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Asset Price Dynamics and Infrequent Trades

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Abstract: We model an economy where stocks and bonds (consols) are traded by two types of agents: speculators, expected utility maximizers always present in the market, and infrequent traders, whose trading motives are not explicitly modeled. A solution technique for equilibrium prices is developed when trades are triggered by stock prices reaching some threshold level, corresponding to a specific value of the dividend flow. Across trade scenarios we find expectations of stock sales to *depress* stock prices relative to the no-trade case, while expectations of stock purchases tend to *inflate* them. Both effects bring about *heteroskedasticity* and *predictability* of stock returns. Our analysis yields insights as to the equilibrium effects of a variety of trading strategies, which mechanically generate market orders in response to changes in stock prices.

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Asset Price Dynamics and Infrequent Trades

1 Introduction

We model an economy where stocks and bonds (consols) are traded by two types of agents: speculators, expected utility maximizers always present in the market, and traders who trade only infrequently and by finite amounts. Infrequent and finite trades would occur, for example, if some agents face fixed transactions costs in rebalancing their portfolios.[1] We do not model infrequent traders' motives nor the structure of the transactions costs they may face, but we take as given their demand of stocks as a function of the stock price.

While trade is given exogenously, infrequent traders' market orders are filled at prices which are determined *endogenously.* This is an important difference with respect to a modified Lucas (1978)-"tree" economy, where the supply of risky and riskless trees were subject to infrequent exogenous changes. In our economy, the endogenous prices at which assets are exchanged determine the after-trade asset allocation.

We discuss a first example where infrequent traders sell stocks in return for all the bonds in the hands of the speculators, when stock prices reach a lower threshold. This behavior has the flavor of a stop-loss strategy on the part of infrequent traders. The example highlights equilibrium effects which may contribute to explain actual stock-price behavior.

Stock prices are *depressed* in anticipation of a wave of price-triggered sales, but after the trade, stock prices may quickly rebound to higher values as the possibility of similar sales in the future disappears. This behavior is reminiscent, for example, of what happened after the 1987 stock-market crash when by Wednesday, October 21st, stock prices had recovered close to half of their October 19th loss.

In this example, dividend growth rates are i.i.d., while before-trade stock returns are more *volatile* than dividend growth rates and *heteroskedastic*, and dividends help *predict* future rates of return on stocks. In our framework, prices react to news on current dividends not only because they affect expected future dividends, but also because, by changing the price, they affect the likelihood and timing of future trades.

We then discuss the robustness of our findings to asset allocations, relative position of the price threshold triggering the trade, and risk aversion. We consider additional scenarios where infrequent traders' behavior has, alternatively, the flavor of a "contrarian" or a "trend-chasing" strategy. Across scenarios, expectations of stock sales *depress* stock prices relative

[1] Even with proportional transactions costs, trades would occur only when the portfolio is sufficiently far "out of line" [Constantinides (1986) and Davis and Norman (1991)]; with continuous trading opportunities, however, the amount traded is finite only if part of the transactions costs is fixed, as in Duffie and Sun (1990).

to the no-trade case, while expectations of stock purchases tend to *inflate* them; and these effects are found to be stronger the higher the speculators' relative rate of risk aversion. Both price depression and price inflation bring about heteroskedasticity and predictability of stock returns. The conditional volatility of stock returns may be higher or lower than in the no-trade case, depending on the relative position of the price threshold triggering the trade.

Closely related to ours, both for techniques and issues considered, is the paper by Brennan and Schwartz (1989) on the effects of portfolio-insurance strategies. Their model is similar to ours in that the pricing kernel is also identified with the marginal utility of the speculators, and the presence of portfolio insurers depresses stock prices and enhances their volatility (as in our first trade scenario). Like our infrequent traders, portfolio insurers' behavior is specified exogenously, although they are allowed to trade continuously to implement their hedging strategy. Basak (1993) develops a model where portfolio insurance schemes are derived endogenously from a constraint on final wealth. Since he takes both speculators' and portfolio insurers' marginal rates of substitution to concur to determine prices, he finds the implications of portfolio insurance to be the opposite of Brennan and Schwartz's: stock prices are *higher*, and *less* volatile relative to an economy with no portfolio insurers.

Another paper which looks at the implications of exogenous trades for asset prices' dynamics is Campbell and Kyle (1993). Their model shares with ours the formal decomposition of asset prices into fundamental value and "noise," where the noise component is induced by current and future expected trade on the part of noise traders. Their empirical findings show that, in order to contribute to the explanation of actual stock price behavior, noise trading needs to be correlated with the dividend process. This feature is captured in the examples of our paper, by assuming a specific level of the dividend flow to trigger the trade (through prices).

Finally, Gennotte and Leland (1990) and Donaldson and Uhlig (1991) are also related to the present work, in that their one-period models are capable to generate discontinuities in prices (crashes) when exogenous traders take positions in stocks which are negatively correlated to the price of stocks themselves. Our model differs from theirs in the obvious sense that it is a dynamic description of asset markets' equilibrium; and in the less obvious sense that it generates price discontinuities (jumps) immediately *after* trade has occurred, rather than at the occurrence of trade.

Section 2 presents the model. Here, we observe that it is useful to identify a component of asset prices which reflects the value of trading the asset at some point in the future. This observation is the basis of a solution technique for equilibrium prices developed in Section 3, in the context of a first example. Section 4 discusses the relevance the example for actual stock price behavior. Section 5 looks at alternative trade scenarios to assess the robustness of our analysis to asset allocations, relative position of the price threshold, and risk aversion. Section 6 concludes, highlighting extensions of our framework to more realistic trade scenarios.

2 Trade and prices

We begin our analysis with a description of the structure of the economy.

Assumption 1. The economy is populated by two groups of investors: speculators and infrequent traders. Speculators are identical, maximize their utility from consumption, and have rational expectations about the dynamics of future cash flows generated by available assets, and about the structure of possible trades. Infrequent traders submit market orders of finite size at discrete points in time. The deeper determinants of these trades are not modeled.

Assumption 2. Two assets exist in positive net supply: stocks, which entitle their owner to a stochastic flow of dividends $\{\xi\}$, and bonds (consols), which entitle to a certain and constant coupon flow $\{r\}$. The time horizon is infinite. We abstract from issues of physical investment and economic growth, and assume that no storage technology is available.

Assumption 3. The state of the economy is summarized by a vector of state variables $[\xi, S, B, Y]$, where ξ is the dividend flow on stocks, and S and B denote per-capita holdings of stocks and bonds, respectively, among speculators. The probabilistic structure of trade is summarized by the state vector Y, which contains information relevant to the speculators' assessment of the likelihood and size of market orders at every point in the economy's state space.

Speculators' optimum. Let c denote the speculators' consumption flow, $U(\cdot)$ an increasing and concave utility function, and ρ the rate of time preference. Also, let P_s and P_b denote the prices of stocks and bonds. The following first-order conditions link optimal speculator's consumption to asset prices (these conditions follow directly from the Bellman equation, as shown in Appendix A):

$$\rho U'(c)P_s = U'(c)\xi + \frac{1}{dt}E_t\{d[U'(c)P_s]\}, \tag{1}$$

$$\rho U'(c)P_b = U'(c)r + \frac{1}{dt}E_t\{d[U'(c)P_b]\}. \tag{2}$$

After adjusting prices and payoffs by the marginal utility of consumption, all assets should yield the same expected rate of return, ρ. Also, speculators' consumption need satisfy the following transversality conditions

$$\lim_{\tau\to\infty} E_t[U'(c(\tau))P_i(\tau)]e^{-\rho(t-\tau)} = 0, \quad i = s,\ b. \tag{3}$$

Equilibrium. As we have assumed away storage or physical investment, speculators consume dividends and coupons generated by the stocks and bonds they own:

$$c = S\xi + Br.$$

Consider next the prices supporting the equilibrium. We define risk-adjusted asset prices p_i and cash-flows f_i,

$$p_s \equiv U'(c)P_s, \quad f_s \equiv U'(c)\xi, \quad p_b \equiv U'(c)P_b, \quad f_b \equiv U'(c)r,$$

and rewrite (1) and (2) as:

$$\rho p_i = f_i + \frac{1}{dt}E_t(dp_i), \quad i = s,\ b. \tag{4}$$

The differential equations (4) must be satisfied at all points in the economy's state space, regardless of the likelihood of trade.

No-expected-jump conditions. Given (4), $E_t(dp_i)$ cannot be of order larger than dt, and this rules out expected jumps in risk-adjusted price paths. It follows that

$$E_t(\Delta p_i) = 0, \quad i = s,\ b, \tag{5}$$

at trading times. Hence, when trade takes place, the discrete change in asset prices is "compensated" by a discrete change in the marginal utility of consumption so that expected risk-adjusted price paths remain continuous.

Trade and prices. Note that unlike in Lucas (1978), consumption and risk-adjusted payoffs $f_i(\tau)$ are affected by trade as well as by dividends. Hence it is useful to decompose risk-adjusted prices in the form

$$p_i(\xi, S, B, Y) = g_i(\xi, S, B) + h_i(\xi, S, B, Y), \quad i = s,\ b, \tag{6}$$

for

$$g_i = \int_t^\infty E_t[f_i(\tau)|\text{no trade}]e^{-\rho(t-\tau)}d\tau, \quad i = s,\ b, \tag{7}$$

where $[f_s(\tau)|\text{no trade}] \equiv U'(S_t\xi_\tau + B_t r)\xi_\tau$, and $[f_b(\tau)|\text{no trade}] \equiv U'(S_t\xi_\tau + B_t r)r$. The function g_i takes the speculators' portfolio composition as given and immutable, and would be the asset's equilibrium value if trade could be disregarded. Hence, h_i reflects the effects of trade on the asset's price, and can be viewed as the value of trading the asset at some point in the future.

3 An application

We now specialize our framework in terms of preferences, cash-flow dynamics, and structure of trade. We confine ourselves to situations where the stock price is a monotonic function of the dividend flow.

Assumption 4. The speculators' instantaneous utility function has the form

$$U(c) = \frac{c^{1-\gamma}}{1-\gamma},$$

where γ is the relative risk aversion parameter.

Assumption 5. Dividends follow a geometric random walk

$$d\xi = \mu\xi dt + \sigma\xi dw,$$

with μ and σ positive constants.

Expanding $E_t(dp_i)$ in (4) by the usual stochastic calculus arguments, we find that in the interior of no-trade regions the p_i functions satisfy the valuation equations

$$\rho p_i = f_i + \mu\xi p_i' + \frac{\sigma^2\xi^2}{2}p_i'', \quad i = s,\ b. \tag{8}$$

No-trade solutions. When either $B = 0$ or $S = 0$, the solution for the no-trade component g_i in (6) is obtained from evaluation of the integral in (7). We have

$$B = 0 \ \Rightarrow \quad g_s = \frac{\xi}{(S\xi)^\gamma d_s}, \quad g_b = \frac{r}{(S\xi)^\gamma d_b}, \tag{9}$$

where $d_s \equiv [\rho + (\gamma-1)\mu - (\gamma-1)\gamma\sigma^2/2]$, $d_b \equiv [\rho + \gamma\mu - \gamma(\gamma+1)\sigma^2/2]$; and

$$S = 0 \Rightarrow \quad g_s = \frac{\xi}{(\rho-\mu)(Br)^\gamma}, \quad g_b = \frac{r}{\rho(Br)^\gamma}. \tag{10}$$

Goods-denominated prices P_i are computed from the risk-adjusted prices p_i by the relation $P_i = p_i c^\gamma$, with $c = S\xi$ in the $B = 0$ case, and $c = Br$ in the $S = 0$ case. To ensure that prices are positive and finite, parameter values must be such that $d_s > 0$, $d_b > 0$, and $\rho > \mu$.

Homogeneous solutions. The second component of asset prices in (6), h_i, can be found as the homogeneous solution of (8). The solution has the form

$$h_i = Q_i(S,B,Y)\xi^{\lambda_1} + N_i(S,B,Y)\xi^{\lambda_2}, \quad i = s,\ b, \tag{11}$$

where λ_1 and λ_2 solve the characteristic equation associated with (8) (see Appendix B), and Q_i and N_i are constants of integration.

In the following we consider an explicit trade scenario where infrequent traders sell off stocks, in exchange for all the bonds held by the speculators, when stock prices reach a lower threshold.

Assumption 6. Initially speculators do not hold stocks: $B = B_0 > 0$, $S = 0$. Infrequent traders buy all of the speculators' bonds, selling S_1 shares of stocks when P_s hits a lower threshold P_{sl}. After the barrier is hit, it disappears and no more trades take place at P_{sl} or any other price.

We do not model infrequent traders' motives and constraints, but we take as given their demand of stocks (and bonds) as a function of stock prices. The infrequent and finite trades postulated in Assumption 6 can be rationalized by considering realistic frictions. With transactions costs, for example, it is not optimal to trade unless the portfolio composition is sufficiently far out of line. The optimal no-trade region when transactions costs are proportional to the amounts traded has been studied by Constantinides (1986) and Davis and Norman (1991) in a continuous-time setting similar to ours. When trade can take place continuously, however, trade will not be finite unless part of the costs is fixed, as in the framework considered by Duffie and Sun (1990).

While the initial and final asset allocations assumed here are extreme, this allows us to use the formulas in (9) and (10) to compute explicit solutions for before- and after-trade prices. Also, trade occurs only once. This may not be taken literally: we may think that after one trade takes place, other trades are expected in the future, but that the likelihood of them occurring soon is *low*.

In equilibrium there is a unique level of dividends ξ_l at which trade takes place. Such level of dividends is implicitly defined by the condition $P_{sl} \equiv P_s(\xi_l, 0, B_0, Y_0)$. The state variable $Y = Y_0$ summarizes the information in Assumption 6 on the possible future trade at ξ_l, and $Y = Y_1$ indicates that no trade is expected to ever take place after the barrier is hit.

At $\xi = \xi_l$ the no-jump conditions (5) reduce to

$$p_i(\xi_l, 0, B_0, Y_0) = p_i(\xi_l, S_1, 0, Y_1), \quad i = s,\ b. \tag{12}$$

As trade occurs with certainty at ξ_l, (12) implies that the risk-adjusted price of each asset cannot jump at the time of trade.

Trade and asset allocation. Since no trade takes place after the barrier is hit, the amount of stocks held by speculators after the trade is endogenously determined by the budget constraint:

$$S_1 = \frac{P_b(\xi_l, S_1, 0, Y_1)}{P_s(\xi_l, S_1, 0, Y_1)} B_0 = \frac{p_b(\xi_l, S_1, 0, Y_1) c^{-\gamma}}{p_s(\xi_l, S_1, 0, Y_1) c^{-\gamma}} B_0 = \frac{p_b(\xi_l, S_1, 0, Y_1)}{p_s(\xi_l, S_1, 0, Y_1)} B_0 = \frac{r d_s}{\xi_l d_b} B_0. \tag{13}$$

Note that because of the boundary conditions (12), the relative price at which trade takes place can be obtained as the ratio between after-trade prices. Equation (13) illustrates how traded quantities are endogenous, and how trade events are not the same as endowment shocks in a modified Lucas (1978)-"tree" economy: the number of stocks received by speculators in exchange for bonds is determined by endogenous prices.

As shown in Appendix C, the transversality conditions (3) require $Q_i = N_i = 0$ after trade takes place, and $Q_i = 0$ before trade takes place. Moreover, the constants N_i in the before-trade price functions must satisfy the following version of (12)

$$\begin{aligned} \frac{\xi_l}{(\rho-\mu)(B_0 r)^\gamma} + N_s \xi_l^{\lambda_2} &= \frac{\xi_l}{d_s(S_1 \xi_l)^\gamma} \\ \frac{r}{\rho(B_0 r)^\gamma} + N_b \xi_l^{\lambda_2} &= \frac{r}{d_b(S_1 \xi_l)^\gamma}, \end{aligned} \tag{14}$$

where the after-trade allocation S_1 can be obtained from (13).

In the rest of the paper we shall focus our discussion on the behavior of stock prices, for explicit choices of parameter values. The behavior of bond prices is similar, if less interesting, due to the absence of fluctuations in the coupon flow.

Risk-adjusted *vs* goods-denominated prices. Figure 1 illustrates the before-trade, after-trade, and no-trade risk-adjusted stock pricing functions p_s, for the choice of parameter values: $\gamma = 1.5$, $\rho = .075$, $\mu = .018$, and $\sigma^2 = .015$. We further set $\xi_l = 1$ and $B_0 = 1$, and normalize $r = 1$.

Figure 1 shows that trade cannot induce discrete changes in the risk-adjusted price p_s. On the other hand, Figure 2 shows that actual prices jump because the speculators' marginal utility of consumption changes at trade times.

For our choice of parameter values, stock prices *rebound* promptly after the trade.

The next section further discusses the implications of the example above, and relates them to actual price behavior.

4 Implications for stock price behavior

About crashes and rebounds

Our Assumption 6 postulates a demand function for stocks on the part of infrequent traders which is positively related to stock prices; this demand resembles that of the "hedgers, rebalancers, and others who use dynamic strategies akin to portfolio insurance" considered by Gennotte and Leland (1990) (p.1003). The trade situation described in the previous section can be viewed as a stop-loss sale by which infrequent traders try to protect the value of their portfolio.

Our infrequent-traders strategy may not ensure that their portfolio value is a "convex function of some underlying [...] reference portfolio," as Brennan and Schwartz (1989) define portfolio insurance, pp.455–456, since they sell stocks in exchange for risky bonds, and bond

Figure 1

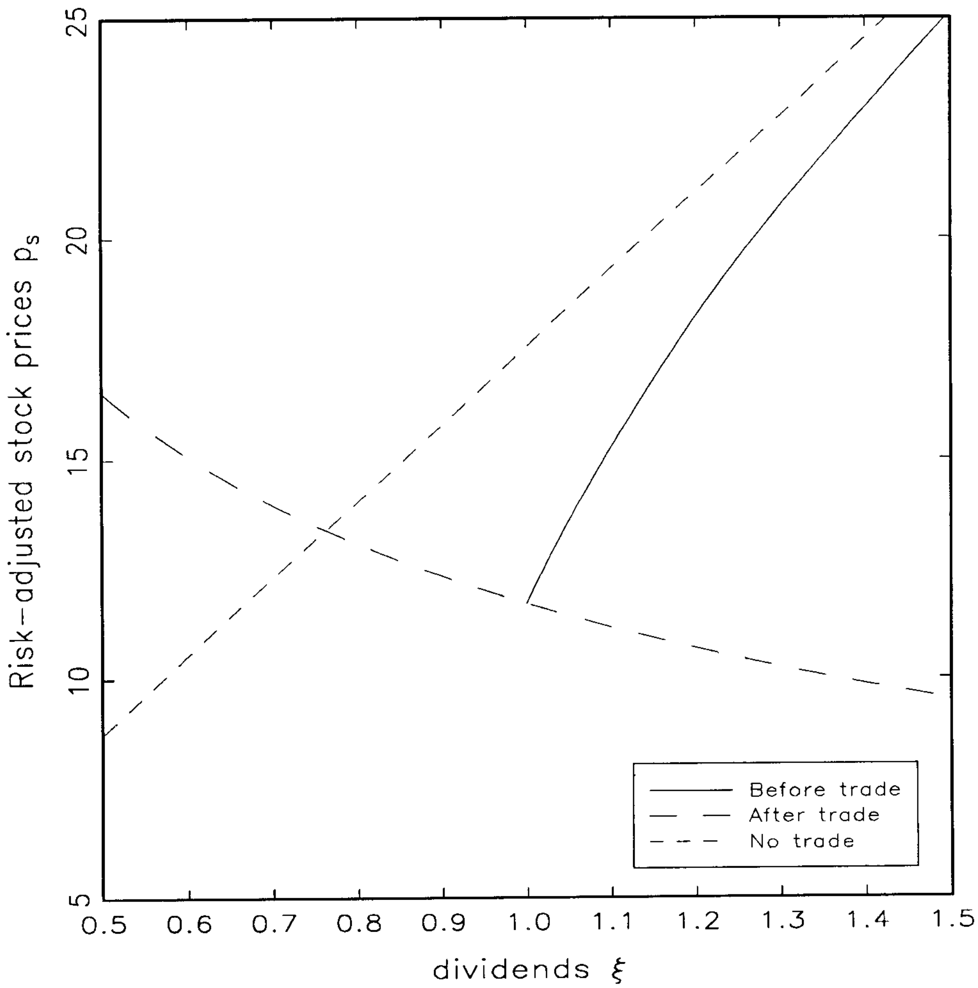

Figure 1 shows the risk-adjusted pricing function for stocks p_s before and after trade, and compares it to the pricing function without trade opportunities. The speculators' initial endowment is $B_0 = 1$ and $S_0 = 0$, and we set $\gamma = 1.5$, $\rho = .075$, $\mu = .018$, and $\sigma^2 = .015$.

Figure 2

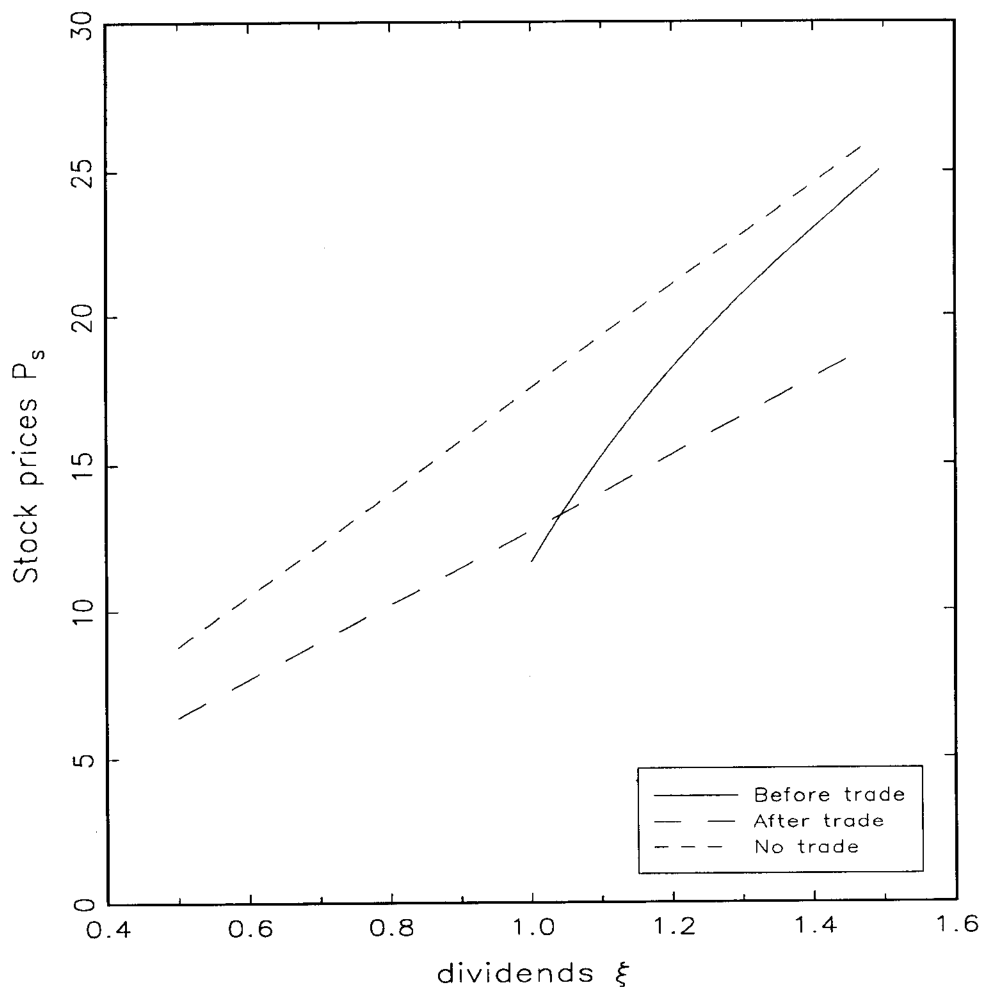

Figure 2 shows the pricing function for stocks P_s before and after trade, and compares it to the pricing function without trade opportunities, where $P_s = p_s/U'(c)$. The speculators' initial endowment is $B_0 = 1$ and $S_0 = 0$, and we set $\gamma = 1.5$, $\rho = .075$, $\mu = .018$, and $\sigma^2 = .015$.

prices may decline after the trade. For the parameter values of Figure 1, however, the bond price does not fall.

In a quite different setting, Gennotte and Leland (1990) show that the uncertainty about price-triggered trades plays a crucial role in reducing market liquidity and argue that even mildly negative news could have been sufficient to cause a stock-market crash like that of October 1987. They find that a rebound, which in their context is a "melt-up" to pre-crash levels, is conceivable, but unlikely.

In our example, stock prices decline *before* the trade as dividends fall. The decline in price due to news on dividends is more pronounced than without infrequent traders (see the no-trade curve of Figure 2), but it is too smooth to have the flavor of a stock-market crash. The price increase in Figure 2, instead, is reminiscent of the increase of stock prices following the 1987 stock-market crash, when, by Wednesday, October 21st, stock prices had recovered close to half of the previous "Black Monday" loss. Our analysis suggests that stock prices may rebound after a wave of price-triggered stock sales if speculators try yet to increase their stock holdings when portfolio insurers stop selling stocks.

Volatility, heteroskedasticity, and predictability of stock returns

The pricing function in Figure 2 is everywhere steeper than its no-trade counterpart, and becomes steeper as dividends approach the trade-trigger point: expectations of trade increase the sensitivity of stock prices to current dividends, and hence the volatility of stock returns. If no trade is expected in the future, the sensitivity of stock prices to dividends is $\partial P_s/\partial\xi = 1/(\rho-\mu)$. When dividends carry information on the likelihood of trade in the future we have

$$\frac{\partial P_s}{\partial \xi} = \frac{1}{\rho-\mu} + N_s\lambda_2(B_0 r)^{\gamma}\xi^{\lambda_2-1}.$$

In fact, one can decompose the elasticity of prices with respect to dividends as in Lucas (1978),

$$\frac{\xi\partial P_s/\partial\xi}{P_s} \equiv -\frac{\xi\partial U'/\partial\xi}{U'} + \frac{\xi\partial p_s/\partial\xi}{p_s},$$

thus making it explicit that a larger dividend has both an income and an "information" effect. The income effect (the first term) is always positive, as rational investors attempt to pass part of a positive windfall over to future dates through purchases of securities. This effect drives securities prices up. In our example, however, before-trade utility does not depend on dividends ($S_0 = 0$) and the income effect is nil. We can then focus exclusively on the second term, which reflects *two* types of information carried by decreasing dividends. First, since dividends are autocorrelated, lower dividends signal lower future cash flows, lowering stock prices. Second, the current size of dividends relative to the trigger level affects the likelihood and timing of future trade.

Moreover, in the no-trade case the rate of return on stocks is given by

$$\frac{dP_s + \xi dt}{P_s} = \rho dt + \sigma\, dw.$$

Absent trade, rates of return on stocks are i.i.d. normal variates, with mean $\rho\tau$ and variance $\sigma^2\tau$ over non-overlapping time-segments of length τ. Thus, rates of return on stocks are as volatile as dividend growth rates, homoskedastic, and current dividends provide no information as to future rates of return.

Conversely, the possibility of trade increases the volatility of rates of return on stocks, and makes them heteroskedastic and predictable. Since $P_s = \xi/(\rho - \mu) + N_s(B_0 r)^\gamma \xi^{\lambda_2}$, the instantaneous return on stocks is

$$\begin{aligned} dP_s + \xi dt &= \left[\frac{\mu\xi}{\rho-\mu} + \mu N_s \lambda_2 (B_0 r)^\gamma \xi^{\lambda_2} + \xi + \frac{1}{2} N_s \lambda_2 (\lambda_2 - 1)(B_0 r)^\gamma \xi^{\lambda_2} \sigma^2\right] dt \\ &\quad + \left[\frac{\sigma\xi}{\rho-\mu} + \sigma N_s \lambda_2 (B_0 r)^\gamma \xi^{\lambda_2}\right] dw. \end{aligned}$$

The diffusion term for the instantaneous before-trade return on stocks (the term in the second square brackets) exceeds its no-trade counterpart by the quantity

$$\sigma N_s \lambda_2 (B_0 r)^\gamma \xi^{\lambda_2}$$

whose sign is positive, since $N_s < 0$ [see equation (14)] and $\lambda_2 < 0$ (see Appendix B). Also, the before-trade stock price is lower than its no-trade counterpart (again, $N_s < 0$). Hence, the conditional volatility of rates of return on stocks is *higher* than in the no-trade case, and *changes over time*, since it depends on the level of dividends.

Also, unlike in the no-trade case, the conditional expectation of rates of return on stocks is a function of the dividend level. Hence, stock returns are *predictable*, since dividends are serially correlated.

The price depression and increased volatility of stock prices are apparent in Figure 2, where the before-trade pricing function is everywhere lower and steeper than the no-trade one.

5 Robustness

The example of Section 3 suggests that rationally anticipated trade may lead to a *depression* of stock prices and an increase in their *volatility* relative to the no-trade case. Here, we discuss the robustness of these two findings to asset allocations, relative position of the price threshold triggering the trade, and risk aversion.

Alternative trade scenarios

For concreteness, we frame our discussion in the context of the four specific trade scenarios illustrated in Figure 3.

We assume infrequent traders to buy all the bonds in the hands of the speculators in exchange for stocks (top two panels), or to buy all the stocks in the hands of the speculators in exchange for bonds (bottom two panels). Also, trade may occur at a *lower* price threshold (left two panels), or at an *upper* price threshold (right two panels). The solution technique is analogous to that of Section 3, and parameter values are the same as in Figures 1 and 2. The constant of integration Q_s has been set as to satisfy the transversality condition (left two panels), or to ensure no jumps in risk adjusted prices (right two panels). Similarly, the constant of integration N_s has been set as to satisfy the no-jump conditions (5) (left two panels), or to ensure that stock prices do not diverge to $\pm\infty$ as dividends tend to zero (right two panels).

The main message from Figure 3 is that stock prices are *depressed*, relative to the no-trade case, when a *sale* of stocks is anticipated; while they are *inflated* when a *purchase* of stocks is anticipated. Both price depression and price inflation become stronger as the market gets closer to the price threshold. Hence, both the conditional volatility and the conditional mean of stock returns depend on the dividend flow, and stock returns are heteroskedastic and predictable.

Moreover, when before-trade stock prices are depressed relative to their no-trade counterparts, they are likely to jump *upwards* after the occurrence of trade, as in the top two panels of Figure 3. On the other hand, when before-trade stock prices are inflated, they are likely to jump *downwards* after the occurrence of trade, as in the bottom two panels of Figure 3.

Finally, volatility can be either higher (Panels a and d) or lower (Panels b and c) than in the no-trade case, depending on the relative position of the price threshold, together with the depression or inflation in prices discussed above.

In the following, we further discuss each of the four trading scenarios of Figure 3:

Panel a. For sake of comparison, here we have replicated the situation of Figure 2. Infrequent traders purchase all the bonds in the hands of the speculators in exchange for stocks, when the stock price hits a *lower* threshold. Stock prices rebound after the trade.

Panel b. The initial and final asset allocations are the same as in Panel a, but trade takes place at an *upper* threshold. Similar to Panel a, the stock price is depressed in anticipation of the trade, but this translates into a *lower* volatility relative to the no-trade case. We can think of infrequent traders as implementing a contrarian strategy, where stocks are sold when their price is increasing. This strategy is rationally anticipated by speculators, and hence the stock price does not increase much with dividends: a wave of sales is expected

Figure 3

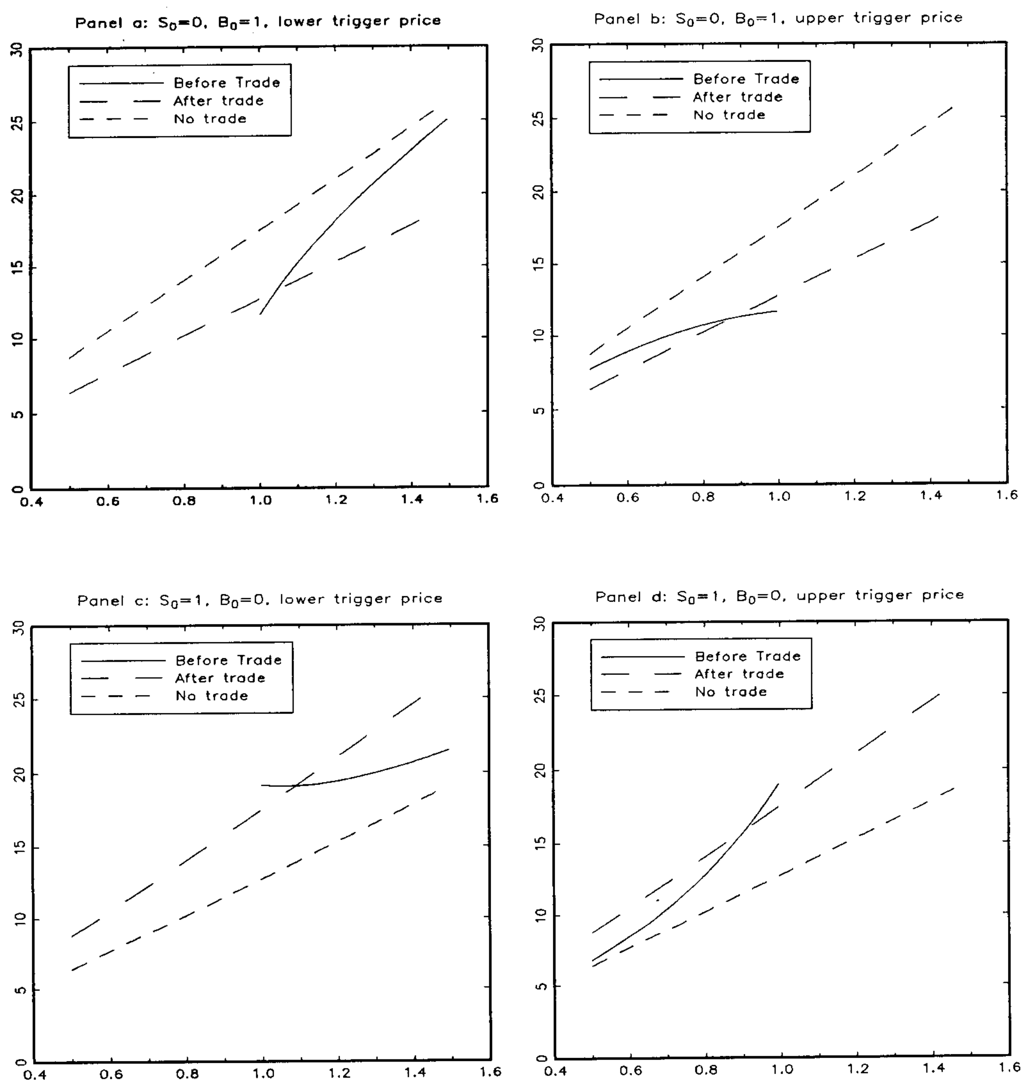

Figure 3 shows the pricing function for stocks P_s before and after trade, and compares it to the pricing function without trade opportunities, where $P_s = p_s/U'(c)$. The speculators' initial endowment is $B_0 = 1$ and $S_0 = 0$ (top two panels), and $B_0 = 0$ and $S_0 = 1$ (bottom two panels). The price threshold is either lower (left two panels), or higher (right two panels) than the before-trade price. We set $\gamma = 1.5$, $\rho = .075$, $\mu = .018$, and $\sigma^2 = .015$.

soon. Again, stock prices rebound after the trade, as no other sell orders are expected.

Panel c. The initial and final asset allocations are the *opposite* of Panels a and b: infrequent traders purchase all the stocks in the hands of the speculators in exchange for bonds; trade takes place at a *lower* threshold. Again, infrequent traders' behavior has the flavor of a contrarian strategy: stocks are purchased when their price is falling. The price depression of the previous two scenarios is here *reversed*: as dividends decrease a wave of purchases is expected soon, and the fall in stock prices reduced. As a consequence, volatility is also reduced. Stock prices fall after the trade since no other buy orders are expected.

Panel d. Initial and final asset allocations are the same as in Panel c, while trade takes place at an *upper* threshold. In this scenario infrequent traders behave like trend-chasers, purchasing stocks as their price is already increasing. Stock prices increase further in anticipation of a buy order, and this increases stock-price volatility in the proximity of the threshold. Again, stock prices fall after the trade.

In summary, the effects of trade on prices and price volatility depend on the specific trade scenario under scrutiny. Stock prices are depressed if stock sales are expected soon, whereas they are inflated if stock purchases are expected. The implications for volatility crucially depend on the relative position of the price threshold, that is on the type of strategy followed by infrequent traders. Stop-loss and trend-chasing strategies exacerbate the trend in stock prices, leading to an increase in volatility. Contrarian strategies, on the other hand, dampen the trend in prices and reduce volatility.

Risk aversion, price effects, and welfare

Experimenting with different values of the speculators' relative risk aversion parameter γ, we found both the price-depression and the price-inflation effect to be stronger the higher γ. In fact the two effects can be so strong that in the scenarios of Panels b and c the monotonicity of the before-trade pricing function can be lost. This is not a problem in general, but it prevents the inversion of the price-dividend correspondence, and the intuition of a *price* threshold (rather than a dividend threshold) triggering the trade would be lost.

The mechanisms driving the price effects above go as follows: Consider, for example, the trade scenario of Panel a in Figure 3, which replicates the scenario discussed in Section 3. Before trade we have $p_s = g_s + N_s\xi^{\lambda_2}$, whereas, if trade were ruled out, we would have $p_s = g_s$. Since $P_s = p_s U'(c)$, the difference between before- and no-trade stock prices is given by the quantity

$$N_s\xi^{\lambda_2}(B_0 r)^{\gamma}. \tag{15}$$

It can be shown that N_s is always negative and decreasing with γ, and hence the quantity in (15) is the more negative the higher γ. A similar argument holds for the trade scenario of Panel b. Here, the difference between before- and no-trade stock prices is given by

$Q_s\xi^{\lambda_1}(B_0r)^\gamma$, which is also negative and decreasing in γ. Hence, the higher γ, the stronger the price-depression effect.

In Panels c and d, the difference between before- and no-trade stock prices is given by $N_s\xi^{\lambda_2}(S_0\xi)^\gamma$ and $Q_s\xi^{\lambda_1}(S_0\xi)^\gamma$, respectively. It can be shown that both N_s and Q_s are here positive, and increasing in γ. Hence, the higher γ, the stronger the price-inflation effect.

The economics behind the price effects of anticipated trade has to do with welfare effects. In Appendix D we show how to calculate the speculators' expected lifetime utility before trade, after trade, and in the no-trade case, for the four trade scenarios of Figure 3. We find that when infrequent traders buy bonds in exchange for stocks (top two panels) speculators' expected lifetime utility is *higher* than in the absence of trade, the more so the closer the trade. Hence, stock prices are depressed because speculators look forward to higher welfare, are less willing to save, and their demand for stocks is weaker. When infrequent traders buy stocks in exchange for bonds (bottom two panels) the opposite holds true: speculators' expected lifetime utility is *lower* than in the absence of trade, and this effect is more pronounced the closer the trade. Stock prices are inflated because speculators expect lower welfare, are more willing to save, and their demand for stocks is stronger.

Whether trade has a positive or a negative effect on welfare depends, in turn, on the relative price at which stocks and bonds are exchanged. Risk-adjusted prices at the time of trade must equal (in expectation) their after-trade counterparts [see (5)]: equilibrium conditions fix the rate at which the two assets are exchanged. A welfare-decreasing trade, at these prices, will be accepted by speculators only if they are somehow "committed" to trade, much like market makers on the floor of an exchange.

The parameter γ regulates the degree of intertemporal substitution as well as risk aversion. Hence, the higher γ, the more speculators try to hedge future changes in welfare and the stronger are the price effects discussed above.

6 Conclusions

This paper studies equilibrium prices when trades are triggered by stock prices reaching some threshold level, corresponding to a specific value of the dividend flow. The trade involves two types of agents: speculators, expected-utility maximizers always present in the market, and infrequent traders, whose trading motives are not explicitly modeled. While we take as given the amount of stocks or bonds that infrequent traders supply when the stock price reaches a threshold level, infrequent traders' market orders are filled at prices which are determined endogenously. Our analysis contributes towards the goal of calculating the equilibrium of an economy with realistic frictions, where it should be optimal for some investors to trade infrequently and by finite amounts. Also, we gain a better understanding of the equilibrium effects of a variety of trading strategies, which mechanically generate market orders in response to changes in stock prices.

For tractability, we limited our analysis to situations where closed-form analytical solutions are available: only one trade may occur, and both initial and final speculators' asset allocations are extreme (either all bonds or all stocks). In an appendix available upon request, we extend our analysis to trade scenarios were the initial asset allocation is *intermediate*, infrequent traders' market orders are *uncertain*, and trade may take place at *two* trigger prices. In the following, we highlight the main features of such extensions.

When we allow for initial speculators' asset allocations to include *both* stocks and bonds, closed-form analytic solutions for asset prices are elusive. Still, approximate solutions can be obtained by approximating speculators' marginal utility with an n-degree polynomial in the dividend flow ξ.

Uncertainty as to the infrequent traders' market orders requires a modification of the no-jump conditions (12). Assume, for example, that when P_s hits the lower threshold P_{sl}, infrequent traders buy all speculators' bonds with probability π, and buy all speculators' stocks with probability $(1-\pi)$. This trading structure is summarized by $Y = Y_0$. When the threshold is reached, the barrier disappears forever: $Y = Y_1$. The no-jump conditions would be modified as follows:

$$p_i(\xi_l, S_0, B_0, Y_0) = \pi p_i(\xi_l, S_1, 0, Y_1) + (1-\pi) p_i(\xi_l, 0, B_1, Y_1), \quad i = s,\ b.$$

Immediately after trade, risk-adjusted prices would be either $p_i(\xi_l, S_1, 0, Y_1)$, if stocks are sold, or $p_i(\xi_l, 0, B_1, Y_0)$, if stocks are purchased. In this case, even risk-adjusted prices *jump* as uncertainty is resolved: the occurrence or absence of trade becomes relevant news itself, and risk-adjusted prices respond to it.

The presence of both an *upper* and a *lower* price threshold can be handled as follows. The four constants of integration Q_s, N_s, Q_b, and N_b must satisfy four boundary conditions of the kind illustrated above, requiring no expected jumps in risk-adjusted prices at both price thresholds.

While the analysis employed in these extensions is more involved, the main implications are the same as those of the scenarios of Sections 3 and 5. Again, we find stock prices being depressed when an imminent *sell* order is likely, while they are inflated when an imminent *buy* order is likely.

Appendix

A First-order conditions

Let $W \equiv SP_s + BP_b$ be the speculator's wealth, and $I \equiv I(W, \xi, S, B, Y)$ denote the value function:

$$I\, e^{-\rho t} = \max_{c(\tau),s(\tau),b(\tau)} \int_t^\infty E_t\left\{U[c(\tau)]\right\} e^{-\rho\tau} d\tau.$$

The speculator's optimal program obeys the relations

$$0 = \max_{c(\tau),s(\tau),b(\tau)} \left\{e^{-\rho t}U(c) + E_t\left[\frac{1}{dt}d\left(I\, e^{-\rho t}\right)\right]\right\} \tag{16}$$

$$s.t. - c\, dt = \Delta s\,(P_s + \Delta P_s) + \Delta b\,(P_b + \Delta P_b) - (s\xi + br)\, dt.$$

The first order condition for optimal equity holdings is given by

$$e^{-\rho t}U'(c)\xi + \frac{\partial}{\partial s}E_t\left[\frac{1}{dt}d\left(I\, e^{-\rho t}\right)\right] = 0.$$

Hence, we obtain

$$U'(c)\xi + \frac{\partial}{\partial s}E_t\left[\frac{1}{dt}(-\rho I dt + dI)\right] = 0,$$

and thus

$$U'(c)\xi - \rho\frac{\partial I}{\partial W}\frac{\partial W}{\partial s} + E_t\left[\frac{1}{dt}d\left(\frac{\partial I}{\partial W}\frac{\partial W}{\partial s}\right)\right] = 0.$$

Using the envelope condition $\partial I/\partial W = U'(c)$ and the definition of wealth, yields equation (1). A similar argument yields (2).

B Homogeneous solutions

The homogeneous solutions of (8) have the form

$$h_i(\xi) = Q_i\xi^{\lambda_1} + N_i\xi^{\lambda_2},$$

where Q_i and N_i are constants of integration and λ_1, λ_2 are the roots of the characteristic equation associated with (8):

$$\lambda_1 = \frac{-(\mu - \sigma^2/2) + \sqrt{(\mu - \sigma^2/2)^2 + 2\rho\sigma^2}}{\sigma^2} > 0,$$

$$\lambda_2 = \frac{-(\mu - \sigma^2/2) - \sqrt{(\mu - \sigma^2/2)^2 + 2\rho\sigma^2}}{\sigma^2} < 0.$$

C Prices and transversality conditions

Consider again the transversality conditions

$$\lim_{\tau\to\infty} E_t[p_i(\xi, s, b, Y)]e^{-\rho(\tau-t)} = 0, \quad i = s,\ b.$$

Writing $p_i = g_i + h_i$ as in (6), we note that the particular solution satisfies the transversality condition by construction. As to h_i, we compute

$$E_t(\xi^\lambda)e^{-\rho(\tau-t)} = \xi_t^\lambda e^{[(\sigma^2/2)\lambda^2+(\mu-\sigma^2/2)\lambda-\rho](\tau-t)},$$

and since λ_1 and λ_2 solve $(\sigma^2/2)\lambda^2 + (\mu - \sigma^2/2)\lambda - \rho = 0$, we have

$$\lim_{\tau\to\infty} E_t[h_i(\xi_\tau)]e^{-\rho(\tau-t)} = Q_i\xi_t^{\lambda_1} + N_i\xi_t^{\lambda_2}, \quad i = s,\ b.$$

Hence, homogeneous solutions h_i satisfy the transversality conditions only if $Q_i = N_i = 0$.

D Welfare and price effects of trade

Here, we demonstrate the correspondence between welfare and price effects of anticipated trades.

From equation (16) we have

$$\rho I = U(c) + \frac{1}{dt}E_t(dI). \tag{17}$$

Given (17), $E_t(dI)$ cannot be of order larger than dt, and this rules out expected jumps in the value function I,

$$E_t(\Delta I) = 0,$$

at trading times.

It is useful to decompose the value function I, in equilibrium, in the form

$$I(\xi, S, B, Y) = \max_{c(\tau),s(\tau),b(\tau)} \int_t^\infty E_t\left\{U[c(\tau)]\right\} e^{-\rho(t-\tau)}d\tau$$
$$= \max_{c(\tau),s(\tau),b(\tau)} \int_t^\infty E_t\left\{U[c(\tau)|\text{no trade}]\right\} e^{-\rho(\tau-t)}d\tau + V(\xi, S, B, Y), \tag{18}$$

where $V(\xi, S, B, Y)$ summarizes the effects of trade on expected-lifetime utility.

Expanding $E_t(I)$ in (17) by the usual stochastic calculus arguments, we find that in the interior of no-trade regions the value function I satisfies

$$\rho I = U(c) + \mu\xi I' + \frac{\sigma^2\xi^2}{2}I''.$$

When either $B = 0$ or $S = 0$, the solution for the no-trade component of the value function is obtained as follows. When $B = 0$ we have

$$\max_{c(\tau),s(\tau),b(\tau)} \int_t^\infty E_t \{U[c(\tau)|\text{no trade}]\} e^{-\rho(\tau-t)} d\tau$$
$$= \frac{S}{1-\gamma}\int_t^\infty E_t \left\{[S\xi(\tau)]^{-\gamma}\xi(\tau)\right\} e^{-\rho(\tau-t)} d\tau = \frac{S}{1-\gamma}\frac{\xi_t}{(S\xi_t)^\gamma d_s},$$

where it is useful to recognize that the integrand function corresponds to $f_s \equiv (S\xi)^{-\gamma}\xi$. When $S = 0$, we have

$$\max_{c(\tau),s(\tau),b(\tau)} \int_t^\infty E_t \{U[c(\tau)|\text{no trade}]\} e^{-\rho(\tau-t)} d\tau$$
$$= \frac{B}{1-\gamma}\int_t^\infty E_t \left\{(Br)^{-\gamma} r\right\} e^{-\rho(\tau-t)} d\tau = \frac{B}{1-\gamma}\frac{r}{\rho(Br)^\gamma},$$

where the integrand function corresponds to $f_b \equiv (Br)^{-\gamma}r$.

The function V is thus analogous to the homogeneous solutions (11), and has the form

$$V = Q_v(S,B,Y)\xi^{\lambda_1} + N_v(S,B,Y)\xi^{\lambda_2}.$$

The constants of integration Q_v and N_v are chosen as to ensure no-expected jumps of the value function at trading times. Moreover, depending on the trade scenario, we impose the before-trade value function to be finite as $\xi \to 0$, and to converge to its no-trade counterpart as $\xi \to \infty$.

It can be shown that the function V is always negative for the trade scenarios illustrated in the two top panels of Figure 3: infrequent traders buy stocks in exchange for bonds. The opposite holds true when infrequent traders buy stocks in exchange for bonds (Figure 3, bottom two panels): the function V is always negative. These findings are stronger the higher γ, and are robust to different values of ξ triggering the trade.

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