Optimal Mortgage Design*

Tomasz Piskorski
Stern NYU
tpiskors@stern.nyu.edu

Alexei Tchistyi
Stern NYU
atchisty@stern.nyu.edu

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Abstract

This paper studies optimal mortgage design. A borrower (a household) with limited liability needs financial support from a lender (a big financial institution) to purchase a home. We characterize the optimal allocation in a continuous time setting in which (i) the borrower's income is volatile and its realization is unobservable to the lender, (ii) the lender has a right to costly foreclose the loan and seize the house, (iii) the borrower's intertemporal consumption preferences are represented by a constant discount factor, (iv) the lender discounts cash flows using a stochastic discount factor that depends on the market interest rate. We show that the optimal allocation can be implemented using either a combination of an interest only mortgage with a home equity line of credit or an option adjustable rate mortgage. Under the optimal contracts, mortgage payments and default rates are higher when the market interest rate is high. However, borrowers benefit from low mortgage payments and low default rates when the market interest rate is low. Thus, our analysis provides theoretical evidence that these alternative mortgages, which have recently generated great controversy, can benefit both lenders and borrowers.

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1 Introduction

Recent years have seen a rapid growth in originations of more sophisticated alternative mortgage products (AMPs), such as option adjustable rate mortgages (option ARMs) and interest only mortgages. In the United States, from 2003 through 2005, the originations of AMPs grew from less than 10% of residential mortgage originations to about 30%.[1] As of the first half of 2006, 37%[2] of mortgage originations were AMPs. Option adjustable rate mortgages experienced particularly fast growth. They accounted for as little as 0.5% of all mortgages written in 2003, but their share soared to at least 12.3% through the first five months of 2006.[3] As AMPs have complemented other forms of housing loans rather than replaced them, these nontraditional mortgages account for a significant part of the recent increase in household mortgage debt in the United States, from about 60% of GDP in 2003 to above 75% of GDP in 2006.[4]

Unlike traditional fixed rate mortgages (FRMs) and adjustable rate mortgages (ARMs), AMPs let borrowers pay only the interest portion of the debt or even less than that, while the loan balance can grow above the amount borrowed initially. Often, these mortgages carry teaser rates and come with a second mortgage, taking the form of a home equity line of credit (HELOC). Interest rates on such loans can increase as interest rates in the economy move higher. As a result of their popularity and the associated increase in the U.S. household debt, AMPs have generated great controversy and criticism. Critics contend that AMPs can hurt borrowers with high interest payments in the future.[5] On the other hand, proponents say that AMPs allow both lenders and borrowers to manage their cash flows intelligently.

Surprisingly, despite of the economic significance of AMPs and the extent of the surrounding controversy, there has so far been no attempt to formally address whether these new mortgages improve benefits to borrowers and lenders relative to traditional mortgages. In this paper, we formally address this question by formulating a general problem of finding the best possible contract between a home buyer and a bank. Instead of considering a particular class of mortgages, we derive an optimal mortgage contract as a solution to a general dynamic contracting problem in a setting with as few assumptions as possible about payments between the borrower and the lender and about circumstances under which the home is repossessed. Then we examine whether features of existing mortgage contracts are consistent with the properties of the best possible contract.

Specifically, we consider a continuous-time setting in which a risk-neutral borrower with limited liability needs outside financial support from a risk-neutral lender in order to purchase a home. Home ownership generates for the borrower a public and deterministic utility stream. While the distribution of the borrower's disposable income is publicly known, its realizations are privately observable by him. There is a liquidation

[1]United States Government Accountability Office (2006).
[2]Inside Mortgage Finance (2006a).
[3]Data from LoanPerformance, an industry tracker unit of First American Real Estate Solutions (FARES).
[4]The mortgage debt data are from Flow of Funds Accounts of the United States, Federal Reserve Board, and the GDP data are from Bureau of Economic Analysis.
[5]See for example Department of the Treasury, Board of Governors of the Federal Reserve System, Federal Deposit Insurance Corporation, National Credit Union Administration (2006), or United States Government Accountability Office (2006).

technology that allows termination of the relationship and transfer of the home to the lender. This transfer of ownership leads to inefficiencies due to associated dead-weight costs. The focus on a risk neutral setup allows us to abstract from any possible insurance role of mortgages and to focus entirely on the fundamental feature of the borrowing-lending relationship with collateral, which is how to efficiently provide a borrower with incentives to repay his debt using a costly liquidation.

An important assumption of our model is that the borrower and the lender have different discount rates. The borrower's discount rate γ represents his intertemporal consumption preferences and is constant over time. On the other hand, the lender, a big financial institution, discounts cash flows using a stochastic discount rate r_t that depends on the market interest rate. To the best of our knowledge, this is the first paper that allows for a stochastic interest rate in an optimal dynamic security design setting. We further assume that r_t follows a two-state Markov process and is smaller than the borrower's discount rate. We assume that the borrower is more impatient than the lender reflecting that a borrowing-constrained household has a higher intertemporal marginal rate of substitution then a financial institution.

An allocation in this environment obligates the borrower to report his disposable income. The allocation specifies transfers between the borrower and the lender, conditional on the history of the borrower's reports and the circumstances under which the lender would foreclose the loan and seize the home. Although the borrower's reports cannot be verified, the threat of losing ownership of the home induces the borrower to pay his debt.

We characterize the optimal allocation using the borrower's continuation payoff a_t and the market interest rate r_t as state variables. Under the optimal allocation, the borrower truthfully reports his income. The home is repossessed when the borrower's continuation payoff a_t hits the borrower's reservation utility A for the first time. The borrower consumes part of his income whenever a_t reaches the upper boundary $a^1(r_t)$. When $a_t \in [A, a^1(r_t)]$, all the income of the borrower is transferred to the lender. The borrower's continuation payoff increases (decreases) when his income realization is high (low).

Interestingly, when the interest rate r_t switches from high to low, the borrower's continuation payoff jumps up. On the other hand, when the interest rate r_t switches from low to high, the borrower's continuation payoff jumps down, which can trigger immediate bankruptcy. This is optimal because the stream of borrower's payments is more valuable for the lender when the interest rate is low. As a result, the chances of home repossession are reduced by moving the borrower's continuation payoff further away from the default boundary A when the interest rate switches to low. However, the threat of repossession must be real enough in order for the borrower to share his income with the lender. As a result, the optimal allocation increases the chances of repossession when the interest rate is high in order to compensate for the weakened threat of repossession in the low state. This is done by moving the borrower's continuation payoff closer to the default boundary A when the interest rate switches to high.

After characterizing the optimal allocation in terms of the continuation payoffs of the borrower and the lender, we examine whether features of existing mortgage contracts are consistent with the properties of optimal allocation. We find that the optimal allocation can be implemented in three different ways using combinations of existing residential mortgage instruments. First, it can be implemented using an interest only mortgage with HELOC and two way balance adjustment. Second, it can be implemented using an interest only mortgage with HELOC with a preferential rate and one way balance adjustment. Third, it can be implemented using an option adjustable rate mortgage with a preferential interest rate. Therefore, our analysis provides theoretical evidence that the alternative mortgage products can be efficiently utilized to mitigate agency cost in the stochastic interest rate environment.

Under the interest only mortgage with HELOC and two way balance adjustment, the borrower owns a home, while being obligated to make interest coupon payments on the interest only mortgage and interest payments on the home equity credit line balance. The parameters of HELOC are reset every time the market interest rate changes. When the market interest rate switches from high to low, the balance on HELOC is automatically reduced by an amount proportional to the outstanding balance and the interest rate charged on HELOC balance is also reduced. On the contrary, the balance and the HELOC interest rate are automatically increased when the market interest rate switches from low to high. The borrower uses his disposable income to make the current interest rate payments on the interest only mortgage and to repay the HELOC balance. When the disposable income realization is low, the borrower can draw on the credit line to make the current debt payments, as long as he does not exceed the credit limit. The borrower is in default if he is unable to make mortgage payments without exceeding the HELOC credit limit. In this case, the lender forecloses the loan and seizes ownership of the home.

Although mortgages with HELOC and two way balance adjustment are interesting from a theoretical point of view, we do not yet observe them in practice. While we actually observe reductions of mortgage debt balance in the form of "cramdown"[6] provisions, the unusual feature of these mortgages is their automatic increase in debt balance in response to a market interest rate increase. The implementation using the interest only mortgage with HELOC with a preferential rate and one way balance adjustment addresses this issue.

The interest only mortgage with HELOC with a preferential rate and one way balance adjustment is similar to the interest only mortgage with HELOC and two way balance adjustment, except that a part of the HELOC balance is subject to a low preferential (teaser) rate and balance adjustment occurs only when the interest rate changes from high to low. This reduction in debt can be interpreted as an automatic "cramdown" provision to be applicable whenever the market interest rate switches to low. When the interest rate changes from low to high, the total amount of the HELOC debt does not change. Instead, the balance subject to the preferential rate shrinks.

The option ARM mortgage charges a low preferential interest rate on a portion of the balance. On the remaining part of the balance, a variable rate is charged which positively correlates with the market interest rate. The balance subject to the preferential rate increases when the interest rate switches from high to low

[6] "Cramdown" is a court-ordered reduction of the secured balance due on a home mortgage loan, granted to a homeowner who has filed for personal bankruptcy protection.

and decreases when the interest rate switches from low to high. Unlike the previous two implementations, here the interest rate changes do not affect the total balance on the loan.

All three optimal mortgage implementations provide financial flexibility for the borrower to cover possible low income realizations. Given the interest only mortgage with HELOC and two way (or one way) balance adjustment, the borrower can draw on HELOC up to its limit, whenever his income is not sufficient to make the coupon payment. Under the option ARM, there is no minimum payment requirement – a low payment from the borrower translates into a higher balance, as long as the balance does not exceed the negative amortization limit. Although home repossession is costly, the borrower does not need to maintain precautionary savings, because the credit commitments by the lender provide a safety net.

The parametrized examples we consider indicate substantial efficiency gains from using mortgage contracts that are contingent on the realization of the lender's interest rate, such as the optimal option ARM or the interest only mortgage with HELOC, compared to contracts that do not depend on the lender's interest rate. These examples also show that the efficiency gains are largest for households that make little or no downpayment.

Critics of alternative mortgage products point out that AMPs seem to be more profitable for lenders than traditional mortgages. They conclude that AMPs allow lenders to profiteer at the expense of homeowners. However, this paper shows that the properties of AMPs are consistent with the properties of the optimal allocation governing the relationship between the borrower and the lender, which represents a Pareto improvement over traditional mortgages. As a consequence, it is possible that both lenders and borrowers benefit from AMPs. Critics of AMPs have also raised concerns that teaser rates and low minimum payments can result in substantially higher mortgage payments and, as a consequence, higher default rates when interest rates in the economy increase. Nevertheless, this paper demonstrates that this possibility does not necessarily contradict optimality of AMPs. Under the optimal mortgage contract, mortgage payments and default rates are indeed higher when the market interest rate is high. However, borrowers benefit from low mortgage payments and low default rates when the interest rate is low.

Related Literature

This paper belongs to the growing literature on dynamic optimal security design, which is a part of the literature on dynamic optimal contracting models using recursive techniques that began with Green (1987), Spear and Srivastava (1987), Abreu, Pearce and Stacchetti (1990), Phelan and Townsend (1991), among many others. Sannikov (2006a) developed continuous-time techniques for a principal-agent problem. The two studies that are most closely related to ours are DeMarzo and Fishman (2004) and its continuous-time formulation by DeMarzo and Sannikov (2006). These papers study long-term financial contracting in a setting with privately observed cash flows, and show that the implementation of the optimal contract involves a credit line with a constant interest rate and credit limit, long-term debt, and equity. Biais et al. (2006)

study the optimal contract in a stationary version of DeMarzo and Fishman's (2004) model and show that its continuous time limit exactly matches DeMarzo and Sannikov's (2006) continuous-time characterization of the optimal contract. Tchistyi (2006) considers a setting with correlated cash flows and shows that the optimal contract can be implemented using a credit line with performance pricing. Sannikov (2006b) shows that an adverse selection problem, due to the borrower's private knowledge concerning quality of a project to be financed, implies that, in the implementation of the optimal contract, a credit line has a growing credit limit. Clementi and Hopenhayn (2006) and DeMarzo and Fishman (2006) offer theoretical analyses of optimal investment and security design in moral hazard environments.

Unlike this paper, none of the above studies considers an environment with a stochastic discount rate. We solve for the optimal allocation in the stochastic discount rate environment and find that its implementation involves a variable interest rate charged on the borrower's debt as well as balance adjustments, adjustable preferential debt or a combination of both. On the technical side, building on the martingale techniques developed for Lévy processes, we extend DeMarzo and Sannikov (2006) characterization of the optimal allocation in a continuous-time setting to a stochastic discount rate environment.

There is a sizeable real estate finance literature that addresses the design of mortgages in the presence of asymmetric information between the borrower and lender. The bulk of this literature focuses on adverse selection and how it affects the menu of mortgages being offered to borrowers with limited insurance possibilities. Chari and Jagannathan (1989) consider a model with two private types of borrowers, who differ in terms of the riskiness of their potential gains from selling the property, and show that the optimal contract to be chosen by borrowers with larger potential gains involves contractual arrangements such as points[7] and prepayment penalties together with a "due-on-sale" clause. Brueckner (1994) develops a model in which borrowers self-select into different loans, and shows that the optimal menu of mortgages will induce longer term borrowers to select loans with higher points and a lower coupon. Unlike these two papers, LeRoy (1996) considers a stochastic interest rate environment and finds that, when borrowers refinance optimally, if interest rates fall, the points/coupon choice can at best serve only to separate the least mobile borrower type from all others. Stanton and Wallace (1998) show that in the presence of transaction costs payable by borrowers on refinancing, it is possible to construct a separating equilibrium in which borrowers with differing mobility select fixed rate mortgages with different combinations of coupon rate and points. Posey and Yavas (2001) study how borrowers with different private levels of default risk would self-select between fixed rate mortgages and adjustable rate mortgages, and show the unique equilibrium may be a separating equilibrium in which the high-risk borrowers choose the adjustable rate mortgages, while low-risk borrowers select the fixed rate mortgages. Unlike these papers that focus on adverse selection, Dunn and Spatt (1985) consider a two-period moral hazard model, where future income realization of borrowers are uncertain and private, and show that the optimal mortgage would involve a due-on-sale clause. In terms of this literature, to our

[7] Points represents the amount paid either to maintain or lower the interest rate charged.

knowledge, our paper is the first study of optimal mortgage design in a dynamic moral hazard environment, and the first study that addresses the optimality of alternative mortgage products.

There is also a large literature that focuses on the choice of mortgage contracts and the risk associated with them (for example, Campbell and Cocco (2003)). Unlike our paper, this literature takes a space of contracts as exogenously given, and studies the household choice within this restricted set of contracts. Another branch of research investigates limited participation models, where housing collateral insulates households from labor income shocks. Lustig and Van Nieuwerburgh (2005) typifies this approach.

The paper is organized as follows. Section 2 presents the continuous-time setting of the model. Section 3 introduces the dynamic contracting model with a stochastic discount rate. Section 4 derives the optimal contract. Section 5 presents the implementations of the optimal contract. Section 6 discusses the approximate implementations of the optimal contract. Section 7 concludes.

2 Set-up

Time is continuous and infinite. There is one borrower and one lender (or a group of lenders). The lender (a big financial institution) is risk neutral, has unlimited capital, and values a stochastic cumulative cash flow $\{f_t\}$ as

$$E\left[\int_0^\infty e^{-R_t} df_t\right],$$

where R_t is the market interest rate at which the lender discounts cash flows that arrive at time t. We assume that

$$R_t = \int_0^t r_s ds,$$

where r is an instantaneous interest rate process, which takes values in the set $\{r_L, r_H\}$, where $0 < r_L < r_H$. We assume that r is a continuous-time process adapted to N, where $N = \{N_t, \mathcal{F}_{1,t}; 0 \leq t < \infty\}$ is a standard compound Poisson process with the intensity $\delta(N_t)$ on a probability space $(\Omega_1, \mathcal{F}_1, m_1)$, such that for $t \geq 0$:

$$r_t(N_t) = \begin{cases} r_0 & \text{if } N_t \text{ is even} \\ r_0^c & \text{if } N_t \text{ is odd} \end{cases},$$

$$\delta(N_t) = \begin{cases} \delta(r_0) & \text{if } N_t \text{ is even} \\ \delta(r_0^c) & \text{if } N_t \text{ is odd} \end{cases},$$

where $r_0 \in \{r_L, r_H\}$ is given, and $r_0^c = \{r_L, r_H\} \setminus \{r_0\}$. The above formulation implies that the interest rate process is a first-order time-invariant continuous Markov chain with an exponential distribution with the rate parameter $\delta(r_t)$ of waiting times between successive changes. That is, for any $t \geq 0$,

$$\begin{aligned} P\left[r_{t+s} = r_L \text{ for all } s \in [t, t+\Delta) \,|r_t = r_L\right] &= e^{-\delta(r_L)\Delta}, \\ P\left[r_{t+s} = r_H \text{ for all } s \in [t, t+\Delta) \,|r_t = r_H\right] &= e^{-\delta(r_H)\Delta}. \end{aligned}$$

The borrower (a household) is also risk neutral, has limited wealth, and values a stochastic cumulative consumption flow $\{C_t\}$ as

$$E\left[\int_0^\infty e^{-\gamma t} dC_t\right].$$

We assume that, for all t, $\gamma \geq r_t$. The borrower can buy a home at date $t = 0$ at price P. At any moment in time, ownership of the home generates to the borrower a public and deterministic instantaneous utility equal to θ. The borrower's initial wealth is $Y_0 \geq 0$. We assume that $P > Y_0$, so that the borrower must obtain funds from the lender to finance the purchase of a home.

A standard Brownian motion $Z = \{Z_t, \mathcal{F}_{2,t}; 0 \leq t < \infty\}$ on $(\Omega_2, \mathcal{F}_2, m_2)$ drives the borrower's disposable income process, where $\{\mathcal{F}_{2,t}; 0 \leq t < \infty\}$ is an augmented filtration generated by the Brownian motion. The borrower's disposable income up to time t, denoted by Y_t, evolves according to

$$dY_t = \mu dt + \sigma dZ_t, \tag{1}$$

where μ is the drift of the borrower's disposable income and σ is the sensitivity of the borrower's income to its Brownian motion component. The borrower's disposable income process, Y, is privately observed by him. In addition, the borrower maintains a private savings account. The private savings account balance, S, grows at the interest rate ρ_t, which is adapted to the process r, and is such that for all t, $\rho_t \leq r_t$. The borrower must maintain a non-negative balance at his account.

At any time, the relationship between the borrower and the lender can be terminated. In this case, the lender receives L, while the borrower receives his reservation value equal to A. We assume that $A \geq \frac{\theta}{\gamma}$ and that

$$r_H L + \gamma A < \theta + \mu,$$

which ensures that the termination of the ongoing relationship is inefficient.

Let $(\Omega, \mathcal{F}, m) := (\Omega_1 \times \Omega_2, \mathcal{F}_1 \times \mathcal{F}_2, m_1 \times m_2)$ be the product space of $(\Omega_1, \mathcal{F}_1, m_1)$ and $(\Omega_2, \mathcal{F}_2, m_2)$.

3 Dynamic Moral Hazard Problem

At time 0, the funds needed to purchase the home in the amount of $P - Y_0$ are transferred from the lender to the borrower. An allocation, (τ, I), specifies a termination time of the relationship, τ, and the transfers between the lender and the borrower that are based on the borrower's report of his income and the realized interest rate process. Let $\hat{Y} = \left\{\hat{Y}_t : t \geq 0\right\}$ be the borrower's report of his income, where $\hat{Y}$ is (Y, r)-measurable. At any time $0 \leq t \leq \tau$, the allocation transfers the reported amount, $\hat{Y}_t$, from the borrower to the lender, and $I_t(\hat{Y}, r)$ from the lender to the borrower. Below we formally define an allocation.

Definition 1 *An allocation, $\xi = (\tau, I)$, specifies a termination time, τ, and transfers from the lender to the borrower, $I = \{I_t : 0 \leq t \leq \tau\}$, that are based on $\hat{Y}$ and r. Formally, τ is a $(\hat{Y}, r)$-measurable stopping time, and I is a $(\hat{Y}, r)$-measurable continuous-time process, which is such that the process*

$$E\left[\int_0^\tau e^{-\gamma s} dI_s \,|\mathcal{F}_t\right]$$

is square-integrable for $0 \leq t \leq \tau$ and $\hat{Y} = Y$.

The borrower can misreport his income. Consequently, under the allocation $\xi = (\tau, I)$, up to time $t \leq \tau$, the borrower receives a total flow of income equal to

$$\underbrace{(dY_t - d\hat{Y}_t)}_{misreporting} + dI_t,$$

and his private savings account balance, S, grows according to

$$dS_t = \rho_t S_t dt + (dY_t - d\hat{Y}_t) + dI_t - dC_t, \tag{2}$$

where dC_t is the borrower's consumption at time t, which must be non-negative. We remember that, for all $t \geq 0$, $S_t \geq 0$ and $\rho_t \leq r_t$.

In response to an allocation (τ, I), the borrower chooses a feasible strategy that consists of his consumption choice and the report of his income in order to maximize his expected utility. Below we formally define the feasible strategy of the borrower.

Definition 2 *Given an allocation $\zeta = (\tau, I)$, a feasible strategy for the borrower is a pair $(C, \hat{Y})$ such that*

(i) $\hat{Y}$ is a continuous-time process adapted to (Y, r), and $(Y - \hat{Y})$ process is of bounded variation,

(ii) C is a nondecreasing continuous-time process adapted to (Y, r),

(iii) the savings process defined by (2) stays non-negative.

The borrower's strategy is incentive compatible if it maximizes his lifetime expected utility in the class of all feasible strategies given an allocation $\zeta = (\tau, I)$. As a result, we have the following definition.

Definition 3 *Given an allocation $\zeta = (\tau, I)$, the borrower's strategy $(C, \hat{Y})$ is incentive compatible if* *(i) given an allocation $\zeta = (\tau, I)$, the borrower's strategy $(C, \hat{Y})$ is feasible,*

(ii) given an allocation $\zeta = (\tau, I)$, the borrower's strategy $(C, \hat{Y})$ provides him with the highest expected utility among all feasible strategies, that is

$$E\left[\int_0^\tau e^{-\gamma t}(dC_t + \theta dt) + e^{-\gamma\tau} A\,|\mathcal{F}_0\right] \geq E_0\left[\int_0^\tau e^{-\gamma t}(dC_t' + \theta dt) + e^{-\gamma\tau} A\,|\mathcal{F}_0\right]$$

for all the borrower's feasible strategies $(C', \hat{Y}')$, given an allocation $\zeta = (\tau, I)$.

The above definition does not explicitly include the participation constraint imposing the condition that the borrower's utility from the continuation of the allocation should be at least as large as the borrower's outside option, A, which he can receive at any time by quitting. As the borrower can always under-report and steal at rate γA until a termination time, any incentive compatible strategy would yield the borrower utility of at least A.

The above definition of an incentive compatible strategy allows us to define the incentive compatible allocation as follows.

Definition 4 *An incentive compatible allocation is an allocation $\zeta = (\tau, I)$, together with the recommendation to the borrower, $(C, \hat{Y})$, where $(C, \hat{Y})$ is a borrower's incentive compatible strategy given an allocation $\zeta = (\tau, I)$.*

The allocation is optimal if it provides the borrower with his initial promised utility a_0 and maximizes the expected profit of the lender in the class of all allocations that are incentive-compatible. Below we provide a formal definition of the optimal allocation.

Definition 5 *Given the promised payoff to the borrower, a_0, an allocation $\zeta = (\tau^*, I^*)$, together with a recommendation to the borrower $(C^*, \hat{Y}^*)$ is optimal if it maximizes the lender's expected utility (expected profit):*

$$E\left[\int_0^\tau e^{-R_t}(d\hat{Y}_t - dI_t) + e^{-R_\tau \tau} L\,|\mathcal{F}_0\right]$$

in the class of all incentive-compatible allocations that satisfy the following promise keeping constraint:

$$a_0 = E\left[\int_0^\tau e^{-\gamma t}(dC_t + \theta_t dt) + e^{-\gamma\tau} A\,|\mathcal{F}_0\right].$$

In the following lemma, we show that searching for optimal allocations, we can restrict our attention to allocations in which truth telling and zero savings are incentive compatible.

Lemma 1 *There exists an optimal allocation in which the borrower chooses to tell the truth and maintains zero savings.*

Proof In the Appendix. ■

The intuition for this result is straightforward. The first part of the result is due to the direct-revelation principle. The second part follows from the fact that it is weakly inefficient for the borrower to save on his private account ($\rho_t \leq r_t$) as any such allocation can be improved by having the lender save and make direct transfers to the borrower. Therefore, we can look for an optimal allocation in which truth telling and zero savings are incentive compatible.

4 Derivation of the Optimal Allocation

In this subsection, we formulate recursively the dynamic moral hazard problem and determine the optimal allocation. First, we consider a problem in which the borrower is not allowed to save and we determine the optimal allocation[8] in this environment. We know from Lemma 1 that it is sufficient to look for optimal allocations in which the borrower reports truthfully and maintains zero savings, and so the optimal allocation of the problem with no private savings, for a given promise to the borrower, yields to the lender at least as much utility as the optimal allocation of the problem when the borrower is allowed to privately save. Finally, we show that the optimal allocation of the problem with no private savings is fully incentive compatible, even when the borrower can maintain undisclosed savings.

Methodologically, our approach is based on continuous-time techniques used by DeMarzo and Sannikov (2006). We extend their techniques to a setting with Lévy processes. Appendix A.2 derives the optimal allocation in a discrete-time version of our model.

4.1 The Optimal Allocation without Hidden Savings

Consider for a moment the dynamic moral hazard problem in which the borrower is not allowed to save. First, we will find a convenient state space for the recursive representation of this problem. For this purpose, we define the borrower's total expected utility received under the allocation $\xi = (\tau, I)$ conditional on his information at time t, from transfers and termination utility, if he tells the truth:

$$V_t = E\left[\int_0^\tau e^{-\gamma s}\left[dI_s + \theta ds\right] + e^{-\gamma\tau} A \,|\mathcal{F}_t\right].$$

Lemma 2 *The process $V = \{V_t, \mathcal{F}_t; 0 \leq t < \tau\}$ is a square-integrable $\mathcal{F}_t$-martingale.*

Proof follows directly from the definition of process V and the fact that this process is square-integrable, which is implied by Definition 1. ■

[8] That is the allocation satisfying the properties of Definition 5 and the additional constraint that $S = 0$.

Below is a convenient representation of the borrower's total expected utility received under the allocation $\xi = (\tau, I)$ conditional on his information at time t, from transfers and termination utility, if he tells the truth. Let $M = \{M_t = N_t - t\delta(N_t), \mathcal{F}_{1,t}; 0 \leq t < \infty\}$ be a compensated compound Poisson process.

Proposition 1 *There exists $\mathcal{F}_t$-predictable processes $(\beta, \psi) = \{(\beta_t, \psi_t); 0 \leq t \leq \tau\}$ such that*

$$\begin{aligned} V_t &= V_0 + \int_0^t e^{-\gamma s} \beta_s dZ_s + \int_0^t e^{-\gamma s} \psi_s dM_s = \\ & V_0 + \int_0^t e^{-\gamma s} \beta_s \underbrace{\left(\frac{dY_s - \mu ds}{\sigma} \right)}_{dZ_s} + \int_0^t e^{-\gamma s} \psi_s (dN_s - \delta(N_s) ds). \end{aligned} \tag{3}$$

Proof We note that the couple (Z, N) is a Brownian-Poisson process, and it is an independent increment process, which is a Lévy processes, on the space $(\Omega, \mathcal{F}, m)$. Then, Theorem III.4.34 in Jacod and Shiryaev (2003) gives us the above martingale representation for a square-integrable martingale adapted to $\mathcal{F}_t$ taking values in a finite dimensional space (the process V). ■

According to the martingale representation (3), the total expected utility of the borrower under the allocation $\xi = (\tau, I)$ and truth telling conditional on his information at time t equals its unconditional expectation plus two terms that represent the accumulated effect on the total utility of, respectively, the income uncertainty revealed up to time t (Brownian motion part), and the interest rate uncertainty that has been revealed up to time t (compensated compound Poisson part).

According to Proposition 1, when the borrower reports truthfully, his total expected utility under the allocation $\xi = (\tau, I)$ conditional on the termination time τ equals

$$V_\tau = V_0 + \int_0^\tau e^{-\gamma s} \beta_s \left(\frac{dY_s - \mu ds}{\sigma} \right) + \int_0^\tau e^{-\gamma s} \psi_s dM_s.$$

As I and τ depend exclusively on the borrower's report $\hat{Y}$ and the public interest rate process r, when the borrower reports $\hat{Y}$, by (3) he gets the expected utility, $a_0(\hat{Y})$, which equals

$$\begin{aligned} a_0(\hat{Y}) = E &\left[V_0 + \int_0^\tau e^{-\gamma t} \beta_t \left(\frac{d\hat{Y}_t - \mu dt}{\sigma} \right) + \int_0^\tau e^{-\gamma t} \psi_t dM_t + \underbrace{\int_0^\tau e^{-\gamma t} (dY_t - d\hat{Y}_t)}_{\text{payoff from stealing}} |\mathcal{F}_0 \right] = \\ E &\left[V_0 + \int_0^\tau e^{-\gamma t} \beta_t \left(\frac{dY_t - \mu dt}{\sigma} \right) + \int_0^\tau e^{-\gamma t} \left(1 - \frac{\beta_t}{\sigma} \right) \left(dY_t - d\hat{Y}_t \right) + \int_0^\tau e^{-\gamma t} \psi_t dM_t \, |\mathcal{F}_0 \right]. \end{aligned} \tag{4}$$

Note that because the process $(\beta, \psi) = \{(\beta_t, \psi_t); 0 \leq t \leq \tau\}$ is $\mathcal{F}_t$−predictable, as for any $t \geq 0$, $s \geq 0$, $E_0 [Z_{t+s} - Z_t \, |\mathcal{F}_0] = E_0 [M_{t+s} - M_t \, |\mathcal{F}_0] = 0$, and given that $E [V_0 \, |\mathcal{F}_0] = V_0$, we have that

$$a_0(\hat{Y}) = V_0 + E \left[\int_0^\tau e^{-\gamma t} \left(1 - \frac{\beta_t}{\sigma} \right) \left(dY_t - d\hat{Y}_t \right) |\mathcal{F}_0 \right]. \tag{5}$$

Representation (5) leads us to the following formulation of incentive compatibility.

Proposition 2 *If the borrower cannot save, truth telling is incentive compatible if and only if* $\beta_t \geq \sigma$ $(m-a.s.)$ *for all* $t \leq \tau$.

Proof Immediately follows from (5). ■

It is important to stress that in providing incentives for truth telling one can neglect an impact of reporting strategies on the magnitude of the adjustments, ψ, in the borrower's promised value that occurs when the lender's interest rate changes. It follows from (4) that, though in principle the reporting strategy of the borrower does affect the magnitude of these adjustments, from the perspective of the borrower such adjustments have zero effect on the borrower's expected utility whatever is his reporting strategy. This property considerably simplifies the formulation of incentive compatibility.

To characterize the optimal allocation recursively, we define the borrower's continuation value at time t if he tells the truth as

$$a_t = E\left[\int_t^\tau e^{-\gamma(s-t)}\left[dI_s + \theta ds\right] + e^{-\gamma(\tau-t)}A\,|\mathcal{F}_t\right].$$

Note that for $t \leq \tau$ we have that

$$V_t = \int_0^t e^{-\gamma s}(dI_s + \theta dt) + e^{-\gamma t}a_t.$$

But this, together with (3), implies the following law of motion of the borrower's continuation value:

$$da_t = \gamma a_t dt - \theta dt - dI_t + \beta_t dZ_t + \psi_t dM_t = \left(\gamma a_t - \theta - \psi_t \delta(r_t)\right)dt - dI_t + \beta_t dZ_t + \psi_t dN_t. \tag{6}$$

Here we discuss informally, using the dynamic programming approach, how to find out the most efficient way to deliver a borrower any promised utility $a \geq A$. The proof of Proposition 3 formalizes our discussion below. Let $b(a,r)$ be the highest expected utility of the lender that can be obtained from an incentive compatible allocation that provides the borrower with utility equal to a given that the current interest rate is equal to r. To simplify our discussion we assume that the function b is concave and C^2 in its first argument. Let b' and b'' denote, respectively, the first and the second derivative of b with respect to the borrower's continuation utility a.

We start by observing that transferring lump-sum dI from the lender to the borrower with promised utility a, moves an allocation to that of the borrower's promised utility of $a - dI$. The efficiency implies that

$$b(a,r) \geq b(a - dI, r) - dI, \tag{7}$$

which shows that for all $(a, r) \in [A, \infty) \times \{r_L, r_H\}$ the marginal cost of delivering the borrower his promised utility can never exceed the cost of an immediate transfer in terms of the lender's utility, that is

$$b'(a, r) \geq -1.$$

Define $a^1(r)$, $r \in \{r_L, r_H\}$, as the lowest value of a such that $b'(a, r) = -1$. Then, it is optimal to pay the borrower as follows

$$dI(a, r) = \max(a - a^1(r), 0).$$

These transfers, and the option to terminate, keep the borrower's promised value between A and $a^1(r)$. But this implies that when $a \in [A, a^1(r)]$, and when the borrower is telling the truth, his promised value evolves according to

$$da_t(r_t) = (\gamma a_t dt - \theta dt - dI_t) + \beta_t dZ_t + \psi_t \left(dN_t - \delta(r_t)dt\right). \tag{8}$$

We need to characterize the optimal choice of process (β_t, ψ_t), where $\frac{\beta_t}{\sigma}$ determines the sensitivity of the borrower's promised value with respect to his report, and ψ_t determines the adjustment of the borrower's promised value due to a change in the interest rate. Using Ito's lemma, we find that

$$\begin{aligned} db(a_t, r_t) \quad &= \quad (\gamma a_t - \theta - \psi_t \delta(r_t)) b'(a_t, r_t) dt \\ &\quad + \frac{1}{2}\beta_t^2 b''(a_t, r_t) dt + \beta_t b'(a_t, r_t) dZ_t + \left[b(a_t + \psi_t, r_t^c) - b(a_t, r_t)\right] dN_t, \end{aligned}$$

where $r_t^c = \{r_L, r_H\} \setminus \{r_t\}$. Using the above equation, we find that the lender's expected cash flows and the change in the value he assigns to the allocation are given as follows:

$$\begin{gathered} E\left[dY_t + db(a_t, r_t) \,|\mathcal{F}_t\right] = \\ \left[\mu + (\gamma a_t - \theta - \psi_t \delta(r_t)) b'(a_t, r_t) + \frac{1}{2}\beta_t^2 b''(a_t, r_t) + \delta(r_t)\left(b(a_t + \psi_t, r_t^c) - b(a_t, r_t)\right)\right] dt. \end{gathered}$$

From Proposition 2, we know that if $\beta_t \geq \sigma$ for all $t \leq \tau$ then the borrower's best response strategy is to report the truth, that is, $\hat{Y} = Y$. Because at the optimum, at any time t, the lender should earn an instantaneous total return equal to the interest rate, r_t, we have the following Bellman equation for the value function

$$r_t b(a_t, r_t) = \max_{\beta_t \geq \sigma,\ \psi_t \geq A - a_t} \left[\begin{array}{c} \mu + (\gamma a_t - \theta - \psi_t \delta(r_t)) b'(a_t, r_t) + \\ \frac{1}{2}\beta_t^2 b''(a_t, r_t) + \delta(r_t)\left(b(a_t + \psi_t, r_t^c) - b(a_t, r_t)\right) \end{array} \right]. \tag{9}$$

Given the concavity of the function $b(\cdot, r_t)$, $b''(a_t, r_t) = \frac{d^2 b(a_t, r_t)}{da_t^2} \leq 0$, setting

$$\beta_t = \sigma$$

for all $t \leq \tau$ is optimal. The concavity of the objective function in ψ_t in the RHS of the Bellman equation (9) also implies that the optimal choice of ψ_t is given as a solution to

$$b'(a_t, r_t) = b'(a_t + \psi_t, r_t^c), \tag{10}$$

provided that $\psi_t > A - a_t$, and otherwise $\psi_t = A - a_t$. Note that this implies that $\psi_t = \psi(a_t, r_t)$.

The lender's value function therefore satisfies the following differential equation

$$\begin{aligned} r_t b(a_t, r_t) &= \mu + (\gamma a_t - \theta - \psi(a_t, r_t)\delta(r_t))b'(a_t, r_t) \\ &\quad + \frac{1}{2}\sigma^2 b''(a_t, r_t) + \delta(r_t)\left(b(a_t + \psi(a_t, r_t), r_t^c) - b(a_t, r_t)\right) \end{aligned} \tag{11}$$

with $b(a_t, r_t) = b(a^1(r_t), r_t) - (a - a^1(r_t))$ for $a_t > a^1(r_t)$ and the function ψ specified above.

We need some boundary conditions to pin down a solution to this equation and the boundaries $a^1(r)$, $r \in \{r_L, r_H\}$. The first boundary condition arises because the relationship must be terminated to hold the borrower's value to A, so $b(A, r_t) = L$. The second boundary condition comes from the fact that the first derivatives must agree at the boundary, so $b'(a^1(r_t), r_t) = -1$. The final boundary condition is the condition for the optimality of $a^1(r_t)$, which requires that the second derivatives match at the boundary. This condition implies that $b''(a^1(r_t), r_t) = 0$, or equivalently, using equation (11), that

$$\begin{aligned} r_t b(a^1(r_t), r_t) &= \mu + \theta - \gamma a^1(r_t) \\ &\quad + \delta(r_t)\left[\psi(a^1(r_t), r_t) + b(a^1(r_t) + \psi(a^1(r_t), r_t), r_t^c) - b(a^1(r_t), r_t)\right]. \end{aligned} \tag{12}$$

By definition, $a^1(r)$ is the lowest value of a such that $b'(a, r) = -1$, thus

$$\psi(a^1(r_L), r_L) = -\psi(a^1(r_H), r_H) = a^1(r_H) - a^1(r_L).$$

This, combined with (12) implies that

$$\begin{aligned} \mu + \theta &= r_L b(a^1(r_L), r_L) + \gamma a^1(r_L) - \delta(r_L)\left[b(a^1(r_H), r_H) - b(a^1(r_L), r_L) + a^1(r_H) - a^1(r_L)\right], \\ \mu + \theta &= r_H b(a^1(r_H), r_H) + \gamma a^1(r_H) - \delta(r_H)\left[b(a^1(r_L), r_L) - b(a^1(r_H), r_H) + a^1(r_L) - a^1(r_H)\right]. \end{aligned}$$

The proposition below formalizes our findings.

Proposition 3 *An optimal allocation that delivers to the borrower the value a_0 takes the following form. There exists boundaries: $a^1(r)$, $r \in \{r_L, r_H\}$, such that* *(i) If $a_0 \in [A, a^1(r_0)]$, $r_0 \in \{r_L, r_H\}$, a_t evolves as*

$$da_t(r_t) = (\gamma a_t dt - \theta dt - dI_t) + (d\hat{Y}_t - \mu dt) + \psi(a_t, r_t)(dN_t - \delta(r_t)dt), \tag{13}$$

and

- *when $a_t \in [A, a^1(r_t))$, $dI_t = 0$,*
- *when $a_t = a^1(r_t)$ the transfers dI_t cause a_t to reflect at $a^1(r_t)$.*

(ii) If $a_0 > a^1(r_0)$ an immediate transfer $a_0 - a^1(r_0)$ is made.

The relationship is terminated at time τ when a_t hits A. The lender's expected utility (expected profit) at any point is given by a concave function $b(a_t, r_t)$, which satisfies:

$$\begin{gathered} r_t b(a_t, r_t) = \\ \mu + (\gamma a_t - \theta - \psi(a_t, r_t)\delta(r_t))b'(a_t, r_t) + \frac{1}{2}\sigma^2 b''(a_t, r_t) + \delta(r_t)\left(b(a_t + \psi(a_t, r_t), r_t^c) - b(a_t, r_t)\right) \end{gathered} \tag{14}$$

when a_t is in the interval $[A, a^1(r_t)]$ and $b'(a_t, r_t) = -1$, when $a_t > a^1(r_t)$, with boundary conditions $b(A, r_t) = L$ and

$$\begin{aligned} \mu + \theta &= r_L b(a^1(r_L), r_L) + \gamma a^1(r_L) - \delta(r_t)\left[b(a^1(r_H), r_H) - b(a^1(r_L), r_L) + a^1(r_H) - a^1(r_L)\right], \\ \mu + \theta &= r_H b(a^1(r_H), r_H) + \gamma a^1(r_H) - \delta(r_t)\left[b(a^1(r_L), r_L) - b(a^1(r_H), r_H) + a^1(r_L) - a^1(r_H)\right]. \end{aligned}$$

The function ψ is defined as follows

$$\psi(a_t, r_t) = \begin{cases} \text{is a solution to } b'(a_t, r_t) = b'(a_t + \psi_t, r_t^c) \text{ for all } (a_t, r_t) \\ \text{for which the solution is such that } \psi(a_t, r_t) > A - a_t, \\ \\ \text{otherwise it is equal to } A - a_t \end{cases}, \tag{15}$$

where $r_t^c = \{r_L, r_H\} \setminus \{r_t\}$.

Proof In the Appendix. ■

The evolution of the promised value (13) implied by the optimal allocation serves three objectives - promise keeping, incentives, and efficiency. The first component of (13) accounts for promise keeping. In order for a_t to correctly describe the lender's promise to the borrower, it should grow at the borrower's discount rate, γ, less the payment, θdt, he receives from owning the home, and less the flow of payments, dI_t, from the lender.

The second term of (13) provides the borrower with incentives to report truthfully his income to the lender. Because of inefficiencies resulting from liquidation, reducing the risk in the borrower's payoff lowers the probability that the borrower's payoff reaches A, and thus lowers the probability of costly liquidation. Therefore, it is optimal to make the sensitivity of the borrower's payoff with respect to its report as small as possible provided that it does not erode his incentives to tell the truth. The minimum volatility of the borrower's promised value with respect to his report of income required for truth-telling equals 1. To understand this, note that, under this choice of volatility, underreporting income by one unit would provide the borrower with one additional unit of current utility through increased consumption, but would also reduce the borrower's promised utility by one unit, so that this volatility provides the borrower with just enough incentives to report a true realization of income. Note that when the borrower reports truthfully, the term $\left(d\hat{Y}_t - \mu dt\right)$ is driftless and equals to σdZ_t.

The last term of (13) captures the effects of changes in the lender's interest rate process on the borrower's promised utility. The optimal adjustments, ψ, in the borrower's promised utility, which are applicable when there is a change in the lender's interest rate, are such that the sensitivity of the lender's expected utility, b, with respect to the borrower's promised utility, a, is equalized just before and after an adjustment is made.[9] This sensitivity represents an instantaneous marginal cost of delivering the borrower his promised payoff in terms of the lender's utility, and so the efficiency calls for equalizing this cost across the states. We note that these adjustments imply the compensating trend in the borrower's promised payoff, $-\delta(r_t)\psi(a_t, r_t)dt$, which exactly offsets the expected effect these adjustments have on the borrower's expected utility.

Below we prove a useful lemma that characterizes the behavior of the optimal allocation when the borrower's promised payoff is close to liquidation and there is an interest rate change.

Lemma 3 *At the optimal allocation, there exists $\bar{a} \in (A, a^1(r_L)]$ such that*

- $\psi(A, r_H) = \bar{a} - A$,
- $\psi(a, r_L) = A - a$ *for* $a \in [A, \bar{a}]$.

Proof From the definition of function b and the fact that $r_L < r_H$ it follows that, for any $a > A$, $b(a, r_L) > b(a, r_H)$. This, together with $b(A, r_L) = b(A, r_H) = L$, implies that $b'(A_+, r_L) > b'(A_+, r_H)$. Let $\bar{a}$ be the smallest $a > A$ such that $b'(a, r_L) = b'(A_+, r_H)$. The existence of such $\bar{a}$ follows from the fact that, for any $a \in [A, a^1(r_t)]$, $b'(a, r_t) \geq -1$ and $b'(a^1(r_t), r_t) = -1$. This combined with (15) yields us the alleged properties of function ψ. ∎

Corollary 1 *Lemma 3 implies that under the optimal allocation, whenever $a_t \in (A, \bar{a}]$, an instantaneous increase of the interest rate, r_t, triggers the termination of the relationship.*

[9] Provided that the solution to (10) is interior.

4.2 The Optimal Allocation with Hidden Savings

So far we have characterized the optimal allocation under the assumption that the borrower cannot save. Now we show that, given the optimal allocation of the problem with no hidden savings, the borrower has no incentive to save at the solution, and thus the allocation of Proposition 3 is also optimal in the environment where the borrower can privately save.

Proposition 4 *Suppose that the process a_t is bounded above and solves*

$$da_t = \gamma a_t dt - \theta dt - dI_t + (d\hat{Y}_t - \mu dt) + \psi_t dM_t \tag{16}$$

until stopping time $\tau = \min\{t\,|a_t = A\}$, where ψ_t is an $\mathcal{F}_t-$predictable process. Then the borrower's expected utility from any feasible strategy in response to an allocation (τ, I) is at most a_0. Moreover, payoff a_0 is attained if the borrower reports truthfully and maintains zero savings.

Proof In the Appendix. ■

The above proposition shows that allocations from a broad class, including the optimal allocation of Proposition 3, remain incentive-compatible even if the borrower is allowed to privately save.

4.3 An Example

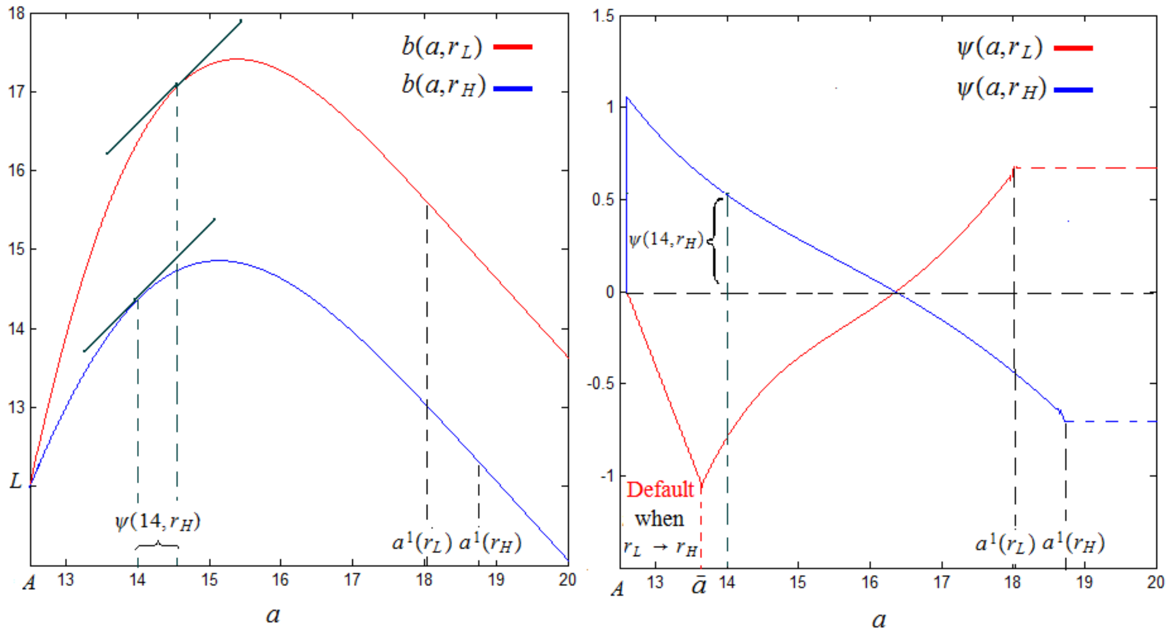
In this section we illustrate the features of the optimal allocation in a parametrized example. Table 1 shows the parameters of the model.

Table 1. Parameters of the model

Interest rate process				Borrower's discount rate	Income process		Utility flow from home	Liquidation values	
r_L	r_H	$\delta(r_L)$	$\delta(r_H)$	γ	μ	σ	θ	A	L
1.5%	6.5%	0.2	0.2	8%	1	1	1	12.5	12

The left hand-side of Figure 1 shows the lender's value function at both interest rates as a function of the borrower's continuation value. For a given promise to the borrower, the value function of the lender at the low interest rate is always above the one at the high interest rate, except at termination when they are equal, as the lender attaches more value to the proceeds from the continuation of the relationship when his discount rate is lower. As we observe, it is optimal to allow the borrower to consume his disposable income earlier when the interest rate is low, that is $a^1(r_L) < a^1(r_H)$. Intuitively, when the lender's interest rate is low, it is more costly to postpone borrower's consumption, as tension between the borrower's valuation of

Figure 1: The lender's value function and the optimal adjustments in the borrower's continuation utility.



future payoffs and that of the lender is larger. To reduce this cost, it is optimal to allow the borrower to consume his excess disposable income earlier.

The right hand-side of Figure 1 shows the optimal adjustments in the borrower's promised utility, ψ, applicable when there is a change in the market interest rate. The borrower's promised utility increases with a decrease in the interest rate and decreases with an interest rate increase, except in the area close to the reflection barriers when this relationship is reversed. The size of these adjustments is proportional to the distance of the borrower's promise from the termination cutoff of A.

The optimal adjustment of the borrower's continuation utility, ψ, is shaped by two competing forces stemming from, respectively, the costly termination of the relationship and the difference in the discount rates. The closer the borrower's continuation utility is to the termination boundary A, the bigger is the role played by the costly termination in shaping the optimal adjustment function. It is efficient to reduce the chances of costly termination when the interest rate falls, as the stream of transfers from the borrower is more valuable for the lender when the interest rate is low. A reduction in the likelihood of termination is engineered by influencing the borrower's promise in two ways. First, it is optimal to instantaneously increase the borrower's promise if the market interest rate falls, and this is even more so the more likely the relationship is to be terminated. Second, it is optimal to introduce a positive trend in the law of motion of the borrower's continuation utility, which reinforces the first adjustment over time to the extent the interest rate stays low. As a result of these adjustments, the chances of costly home repossession are reduced by moving the borrower's continuation payoff further away from the termination boundary A. However, the threat of repossession must be real enough in order for the borrower to share his income with the lender. As a result, the optimal allocation increases the chances of repossession when the interest rate is high in order to compensate for the weakened threat of repossession in the low-interest state, both by instantaneously decreasing the borrower's continuation utility and by introducing a negative trend in its law of motion.

If the borrower's continuation utility is distant from the termination boundary A, then, intuitively, the discrepancy in the discount rates begins to play the dominant role in shaping the optimal adjustment function, as the likelihood of termination is small. When the lender's interest rate switches to low, there is more tension between the borrower's valuation of future payoffs and that of the lender, and thus it is more costly to postpone the borrower's consumption, the more so the bigger is his promise. To reduce this cost, it is optimal to decrease the borrower's promise when the interest rate falls, by both an instantaneous adjustment and a negative time trend, provided that his prior promise was sufficiently large. In order to compensate for this reduction in the borrower's promise when the interest rate switches to low, his continuation utility is increased to a range of high values of the borrower's promise when the interest rate increases. It is important to observe that the adjustment of the borrower's promise in this region has second order welfare effects. This is because there is less difference between the slopes of the lender's value function at the low and at the high interest rate state, the further away the borrower's promise is from the termination boundary A. We will use this fact in Section 6, where we simply ignore the adjustments of the borrower's promise in a region close to the reflection barriers.

5 Implementations of the Optimal Allocation

So far, we have characterized the optimal allocation in terms of the transfers between the borrower and the lender and the liquidation time of their relationship. In this section, we show that the optimal allocation can be implemented using financial arrangements that resemble the ones used in the residential mortgage market. We start with the following definition.

Definition 6 *The mortgage contract is optimal if it implements the optimal allocation of Proposition 3.*

5.1 Interest Only Mortgage with Home Equity Line of Credit (HELOC) and Two Way Balance Adjustment

In this section we consider a loan contract, which is a combination of two forms of debt - an interest only mortgage and a second "piggyback"[10] mortgage that closes simultaneously with the first. Recently, there has been a noticeable increase in the use of "piggyback" mortgages, and many lenders structure a second "piggyback" loan as a home equity line of credit. These lines are revolving lines of credit like credit cards, yet

[10]Named as such in the housing finance industry because a second mortgage is "piggybacked" onto the original mortgage loan.

they are secured by the borrower's home collateral. Homeowners who pay off the line of credit can continue to draw upon it and use the funds for other purposes. A definition below formally describes a contract that consists of an interest only mortgage with a home equity line of credit.

Definition 7 *Interest only mortgage with home equity line of credit and two way balance adjustment consists of:*

- *Home equity line of credit with a time-t limit equal to C_t^L. The initial balance equals B_0. At any time t, an instantaneous interest rate on the time-t balance, B_t, is equal to $\bar{r}_t$. If the balance on the credit line exceeds its limit, default occurs.*

- *Balance adjustment, that is, an adjustment of the borrower's balance on the home equity line of credit by BA_t, applicable when there is an interest rate change.*

- *Interest only mortgage with a required coupon (interest payment) equal to x_t. If the coupon is not paid default occurs.*

- *When default happens, the lender receives the liquidation value of the home equal to L, and the borrower obtains the value of his outside option equal to A.*

The proposition below shows that the optimal allocation can be implemented with a mortgage contract belonging to the class of contracts defined above.

Proposition 5 *There exists an optimal interest only mortgage with HELOC and two way balance adjustment that has the following features:*

$$\bar{r}_t(B_t, r_t) = \gamma + \delta(r_t)\frac{\left[\psi(a^1(r_t) - B_t, r_t) - \psi(a^1(r_t), r_t)\right]}{B_t}, \tag{17}$$

$$C_t^L(r_t) = a^1(r_t) - A, \tag{18}$$

$$x_t(r_t) = \theta + \mu - \gamma a^1(r_t) + \delta(r_t)\psi(a^1(r_t), r_t), \tag{19}$$

$$BA(B_t, r_t) = -\psi(a^1(r_t) - B_t, r_t) + (a^1(r_t^c) - a^1(r_t)). \tag{20}$$

Under this mortgage contract, it is incentive compatible for the borrower to refrain from stealing. Once the borrower balance reaches zero, all excess disposable income is consumed by the borrower. With this mortgage contract, the borrower's expected payoff, a_t, is determined by the current HELOC balance, B_t, as follows:

$$a_t = A + \left[C_t^L(r_t) - B_t\right] = a^1(r_t) - B_t. \tag{21}$$

Proof In the Appendix. ■

How does the above implementation insure that the borrower refrains from stealing and consumes all excess disposable income only when his HELOC balance reaches zero? Given a time-t balance B_t on the

HELOC, the borrower can immediately consume all his available credit in the amount of $C_t^L(r_t) - B_t$ and default, which allows him to receive his outside option of A. But (21) implies that the payoff from this strategy is equal to a_t, which is the expected utility he would obtain by postponing consumption until his HELOC balance is zero.

In the implementation of Proposition 5, the balance on the home equity line of credit can be considered as a memory device that summarizes all the relevant information regarding the past cash flow realizations revealed by the borrower through repayments. The interest rate along with the required mortgage coupon payment, balance adjustment, and the credit line limit, determine the dynamics of the balance on the HELOC and the timing of default.

The adjustable features of the above mortgage contract are needed to implement the effects of the changes in the interest rate on the borrower's continuation utility. We remember that these adjustments take two forms - the instantaneous adjustment when the interest rate changes, and the compensating trend in the law of motion of the borrower's utility. In the above implementation, the balance adjustment (20) implements the instantaneous adjustments in the borrower's promised utility that are applicable when there is a change in the interest rate. The variable part of the interest rate (17) guarantees that a change in the borrower's promised utility implied by the mortgage contract includes the trend that compensates the borrower, in expectation, for the instantaneous adjustments in his promise utility that happen when the interest rate changes.

The fixed component of the variable interest rate (17) on the HELOC insures that under the optimal strategy of the borrower, given the above mortgage contract, his promised utility increases at the rate of γ, as in the optimal allocation of Proposition 3. The mortgage coupon (19) guarantees that the change in the borrower's promised utility implied by the mortgage contract reflects the reduction by the payments the borrower receives from owning the home. It also insures that an above-average income realization, and so an above-average repayment, increases the borrower's promised utility, which corresponds here to a decrease in his HELOC balance, and vice versa. Finally, the dependence of the credit line limit (18) on the current interest rate mirrors the dependence of the reflection barriers, a^1, on the interest rate in the optimal allocation.

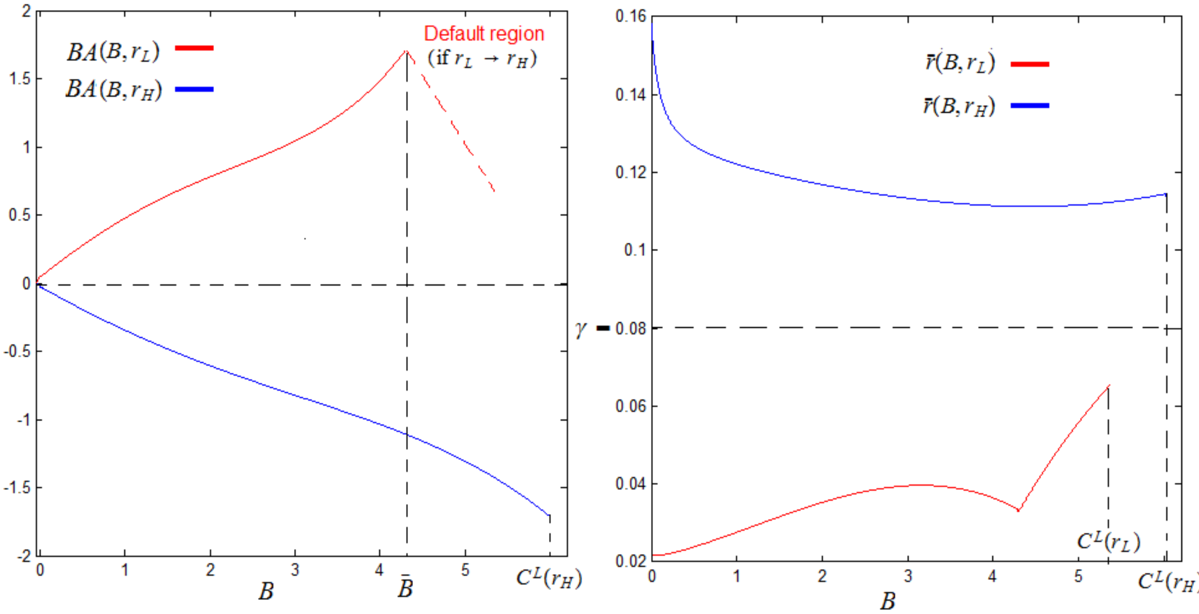
To further characterize the above mortgage contract we, will restrict our attention to the environment in which the optimal contract satisfies the following condition.[11]

Condition 1 *The function ψ implied by the optimal allocation is such that $\psi(a, r_L)$ is strictly increasing in a for $a \in [\bar{a}, a^1(a_L)]$, and so $\psi(a, r_H)$ is strictly decreasing in a for $a \in [A, a^1(a_H)]$, where $\bar{a}$ is defined as in Lemma 3.*

Proposition 5, together with Lemma 3, implies the following properties of the above mortgage contract.

[11] This condition holds in all parametrized examples we considered.

Figure 2: Optimal balance adjustment and the variable interest rate on the HELOC debt.



Corollary 2 *The optimal interest only mortgage with HELOC and two way balance adjustment has the following features:*

i) Let $\bar{B} = a^1(r_L) - \bar{a}$ *where* $\bar{a}$ *is defined in Lemma 3. Then, whenever* $B_t \in [\bar{B}, C_t^L(r_L))$*, an instantaneous change of the interest rate from* r_L *to* r_H *triggers the default of the mortgage;*

ii) $BA(B, r_t) = 0$ *for* $B = 0$*. Suppose further that the optimal function* ψ *satisfies the properties of Condition 1. Then,*

- $BA(B, r_L)$ *is positive and strictly increasing in* B *for* $B \in (0, \bar{B}]$,
- $BA(B, r_H)$ *is negative and strictly decreasing in* B *for* $B \in (0, C_t^L(r_H)]$,
- $\bar{r}_t(B', r_L) < \gamma < \bar{r}_t(B'', r_H)$*, for any* $B' \in [0, C_t^L(r_L)]$*,* $B'' \in [0, C_t^L(r_H)]$.

As the above corollary shows, under the optimal interest only mortgage with HELOC and two way balance adjustment, whenever the HELOC balance is close to the credit limit, an increase in the interest rate would cause the liquidation of the mortgage. Provided that the optimal adjustment function, ψ, satisfies the properties of Condition 1, a decrease in the interest rate causes a decrease in the borrower's HELOC balance and vice versa. The magnitude of these adjustments is proportional to the HELOC balance. The variable interest rate on the HELOC balance positively correlates with the lender's interest rate. It is optimal to reduce mortgage payments, and as a result default rates, when the market interest rate is low because, in this case, the stream of borrower's payments is more valuable for the lender. However, the threat of repossession must be real enough in order for the borrower to share his income with the lender. As a result the optimal mortgage increases the chances of repossession when the interest rate is high in order to compensate for the weakened threat of repossession in the low state by requiring higher mortgage payments and default rates. Figure 2 presents the optimal balance adjustment and the variable interest rate on the HELOC debt in the parametrized environment of Section 4.3.

Although mortgages with HELOC and two way balance adjustment are interesting from the theoretical point of view, we do not yet observe anything like that in practice. While we actually observe reductions of mortgage debt balance in the form of "cramdown" provisions, the unusual feature of these mortgages is the automatic increase in debt balance in response to a market interest rate increase. Below we discuss an implementation using the interest only mortgage with HELOC with a preferential rate and one way balance adjustment that addresses this issue.

5.2 Interest Only Mortgage with HELOC with Preferential Rate and One Way Balance Adjustment

In this section we consider a combination of an interest only mortgage with HELOC, where a part of the HELOC balance is subject to a preferential interest rate. The adjustment of the HELOC debt is only allowed when the lender's interest rate declines. The definition below provides a formal description of this class of mortgage contracts.

Definition 8 *The interest only mortgage with HELOC with preferential rate and one way balance adjustment consists of:*

- *HELOC with a time-t limit equal to $C_t^L(r_t)$. The initial balance equals B_0. At any time t, an instantaneous interest rate on a time-t balance, B_t, is equal to $\bar{r}_t^p$ on the portion of the balance below a preferential range, $p_t \geq 0$, and $\bar{r}_t$ on the portion of the balance above p_t. If the amount of debt subject to the preferential rate falls to zero, the mortgage is reset to other contract;*
- *Negative balance adjustment, that is the adjustment of the HELOC debt by BA_t^- when the lender's interest rate decreases;*
- *Interest only mortgage with a required coupon payment equal to x_t. If the coupon is not paid, default occurs;*
- *When default happens, the lender receives the liquidation value of the home equal to L, and the borrower obtains the value of his outside option, equal to A.*

The proposition below shows that the optimal allocation can be implemented with a mortgage contract belonging to the class of contracts defined above.

Proposition 6 *There exists an optimal interest only mortgage with HELOC with preferential rate and one way balance adjustment that has the following features:*

$$\bar{r}_t^p = 0, \tag{22}$$

$$\bar{r}_t(B_t - p_t, r_t) = \gamma + \delta(r_t)\frac{\left[\psi(a^1(r_t) - (B_t - p_t), r_t) - \psi(a^1(r_t), r_t)\right]}{B_t - p_t}, \tag{23}$$

$$dp_t = \begin{cases} \left[\psi(a^1(r_L) - (B_t - p_t), r_L) - \left(a^1(r_H) - a^1(r_L)\right)\right] I_{(r_{t\_}=r_L)} \ if \ B_t \geq p_t \\ 0, \ if \ B_t < p_t \end{cases}, \tag{24}$$

$$BA^-(B_t - p_t) = -\psi(a^1(r_H) - (B_t - p_t), r_H) + \left[a^1(r_L) - a^1(r_H)\right], \tag{25}$$

$$x_t(r_t) = \theta + \mu - \gamma a^1(r_t) + \delta(r_t)\psi(a^1(r_t), r_t), \tag{26}$$

$$C_t^L(p_t, r_t) = p_t + a^1(r_t) - A. \tag{27}$$

Under this mortgage contract, it is incentive compatible for the borrower to refrain from stealing. Once the borrower's balance falls to the preferential debt limit, p, all excess disposable income is consumed by the borrower. For the debt balance $B_t \geq p_t$, the borrower's expected payoff, a_t, is determined by the current HELOC balance above the preferential debt limit, as follows:

$$a_t = A + \left[C_t^L(p_t, r_t) - B_t\right] = a^1(r_t) - (B_t - p_t). \tag{28}$$

If the amount of debt subject to the preferential rate falls to zero, the mortgage is reset to a contract that implements the continuation of the optimal allocation.

Proof In the Appendix. ■

The above implementation insures that the borrower refrains from stealing and consumes all excess disposable income only when his time-t HELOC balance, B_t, falls to the debt limit, p_t, which is subject to the preferential interest rate. Intuitively, given a time-t balance B_t on the HELOC, the borrower can immediately consume all his available credit in the amount of $C_t^L(p_t, r_t) - B_t$ and default, which allows him to receive his outside option of A. But (28) implies that the payoff from this strategy is equal to a_t, which is the expected utility the borrower would obtain by postponing consumption until his HELOC balance falls to the preferential debt limit.

In the implementation of Proposition 6, the balance on the HELOC above the debt limit subject to the preferential interest rate can be considered as a memory device that summarizes all the relevant information regarding past cash flow realizations revealed by the borrower through repayments. The interest rates, along with the required mortgage coupon payment, the negative balance adjustment, the preferential debt limit, and the credit line limit determine the dynamics of the balance on the HELOC and the timing of default.

As in the previous implementation, the adjustable features of the above mortgage contract are needed to implement the effects of the changes in the interest rate on the borrower's continuation utility. In the above implementation, the balance adjustment (25) implements the instantaneous adjustments in the borrower's promised utility that are applicable when the lender's interest rate decreases. The adjustments of the preferential debt limit (24) implement the instantaneous adjustments in the borrower's promised utility that are applicable when the lender's interest rate increases. The variable component of the interest rate (23) guarantees that a change in the borrower's promised utility implied by the mortgage contract includes the trend that compensates the borrower, in expectation, for the instantaneous adjustments in his promise utility that happen when the interest rate changes.

The fixed component of the interest rate (23) on the HELOC balance above the preferential debt limit insures that, under the optimal strategy of the borrower, given the above mortgage contract, his promised utility would be increased at the rate of γ, as in the optimal allocation. The mortgage coupon (26) guarantees that a change in the borrower's promised utility implied by the mortgage contract reflects the reduction by the payments the borrower receives from owning the home. The coupon also insures that an above average income realization, and so an above average repayment, increases the borrower's promised utility, which corresponds here to a decrease in his HELOC balance, and vice versa. Finally, the dependence of the credit line limit (27) on the current interest rate mirrors the dependence of the reflection barriers, a^1, on the current interest rate in the optimal allocation.

In the proposed implementation, parameter $p_0 \geq 0$ at time zero can be chosen arbitrary. One way to initialize the mortgage is to set the market value of the mortgage equal to the book value:

$$B_0 = b\left(r_0, p_0 + a^1(r_0) - B_0\right).$$

Proposition 6, together with Lemma 3, imply the following properties of the above mortgage contract.

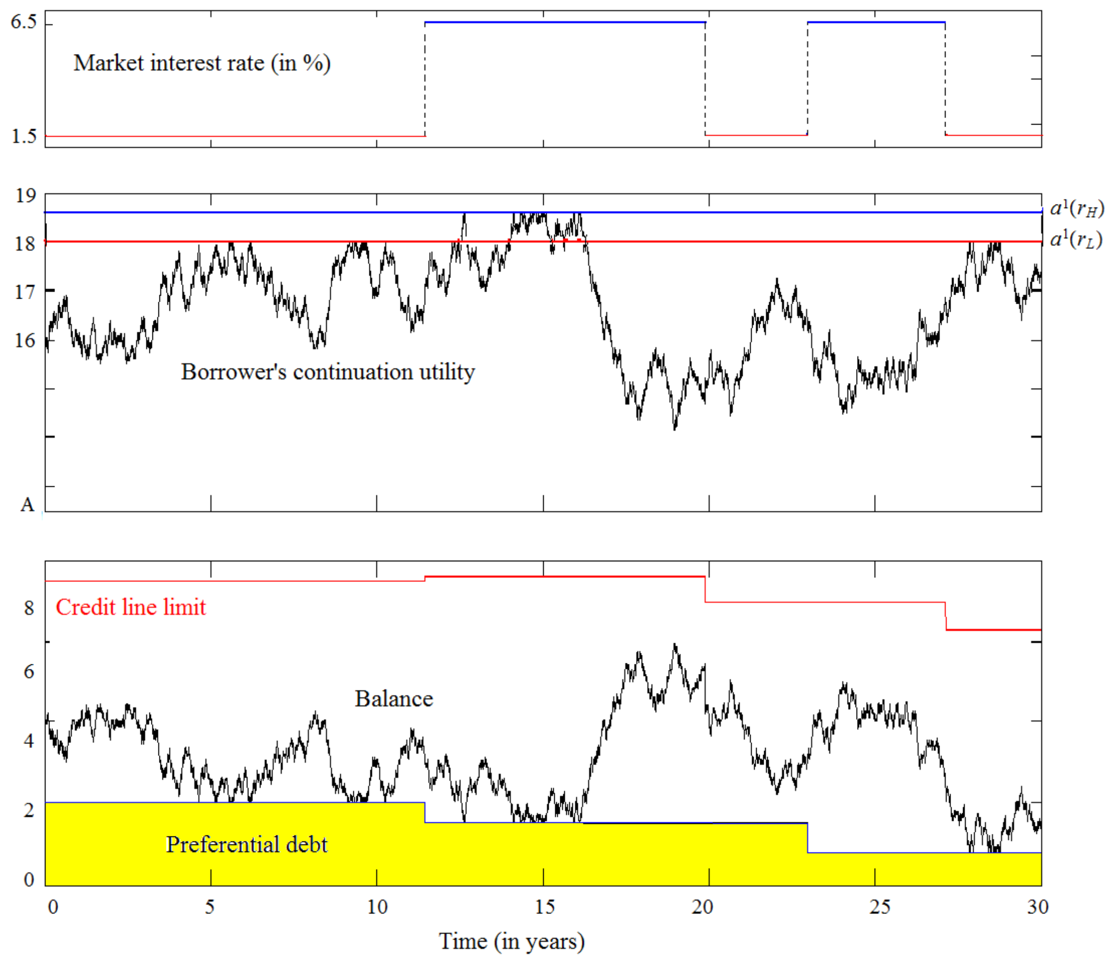
Corollary 3 *The optimal interest only mortgage with HELOC with preferential rate and one way balance adjustment has the following features:*

i) Let $\bar{B}_t = p_t + a^1(r_L) - \bar{a}$ where $\bar{a}$ is defined as in Lemma 3. Then, whenever $B_t \in [\bar{B}_t, C_t^L(p_t, r_L))$, an instantaneous increase in the lender's interest rate triggers the default of the mortgage;

ii) $BA^-(B_t - p_t) = 0$ for $B_t = p_t$. Suppose further that the optimal function ψ optimal contract satisfies the properties of Condition 1. Then,

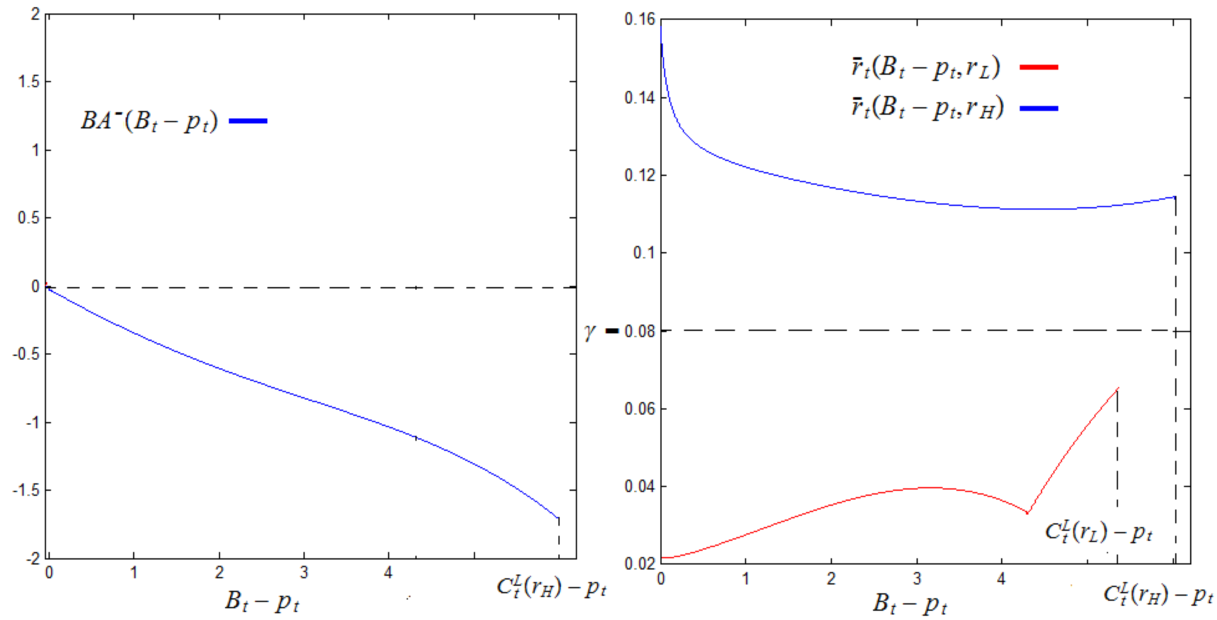
- *$BA^-(B_t - p_t)$ is negative and strictly decreasing in $(B_t - p_t)$ for $B_t \in (p_t, C_t^L(r_H)]$,*
- *$dp_t \leq 0$ for any $B_t \geq p_t$, with strict inequality whenever the interest rate, r_t, increases and $B_t > p_t$,*
- *$\bar{r}_t(B' - p'_t, r_L) < \gamma < \bar{r}_t(B'' - p''_t, r_H)$, for any $B' \in [p'_t, C_t^L(r_L)]$, $B'' \in [p''_t, C_t^L(r_H)]$.*

Figure 3: A simulated path of the optimal interest only mortgage with HELOC.



As the above corollary shows, under the optimal interest only mortgage with HELOC with preferential rate and one way balance adjustment, whenever the HELOC balance is close to the credit limit, an increase in the interest rate would cause the liquidation of the mortgage. If the optimal adjustment function, ψ, satisfies the properties of Condition 1, a decrease in the interest rate causes a decrease in the borrower's HELOC balance. The magnitude of this adjustment is proportional to the HELOC balance. This adjustment can be interpreted as offering the borrower an automatic "cramdown" provision, whenever the interest rate switches to low. An increase in the interest rate causes a drop in the amount of debt subject to the preferential interest rate. Consequently, under this contract, it is optimal to reduce the preferential treatment of the HELOC debt over time. We note that a declining preferential treatment of debt over time is a typical feature of many mortgage contracts currently offered in the housing finance market. As in the mortgage contract with variable interest rate and two way balance adjustment, the variable interest rate on the HELOC balance (23) positively correlates with the lender's interest rate.

Figure 4: The optimal negative balance adjustment and the variable interest rate on the HELOC debt.



The top part of Figure 3 presents a simulated path of the market interest rate, the middle one presents a simulated path of the borrower's continuation value under the optimal allocation, and the bottom one presents the behavior of credit line, the preferential debt range, and the HELOC balance implied by the optimal mortgage contract of Proposition 6, where the parameters of the model are set as in Section 4.3. Figure 4 presents the optimal negative balance adjustment and the variable interest rate on the HELOC debt in this parametrized example.

The implementation with an interest only mortgage and HELOC with one way balance adjustment avoids increasing the borrower's debt when the interest rate changes from low to high by decreasing instead the amount of balance subject to the preferential rate. Similarly, one could avoid the reduction of the borrower's debt (negative balance adjustment) when the interest rate decreases, by considering an implementation where the total amount of debt is left unchanged and instead, the balance subject to the preferential rate is increased. Below we discuss an implementation using the option ARM that exploits this idea.

5.3 Option Adjustable Rate Mortgage

In this section we consider an option ARM. This is an adjustable rate mortgage on which the borrower is offered an option on how large a payment to make. A part of the mortgage debt is subject to the preferential interest rate. The definition below provides a formal description of this class of mortgage contracts.

Definition 9 *An option adjustable rate mortgage with preferential interest rate consists of:*

- *Mortgage debt with a time-t negative amortization limit equal to C_t^L. If the debt exceeds the negative amortization limit, default occurs. The initial balance is equal to $B_0 \geq p_0$;*

- *At any time t, an instantaneous interest rate on a time-t debt balance, B_t, is equal to a preferential rate $\bar{r}_t^p \leq \gamma$ on a part of the balance below p_t, and $\bar{r}_t$ on a part of debt balance above p_t. If the preferential rate reaches its upper boundary γ, the mortgage is reset to other contract;*

- *When default happens, the lender receives the liquidation value of the home equal to L, and the borrower obtains the value of his outside option equal to A.*

The proposition below shows that the optimal allocation can be implemented with a mortgage contract belonging to the class of contracts defined above.

Proposition 7 *There exists an optimal option adjustable rate mortgage with preferential interest rate that has the following features*

$$\bar{r}_t(B_t - p_t, r_t) = \gamma + \delta(r_t)\frac{\left[\psi(a^1(r_t) - (B_t - p_t), r_t) - \psi(a^1(r_t), r_t)\right]}{B_t - p_t}, \; if \;\; B_t \geq p_t \tag{29}$$

$$\bar{r}_t^p(p_t, r_t) = \frac{\theta + \mu - \gamma a^1(r_t) + \delta(r_t)\psi(a^1(r_t), r_t)}{p_t} \tag{30}$$

$$C_t^L(p_t, r_t) = p_t + a^1(r_t) - A \tag{31}$$

$$dp_t = \begin{cases} \left[\psi(a^1(r_t) - (B_t - p_t), r_t) - \left(a^1(r_t^c) - a^1(r_t)\right)\right] dN_t, \; if \;\; B_t \geq p_t \\ 0, \; if \;\; B_t < p_t \end{cases}. \tag{32}$$

Under the terms of this mortgage, it is incentive compatible for the borrower to refrain from stealing and maintain balance B_t above p_t. The borrower uses all available cash flows to pay the balance when $B_t > p_t$, and consumes all excess cash flows once the balance drops to p_t. For the debt balance $B_t \geq p_t$, the borrower's expected payoff, a_t, is determined by the current balance above the preferential debt limit as follows:

$$a_t = A + \left[C_t^L(p_t, r_t) - B_t\right] = a^1(r_t) - (B_t - p_t) \tag{33}$$

If the preferential rate reaches its upper boundary γ, the mortgage is reset to a contract that implements the continuation of the optimal allocation.

Proof In the Appendix. ■

How does the above implementation insure that the borrower refrains from stealing and consumes all excess disposable income only when his time-t debt balance, B_t, falls to the debt limit, p_t, subject to the preferential interest rate given by (30)? Given a time-t balance B_t, the borrower can immediately consume all his available credit in the amount of $C_t^L(p_t, r_t) - B_t$ and default, which allows him to receive his outside option of A. But (33) implies that the payoff from this strategy is equal to a_t, that is the expected utility the borrower would obtain by postponing consumption until his debt balance falls to the preferential debt limit.

As in the implementation with interest only mortgage and HELOC with preferential interest rate, the debt balance above the debt limit subject to the preferential interest rate can be considered as a memory device that summarizes all the relevant information regarding the past cash flow realizations revealed by the borrower through repayments. The interest rates, along with the preferential debt limit, and the credit line limit, determine the dynamics of the debt balance and the timing of default.

As in the previous implementations, the adjustable features of the above mortgage contract are needed to implement the effects of the changes in the interest rate on the borrower's continuation utility. In the optimal option ARM, the adjustments of the debt subject to the preferential rate (32) implement all instantaneous adjustments in the borrower's promised utility that are applicable when the lender's interest rate changes. The variable component of the interest rate (23) guarantees that a change in the borrower's promised utility implied by the mortgage contract includes the trend that compensates the borrower, in expectation, for the instantaneous adjustments in his promise utility that happen when the interest rate changes.

The fixed component of the interest rate (29) on the debt above the preferential debt limit insures that under the optimal strategy of the borrower, given the above mortgage contract, the borrower's promised utility would be increased at the rate of γ as in the optimal allocation. The preferential interest rate insures that an above-average income realization and so an above-average repayment increases the borrower's promised utility, which corresponds here to a decrease in his debt balance, and vice versa. Finally, the dependence of the credit line limit (31) on the current interest rate mirrors the dependence of the reflection barriers, a^1, on the current interest rate in the optimal allocation.

In the proposed implementation, parameter p_0 at time zero can be chosen arbitrarily, provided interest rate $\bar{r}_0^p$ given by (30) is no greater than γ. One way to initiate the mortgage is to have the market value of the mortgage be equal to the book value:

$$B_0 = b\left(r_0, p_0 + a^1(r_0) - B_0\right).$$

Proposition 7, together with Lemma 3, implies the following properties of the above mortgage contract.

Corollary 4 *The optimal option ARM with preferential interest rate has the following features:*

i) Let $\bar{B}_t = p_t + a^1(r_L) - \bar{a}$ where $\bar{a}$ is defined as in Lemma 3. Then, whenever $B_t \in [\bar{B}_t, C_t^L(p_t, r_L))$, an instantaneous increase in the lender's interest rate triggers the default of the mortgage;

ii) Suppose further that the optimal function ψ optimal contract satisfies the properties of Condition 1, then,

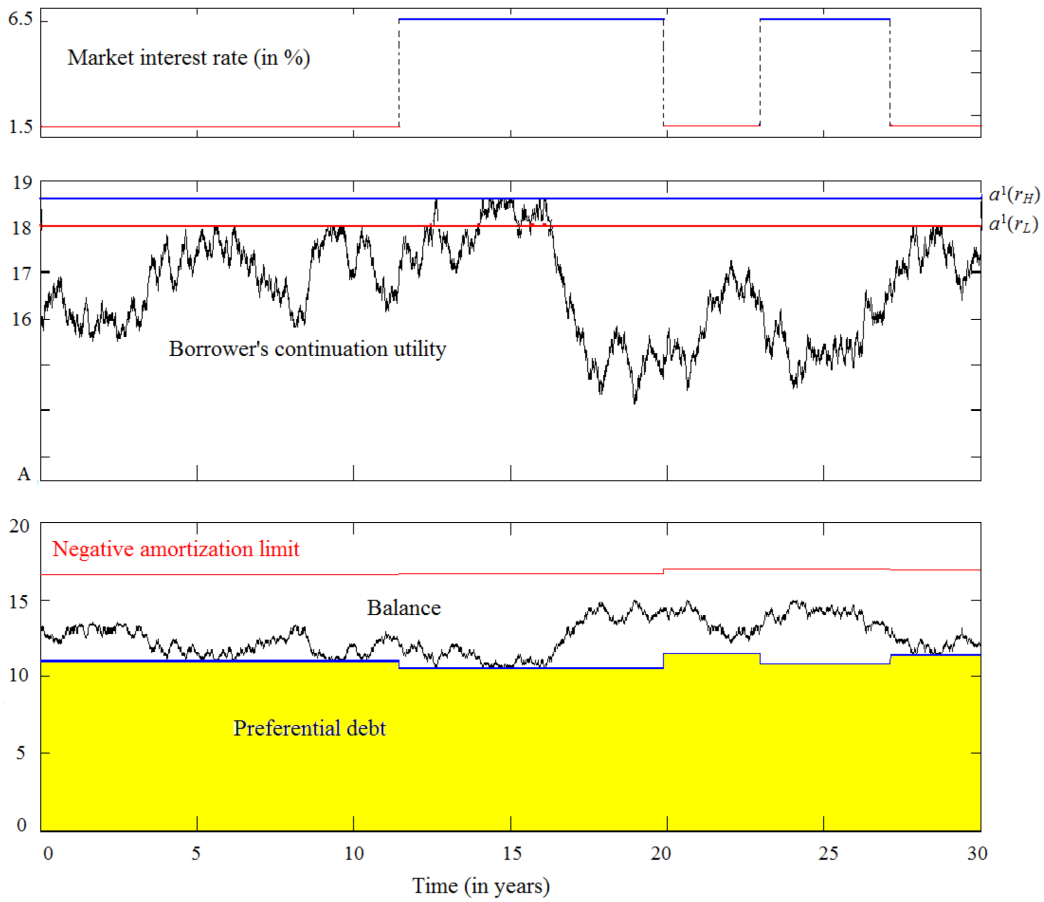
- *$dp_t < 0$, whenever interest rate, r_t, increases and $B_t > p_t$,*

– $dp_t > 0$, *whenever interest rate,* r_t*, decreases and* $B_t > p_t$,

– $\bar{r}_t(B' - p'_t, r_L) < \gamma < \bar{r}_t(B'' - p''_t, r_H)$*, for any* $B' \in [p'_t, C^L_t(r_L)]$, $B'' \in [p''_t, C^L_t(r_H)]$.

As the above corollary shows, under the optimal option ARM with preferential interest rate, whenever the debt balance is close to the negative amortization limit, an increase in the interest rate would cause the liquidation of the mortgage. If the optimal adjustment function, ψ, satisfies the properties of Condition 1, a decrease in the lender's interest rate causes an increase in the amount of debt subject to the preferential rate and vice versa. As in the previous mortgage contracts with variable interest rate, the interest rate on the debt balance, (29), positively correlates with the lender's interest rate.

Figure 5: A simulated path of the optimal option ARM.



The top part of Figure 5 presents a simulated path of the market interest rate, the middle one presenting a simulated path of the borrower's continuation value under the optimal allocation, while the bottom one presenting the behavior of credit line, the preferential debt limit, and the debt balance implied by the optimal option ARM, where the parameters of the model are set as in Section 4.3. The variable interest rate on the debt above the preferential debt limit is the same as shown in the right hand side of Figure 4.

6 Approximate Implementations

In this section we consider simpler mortgage contracts that implement the optimal allocation approximately. In order to define an approximate implementation, we consider the following change in the optimal allocation of Proposition 3. Replace the optimal function ψ in Proposition 3 by any function $\hat{\psi} : [A, \infty) \times \{r_L, r_H\} \to R$, such that, $\hat{\psi}(a, r) + a \geq A$ for any $a \geq A$, and replace the reflection barriers $a^1(r)$, $r \in \{r_L, r_H\}$, by any finite $\hat{a}^1(r) \geq A$, $r \in \{r_L, r_H\}$. Under this allocation, the borrower's continuation utility process, $\hat{a}_t$, solves:

$$d\hat{a}_t = \gamma \hat{a}_t dt - \theta dt - d\hat{I}_t + (d\hat{Y}_t - \mu dt) + \hat{\psi}(\hat{a}_t, r_t) dM_t, \tag{34}$$

given initial $\hat{a}_0 = a_0$, where $\hat{I}_t = \max(0, \hat{a}_t - \hat{a}_1(r_t))$, for any $0 \leq t \leq \hat{\tau} = \inf\{t : \hat{a}_t = A\}$. Proposition 4 implies that the borrower's expected utility from any feasible strategy in response to the above allocation is, at most, a_0, and is attained if the borrower reports truthfully and maintains zero savings. The lender's expected value under this allocation and the borrower's optimal strategy is equal to

$$E\left[\int_0^{\hat{\tau}} e^{-R_t}(dY_t - d\hat{I}_t) + e^{-R_{\hat{\tau}}\hat{\tau}} L \,|\mathcal{F}_0\right], \tag{35}$$

which is by definition less than or equal to the lender's value under the optimal allocation, given a_0.

In what follows, we will focus on the following approximation to the optimal functions ψ and a^1.

Definition 10 *The approximately optimal function $\hat{\psi}$ and $\hat{a}^1$ satisfy*

- $\hat{a}^1 = \hat{a}^1(r_L) = \hat{a}^1(r_H)$ *and* $\hat{\psi}(\hat{a}^1, r_L) = \hat{\psi}(\hat{a}^1, r_H) = 0$.

  Whenever $a^1(r_L) \leq a^1(r_H)$, $\hat{a}^1 = \inf\{a \geq A : \ \psi(a, r_L) = \psi(a, r_H) = 0\}$,

- $\hat{\psi}(a, r_L) = \begin{cases} A - a & \text{for } a \in [A, \bar{a}], \text{where } \bar{a} \text{ is defined in Lemma 3,} \\ -\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right)(\hat{a}^1 - a) & \text{for } a \in [\bar{a}, \hat{a}^1], \end{cases}$

- $\hat{\psi}(a, r_H) = \left(\frac{\bar{a}-A}{\hat{a}^1-A}\right)(\hat{a}^1 - a) \quad \text{for } a \in [A, \hat{a}^1]$,

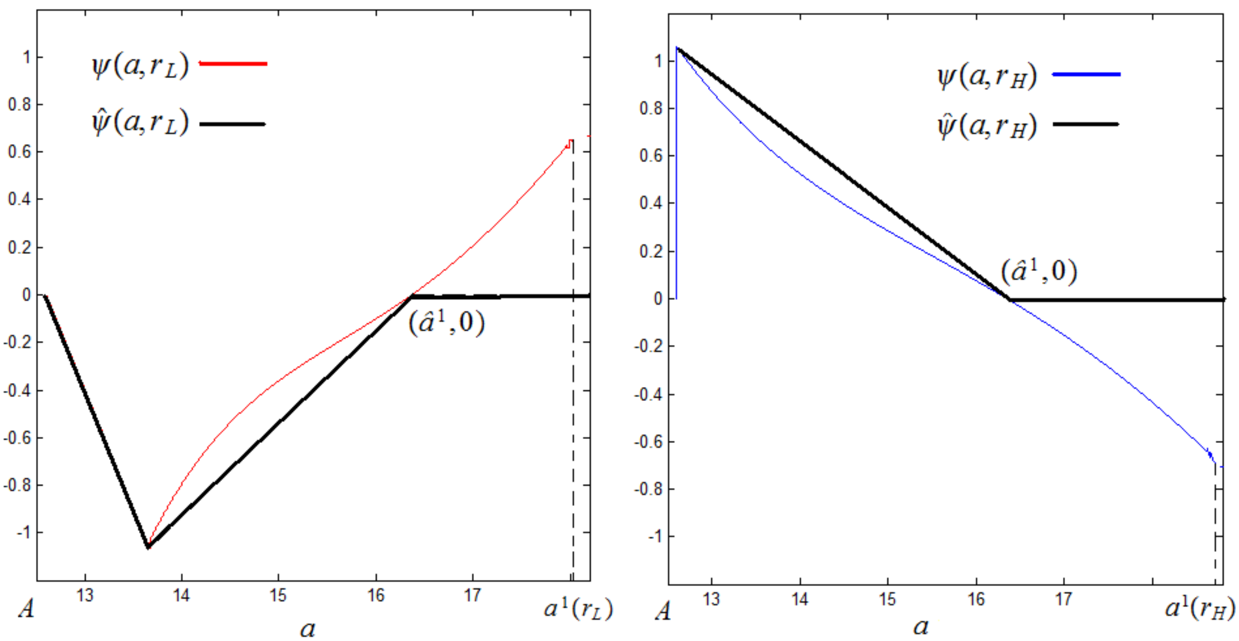
where ψ and a^1 are the functions from the optimal contract of Proposition 3.

Figure 6 presents the approximately optimal function $\hat{\psi}$ and $\hat{a}^1$, together with their optimal counterparts in the parametrized environment of Section 4.3.

Consequently, we have the following definitions of, respectively, the approximately optimal allocation and the approximately optimal mortgage contract.

Definition 11 *Given an initial promise to the borrower, $\hat{a}_0 = a_0$, the approximately optimal allocation is an allocation where the borrower's continuation utility process solves (34) and the lender's value is given by (35), where $\hat{\psi}$ and $\hat{a}$ are approximately optimal functions in the sense of Definition 10.*

Figure 6: The approximately optimal function $\hat{\psi}$ and $\hat{a}^1$.



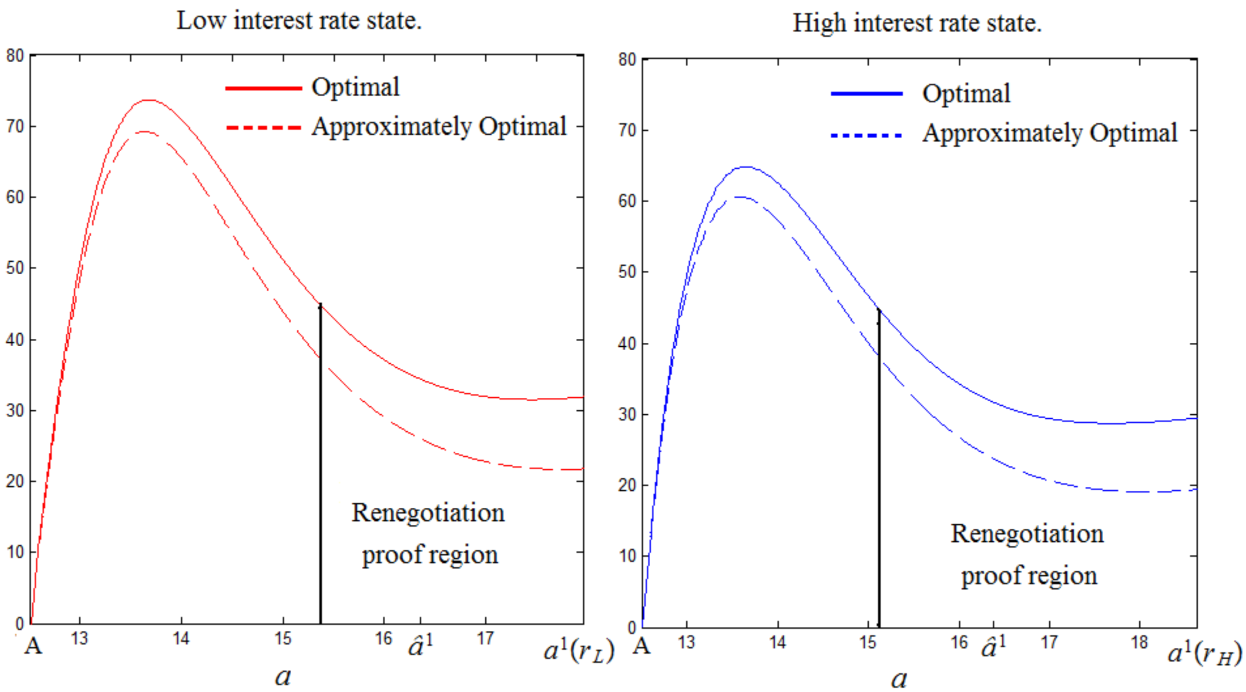
Definition 12 *The approximately optimal mortgage contract is a contract that implements the approximately optimal allocation of Definition 11.*

We note that none of Propositions 5 - 7 concerning the implementation of the optimal allocation rely on any particular properties of functions ψ and a^1 in establishing the incentive compatibility of the postulated response of the borrower to these contracts. This implies that the mortgage contracts of Propositions 5 - 7, where the function ψ is replaced by $\hat{\psi}$ and the reflation barriers $a^1(r)$ are replaced by $\hat{a}^1(r)$, $r \in \{r_L, r_H\}$, implements an allocation where the borrower's continuation utility process solves (34) and the lender's value is given by (35).

6.1 Efficiency Gains Due to Optimal and Approximately Optimal Contracts

How close is the expected lender's value function under the approximately optimal mortgage contract to the value he would obtain under the optimal contract? What are the gains in terms of the lender's value from using the contracts that adjust the borrower's promise when the lender's interest rate changes? To shed some light on these questions, we compare the lender's value under, respectively, the approximately optimal contract and the optimal contract, with a best value achievable, for a given borrower's promise, under a simple contract, where no adjustments in the borrower's continuation value are allowed due to changes in

Figure 7: Gains in basis points of the lender's value under the optimal and the approximately optimal contract.



the lender's interest rate.[12]

Figure 7 presents the percentage improvement (in basis points) in the lender's value across the borrower's promise, which comes from switching from the best contract, where no adjustments in the borrower's continuation value are allowed due to changes in the lender's interest rate to, respectively, the approximately optimal contract, and the optimal contract. The computations are performed in the parametrized environment of Section 4.3.

As we observe from Figure 7, the value of the lender under the approximately optimal contract is close to that under the optimal contract with loss ranging from zero to just above 10 basis points of the value to the lender. Both contracts yield much better performance compared to the contract that sets $\psi = 0$. The gain can be as high as 70 basis points and, in the renegotiation proof region, the gain can be well above 40 basis points for the optimal contract and well above 30 basis points for the approximately optimal contract.

Many reasonable models of determination of initial starting point in terms of the borrower's promised utility will have a property that the borrower's promise increases with the amount of downpayment (Y_0). Figure 7 indicates that, if this is the case, the largest efficiency gains in the renegotiation proof region are

[12] This contract is found by solving, for a given promise to the borrower, the problem of maximizing the lender's expected utility subject to incentive compatibility and promise keeping constraints, and subject to an additional constraint that forbids any adjustments in the borrower's continuation value due to changes in the lender's interest rate.

to be realized on the optimal mortgages given to households that make little or no downpayment.

This comparison suggests that there are substantial efficiency gains from using mortgage contracts that are contingent on the realization of the lender's interest rate, such as the optimal option ARM or the interest only mortgage with HELOC described in Section 5, compared to the contracts that do not depend on the lender's interest rate. At the same time, the implementation of the optimal contract can be considerably simplified by using the approximately optimal contract with little loss of efficiency.[13]

6.2 Approximately Optimal Interest Only Fixed Rate Mortgage (FRM) with HELOC and Two Way Balance Adjustment

In this section, we consider an implementation of the approximately optimal contract using an interest only mortgage with HELOC and two way balance adjustment.

Proposition 8 *There exists an approximately optimal interest only mortgage with HELOC and two way balance adjustment that has the following features:*

$$x_t = x = \theta + \mu - \gamma \hat{a}^1, \tag{36}$$

$$C_t^L = C^L = \hat{a}^1 - A, \tag{37}$$

$$\bar{r}_t(B_t, r_t) = \begin{cases} \gamma - \delta(r_L)\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right) & \text{for } B_t \in [0, \hat{B}] \quad \text{and } r_t = r_L \\ \gamma - \delta(r_L)\left(\frac{\hat{a}^1-A-B_t}{B_t}\right) & \text{for } B_t \in [\hat{B}, C^L] \quad \text{and } r_t = r_L \\ \gamma + \delta(r_H)\left(\frac{\bar{a}-A}{\hat{a}^1-A}\right) & \text{for } B_t \in [0, C^L] \quad \text{and } r_t = r_H \end{cases}, \tag{38}$$

$$BA(B_t, r_t) = \begin{cases} \left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right) B_t & \text{for } B_t \in [0, \hat{B}] \quad \text{and } r_t = r_L \\ -\left(\frac{\bar{a}-A}{\hat{a}^1-A}\right) B_t & \text{for } B_t \in [0, C^L] \quad \text{and } r_t = r_H \end{cases}, \tag{39}$$

where $\hat{B} = \hat{a}^1 - \bar{a}$. Under this mortgage contract, it is incentive compatible for the borrower to refrain from stealing. Once balance reaches zero, all excess disposable income is consumed by the borrower. Whenever $B_t \in [\hat{B}, C^L)$, an instantaneous change of the interest rate from r_L to r_H triggers the default of the mortgage. Under this mortgage, the borrower's expected payoff, $\hat{a}_t$, is determined by the current HELOC balance as follows:

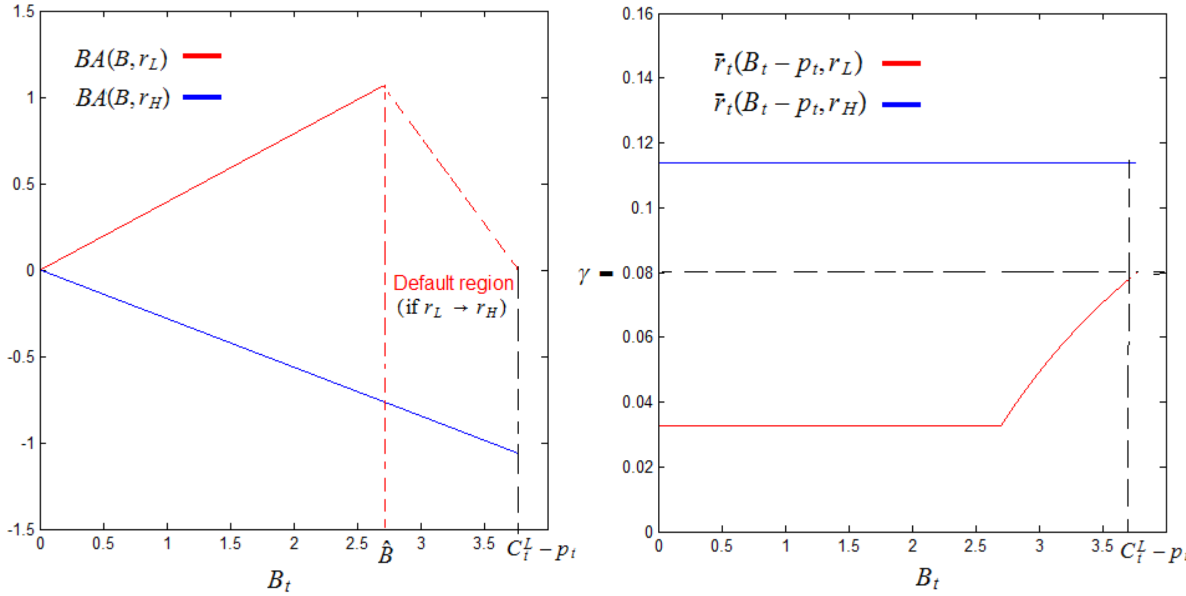
$$\hat{a}_t = A + \left[C_t^L(r_t) - B_t\right] = \hat{a}^1 - B_t. \tag{40}$$

Proof Immediately follows from Proposition 5, where the function ψ is replaced by $\hat{\psi}$ and where the reflection barriers $a^1(r)$, $r \in \{r_L, r_H\}$, are replaced by $\hat{a}^1$. ■

The intuition behind incentive compatibility of the postulated strategy of the borrower under the above mortgage contract is the same as in the case of the optimal mortgage contract of Proposition 5. The coupon

[13] This result holds across all parameterizations we considered.

Figure 8: Approximately optimal balance adjustment and the variable interest rate on the HELOC debt.



(36), the HELOC limit (37), the variable interest rate (38), and the HELOC balance adjustment (39) play the same role in implementing the approximately optimal allocation as their counterparts in Proposition 5 in implementing the optimal allocation.

As we observe from Proposition 8, the approximately optimal interest only mortgage with HELOC takes the simple form of the interest only fixed rate mortgage with constant interest coupon payment of (36). The HELOC has a constant credit limit given by (37), and a simple variable rate given by (38). It follows from Proposition 8 that this mortgage contract has the following properties.

Corollary 5 *The approximately optimal interest only FRM with HELOC and two way balance adjustment has the following features:*

i) $BA(B, r_t) = 0$ *for* $B = 0$*, and*

- $BA(B, r_L)$ *is positive and strictly increasing in* B *for* $B \in (0, \hat{B}]$,
- $BA(B, r_H)$ *is negative and strictly decreasing in* B *for* $B \in (0, C^L]$,

(ii) $\bar{r}_t(B', r_L) < \gamma < \bar{r}_t(B'', r_H)$*, for any* $B' \in [0, C^L]$, $B'' \in [0, C^L]$.

As the above corollary shows, under the approximately optimal interest only FRM with HELOC and two way balance adjustment, a decrease in the interest rate causes a decrease in the borrower's HELOC balance and vice versa. The magnitude of these adjustments is linearly proportional to the HELOC balance. The variable interest rate on the HELOC balance positively correlates with the lender's interest rate, and is independent of the borrower's debt balance, except the debt region $[\hat{B}, C^L]$, where the HELOC interest rate increases with the balance, provided that the lender's interest rate is low ($r_t = r_L$). Figure 8 presents the approximately optimal balance adjustment and the variable interest rate on the HELOC debt in the parametrized environment of Section 4.3.

6.3 Approximately Optimal Interest Only FRM with HELOC with Preferential Interest Rate and One Way Balance Adjustment

In this section, we consider an implementation of the approximately optimal allocation by an interest only mortgage with HELOC with preferential interest rate and one way balance adjustment.

Proposition 9 *There exists an approximately optimal interest only FRM with HELOC with preferential rate, and one way balance adjustment that has the following features:*

$$x_t = x = \theta + \mu - \gamma \hat{a}^1, \tag{41}$$

$$C_t^L(p_t) = p_t + \hat{a}^1 - A, \tag{42}$$

$$\bar{r}_t^p = 0, \tag{43}$$

$$\bar{r}_t(B_t - p_t, r_t) = \begin{cases} \gamma - \delta(r_L)\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right) & \text{for } B_t \in [p_t, \hat{B}_t] \quad \text{and } r_t = r_L \\ \gamma - \delta(r_L)\left(\frac{\hat{a}^1-A-B_t+p_t}{B_t-p_t}\right) & \text{for } B_t \in [\hat{B}_t, C_t^L] \quad \text{and } r_t = r_L \\ \gamma + \delta(r_H)\left(\frac{\bar{a}-A}{\hat{a}^1-A}\right) & \text{for } B_t \in [p_t, C_t^L] \quad \text{and } r_t = r_H \end{cases}, \tag{44}$$

$$dp_t = \begin{cases} -\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right)(B_t - p_t) & \text{for } B_t \in [p_t, \hat{B}_t] \quad \text{and } r_t = r_L \\ 0 & \text{for } B_t < p_t \end{cases}, \tag{45}$$

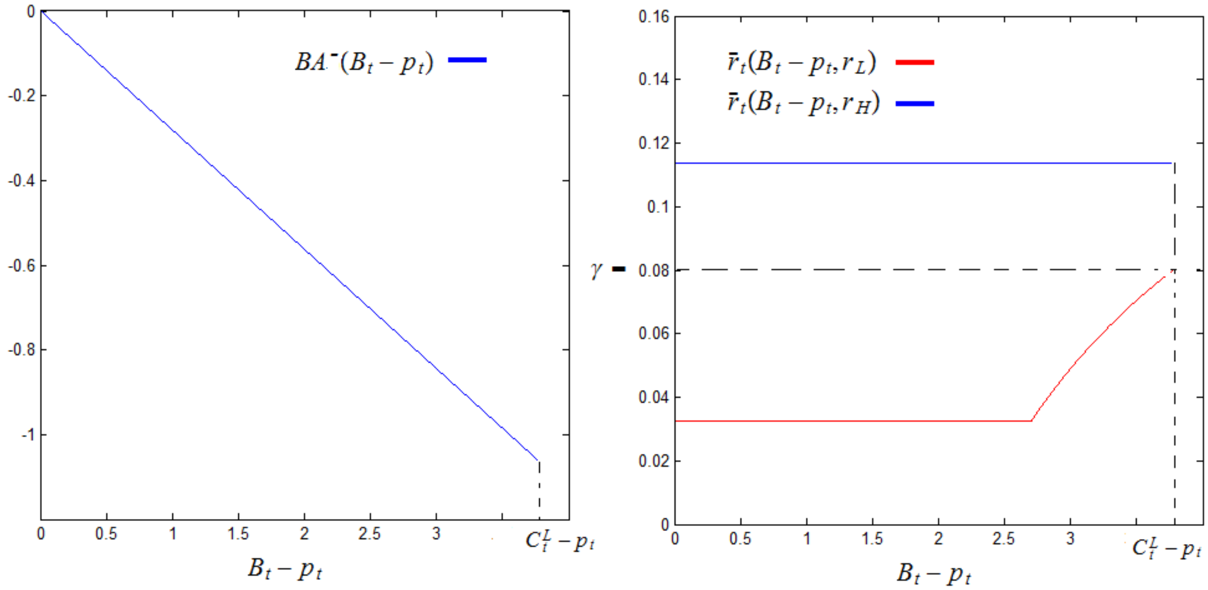
$$BA^-(B_t - p_t) = -\left(\frac{\bar{a} - A}{\hat{a}^1 - A}\right)(B_t - p_t), \tag{46}$$

where $\hat{B}_t = p_t + \hat{a}^1 - \bar{a}$. Under this mortgage contract, it is incentive compatible for the borrower to refrain from stealing. Once the borrower's balance falls to the preferential debt limit, p, all excess disposable income is consumed by the borrower. Whenever $B_t \in [\hat{B}_t, C_t^L(p_t))$, an instantaneous change of the interest rate from r_L to r_H triggers the default of the mortgage. For the balance $B_t \geq p_t$, the borrower's expected payoff, $\hat{a}_t$, is determined by the current HELOC balance above the preferential debt limit as follows:

$$a_t = A + \left[C_t^L(p_t, r_t) - B_t\right] = \hat{a}^1 - (B_t - p_t). \tag{47}$$

If the amount of debt subject to the preferential rate falls to zero, the mortgage is reset to a contract that implements the continuation of the approximately optimal allocation.

Figure 9: The approximately optimal negative balance adjustment and the interest rate on the HELOC debt.



Proof Immediately follows from Proposition 7 where the function ψ is replaced by $\hat{\psi}$ and the reflection barriers $a^1(r)$, $r \in \{r_L, r_H\}$, are replaced by $\hat{a}^1$. ■

The intuition behind incentive compatibility of the postulated strategy of the borrower under the above mortgage contract is the same as in the case of the optimal mortgage contract of Proposition 6. The coupon (41), the HELOC limit (42), the interest rates (43) and (44), the preferential debt limit (45), and the negative balance adjustment (46) play the same role in implementing the approximately optimal allocation as do their counterparts in Proposition 7 in implementing the optimal allocation.

As we observe from Proposition 9, the approximately optimal interest only mortgage with HELOC with preferential interest rate and one way balance adjustment takes the simple form of the interest only fixed rate mortgage, with the constant interest coupon payment of (41), combined with the HELOC with the credit limit of (42) and the simple variable rate given by (43). It follows from Proposition 9 that this mortgage contract has the following properties.

Corollary 6 *The approximately optimal interest only FRM with HELOC with preferential rate and one way balance adjustment has the following features:*

i) $BA^{-}(B_t - p_t) = 0$ *for* $B_t = p_t$, *and*

- $BA^{-}(B_t - p_t)$ *is negative and strictly decreasing in* $(B_t - p_t)$ *for* $B_t \in (p_t, C_t^L(r_H)]$,

– $dp_t \leq 0$ *for any* $B_t \geq p_t$, *with strict inequality whenever the interest rate,* r_t, *increases and* $B_t > p_t$,

(ii) $\bar{r}_t(B' - p'_t, r_L) < \gamma < \bar{r}_t(B'' - p''_t, r_H)$, *for any* $B' \in [p'_t, C_t^L(r_L)]$, $B'' \in [p''_t, C_t^L(r_H)]$.

As the above corollary shows, a decrease in the lender's interest rate causes a decrease in the borrower's HELOC balance. The magnitude of this adjustment is linearly proportional to the HELOC balance, and, as before, can be interpreted as offering the borrower an automatic "cramdown" provision. An increase in the lender's interest rate causes a drop in the amount of debt subject to the preferential interest rate. Consequently, under this contract, the preferential HELOC debt treatment is reduced over time. The variable interest rate on the HELOC balance positively correlates with the lender's interest rate, and is independent of the borrower's debt balance, except the debt region $[\hat{B}, C^L]$, where the HELOC interest rate increases with the balance provided that the lender's interest rate is low $(r_t = r_L)$.

Figure 9 presents the approximately optimal negative balance adjustment and the variable interest rate on the HELOC debt as a function of the borrower's mortgage debt above the preferential range in the parametrized environment of Section 4.3.

6.4 Approximately Optimal Option ARM with Preferential Rate

In this section we consider an implementation of the approximately optimal contract by an option ARM.

Proposition 10 *There exists an approximately optimal adjustable rate mortgage with preferential interest rate and negative amortization that has the following features*

$$C_t^L(p_t) = p_t + \hat{a}^1 - A \tag{48}$$

$$\bar{r}_t^p(p_t) = \frac{\theta + \mu - \gamma\hat{a}^1}{p_t} \tag{49}$$

$$\bar{r}_t(B_t - p_t, r_t) = \begin{cases} \gamma - \delta(r_L)\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right) & \text{for } B_t \in [p_t, \hat{B}_t] \quad \text{and } r_t = r_L \\ \gamma - \delta(r_L)\left(\frac{\hat{a}^1-A-B_t+p_t}{B_t-p_t}\right) & \text{for } B_t \in [\hat{B}_t, C_t^L] \quad \text{and } r_t = r_L \\ \gamma + \delta(r_H)\left(\frac{\bar{a}-A}{\hat{a}^1-A}\right) & \text{for } B_t \in [p_t, C_t^L] \quad \text{and } r_t = r_H \end{cases}, \tag{50}$$

$$dp_t = \begin{cases} -\left(\frac{\bar{a}-A}{\hat{a}^1-\bar{a}}\right)(B_t - p_t) & \text{for } B_t \in [p_t, \hat{B}] \quad \text{and } r_t = r_L \\ \left(\frac{\bar{a}-A}{\hat{a}^1-A}\right)(B_t - p_t) & \text{for } B_t \in [p_t, C_t^L] \quad \text{and } r_t = r_H \\ 0 & \text{for } B_t < p_t \end{cases}, \tag{51}$$

where $\hat{B}_t = p_t + \hat{a}^1 - \bar{a}$. *Under the terms of this mortgage, it is incentive compatible for the borrower to refrain from stealing and maintain balance* B_t *above* p_t. *The borrower uses all available cash flows to pay the balance when* $B_t > p_t$, *and consumes all excess cash flows once the balance drops to* $p_t(r_t)$. *Whenever* $B_t \in [\hat{B}_t, C_t^L(p_t))$, *an instantaneous change of the interest rate from* r_L *to* r_H *triggers the default of the* *mortgage. For the balance $B_t \geq p_t$, the borrower's expected payoff, $\hat{a}_t$, is determined by the current HELOC balance above the preferential debt limit as follows:*

$$\hat{a}_t = A + C_t^L - B_t = \hat{a}^1 - (B_t - p_t) \tag{52}$$

If the preferential rate reaches its upper boundary γ, the mortgage is reset to a contract that implements the continuation of the approximately optimal allocation.

Proof In the Appendix. ■

The intuition behind incentive compatibility of the postulated strategy of the borrower under the above mortgage contract is the same as in the case of the optimal mortgage contract of Proposition 7. The negative amortization limit (48), the interest rates (49) and (50), and the preferential debt limit (51) play the same role in implementing the approximately optimal allocation as their counterparts from Proposition 7 do in implementing the optimal allocation.

Proposition 10 implies the following properties of the above mortgage contract.

Corollary 7 *The mortgage contract of Proposition 10 has the following features:*

(i) $\bar{r}_t(B' - p'_t, r_L) < \gamma < \bar{r}_t(B'' - p''_t, r_H)$, *for any* $B' \in [p'_t, C_t^L(r_L)]$, $B'' \in [p''_t, C_t^L(r_H)]$.

(ii) $dp_t < 0$, $d\bar{r}_t^p > 0$ *whenever the interest rate,* r_t, *increases and* $B_t > p_t$,

$dp_t > 0$, $d\bar{r}_t^p < 0$ *whenever the interest rate,* r_t, *decreases and* $B_t > p_t$,

As the above corollary shows, a decrease in the interest rate causes an increase in the amount of debt subject to the preferential rate, and a fall in the preferential interest rate and the variable rate charged on the debt balance above the preferential debt range. An increase in the interest rate causes a drop in the amount of debt subject to the preferential interest rate, and an increase in the preferential interest rate and the variable rate charged on the debt balance above the preferential debt range. The variable interest rate on the debt balance above the preferential debt range does not depend the borrower's debt balance, except for the debt region $[\hat{B}, C^L]$, where this interest rate increases with the balance, provided that the market interest rate is low ($r_t = r_L$). The variable interest rate on the debt above the preferential debt limit, in the parametrized example of Section 4.3, is the same as shown on the right hand side of Figure 9.

7 Concluding Remarks

Recent years have seen a rapid growth in originations of more sophisticated alternative mortgage products (AMPs), such as option adjustable rate mortgages (option ARMs) and interest only mortgages. Critics of AMPs point out that they seem to be more profitable for lenders than traditional mortgages. They conclude that AMPs allow lenders to profiteer at the expense of homeowners. However, this paper shows that the properties of AMPs are consistent with the properties of the optimal allocation governing the relationship between the borrower and the lender, which represents a Pareto improvement over traditional mortgages. As a consequence, it is possible that both lenders and borrowers can benefit from AMPs. Critics of AMPs have raised the concern that teaser rates and low minimum payments can result in substantially higher mortgage payments and, as a consequence, higher default rates when the market interest rate increases. Nevertheless, this paper demonstrates that this does not necessarily contradict the optimality of AMPs. Under the optimal mortgage contract, mortgage payments and default rates are indeed higher when the market interest rate is high. However, borrowers benefit from low mortgage payments and low default rates when the market interest rate is low.

In this paper, we ignored inflation, which is an important consideration for home buyers choosing between ARMs and FRMs.[14] However, as long as inflation affects the borrower's income and the liquidation value of the home equally, it would not change the properties of the optimal mortgage in real terms. We also did not allow for contract renegotiations, because a possibility of renegotiation would lead to a suboptimal contract. In practice, lenders should be able to commit to the terms of a mortgage contract, or make renegotiation very costly for borrowers.

For the sake of tractability of our dynamic contracting problem, we had to assume risk-neutrality of the borrower and the lender. The properties of the optimal mortgage are determined by the conflict of interest between the borrower and the lender and by the gains from trade based on the differences between the borrower's and the lender's discount factors. We conjecture that the properties of the optimal mortgage will be preserved if we allow for risk-aversion in our model. However, solving the model with risk-aversion would require development of completely new dynamic

ing techniques.

There are a number of research directions one might pursue from here. In this paper we have considered time-homogeneous setting, in which agents are infinitely lived and the borrower's average income and the liquidation values of the home do not change over time. Relaxing this assumptions would allow us to study the effects of home appreciation trends and households' life-cycle income profiles on optimal mortgage design. Another avenue of research would be to extend our analysis to a general equilibrium framework and to study what effects the presence of private information in the mortgage origination market have on equilibrium home prices, and how this varies over the business cycle.

[14] See, for example, Campbell and Cocco (2003).

Appendix

A.1 Proofs of Lemmas and Propositions

Proof of Lemma 1

Consider any incentive compatible allocation $(\tau, I, C, \hat{Y})$. We prove the lemma by showing the existence of the new incentive-compatible allocation that that has the following properties:

(i) the borrower gets the same expected utility as under the old allocation (τ, I),

(ii) the borrower chooses to reveal the cash flows truthfully,

(iii) the borrower maintains zero savings,

(iv) the lender gets the same or greater expected profit as under the old allocation (τ, I).

Consider the candidate incentive compatible allocation (τ', I', C, Y) where

$$\begin{aligned} \tau'(Y, r) &= \tau(\hat{Y}(Y, r), r), \\ I'(Y, r) &= C(Y, r). \end{aligned}$$

We observe that the borrower's consumption and the termination time under the new allocation and the proposed borrower's response strategy, (C, Y), are the same as under the old allocation, so he earns the same payoff, which establishes property (i). Also, by construction, the proposed response of the borrower to the allocation (τ', I') involves truth-telling and zero savings, which establishes properties (ii) and (iii).

Now we will show that (C, Y) is the borrower's incentive compatible strategy under the allocation (τ', I'). We note that the strategy (C, Y) yields the same utility to the borrower under the allocation (τ', I') as the incentive compatible strategy associated with the allocation (τ, I). Therefore, to show that (C, Y) is the borrower's incentive compatible strategy under the allocation (τ', I'), it is enough to show that if any alternative strategy (C', Y') is feasible under the allocation (τ', I'), then C' is also feasible under the old allocation (τ, I).

It follows that if C' is feasible under the new allocation, then the borrower has nonnegative savings if he reports $\hat{Y}(Y'(Y, r), r)$ and consumes C' under the old allocation, and thus C' is also feasible under the old allocation (τ, I). To see this we note that that the borrower's savings at any time $t \leq \tau(\hat{Y}(Y'(Y, r), r) =$

$\tau'(Y'(Y,r),r)$ under the old allocation (τ, I) and the borrower's strategy $(C', \hat{Y}(Y'(Y,r),r))$ are equal to

$$\underbrace{\int_0^t e^{\rho_t(t-s)}\left[dY_s - d\hat{Y}_s(Y'(Y,r),r) + dI_s(\hat{Y}(Y'(Y,r),r) - dC'_s(Y,r)\right]}_{\text{Savings under the old allocation, the borrower's strategy } (C',\hat{Y}(Y'(Y,r),r)), \text{ and the realized } (Y,\ r)} =$$

$$\underbrace{\int_0^t e^{\rho_t(t-s)}\left[dY'_s(Y,r) - d\hat{Y}_s(Y'(Y,r),r) + dI_s(\hat{Y}(Y'(Y,r),r) - dC_s(Y'(Y,r),r)\right]}_{(\geq 0) \text{ Savings under the old allocation given the borrower's strategy } (C,\hat{Y}(Y'(Y,r),r)), \text{ and the realized } (Y'(Y,r),r)}$$

$$\underbrace{+\int_0^t e^{\rho_t(t-s)}\left[dY_s - dY'_s(Y,r) + \underbrace{dC_s(Y'(Y,r),r)}_{=I'(Y'(Y,r),r)} - dC'_s(Y,r)\right]}_{(\geq 0) \text{ Savings under the new allocation, the borrower's strategy } (C,Y'(Y,r)), \text{ and the realized } (Y,\ r)} \geq 0.$$

Finally, to complete the proof, we need to show that under the new allocation (τ', I') the lender gets the same or greater expected profit as under the allocation (τ, I). Note that under the new allocation the lender does savings for the borrower. As by assumption the lender's interest rate process is always greater or equal from the saving's interest rate available to the borrower (i.e., for all t, $r_t \geq \rho_t$), the lender's expected profit improves by

$$E_0\left[\int_0^\tau e^{-R_t}\left(r_t - \rho_t\right) S_t dt\right] \geq 0,$$

which shows (iv). $\square$

Proof of Proposition 3

First let $b(a,r)$ be a concave function[15] that solves the second-order differential equation of the proposition, i.e.:

$$\begin{aligned} &rb(a,r) = \\ &\mu + (\gamma a - \theta - \psi(a,r)\delta(r))b'(a_t,r_t) + \frac{1}{2}\sigma^2 b''(a,r) + \delta(r_t)\left(b(a_t + \psi(a,r), r^c) - b(a,r)\right) \end{aligned} \tag{53}$$

when a is in the interval $[A, a^1(r)]$, and $b'(a,r) = -1$ when $a > a^1(r)$, with boundary conditions $b(A,r) = L$ and

$$\begin{aligned} \mu + \theta &= r_L b(a^1(r_L), r_L) + \gamma a^1(r_L) - \delta(r_L)\left[b(a^1(r_H), r_H) - b(a^1(r_L), r_L) + a^1(r_H) - a^1(r_L)\right], \\ \mu + \theta &= r_H b(a^1(r_H), r_H) + \gamma a^1(r_H) - \delta(r_H)\left[b(a^1(r_L), r_L) - b(a^1(r_H), r_H) + a^1(r_L) - a^1(r_H)\right], \end{aligned}$$

[15] We establish concavity of the lender's value function in a discrete time approximation to our model. See Appendix A.2.

where function ψ is defined as follows

$$\psi(a,r) = \begin{cases} \text{is a solution to } b'(a,r) = b'(a+\psi, r^c) \text{ for all } (a,r) \\ \text{for which the solution is such that } \psi(a,r) > A - a, \\ \\ \text{otherwise it is equal to } A - a. \end{cases},$$

where $r \in \{r_L, r_H\}$ and $r^c = \{r_L, r_H\} \setminus \{r\}$.

For any incentive compatible allocation (τ, I, C, Y) we define:

$$G_t = \int_0^t e^{-R_s}(dY_s - dI_s) + e^{-R_t} b(a_t, r_t), \tag{54}$$

where a_t evolves according to (6). We note that the process G is such that G_t is $\mathcal{F}_t-$measurable.

We remember that under an arbitrary incentive compatible allocation, (τ, I, C, Y), a_t evolves as

$$da_t(r_t) = (\gamma a_t - \theta - \psi_t \delta(r_t))\, dt - dI_t + \beta_t dZ_t + \psi_t dN_t.$$

where $\beta_t \geq \sigma$ m-a.s. for any $0 \leq t \leq \tau$. From Ito's lemma we get that

$$\begin{aligned} db(a_t, r_t) = &\left[(\gamma a_t - \theta - \psi_t \delta(r_t)) b'(a_t, r_t) + \frac{1}{2}\beta_t^2 b''(a_t, r_t)\right] dt - b'(a_t, r_t) dI_t \\ &+ \beta_t b'(a_t, r_t) dZ_t + \left[b(a_t + \psi_t, r_t^c) - b(a_t, r_t)\right] dN_t. \end{aligned}$$

Then combining the above with (54) yields

$$\begin{aligned} e^{R_t} dG_t \quad = \quad &\left[\mu + (\gamma a_t - \theta - \psi_t \delta(r_t)) b_1(a_t, r_t) + \frac{1}{2}\beta_t^2 b_2(a_t, r_t) - r_t b_1(a_t, r_t)\right] dt \\ &-(1 + b_1(a_t, r_t)) dI_t + (\sigma + \beta_t b_1(a_t, r_t))\, dZ_t + \left[b(a_t + \psi_t, r_t^c) - b(a_t, r_t)\right] dN_t \end{aligned}$$

Combining the above with (53) yields

$$\begin{aligned} e^{R_t} dG_t \quad \leq \quad &\left[\frac{1}{2}\left(\beta_t^2 - \sigma^2\right) b''(a_t, r_t) + \delta(r_t) b'(a_t, r_t) \left[\psi(a_t, r_t) - \psi_t\right]\right] dt - (1 + b'(a_t, r_t)) dI_t \\ &+ (\sigma + \beta_t b'(a_t, r_t))\, dZ_t + \left[b(a_t + \psi_t, r_t^c) - b(a_t + \psi(a_t, r_t), r_t^c)\right] dN_t, \end{aligned}$$

with equality whenever $a \in [A, a^1(r_t)]$. From the above we have that for any $0 \leq t < \tau$:

$$
\begin{aligned}
e^{R_t} dG_t \quad \leq \quad & \underbrace{\left[\frac{1}{2}\left(\beta_t^2 - \sigma^2\right) b''(a_t, r_t)\right]}_{\leq 0} dt \underbrace{-(1 + b'(a_t, r_t)) dI_t}_{\leq 0} \\
& + \underbrace{\delta(r_t)\left([b(a_t + \psi_t, r_t^c) - \psi_t b'(a_t, r_t)] - [b(a_t + \psi(a_t, r_t), r_t^c) - \psi(a_t, r_t) b'(a_t, r_t)]\right)}_{\leq 0} dt \\
& + (\sigma + \beta_t b'(a_t, r_t)) dZ_t + [b(a_t + \psi_t, r_t^c) - b(a_t + \psi(a_t, r_t), r_t^c)] dM_t, \qquad (55)
\end{aligned}
$$

with equality whenever $a \in [A, a^1(r_t)]$. The first component of the RHS of the above inequality is less or equal to zero because the function b is concave and $\beta_t \geq \sigma$ for any $t \leq \tau$. The second component is less or equal to zero because $b' \geq -1$ and $dI_t \geq 0$. The third component is less or equal to zero because, by definition, the function ψ is a solution to

$$
\max_{\psi \geq A - a} \left[b(a + \psi, r^c) - \psi b'(a, r))\right].
$$

The condition (55) implies that the process G is an $\mathcal{F}_t-$supermartingale up to time $t = \tau$, where we recall that Z and M are martingales. It will be an $\mathcal{F}_t-$martingale if and only if, for $t > 0$, $a_t \leq a^1(r_t)$, $\beta_t = \sigma$ m-a.s., $\psi_t = \psi(a_t, r_t)$, and I_t is increasing only when $a_t \geq a^1(r_t)$.

We now evaluate the lender's payoff for an arbitrary incentive compatible allocation (τ, I, C, Y), which equals

$$
E\left[\int_0^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau} L\right].
$$

We note that $b(a_\tau, r_\tau) = L$ as, from the definition of a, $a_\tau = A$. Using this, and the definition of process G, we have that under any arbitrary incentive compatible allocation (τ, I, C, Y) and any $t \in [0, \infty)$:

$$
\begin{gathered}
E\left[\int_0^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau} L\right] = \\
E\left[G_{t\wedge\tau}\right] + E\left[1_{t\leq\tau}\left(\int_t^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau} L - e^{-R_t} b(a_t, r_t)\right)\right] \leq \\
b(a_0, r_0) + E\left[1_{t\leq\tau}\left(\int_t^\tau e^{-R_s}(dY_s - dI_s) + e^{-R_\tau} L - e^{-R_t} b(a_t, r_t)\right)\right] = \\
b(a_0, r_0) + e^{-R_t} E\left[1_{t\leq\tau}\left(E\left[\int_t^\tau e^{R_t - R_s}(dY_s - dI_s) + e^{R_t - R_\tau} L \,|\mathcal{F}_t\right] - b(a_t, r_t)\right)\right], \qquad (56)
\end{gathered}
$$

where, the inequality follows from the fact that $G_{t\wedge\tau}$ is supermartingale and $G_0 = b(a_0, r_0)$. We note that in the above

$$E\left[\int_t^{\tau} e^{R_t-R_s}(dY_s-dI_s)+e^{R_t-R_\tau}L\,|\mathcal{F}_t\right]<\frac{\mu}{r_L}+\frac{\theta}{\gamma}-a_t,$$

as the RHS of the above inequality is the upper bound on the lender's expected profit under the first-best (public information) allocation. Using the above inequality in (56) we have that

$$E\left[\int_0^{\tau} e^{-R_s}(dY_s-dI_s)+e^{-R_\tau}L\right]\le b(a_0,r_0)+e^{-R_t}E\left[1_{t\le\tau}\left(\frac{\mu}{r_L}+\frac{\theta}{\gamma}-a_t-b(a_t,r_t)\right)\right].$$

Using $b'(a,r)\ge -1$, we have that, for any $a\ge A$, $-a-b(a,r)\le -A-L$. Applying this to the above inequality yields

$$E\left[\int_0^{\tau} e^{-R_s}(dY_s-dI_s)+e^{-R_\tau}L\right]\le b(a_0,r_0)+e^{-R_t}E\left[1_{t\le\tau}\left(\frac{\mu}{r_L}+\frac{\theta}{\gamma}-A-L\right)\right].$$

Taking $t\to\infty$ yields

$$E\left[\int_0^{\tau} e^{-R_s}(dY_s-dI_s)+e^{-R_\tau}L\right]\le b(a_0,r_0).$$

Let (τ^*,I^*,C^*,Y) be an allocation satisfying the conditions of the proposition. We remember that this allocation is incentive compatible as it is feasible and $\beta_t=\sigma\ge\sigma$ for any $t\le\tau$. Also under this allocation the process G_t is a martingale until time τ (note that $b'(a,r)$ is bounded). So we have that

$$E\left[\int_0^{\tau^*} e^{-R_s}(dY_s-dI_s^*)+e^{-R_{\tau^*}}L\right]=$$

$$b(a_0,r_0)+e^{-R_t}E\left[1_{t\le\tau^*}\left(E\left[\int_t^{\tau^*} e^{R_t-R_s}(dY_s-dI_s^*)+e^{R_t-R_{\tau^*}}L\,|\mathcal{F}_t\right]-b(a_t,r_t)\right)\right].$$

Taking $t\to\infty$ and using

$$\lim_{t\to\infty}e^{-R_t}E\left[1_{t\le\tau^*}\left(E\left[\int_t^{\tau^*} e^{R_t-R_s}(dY_s-dI_s^*)+e^{R_t-R_{\tau^*}}L\,|\mathcal{F}_t\right]-b(a_t,r_t)\right)\right]=0,$$

yields

$$E\left[\int_0^{\tau^*} e^{-R_s}(dY_s-dI_s^*)+e^{-R_{\tau^*}}L\right]=b(a_0,r_0).\quad\square$$

Proof of Proposition 4

Let $(C, \hat{Y})$ be any borrower's feasible strategy given the allocation (τ, I). The borrower's private saving's account balance, S, under the strategy $(C, \hat{Y})$ and the allocation (τ, I) grows, for $t \in [0, \tau]$, according to

$$dS_t = \rho_t S_t dt + (dY_t - d\hat{Y}_t) + dI_t - dC_t, \tag{57}$$

where we remember that $\rho_t \leq r_t$. Define the process $\hat{V}$ as

$$\hat{V}_t = \int_0^t e^{-\gamma s} dC_s + \int_0^t e^{-\gamma s} \theta ds + e^{-\gamma t}(S_t + a_t),$$

From the above it follows that

$$e^{\gamma t} d\hat{V}_t = dC_t + \theta dt + dS_t - \gamma S_t dt + da_t - \gamma a_t dt$$

Using (16) and (57) yields

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= (\rho_t - \gamma) S_t dt + (dY_t - \mu dt) dt + \psi_t dM_t = \\ & (\rho_t - \gamma) S_t dt + \sigma dZ_t + \psi_t dM_t. \end{aligned} \tag{58}$$

Noting that $e^{\gamma t} \geq 1$ for any $t \geq 0$, we have that

$$d\hat{V}_t \leq (\rho_t - \gamma) S_t dt + \sigma dZ_t + \psi_t dM_t.$$

Recall that Z and M are martingales, $\rho_t < \gamma$, and that the process S is nonnegative. So it follows from the above that the process $\hat{V}$ is supermartingale up to time τ (note that a is bounded from below). Using this and the fact that by definition $a_\tau = A$, we have that for any feasible strategy of the borrower,

$$a_0 = \hat{V}_0 \geq E\left[\hat{V}_\tau\right] = E\left[\int_0^\tau e^{-\gamma s} dC_s + \int_0^\tau e^{-\gamma s} \theta ds + e^{-\gamma \tau}(S_\tau + A)\right], \tag{59}$$

The right-hand-side of (59) represents the expected future payoff for the borrower under any feasible $\left(C, \hat{Y}, S\right)$. This payoff is bounded by a_0. If the borrower maintains zero savings, $S_t = 0$, reports cash flows truthfully, $d\hat{Y}_t = dY_t$, then $\hat{V}$ is a martingale up to time τ, which means that (59) holds with equality and the borrower's expected future payoff is a_0. Thus, this is the optimal strategy for the borrower. $\square$

Proof of Proposition 5

Consider the candidate mortgage contract. Under this contract the borrower's balance on the credit line evolves according to

$$dB_t = \bar{r}_t(B_t, r_t)B_t dt + x_t(r_t)dt - (d\hat{Y}_t - dI_t) + BA(B_t, r_t)dN_t, \tag{60}$$

when $B_t \leq C_t^L(r_t)$, while the borrower's savings evolve according to

$$dS_t = \rho_t S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) - dC_t, \tag{61}$$

where I_t represents cumulative withdrawal of money from the credit line by the borrower.

Let $(C, \hat{Y}, S)$ be any borrower's feasible strategy under the proposed mortgage contract. For any feasible borrower's strategy $(C, \hat{Y}, S)$ define a process $\hat{V}$ as

$$\hat{V}_t = \int_0^t e^{-\gamma s}(dC_s + \theta ds) + e^{-\gamma t}\left(\tilde{a}_t + S_t\right), \tag{62}$$

where

$$\tilde{a}_t = a^1(r_t) - B_t \tag{63}$$

It follows from (60), (63), and (17)-(20) that $\tilde{a}$ evolves as

$$\begin{aligned}
d\tilde{a}_t &= \left[a^1(r_t^c) - a^1(r_t)\right] dN_t - dB_t \\
&= \left[a^1(r_t^c) - a^1(r_t)\right] dN_t - \bar{r}_t(B_t, r_t)B_t dt - x_t(B_t, r_t)dt - BA(B_t, r_t)dN_t + d\hat{Y}_t - dI_t \\
&= \left[a^1(r_t^c) - a^1(r_t)\right] dN_t - \left[\gamma\left(a^1(r_t) - \tilde{a}_t\right) - \delta(r_t)\left[\psi(a^1(r_t), r_t) - \psi(\tilde{a}_t, r_t)\right]\right] dt \\
&\quad - \left[\theta + \mu - \gamma a^1(r_t) + \delta(r_t)\psi(a^1(r_t), r_t)\right] dt - \left[-\psi(\tilde{a}_t, r_t) + (a^1(r_t^c) - a^1(r_t))\right] dN_t \\
&\quad + d\hat{Y}_t - dI_t \\
&= \left(\gamma\tilde{a}_t - \theta - \delta(r_t)\psi(\tilde{a}_t, r_t)\right) dt + (d\hat{Y}_t - \mu dt) - dI_t + \psi(\tilde{a}_t, r_t)dN_t
\end{aligned} \tag{64}$$

Using (1), (61), (62), (64) yields

$$\begin{aligned}
e^{\gamma t} d\hat{V}_t &= dC_t + \theta dt + d\tilde{a}_t + dS_t - \gamma\tilde{a}_t dt - \gamma S_t dt \\
&= \sigma dZ_t + \psi(\tilde{a}_t, r_t)dM_t + (\rho_t - \gamma)S_t dt
\end{aligned}$$

Recall that Z and M are martingales, $\rho_t < \gamma$, and that the process S is nonnegative. So it follows from the above that the process $\hat{V}$ is a supermartingale up to time $\tau\left(C, \hat{Y}, S\right) = \inf\left\{t : B_t = C_t^L\right\}$ (note that $\tilde{a}$ is bounded from below). Using this and the fact that by definition $\tilde{a}_\tau = A$, we have that for any feasible strategy of the borrower, $(C, \hat{Y}, S)$,

$$A + CL_0(r_0) - B_0 = a_0 = \tilde{a}_0 = \hat{V}_0 \geq E\left[\hat{V}_{\tau\left(C,\hat{Y},S\right)}\right]$$
$$= E\left[\int\limits_0^{\tau\left(C,\hat{Y},S\right)} e^{-\gamma s}(dC_s + \theta ds) + e^{-\gamma\tau\left(C,\hat{Y},S\right)}(S_{\tau\left(C,\hat{Y},S\right)} + A)\right] \quad (65)$$

The right-hand-side of (65) represents the expected future payoff for the borrower under any feasible strategy $\left(C, \hat{Y}, S\right)$, given the terms of the mortgage. This payoff is bounded by $A + C_0^L(r_0) - B_0$, where B_0 is the initial draw on the credit line. If the borrower maintains zero savings, $S_t = 0$, reports cash flows truthfully, $d\hat{Y}_t = dY_t$, and consumes all excess cash flows once the balance on the credit line reaches 0, so that $C = I = I^* = \max(0, -B_t) = \max(0, \tilde{a}_t - a^1(r_t))$, then $\hat{V}$ is a martingale, which means that (65) holds with equality and the borrower's expected future payoff is $A + C_0^L(r_0) - B_0$. Thus, this is the optimal strategy for the borrower.

Reproducing the above argument for the borrower's optimal strategy, $(C, \hat{Y}, S) = (I^*, Y, 0)$, and the process $\hat{V}_{t'}$, $t' \leq \tau(I^*, Y, 0)$, defined as

$$\hat{V}_{t',t} = \int\limits_{t'}^{t} e^{-\gamma(s-t')}(dC_s + \theta ds) + e^{-\gamma(t-t')}\tilde{a}_t, \quad t \geq t', \quad (66)$$

yields that, for any $0 \leq t \leq \tau(I^*, Y, 0)$, $\tilde{a}_t$ is equal to the borrower's continuation payoff under the proposed mortgage contract with the initial payoff for the borrower given by $a_0 = A + C_0^L(r_0) - B_0$, which establishes (21).

Under the proposed mortgage contract and the borrower's optimal strategy, the lender's payoff equals

$$E\left[\int\limits_0^{\tau(I^*,Y,0)} e^{-R_t}(dY_t - dI_t^*) + e^{-R_{\tau(I^*,Y,0)}\tau(I^*,Y,0)} L \,|\mathcal{F}_0\right],$$

where

$$\tau(I^*, Y, 0) = \inf\left\{t : B_t = C_t^L\right\} = \inf\left\{t : \tilde{a}_t = A\right\} = \tau^*(Y),$$

as the borrower's continuation payoff, $\tilde{a}$, evolve according to the equation (64), e.g. as in the optimal allocation. Therefore, we conclude that the proposed mortgage contract implements the optimal allocation. $\square$

Proof of Proposition 6

Define $\tilde{a}_t$ as follows:

$$\begin{aligned} \tilde{a}_t &= A + C_t^L(r_t) - B_t \quad (67) \\ &= p_t + a^1(r_t) - B_t. \quad (68) \end{aligned}$$

Under the candidate mortgage contract the balance on the HELOC evolves according to

$$dB_t = \left(\bar{r}_t^p B_t + (\bar{r}_t - \bar{r}_t^p)(B_t - p_t)^+\right) dt + x_t(r_t)dt + BA^-(B_t - p_t)dN_t - d\hat{Y}_t + dI_t \quad (69)$$

when $B_t \leq C_t^L$, where I_t represents cumulative withdrawal of money from the credit line by the borrower. In addition,

$$\begin{aligned} dC_t^L &= dp_t + da^1(r_t) \\ &= \left(\psi(a^1(r_t) - (B_t - p_t, r_t)I_{r_{t\_}=r_L} + da^1(r_t)I_{r_{t\_}=r_H}\right) dN_t \quad (70) \end{aligned}$$

Using (68)-(70) and (22)-(27), for $B_t \geq p_t$, we can write

$$\begin{aligned} d\tilde{a}_t &= dC_t^L(r_t) - dB_t \\ &= \left(\psi(p_t + a^1(r_t) - B_t, r_t)I_{r_{t\_}=r_L} dN_t + da^1(r_t)I_{r_{t\_}=r_H}\right) dN_t - \left(\bar{r}_t^p B_t + (\bar{r}_t - \bar{r}_t^p)(B_t - p_t)\right) dt \\ &\quad -x_t(r_t) - BA^-(B_t - p_t)dN_t + d\hat{Y}_t - dI_t \\ &= \left(\psi(p_t + a^1(r_t) - B_t, r_t)I_{r_{t\_}=r_L} dN_t + da^1(r_t)I_{r_{t\_}=r_H}\right) dN_t - \left(\bar{r}_t^p B_t + (\bar{r}_t - \bar{r}_t^p)(B_t - p_t)\right) dt \\ &\quad -x_t(r_t) - (-\psi(p_t + a^1(r_t) - B_t, r_t)I_{r_{t\_}=r_H} dN_t + da^1(r_t)I_{r_{t\_}=r_H})dN_t + d\hat{Y}_t - dI_t \\ &= -\left(\theta + \mu - \gamma a^1(r_t) + \gamma(B_t - p_t)\right) dt + d\hat{Y}_t - dI_t + \psi(p_t + a^1(r_t) - B_t, r_t)dM_t \\ &= \gamma \tilde{a}_t dt - \mu dt - \theta dt + d\hat{Y}_t - dI_t + \psi(\tilde{a}_t, r_t)dM_t \quad (71) \end{aligned}$$

The borrower's savings evolve according to

$$dS_t = \rho_t S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) - dC_t. \quad (72)$$

Consider

$$\hat{V}_t = \int_0^t e^{-\gamma s}(\theta dt + dC_s) + e^{-\gamma t}(\Omega_t + S_t)$$

where

$$\Omega_t = \begin{cases} a^1(r_t) + (p_t - B_t), \; if \; B_t < p_t \\ \tilde{a}_t, \; if \; B_t \geq p_t \end{cases}. \quad (73)$$

We will show that for any feasible strategy $\left(C, \hat{Y}, S\right)$ of the borrower, $\hat{V}_t$ is a supermartingale. Note that

$$d\Omega_t = \begin{cases} \underbrace{[a^1(r_t^c) - a^1(r_t)]}_{\psi(a^1(r_t),r_t)} dN_t - dB_t, \; if \; B_t < p_t \\ d\tilde{a}_t, \; if \; B_t \geq p_t \end{cases} . \tag{74}$$

Using (72),

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= \theta dt + dC_t + dS_t - \gamma S_t dt + d\Omega_t - \gamma \Omega_t dt \\ &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) + d\Omega_t - \gamma \Omega_t dt. \end{aligned}$$

First, we consider the case with $B_t \geq p_t$. Using (1), (71), (73)-(74),

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) + d\tilde{a}_t - \gamma \tilde{a}_t dt \\ &= (\rho_t - \gamma) S_t dt + \sigma dZ_t + \psi(\tilde{a}_t, r_t) dM_t. \end{aligned} \tag{75}$$

Now, let $B_t < p_t$. Using (1), (22)-(26), (69)-(74) yields

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) \\ &\quad + \psi(a^1(r_t), r_t) dN_t - dB_t - \gamma \left(a^1(r_t) + (p_t - B_t)\right) dt \\ &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) \\ &\quad + \psi(a^1(r_t), r_t) dN_t \\ &\quad - \left[\left(\bar{r}_t^p B_t + (\bar{r}_t - \bar{r}_t^p)(B_t - p_t)^+\right) dt + x_t(r_t) dt + BA^-(B_t - p_t) dN_t - d\hat{Y}_t + dI_t\right] \\ &\quad - \gamma \left(a^1(r_t) + (p_t - B_t)\right) dt \\ &= \theta dt - (\gamma - \rho_t) S_t dt + dY_t \\ &\quad + \psi(a^1(r_t), r_t) dN_t - [\theta + \mu - \gamma a^1(r_t) + \delta(r_t)\psi(a^1(r_t), r_t)] dt \\ &\quad - \gamma \left(a^1(r_t) + (p_t - B_t)\right) dt \\ &= -\gamma (p_t - B_t) dt - (\gamma - \rho_t) S_t dt + \psi(a^1(r_t), r_t) dM_t + \sigma dZ_t \end{aligned} \tag{76}$$

Recall that Z and M are martingales and that $\rho_t < \gamma$. Thus, it follows from (75), (76), and the fact that $\tilde{a}_t$ is bounded from below, that for any feasible strategy $\left(C, \hat{Y}, S\right)$ of the borrower $\hat{V}_t$ is a supermartingale until default time $\tau\left(C, \hat{Y}, S\right) = \inf\left\{t : B_t = C_t^L\right\}$. Since $\Omega_\tau = A$,

$$\begin{aligned} A + C_0^L(r_0) - B_0 &= \tilde{a}_0 + S_0 = \hat{V}_0 \geq E\left[\hat{V}_{\tau\left(C,\hat{Y},S\right)}\right] \\ &= E\left[\int_0^{\tau\left(C,\hat{Y},S\right)} e^{-\gamma s}\left(\theta dt + dC_s\right) + e^{-\gamma\tau\left(\hat{Y},C,S\right)}\left(A + S_{\tau\left(C,\hat{Y},S\right)}\right)\right], \end{aligned} \tag{77}$$

where B_0 is the time-zero draw on the credit line.

The right-hand-side of (77) represents the expected future payoff for the borrower under strategy $\left(C, \hat{Y}, S\right)$, given the terms of the mortgage. This payoff is bounded by $A + C_0^L(r_0) - B_0' + S_0$. If the borrower maintains zero savings, $S_t = 0$, reports cash flows truthfully, $d\hat{Y}_t = dY_t$, and consumes all excess cash flows once the balance on the credit line reaches $p_t(r_t)$, so that $B_t \geq p_t$ and $C_t = I_t^* = \max(0, p_t - B_t) = \max(0, \tilde{a}_t - a_t^1)$, then $\hat{V}_t$ is a martingale, which means that (88) holds with equality and the borrower's expected future payoff is $A + C_0^L(r_0) - B_0$. Thus, this is the optimal strategy for the borrower.

Reproducing the above argument for the borrower's optimal strategy, $(C, \hat{Y}, S) = (I^*, Y, 0)$, and the process $\hat{V}_{t'}$, $t' \leq \tau(I^*, Y, 0)$, defined in (66) yields that, for any $0 \leq t \leq \tau(C, \hat{Y}, 0)$, $\tilde{a}_t$ is equal to the borrower's continuation payoff under the proposed mortgage contract with the initial payoff for the borrower given by $a_0 = A + C_0^L(r_0) - B_0$, which establishes (28).

Under the proposed mortgage contract and the borrower's optimal strategy $(C, \hat{Y}, S) = (I^*, Y, 0)$, the lender's payoff equals

$$E\left[\int_0^{\tau(I^*,Y,0)} e^{-R_t}(dY_t - dI_t^*) + e^{-R_{\tau(I^*,Y,0)}\tau(I^*,Y,0)} L \,|\mathcal{F}_0\right],$$

where

$$\tau(I^*, Y, 0) = \inf\left\{t : B_t = C_t^L\right\} = \inf\left\{t : \tilde{a}_t = A\right\} = \tau^*(Y),$$

as the borrower's continuation payoff, $\tilde{a}$, evolve according to the equation (71), e.g. as in the optimal allocation. Therefore, we conclude that the proposed mortgage contract implements the optimal allocation. $\square$

Proof of Proposition 7

Define $\tilde{a}_t$ as follows:

$$\tilde{a}_t = A + C_t^L(r_t) - B_t \tag{78}$$

$$= p_t + a^1(r_t) - B_t. \tag{79}$$

Under the candidate mortgage contract the debt balance evolves according to

$$dB_t = \left(\bar{r}_t^p B_t + \left(\bar{r}_t - \bar{r}_t^p\right)\left(B_t - p_t\right)^+\right) dt - d\hat{Y}_t + dI_t, \tag{80}$$

when $B_t \leq C_t^L$, where I_t represents cumulative withdrawal of money by the borrower. In addition,

$$\begin{aligned} dC_t^L &= dp_t + da^1(r_t) \\ &= \psi(p_t + a^1(r_t) - B_t, r_t)dN_t \end{aligned} \tag{81}$$

Using (29)-(31), (79)-(81), for $B_t \geq p_t$ we can write

$$\begin{aligned} d\tilde{a}_t &= dC_t^L(r_t) - dB_t \\ &= \psi(p_t + a^1(r_t) - B_t, r_t)dN_t - \left(\bar{r}_t^p B_t + \left(\bar{r}_t - \bar{r}_t^p\right)\left(B_t - p_t\right)\right) dt + d\hat{Y}_t - dI_t \\ &= -\left(\bar{r}_t^p p_t + \bar{r}_t\left(B_t - p_t\right) - \delta\psi(p_t + a^1(r_t) - B_t, r_t)\right) dt + d\hat{Y}_t - dI_t \\ &\quad + \psi(p_t + a^1(r_t) - B_t, r_t)dM_t \\ &= -\left(\theta + \mu - \gamma a^1(r_t) + \gamma\left(B_t - p_t\right)\right) dt + d\hat{Y}_t - dI_t + \psi(p_t + a^1(r_t) - B_t, r_t)dM_t \\ &= \gamma\tilde{a}_t dt - \mu dt - \theta dt + d\hat{Y}_t - dI_t + \psi(\tilde{a}_t, r_t)dM_t \end{aligned} \tag{82}$$

The borrower's savings evolve according to

$$dS_t = \rho_t S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) - dC_t. \tag{83}$$

Consider

$$\hat{V}_t = \int_0^t e^{-\gamma s}\left(\theta dt + dC_s\right) + e^{-\gamma t}\left(\Omega_t + S_t\right)$$

where

$$\Omega_t = \begin{cases} a^1\left(r_t\right) + \left(p_t - B_t\right),\ if\ B_t < p_t \\ \tilde{a}_t,\ if\ B_t \geq p_t \end{cases}. \tag{84}$$

We will show that for any feasible strategy $\left(C, \hat{Y}, S\right)$ of the borrower, $\hat{V}_t$ is a supermartingale. Note that

$$d\Omega_t = \begin{cases} \underbrace{\left[a^1(r_t^c) - a^1(r_t)\right]}_{\psi(a^1(r_t), r_t)} dN_t - dB_t,\ if\ B_t < p_t \\ d\tilde{a}_t,\ if\ B_t \geq p_t \end{cases}. \tag{85}$$

Using (83),

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= \theta dt + dC_t + dS_t - \gamma S_t dt + d\Omega_t - \gamma \Omega_t dt \\ &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) + d\Omega_t - \gamma \Omega_t dt. \end{aligned}$$

First, we consider the case with $B_t \geq p_t$. Using (1), (82), and (84)-(85),

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) + d\tilde{a}_t - \gamma \tilde{a}_t dt \\ &= (\rho_t - \gamma) S_t dt + \sigma dZ_t + \psi(\tilde{a}_t, r_t) dM_t. \end{aligned} \tag{86}$$

Now, let $B_t < p_t$. Using (1), (30), (80)-(85) yields

$$\begin{aligned} e^{\gamma t} d\hat{V}_t &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) + d\Omega_t - \gamma \Omega_t dt \\ &= \theta dt - (\gamma - \rho_t) S_t dt + dI_t + \left(dY_t - d\hat{Y}_t\right) \\ &\quad + \psi(a^1(r_t), r_t) dN_t - dB_t - \gamma \left(a^1(r_t) + (p_t - B_t)\right) dt \\ &= -\left(\bar{r}^p_t B_t + \gamma (p_t - B_t) + \gamma a^1(r_t) - \theta - \mu + (\gamma - \rho_t) S_t\right) dt \\ &\quad + \psi(a^1(r_t), r_t) dN_t + \sigma dZ_t \\ &= -\left(\bar{r}^p_t B_t + \gamma (p_t - B_t) - \bar{r}^p_t p_t + \delta \psi(a^1(r_t), r_t) + (\gamma - \rho_t) S_t\right) dt \\ &\quad + \psi(a^1(r_t), r_t) dN_t + \sigma dZ_t \\ &= -(\gamma - \bar{r}^p_t)(p_t - B_t) dt - (\gamma - \rho_t) S_t dt + \psi(a^1(r_t), r_t) dM_t + \sigma dZ_t \end{aligned} \tag{87}$$

Recall that Z and M are martingales, $\bar{r}^p_t \leq \gamma$, $\rho_t < \gamma$. Thus, it follows from (86), (87), and the fact that $\tilde{a}_t$ is bounded from below, that for any feasible strategy $\left(C, \hat{Y}, S\right)$ of the borrower $\hat{V}_t$ is a supermartingale until default time $\tau\left(C, \hat{Y}, S\right) = \inf\left\{t : B_t = C^L_t\right\}$. Since $\Omega_\tau = A$,

$$\begin{aligned} A + C^L_0(r_0) - B_0 &= \tilde{a}_0 = \hat{V}_0 \geq E\left[\hat{V}_{\tau(C,\hat{Y},S)}\right] \\ &= E\left[\int_0^{\tau(C,\hat{Y},S)} e^{-\gamma s} (\theta dt + dC_s) + e^{-\gamma \tau(C,\hat{Y},S)} \left(A + S_{\tau(C,\hat{Y},S)}\right)\right], \end{aligned} \tag{88}$$

where B_0 is the time-zero draw on the credit line.

The right-hand-side of (88) represents the expected future payoff for the borrower under strategy $\left(C, \hat{Y}, S\right)$, given the terms of the mortgage. This payoff is bounded by $A + C^L_0(r_0) - B_0$. If the borrower maintains zero savings, $S_t = 0$, reports cash flows truthfully, $d\hat{Y}_t = dY_t$, and consumes all excess cash flows once the balance on the credit line reaches $p_t(r_t)$, so that $B_t \geq p_t$ and $C_t = I^*_t = \max(0, p_t - B_t) = \max(0, \tilde{a}_t - a^1_t)$, then $\hat{V}_t$ is a martingale, which means that (88) holds with equality and the borrower's expected future payoff is $A + C^L_0(r_0) - B_0$. Thus, this is the optimal strategy for the borrower.

Reproducing the above argument for the borrower's optimal strategy, $(C, \hat{Y}, S) = (I^*, Y, 0)$, and the process $\hat{V}_{t'}$, $t' \leq \tau(I^*, Y, 0)$, defined in (66) yields that, for any $0 \leq t \leq \tau(C, \hat{Y}, 0)$, $\tilde{a}_t$ is equal to the borrower's continuation payoff under the proposed mortgage contract with the initial payoff for the borrower given by $a_0 = A + C_0^L(r_0) - B_0$, which establishes (33).

Under the proposed mortgage contract and the borrower's optimal strategy $(C, \hat{Y}, S) = (I^*, Y, 0)$, the lender's payoff equals

$$E\left[\int\limits_0^{\tau(I^*,Y,0)} e^{-R_t}(dY_t - dI_t^*) + e^{-R_{\tau(I^*,Y,0)}\tau(I^*,Y,0)} L \,|\mathcal{F}_0\right],$$

where

$$\tau(I^*, Y, 0) = \inf\left\{t : B_t = C_t^L\right\} = \inf\left\{t : \tilde{a}_t = A\right\} = \tau^*(Y),$$

as the borrower's continuation payoff, $\tilde{a}$, evolve according to the equation (82), e.g. as in the optimal allocation. Therefore, we conclude that the proposed mortgage contract implements the optimal allocation. $\square$

A.2 Discrete-Time Formulation

In this section we present a discrete-time version of our model.

Set-up

Time is discrete and the period has a length equal to $\Delta > 0$. There is one borrower and one lender. lender is risk neutral, have unlimited capital, and values a stochastic cash flow sequence $\{C_t\}$ as $\sum_t E\left[e^{-R_t} C_t\right]$. We assume that $R_t = \sum_{s=1}^{t} \Delta r_s$, where $\{r_t\}$ is the lender's stochastic discount rate sequence (interest rate sequence). The lender's stochastic discount rate sequence is a first-order time-invariant Markov chain. We further assume that, for any t, $r_t \in \{r_L, r_H\}$, $0 \leq r_L < r_H$,

$$
\begin{aligned}
\Pr\left[r_{t+1} = r_L \,|r_t = r_L\right] &= e^{-\Delta\delta(r_L)}, \\
\Pr\left[r_{t+1} = r_H \,|r_t = r_H\right] &= e^{-\Delta\delta(r_H)},
\end{aligned}
$$

Note that this implies that $P\left[r_{t+1} = r_L \,|r_t = r_H\right] = 1 - e^{-\Delta\delta(r_H)}$ and $P\left[r_{t+1} = r_H \,|r_t = r_L\right] = 1 - e^{-\Delta\delta(r_L)}$.

The borrower is also risk neutral, has limited wealth, and values a stochastic cash flow sequence $\{C_t\}$ as $\sum_t e^{-\Delta\gamma t} E\left[C_t\right]$. We assume that, for all t, $\gamma \geq r_t$. The borrower can buy a home at date $t = 0$, which requires an initial investment in assets of P. The borrower initial wealth is $Y_0 \geq 0$. We assume that $P > Y_0$, so that the borrower must borrow from the lender (sign a contract with the lender) to finance the purchase of a home. In every period, the home ownership generates to the borrower a public deterministic utility stream equal to $\{\Delta\theta_t\}$.

The borrower's income at date t is given by a random variable $Y_t \in \mathcal{Y}_t$. The borrower's income realizations, $\{Y_t\}$, are jointly independent and $E_s[Y_t] = E[Y_t] = \Delta\mu_t$ for all $s < t$. We note that the independence assumption implies no learning about future income flows. For all t, denote the minimum element of the support of Y_t by $Y_t^0 \geq 0$. The minimal cash flow Y_t^0 is collectible by the lender. The excess income realizations $Y_t - Y_t^0$, are privately observable by the borrower.

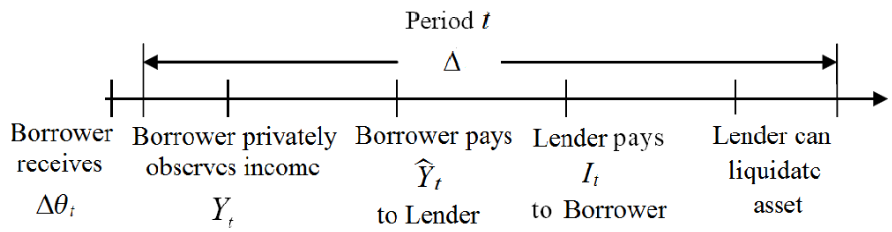
The borrower also maintains a private savings account. The borrower's balance grows at the interest rate $\Delta\rho_t$, such that $\rho_t \leq r_L$. The borrower must maintain a nonnegative balance at his account.

At any time the lender can liquidate the project. In case of liquidation at time t, the lender receives L_t, while the borrower receives his time-t reservation value equal to A_t.

Optimal Contract (Optimal Allocation)

If the lender agrees to fund the project at date 0, the borrower and the lender sign a contract that will govern their relationship until time T. Let $\hat{Y}_t$ be a borrower's report of cash flow realization at time t and

Figure 10: Sequence of events.



let $\hat{Y}^t = \left\{\hat{Y}_0, ..., \hat{Y}_t\right\}$ be the history of borrower's reports up to time t. Let $r^t = \{r_0, r_1, ..., r_t\}$ be the history of interest rates. Consequently we have the following definition.

Definition 13 *A contract $\zeta = (p, I)$ is a sequence of two functions $\{p_t, I_t\}_{t=0}^T$, such that, given the history of borrower's reports of income, $\hat{Y}^t = \left\{\hat{Y}_0, ..., \hat{Y}_t\right\}$, and the history of interest rates, $r^t = \{r_0, r_1, ..., r_t\}$, the contract obliges lender to make, at any time t, payment $I_t(\hat{Y}^t, r^t) \geq 0$ and liquidate the project with the probability $p_t(\hat{Y}^t, r^t) \in [0, 1]$.*

Figure 10 presents the sequence of events.

The borrower can misreport his income. Consequently, under the contract $\xi = (p, I)$, the borrower's income at time t equals

$$\underbrace{(Y_t - \hat{Y}_t)}_{misreporting} + I_t,$$

The borrower's private saving's account balance, S, grows according to

$$S_{t+1} = e^{\Delta \rho_t} S_t + (Y_t - \hat{Y}_t) + I_t - C_t,$$

where C_t is the borrower's consumption at time t, which must be nonnegative. We remember that, for all $t \geq 0$, $S_t \geq 0$ and $\rho_t \leq r_t$.

Definition 14 *Given a contract $\zeta = (p, I)$, the borrower's strategy is a consumption-report pair $(C, \hat{Y}) = \left\{C_t, \hat{Y}_t\right\}_{t=0}^T$, such that, given the history of the borrower's income realizations, Y^t, and the history of interest rates, r^t, the borrower consumes $C_t(Y^t, r^t)$ and reports $\hat{Y}_t(Y^t, r^t)$ at time t.*

Definition 15 *Given a contract $\zeta = (p, I)$, the borrower's strategy, $(C, \hat{Y})$, is feasible if the borrower's consumption and savings are nonnegative under this strategy.*

Let

$$P_t(\hat{Y}^t, r^t) = \begin{cases} 1 & \text{for } t = 0 \\ \prod_{k=1}^{t-1} (1 - p_k(\hat{Y}^k, r^k)) & \text{for } t \geq 1 \end{cases},$$

be the probability that the project is active at the beginning of period t under the contract $\zeta = (p, I)$, given the history of reports, $\hat{Y}^t$, and the history of interest rates, r^t.

Definition 16 *A contract $\zeta = (p, I)$ together with the borrower's strategy $(C, \hat{Y})$ is incentive compatible if:*

(i) given a contract $\zeta = (p, I)$, the borrower's strategy $(C, \hat{Y})$ is feasible,

(ii) given a contract $\zeta = (p, I)$, the borrower's strategy $(C, \hat{Y})$ provides him with the highest expected utility among all feasible strategies, i.e.:

$$E\left[\sum_{t=0}^{T} e^{-\gamma\Delta t} P_t(\hat{Y}^t, r^t)\left[C_t(Y^t, r^t) + \Delta\theta_t + p_t(\hat{Y}^t, r^t)A_t\right]|r_0\right] \geq$$
$$E\left[\sum_{t=0}^{T} e^{-\gamma\Delta t} P_t(\hat{Y}'^t, r^t)\left[C'_t(Y^t, r^t) + \Delta\theta_t + p_t(\hat{Y}'^t, r^t)A_t\right]|r_0\right]$$

for any feasible strategy $(C', \hat{Y}')$ given a contract $\zeta = (p, I)$,

(iii) the borrower's continuation payoff under the contract $\zeta = (p, I)$ and the strategy $(C, \widehat{Y})$ is greater or equal from the payoff he could guarantee himself by leaving the contract, e.g., for any (Y^s, r^s), such that the contract has not been terminated at time $s \leq T$:

$$E\left[\sum_{t=s+1}^{T} e^{-\Delta\gamma(s-t)} P_t(\hat{Y}^t, r^t)\left[C_t(Y^t, r^t) + \Delta\theta_t + p_t(\hat{Y}^t, r^t)A_t\right]|(Y^s, r^s)\right] \geq A_s.$$

Definition 17 *Given the borrower's initial utility, a_0, and the initial interest rate for the lender, r_0, the contract, $\zeta^* = (p^*, I^*)$, together with the recommendation to the borrower, $(C^*, \hat{Y}^*)$, is optimal if it maximizes the lender's payoff:*

$$E\left[\sum_{t=0}^{T} e^{-R_t} P_t(\hat{Y}^{*t}, r^t)\left(\hat{Y}^*_t - I^*_t(\hat{Y}^{*t}, r^t) + p^*_t(\hat{Y}^{*t}, r^t)L_t\right)|r_0\right]$$

in the class of all incentive-compatible contracts that satisfy the following promise keeping constraint:

$$a_0 = E\left[\sum_{t=0}^{T} e^{-\gamma\Delta t} P^*_t(\widehat{Y}^{*t}, r^t)\left[C^*_t(Y^t, r^t) + \Delta\theta_t + p^*_t(\widehat{Y}^{*t}, r^t)A_t\right]|r_0\right]$$

As we will see, in the search for an optimal contract, we can focus our attention on the direct-revelation contracts with no private savings by the borrower. The reasoning behind this result is simple. Consider any contract such that the borrower's optimal response entails concealing cash flows ($\hat{Y}_t \leq Y_t$). We can design a new contract in which the borrower gives the diverted cash flows to the lender ($\hat{Y}_t = Y_t$), and the lender then pays the borrower (through I_t) an amount equal to $(Y_t - \hat{Y}_t)$. Similarly, rather than privately save, the borrower can give income to the lender, and receive it back in the future with interest ρ_t. This leaves the borrower's utility unchanged, but leads to a weakly higher payoff for the lender since private savings are inefficient (as $\rho_t \leq r_L < r_H$). This leads us to the following result.

Lemma 4 *Let $(\zeta, C, \hat{Y})$ be an incentive compatible contract with an borrower's strategy. Then there exists an incentive compatible contract with an borrower's strategy $(\zeta', C', \hat{Y}')$ such that:*

- *the borrower's payoff under the contract ζ' is the same as under the contract ζ,*
- *the lender's payoff under the contract ζ' is weakly higher from those under the contract ζ,*
- *the borrower reports its income truthfully and maintains no savings under the contract ζ'.*

Recursive Formulation of the Contracting Problem

We formulate recursively the contracting problem using dynamic programming approach similarly to DeMarzo and Fishman (2004). First, we characterize an optimal contract under the assumption that private saving is impossible. We know from Lemma 4, that it is sufficient to look for the optimal contracts in which the borrower reports truthfully and maintains zero savings. But this implies that the optimal contract in the environment with no private savings yields to the lender, for a given promise to the borrower, at least as much profit as the optimal contract of the problem when borrower is allowed to privately save. Finally, we will show that, given the optimal contract of the relaxed problem, the borrower has no incentive to save at the solution, and thus this contract is also optimal in the environment with private savings by the borrower.

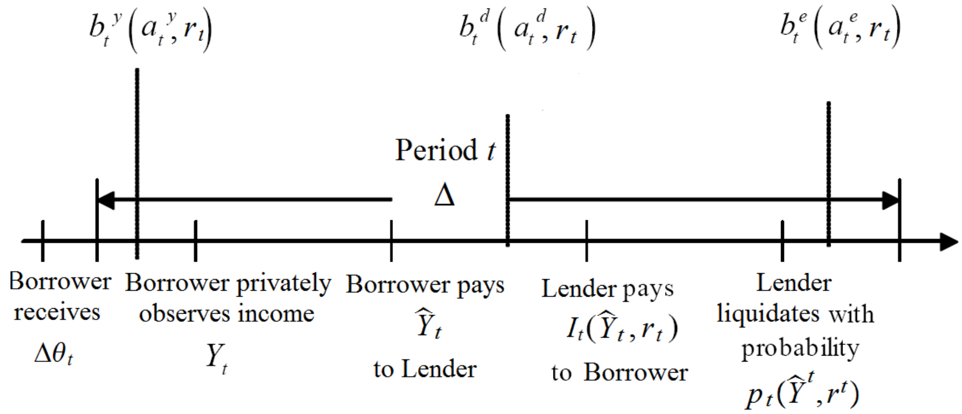
Consider the subgame that begins at the end of period t in which interest for the lender equals r_t. For this subgame, let ζ_t denote a contract governing the relationship between lenders and the borrower, and let $\hat{Y}_t$ be a report strategy for the borrower, A_t and B_t be the continuation payoffs, respectively, to the borrower, and to the lender. Define $\Gamma_t(r_t)$ to be the set of incentive compatible contract strategy pairs. Because cash flows are independent over time, and for a moment we assume there is no private savings, the set of incentive compatible contract strategy pairs $\Gamma_t(r_t)$ are common knowledge at time t and independent of the prior history. This allows us, along the lines of Spear and Srivastava (1987), Green (1987), and Abreu et al. (1990), to characterize the optimal contract by using the borrower's continuation utility as a state variable.

Define the payoff possibility set as

$$\chi_t(r_t) = \left\{ (a,b) \,\middle|\, a = A_t(\zeta_t, \hat{Y}_t),\ b = A_t(\zeta_t, \hat{Y}_t) \text{ for some } (\zeta_t, \hat{Y}_t) \in \Gamma_t(r_t) \right\}.$$

This set describes the payoff combinations that can be achieved by operating the project beyond date t. The payoff combinations, corresponding to optimal contracts, are on the frontier of this set, and can be described by the end-of-period continuation payoff function, that is the continuation payoff for the lender, as a function

Figure 11: Continuation functions.



of the borrower's continuation payoff

$$b_t^e(a, r_t) = \max\{b \,|(a, b) \in \chi_t(r_t)\}$$

For each period, we consider the start-of-period, the intra-period (just prior to the termination decision), and the end-of-period continuation functions for the lender, denoted by, respectively, $b_t^y(a_t^y(r_t), r_t)$, $b_t^d(a_t^d(r_t), r_t)$, $b_t^e(a_t^e(r_t), r_t)$, where $a_t^y(r_t), a_t^d(r_t)$, and $a_t^e(r_t)$ denote the borrower's continuation payoffs at the start, the middle, and the end of period $t \leq T$, respectively, as shown in Figure 11.

Derivation of the Optimal Contract

Here, we present an algorithm that will allow us to solve for the optimal contract, given that the project lasts up to a finite time T. This allows us to define the continuation function for date T, and then solve for earlier dates recursively. The properties of the optimal contract in finite time will carry over to an infinitely-lived relationship.[16] The algorithm consists of the following three steps.

Terminal Value

First, we note that, after time T, any payments to the borrower are transfers from the lender. Since $\gamma \geq r_t$ for all t, it is weakly efficient to make such payments immediately. Hence the continuation function at the end of the last period is given by

$$b_T^e(a_T^e(r_T), r_T) = \begin{cases} -a_T^e(r_T) & \text{for } a_T^e \geq A_T \\ -\infty & \text{for } a_T^e < A_T \end{cases} . \tag{89}$$

[16] In the stationary setting the optimal infinite horizon contract can be derived by taking the limit as $T \to \infty$.

Given the continuation function (89), we will work backwards to determine recursively continuation function $b_t^e(\cdot, r_t)$ for dates $t < T$. Note that, for $a_T^e(r_T) \geq A_T$, the continuation function $b_T^e(\cdot, r_T)$ is decreasing and concave (it is a linear function). In what follows we will show inductively that $b_t^e(\cdot, r_t)$ is concave as well.

Step One: Liquidation Problem

Assume that $b_t^e(\cdot, r_t)$ is concave (note that the payoff function at the end of last period, $b_T^e(\cdot, r_T)$, is indeed concave). At any time t the borrower has reservation value A_t. Thus prior to the termination decision the lowest feasible payoff for the borrower is equal to A_t. So $b_t^d(\cdot, r_t)$ will be defined for $a_t^d(r_t) \geq A_t$. Also note that because the contract may terminate probabilistically at any t, all payoff within convex hull of (L_t, A_t) and the payoff possibilities defined by $b_t^e(\cdot, r_t)$ are possible. Consider a line passing by point (L_t, A_t), which is tangent to payoff possibility frontier given by $b_t^e(\cdot, r_t)$. Let $a_t^L(r_t)$ be the payoff at the point of tangency of this line. An borrower's payoff level $a_t^d(r_t) \in [A_t, a_t^L(r_t)]$ is achieved by terminating with probability

$$p_t(a_t^d(r_t), r_t) = \frac{a_t^L(r_t) - a_t^d(r_t)}{a_t^L(r_t) - A_t} \tag{90}$$

If the borrower's payoff a_t^d is larger than $a_t^L(r_t)$ it is optimal to continue with this payoff, so the probability of termination is zero in this case.

Note that paying one dollar to an borrower costs lender one dollar. Since $b_t^e(\cdot, r_t)$ is assumed to be concave, there will be a threshold level of the borrower's payoff $a_t^1(r_t)$, such that cash payments will be used above this threshold. This threshold point $a_t^1(r_t)$ is the point in which the slope of the continuation function $b_t^e(\cdot, r_t)$ is below -1. That implies the following characterization of the payment function

$$I_t(a_t^d(r_t), r_t) = \max(a_t^d(r_t) - a_t^1(r_t), 0) \tag{91}$$

Note that as a result of these transformations, as Figure 12 indicates, the continuation payoff function has the following properties

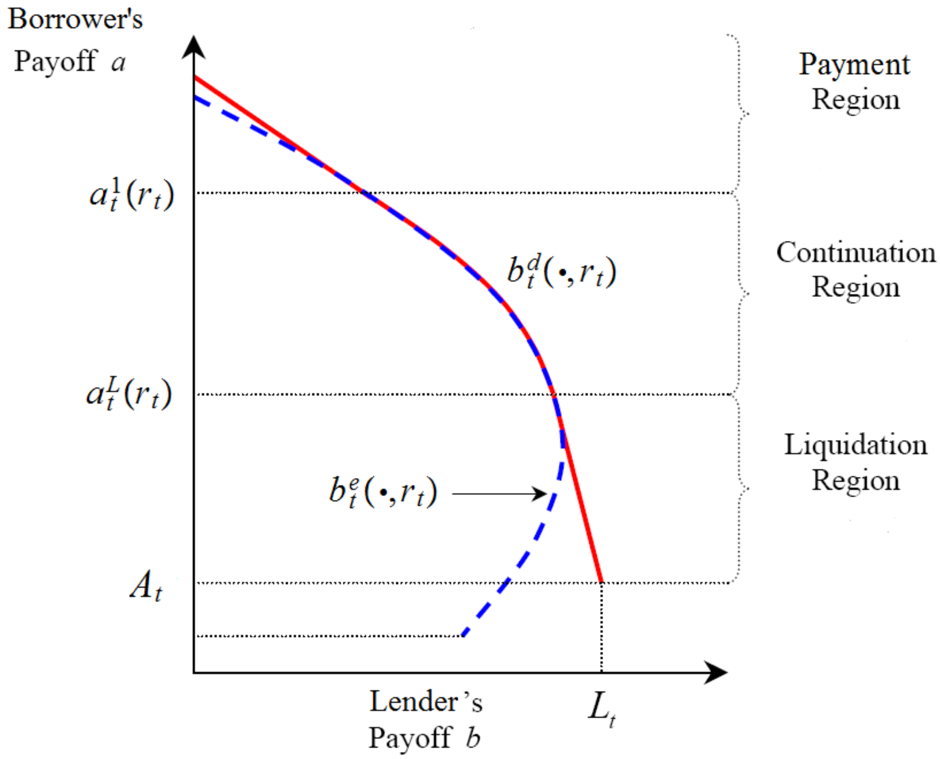
$$b_t^d(\cdot, r_t) \text{ is concave with } \frac{db_t^d(a, r_t)}{da} \geq -1 \tag{92}$$

In the final period, by definition, termination is optimal. In this case we set $a_T^L(r_t) = -\infty$. Also because $\gamma \geq r_t$ for all t, it is weakly efficient to make any payments immediately. Thus, $a_T^1(r_T) = A_T$, and as a result we have that $p_T(a_T^d(r_T), r_T) = 1$, $I_T(a_T^d(r_T), r_T) = a_T^d(r_T) - A_t$, and $b_T^d(a_T^d, r_T) = L_T - (a_T^d(r_T) - A_T)$. We summarize our findings in the proposition below.

Proposition 11 *Given $b_t^e(\cdot, r_t)$ concave, let*

$$l_t(r_t) = \sup\left\{\frac{b_t^e(a, r_t) - L_t}{a - A_t} : a \geq A_t\right\}.$$

Figure 12: Continuation payoff function $b_t^d(\cdot, r_t)$.



Then, if $l_t(r_t) > -1$, *define :*

$$\begin{aligned} a_t^L(r_t) &= \inf\left\{a > 0 : \frac{db_t^e(a, r_t)}{da} \leq l_t(r_t)\right\}, \\ a_t^1(r_t) &= \inf\left\{a > 0 : \frac{db_t^e(a, r_t)}{da} \leq -1\right\}. \end{aligned}$$

and then we have that

$$b_t^d(a_t^d(r_t), r_t) = \begin{cases} b_t^e(a_t^1(r_t), r_t) - (a_t^d(r_t) - a_t^1(r_t)) & \textit{for } a_t^d(r_t) \geq a_t^1(r_t) \\ b_t^e(a_t^d, r_t) & \textit{for } a_t^L(r_t) \leq a_t^d(r_t) < a_t^1(r_t) \\ b_t^e(a_t^L(r_t), r_t) - l_t(r_t)(a_t^L(r_t) - a_t^d(r_t)) & \textit{for } A_t \leq a_t^d(r_t) < a_t^L(r_t) \\ -\infty & \textit{for } a_t^d(r_t) < A_t \end{cases},$$

If $l_t(r_t) \leq -1$, *the termination of the contract is optimal. In this case, define* $a_t^L(r_t) = \infty$, $a_t^1(r_t) = A_t$ *and*

$$b_t^d(a_t^d(r_t), r_t) = \begin{cases} L_t - (a_t^d(r_t) - A_t) & \textit{for } a_t^d(r_t) \geq A_t \\ -\infty & \textit{for } a_t^d(r_t) < A_t \end{cases}$$

Finally, we note that the above implies that $b_t^d(\cdot, r_t)$ satisfies (92).

Proof We construct $b_t^d(\cdot, r_t)$ as in Figure 12. First, we consider the termination option (L_t, A_t). Since the borrower can always terminate and receive A_t, payoffs below this are infeasible: $b_t^d(\cdot, r_t) = -\infty$ for $a < A_t$. For payoffs above A_t, we need to find the line from (L_t, A_t) to the curve $b_t^e(\cdot, r_t)$ with the highest slope. This highest slope is given by $l_t(r_t)$.

If $l_t(r_t) > -1$ the line with the highest slope connects to $b_t^e(\cdot, r_t)$ at $a_t^L(r_t)$. Thus, payoffs $a_t^d(r_t) \in [A_t, a_1^L(r_t)]$ can be achieved by mixing between the borrower's termination value of A_t and the minimal value of continuation with $a_1^L(r_t)$. The probability of termination, $p_t(a_t^d(r_t), r_t)$, is given by (90) which solves

$$p_t A_t + (1 - p_t) a_t^L(r_t) = a_t^d(r_t)$$

In this case the lenders' expected payoff is

$$p_t(a_t^d(r_t), r_t) L_t + (1 - p_t(a_t^d(r_t), r_t)) b_t^e(a_t^L(r_t), r_t) =$$
$$b_t^e(a_t^L(r_t), r_t) + \frac{a_t^L(r_t) - a_t^d(r_t)}{a_t^L(r_t) - A_t}(L_t - b_t^e(a_t^L, r_t)) = b_t^e(a_t^L(r_t), r_t) - l_t(r_t)\left(a_t^L(r_t) - a_t^d(r_t)\right)$$

Since we assumed that $l_t(r_t) > -1$, there is $a_t^1(r_t) \geq a_t^L(r_t)$, such that, above $a_t^1(r_t)$ it is cheaper to compensate the borrower directly with the payment $I_t(a_t^d(r_t), r_t)$ given by (91). In this case an lender's payoff is

$$b_t^d(a_t^d(r_t), r_t) = b_t^e(a_t^1(r_t), r_t) - I_t(a_t^d(r_t), r_t) = b_t^e(a_t^1(r_t), r_t) - (a_t^d(r_t) - a_t^1(r_t))$$

In the region $\left[a_t^L(r_t), a_t^1(r_t)\right]$, it is efficient not to pay the borrower. Note that for $a_t^d(r_t) \geq a_t^L(r_t)$, $p_t(a_t^d(r_t), r_t) = 0$.

Now suppose that $l_t(r_t) \leq -1$. In this case paying the borrower at once is cheaper for any payoff above A_t. Therefore, it is optimal to terminate with probability 1, which corresponds to setting $a_t^L(r_t) = \infty$ in (90), and lender's payoff is

$$b_t^d(a_t^d(r_t), r_t) = L_t - (a_t^d(r_t) - A_t)$$

for $a_t^d(r_t) \geq A_t$. The above properties imply that $b_t^d(r_t)$ satisfies (92). ■

Step Two: The Intra-Period Agency Problem

In this subsection, we solve for the continuation function, $b_t^y(\cdot, r_t)$, before the income Y_t is realized, given the continuation function $b_t^d(\cdot, r_t)$. To do so, we must make sure that the borrowers is provided with incentives to reveal its income to the lender. Consider the borrower's problem at the start of period t. After realizing the cash flow Y_t, the borrower must choose a report $\hat{Y}_t$ to make. Contingent upon report $\hat{Y}_t$, according to the contract, the borrower will receive a continuation payoff $a_t^d(r_t)$. An optimal contract specifies the continuation payoff a_t^d as a function of the reported income so as to provide incentives for the borrower to report truthfully (that is to choose $\hat{Y}_t = Y_t$). This must be done in a way that maximizes the lender's expected payoff. Formally, this problem is equivalent to the following optimization problem:

$$\begin{aligned} b_t^y(a_t^y(r_t), r_t) = & \max_{a_t^d(r_t) \geq A_t} E[Y_t + b_t^d(a_t^d(r_t), r_t)] \;\; s.t. \\ & (IC): \; a_t^d(r_t, Y_t) \geq a_t^d(r_t, \hat{Y}_t) + (Y_t - \hat{Y}_t) \text{ for any } \hat{Y}_t \leq Y_t, \; \widehat{Y}_t \in \mathcal{Y}_t \\ & (PK): E[a_t^d(r_t, Y_t)] = a_t^y(r_t) \end{aligned} \tag{93}$$

The objective function is the expected payoff of lender at the beginning of period t. The lender receives the cash flows and the highest possible continuation payoff $b_t^d(\cdot, r_t)$ given that the borrower receives $a_t^d(r_t)$. The first constraint, (IC), is the incentive compatibility constraint for the borrower. It insures that it is optimal for the borrower to report all its income to the lender, rather than under-report and consume some of the cash flow himself. The second constraint, (PK), is the "promise-keeping" constraint. This constraint guarantees the borrower's expected continuation payoff is consistent with his promised continuation payoff $a_t^y(r_t)$ at the beginning of the period t.

To solve (93), we first note that the (IC) constraint is equivalent to $a_t^d(r_t, Y) - Y$ being weakly increasing in Y. The promise-keeping constraint fixes the mean payoff to the borrower, so different choices of $a_t^d(r_t)$ affect its variability. But as $b_t^d(\cdot, r_t)$ is concave, it is optimal to minimize the variability of $a_t^d(r_t, y)$. This is done by setting $\frac{da_t^d(r_t,y)}{dy} = 1$, so that the incentive constraints just bind. We summarize this discussion formally in the proposition below.

Proposition 12 *Given $b_t^d(\cdot, r_t)$, which satisfies (92), the optimal continuation payoff for the borrower contingent on the reported income Y_t is given by*

$$a_t^d(r_t, Y_t) = a_t^y(r_t) + (Y_t - \Delta\mu_t)$$

This implies that the beginning of period continuation function, b_t^d, equals

$$b_t^y(a_t^y(r_t), r_t) = \Delta\mu_t + E[b_t^d\left(a_t^y(r_t) + (Y_t - \Delta\mu_t), r_t\right)],$$

and $b_t^y(\cdot, r_t)$ is concave.

Proof We start by verifying that it is without loss of generality to assume that the borrower reveals its entire income Y_t at the solution. Suppose there is a solution in which the borrower reveals $Y_t^0 \leq \hat{Y}_t(r_t, Y_t) \leq Y_t$ and receives $a_t^d(r_t, \hat{Y}_t)$. Consider the new continuation payoff $a_t^{d*}(r_t, Y_t) = a_t^d(r_t, \widehat{Y}_t(r_t, Y_t)) + Y_t - \hat{Y}_t(r_t, Y_t)$. Given this continuation payoff, it is easy to see that truthful reporting is optimal, and the borrower's payoffs are unchanged. The change in the lender's payoff is

$$[Y_t + b_t^d(a_t^{d*}(r_t, Y_t), r_t)] - [\hat{Y}_t + b_t^d(a_t^d(r_t, \widehat{Y}_t), r_t)] =$$
$$Y_t - \widehat{Y}_t + b_t^d(a_t^d(r_t, \widehat{Y}_t) + Y_t - \hat{Y}_t(r_t, Y_t), r_t) - b_t^d(a_t^d(r_t, \hat{Y}_t), r_t) \geq 0$$

where the last inequality follows since $\frac{db_t^d(a, r_t)}{da} \geq -1$.

Note that, given truthful reporting, the (IC) constraint is equivalent to $g(Y, r_t) = a_t^d(Y, r_t) - Y$ being weakly increasing in Y. The promise keeping constraint, (PK), then becomes

$$E[g(Y_t, r_t)] = a_t^y(r_t) - \Delta\mu_t$$

and the continuation payoff at the beginning of period t is

$$E[Y_t + b_t^d(Y_t + g(Y_t), r_t)] = \Delta\mu_t + E[b_t^d(Y_t + g(Y_t), r_t)]$$

As the mean of function $g(\cdot, r_t)$ is fixed by (PK), and since b_t^d is concave, the optimal choice of $g(\cdot, r_t)$ is to minimize the variability of $Y_t + g(Y_t, r_t)$ subject to the constraint that $g(\cdot, r_t)$ is weakly increasing. The solution of this problem is to make $g(\cdot, r_t)$ constant and equal to its mean $a_t^y(r_t) - \Delta\mu_t$. Therefore,we have that

$$a_t^d(Y, r_t) = a_t^y(r_t) + (Y - \Delta\mu_t)$$

Finally, we observe that, as the expectation operator is a linear operator, the concavity of $b_t^d(\cdot, r_t)$ implies the concavity of $b_t^y(\cdot, r_t)$. ■

We note that the above result implies that, the borrower has no incentive to use private savings, justifying our solution methodology. As the above proposition shows, the marginal benefit to the borrower from reporting a higher cash flow is constant. As a result, since for any t the borrower's discount rate, γ, exceeds the return to private savings, ρ_t, there is no benefit to hiding cash flows today in order to report higher cash flows in the future (or increase future consumption).

Lemma 5 *Function $b_t^y(\cdot, r_t)$ satisfies*

$$\frac{db_t^y(\cdot, r_t)}{da} \geq -1 \tag{94}$$

Proof From Proposition 12 we have that $b_t^y(a_t^y(r_t), r_t) = \Delta\mu_t + E[b_t^d\,(a_t^y(r_t) + (Y_t - \Delta\mu_t), r_t)]$. This combined with (92) and the properties of the expectation operator yields (94). ■

Step Three: Discounting Between the Periods

So far, we have described how to compute the continuation function $b_t^y(\cdot, r_t)$ at the start of period t given the continuation function $b_t^e(\cdot, r_t)$ at the end of period t. In this subsection, we derive the continuation function at the end of the prior period, $b^e_{t-1}(\cdot, r_{t-1})$, and combine our results to complete our recursive characterization of the optimal contract.

Moving from the start of period t to the end of the prior period, $t-1$, involves discounting the payoffs of the borrower and the lender. To provide the borrower with a payoff of a at the end of period $t-1$, the borrower's expectation of what he is to be given, at the beginning of period t, must be equal the value of a plus increase at his subjective rate γ less the amount of deterministic utility, $\Delta\theta_t$, he derives at the beginning of period t from home ownership. The lender's continuation payoff at the end of time $t-1$ is the expected continuation payoff at time t discounted at rate r_{t-1}. Therefore, given the end of period continuation value $a^e_{t-1}(r_{t-1})$, the lender optimally chooses the beginning of next period continuation values for the borrower $a^y_t(r_t)$ in order to maximize the expected discounted continuation payoff function, subject to keeping his promises to the borrower. From Proposition 12 we know that $a^d_t(r_t, Y) = a^y_t(r_t) + (Y - \Delta\mu_t)$. As $a^d_t(r_t) \geq A_t$, it implies that, in choosing $a^y_t(r_t)$, the lender will have to satisfy the following condition

$$a^y_t(r_t) \geq A_t - (Y^0_t - \Delta\mu_t)$$

Our discussion above implies the following characterization of continuation function $b^e_{t-1}(\cdot, r_{t-1})$.

Proposition 13 *Given $b^y_t(\cdot, r_t)$, $a^e_{t-1}(r_{t-1}) \in \left[a^L_{t-1}(r_{t-1}), a^1_{t-1}(r_{t-1})\right]$, and r_{t-1}, the continuation function at the end of the prior period, $b^e_{t-1}(a^e_{t-1}(r_{t-1}), r_{t-1})$, is given by*

$$b^e_{t-1}(a^e_{t-1}(r_{t-1}), r_{t-1}) = \max_{a^y_t(a^e_{t-1}(r_{t-1}), r_{t-1}, r_t) \geq A_t - (Y^0_t - \Delta\mu_t)} e^{-\Delta r_{t-1}} E\left[b^y_t(a^y_t(r_t), r_t)\,|r_{t-1}\right]$$

$$\textit{subject to}$$

$$E(a^y_t(r_t)\,|r_{t-1}) = e^{\Delta\gamma} a^e_{t-1}(r_{t-1}) - \Delta\theta_t$$

Proof Immediate from the above discussion. ■

Lemma 6 *The end of period continuation function $b^e_{t-1}(\cdot, r_{t-1})$ is concave.*

Proof From Proposition 13 we know that $a^e_{t-1}(r_{t-1}) \in \left[a^L_{t-1}(r_{t-1}), a^1_{t-1}(r_{t-1})\right]$. Let $\bar{a}^y_t(a^e_{t-1}(r_{t-1}), r_{t-1}, r_t)$, for $r_t \in \{r_L, r_H\}$, be the solution to the problem defined in Proposition 8. We consider two cases:

Case 1: Suppose that the solution is interior, that is for $r_t \in \{r_L, r_H\}$, we have that

$$\bar{a}^y_t(a^e_{t-1}(r_{t-1}), r_{t-1}, r_t) > A_t - (Y^0_t - \Delta\mu_t).$$

Let $r_{t-1} = r_i$, where $i \in \{L, H\}$. Let $r_{t-1} = r_i$, where $i \in \{L, H\}$. From the promise keeping constraint

$$(1 - e^{-\Delta\delta(r_i)}) a^y_t(r_{-i}) + e^{-\Delta\delta(r_i)} a^y_t(r_i) = e^{\Delta\gamma} a^e_{t-1}(r_i) - \Delta\theta_t \tag{95}$$

we have that

$$a_t^y(r_{-i}) = \frac{e^{\Delta\gamma} a_{t-1}^e(r_i) - \Delta\theta_t - e^{-\Delta\delta(r_i)}\overline{a}_t^y(r_i)}{(1-e^{-\Delta\delta(r_i)})} \tag{96}$$

Using this we can express $b_{t-1}^e(a_{t-1}^e(r_{t-1}), r_{t-1})$ as

$$b_{t-1}^e(a_{t-1}^e(r_i), r_i) =$$
$$\max_{a_t^y(r_t)\geq A_t-(Y_0-\Delta\mu_t)} e^{-\Delta r_i} \left[\begin{array}{c} e^{-\Delta\delta(r_i)} b_t^y(a_t^y(r_i), r_i) + \\ (1-e^{-\Delta\delta(r_i)}) b_t^y\left(\frac{e^{\Delta\gamma} a_{t-1}^e(r_i) - \Delta\theta_t - e^{-\Delta\delta(r_i)}\overline{a}_t^y(r_i)}{(1-e^{-\Delta\delta(r_i)})}, r_{-i}\right) \end{array} \right]$$

subject to $\frac{e^{\Delta\gamma} a_{t-1}^e(r_i) - \Delta\theta_t - e^{-\Delta\delta(r_i)}\overline{a}_t^y(r_i)}{(1-e^{-\Delta\delta(r_i)})} \geq A_t - (Y_0 - \Delta\mu_t)$. Suppose that at the optimal choice the appropriate derivatives exist and so we have that

$$\frac{db_t^y(\overline{a}_t^y(r_i), r_i)}{da} = \frac{db_t^y\left(\frac{e^{\Delta\gamma} a_{t-1}^e(r_i) - \Delta\theta_t - e^{-\Delta\delta(r_i)}\overline{a}_t^y(r_i)}{(1-e^{-\Delta\delta(r_i)})}, r_{-i}\right)}{da}. \tag{97}$$

Now we note that

$$\frac{db_{t-1}^e(a_{t-1}^e(r_i), r_i)}{da_{t-1}^e(r_i)} = e^{-\Delta(r_i+\delta(r_i))} \frac{db_t^y(\overline{a}_t^y(r_i), r_i)}{da} \frac{d\overline{a}_t^y(r_i)}{da_{t-1}^e(r_i)}$$
$$+ \left[e^{\Delta(\gamma - r_i)} - e^{-\Delta(r_i+\delta(r_i))} \frac{d\overline{a}_t^y(r_i)}{da_{t-1}^e(r_i)} \right] \frac{db_t^y\left(\frac{e^{\Delta\gamma} a_{t-1}^e(r_i) - \Delta\theta_t - e^{-\Delta\delta(r_i)}\overline{a}_t^y(r_i)}{(1-e^{-\Delta\delta(r_i)})}, r_{-i}\right)}{da}.$$

Using (97) in the above implies that

$$\frac{db_{t-1}^e(a_{t-1}^e(r_i), r_i)}{da_{t-1}^e(r_i)} = e^{\Delta(\gamma - r_i)} \frac{db_t^y\left(\frac{e^{\Delta\gamma} a_{t-1}^e(r_i) - \Delta\theta_t - e^{-\Delta\delta(r_i)}\overline{a}_t^y(r_i)}{(1-e^{-\Delta\delta(r_i)})}, r_{-i}\right)}{da}.$$

From the above we have that

$$\frac{d^2 b_{t-1}^e(a_{t-1}^e(r_i), r_i)}{d\left(a_{t-1}^e(r_i)\right)^2} =$$
$$e^{\Delta(\gamma - r_i)} \frac{d^2 b_t^y\left(\frac{e^{\Delta\gamma} a_{t-1}^e(r_i) - \Delta\theta_t - e^{-\Delta\delta(r_i)}\overline{a}_t^y(r_i)}{(1-e^{-\Delta\delta(r_i)})}, r_{-i}\right)}{da^2} \left[e^{\Delta\gamma} + e^{-\Delta\delta(r_i)} \frac{d\overline{a}_t^y(r_i)}{da_{t-1}^e(r_i)} \right] \tag{98}$$

It follows from the properties of functions $b_t^y(\cdot, r_t)$, and from (95), that

$$\frac{d\overline{a}_t^y(r_i)}{da_{t-1}^e(r_i)} \leq \frac{e^{\Delta\gamma}}{e^{-\Delta\delta(r_i)}}.$$

But this combined with the condition (98) and the fact that $b_t^y(\cdot, r_t)$ is concave implies that

$$\frac{d^2 b_{t-1}^e(a_{t-1}^e(r_i), r_i)}{d\left(a_{t-1}^e(r_i)\right)^2} \leq 0$$

so we conclude that $b_{t-1}^e(a_{t-1}^e(r_{t-1}), r_{t-1})$ is concave for $a_{t-1}^e(r_{t-1})$ and r_{t-1} such that $\bar{a}_t^y(r_t) > A_t - (Y_t^0 - \Delta\mu_t)$ for $r_t \in \{r_L, r_H\}$ if at the optimal choice the appropriate derivatives exist.

If the appropriate derivatives do not exist around the solution then as we increase $a_{t-1}^e(r_i)$ either $\bar{a}_t^y(r_i)$ or $\bar{a}_t^y(r_{-i})$ must increase. Without loss in generality suppose that it is $\bar{a}_t^y(r_i)$ that locally increases. Then around the solution we will have that

$$\begin{aligned} \frac{d\bar{a}_t^y(a_{t-1}^e(r_i), r_i, r_{-i})}{da_{t-1}^e(r_i)} &= 0, \\ \frac{d\bar{a}_t^y(a_{t-1}^e(r_i), r_i, r_i)}{da_{t-1}^e(r_i)} &= \frac{e^{\Delta\gamma}}{e^{-\Delta\delta(r_i)}}. \end{aligned}$$

But then

$$\begin{aligned} \frac{db_{t-1}^e(a_{t-1}^e(r_i), r_i)}{da_{t-1}^e(r_i)} &= e^{-\Delta(r_i+\delta(r_i))} \frac{db_t^y(\bar{a}_t^y(a_{t-1}^e(r_i), r_i, r_i), r_i)}{da} \frac{d\bar{a}_t^y(a_{t-1}^e(r_i), r_i, r_i)}{da_{t-1}^e(r_i)} \\ &= e^{\Delta(\gamma - r_i)} \frac{db_t^y(\bar{a}_t^y(a_{t-1}^e(r_i), r_i, r_i), r_i)}{da}. \end{aligned}$$

Since $\bar{a}_t^y(a_{t-1}^e(r_i), r_i, r_i)$ increases in $a_{t-1}^e(r_i)$ and $\frac{db_t^y(\cdot, r_i)}{da}$ decreases in a, $\frac{db_{t-1}^e(a_{t-1}^e(r_i), r_i)}{da_{t-1}^e(r_i)}$ decreases in $a_{t-1}^e(r_i)$. The same argument can be repeated for the case when it is $\bar{a}_t^y(r_{-i})$ that increases with $a_{t-1}^e(r_i)$ around the solution when the appropriate derivatives do not exist.

But this altogether implies that the value function for the lender is concave in a whenever $\bar{a}_t^y(a_{t-1}^e(r_{t-1}), r_{t-1}, r_t) > A_t - (Y_t^0 - \Delta\mu_t)$, $r_t \in \{r_L, r_H\}$.

Case 2: Suppose that the solution is not interior, that is we have that $\bar{a}_t^y(r_t) = A_t - (Y_t^0 - \mu_t)$ for some $r_t \in \{r_L, r_H\}$. First suppose that in the neighborhood of $a_{t-1}^e(r_i)$, $\bar{a}_t^y(r_{-i}) = A_t - (Y_t^0 - \mu_t)$. Then we have that

$$\begin{aligned} \frac{d\bar{a}_t^y(r_i)}{da_{t-1}^e(r_i)} &= \frac{e^{\Delta\gamma}}{e^{-\Delta\delta(r_i)}}, \\ \frac{d\bar{a}_t^y(r_{-i})}{da_{t-1}^e(r_i)} &= 0. \end{aligned}$$

But then

$$\frac{db_{t-1}^e(a_{t-1}^e(r_i), r_i)}{da_{t-1}^e(r_i)} = e^{-\Delta(r_i+\delta(r_i))} \frac{db_t^y(\bar{a}_t^y(r_i), r_i)}{da} \frac{d\bar{a}_t^y(r_i)}{da_{t-1}^e(r_i)} = e^{\Delta(\gamma - r_i)} \frac{db_t^y(\bar{a}_t^y(r_i), r_i)}{da}.$$

From the above we have that

$$\frac{d^2 b_{t-1}^e(a_{t-1}^e(r_i), r_i)}{d\left(a_{t-1}^e(r_i)\right)^2} = e^{\Delta(\gamma - r_i)} \frac{e^{\Delta\gamma}}{e^{-\Delta\delta(r_i)}} \frac{d^2 b_t^y(\bar{a}_t^y(r_i), r_i)}{da^2} \leq 0, \tag{99}$$

where the last inequality follows from the concavity of function $b_t^y(\cdot, r_t)$.

If in the neighborhood of $a_{t-1}^e(r_i)$, $\bar{a}_t^y(r_i) = A_t - (Y_t^0 - \mu_t)$, we have that

$$\begin{aligned} \frac{d\bar{a}_t^y(r_i)}{da_{t-1}^e(r_i)} &= 0, \\ \frac{d\bar{a}_t^y(r_{-i})}{da_{t-1}^e(r_i)} &= \frac{e^{\Delta\gamma}}{1 - e^{-\Delta\delta(r_i)}}. \end{aligned}$$

But then

$$\frac{db_{t-1}^e(a_{t-1}^e(r_i), r_i)}{da_{t-1}^e(r_i)} = (1 - e^{-\Delta\delta(r_i)}) e^{-\Delta r_i} \frac{db_t^y(\bar{a}_t^y(r_{-i}), r_{-i})}{d\bar{a}_t^y(r_{-i})} \frac{d\bar{a}_t^y(r_{-i})}{da_{t-1}^e} = e^{\Delta(\gamma - r_i)} \frac{db_t^y(\bar{a}_t^y(r_{-i}), r_{-i})}{d\bar{a}_t^y(r_{-i})}.$$

From the above we have that

$$\frac{d^2 b_{t-1}^e(a_{t-1}^e(r_i), r_i)}{d\left(a_{t-1}^e(r_i)\right)^2} = e^{\Delta(\gamma - r_i)} \frac{e^{\Delta\gamma}}{1 - e^{-\Delta\delta(r_i)}} \frac{db_t^y(\bar{a}_t^y(r_{-i}), r_{-i})}{d\bar{a}_t^y(r_{-i})} \leq 0, \tag{100}$$

where the last inequality follows from the concavity of function $b_t^y(\cdot, r_t)$.

The properties (99)-(100) imply that $b_{t-1}^e(a_{t-1}^e(r_{t-1}), r_{t-1})$ is concave for a_{t-1}^e and r_{t-1} such that $\bar{a}_t^y(r_t) = A_t - (Y_t^0 - \Delta\mu_t)$ for some $r_t \in \{r_L, r_H\}$.

Combining our discussion of Case 1 and Case 2, we conclude that the end of period continuation function $b_{t-1}^e(\cdot, r_{t-1})$ is concave. ■

Starting from the terminal continuation b_T^e defined by (89), the Propositions 11-13 allow us to recursively solve for the continuation function at all earlier points in the contract. Note that as $b_T^e(\cdot, r_T)$ is concave, the Propositions 11-13, and Lemma 6 imply that functions $b_t^y(\cdot, r_t), b_t^d(\cdot, r_t), b_t^e(\cdot, r_t)$ are concave for all $t \leq T$, $r_t \in \{r_L, r_H\}$.

The Dynamics of the Optimal Contract

Having solved for the optimal contract recursively, we can now describe the dynamics of the optimal contract. Let $\bar{a}_t^y$ be the optimal function solving the problem of the Proposition 8. Define

$$\psi_t(a_{t-1}^e(r_{t-1}), r_{t-1}, r_t) = \bar{a}_t^y(a_{t-1}^e(r_{t-1}), r_{t-1}, r_t) - e^{\Delta\gamma} a_{t-1}^e(r_{t-1}) + \Delta\theta_t$$

From the definition of $\overline{a}_t^y$ we have that

$$e^{-\Delta\delta(r_{t-1})}\psi_t(a_{t-1}^e(r_{t-1}), r_{t-1}, r_{t-1}) + (1 - e^{-\Delta\delta(r_{t-1})})\psi_t(a_{t-1}^e(r_{t-1}), r_{t-1}, r_{t-1}^c) = 0,$$

where we remember that $r_{t-1}^c = \{r_L, r_H\} \setminus \{r_{t-1}\}$. The above implies that

$$\psi_t(a_{t-1}^e(r_{t-1}), r_{t-1}, r_{t-1}) = -(e^{\Delta\delta(r_{t-1})} - 1)\psi_t(a_{t-1}^e(r_{t-1}), r_{t-1}, r_{t-1}^c).$$

The behavior of the contract is governed by the current promised continuation payoff for the borrower. From our above results, the evolution of this state variable prior to termination can be described as follows:

$$\begin{array}{lll}
a_{t-1}^e(r_{t-1}) & \rightarrow & \begin{array}{l} a_t^y(r_t = r_{t-1}) = e^{\Delta\gamma}a_{t-1}^e(r_{t-1}) - \Delta\theta_t - (e^{\Delta\delta(r_{t-1})} - 1)\psi_t(a_{t-1}^e(r_{t-1}), r_{t-1}, r_{t-1}^c) \\ a_t^y(r_t = r_{t-1}^c) = e^{\Delta\gamma}a_{t-1}^e(r_{t-1}) - \Delta\theta_t + \psi_t(a_{t-1}^e(r_{t-1}), r_{t-1}, r_{t-1}^c) \end{array} \\
 & & \\
a_t^y(r_t) & \rightarrow & a_t^d(r_t) = a_t^y(r_t) + (Y_t - \Delta\mu_t), \\
 & & \\
a_t^d(r_t) & \rightarrow & a_t^e(r_t) = \min(a_t^1(r_t), \max(a_t^L(r_t), a_t^d(r_t))).
\end{array}$$

Given the borrower's promised payoff, the payments to the borrower and the termination probability at each period t are given by

$$\begin{aligned}
I_t(a_t^d(r_t), r_t) &= \max(a_t^d(r_t) - a_t^1(r_t), 0), \\
p_t(a_t^d(r_t), r_t) &= \frac{\max(a_t^L(r_t) - a_t^d(r_t), 0)}{a_t^L(r_t) - A_t}.
\end{aligned}$$

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