Stalking the "Efficient Price" in Market Microstructure Specifications: An Overview

Joel Hasbrouck

May 25, 2000
This version: June 21, 2000

Preliminary Draft
Not for Quotation
Comments Welcome

Professor of Finance
Stern School of Business
New York University
Suite 9-190 Mail Code 0268
44 West Fourth St.
New York, NY 10012-1126

Tel: (212) 998-0310
Fax: (212) 995–4901
E-mail: jhasbrou@stern.nyu.edu

The most recent version of this paper and Matlab programs used in the analysis are available at http://www.stern.nyu.edu/~jhasbrou

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Abstract

The principle that revisions to the expectation of a security's value should be unforecastable identifies this expectation as a martingale. When price changes can plausibly be assumed covariance stationary, this in turn motivates interest in the random walk. In the presence of the market frictions featured in many microstructure models, however, this expectation does not invariably coincide with observed security prices such as trades and quotes. Accordingly, the random walk becomes an implicit, unobserved component. This paper is an overview of econometric approaches to characterizing this important component in single- and multiple-price applications.

1. Introduction

In modeling security price dynamics, the martingale and its statistically expedient variant, the random walk, have long figured prominently. Historically, the martingale and random-walk models have usually been employed in security price studies at daily and longer horizons. At long horizons, it is common (and often appropriate) to ignore the finer details of the trading process. A random-walk specification estimated directly for monthly closing stock prices, for example, is a reasonable point of reference or departure.

Over shorter intervals (e.g., trade-to-trade), however, market frictions and effects attributable to trading process often introduce short-term, transient effects. The most prominent example is perhaps the "bounce" arising from trades that randomly occur at the bid or ask prices. Such effects introduce dependencies into the price dynamics. In consequence the unadorned random walk is no longer an attractive or reasonable specification.

Despite this, it is grossly incorrect to conclude that martingales have no further relevance. Traders still rely on their beliefs about what the security is worth. In virtually all microstructure models, the expectation of terminal security value conditional on current information plays a crucial role. Broadly speaking, when the information set is increasingly refined ("a filtration") this (indeed, any) sequence of conditional expectations is a martingale. In recognition of this, many structural microstructure models are formed by adding trading-related effects to a random walk that is simply called the "efficient price". Consistent with this usage, the present paper will use the term "efficient price" to refer to a martingale expectation of future (perhaps terminal) prices.

When we introduce microstructure effects into a price model, we break the identity between the observed price and the underlying expected-value martingale. To address this difficulty, we need econometric approaches that allow us to characterize this structural, implicit, unobserved martingale. This note seeks to summarize and review such approaches.[1]

These approaches are mostly drawn from the econometric literature that focuses on *macroeconomic* time series. Statistical theorems and proofs are valid, of course, whether *t* indexes years or seconds. Nevertheless, model specifications and identification restrictions are often highly dependent on the assumed underlying structural economic model. Restrictions that are reasonable for macroeconomic time series might well be objectionable in analyses of microstructure data (and vice versa). There are no one-size-fits-all statistical models.

2. Random walk decompositions

a. Martingales in market microstructure

In the usual theoretical models, the martingale property of security prices is a manifestation of "market efficiency", a property of the dynamic economic equilibrium. Although the original formulations relied on frictionless perfect market assumptions, the martingale property is robust to certain imperfections that fall under the microstructure purview. Perhaps most importantly, asymmetric information does not inevitably cause non-martingale behavior (Easley and O'Hara (1987); Glosten and Milgrom (1985); Kyle (1985)).

The effects of fixed transaction costs, inventory control and price discreteness are more complicated. In the Roll (1984) model, for example, "bid-ask bounce" leads to transaction price changes that are negatively dependent. The model is typical, however, in that long-run changes in transaction prices are dominated by an implicit martingale, the midpoint of the bid and ask quotes (which is, in Roll's application, unobserved). The

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[1] Hamilton (1994) is a textbook reference for most of the standard time-series results invoked in this paper. Hasbrouck (1996) discusses related microstructure applications of this material.

question then becomes, how can we make inferences about the implicit, unobserved martingale, based on the observed data.

Analysis of similar situations in macroeconomic time series yields some useful results. We summarize these below, but first note a key assumption. In analyzing long-horizon data, the empirical studies of the martingale behavior generally estimate random-walk specifications. The random-walk is a martingale that possesses sufficient regularity (specifically stationary, independent increments) to be amenable to estimation on the basis of a single time series realization. We adopt a corresponding assumption here. That is, it will be assumed that price changes are covariance stationary and ergodic.

b. The univariate random-walk decomposition

Suppose that the (unobservable) efficient price, m_t, follows a random walk:

$$m_t = m_{t-1} + u_t \tag{1}$$

where $Eu_t = 0$; $Eu_t^2 = \sigma_u^2$; $Eu_t u_s = 0$ for $t \neq s$. The (observed) transaction price is equal to this plus a component that impounds various microstructure effects:

$$p_t = m_t + s_t \tag{2}$$

How might s_t be characterized? If we view s_t as arising from bid-ask bounce, discreteness, inventory effects and the like, the defining feature of s_t appears to be transience, i.e., an absence of permanent effect on prices. We formalize this by requiring s_t to be a zero-mean covariance-stationary process: $Es_t = 0$ and the autocovariances $Es_t s_{t-k}$ depend only on *k*.

It is perhaps unsurprising that this decomposition given in equation (2) is unidentified. For example, given that $p_t = 100$, there is nothing in the model that allows us to assert $m_t = 99$ and $s_t = 1$, as opposed to, say, $m_t = 102$ and $s_t = -2$. It turns out, however, that the properties of the *observed* price process do suffice to identify the *variance* of the random walk component, σ_u^2.

Specifically, consider the innovations (moving average) representation of Δp_t is:

$$\Delta p_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots \tag{3}$$

Since Δp_t is assumed to be covariance stationary, the MA representation exists by virtue of the Wold Theorem. In estimation situations, it is typically obtained by inverting estimates of the corresponding autoregressive model. It can be shown that the random-walk variance is given by:

$$\sigma_u^2 = \left(1 + \psi_1 + \psi_2 + \cdots\right)^2 \sigma_\varepsilon^2 \tag{4}$$

(See Watson (1986).) Intuitively, the random-walk variance is identified because in a long sample of data, price changes are dominated by the random-walk component.

Now while this is a useful result, one might well seek to go further. We therefore ask, are there economic features of microstructure models that support stronger identifying restrictions?

c. Analysis of two structural models

The macroeconomic literature identifies two identifying restrictions that are, in a sense, polar. These restrictions suggest economic microstructure models, which are described below.

A special case: The pure asymmetric information model

In the macroeconomic literature, it was first suggested that s_t be driven entirely by the random-walk innovation, u_t (Beveridge and Nelson (1981)). A microstructure model along these lines can be constructed as follows. First, drawing on the asymmetric information models, we assume that the efficient price is driven entirely by the trade direction

$$\begin{aligned} m_t &= m_{t-1} + u_t \\ q_t &= \begin{cases} -1, & \text{if the trade is a sale to the dealer} \\ +1, & \text{if the trade is a buy from the dealer} \end{cases} \\ u_t &= \lambda q_t \end{aligned} \tag{5}$$

The q_t are trade-direction indicators (identically and independently distributed with $\Pr\left(q_t = -1\right) = \Pr\left(q_t = +1\right) = 1/2$). The λ coefficient is a trade impact parameter that reflects what the dealer learns from the trade. Notice that there is no component to $\Delta m_t = u_t$ beyond that induced by trades: there is no non-trade "public" information.

Next, we assume that the dealer must recover a fixed cost of trading *c*, so that

$$p_t = m_t + c\,q_t \tag{6}$$

All randomness in this model derives from one variable (q_t). In consequence, strong identification is possible. The econometrician (or a market participant) who knows the price record $p_t, p_{t-1}, p_{t-2}, \ldots$ can compute m_t (and $s_t = c\,q_t$). It is emphasized here that s_t and u_t are perfectly (and positively) correlated.

A special case: a pure "public information" model

The assumption of perfect correlation between the m_t and s_t innovations is a strong restriction. Watson (1986) discusses the case where these two processes are completely *uncorrelated*. This is the case with the Roll (1984) model discussed above, wherein the trade direction is assumed independent of the efficient price increment. The model now comprises, however, two sources of randomness. In consequence, the econometrician (or market participant) can only estimate m_t subject to error, *even if the full price record* $\{\ldots, p_{t+1}, p_t, p_{t-1}, \ldots\}$ *is known.*

From a microstructure perspective, an either/or choice between these two polar identification restrictions is an unattractive one. Models with public and private information, that also feature trade-related transient price effects, are unlikely to fit neatly into either framework.

d. Permanent-transitory decompositions

The random-walk decomposition presented in equations (1) and (2) belongs to the broader class of permanent-transitory (PT) decompositions (Quah (1992)). The generality afforded by this broader class may be described as follows. A random walk (such as m_t in (1)) is said to be integrated of order one, denoted *I*(1). More generally, an *I*(1) time series is one that contains a random walk and possesses covariance-stationary first-differences. In equation (2), for example, p_t is also *I*(1). Analogously to the random-walk decomposition for the security price given in (2), a permanent-transitory decomposition is:

$$p_t = f_t + s_t \tag{7}$$

where f_t is $I(1)$ and s_t is covariance-stationary, that is, $I(0)$. The essential generalization here is that the permanent component f_t need not be a random walk. This is frequently useful in macroeconomic time series.

From a microstructure perspective, however, the usefulness of this generality is questionable. If f_t is not a random-walk, then it can't be a martingale, nor therefore can it be an unbiased conditional expectation of the security's eventual value. Nor will its variance generally equal the long-run variance of the security price. One cannot go so far as to say that such a component could never be of interest to a trader (or econometrician), but a justification could only be based on the particulars of well-defined structural model. Outside of such a model, it is difficult to conjecture why a permanent non-martingale price component warrants general interest.

Furthermore, the examples in the last section illustrate that identification of simple random-walk decompositions in microstructure settings is problematic. The additional generality implicit in a PT decomposition renders identification even more difficult.

3. Multiple prices and cointegration

a. Cointegration: a microstructure perspective

A single security often exhibits multiple "prices". The most recent transaction price, the bid quote, the offer quote, to say nothing of trade and quote prices that arise in different trading venues, can each lay a claim to being the "true price" (at least for some set of traders or potential traders). Indeed, most traders would prefer to form beliefs about the security from the full set of prices. The electronic trading screens available for many securities present such a set.

As in the one-price case, the trader (and the econometrician) is forming a belief about the security value. Such beliefs, if defined by a sequence of conditional expectations, must still evolve as a martingale. What is new here is that this expectation is now conditional on multiple prices. From an econometric perspective, we are trying to characterize a *single* implicit random walk that is common to all of the prices.

A statistical model of the joint behavior of such a set of prices must reflect two considerations. First, each price (considered individually) is $I(1)$ (contains a random walk). Second, pairs of prices are linked in the long run by arbitrage and/or equilibrium relationships. Therefore, any two prices will not arbitrarily diverge over time. These considerations imply that the set of prices embodies a single long-term component. Formally, we say that the set of prices is cointegrated.

Cointegration is related to, but quite distinct from, correlation. The daily high-water mark of the Hudson River at 96th Street is highly correlated with the measurement taken at 14th Street. But the two series of measurements are not cointegrated because neither is individually *integrated*. (Neither measure will tend to wander off over time without bound.) Moreover, cointegration is a stronger restriction than correlation. The daily price changes of Ford and GM are positively correlated due to common industry effects. But there are no obvious equilibrium or arbitrage relationships that tie the two firms together: one might eventually go bankrupt while the other thrives. On the other hand, the bid and offer quotes for GM *are* almost certainly cointegrated. The difference between these two prices (the ask less the bid) is the spread. The spread cannot go negative or explode without bound (given any reasonable economic model of competitive liquidity provision).

The analysis of macroeconomic time series has yielded many useful results in the specification and estimation of cointegrated models. Relative to macro applications, however, microstructure cointegrated models are usually much *simpler*. Macroeconomic analyses are complicated because the precise nature of the cointegrating relations is unknown. In the long run, for example, the proportion of consumption to national income is likely to be stationary. (That is, the log of consumption and the log of income are cointegrated.) But since there is no obvious way to specify the proportion a priori, it must be estimated. Microstructure models are more precise. Although the price of GM on the New York and Boston Exchanges might differ at any given time, it is sensible to assume that this difference is stationary. Formally, this identifies a basis for the cointegrating vectors.

b. A dual market example

Suppose that we are interested in a security that trades in two different markets. It is convenient to generalize the random-walk decomposition discussed in Section 2.b to allow for two prices: $p_t = \begin{bmatrix} p_{1t} & p_{2t} \end{bmatrix}'$. That these two prices are driven by the same implicit efficient price can be expressed as:

$$p_t = \begin{bmatrix} p_{1t} \\ p_{2t} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} m_t + \begin{bmatrix} s_{1t} \\ s_{2t} \end{bmatrix} \tag{8}$$

As in equation (1) above m_t follows a random walk. The price equation is a bivariate generalization of equation (2)

As in the univariate case, the variance of the random walk is identified. The moving average representation for Δp_t is notationally the same as the one given in equation (3):

$$\Delta p_t = \varepsilon_t + \psi_1 \varepsilon_{t-1} + \psi_2 \varepsilon_{t-2} + \cdots \tag{9}$$

Here, however, $\varepsilon_t = \begin{bmatrix} \varepsilon_{1t} & \varepsilon_{2t} \end{bmatrix}$ is a vector process, where the two components reflect the innovations ("new information") revealed in the first and second markets. The ψ_i are two-by-two matrices. This *vector* moving average (VMA) representation is usually obtained by inverting a vector error correction model.

Define the matrix sum of the moving average coefficients by $\psi(1) = I + \psi_1 + \psi_2 + \cdots$, where I is the 2×2 identity matrix. It is a property of this model that the rows of $\psi(1)$ are identical. The intuition here is that this sum reflects the impact of an initial disturbance on the long-run component which is common to all prices. Let Ψ denote one of the (identical) rows of $\psi(1)$. Then the variance of the random-component is:

$$\sigma_u^2 = \Psi \Omega \Psi' \tag{10}$$

where $\Omega = Var(\varepsilon_t)$. (This is the multivariate analog of equation (4)).

c. Price discovery: the information share (variance) approach

The random-walk variance reflects contributions from both markets. To see this, we may write out equation (10) in full as:

$$\sigma_u^2 = \begin{bmatrix} \Psi_1 & \Psi_2 \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \sigma_{1,2} \\ \sigma_{1,2} & \sigma_2^2 \end{bmatrix} \begin{bmatrix} \Psi_1 \\ \Psi_2 \end{bmatrix} \tag{11}$$

where $\sigma_1^2 = Var(\varepsilon_{1t})$; $\sigma_2^2 = Var(\varepsilon_{2t})$; and $\sigma_{1,2} = Cov(\varepsilon_{1t}, \varepsilon_{2t})$. Now if the covariance matrix happened to be diagonal (that is, if $\sigma_{1,2} = 0$), then equation (11) would imply a clean decomposition of the random-walk variance between the two markets. Hasbrouck (1995) suggests that these relative contributions be considered the "information shares" of the two markets.

More generally, when the innovations in the two markets are correlated (that is, when $\sigma_{1,2} \neq 0$), the covariance terms in equation (11) can't be attributed to either market. In such situations, Hasbrouck suggests constructing upper and lower bounds for the information shares by orthogonalizing (rotating) the covariance matrix to maximize and minimize the explanatory power of a particular market. These bounds, it might be emphasized, are not the usual estimation confidence intervals. They arise from insufficient identification, and so might be large even with a data sample of "infinite" size.[2]

This characterization of the random-walk variance is very useful, but also somewhat limited. Are there reasonable identification restrictions that might offer a more detailed picture? Unfortunately, from an economic perspective, further identification restrictions are generally no more attractive here than they were in the univariate case discussed above.

d. Price discovery: The permanent-transitory (PT) approach

The permanent-transitory decomposition discussed in connection with the single price framework (Section 2.d) can also be extended to cointegrated systems. That is,

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[2] In practice, these bounds will be tighter as the correlation between the innovations in the two markets approaches zero. This in turn is largely a function of the time resolution in the analysis. GM bid-price changes computed for the New York and Boston Exchanges over a one-month horizon have a correlation near one; the corresponding price changes computed over a one-second resolution have a correlation near zero.

$$p_t = \begin{bmatrix} p_{1t} \\ p_{2t} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} f_t + \begin{bmatrix} s_{1t} \\ s_{2t} \end{bmatrix} \quad (12)$$

where f_t is $I(1)$, *but not necessarily a random walk.* As in the univariate case, this additional generality aggravates the identification problems. In the context of macroeconomic time series, Gonzalo and Granger (1995) suggest identifying f_t (up to a normalization) by two conditions:

a) f_t is an exact linear function of the *current* variables, in this case the p_t.

b) The residual transitory component, $\begin{bmatrix} s_{1t} & s_{2t} \end{bmatrix}'$, has no permanent effect on p_t.

Harris, McInish, and Wood (2000) apply this approach to trade price data from multiple equity markets.

In the two-price application, the Harris et al common factor approach is tantamount to specifying $f_t = a\, p_t$ where $a = \begin{bmatrix} a_1 & a_2 \end{bmatrix}$, subject to a normalization: $a_1 + a_2 = 1$. Intuitively, f_t in this model is a weighted average of current trade prices. [3] Harris et al suggest that a_1 and a_2 are useful measures of the price discovery originating in the two markets. These parameters can be estimated easily from the vector error correction model.

Relative to the information shares discussed in connection with the random-walk decomposition model, the a's possess a very appealing feature. Whereas the information shares can only be determined within a range, the a's are exactly identified. In the application presented in Harris et al, moreover, the estimates are generally reasonable.

The identification, however, comes at the cost of imputing to agents in the economy a dubious objective. Specifically, why should a trader care about an average of current prices? The average won't generally be an optimal prediction of the price ten trades or ten minutes in the future. If forward-looking traders rely on the conditional expectation of terminal value, they will prefer an information set that is as large as

[3] There is nothing in the statistical specification, however, that requires the a_i to be nonnegative, so perhaps f_t is more accurately characterized as a portfolio.

possible. If past prices contain useful information, why should they be excluded from the conditioning set?

e. Information-share and common-factor measures of price discovery: a comparison

The relative merits of the information share (IS) and permanent-transitory (PT) measures of price discovery might be summarized as follows. The information shares arise from a random-walk decomposition subject to minimal identification restrictions. The implicit random walk, being a martingale, is consistent with rational updating of conditional expectations. Estimates of the information shares, however, can be determined only within bounds that are likely to be uncomfortably large in many applications. The permanent factor coefficients (the a) in the PT approach, on the other hand, may be estimated much more precisely. Their economic justification, however, rests on the presumption that the representative trader's objective is an average of current prices.

The contrast has to this point, been made at an abstract level of economic and econometric principle. The reader might well be wondering if material differences between the two approaches are likely to arise in practice. To answer this question, I consider the implications of both approaches for three simple structural models. That is, I consider the behavior of these specifications in situations when the structural models are known and the estimation procedures are applied to data generated by artificial, but nonetheless plausible, economic models.

Price discovery example I: A two-market "Roll" model.

In this model, the quote midpoint m_t is assumed to be driven by non-trade public information, which is available to all market participants. The price in Market $i=1,2$ is simply equal to the (common) efficient price, plus bid-ask bounce:

$$\begin{aligned} &m_t = m_{t-1} + u_t \\ &q_{it} = \begin{cases} -1, \text{ with pr. } 1/2 \\ +1, \text{ with pr. } 1/2 \end{cases}, \text{ for } i = 1,2 \\ &p_{it} = m_t + c\, q_{it}, \text{ for } i = 1,2 \end{aligned} \tag{13}$$

I assume that $\{u_t, q_{1t}, q_{2t}\}$ are mutually uncorrelated. The two markets in this model are (statistically) identical. This symmetry suggests that, in a structural sense, neither market should dominate. Any sensible attributions of price discovery should either be indeterminate or of equal share (0.5). With parameter values $c = 1$ and $\sigma_u = 1$, I generated 100,000 observations, and analyzed the data according the PT and IS approaches.

Table 1 summarizes the results. The PT approach determines an attribution of price discovery to Market 1 that is (to reported precision) identical to the "correct" (that is, structural) value, 0.5. The bounds implied by the information share approach certainly contain this value, but the range is a wide one (0.21, 0.79).

All else equal, one is drawn to the PT approach here because it appears to arrive at the correct value with little uncertainty. There are, however, some drawbacks to this attribution. The statistical properties of the PT common price factor, in this case $f_t = 0.5\, p_{1t} + 0.5\, p_{2t}$ differ dramatically from those of the structural efficient price. With the parameter values used here, $Var(\Delta m_t) = \sigma_u^2 = 1$ and $Corr(\Delta m_t, \Delta m_{t-1}) = 0$. In the simulated data, however, $Var(\Delta f_t) = 2.00$ and $Corr(\Delta f_t, \Delta f_{t-1}) = -0.25$. Thus, an analysis that takes f_t as a proxy for the structural efficient price will grossly overestimate volatility and autocorrelation. Under the IS approach, however, the estimated variance of the common price factor (the random walk) component is 1.01, with first-order autocorrelation that is zero (by construction).

Price discovery example II: Two markets with private information.

Suppose that Market 1 is identical to the asymmetric information market considered in Section 2.c. That is, the quote midpoint m_t is driven purely by Market 1's trades:

$$m_t = m_{t-1} + \lambda\, q_{1t} \tag{14}$$

The transaction price in this market is

$$p_{1t} = m_t + c_1\, q_{1t}\,.$$

Market 2 is the derivative (satellite) market, with trade prices given by

$$p_{2t} = m_{t-1} + c_2\, q_{2t}\,.$$

Note that Market 2 relies on the lagged (stale) value of m_t. Thus, from a structural viewpoint, it is clear that all price discovery occurs in the first market. For parameter values $c_1 = c_2 = 1$ and $\lambda = 1$, I simulated and analyzed 100,000 observations.

Table 2 summarizes the results. As in the previous example, the PT approach attributes nearly all of the price discovery to the Market 1 (the structurally correct answer). The IS approach also makes this determination. In contrast to the last example, the bounds of the price discovery share are very tight. (This is a consequence of there being only one source of randomness in the Market 1 price. Because the p_{1t} dynamics are driven entirely by q_{1t}, m_t can be recovered exactly from current and past prices.)

Although the PT and IS approaches give similar determinations of price discovery, the behavior of the common price factors differs strongly (as in the last example). The random-walk component in the IS approach has moments that are very similar to those of the efficient price. The PT price factor f_t has a variance that is more than four times as large as that of the efficient price, however. The factor also exhibits strong negative autocorrelation.

Price discovery example III: Two markets with public and private information.

Example III combines features of Examples I and II. The quote midpoint m_t is driven by Market 1's trades and a non-trade ("public information") component:

$$m_t = m_{t-1} + \lambda\, q_{1t} + u_t \text{ where } u_t \sim N\left(0, \sigma_u^2\right) \tag{15}$$

The transaction price in this market is

$$p_{1t} = m_t + c_1\, q_{1t}\,.$$

As in example II, Market 2 is the derivative (satellite) market, with trade prices based stale (lagged) information.

$$p_{2t} = m_{t-1} + c_2\, q_{2t}\,.$$

Once again, Market 1 is in a structural sense, the clear leader.

The parameter values used for the simulations were $c_1 = 1$, $c_2 = 0$ and $\lambda = 1$.

A supporting economic story here might be that $c_1 > c_2$ because the costs of market-making (including monitoring and regulatory systems) in an environment with informed trading are larger than the costs of a passive system. Market 2 crosses trades cheaply (costlessly) at stale prices.[4] I simulated and analyzed 100,000 observations.

Table 3 reports the results. The PT approach attributes 60% of the price discovery to Market 1; the bounds of the IS attribution are (90%, 98%). Thus (up to an estimation error) the IS bounds contain the correct value. The PT attribution is a substantial underestimate, and also lies below the lower bound of the IS range. As in the earlier examples, the statistical properties of the random-walk component in the IS approach are quite close to those of the structural efficient price. In contrast with the earlier examples, this also characterizes the PT common factor.

Summary

The examples studied here cover a range of structural models: one in which all information is public; one in which all information is private (trade-related); and one with a mix of public and private information.

In the analysis of these examples, neither the PT nor the IS approaches always arrives precisely at the structurally-correct determination. The bounds generated by the IS approach usually contain (up to estimation error) the true value. This cannot be said of the PT approach, which may be quite misleading (as in example III).

It might be alleged in favor of the PT approach that it sometimes achieves a precise identification when the IS bounds are vague (as in example I). But the price factor so constructed in this situation is substantially more volatile and autocorrelated than the structural efficient price. This raises questions about the extent to which the PT factor approximates the structural efficient price.

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[4] In the analysis of example II, the parameterization was $c_1 = c_2 = 1$. Setting $c_1 = 1$ and $c_2 = 0$ in Example II yields a system that is noninvertible, and therefore cannot be estimated by a VECM.

On balance, therefore, the IS approach appears to support more reliable inference. As long as the price component of interest is presumed to follow a random walk, the IS analysis offers the most accurate characterization. The random-walk restriction on the efficient price is motivated by the economic logic that this component should behave as a martingale. It might well be the case that other microstructure considerations motivate interest in non-random-walk price components. But absent such structural economic considerations, the rationale for alternative statistical restrictions (as in the PT approach) is, in microstructure applications, unclear.

4. Multiple markets with different trading frequencies

The discussion of the multiple-market models in the previous section made no note of a particular feature that greatly simplified the analysis. Specifically, the data record contained contemporaneously-determined prices for each market for each time t. This feature is unfortunately quite problematic in practice. Transaction frequency often differs dramatically across the various markets in which a security is traded. Obtaining a multivariate transaction price series in which the component prices are determined approximately contemporaneously, therefore, requires thinning the data set.

Thinning is not an innocuous procedure. By discarding prices established in the higher-frequency market that are not close in time to trades in the low-frequency market, the analyst suppresses any economic value these "intermediate" prices might have. Furthermore, inferences about price discovery that includes these intermediate times might be reversed in an analysis that focuses only on times with coincident (or nearly coincident) prices. Two examples illustrate these points.

Data thinning example I

Suppose that the efficient price evolves as a random walk,

$$m_t = m_{t-1} + u_t \,, \tag{16}$$

where t indexes "minutes". Market 1 (the "high-frequency market") has a trade every minute at price $p_{1t} = m_t$. Market 2 (the "low-frequency market") has a trade every 100 minutes:

$$p_{2t} = \begin{cases} m_t \text{ if } \operatorname{mod}(t,100) = 0 \\ \textit{Undefined}, \text{ otherwise} \end{cases} \tag{17}$$

From a structural perspective, *all* the price discovery at intermediate times (that is, times at which $\operatorname{mod}(t,100) \neq 0$) is occurring in Market 1. If we thin the data set to the "endpoint" times $t = 100i$ for $i = 1, 2, \ldots$, then the two markets will appear to be informationally equivalent.

Data thinning example II:

This example, although perhaps extreme, illustrates the problem with relying solely on endpoint prices. We extend the previous example to "penalize" Market 1 as follows:

$$p_{1t} = \begin{cases} m_{t-1}, \text{ if } \operatorname{mod}(t,100) = 0 \\ m_t, \text{ otherwise} \end{cases} \tag{18}$$

That is, Market 1 still performs all of the price discovery during intermediate times, but uses stale prices at the endpoints. Viewed over all *t*, Market 1 performs 99% of the price discovery. But viewed solely from the endpoints, Market 1 appears completely redundant.[5]

Thinning: a summary

Market data are not always conveniently timed, and it is probably too strong a pronouncement to assert that thinning is never justified. But the above examples should introduce an element of doubt or qualification to the practice. Thinning is essentially censoring, and price patterns that clearly characterize a full data record may be obscured or reversed in a censored sample.

Moreover, the examples featured exogenous trading occurrences. If trade occurrence is endogenous to the information process, the possibilities of misleading inference increase further. Suppose, for example, that in reaction to new information,

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[5] That is, a time-series analysis of endpoint price changes would (in a sufficiently large sample) find Granger-Sims causality running from Market 2 to Market 1, but not the reverse.

traders in the satellite market refrain from transacting until prices in the dominant market have "settled down".

5. Conclusions

From an econometric perspective, market frictions such as bid-ask bounce, discreteness, inventory control, etc., generally introduce transient components into security price processes. In the presence of such components, these prices aren't martingales. The martingale still figures prominently in the analysis because this property characterizes a sequence of conditional expectations, such as those formed by traders regarding a security's ultimate value. This sequence of expectations is unobserved, however, and so the martingale must be regarded as an implicit one.

With this perspective, the present paper summarizes econometric approaches to characterizing the unobserved random walk component of a security price. These techniques mostly originated in the analysis of macroeconomic time series. While the basic results are invariant to time scale, however, there are structural economic features of microstructure settings that must be taken into account.

Thus, decompositions that characterize the random-walk component implicit in a price series with covariance-stationary first differences play a prominent role. It is generally possible to compute the variance of this component, and to characterize contributions to this variance. When the supporting data comprise prices of the same security from multiple markets, these techniques support qualified attributions of price discovery.

Stronger results, however, such as determining precisely the location of the implicit random-walk component at a given time, rely on identification restrictions that are not plausible in microstructure settings, irrespective of their merits in macroeconomic applications. Moreover, general permanent-transitory decompositions used in macroeconomics, when applied to microstructure price data, characterize non-martingale factors. While non-martingale factors might be of interest in a particular structural model, they are poor proxies for optimally formed and updated expectations of security value.

The paper considers the attribution of price discovery in multiple markets using simulated data from simple structural models of trading. The analysis contrasts the information share approach discussed in Hasbrouck (1995), in which the price component of interest is forced to be a random walk, with the approach suggested by Harris, McInish, and Wood (2000), in which the price component is permanent in the sense of Gonzalo and Granger (1995). Although the latter approach can sometimes appear to offer greater precision, the information share computation is more reliable.

Finally, multiple-market analyses often compare venues with differing trading frequencies. The desire for contemporaneously-determined prices motivates a thinning of the data, to include only those times (or small time windows) in which trades occurred in all markets. This paper shows that this censoring can drastically affect the inferences concerning price discovery. Clear patterns of price leadership in the full data set can be obscured or even reversed by the censoring.

6. References

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Table 1. Price discovery example I: A two-market "Roll" model

The model is:

Efficient price: $m_t = m_{t-1} + u_t;\ u_t \sim N\left(0, \sigma_u^2\right)$

Trade direction: $q_{it} = \pm 1$, each with pr. $1/2$, for $i = 1, 2$

Transaction price: $p_{it} = m_t + c\, q_{it}$, for $i = 1, 2$

The model was simulated using parameter values $c = 1$ and $\sigma_u = 1$ for 100,000 observations. A vector error correction model (VECM) was estimated for 10 lags. The common price factor under the permanent/transitory approach is computed directly from the VECM (using the Granger-Gonzalo identification restriction). In this approach, Market 1's price discovery share is equal to the coefficient of Market 1's price in the common factor. The efficient price properties under the information share approach are computed by inverting the VECM to obtain a vector moving average (VMA) representation through 60 lags. In this approach, Market 1's price discovery share is equal to the proportion of variance (in the implicit random-walk price factor) that can be attributed to Market 1's price innovations. This is only identified within the given range.

	Structural Model	Permanent/ Transitory	Information Share
Price discovery (Market 1's share)	0.5	0.50	(0.21, 0.79)
Price component analyzed	Efficient price (m_t)	Permanent factor (f_t)	Random Walk
Variance of (first-difference of) price component	1.	2.00	1.01
Autocorrelation of (first-difference of) price component	0.	-0.25	0.

Table 2. Price discovery example II: Two markets with private information

The model is:

Efficient price:	$m_t = m_{t-1} + \lambda\, q_{1it}$
Trade direction:	$q_{it} = \pm 1$, each with pr. $1/2$, for $i = 1,2$
Transaction prices:	$p_{1t} = m_t + c_1\, q_{1t}$ $p_{2t} = m_{t-1} + c_2\, q_{2t}$

The model was simulated using parameter values $c_1 = c_2 = 1$ and $\lambda = 1$ for 100,000 observations. A vector error correction model (VECM) was estimated for 10 lags. The common price factor under the permanent/transitory approach is computed directly from the VECM (using the Granger-Gonzalo identification restriction). In this approach, Market 1's price discovery share is equal to the coefficient of Market 1's price in the common factor. The efficient price properties under the information share approach are computed by inverting the VECM to obtain a vector moving average (VMA) representation through 60 lags. In this approach, Market 1's price discovery share is equal to the proportion of variance (in the implicit random-walk price factor) that can be attributed to Market 1's price innovations. This is only identified within the given range.

	Structural Model	Permanent/ Transitory	Information Share
Price discovery (Market 1's share)	1.	0.98	(1.00, 1.00)[1]
Price component analyzed	Efficient price (m_t)	Permanent factor (f_t)	Random Walk
Variance of (first-difference of) price component	1.	4.74	1.01
Autocorrelation of (first-difference of) price component	0.	-0.39	0.

[1] The bounds are approximately equal.

Table 3. Price discovery example III:

Two markets, with public and private information

The model is:

Efficient price:	$m_t = m_{t-1} + \lambda\, q_{1it} + u_t$; where $u_t \sim N\left(0, \sigma_u^2\right)$
Trade direction:	$q_{it} = \pm 1$, each with pr. $1/2$, for $i = 1, 2$
Transaction prices:	$p_{1t} = m_t + c_1\, q_{1t}$ $p_{2t} = m_{t-1} + c_2\, q_{2t}$

The model was simulated using parameter values $c_1 = 1$, $c_2 = 0$, $\sigma_u = 1$ and $\lambda = 1$ for 100,000 observations. A vector error correction model (VECM) was estimated for 10 lags. The common price factor under the permanent/transitory approach is computed directly from the VECM (using the Granger-Gonzalo identification restriction). In this approach, Market 1's price discovery share is equal to the coefficient of Market 1's price in the common factor. The efficient price properties under the information share approach are computed by inverting the VECM to obtain a vector moving average (VMA) representation through 60 lags. In this approach, Market 1's price discovery share is equal to the proportion of variance (in the implicit random-walk price factor) that can be attributed to Market 1's price innovations. This is only identified within the given range.

	Structural Model	Permanent/ Transitory	Information Share
Price discovery (Market 1's share)	1.	0.60	(0.90, 0.98)
Price component analyzed	Efficient price (m_t)	Permanent factor (f_t)	Random Walk
Variance of (first-difference of) price component	2.	1.98	2.01
Autocorrelation of (first-difference of) price component	0.	0.00	0.