The Term Structure of Interest-Rate Futures Prices.[1]

Richard C. Stapleton[2], and Marti G. Subrahmanyam[3]

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[2]Department of Accounting and Finance, University of Strathclyde, 100, Cathedral Street, Glasgow, Scotland. Tel:(44)524–381172, Fax:(44)524–846874, Email:rcs@staplet.demon.co.uk

[3]Leonard N. Stern School of Business, New York University, Management Education Center, 44 West 4th Street, Suite 9–190, New York, NY10012–1126, USA. Tel: (212)998–0348, Fax: (212)995–4233, Email:msubrahm@stern.nyu.edu

Abstract

The Term Structure of Interest-Rate Futures Prices

We derive general properties of two-factor models of the term structure of interest rates and, in particular, the process for futures prices and rates. Then, as a special case, we derive a no-arbitrage model of the term structure in which any two futures rates act as factors. The term structure shifts and tilts as the factor rates vary. The cross-sectional properties of the model derive from the solution of a two-dimensional autoregressive process for the short-term rate, which exhibits both mean reversion and a lagged persistence parameter. We show that the correlation of the futures rates is restricted by the no-arbitrage conditions of the model. In addition, we investigate the determinants of the volatility of the futures rates of various maturities. These are shown to be related to the volatilities of the short rate, the volatility of the second factor, the degree of mean reversion and the persistence of the second factor shock. We obtain specific results for futures rates in the case where the logarithm of the short-term rate [e.g., the London Inter-Bank Offer Rate (Libor)] follows a two-dimensional process. Our results lead to empirical hypotheses that are testable using data from the liquid market for Eurocurrency interest rate futures contracts.

1 Introduction

Theoretical models of the term structure of interest rates are of interest to practitioners and financial academics alike, both for the pricing of interest-rate sensitive derivative contracts, and for the measurement of the interest-rate risk arising from holding portfolios of these contracts. The term structure exhibits several patterns of changes over time. In some periods, it shifts up or down, perhaps in response to higher expectations of future inflation. In other periods, it tilts, with short rates rising and long rates falling, perhaps in response to a tightening of monetary policy. Sometimes its shape changes to an appreciable extent, affecting its curvature. Hence a desirable feature of a term-structure model is that it should be able to capture at least the shifts and tilts of the term structure.

One early, intuitively appealing, two-factor model, that captured these empirical features of the term structure is the long rate-spread model of Brennan and Schwartz (1979). Although this model has the attractive feature of modelling term structure movements in terms of two key rates, it is not presented in the "no-arbitrage" setting first proposed by Ho and Lee (1986). Today, it is recognised that a highly desirable, if not a necessary condition for a model to satisfy is the no-arbitrage condition. In this paper, we develop a model of the term structure of futures rates that is consistent with the principle of no-arbitrage. Our approach yields a two-factor shift-tilt model similar in many respects to the Brennan and Schwartz model of spot rates.

The no-arbitrage condition, when applied to the term structure requires the price of a long-term bond to be related to the expected value, under the equivalent martingale measure (EMM), of the future relevant short-term bond prices. This requirement links the cross-sectional properties of the term structure at each point in time to the time-series properties of bond prices and interest rates. This point has been well recognized, in a one-factor setting, since the work of Vasicek (1977). In this paper, we extend this analysis to a two-factor setting. In the context of our two-factor model, we show that, if the short rate follows a mean-reverting two-dimensional process (a process generated by two state variables), then the no-arbitrage condition implies a short rate-long rate model of the term structure of futures rates. In our model, the correlation between the long and short maturity futures rates is restricted by the degree of mean reversion of the short rate and the relative volatilities of the long and short-maturity futures rates.

Also, the volatility structure of futures rates of various maturities can be derived explicitly from the assumed process for the spot short rate.

We suggest a time series model in which the conditional mean of the short rate follows a two-dimensional process, similar to that proposed by Hull and White (1994). This assumption allows us to nest the popular AR(1) single-factor model as a special case. It is also general enough to produce stochastic no-arbitrage term structures with shapes that capture most of those observed empirically. A similar model in which the conditional mean of the short rate is stochastic has been suggested by Balduzzi, Das and Foresi (1998) and analysed by Gong and Remolona (1997).

Previous work on the term structure of interest rates has concentrated mainly on bond yields of varying maturities or, more recently, on forward rates. In contrast, this paper concentrates on futures rates, partly motivated by the relative lack of previous theoretical models of interest-rate futures prices. However, the main reason for focussing on futures rates is analytical tractability. Futures prices are simple expectations of spot prices under the EMM, whereas forward prices and spot rates involve more complex relationships. It follows that futures prices and futures rates are fairly simple to derive from the dynamics of the spot rate. In contrast, closed-form solutions for forward rates have been obtained only under rather restrictive (e.g. Gaussian) assumptions. Further, from an empirical perspective, since forward and futures rates differ only by a convexity adjustment, it is likely that most of the time series and cross-sectional properties of futures rates are shared by forward rates, to a close approximation, at least for short maturity contracts. It makes sense, therefore, to analyse these properties, even if the ultimate goal is knowledge of the term-structure behaviour of forward or spot prices. Finally, the analysis of futures rates is attractive because of the availability of data from trading on organized futures exchanges. Hence, the models derived in the paper are directly testable, using data from the liquid market for Eurocurrency interest rate futures contracts.

Recent literature, mainly inspired by the practical need to price various interest rate derivative contracts, has produced a rich variety of term structure models. In section 2 of this paper we discuss this literature, relate our analysis to previously proposed models and explain the incremental contribution of our work. One of the most difficult aspects of term structure modelling is notation and definition of the relevant variables and parameters. For this reason, we devote much of section 3 to a description of the set-up of the problem, the variables and our notation. In this section, we then derive some general properties that characterise two-factor models. In particular, we show that if a price of a zero-coupon bond follows a two-dimensional process, then its conditional expectation is generated by a two-factor model. We also analyse futures prices of zero-coupon bonds and interest-rate futures for the general case where the spot price or rate follows a two-dimensional process. In section 4 we assume that the logarithm of the London Inter-Bank Offer Rate (Libor) follows a two-dimensional process and derive our main result for futures contracts on the Libor. Numerical simulations of the results for the term structure of futures rates, the futures volatility structure and the correlation of futures and spot rates are shown in section 5. The conclusions and possible applications of our model to the valuation of interest rate options and to risk management are discussed in section 6.

2 Term Structure Models : The Literature

The literature on the pricing of futures contracts was pioneered by Cox, Ingersoll and Ross (1981) [CIR], who characterized the futures price of an asset as the expectation, under the risk-neutral measure, of the spot price of the asset on the expiration date. Although futures prices, in general, have been considered by many other papers in the literature, there are few that have dealt specifically with the pricing of interest rate futures contracts. This gap in the literature is striking, given that short-term interest rate futures contracts based on the Libor are traded in many markets and are among the most liquid futures contracts. An important exception is the paper by Sundaresan (1991) that uses the general CIR characterization to price Libor-based futures contracts. Sundaresan shows that, under the risk-neutral measure, the futures interest rate is the expectation of the spot interest rate in the future. This follows from the fact that the Libor futures contract is written on the three-month Libor itself, rather than on the price of a zero-coupon instrument. This fact is used and its implications are derived in Brace, Gatarek and Musiela (1997) [BGM]. In the present paper, we use this result to obtain closed-form results for the term structure of futures interest rates.

In a comprehensive paper on the term structure of futures rates that presents both theoretical and empirical results, Jegadeesh and Pennacchi (1996) [JP] provide a model of futures rates based on a two-factor extension of the Vasicek (1977) model. Similar in spirit to the Vasicek model, they assume that the (continuously-compounded) interest rate is normally distributed, and derive bond prices and Libors in a two-factor equilibrium model, that involves the market price of risk. They then estimate the model using futures prices of Libor contracts, backing out estimates of the coefficients of mean-reversion of the short rate as well as the second stochastic conditional-mean factor. Our general model is closely related to the JP paper, with the important distinction that it is embedded in an arbitrage-free, rather than an equilibrium framework, thus eliminating the need for explicitly incorporating the market price of risk. Although our analysis is based on weaker assumptions, we are able to derive quite general, distribution-free results for futures rates. We then include, as a significant special case, a model in which the interest rate is lognormal. This is an assumption that is widely used in the modern term-structure literature. Our main result is that a cross-sectional relationship exists for futures rates, where a futures rate is log-linear in any two futures rates.

The work of Gong and Remolona (1997) is similar, in some respects, to that of JP. They also employ a two-factor model, in which the second factor is the conditional mean of the short-rate process. However, they focus on the yield of long-dated bonds rather than on the futures rates. In their model, the short-term rate is linear in the two factors. Also, in a manner similar to Vasicek (1977), they assume a market price of risk, solve for bond prices, and back-out the long-term rates and the variances of the two factors. In contrast, we work under the equivalent martingale measure and directly derive futures rates for all maturities. We are also able to compute the variances of the futures rates of different maturities and the correlations between them.

While the literature on futures rates is somewhat sparse, the same is not true for forward rates. Indeed, much of the recent literature, dating back to the work of Ho and Lee (1986), has been concerned with the evolution of forward rates. The most widely cited work in this area is by Heath, Jarrow and Morton [HJM] (1990a, 1990b, 1992). HJM provide a continuous-time limit to the Ho-Lee model and generalize their results to a forward rate which evolves as a generalized Ito-process with multiple factors. The HJM paper can be distinguished from our paper in terms of the inputs to the two frameworks. The required input to the HJM-type models is the term structure of the volatility of forward rates. In contrast, in our paper, we derive the term structure of volatility of futures rates from a more basic assumption regarding the process for the spot rate. To the extent that the futures and forward volatility structures are related, our analysis in this paper provides a link between the spot-rate models of the Vasicek type and the extended HJM-type forward rate models.

The two-factor models developed in this paper are related also to the exponential affine-class of term-structure models introduced by Duffie and Kan (1994). This class is defined as the one where the continuously- compounded spot rate is a linear function of any n factors or spot rates. In an interesting special case of our model, where the logarithm of the Libor evolves as a two-dimensional linear process, it is the logarithm of the futures rate that is linear in the logarithm of any two futures rates.

Finally, our analysis derives from previous papers that have assumed a two-dimensional process for the spot interest rate such as Hull and White (1994). Following Vasicek (1977), Hull and White investigate models where some general function of the price of a zero-coupon follows a two-dimensional process with a stochastic conditional mean. Similar models have been proposed in Balduzzi, Das and Foresi (1998). In section 4 of this paper we investigate the properties of a model in which the short-term rate of interest, defined on a Libor basis, is lognormally distributed. Models of this type have been investigated by Miltersen, Sandmann and Sondermann (1997) and by BGM. This assumption has the advantage that the variance is dependent on the level of the rate. Thus, rates are skewed to the right in our model, which may be empirically realistic for the short term interest rate markets in several of the major currencies. Also, the Black, Derman and Toy (1990)(BDT) and Black and Karasinski (1990) models have similar assumptions. However all these models are single-factor models. Our incremental contribution to this literature is that we analyze a particularly simple two-factor extension of the BDT model. We also provide a set of necessary and sufficient conditions for the cross-sectional two-factor model to hold in a no-arbitrage setting.

3 Some general properties of two-factor models

In this section, we first introduce the notation that we will employ to denote zero-coupon bond prices, short-term interest rates, and futures rates. We then establish two statistical results, that hold for any two-factor process of the form that we assume for the short-term rate. These results are used to establish a general proposition, that holds for the conditional expectation of any function of the zero-coupon bond price. The conditional expectation is of key significance, since the futures price (or rate) is closely related to the conditional expectation of the future spot interest rate. These results are directly applied later in the section to establish futures prices and futures rates.

3.1 Definitions and notation

We denote P_t as the time-t price of a zero-coupon bond paying \$1 with certainty at time $t+m$, where m is measured in years. The short-term interest rate is defined in relation to this m-year bond, where m is fixed. The short-term interest rate for m-year money at time t is denoted as i_t, where i_t is a function of P_t. The conventional definition of the interest rate is the continuously compounded rate, where $i_t = -ln(P_t)/m$. In this paper, we also investigate alternative definitions of the interest rate function. The other difference between this spot rate and the interest rate in the paper of HJM is that m, as in BGM, is not necessarily a very short (instantaneous) period. However, as in HJM, m does not vary.

We are concerned with interest rate contracts for delivery at a future date T. We denote the futures rate as $F_{t,T}$, the rate contracted at t for delivery at T of an m-period loan. We denote the logarithm of the futures rate as $f_{t,T} = \ln\,[F_{t,T}]$. Note that under this notation, which is broadly consistent with HJM, $F_{t,t} = i_t$ and $f_{t,t} = \ln\,(i_t)$.

The mean and annualised standard deviation at time t (of the logarithm) of the spot rate at time T, under the EMM, are denoted

$$\begin{aligned}\mu(t,T,T) &= E_t[f_{T,T}] \\ \sigma(t,T,T) &= [\mathrm{var}_t[f_{T,T}]/(T-t)]^{\frac{1}{2}}\end{aligned}$$

respectively.

In particular, at time 0, the mean and standard deviation of the log-spot rates at t and T respectively can be written as

$$\mu(0,t,t) = E_0[f_{t,t}]$$

$$\sigma(0,t,t) = [\text{var}_0[f_{t,t}]/t]^{\frac{1}{2}}$$

and

$$\begin{aligned}\mu(0,T,T) &= E_0[f_{T,T}] \\ \sigma(0,T,T) &= [\text{var}_0[f_{T,T}]/T]^{\frac{1}{2}}\end{aligned}$$

In the case of futures rates, we define the mean and standard deviation at time-0 of the log-futures at time-t for delivery at time-T as

$$\begin{aligned}\mu(0,t,T) &= E_0[f_{t,T}] \\ \sigma(0,t,T) &= [\text{var}_0[f_{t,T}]/t]^{\frac{1}{2}}\end{aligned}$$

Table 1 summarizes the notation used in the paper.

Note that the mean and variance of the spot rate are statistics of a time-t or a time-T measurable random variable. In the case of the futures rates, these statistics relate to a time-t measurable random variable.

3.2 General properties of two-factor models

We now establish that, if a variable follows a two-dimensional process, the conditional expectation of the variable is necessarily governed by a two-factor cross-sectional model.[1] We begin by proving this result quite generally, and then apply it to the case of bond prices and interest rates. We first state and prove the following lemmas regarding the conditional mean and the variance of the conditional mean for a general two-dimensional process:

Lemma 1 *The variable x_t follows the process*

$$x_t = (1-c)x_{t-1} + y_{t-1} + \epsilon_t$$

where

$$y_t = (1-\alpha)y_{t-1} + \nu_t$$

if and only if the conditional expectation of x_{t+k} is of the form

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1})$$

[1]Hull and White (1994) and Jegadeesh and Pennacchi (1996), for example, assume such a two-dimensional process for short-term interest rates.

where

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Proof. See appendix 1.

Lemma 2 *If the variable x_t follows the process*

$$x_t = (1-c)x_{t-1} + y_{t-1} + \epsilon_t$$

where $E(x_{t-1}y_{t-1}) = 0$ and where

$$y_t = (1-\alpha)y_t + \nu_t$$

then the conditional variance of $E(x_{t+k})$ is given by

$$\begin{aligned} \text{var}_{t-1}[E_t(x_{t+k})] &= (1-c)^{2k}\text{var}_{t-1}[x_t] \\ &\quad + \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}\right]^2 \text{var}_{t-1}(\nu_t) \end{aligned}$$

Proof. See appendix 1.

We can now apply these results to the case of bond prices and interest rates. We assume that some function of the zero-coupon bond price, P_t, follows the process assumed in Lemma 1. Specifically, let

$$x_t = g(P_t) - E_0[g(P_t)]$$

where $g(P_t)$ is any function and $E_0[.]$ is its expectation at time 0. Note that this allows for any specification of the relationship of interest rates to bond price, covering alternative definitions, including continuous or discrete compounding. It follows immediately from Lemmas 1 and 2 that:

Proposition 1 *A function of the price of an m-year zero-coupon bond P_t follows a two-dimensional process:*

$$g(P_t) = E_0[g(P_t)] + (1-c)\{g(P_{t-1}) - E_0[g(P_{t-1})]\} + y_{t-1} + \epsilon_t \qquad (1)$$

where

$$y_t = (1-\alpha)y_{t-1} + \nu_t$$

if and only if the conditional expectation, $E_t[g(P_{t+k})]$, is given by

$$E_t[g(P_{t+k})] - E_0[g(P_{t+k})] = a_k\,[g(P_t) - E_0[g(P_t)]] + b_k\,[E_t[g(P_{t+1})] - E_0[g(P_{t+1})]]$$

where

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Also, the variance of the conditional expectation is given by

$$\begin{aligned}\mathrm{var}_{t-1}[E_t(g(P_{t+k}))] &= (1-c)^{2k}\mathrm{var}_{t-1}[g(P_t)] \\ &\quad + \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}\right]^2 \mathrm{var}_{t-1}(\nu_t).\end{aligned}$$

Proposition 1 states the implications of a two dimensional, stochastic conditional mean, process for an arbitrary function of the zero-coupon bond price. The function could be a rate of interest, such as the continuously compounded rate (as in HJM) or the Libor rate (as in BGM), or it could be the price of the zero-coupon bond itself. The first part of the proposition restricts the cross-sectional properties of the conditional expectation. The second part gives an estimate of the volatility of the derived process for the conditional expectation. As we will see in the next section, these properties are directly relevant for the investigation of futures prices and rates. The intuition behind Proposition 1 is that the two influences on the function of the zero-bond prices, one of which is lagged, yield a cross-sectional structure with two factors. This contrasts with the single factor case, where there is a simple correspondence between what is driving the time series process and the cross-sectional factor structure.

We first look at an important special case of the Proposition 1, where the logarithm of the short-term interest rate follows the two-factor process. This is also a special case of Hull and White (1994). In Hull and White, the function $g(P_t) = \ln(i_t)$, where i_t is the continuously compounded short rate, follows the continuous-time stochastic process

$$d \ln i_t = [\theta(t) - c \ln i_t + u]dt + \sigma_1 dz_1 \tag{2}$$

where $\theta(t)$ is a parameter chosen to match the model parameters to the initial term structure, u is a stochastic conditional mean, and c is the mean-reversion coefficient. σ_1 is the instantaneous standard deviation of the Weiner process, dz_1. The variable u itself follows the stochastic process

$$du = \alpha u dt + \sigma_2 dz_2$$

where α is the mean reversion of u and σ_2 is the instantaneous standard deviation of the Weiner process, dz_2. The two Weiner processes have an instantaneous correlation of ρ. Given the notation introduced earlier, the above model, in discrete form, leads to:[2]

$$f_{t,t} - \mu(0,t,t) = [f_{t-1,t-1} - \mu(0,t-1,t-1)](1-c) + y_{t-1} + \varepsilon_t, \ \ \forall t, \tag{3}$$

where

$$y_t = (1-\alpha)y_{t-1} + \nu_t$$

We assume for simplicity that ε_t and ν_t are uncorrelated.[3] In this case, we have, as a corollary of Proposition 1:

Corollary 1 *The logarithm of the spot rate follows the process*

$$f_{t,t} - \mu(0,t,t) = [f_{t-1,t-1} - \mu(0,t-1,t-1)](1-c) + y_{t-1} + \varepsilon_t, \ \ \forall t, \tag{4}$$

where

$$y_t = y_{t-1}(1-\alpha) + \nu_t$$

[2] To show this, take the unconditional expectation of the discrete version of equation (2) and the expression for the stochastic mean factor, and substitute for the means of the log-interest rate and the stochastic mean factors.

[3] Note that this assumption is easily generalized to the case of correlated errors at the cost of some algebraic complexity. However, little true generality would be achieved, since the model can be converted into one with uncorrelated errors by orthogonalising the factors.

if and only if the expectation of the logarithm of the interest rate i_{t+k} at time t is

$$\begin{aligned}\mu(t,t+k,t+k)-\mu(0,t+k,t+k) &= a_k[f_{t,t}-\mu(0,t,t)] \\ &+ b_k[\mu(t,t+1,t+1)-\mu(0,t+1,t+1)]\end{aligned}$$

Also, the variance of the conditional expectation is given by

$$\begin{aligned}\mathrm{var}_{t-1}[E_t(f_{t+k,t+k})] &= (1-c)^{2k}\mathrm{var}_{t-1}[\mu(t,t+1,t+1)] \\ &+ \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}\right]^2 \mathrm{var}_{t-1}(\nu_t)\end{aligned}$$

Proof. The corollary follows directly from Proposition 1 where $g(P_t) = ln(i_t)$ and i_t is any interest rate function of P_t. □

In this case, the spot rate follows a logarithmic, mean reverting process with a stochastic conditional mean. The implication is that the conditional expectation of the logarithmic rate for maturity $t+k$ is generated by a two-factor cross-sectional model. Also, the variance of the conditional expectation is determined by the mean reversion of the short rate, c, the variance of the short rate, the variance of the stochastic mean factor and its mean reversion, α. The corollary has direct implications for the behaviour of futures rates in a logarithmic short-rate model. These are explored in section 4, where we assume that the interest rate function is the m-year Libor, rather than the continuously compounded rate.

3.3 Futures prices and rates in a no-arbitrage economy

In this sub-section, we apply the results in the previous sub-section 3.2 to derive futures prices and futures interest rates in a no-arbitrage setting. We assume here that the two-dimensional process, for prices or rates defined above, holds under the EMM. The EMM is the measure under which all zero-coupon bond prices, normalised by the money market account, follow martingales.

Cox, Ingersoll and Ross (1981) and Jarrow and Oldfield (1981) established the proposition that the futures price, of any asset, is the expected value of the future spot price, where the expected value is taken with respect to the equivalent martingale measure. We can now apply this result to determine, for example, the behaviour of the futures prices of zero-coupon bonds, assuming that the bond prices are generated by the two-dimensional process analysed in sub-section 3.2. Since there is a one-to-one relationship between zero-coupon bond prices and short-term interest rates, defined in a particular way, we can then proceed to derive a model for futures interest rates.

Initially, we make no specific distributional assumptions. We assume only a) the existence of a no-arbitrage economy in which the EMM exists, and b) that a function of the time t price of an m-year zero-coupon bond, $g(P_t)$, follows a two-dimensional process of the general form assumed in Lemma 1, and c) that a market exists for trading futures contracts on $g(P_t)$, where the contracts are marked-to-market at the same frequency as the definition of the discrete time-period from t to $t+1$. We first establish a general result, for any function $g(P_t)$, and then illustrate it with some familiar examples. We denote the futures price, at t, for delivery of $g(P_{t+k})$, at $t+k$, as $g(P_{t,t+k})$. We have

Proposition 2 *Assume that equation (1) holds for $g(P_t)$ under the EMM, then*

$$g(P_{t,t+k}) - g(P_{0,t+k}) = a_k\left[g(P_t) - g(P_{0,t})\right] + b_k\left[g(P_{t,t+1}) - g(P_{0,t+1})\right]$$

where

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Also, the variance of the futures price is given by

$$\begin{aligned}\mathrm{var}_{t-1}[g(P_{t,t+k})] &= (1-c)^{2k}\mathrm{var}_{t-1}[g(P_t)] \\ &\quad + \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}\right]^2 \mathrm{var}_{t-1}(\nu_t)\end{aligned}$$

Proof. From CIR (1981), proposition 2 and Pliska (1997), the futures price of any payoff is its expected value, under the EMM. Applying this result to $g(P_{t+k})$, and using Proposition 1, yields Proposition 2 .□

The rather general result in Proposition 2 is of interest because of two special cases. The first is the case where the futures contract is on the zero-coupon bond itself. The second is the case of a futures contract on an interest rate, which is a function of the zero-coupon bond price. We consider these cases in the corollaries below.

We first have, as an implication of Proposition 2:

Corollary 2 (A Linear Process for the Zero-Bond Price) *The price of an m-year zero-coupon bond P_t follows a two-dimensional process under the equivalent martingale measure (EMM):*

$$P_t = E_0(P_t) + (1-c)[P_{t-1} - E_0(P_{t-1})] + y_{t-1} + \epsilon_t$$

where

$$y_{t-1} = (1-\alpha)y_{t-2} + \nu_{t-1}$$

if and only if the kth futures price $P_{t,t+k}$ is given by

$$P_{t,t+k} - P_{0,t+k} = a_k[P_t - P_{0,t}] + b_k[P_{t,t+1} - P_{0,t+1}]$$

where

$$b_k = \sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Also, the variance of the futures price is given by

$$\begin{aligned}\text{var}_{t-1}[P_{t,t+k}] &= (1-c)^{2k}\text{var}_{t-1}[P_t] \\ &\quad + \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}\right]^2 \text{var}_{t-1}(\nu_t)\end{aligned}$$

Proof. This follows as a special case of Proposition 2 with $g(P_t) = P_t$, and $g(P_{t,t+k}) = P_{t,t+k}$. □

The above results show that futures prices at time t are generated by a linear two-factor model if and only if the zero-bond price follows a process of the Hull-White type. Note that the two factors generating the kth futures price are the spot price of the bond and the first futures price, i.e., the futures with maturity equal to $t+1$. Similarly, the variance of the kth futures price is determined by the variance of the spot bond price, the variance of the conditional mean and the mean reversion coefficients.

Corollary 2 is helpful in understanding the conditions under which the term structure follows a two-factor process. Essentially, if futures prices of long-dated futures contracts are given by the cross-sectional model in Proposition 2, then forward prices, and also futures and forward rates will follow two-factor models. The relationship for interest rates, however, is in general complex, since the function $i_t(P_t)$ is, in general, non-linear.

We next illustrate the use of Proposition 2 in the case of interest rate (as opposed to price) futures. Instead of assuming that the price of a zero-coupon bond follows a two-dimensional, linear process, we now assume that the interest rate, defined as any function of the zero-coupon bond price, follows a two-dimensional, linear process. We have the following corollary of Proposition 2:

Corollary 3 (A Linear Process for the Interest Rate) *In a no-arbitrage economy the short-term rate of interest follows a process of the form*

$$F_{t,t} = E_0[F_{t,t}] + (1-c)[F_{t-1,t-1} - E_0[F_{t-1,t-1}] + y_{t-1} + \epsilon_t$$

where

$$y_t = (1-\alpha)y_{t-1} + \nu_t$$

if and only if the term structure of futures rates at time t is generated by a two-factor model, where the kth *futures rate is given by*

$$\begin{aligned} F_{t,t+k} - F_{0,t+k} &= a_k[F_{t,t} - F_{0,t}] \\ &\quad + b_k[F_{t,t+1} - F_{0,t+1}] \end{aligned} \tag{5}$$

where

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Also, the variance of the kth futures is

$$\begin{aligned} \mathrm{var}_{t-1}[F_{t,t+k}] &= (1-c)^{2k}\mathrm{var}_{t-1}[F_{t,t}] \\ &+ \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}\right]^2 \mathrm{var}_{t-1}(\nu_t)n \end{aligned}$$

Proof. The proof of the corollary again follows as a special case of Proposition 2, where $g(P_t) = i_t$. □

The corollary illustrates the simple two-factor structure of futures rates that is implied by the two-dimensional process for the spot rate. Note that the mean reversion coefficients are embedded in the cross-sectional coefficients, a_k and b_k. Also, it follows from (5), given the linear structure, that the futures rates will be normally distributed, if the spot rate and the first futures rate are normally distributed. Hence, the corollary could be helpful in building a Gaussian model of the term structure of futures rates.[4]

4 Libor futures prices in a lognormal short-rate model

In the previous section we showed that if either the price of a zero-coupon bond, or a short-term interest rate, evolves as a two-dimensional mean-reverting process under the risk-neutral measure, then a simple cross-sectional

[4]In a two-factor extension of the Vasicek (1977) framework, Jegadeesh and Pennacchi (1996) estimate a two-factor term structure model similar to that in equation (5)under the assumption of normally distributed interest rates. They show that their model fits the level of Eurodollar short-term interest rates contracts rather well for maturities of up to two years, and *changes* in the rates for longer-dated contracts. It is possible that this is because of ignoring the skewness effect (due to the normality assumption), which becomes significant for longer-dated contracts.

relationship exists between futures prices (or rates) of various maturities. In principle, these models could be applied to predict relationships between the prices of Eurocurrency futures contracts, based on Libor or some other similar reference rate, which are the most important short-term interest rate futures contract traded on the markets. However, in the case of Libor, the consensus in the academic literature and in market practice is that changes in interest rates are dependent on the level of interest rates. In particular, a lognormal distribution for short-term interest rates is commonly assumed.[5]

When the logarithm of the short-term interest rate follows a linear process, the results of the analysis of futures prices in section 3 cannot be used, since the market does not trade futures on the logarithm of the Libor. However, if it is assumed that the Libor follows a lognormal process, standard results relating the mean of the lognormal variable to its logarithmic mean can be used to derive results for futures prices in this case, using Corollary 1, from section 3.

The standard Eurodollar futures contract is defined on the Libor. We now assume that the the function $g(P_t)$ in Proposition 1 gives us the logarithm of the Libor. Since the Libor, i_t, is defined on an "add-on basis", it is related to the zero-coupon bond price, P_t, by the relation

$$P_t = 1/(1 + i_t m),$$

where m is the proportion of a year.[6] The logarithm of i_t is therefore given by

$$f(t,t) = g(P_t) = \ln\{[(1/P_t) - 1]/m\}.$$

We assume now that the logarithm of the Libor follows a two-dimensional lognormal process, under the equivalent martingale measure. We make use of the following Lemma:

Lemma 3 *In a no-arbitrage economy, if the Libor follows a lognormal process under the equivalent martingale measure, then the k-period Libor futures*

[5]This is borne out by the empirical research of Chan et.al.,(1992) and more recently of Eom (1994) and Bliss and Smith (1998). There continues to be debate over the elasticity parameter of the changes in interest rates with respect to the level.

[6]In the Libor contract, m has to be adjusted for the day-count convention. Hence, m becomes the actual number of days of the loan contract divided by the day-count basis (usually 360 days).

rate at time t is

$$F_{t,t+k} = \exp[\mu(t,t+k,t+k) + \frac{kn}{2}\sigma^2(t,t+k,t+k)] \tag{6}$$

where n is the length, in years, of the period t to $t+1$. Also, $F_{t,t+k}$ is lognormal, with logarithmic mean

$$\mu(0,t,t+k) = \mu(0,t+k,t+k) + \frac{kn}{2}\sigma^2(t,t+k,t+k) \tag{7}$$

Proof. See Appendix 2.

Lemma 3 establishes that the lognormality of the spot Libor implies lognormality of the Libor futures. This is important for our analysis of the behaviour of the futures rate, in this section. This property follows from the CIR (1981) result that the futures price is the expectation, under the EMM, of the spot price. Secondly, the lemma establishes a useful relationship between the logarithmic mean of the futures rate and that of the corresponding spot rate. We will use this relationship, which itself follows from the lognormality of the futures and spot rates, in the proof of the following proposition.

Proposition 3 *In a no-arbitrage economy, in which the Libor follows a two-dimensional lognormal process, under the equivalent martingale measure, of the form*

$$f_{t,t} = \mu(0,t,t) + [f_{t-1,t-1} - \mu(0,t-1,t-1)](1-c) + y_{t-1} + \epsilon_t \tag{8}$$

where

$$y_t = (1-\alpha)y_{t-1} + \nu_t,$$

where ϵ_t and ν_t are independent, normally distributed variables, the term structure of futures rates at time t is generated by a two-factor model. The kth *futures rate is given by*

$$\begin{aligned} f_{t,t+k} - \mu(0,t,t+k) &= a_k[f_{t,t} - \mu(0,t,t)] \\ &\quad + b_k[f_{t,t+1} - \mu(0,t,t+1)] \end{aligned} \tag{9}$$

where

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Also, the volatility of the kth futures is given by $\sigma(t-1, t, t+k)$ *in*

$$\begin{aligned}\sigma^2(t-1,t,t+k)n &= (1-c)^{2k}\sigma^2(t-1,t,t)n \\ &\quad + \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}\right]^2 \mathrm{var}_{t-1}(\nu_t)\end{aligned}$$

Proof. From Corollary 1

$$\begin{aligned}\mu(t,t+k,t+k) - \mu(0,t+k,t+k) &= a_k[f_{t,t} - \mu(0,t,t)] \\ &+ b_k[\mu(t,t+1,t+1) - \mu(0,t+1,t+1)]\end{aligned}$$

is a necessary and sufficient condition. Substituting the results of Lemma 3 then yields the statement in the proposition. Again, substituting Lemma 3 in the expression for the conditional variance of the conditional expectation in Corollary 1, yields the volatility of the kth futures contract. □

Proposition 3 is the main result of this paper. Equation (9) shows the conditions under which a simple log-linear relationship exists for futures rates of various maturities. In this cross-sectional model, futures rates are related to the spot Libor and the first Libor futures. The result extends to the lognormal Libor case the prior results on the term structure of Duffie and Kan (1993) and Gong and Remolona (1997). Proposition 3 relates the kth futures rate, (i.e., the one expiring in k periods) to the spot rate $f_{t,t}$ and the first futures rate, $f_{t+1,t+1}$.[7] For example, this means that the kth

[7] The equation can be re-written by substituting the no-arbitrage relationship between the time-0 futures rates and the mean terms from equations (6) and (7) in Lemma 3 into (9). Specifically, the mean terms ($\mu(0,t,t+k)$, $\mu(0,t,t)$ and $\mu(0,t,t+1)$) can be replaced by the futures rates yielding an expression for changes in the futures rates, which is again linear in $f_{t,t}$ and $f_{t,t+1}$. However, the expression is complex due to the presence of drift terms, which depend on the volatilities of these rates.

three-month futures rate is related to the spot three-month rate and the one-period, three-month futures rate. However, following Duffie and Kan (1993), if the model is linear in two such rates, it can always be expressed in terms of *any* two futures rates. In the present context, therefore, the kth futures rate can be expressed as a function of the spot rate and the Nth futures rate. We have the following implication of Proposition 3:

Corollary 4 *Suppose any two futures rates are chosen as factors, where N_1 and N_2 are the maturities of the factors, then the following linear model holds for the kth futures rate:*

$$\begin{aligned} f_{t,t+k} &= \mu(0,t,t+k) + A_k(N_1,N_2)[f_{t,t+N_1} - \mu(0,t,t+N_1)] \\ &+ B_k(N_1,N_2)[f_{t,t+N_2} - \mu(0,t,t+N_2)] \end{aligned} \tag{10}$$

where

$$B_k(N_1,N_2) = (a_k b_{N_1} - b_k a_{N_1})/(a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

$$A_k(N_1,N_2) = (-a_k b_{N_1} + b_k a_{N_1})/(a_{N_2} b_{N_1} - b_{N_2} a_{N_1}),$$

and

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Proof. Corollary 4 follows by solving equation (5) for $k = N_1$, and $k = N_2$ and then substituting back into equation (5).□

The kth futures rate is log-linear in any two futures rates. The meaning of the result is illustrated by the following special case, where there is no mean reversion in the short rate, i.e. the logarithm of the Libor follows a random walk.

Corollary 5 The Random Walk Case

Suppose that $c = 0$, i.e., the logarithm of the Libor follows a random walk. In this case, the kth futures Libor is

$$f_{t,t+k} = \mu(0,t,t+k) + \left(\frac{N-k}{N}\right)[f_{t,t} - \mu(0,t,t)] + \left(\frac{k}{N}\right)[f_{t,t+N} - \mu(0,t,t+N)]. \tag{11}$$

Proof. Corollary 5 follows directly from Corollary 4 with

$$b_{k,N} = \frac{k}{N},$$

and hence,

$$a_{k,n} = \frac{N-k}{N}.$$

□

Here, the kth futures is affected by changes in the Nth futures according to how close k is to N. Equation (11) is a simple two-factor "duration-type" model, in which the term structure of futures rates shifts and tilts. This and other special cases are illustrated, using numerical examples, in the next section.

We now derive the correlation between the futures rates and the spot rate. This is important for two reasons. First, the correlation between any two futures rates, which may be taken as factors in the above model, cannot be determined independently of the mean-reversion of the short rate, c, and the persistence of the conditional mean shock factor, α. Second, the correlation is an important determinant of the value of certain derivatives, whose payoff depends on the difference between various rates of interest in the term structure. From Proposition 3, the conditional variance of the futures rate is

$$\begin{aligned}\sigma^2(t-1,t,t+k)n &= (1-c)^{2k}\mathrm{var}_{t-1}[f_{t,t}] \\ &\quad + \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}\right]^2 \mathrm{var}_{t-1}(\nu_t) \quad (12)\end{aligned}$$

and since the variance of the spot rate is

$$\sigma^2(t-1,t,t)n = \mathrm{var}_{t-1}[f_{t,t}]$$

it follows that the covariance of the spot and the (logarithm of the) k-th futures rate is

$$\begin{aligned}\mathrm{cov}_{t-1}[f_{t,t},f_{t,t+k}] &= (1-c)^k\mathrm{var}_{t-1}[f_{t,t}] \\ &= (1-c)^k\sigma^2(t-1,t,t)n. \quad (13)\end{aligned}$$

The correlation of the spot and futures rates is therefore given by

$$\rho(t-1,t,t+k) = \frac{(1-c)^k\sigma(t-1,t,t)}{\sigma(t-1,t,t+k)} \quad (14)$$

This expression for the correlation of the short rate and the kth futures rate illustrates an important implication of the no-arbitrage model. Given the volatilities of the spot and futures rates, we cannot independently choose both the correlation and the degree of mean-reversion. The no-arbitrage model restricts the correlation between the two factors to be a function of the degree of mean-reversion of the short rate.[8] Further, because the futures volatility depends, in addition, on the degree of persistence of the premium factor shock, all the parameters affect the correlation.

[8]This would imply that one cannot arbitrarily specify a two-factor model such as Brennan and Schwartz (1979) or Bühler et.al., (1999) without restricting the correlation between the short and long rates.

5 Libor Futures, Volatilities and Correlation

In this section we present some examples that illustrate the effect of parameter values on the Libor futures rates produced by the model. In these examples, we use different values of the mean-reversion coefficients of the two factors. We refer to the mean reversion of the second factor (the stochastic conditional mean) as persistence. This is because $(1-\alpha)$ determines the persistence or memory effect of a conditional mean shock. Also, referring to $(1-\alpha)$ as persistence serves to distinguish it clearly from the mean reversion of the short rate (c) which we refer to simply as mean reversion.

5.1 The Cross-sectional properties of Libor futures rates

What types of term structures are the possible results of the two-factor model derived in Proposition 3? This question is best answered by considering numerical examples, where we choose different parameter values. According to equation (8), the mean reversion coefficient (c) and the persistence parameter (α), together with the changes in the short rate and the conditional mean factor should determine the cross-sectional shape of the term structure. In Figures 1-5, we show a series of examples, for different parameter values, assuming an initial term structure where all futures rates are 5%. In all cases the short rate shifts up or down by 1%. The conditional mean factor moves up or down by 0.3% in the same direction in Figures 1-4 and in the opposite direction in Figure 5. Figures 1 and 2 illustrate the case of no mean reversion with different degrees of persistence. Figures 3 and 4 illustrate cases of mean-reversion.

The Figures 1-5 illustrate the effects of 1) persistence $(1-\alpha)$, 2) mean reversion coefficient of the short rate (c), and 3) the shock to the conditional mean (y). We use two values each for the persistence $(1-\alpha)$ (0.25 and 0.75) and mean reversion coefficient (c) (0 and 0.05). We consider the effect of a shock to the conditional mean of 0.3% up and down. We consider these as follows:

1. *The persistence factor*

   In Figures 1 and 2, we compare a one-factor model, where the short rate moves by 1%, up or down, with a two-factor model, where the first futures rate moves by a further 0.3%. The one-factor model, with mean reversion, $c = 0$, yields a parallel shift in the term structure of futures rates. In the two-factor model, high persistence of the premium shock (low α) implies relatively large changes in long-maturity futures rates. Figure 2 is similar to Figure 1 except that the persistence is low (high α). With higher persistence, as in Figure 1, the shock to the stochastic mean lasts longer, resulting in a large effect for longer maturity rates. In contrast, in Figure 2, the shock dies out more quickly, resulting is a smaller effect for longer maturity rates.

2. *Mean reversion of the Libor*

   In Figures 3 and 4 we introduce mean reversion, with $c = 0.05$. In the one-factor case, the effect is to produce continuously downward or upward sloping term structures according to whether the initial shock to the Libor is positive or negative. In the two-factor case, the result depends also on the degree of persistence. If persistence is relatively high, as in Figure 3, the net effect can be a rising and then falling futures curve. When persistence is low, as in Figure 4, the effect of mean reversion dominates and the futures curve is downward sloping over most of its range.

3. *The stochastic mean factor*

   The difference between the two-factor and one-factor models in Figures 1-4 depends upon the size and direction of the shock to the first futures premium. In these cases, the shock was in the same direction as the shock to the Libor. However these shocks could have opposite signs, a case considered in Figure 5. Here, the one-factor model yields a continuously rising or falling curve, while the two-factor model produces rates which can have positive or negative changes at different points of the yield curve. In this case, the term structure of futures rates can tilt as well as shift. For the parameter values in Figure 5, there is no change in the fifth futures rate, where the effects of the two factor changes just happen to cancel out.

5.2 The volatility of Libor futures rates

Proposition 3 also allows us to investigate the properties of the volatility of futures rates. Again, the shape of the volatility structure of futures Libors depends on the persistence parameter (α) and the mean-reversion parameter (c). In addition, it also depends on the relative volatilities of the short rate and the stochastic mean factor, defined as σ_1 and σ_2 respectively, for simplicity. Figures 6-9 correspond to the cases in Figures 1-4 respectively.

1. *The persistence factor*

   In Figures 6 and 7, we compare a one-factor model, where the short rate moves have a volatility of 10%, with a two-factor model, where there is, in addition, a stochastic mean factor with a volatility of 6%. The one-factor model, with no mean reversion yields a flat volatility structure, which mirrors the parallel shift in the term structure of futures rates, shown in Figure 1. In contrast, the two-factor model produces an increasing volatility curve. The persistence factor also affects the shape of the term structure of volatility. High persistence of the stochastic mean shock (low α) implies relatively large volatilities for long-maturity futures rates. This can be seen by comparing Figure 6 with Figure 7, where persistence is low (high α). Note that when persistence of the stochastic mean shock is low (Figure 7), the two factor model has a volatility structure close to that of a one-factor model with higher volatility.

2. *Mean reversion of the Libor*

   In Figures 8 and 9, we introduce mean reversion, with $c = 0.05$. In the one-factor case, the effect is to produce a continuously downward sloping volatility structure. This illustrates the well-known effect of mean-reversion of the short rate. In the two-factor case, the result again depends also on the degree of persistence. If persistence is relatively high, as in Figure 8, then the net effect can be a humped futures volatility curve. When persistence is low, as in Figure 9, the effect of mean reversion tends to dominate and the futures volatility curve is downward sloping over most of its range. In any case, given the degree of mean reversion, the point at which the volatility curve starts declining, for a given volatility of the second factor, depends on the degree of persistence of the stochastic mean factor.

3. *The (relative) volatility of the stochastic mean*

   The difference between the two-factor and one-factor models in Figures 6-9 depends upon the volatility of conditional mean factor, σ_2, in relation to the persistence and mean-reversion parameters. This is illustrated by the humped shape of the term structure of volatility in Figures 8 and 9. A different ratio of the volatilities of the two factors would change both the slope of the hump and its location in the term structure of volatility. It also depends on the interaction between this effect, the mean reversion of the short rate and the persistence of the shock. For instance, in Figure 8, the volatility peaks at period 6, as the persistence is high, whereas in Figure 9 it peaks earlier at period 2, since the shock decays more quickly when the persistence is low.

5.3 The correlation of Libor spot and futures rates

A further implication of Proposition 3 concerns the correlation of spot and futures rates. The degree of correlation of the logarithmic rates depends on the relative importance of the second factor. In a one-factor model all futures rates are perfectly correlated with each other and with the spot rate. In our two-factor model, the correlation structure depends on the mean reversion and persistence parameters, in addition to the ratio of the volatilities of the two factors. Specifically, given the volatility of the first factor, we need to examine the effect of mean reversion (c), persistence ($1 - \alpha$), and the volatility of the second factor, σ_2, on the correlation of the spot Libor with the kth futures rate. Again, this is best analysed with the help of numerical examples. We now look at the effect of c, α, and σ_2 on the correlation of the spot Libor with the kth futures rate, using numerical examples similar to those illustrated in Figures 1-9 above.

1. *The persistence factor*

   In Figure 10, we compare a two-factor model, where the short rate moves have a volatility of 10% and the second, stochastic mean factor has a volatility of 6%, with a two-factor model, where the short rate moves have a volatility of 10% and the second factor has a volatility of 3%. In each case, the mean reversion, c, is zero, and the persistence is relatively high (α is low), $\alpha = 0.25$. As expected, the correlation declines with the maturity of the futures contract, in both cases. In the 6% and the 3% volatility cases respectively, the correlation falls to about $\rho = 0.4$ and $\rho = 0.65$ for the longest maturity futures contract. Figure 11 has the comparable graphs for the low persistence case (α is high). Comparing the correlation structure in Figures 10 and 11, we notice the influence of the persistence parameter, α. When persistence is low (α high) as in Figure 11, the size of the premium shock is far lower in later periods, and consequently, the correlation of the longer maturity futures rates with the spot rate is higher, only falling to about $\rho = 0.8$ and $\rho = 0.9$ for the 6% and the 3% volatility cases respectively for long maturity futures contracts.

2. *Mean reversion of the Libor*

   In Figures 12 and 13 we introduce mean reversion, with $c = 0.05$. Comparing Figures 10 and 12, we see that the correlation function falls more steeply with positive mean reversion. It also falls to a significantly lower level. In the case of low persistence, comparing Figures 11 and 13, the effect of mean reversion is somewhat marginal. This again highlights the important role of the persistence parameter in two-factor models.

6 Conclusions

This paper has explored the relationships between models of the extended Vasicek type, such as the two-factor model of Hull and White (1994), and models of the term structure of the Brennan and Schwartz (1979)-type. It has done so in the context of futures prices and rates. Basically, if we assume that the price of a zero-coupon bond (or, indeed, any function of the price) follows a two-dimensional process, then the term structure of futures prices or rates is given by a two-factor cross-sectional model. As an important special case, assuming that the logarithm of the Libor interest rate follows a two-dimensional, mean-reverting process, we find that the term structure of futures rates can be written as a log-linear function of any two rates.

The interest rate process assumed in the lognormal model is relatively simple to compute. It is possible to calibrate the model to provide estimates of futures rates and volatilities from cap-floor and swaption prices. It can then be used either to value American-style or path-dependent options. Alternatively, the model can be used to generate interest-rate scenarios, which can in turn be used to evaluate the risk of interest-rate dependent portfolios.

Perhaps the most important theoretical implications of the paper concern the relationship between HJM type forward rate models and Vasicek-Hull-White type models of the spot rate process. We have shown in particular that the degree of persistence of the second, conditional mean, factor shock is a critical determinant of the futures volatility structure. Given the close relationship of futures and forward rates, it must also be an important determinant of the forward volatility structure, which is an input to the HJM type models. The well known humped volatility structure has been reproduced in a two-factor model with mean reversion of the short rate and persistence of the conditional-mean factor shock.

The results in the paper also have some interesting empirical implications. Mean-reversion of short term interest rates is a crucial determinant of the pricing of interest rate contingent claims, in general, and interest rate caps, floors and swaptions, in particular. It is well-known that it is extremely difficult to estimate the coefficient of mean-reversion of short term interest rates from historical data, due to low power. Our model provides an alternative method of estimating the mean-reversion and persistence factors using futures rather than spot data, and using both cross-sectional and time-series data rather than time-series data alone. This derives from the fact that the mean-reversion coefficient, together with the volatility and persistence of the second factor, determines the shape of the futures volatility curve. Hence, observation of the futures volatility curve could lead to improved estimation of mean-reversion. In addition, the model provides the inputs required to judge when a two-factor model may substantially change the pricing and hedging of interest rate contingent claims, and when a one-factor model is sufficient.

Appendix 1: Properties of the conditional mean for two-dimensional time-series processes: Proofs of Lemmas 1 and 2

In this appendix we prove Lemmas 1 and 2. Lemma 1 establishes the cross-sectional linear property of the process. Lemma 2 establishes the conditional variance of the expectation of a variable which follows the same two-dimensional process.

Lemma 1

The variable x_t follows the time series process

$$x_t = (1-c)x_{t-1} + y_{t-1} + \epsilon_t$$

where $E(x_{t-1}y_{t-1}) = 0$ and where

$$y_t = (1-\alpha)y_{t-1} + \nu_t$$

if and only if the conditional expectation of x_{t+k} is of the form

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1})$$

where

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Proof.

Sufficiency

Successive substitution x_1, x_2, ... ,x_{t+k} and taking the conditional expectation yields

$$E_t(x_{t+k}) = x_t(1-c)^k + V_t \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1} \tag{15}$$

where

$$V_t = \sum_{\tau=0}^{t-1} \nu_{t-\tau}(1-\alpha)^\tau$$

Substituting the corresponding expression for $E_t(x_{t+1})$:

$$E_t(x_{t+1}) = x_t(1-c) + V_t$$

yields

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1}), \tag{16}$$

where

$$b_k = \sum_{\tau=1}^{k} (1-c)^{k-\tau}(1-\alpha)^{\tau-1}$$

and

$$a_k = (1-c)^k - (1-c)b_k.$$

Necessity

Assume

$$E_t(x_{t+k}) = a_k x_t + b_k E_t(x_{t+1})$$

where a_k and b_k are defined by (16) above, and x_t and $E_t(x_{t+1})$ are not perfectly correlated. Consider the orthogonal component z_t from

$$E_t(x_{t+1}) = \gamma x_t + z_t \tag{17}$$

Then

$$E_t(x_{t+1}) = (a_1 + b_1\gamma)x_t + b_1 z_t$$

and hence, since $a_1 = 0$ and $b_1 = 1$

$$x_{t+1} = \gamma x_t + z_t + \epsilon_{t+1} \tag{18}$$

where $\mathrm{E}_t(\epsilon_{t+1}) = 0$. Hence x_t follows a two-dimensional process with innovations z_t, ϵ_{t+1}.

We now show that $\gamma = (1-c)$ and also that z_t follows a mean reverting process with mean reversion α. Suppose by way of contradiction, that $\gamma =$

$(1 - c')$. Also, suppose there is a shock such that x_t changes while the difference, $E_t(x_{t+1}) - x_t$, is constant; then, $E_t(x_{t+k})$ will not be given by equation (16), since $c \neq c'$. It follows that we must have $\gamma = (1 - c)$. Second, suppose that $\gamma = (1 - c)$, but z_t mean reverts at a rate different from α. Then, if the difference, $E_t(x_{t+1}) - x_t$, changes, while x_t is constant, then again $E_t(x_{t+k})$ will not be given by equation (16). Hence, a necessary condition is that z_t mean reverts at a rate α. □

Lemma 2

If the variable x_t follows the process

$$x_t = (1 - c)x_{t-1} + y_{t-1} + \epsilon_t$$

where $E(x_{t-1}y_{t-1}) = 0$ and where

$$y_t = (1 - \alpha)y_t + \nu_t$$

then the conditional variance of $E(x_{t+k})$ is given by

$$\begin{aligned} \mathrm{var}_{t-1}[E_t(x_{t+k})] &= (1-c)^{2k}\mathrm{var}_{t-1}[x_t] \\ &+ \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}\right]^2 \mathrm{var}_{t-1}(\nu_t) \end{aligned} \tag{19}$$

Proof. From equation (15), the variance at $t - 1$, of the time t conditional expectation of x_{t+k}, is

$$\begin{aligned} \mathrm{var}_{t-1}[E_t(x_{t+k})] &= (1-c)^{2k}\mathrm{var}_{t-1}[x_t] \\ &+ \left[\sum_{\tau=1}^{k}(1-c)^{k-\tau}(1-\alpha)^{\tau-1}\right]^2 \mathrm{var}_{t-1}(V_t) \end{aligned} \tag{20}$$

It follows, since

$$\mathrm{var}_{t-1}(V_t) = \mathrm{var}_{t-1}(\nu_t),$$

that the conditional variance is as stated in lemma 2. □

Appendix 2: Properties of Lognormal Libor Rates: Proof of Lemma 3

Lemma 3

In a no-arbitrage economy, if the Libor rate follows a lognormal process under the equivalent martingale measure, then the k-period Libor futures rate at time t is

$$F_{t,t+k} = \exp[\mu(t, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k)] \quad (21)$$

where n is the length, in years, of the period t to t + 1. Also, $F_{t,t+k}$ *is lognormal, with logarithmic mean*

$$\mu(0, t, t+k) = \mu(0, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k) \quad (22)$$

Proof.

From CIR (1981), the futures rate is equal to the expectation of the Libor rate under the equivalent martingale measure, $F_{t,t+k} = E_t(i_{t+k})$. Since by assumption i_{t+k} is lognormal, under the EMM, with a conditional logarithmic mean and annualised volatility of $\mu(t, t+k, t+k)$ and $\sigma(t, t+k, t+k)$, we have

$$F_{t,t+k} = E_t(i_{t+k}) = \exp\left[\mu(t, t+k, t+k) + \frac{kn}{2}\sigma^2(t, t+k, t+k)\right]$$

Now since

$$F(t, t+k) = E_t(i_{t+k})$$

the expectation of the futures rate is

$$\mathrm{E}_0[F(t, t+k)] = E_0(i_{t+k}), \quad (23)$$

by the law of iterated expectations.

Taking the logarithm of equation (23) and using the relationship of the mean and variance of lognormal variables , we have

$$\mu(0,t,t+k)+\frac{tn}{2}\sigma^2(0,t,t+k)=\mu(0,t+k,t+k)+\frac{(t+k)n}{2}\sigma^2(0,t+k,t+k). \tag{24}$$

From the lognormality of i_{t+k},

$$(t+k)n\sigma^2(0,t+k,t+k)=\text{var}_0[\mu(t,t+k,t+k)]+kn\sigma^2(t,t+k,t+k). \tag{25}$$

Moreover,

$$F_{t,t+k}=\exp[\mu(t,t+k,t+k)+\frac{kn}{2}\sigma^2(t,t+k,t+k)]$$

$$\text{var}_0[\mu(t,t+k,t+k)]=nt\sigma^2(0,t,t+k). \tag{26}$$

Substituting equations (26) into (25), and then (25) into (24), yields

$$\mu(0,t,t+k)=\mu(0,t+k,t+k)+\frac{kn}{2}\sigma^2(t,t+k,t+k).\Box$$

Table 1
Notation for the Mean and Volatility of Spot and Futures Rates

	(1)		(2)		(3)	
Time Period	0		t		T	
Spot prices and interest rates for m-year money	$\mu(0,t,t)$	Unconditional logarithmic mean of i_t	P_t	Zero bond price at t for delivery of \$1 at $(t+m)$	P_T	Zero bond price at time T for delivery of \$1 at time $T+m$
	$\sigma(0,t,t)$	Unconditional (annualised) volatility of i_t	$i_t = F_{t,t}$	m–year interest rate at time t	$i_T = F_{T,T}$	m–year interest rate at time T
Futures interest rates for bonds maturing at time $T+m$	$\mu(0,t,T)$	Mean of $f_{t,T}$				
	$\sigma(0,t,T)$	Unconditional (annualised) volatility of $F_{t,T}$				
			$F_{t,T}$	futures interest rate at t for delivery at T (m-year money)		
			$f_{t,T}$	Logarithm of $F_{t,T}$		
			$\mu(t,T,T)$	Conditional mean of $f_{T,T}$		
			$\sigma(t,T,T)$	Conditional (annualised) volatility of $F_{T,T}$		

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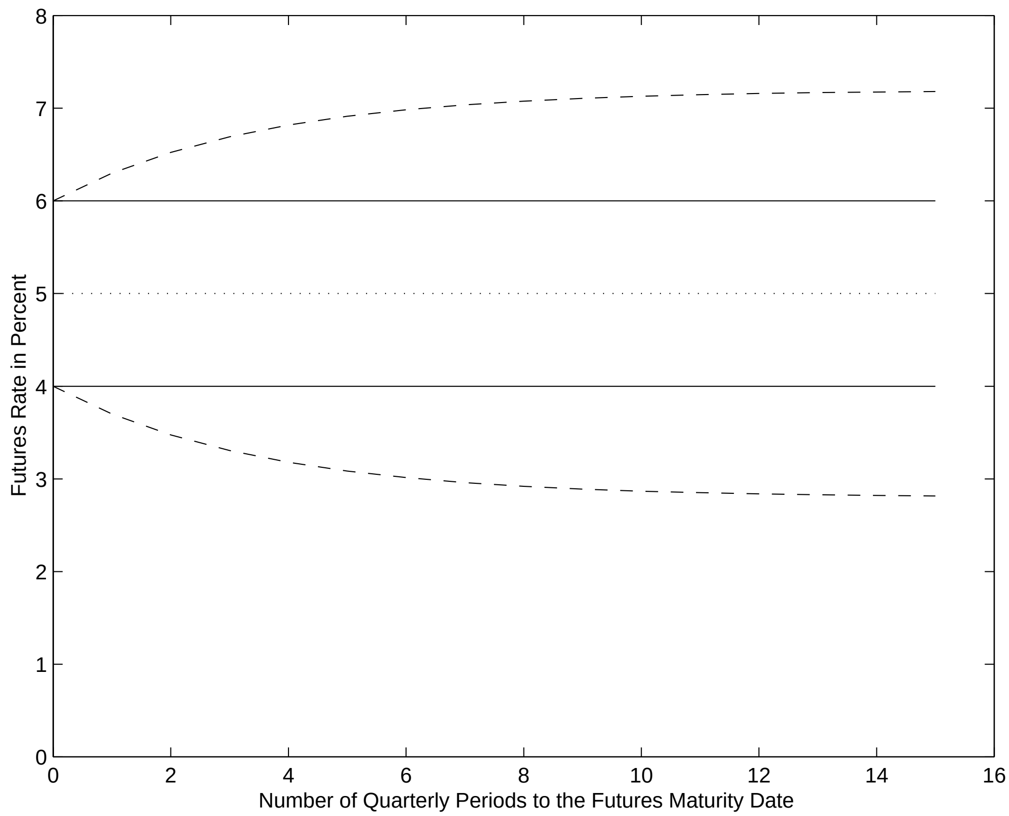
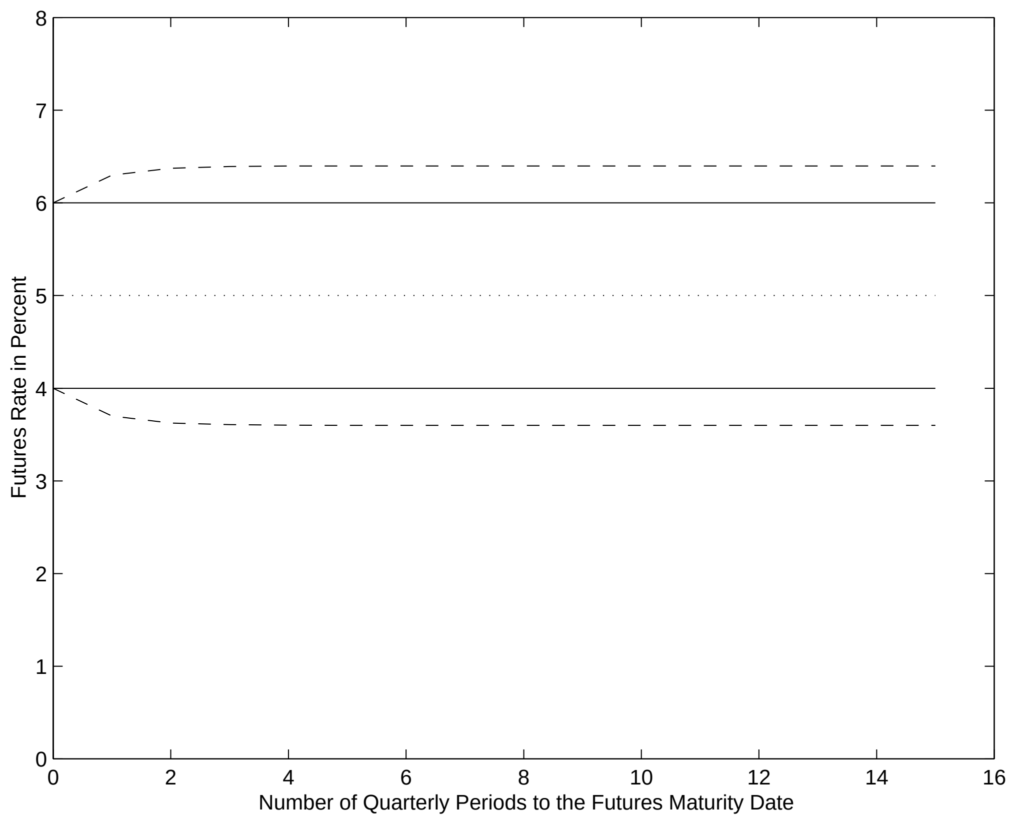
Figure 1: The Futures Term Structures: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.25$.



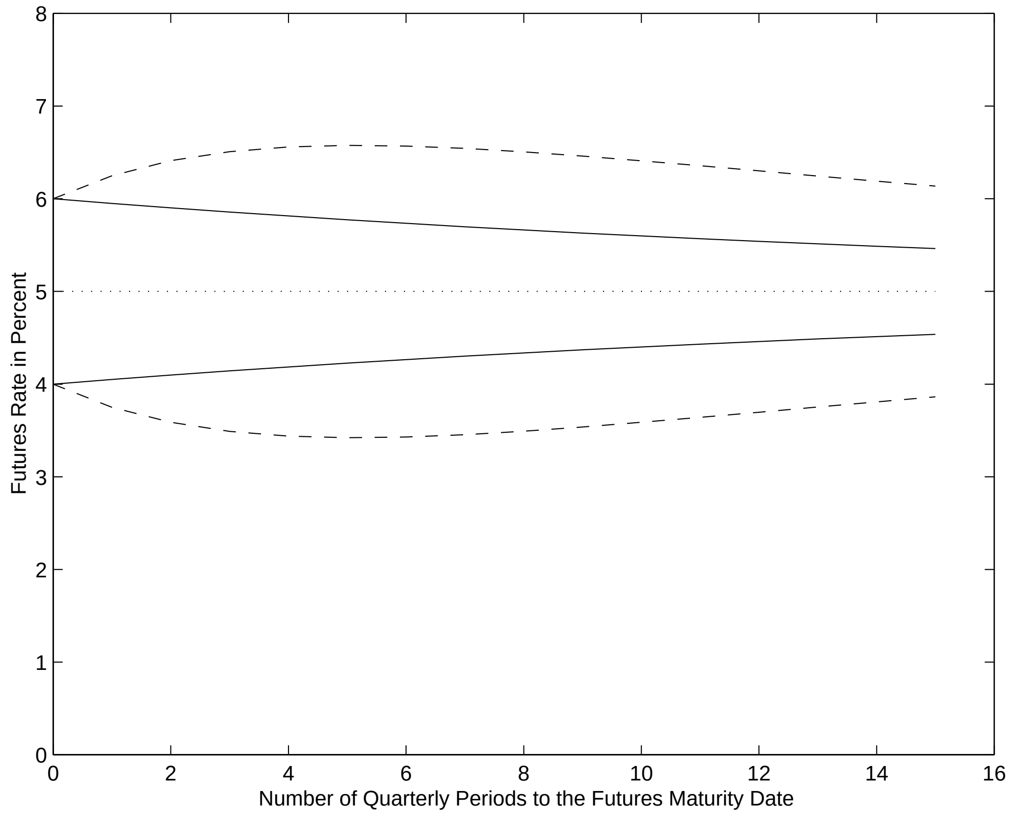
The figure shows the term structures of futures rates for the base case, where the rates are 5% for all maturities, and the cases where the changes in the logarithms of the short rate and the premium factors are the same in the shifts up and down. In the one-factor case, the logarithm of the short rate shifts up by 1%. The shift down is similar in logarithmic terms. In the two-factor case, the premium factor also moves up by 0.3% from 1, in the same direction. The shift down is similar in logarithmic terms. The dotted line indicates the base case, the solid lines the term structures with the one-factor model, and the dashed lines the term structures with the two-factor model.

Figure 2: The Futures Term Structures: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.75$.



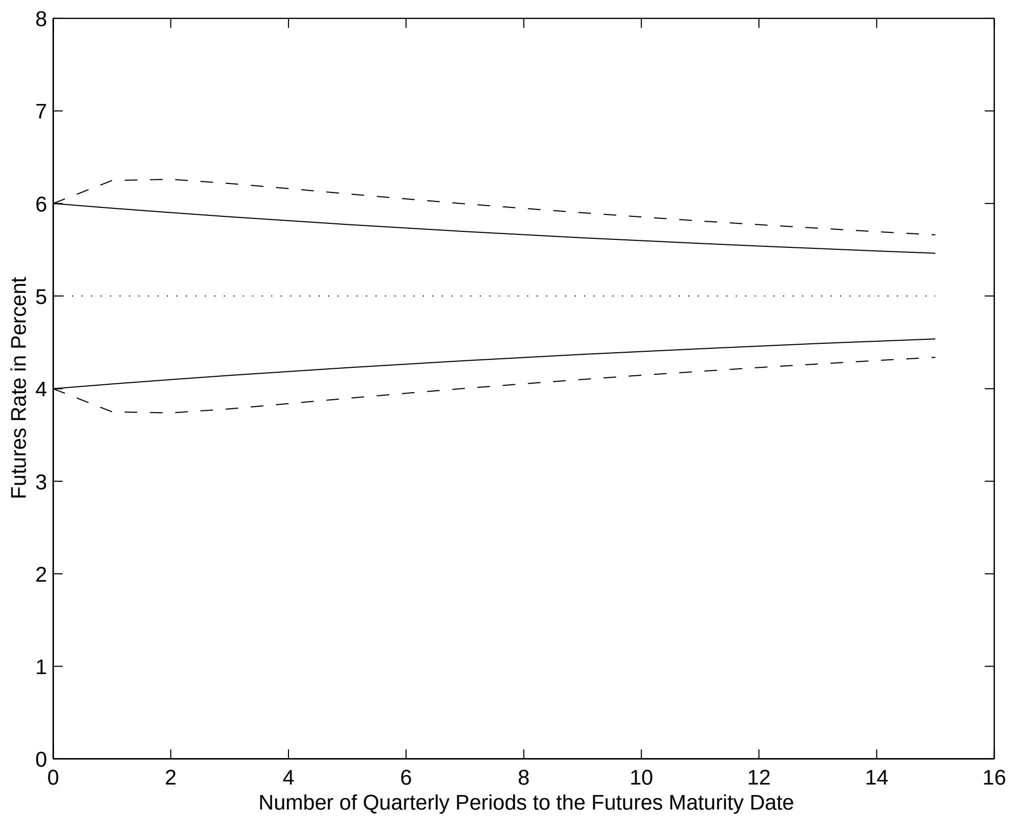
The figure shows the term structures of futures rates for the base case, where the rates are 5% for all maturities, and the cases where the changes in the logarithms of the short rate and the premium factors are the same in the shifts up and down. In the one-factor case, the logarithm of the short rate shifts up by 1%. The shift down is similar in logarithmic terms. In the two-factor case, the premium factor also moves up by 0.3% from 1, in the same direction. The shift down is similar in logarithmic terms. The dotted line indicates the base case, the solid lines the term structures with the one-factor model, and the dashed lines the term structures with the two-factor model.

Figure 3: The Futures Term Structures: Mean Reversion, $c = 0.05$; Persistence, $\alpha = 0.25$.



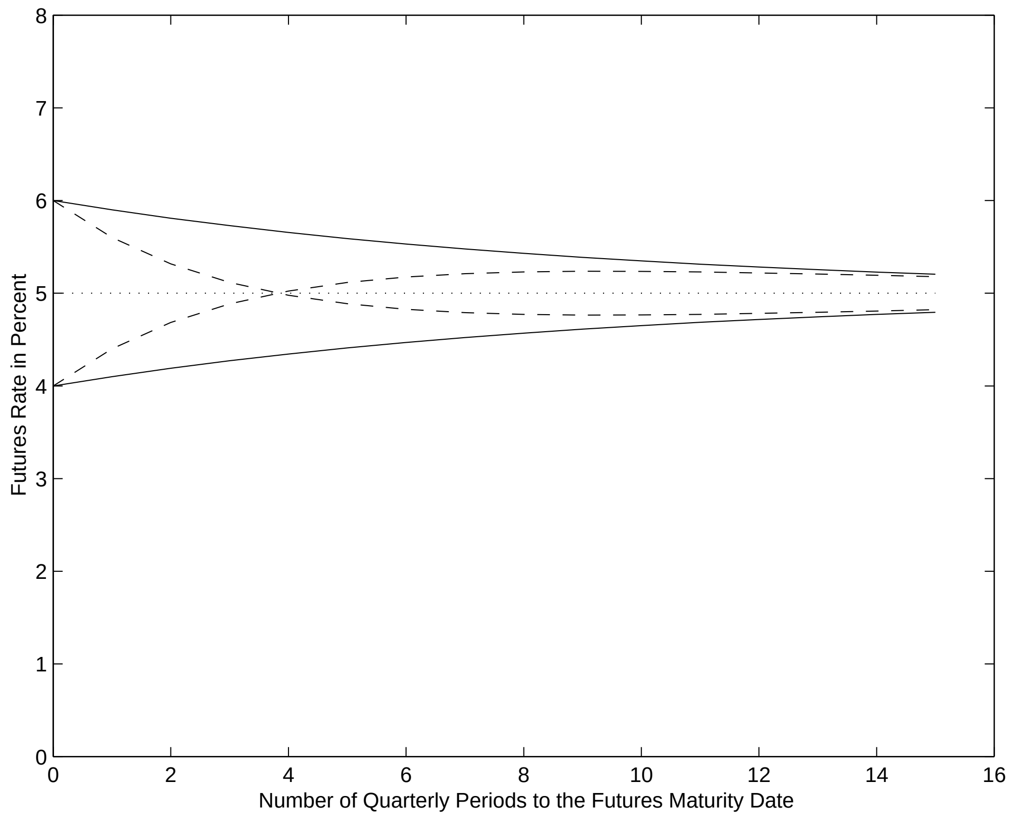
The figure shows the term structures of futures rates for the base case, where the rates are 5% for all maturities, and the cases where the changes in the logarithms of the short rate and the premium factors are the same in the shifts up and down. In the one-factor case, the logarithm of the short rate shifts up by 1%. The shift down is similar in logarithmic terms. In the two-factor case, the premium factor also moves up by 0.3% from 1, in the same direction. The shift down is similar in logarithmic terms. The dotted line indicates the base case, the solid lines the term structures with the one-factor model, and the dashed lines the term structures with the two-factor model.

Figure 4: The Futures Term Structures: Mean Reversion, $c = 0.05$; Persistence, $\alpha = 0.75$.



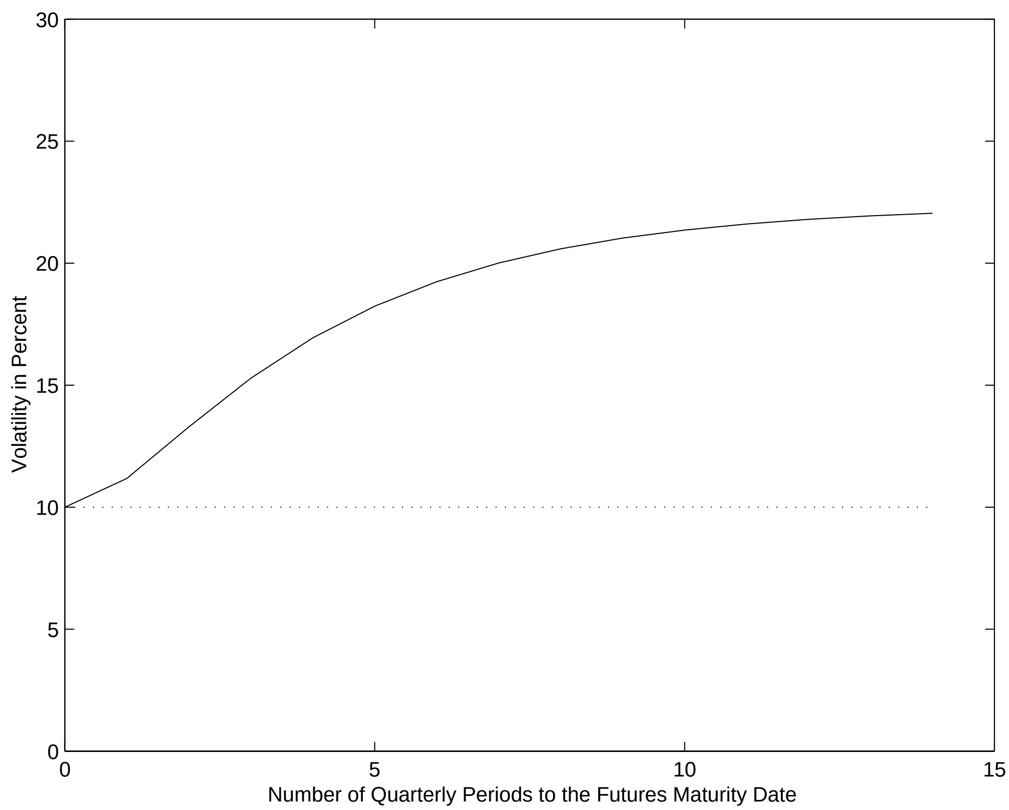
The figure shows the term structures of futures rates for the base case, where the rates are 5% for all maturities, and the cases where the changes in the logarithms of the short rate and the premium factors are the same in the shifts up and down. In the one-factor case, the logarithm of the short rate shifts up by 1%. The shift down is similar in logarithmic terms. In the two-factor case, the premium factor also moves up by 0.3% from 1, in the same direction. The shift down is similar in logarithmic terms. The dotted line indicates the base case, the solid lines the term structures with the one-factor model, and the dashed lines the term structures with the two-factor model.

Figure 5: Futures Term Structures: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.25$.



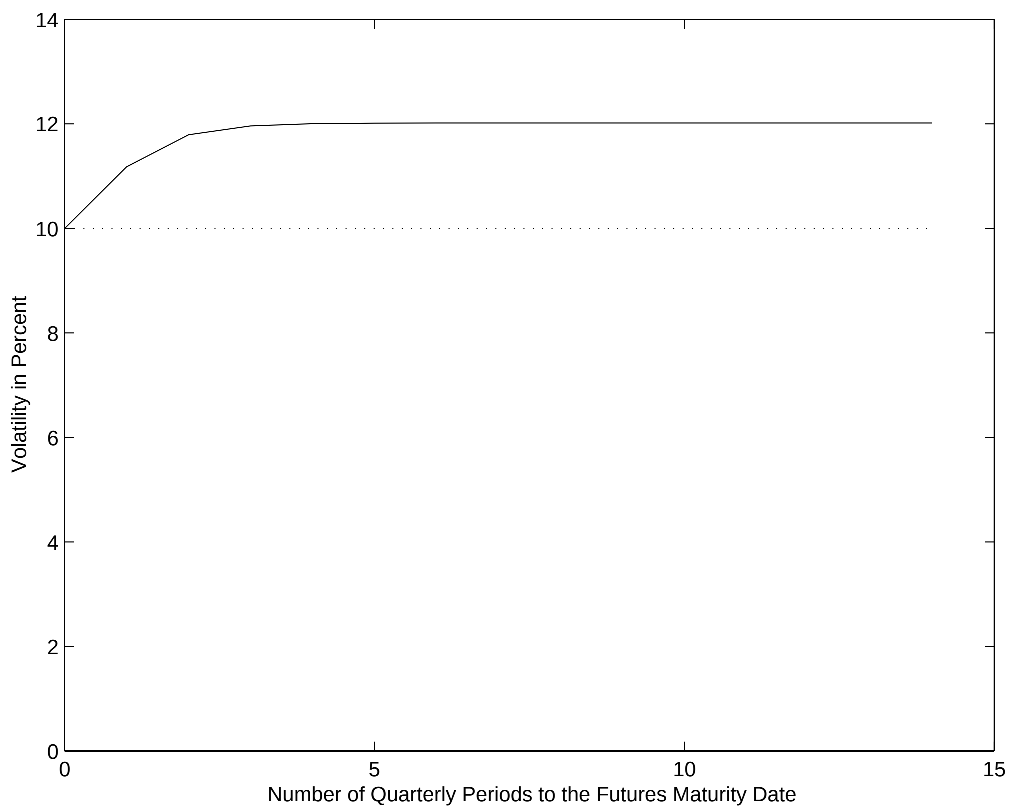
The figure shows the term structures of futures rates for the base case, where the rates are 5% for all maturities, and the cases where the changes in the logarithms of the short rate and the premium factors are the same in the shifts up and down. In the one-factor case, the logarithm of the short rate shifts up by 1%. The shift down is similar in logarithmic terms. In the two-factor case, the premium factor also moves up by 0.3% from 1, in the opposite direction. The shift down is similar in logarithmic terms. The dotted line indicates the base case, the solid lines the term structures with the one-factor model, and the dashed lines the term structures with the two-factor model.

Figure 6: The Futures Volatility Structure: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.25$.



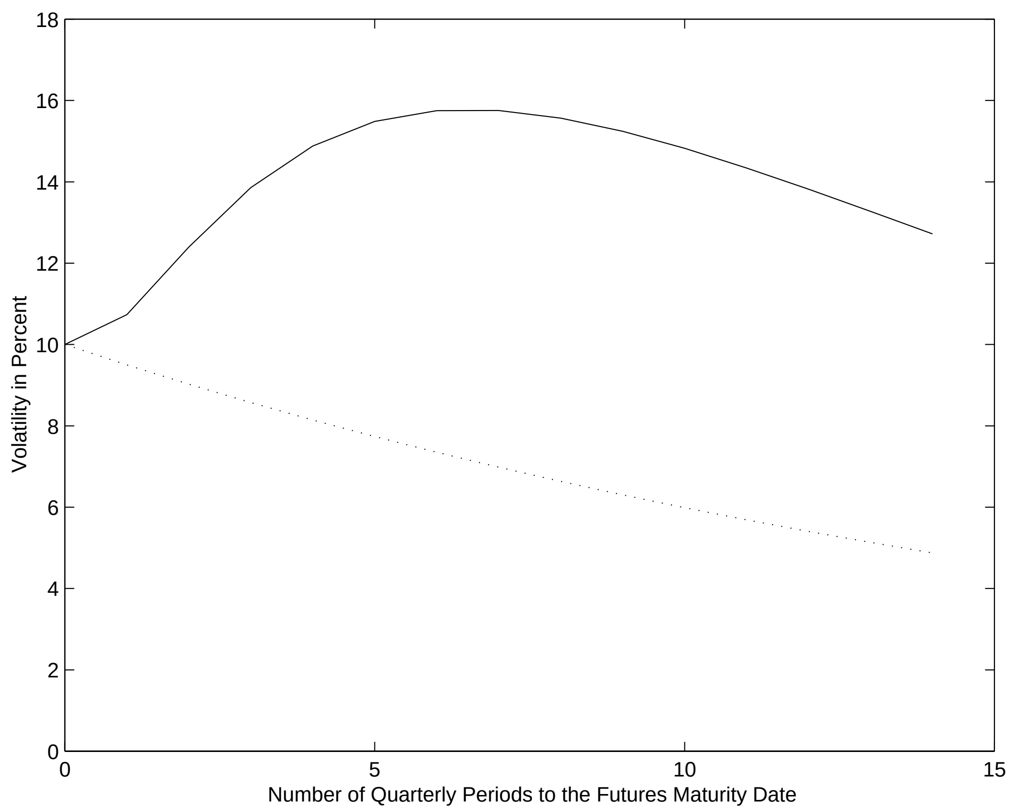
The figure shows the term structure of volatilities for the one-factor base case, where the volatilities are 10% for all maturities, and the two-factor case where the volatilities take into account the effects of mean-reversion and persistence in the short term interest rate (using Proposition (3)). In the one-factor case, the volatility of the premium factor is zero. In the two-factor case, the volatility of the premium factor is 6%. The dotted line indicates the futures volatility term structure for the one-factor model, and the solid line indicates the futures volatility term structure for the two-factor model.

Figure 7: The Futures Volatility Structure: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.75$.



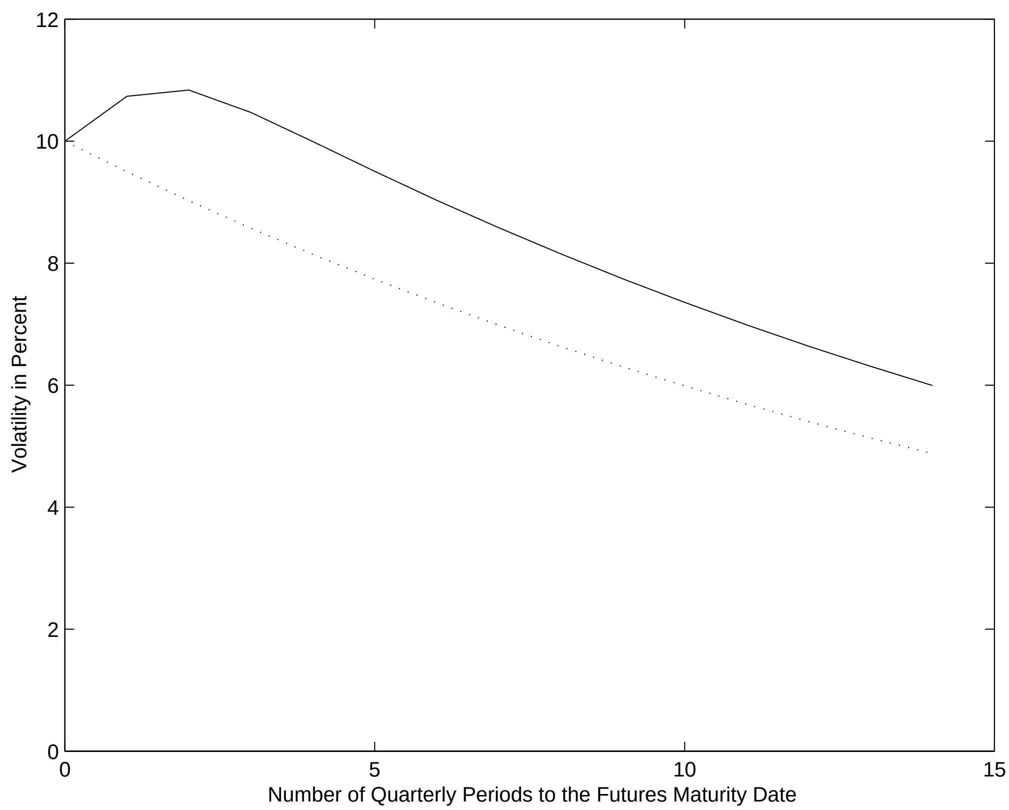
The figure shows the term structure of volatilities for the one-factor base case, where the volatilities are 10% for all maturities, and the two-factor case where the volatilities take into account the effects of mean-reversion and persistence in the short term interest rate (using Proposition (3)). In the one-factor case, the volatility of the premium factor is zero. In the two-factor case, the volatility of the premium factor is 6%. The dotted line indicates the futures volatility term structure for the one-factor model, and the solid line indicates the futures volatility term structure for the two-factor model.

Figure 8: The Futures Volatility Structure: Mean Reversion, $c = 0.05$; Persistence, $\alpha = 0.25$.



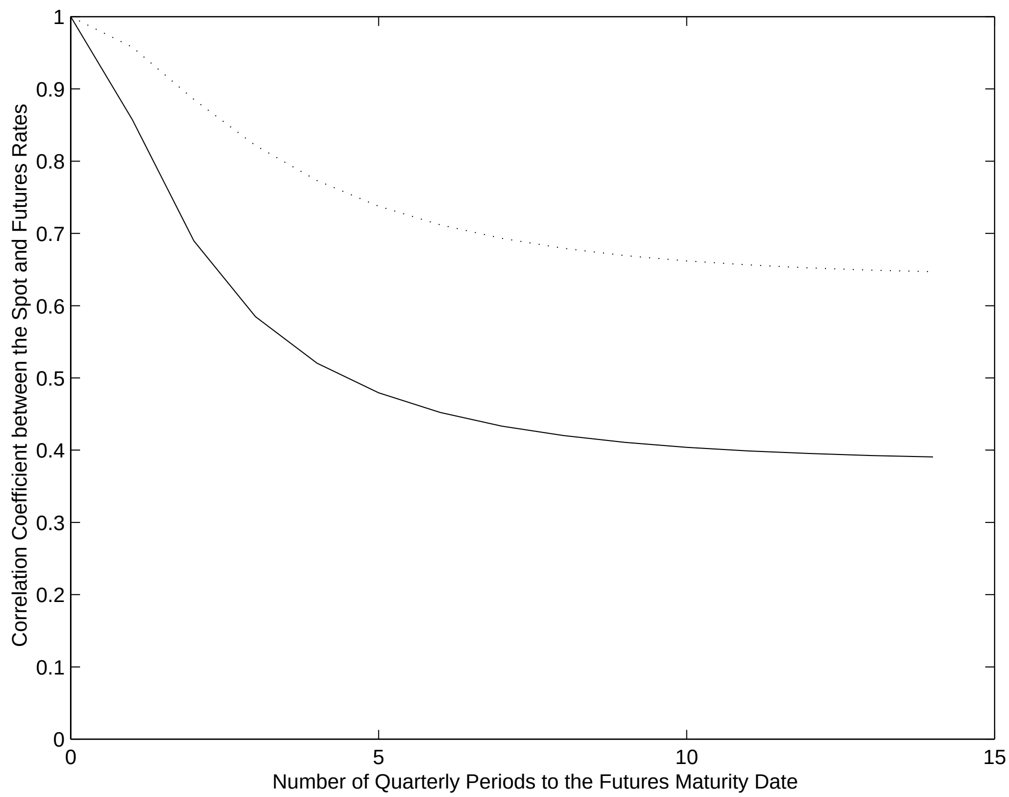
The figure shows the term structure of volatilities for the one-factor base case, where the volatilities are 10% for all maturities, and the two-factor case where the volatilities take into account the effects of mean-reversion and persistence in the short term interest rate (using Proposition (3)). In the one-factor case, the volatility of the premium factor is zero. In the two-factor case, the volatility of the premium factor is 6%. The dotted line indicates the futures volatility term structure for the one-factor model, and the solid line indicates the futures volatility term structure for the two-factor model.

Figure 9: The Futures Volatility Structure: Mean Reversion, $c = 0.05$; Persistence, $\alpha = 0.75$.



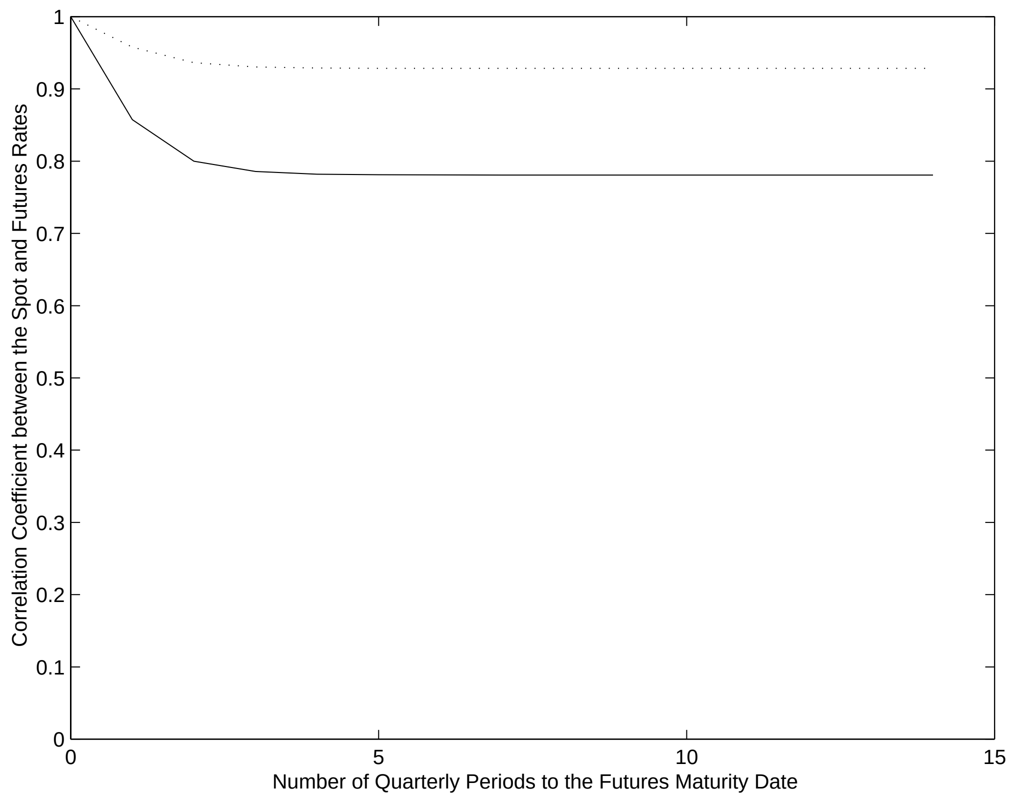
The figure shows the term structure of volatilities for the one-factor base case, where the volatilities are 10% for all maturities, and the two-factor case where the volatilities take into account the effects of mean-reversion and persistence in the short term interest rate (using Proposition (3)). In the one-factor case, the volatility of the premium factor is zero. In the two-factor case, the volatility of the premium factor is 6%. The dotted line indicates the futures volatility term structure for the one-factor model, and the solid line indicates the futures volatility term structure for the two-factor model.

Figure 10: The Futures Correlation Structure: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.25$.



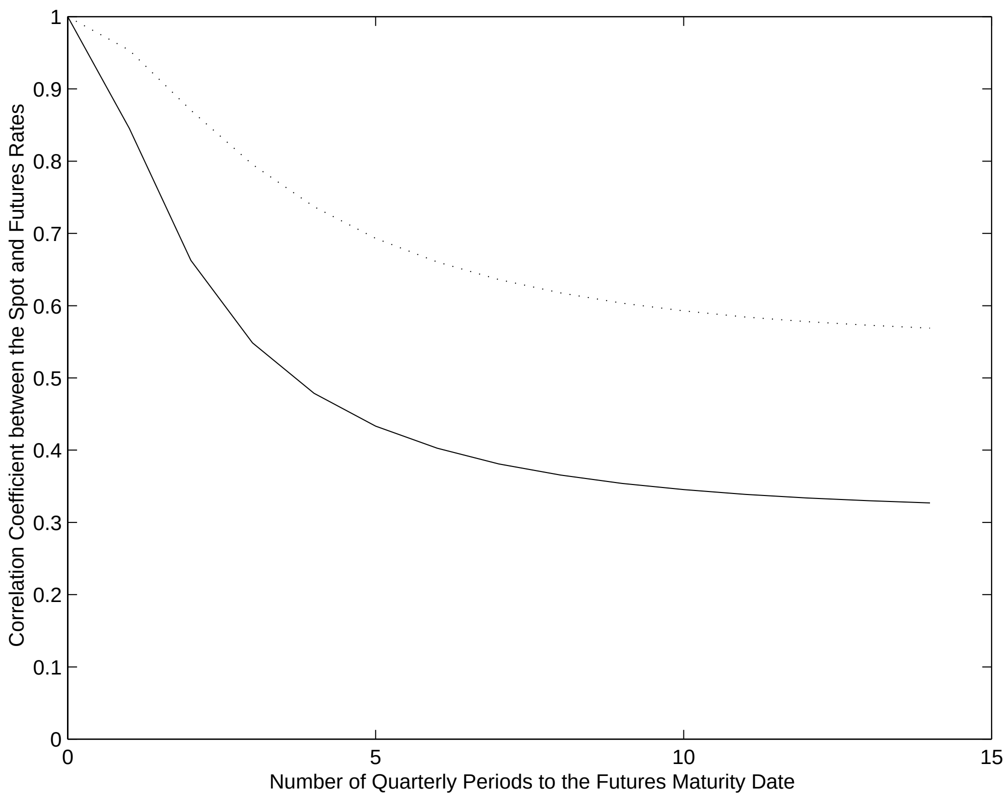
The figure shows the term structures of correlations between the spot rate and the k^{th} futures rate for the two-factor case. The correlations take into account the effects of mean-reversion and persistence in the short term interest rate, using equation (10). The dotted line and the solid line indicate the futures correlation term structure when the volatility of the premium factor are 3% and 6% respectively.

Figure 11: The Futures Correlation Structure: Mean Reversion, $c = 0$; Persistence, $\alpha = 0.75$.



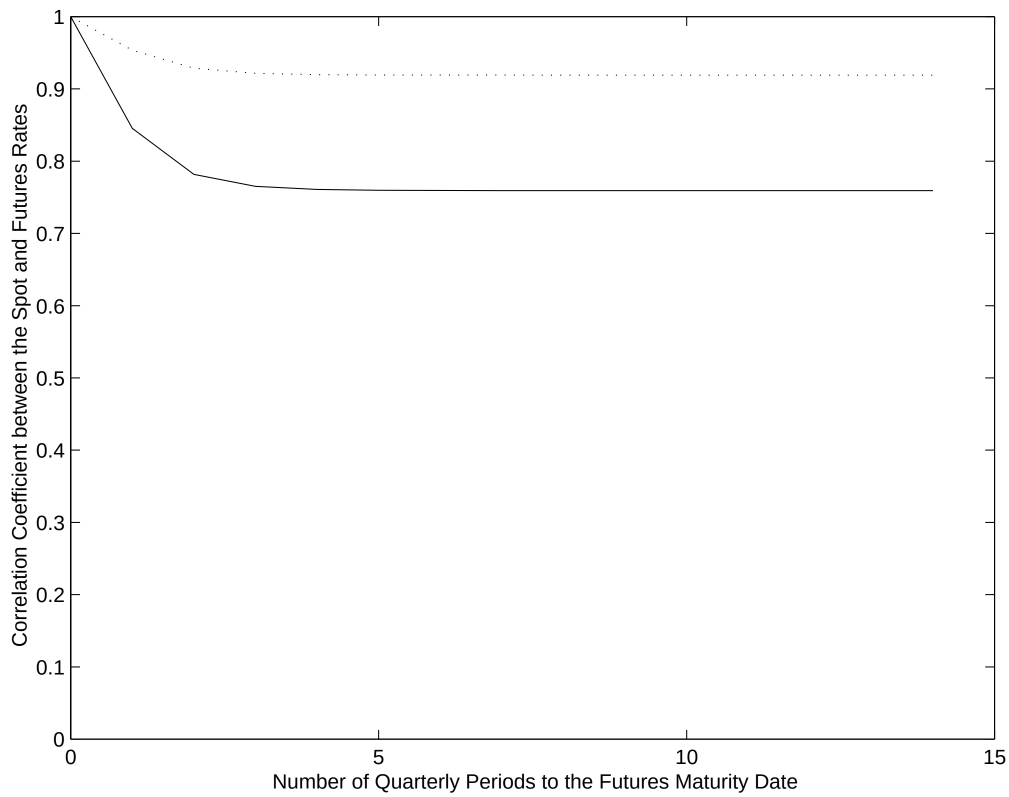
The figure shows the term structures of correlations between the spot rate and the k^{th} futures rate for the two-factor case. The correlations take into account the effects of mean-reversion and persistence in the short term interest rate, using equation (10). The dotted line and the solid line indicate the futures correlation term structure when the volatility of the premium factor are 3% and 6% respectively.

Figure 12: The Futures Correlation Structure: Mean Reversion, $c = 0.05$; Persistence, $\alpha = 0.25$.



The figure shows the term structures of correlations between the spot rate and the k^{th} futures rate for the two-factor case. The correlations take into account the effects of mean-reversion and persistence in the short term interest rate, using equation (10). The dotted line and the solid line indicate the futures correlation term structure when the volatility of the premium factor are 3% and 6% respectively.

Figure 13: The Futures Correlation Structure: Mean Reversion, $c = 0.05$; Persistence, $\alpha = 0.75$.



The figure shows the term structures of correlations between the spot rate and the k^{th} futures rate for the two-factor case. The correlations take into account the effects of mean-reversion and persistence in the short term interest rate, using equation (10). The dotted line and the solid line indicate the futures correlation term structure when the volatility of the premium factor are 3% and 6% respectively.