dt + \sigma_1 dW_1 \\ dS_2 = \mu_2 dt + \sigma_2 dW_2 \\ dS_3 = \mu_3 dt + \sigma_3 dW_3 \end{cases},$$

where each W_i is a standard Brownian motion and $\operatorname{corr}(W_i, W_j) = \rho_{ij}$ if $i \neq j$. Consider a bet that pays \$1 at time T if $S_1 \geq \max(S_2, S_3)$. When does such event happen? Easy to show that this event will happen if and only if both of the following are true:

$$\frac{1}{\sigma_{12}\sqrt{T-t}} \left(W_1^{t,T} - W_2^{t,T}\right) > \frac{1}{\sigma_{12}\sqrt{T-t}} \left(S_2(t) - S_1(t) + (\mu_2 - \mu_1)(T-t)\right),$$

and

$$\frac{1}{\sigma_{13}\sqrt{T-t}} \left(W_1^{t,T} - W_3^{t,T}\right) > \frac{1}{\sigma_{13}\sqrt{T-t}} \left(S_3(t) - S_1(t) + (\mu_3 - \mu_1)(T-t)\right),$$

where $W_i^{T,t} = W_i(T) - W_i(t)$, $\sigma_{ij} = \sqrt{\sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j}$. It will be convenient to introduce extra variables $A_{ij}(t) \equiv S_j(t) - S_i(t) + (\mu_j - \mu_i)(T-t)$ to represent the right side of each expression. Note that because of the normalization chosen, the left side of each expression is a standard normal variable; moreover, both variables are **jointly** normal. The correlation between them is

$$\rho_{123} = \frac{\sigma_1^2 - \rho_{12}\sigma_1\sigma_2 - \rho_{13}\sigma_1\sigma_3 + \rho_{23}\sigma_2\sigma_3}{\sigma_{12}\sigma_{13}}.$$

Define ϕ_{123} as the probability density function of these two variables:

$$\phi_{123}(x,y) = \frac{1}{2\pi\sqrt{1-\rho_{123}^2}} \exp\left(-\frac{1}{2(1-\rho_{123}^2)}(x^2 + y^2 - 2\rho_{123}xy)\right).$$

The price of a bet on the win of first party is just:

$$\pi_1(t) = \int_{A_{12}(t)}^{\infty} \int_{A_{13}(t)}^{\infty} \phi_{123}(x,y)\,dx\,dy.$$

We can apply Ito's formula to the bet price. The result will have zero drift, however the quantity of interest to us is its diffusion term which is

$$\left\{-\frac{1}{\sqrt{T-t}} \int_{A_{13}(t)}^{\infty} \phi_{123}(A_{12}(t), y)\,dy\right\} dW_{12} + \left\{-\frac{1}{\sqrt{T-t}} \int_{A_{12}(t)}^{\infty} \phi_{123}(x, A_{13}(t))\,dx\right\} dW_{13},$$

where $W_{ij} = \frac{1}{\sigma_{ij}}(\sigma_j W_j - \sigma_i W_i)$ is a standard Brownian motion. It can be further simplified if we note the following fact about two-dimensional normal distribution. Let x, y be jointly normal with correlation ρ, zero means and unit variances. Then we can write $x = \rho y + z\sqrt{1-\rho^2}$ where z is independent from y and has zero mean. So,

$$P\{x > A | y = B\} = P\left\{z > \frac{A - \rho B}{\sqrt{1-\rho^2}}\right\} = \mathcal{N}\left(\frac{\rho B - A}{\sqrt{1-\rho^2}}\right).$$

Now note that $\phi_{123}(A_{12}(t), y) = \phi(A_{12}(t))\phi(y|A_{12}(t))$, where the last multiplier represents density of y conditional on $x = A_{12}(t)$. Taking this into account, the diffusion term can be rewritten as

$$-\frac{1}{\sqrt{T-t}}\mathcal{N}\left(\frac{\rho_{123}A_{12}(t) - A_{13}(t)}{\sqrt{1-\rho_{123}^2}}\right)\phi(A_{12}(t))dW_{12} - \frac{1}{\sqrt{T-t}}\mathcal{N}\left(\frac{\rho_{123}A_{13}(t) - A_{12}(t)}{\sqrt{1-\rho_{123}^2}}\right)\phi(A_{13}(t))dW_{13},$$

It immediately follows that instantaneous volatility of the first contract is

$$\Sigma_{11}(t) = \frac{1}{\sqrt{T-t}}\left\{\left(\mathcal{N}\left(\frac{\rho_{123}A_{12}(t) - A_{13}(t)}{\sqrt{1-\rho_{123}^2}}\right)\right)^2\phi(A_{12}(t))^2 + \left(\mathcal{N}\left(\frac{\rho_{123}A_{13}(t) - A_{12}(t)}{\sqrt{1-\rho_{123}^2}}\right)\right)^2\phi(A_{13}(t))^2 + 2\rho_{123}\mathcal{N}\left(\frac{\rho_{123}A_{12}(t) - A_{13}(t)}{\sqrt{1-\rho_{123}^2}}\right)\mathcal{N}\left(\frac{\rho_{123}A_{13}(t) - A_{12}(t)}{\sqrt{1-\rho_{123}^2}}\right)\phi(A_{12}(t))\phi(A_{13}(t))\right\}^{\frac{1}{2}}.$$

Moreover, we can easily apply this technique to calculate covariances $\Sigma_{ij}(t)$ between any two claims π_i and π_j. This result, however, is not very satisfactory as variables $A_{12}(t)$ and $A_{23}(t)$ depend on unobserved (to researchers) state of the latent ability processes. We now show that the volatility is uniquely determined if we take into account prices of all three contigent claim (π_1, π_2 and π_3)[33].

THEOREM 5. (Prices define volatility for ternary claims)
Fix all ρ and σ parameters and define ϕ_{ijk} as above. For each vector of prices (π_1, π_2, π_3) such that $\pi_i > 0,\ \pi_i < 1,\ \sum \pi_i = 1$, there exists a unique solution of the following system of equations:

$$\pi_1 = \int_{A_{12}}^{\infty}\int_{A_{13}}^{\infty}\phi_{123}(x,y)dxdy,$$
$$\pi_2 = \int_{A_{23}}^{\infty}\int_{A_{21}}^{\infty}\phi_{231}(x,y)dxdy,$$
$$\pi_3 = \int_{A_{31}}^{\infty}\int_{A_{32}}^{\infty}\phi_{312}(x,y)dxdy,$$
$$\sigma_{12}A_{12} + \sigma_{23}A_{23} + \sigma_{31}A_{31} = 0. \tag{12}$$
$$A_{ij} = -A_{ji}\ \forall i,\ j \in \{1,2,3\},\ i \neq j.$$

Proof:
Uniqueness: Assume that there are two different solutions (A_{12}, A_{23}, A_{31}) and $(\hat{A}_{12}, \hat{A}_{23}, \hat{A}_{31})$. Without loss of generality we assume that $\hat{A}_{12} > A_{12}$. From the equation $\pi_1 = \int_{A_{12}}^{\infty}\int_{A_{13}}^{\infty}\phi_{123}(x,y)dxdy$ it must be that $\hat{A}_{13} < A_{13}$ and so $\hat{A}_{31} > A_{31}$ (here, and in many other places, we use the fact that $A_{ij} = -A_{ji}$). From the equation $\pi_2 = \int_{A_{23}}^{\infty}\int_{A_{21}}^{\infty}\phi_{231}(x,y)dxdy$ we conclude that $\hat{A}_{23} > A_{23}$. By summing all three inequalities with proper multipliers we get $\sigma_{12}\hat{A}_{12} + \sigma_{31}\hat{A}_{31} + \sigma_{23}\hat{A}_{23} > \sigma_{12}A_{12} + \sigma_{31}A_{31} + \sigma_{23}A_{23}$. But that can't be as both sides are equal to zero according to Equation 12. We obtained a contradiction so there can be at most one solution of this system of equations.

Algorithm 1 Algorithm to restore A-values from prices

$M_{12} \Leftarrow \mathcal{N}^{-1}(1-\pi_1)$ {Define M_{12} by $\pi_1 = \int_{M_{12}}^{\infty}\int_{-\infty}^{\infty}\phi_{123}(x,y)dxdy = 1-\mathcal{N}(M_{12})$.}

$m_{12} \Leftarrow \mathcal{N}^{-1}(\pi_2)$ {Define m_{12} by $\pi_2 = \int_{-\infty}^{\infty}\int_{-m_{12}}^{\infty}\phi_{231}(x,y)dxdy = 1-\mathcal{N}(-m_{12}) = \mathcal{N}(m_{12})$.}

Ensure: $m_{12} < M_{12}$ {Note, as $\pi_2 < 1-\pi_1$, we must have $m_{12} < M_{12}$.}

$a \Leftarrow m_{12},\ b \Leftarrow M_{12}$ {Prepare for binary search for A_{12} on interval (m_{12}, M_{12}).}

repeat

$x \Leftarrow \frac{1}{2}(a+b)$

$A_{13} : \int_{x}^{\infty}\int_{A_{13}}^{\infty}\phi_{123}(x,y)dxdy = \pi_1$ {As $x < M_{12}$ we can find a unique A_{13} such that $\int_{x}^{\infty}\int_{A_{13}}^{\infty}\phi_{123}(x,y)dxdy = \pi_1$.}

$A_{23} : \int_{A_{23}}^{\infty}\int_{-x}^{\infty}\phi_{231}(x,y)dxdy, = \pi_2$ {As $x > m_{12}$ we can find a unique A_{23} such that $\int_{A_{23}}^{\infty}\int_{-x}^{\infty}\phi_{231}(x,y)dxdy, = \pi_2$.}

$r \Leftarrow \sigma_{12}x + \sigma_{23}A_{23} + \sigma_{31}A_{31}$

if $r < 0$ **then**

$a \Leftarrow x$

else

$b \Leftarrow x$

end if

until $|r| < \varepsilon$

Existence:

The proof of the uniqueness part suggests the Algorithm 1 to find the solution. Note that we don't need to satisfy the equation $\pi_3 = \int_{A_{31}}^{\infty}\int_{A_{32}}^{\infty}\phi_{312}(x,y)dxdy$, because $\sum \pi_i = 1$ it will be satisfied automatically as soon as all other equations are satisfied. The proposed algorithm proves existence of the solution as we can easily check that $r(x)$ is a continous and strictly increasing function of x (taking into account dependency of A_{13} and A_{23} on x) and $m_{12} < M_{12}$ and $\lim_{x\to M_{12}-0} r(x) = +\infty$ (note that for M_{12} we have $A_{13} = -\infty$) and $\lim_{x\to m_{12}+0} r(x) = -\infty$ (note that for m_{12} we have $A_{23} = -\infty$). This completes the proof of the Theorem 2. Q.E.D.

Moreover, we obtained the following simple corollary:

COROLLARY 3. *Conditional on prices π_i, A_{ij} values (and therefore $\Sigma(t)\sqrt{T-t}$) don't depend on time until*

[33] Obviously, two of them are sufficient as all three will always sum to 1.

expiration of the claim and drifts of the latent ability processes. Moreover, they will not change if we scale all volatilities $(\sigma_1,\ \sigma_2,\ \sigma_3)$ *by the same multiplier.*

Example

Assume that current claim prices in the Presidential elections market for three Presidential candidates are $\pi_1 = 0.2,\ \pi_2 = 0.3,\ \pi_3 = 0.5$. Also assume that all three candidates are equally volatile ($\sigma_i \equiv 1$), there is a high positive correlation between abilities of the first two candidates ($\rho_{12} = 0.5$)[34] and low negative correlations with the third candidate ($\rho_{13} = \rho_{23} = -0.1$).

Straightforward to calculate that $\sigma_{12} = 1,\ \sigma_{13} = \sigma_{23} = \sqrt{2.2}$ and $\rho_{123} = \rho_{231} \approx 0.337,\ \rho_{321} \approx 0.772$. By applying our binary search algorithm we obtained the following A values: $A_{12} \approx 0.225,\ A_{13} \approx 0.350,\ A_{23} \approx 0.198$. Finally, the instantaneous volatilities are $\frac{0.2624}{\sqrt{T-t}}$, $\frac{0.3259}{\sqrt{T-t}}$ and $\frac{0.3904}{\sqrt{T-t}}$. These volatility levels are close to what we would expect if we considered each claim as a contract for an independent binary event. In the binary case, our estimates would have been $\frac{0.28}{\sqrt{T-t}}$, $\frac{0.3477}{\sqrt{T-t}}$ and $\frac{0.3989}{\sqrt{T-t}}$.[35] One may ask, whether ternary model always gives results "close" to the binary case. The answer is "no" and here is one of the counterexamples: $\pi_1 = 0.35,\ \pi_2 = 0.25,\ \pi_3 = 0.40,\ \rho_{12} = 0.5,\ \rho_{13} = 0.2,\ \rho_{23} = 0.35,\ \sigma_i \equiv 1$. The ternary model predicts volatility of $\frac{0.08746}{\sqrt{T-t}}$ for the second claim, while the binary model overestimates it ($\frac{0.3178}{\sqrt{T-t}}$).

The last example clearly shows that a multi-party nature of the claim can have influence on its volatility. Unfortunately, the multi-party nature makes volatility much harder to calculate. Remember, in the binary scenario all we needed to know in order to calculate the claim volatility was its current price and time to expiration: none of the parameters of the underlying ability processes mattered if we knew the price. In the ternary case we need more data. At first, we need all three claim prices π_1, π_2, π_3 (strictly speaking, at least two of them) to determine volatility of any claim. At second, we need to know *relative* volatilities of each ability process (say $\lambda_2 = \frac{\sigma_2}{\sigma_1}$ and $\lambda_3 = \frac{\sigma_3}{\sigma_1}$). At last, we need to know correlations between Brownian motions driving each ability process (values ρ_{ij}). This result may leave us wondering whether the case of $N > 3$ is even harder and makes the volatility depend on some even more obscure parameters. Fortunately, the answer is no - all we need to know are the current prices, volatilities and pairwise correlations between Brownian motions. The result is stated by the following theorem.[36]

THEOREM 6. (Contigent claim pricing in a simple prediction market with N competing parties)

Consider a complete probability space $(\Omega, \mathscr{F}, P)$ *on which we have a N-dimensional Brownian motion* $(W_1(t), ...W_N(t);\ \ \mathscr{F}_t)$; $0 \le t \le T$, *where* $\mathscr{F}_t$ *is a natural filtration for* $W_{1..N}$ *i.e.* $\mathscr{F}_t = \sigma((W_1(s), ...W_N(s));\ 0 \le s \le t)$. *Assume that each* $(W_i(t);\ \ \mathscr{F}_t)$ *is a standard Brownian motion and* $\mathrm{corr}(W_i(t),\ W_j(t)) = \rho_{ij},\ |\rho_{ij}| < 1\ (i \neq j)$. *Consider an efficient and accurate prediction market for a competitive event such that underlying ability processes* S_i *are* $\mathscr{F}_t$*-measurable and satisfy diffusion equation*

$$dS_i = \mu_i(t)(\alpha S_i + \beta)dt + \sigma_i(\alpha S_i + \beta)dW_i,$$

[34] For example, because they belong to the same party.

[35] Note that they are **not** equal to ternary estimates assuming independence of the underlying Brownian motions ($\rho_{ij} = 0$). In our example, independence assumption gives the following volatilities: $\frac{0.2678}{\sqrt{T-t}}$, $\frac{0.3333}{\sqrt{T-t}}$ and $\frac{0.3780}{\sqrt{T-t}}$.

[36] In order to save paper we present constant volatility case. Readers are welcome to extend this result to time-varying volatilities.

where $\mu_i(t)$ is a continuous function, $\sigma_i \geq 0$, $\alpha \geq 0$ and $P\{\alpha S_i + \beta > 0\} = 1$. Define a contigent claim price processes $\pi_i(t)$ as

$$\pi_i(t) = \mathbf{E}\left[\mathbb{I}\left(\forall j \neq i\ S_i(T) > S_j(T)\right) | \mathscr{F}_t\right].$$

Under conditions above $\pi_i(t)$ is an Ito's diffusion with zero drift and covariance matrix Γ given by the following expression

$$\Gamma_{ij}(t) = \frac{1}{\sqrt{T-t}}\left\{\sum_{k\neq i}\sum_{l\neq j}\rho_{ikjl}\Phi\left(A_{i\ -k};\mu^{ik};\Omega^{ik}\right)\phi(A_{ik})\Phi\left(A_{j\ -l};\mu^{jl};\Omega^{jl}\right)\phi(A_{jl})\right\},$$

where

$$\rho_{ikjl} = \frac{\sigma_k\sigma_l\rho_{kl} - \rho_{ij}\sigma_i\sigma_j - \rho_{kl}\sigma_k\sigma_l + \rho_{jl}\sigma_j\sigma_l}{\sigma_{ik}\sigma_{jl}},$$

$$\rho_{ijk} = \rho_{ijik},$$

$$\sigma_{ij} = \sqrt{\sigma_i^2 + \sigma_j^2 - 2\rho_{ij}\sigma_i\sigma_j},$$

$A_{i\ -j}$ *stands for* A_i *without element* $A_{i\ j}$

A_i *stands for* $A_{i\ 1}, A_{i\ 2}, \ldots A_{i\ i-1}, A_{i\ i+1}, \ldots A_{i\ N}$,

$\Phi\left(A_{i\ -j};\mu^{ij};\Omega^{ij}\right)$ *stands for cumulative distribution function of multivariate normal with mean μ^{ij} and covariance matrix Ω^{ij},*

$$\mu^{ij} = A_{ij}\Omega^i_{j\ -j},$$

$$\Omega^{ij} = \Omega^i_{-j\ -j} - \Omega^i_{j\ -j}\left(\Omega^i_{j\ -j}\right)^T,$$

$\Omega^i_{j\ -j}$ *stands for matrix Ω^i where we removed j-th row and all columns except for j-th column (so the result is a single column),*

$\Omega^i_{-j\ -j}$ *stands for matrix Ω^i where we removed j-th row and j-th column (so the result is a square matrix),*

$$\Omega^i = \begin{pmatrix} 1 & \rho_{i\ 1\ 2} & \ldots & \rho_{i\ 1\ i-1} & \rho_{i\ 1\ i+1} & \ldots \\ \rho_{i\ 2\ 1} & 1 & \ldots & \rho_{i\ 2\ i-1} & \rho_{i\ 2\ i+1} & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

and values A_{ij} are chosen to solve the following system of equations:

$$\forall i\ : \pi_i = \Phi\left(A_i; \mathbf{0}; \Omega^i\right),$$

$$\forall i, j\ i \neq j\ :\ A_{ij} = -A_{ji},$$

$$\forall i, j\ i \neq j \neq k,\ i \neq k\ :\ \sigma_{ij}A_{ij} + \sigma_{jk}A_{jk} + \sigma_{ki}A_{ki} = 0.$$

Briefly, $d\pi_i(t) = \Sigma_i(t)dW$ where W is some standard Brownian motion with respect to filtration $\mathscr{F}_t$.

Proof: See Appendix.

Appendix B: Proofs of Theorems

Proof of Theorem 1

Define the function $f(S) = \int_{x_0}^{S} \frac{1}{\alpha x + \beta} dx$, where x_0 is some fixed value. Let $Y_i(t) = f(S_i(t))$ be a new stochastic process. Note that f is strictly increasing on the domain of S, so we have that $P\{Y_1(T) > Y_2(T)\} = P\{S_1(T) > S_2(T)\}$ i.e. instead of original processes we can consider transformed ones. We can apply Ito's formula to find drift and volatility of the transformed process. First calculate the derivatives:

$$\frac{\partial f}{\partial S} = \frac{1}{\alpha S + \beta},$$

$$\frac{\partial^2 f}{\partial S^2} = -\frac{\alpha}{(\alpha S + \beta)^2}.$$

Now Ito's formula says:

$$df(S_i, t) = \frac{\partial f}{\partial S} dS_i + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS_i)^2.$$

Substituting everything gives us:

$$dY_i = \left(\mu_i(t) - \frac{\alpha}{2}\sigma_i^2(t)\right) dt + \sigma_i(t) dW_i.$$

Remember that $P\{Y_1(T) > Y_2(T)\} = P\{S_1(T) > S_2(T)\}$. Now,

$$Y_i(T) = Y_1(t) + \int_t^T \left(\mu_i(u) - \frac{\alpha}{2}\sigma_i^2(u)\right) du + \int_t^T \sigma_i(u) dW_i(u)$$

So $Y_1(T) > Y_2(T)$ if and only if $\int_t^T \sigma_1(u) dW_1(u) - \int_t^T \sigma_2(u) dW_2(u) > Y_2(t) - Y_1(t) + \int_t^T \left(\mu_2(t) - \frac{\alpha}{2}\sigma_2^2(t)\right) dt - \int_t^T \left(\mu_1(t) - \frac{\alpha}{2}\sigma_1^2(t)\right) dt$. We already know that the left side is a normal variable with mean zero and variance:

$$\sigma_{t,T}^2 = \int_t^T \left(\sigma_1^2(u) + \sigma_2^2(u) - 2\rho_{12}\sigma_1(u)\sigma_2(u)\right) du,$$

so

$$\pi(t) = \mathcal{N}\left(\frac{Y_1(t) - Y_2(t) + \int_t^T (\mu_1(u) - \mu_2(u))\, du - \frac{\alpha}{2}\int_t^T (\sigma_1^2(u) - \sigma_2^2(u))\, du}{\sigma_{t,T}}\right).$$

As before we're not so much interested in the formula itself as in the diffusion equation. By applying Ito's formula to $\pi(t)$ we can easily calculate its drift and volatility; we can save time, however, by noting that drift must be zero as in the simple case (expectations of stochastic processes must have zero drift). The volatility term is just $\frac{\partial \pi}{\partial Y_1}\sigma_1(t) dW_1 + \frac{\partial \pi}{\partial Y_2}\sigma_2(t) dW_2$ which can be written as

$$\frac{\sigma(t)}{\sigma_{t,T}} \phi\left(\mathcal{N}^{-1}(\pi(t))\right) dW$$

where

$$\sigma(t) = \sqrt{\sigma_1^2(t) + \sigma_2^2(t) - 2\rho_{12}\sigma_1(t)\sigma_2(t)},$$

is instantaneous volatility (compare it with expression for $\sigma_{t,T}$) and

$$W(t) = \frac{\sigma_1(t)W_1(t) - \sigma_2(t)W_2(t)}{\sigma(t)}$$

is a standard Brownian motion. That proves the Theorem. To prove the Corollary just note that if $\sigma(u) \equiv c$ on $[t, T]$, then $\sigma_{t,T} = c\sqrt{(T-t)}$. Q.E.D.

Proof of Theorem 2

We will use the notation from the Proof of Theorem 1, in particular, definitions of $\sigma(t)$ and $\sigma_{t,T}$. Let $V(t) = \Sigma(t)\frac{\sigma_{t,T}}{\sigma(t)} = \phi\left(\mathcal{N}^{-1}(\pi(t))\right)$. This can be written as $V(t) = U(\pi(t))$, where $U(x) = \phi\left(\mathcal{N}^{-1}(x)\right)$. Note that $U'(x) = -\mathcal{N}^{-1}(x)$ and $U''(x) = -\frac{1}{U(x)}$. We are now ready to apply Ito's formula in integral form which gives us:

$$V(s) = V(t) - \int_t^s \mathcal{N}^{-1}(\pi(u))d\pi(u) - \int_t^s \frac{1}{2U(\pi(u))}(d\pi(u))^2. \tag{13}$$

Note, that $\int_t^s \mathcal{N}^{-1}(\pi(u))d\pi(u)$ is an Ito integral and its integrand is square intergrable (follows from the construction of the "ability" processes), so its expectation is zero. The second integral is

$$\int_t^s \frac{1}{2U(\pi(u))}(d\pi(u))^2 = \int_t^s \frac{1}{2U(\pi(u))}\frac{\sigma^2(u)}{\sigma^2_{u,T}}U^2(\pi(u))du = \int_t^s \frac{\sigma^2(u)}{2\sigma^2_{u,T}}V(u)du.$$

After applying conditional expectation to the Equation 13 we get

$$\mathbf{E}\left[V(s)\,|\mathscr{F}_t\right] = V(t) - \frac{1}{2}\int_t^s \frac{\sigma^2(u)}{\sigma^2_{u,T}}\mathbf{E}\left[V(u)\,|\mathscr{F}_t\right]du,$$

here we used Fubini's theorem to put the expectation operator under the integral. Now, if we define $f(u) = \mathbf{E}\left[V(u)\,|\mathscr{F}_t\right]$, we have an integral equation

$$f(s) = f(t) - \frac{1}{2}\int_t^s \frac{\sigma^2(u)}{\sigma^2_{u,T}}f(u)du.$$

After taking the derivative of both sides with respect to s and rearranging terms we get:

$$\frac{\partial}{\partial s}(\log f(s)) = \frac{1}{2}\frac{\partial}{\partial s}\left(\log\left(\sigma^2_{s,T}\right)\right).$$

Here we use the fact that $\frac{\partial}{\partial s}\log\left(\sigma^2_{s,T}\right) = -\frac{\sigma^2(s)}{\sigma^2_{s,T}}$. Easy to see that all solutions of this equation are of the form $f(s) = C\sigma_{s,T}$. But then,

$$\mathbf{E}\left[\Sigma(s)\,|\mathscr{F}_t\right] = \frac{\sigma(s)}{\sigma_{s,T}}\mathbf{E}\left[V(s)\,|\mathscr{F}_t\right] = \frac{\sigma(s)f(s)}{\sigma_{s,T}} \equiv C\sigma(s).$$

Q.E.D.

Proof of Theorem 3

We will use the notation from the Proof of Theorem 1, in particular, definitions of $\sigma(t)$ and $\sigma_{t,T}$. First, note that

$$\mathbf{E}\left[(\pi(r) - \pi(t))^2\,|\mathscr{F}_t\right] = \mathbf{E}\left[\left(\int_t^r \Sigma(u)dW\right)^2|\mathscr{F}_t\right] = \int_t^r \mathbf{E}\left[\Sigma(u)^2\,|\mathscr{F}_t\right]du, \tag{14}$$

where the last step is obtained by applying Ito's isometry (Oksendal 2005). Now, the easiest proof is obtained by noting that the evolution process of the prediction market contract

$$d\pi(t) = 0\;dt + \frac{\sigma(t)}{\sigma_{t,T}}\phi\left(\mathcal{N}^{-1}\left(\pi(t)\right)\right)dW,$$

does not depend on parameters of the "ability" processes except for σ. As we will obtain the same result with any valid values of μ, α and β, we can as well take the simplest possible set: $\mu \equiv 0$, $\alpha \equiv 0$, $\beta \equiv 1$. With these settings $Y(t) = Y_1(t) - Y_2(t)$ is just a Brownian motion and $\pi(u) = \mathcal{N}\left(\frac{Y(u)}{\sigma_{u,T}}\right)$ and $\Sigma(u) = \frac{\sigma(u)}{\sigma_{u,T}}\phi\left(\frac{Y(u)}{\sigma_{u,T}}\right)$. By using the normality of increments of a standard Brownian motion:

$$\mathbf{E}\left[\Sigma(u)^2\,|\mathscr{F}_t\right] = \int_{-\infty}^{\infty}\frac{1}{\sigma_{t,u}}\phi\left(\frac{\delta}{\sigma_{t,u}}\right)\left[\frac{\sigma(u)}{\sigma_{u,T}}\phi\left(\frac{Y(t)+\delta}{\sigma_{u,T}}\right)\right]^2 d\delta.$$

Simple algebraic calculations give

$$\phi\left(\frac{\delta}{\sigma_{t,u}}\right)\phi^2\left(\frac{Y(t)+\delta}{\sigma_{u,T}}\right)=\phi\left(\delta\frac{\sqrt{2\sigma_{t,u}^2+\sigma_{u,T}^2}}{\sigma_{t,u}\sigma_{u,T}}+C\right)\phi^2\left(\frac{Y(t)}{\sqrt{2\sigma_{t,u}^2+\sigma_{u,T}^2}}\right).$$

Only the first term depends on δ, so we can integrate to obtain

$$\mathbf{E}\left[\Sigma(u)^2\,|\mathscr{F}_t\right]=\frac{\sigma^2(u)\phi^2\left(\frac{Y(t)}{\sqrt{2\sigma_{t,u}^2+\sigma_{u,T}^2}}\right)}{\sigma_{u,T}\sqrt{2\sigma_{t,u}^2+\sigma_{u,T}^2}}=\frac{\sigma^2(u)}{\sigma_{u,T}\sqrt{2\sigma_{t,u}^2+\sigma_{u,T}^2}}\phi^2\left(\frac{\mathscr{N}^{-1}\left(\pi(t)\right)\sigma_{t,T}}{\sqrt{2\sigma_{t,u}^2+\sigma_{u,T}^2}}\right).$$

This can be rewritten as

$$\mathbf{E}\left[\Sigma(u)^2\,|\mathscr{F}_t\right]=\frac{\sigma^2(u)/\sigma_{t,T}^2}{\sqrt{1-\lambda(u)}\sqrt{1+\lambda(u)}}\phi^2\left(\frac{\mathscr{N}^{-1}\left(\pi(t)\right)}{\sqrt{1+\lambda(u)}}\right),$$

where

$$\lambda(u)=\frac{\sigma_{t,u}^2}{\sigma_{t,T}^2}.$$

Note that $d(\lambda(u))=\frac{\sigma(u)^2}{\sigma_{t,T}^2}$. If we substitute this expression to the Equation 14 and change the variable as $u\longmapsto\lambda$ we immediately obtain the Equation 3. Q.E.D.

Proof of Theorem 4

We will use the notation from the Proof of Theorem 1. The easiest proof is obtained by noting that the evolution process of the prediction market contract

$$d\pi(t)=0\;dt+\frac{\sigma(t)}{\sigma_{t,T}}\phi\left(\mathscr{N}^{-1}\left(\pi(t)\right)\right)dW,$$

does not depend on parameters of the "ability" processes. As we will obtain the same result with any valid values of μ,α and β, we can as well take the simplest possible set: $\mu\equiv 0$, $\alpha\equiv 0$, $\beta\equiv 1$. With these settings $Y(t)=Y_1(t)-Y_2(t)$ is just a Brownian motion and $\pi(t)=\mathscr{N}\left(\frac{Y(t)}{\sigma_{t,T}}\right)$, so

$$P\{\pi(T')>K|\pi(t)\}=P\left\{\mathscr{N}\left(\frac{Y(T')}{\sigma_{T',T}}\right)>K\,\middle|\,\mathscr{N}\left(\frac{Y(t)}{\sigma_{t,T}}\right)=\pi(t)\right\}.$$

The resulting formula can be obtained by simple algebraic manipulations and the fact that $Y(T')-Y(t)$ is a normal random variable with mean zero and standard deviation $\sigma_{t,T'}$. Q.E.D.

Proof of Theorem 6

Proof is a straightforward but tedious extension of the 3-dimensional argument to higher dimensions. We'll sketch it below.

At first, as in proof of the binary case, define the function $f(S)=\int_{x_0}^{S}\frac{1}{\alpha x+\beta}dx$, where x_0 is some fixed value. Let $Y_i(t)=f(S_i(t))$ be a new stochastic process. Note that $P\left\{Y_i(T)>\max_{j\neq i}P_j(T)\right\}=P\left\{S_i(T)>\max_{j\neq i}S_j(T)\right\}$. After applying Ito's formula we get

$$dY_i=\left(\mu_i(t)-\frac{\alpha}{2}\sigma_i^2\right)dt+\sigma_i dW_i.$$

Remember that π_i is the price of a bet that pays \$1 at time T if $Y_i(T)\geq\max_{j\neq i}Y_j(T)$. This can happen if and only if $\forall j\neq i$ the following is true:

$$\frac{1}{\sigma_{ij}\sqrt{T-t}}\left(W_i^{t,T}-W_j^{t,T}\right)>\frac{1}{\sigma_{ij}\sqrt{T-t}}\left(Y_j(t)-Y_i(t)+\int_t^T(\mu_j-\mu_i)(u)du\right)\equiv -A_{ij}(t). \quad (15)$$

From this representation we can see that

$$\forall i, j \; i \neq j \; : \; A_{ij} = -A_{ji},$$

$$\forall i, j \; i \neq j \neq k, \; i \neq k \; : \; \sigma_{ij} A_{ij} + \sigma_{jk} A_{jk} + \sigma_{ki} A_{ki} = 0.$$

Moreover, the left side of the equation 15 is a standard normal variable. Moreover, all these variables are jointly normal and their covariance matrix is Ω^i. It immediately follows that $\pi_i = \Phi\left(A_i; \mathbf{0}; \Omega^i\right)$.

Ito's formula again gives us drift of π_i:

$$\sum_{j \neq i} \frac{\partial \Phi\left(A_i; \mathbf{0}; \Omega^i\right)}{\partial A_{ij}} \frac{1}{\sqrt{T-t}} dW_{ij}.$$

To complete the proof just note that $\frac{\partial \Phi\left(A_i; \mathbf{0}; \Omega^i\right)}{\partial A_{ij}}$ is the cumulative distribution function of $A_{i\ -j}$ conditional on A_{ij} multiplied by the marginal density at A_{ij} (i.e. $\phi(A_{ij})$). But conditioning a multivariate normal variable on one of its components also gives a normal variable and its mean and variance are given by μ^{ij} and Ω^{ij} (Wikipedia 2008). Q.E.D.