# Optimal investment with taxes : an existence result

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**Abstract**

We study the deterministic control problem of maximizing utility from consumption of an agent who seeks to optimally allocate his wealth between consumption and investment in a financial asset subject to taxes on benefits with first-in-first-out priority rule on sales. Short sales are prohibitted and consumption is restricted to be non-negative. Such a problem has been introduced in a previous paper by the same authors where the first order conditions have been derived. In this paper, we establish an existence result for this non-classical optimal control problem.

**Key words.** optimal investment with taxes, weak compactness, Levy convergence.

**JEL classification.** C69, D11, D91.

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# 1 Introduction

The simplest consumption-investment problem can be formulated as follows; see e.g. Ando and Modigliani (1963). There is an economic agent with preferences described by a utility function function $U(c) = \int_0^T u(t, c(t))dt$, where $c$ is the consumption path in the time interval $[0, T]$. The agent has an income function $\omega$ defined on $[0, T]$. The financial market consists of one asset with price function $S$. At each time $t$, the agent receives an income $\omega(t)$, rebalances his portfolio (by buying or selling some financial assets) and spends the rest for consumption.

In this paper, we study the case where the portfolio rebalancement involves the payment of taxes on benefits. Then, the purchasing time of the asset to be sold has to be recorded in order to compute the amount of tax to be paid. Also, the sales may be submit to some priority rule imposed by the tax administration. Dermody and Rockafellar (1991, 1995) studied the problem of hedging and utility maximization in a deterministic and finite discrete-time model, without priority rule on the sales.

Instead, we consider a deterministic continuous-time model. Notice that Dermody and Rockafellar's framework is not enbedded in our formulation, since our portfolio strategies are supposed to be absolutely continuous with repect to the Lebesgue measure. In addition to the no-short-selling constraint, our model assumes that sales are subject to the fist-in-first-out priority rule on sales. The agent's problem turns out to be a nonclassical optimal control problem with *endogenous delay* and with a complex nonnegativity constraint on consumption. In particular, we do not know wether the function to be maximized is convex in the control variables.

Such a problem has been introduced by Jouini, Koehl and Touzi (1999). In the latter paper, we have derived the first order conditions of the problem as well as the following economically appealing result : an optimal strategy (if it exists) can always be chosen such that the agent never sells out from his portfolio and buys new financial assets simultaneously. Such a (surprisingly difficult) result allows to simplify the non-negativity constraint on consumption and provides an $L^p$ bound on purchases ($p > 1$) and an $L^1$ bound on sales. We exploit these bounds in order to establish an existence result without appealing to convexity of the function to be maximized.

Before closing this introductory section let us relate our problem to the classical optimal investment problem without taxes. In the latter problem, the existence result follows easily from the fact that the objective function is continuous, and the budget set is compact. In our problem, the budget set may be identified to a subset of the previous compact budget set. Nevertheless, the constraints induced by the first-in-first-out priority rule on sales involves the delay function, and therefore the whole past of the portfolio strategy. In this setting, closedness of the budget set is far from being straightforward, as one can see from the precise problem formulation of the next section. This is the reason why our existence result relies on extremely demanding tools from functional analysis.

The paper is organized as follows. Section 2 provides a precise description of the model and recalls some basic results from Jouini, Koehl and Touzi (1999). The existence result for the optimal control problem with endogenous delay is reported in section 3.

# 2 The model

## 2.1 The financial market

There is a single consumption commodity available for consumption through $[0, T]$ where $T$ is a finite time horizon. The financial market consists of one riskless asset, called bond, whose price function is given by :

$$S(t) \quad = \quad S(0) \exp \int_0^t r(s) ds, \qquad t \in [0, T],$$

where $r(.)$ is a continuous nonnegative function defined on $[0, T]$; $r(t)$ is the instantaneous interest rate at time $t$.

## 2.2 Taxation rule

Following Jouini, Koehl and Touzi (1999), we assume that sales are subject to taxes on benefits[1]. More precisely, we shall consider the usual *first-in-first-out* rule according to which any bond sold at some time $t$ should be the oldest one in the time $t$ portfolio.

We introduce the set $\Delta = \{(t, u) \in I\!R^2 : 0 \leq u \leq t \leq T\}$. Fix some $(t, u)$ in $\Delta$. For each monetary unit invested at time $u$ and sold out at time $t$, we denote by $\varphi(t, u)$ the after tax amount received at time $t$, i.e. the amount of tax paid by the investor is

$$\frac{S(t)}{S(u)} - \varphi(t, u).$$

The after tax return function $\varphi$ defined on $\Delta$ is assumed to satisfy the following standing conditions.

**Assumption 2.1** *$\varphi$ is a $C^1$ function mapping $\Delta$ into $[1, +\infty)$ with $\varphi(t, t) = 1$, for all $t \in [0, T]$,*

$$\frac{\varphi_t}{\varphi}(t, .) \quad \text{is decreasing for any} \quad t \in [0, T] \tag{2.1}$$

[1] Since the instantaneous interest rate is nonnegative, sales always yield some nonnegative benefit.

*and*

$$f(t,.)\ :\ u \longmapsto \varphi(t,u)S(u) \quad \textit{is nondecreasing;} \quad t \in [0,T]. \tag{2.2}$$

The fact that $\varphi \geq 1$ is a natural condition on the after tax return function $\varphi$ since the asset price $S(t)$ is nondecreasing and the tax is a (possibly varying) proportion of the capital gains. The restriction $\varphi(t,t) = 1$ is a natural condition which expresses the fact that there is no benefit from selling and buying a financial asset at the same time $t$. The technical condition (2.1) is a needed for the proof of the basic result of Jouini, Koehl and Touzi (1999) which will be recalled later on. The last condition (2.2) is not assumed in Jouini, Koehl and Touzi (1999) but is needed here in order to establish our existence result. The simplest taxation rule is given by the following example.

**Example 2.1** Constant tax rate. Suppose that the tax to be paid for one asset bought at time $u$ and sold at time $t$ is given by $\tau[S(t) - S(u)]$. Therefore the investor return from such a strategy is $\varphi(t,u) = [S(t) - \tau(S(t) - S(u))]/S(u) = \tau + (1-\tau)S(t)/S(u) = \tau + (1-\tau)exp\int_u^t r(s)ds$. It is easily checked that $\varphi$ satisfies the conditions of Assumption 2.1.

**Remark 2.1** We recall from Jouini, Koehl and Touzi that condition (2.1) implies that the after tax return function $\varphi$ is nondecreasing in $t$ and nonincreasing in $u$.

## 2.3 Trading strategies

We denote by $L^1_+$ the set of all nonnegative $L^1[0,T]$ functions. Let $(x,y)$ be a pair of $L^1_+$ functions. Here $x(t)$ (resp. $y(t)$) is the investment (resp. disinvestment) rate in units of the bond at time $t$. In other words, $\int_0^t x(s)ds$ (resp. $\int_0^t y(s)ds$) is the cumulated number of assets purchased (resp. sold out) up to time $t$. Such a pair $(x,y)$ is said to be a trading strategy if the no short selling constraint

$$\int_0^t y(s)ds \ \leq\ \int_0^t x(s)ds, \quad 0 \leq t \leq T \tag{2.3}$$

holds. Condition (2.3) says that sales must not exceed purchases at any time. Given a trading strategy $(x,y)$, we define the delay function $\theta^{x,y}$ by :

$$\theta^{x,y}(t) \ = \ \sup\left\{ s \in [0,t]\ :\ \int_0^s x(u)du \leq \int_0^t y(u)du \right\}.$$

In the sequel, we shall write $\theta$ for $\theta^{x,y}$ for notational simplicity. As defined, $\theta$ is nondecreasing and whenever $\int_0^t y(s)ds > 0$, $\theta(t)$ is the purchasing date of the oldest asset in the portfolio.

If $\int_0^t x(s)ds = \int_0^t y(s)ds = 0$ (no market participation up to time $t$ a.e.), then $\theta(s) = s$ for all $s \in [0,t]$. Furthermore, from the no short sales constraint (2.3), we have

$$\theta(0) \ = \ 0 \ \le \ \theta(t) \ \le \ t, \quad 0 \le t \le T. \tag{2.4}$$

We recall the following properties from Jouini, Koehl and Touzi (1999).

**Lemma 2.1** (i) *$\theta$ is right-continuous on $[0,T]$, i.e. $\theta(t^+) = \theta(t)$ for all $t \in [0,T]$,*
(ii) *for all $t \in [0,T]$, $\int_0^{\theta(t)} x(s)ds = \int_0^t y(s)ds$,*
(iii) *for all $t \in [0,T]$, $\int_{\theta(t^-)}^{\theta(t)} x(s)ds = 0$.*

Part (i) of the above lemma states that delay function $\theta$ are non-decreasing right-continuous functions, as a direct consequence of its definition through some strategy $(x,y)$. Part (ii) provides an economic interpretation of $\theta$. Namely, loosely speaking, the cumulated sales at time $t$ correspond to the cumulated pursahed shares at time $\theta(t)$. Finally, (iii) says that the jumps of the delay function $\theta$ are located in the regions (of positive measures) with no investment in the bond. This is a natural property of $\theta$ which expected from its definition.

In this paper, we need the extend part (ii) of the last lemma by replacing the interval $[0,t]$ by any Borel subset $A$ of $\mathcal{B}([0,T])$. The following result says that we have a similar result : the sales of $A$ correspond to purshases at corresponding dates in $\theta(A)$.

**Lemma 2.2** *For any set $A \subset \mathcal{B}([0,T])$, we have :*

$$\int_A y(s)ds \ = \ \int_{\theta(A)} x(s)ds$$

**Proof.** (i) We first prove that the mapping $\mu$ defined on $\mathcal{B}([0,T])$ by :

$$\mu(A) \ := \ \int_{\theta^{x,y}(A)} x(t)dt$$

defines a measure on $[0,T]$. To see this this, we only have to check that $\mu$ is $\sigma$-additive. Let $(A_i)_{i\ge 0}$ be a sequence of non-intersecting sets of $\mathcal{B}([0,T])$. Then, for all $n$, we have :

$$\sum_{i=0}^n \mu(A_i) \ \ge \ \sum_{i=0}^n \int_{\theta^{x,y}(A_i)} x(t)dt \ = \ \int x 1_{\{\cup_{i=1}^n \theta^{x,y}(A_i)\}}. \tag{2.5}$$

By the dominated convergence Theorem, this provides :

$$\sum_{i=0}^\infty \mu(A_i) \ \ge \ \int_{\theta^{x,y}(\cup_{i=1}^\infty A_i)} x.$$

In order to have equality, we have to prove equality in (2.5). To see this, it suffices to prove that

$$\lambda\left(\theta^{x,y}(A_i) \cap \theta^{x,y}(A_j)\right) \;=\; 0 \;\text{ for }\; i \neq j, \tag{2.6}$$

where $\lambda$ is the Lebesgue measure on $\mathcal{B}([0,T])$. To prove (2.6), let $\alpha = \theta^{x,y}(t_i) = \theta^{x,y}(t_j)$ with $(t_i, t_j) \in A_i \times A_j$. Then since $A_i \cap A_j = \emptyset$, we can assume $t_i < t_j$. Since $\theta$ is nondecreasing, we must have $\theta^{x,y}([t_i, t_j]) = \{\alpha\}$ and $(\theta^{x,y})^{-1}(\{\alpha\})$ has a nonempty interior. Now, it is clear that there is at most a countable number of non-intersecting intervals with nonempty interior and therefore a countable number of such $\alpha$'s.
(ii) From Lemma 2.1 (ii), we have $\int_0^{\theta^{x,y}(t)} x(s)ds = \int_0^t y(s)ds$. We intend to prove that $\mu([0,t]) = \int_0^{\theta^{x,y}(t)} x(s)ds$ in order to obtain that $\mu$ coincides with the measure with density $y$. To see this, notice that

$$\mu([0,t]) \;=\; \int_0^{\theta^{x,y}(t)} x(s)ds - \sum_{u \in \mathcal{J}_t(\theta^{x,y})} \int_{\theta^{x,y}(u-)}^{\theta^{x,y}(u)} x(s)ds$$

where $\mathcal{J}(\theta^{x,y})$ is the set of (at most countable) jumps of $\theta^{x,y}$ prior to $t$. Now, by Lemma 2.1 (iii), we have $\int_{\theta^{x,y}(u-)}^{\theta^{x,y}(u)} x(s)ds = 0$ for all $u \in [0,T]$ which ends the proof. □

## 2.4 The Agent's Problem

At each time $t \in [0,T]$, the agent is endowed with an income rate $\omega(t)$ in units of the consumption good. Here $\omega$ is a given positive continuous function on $[0,T]$. Then, given a trading strategy $(x,y)$, the agent's consumption rate function is given by :

$$c^{x,y}(t) \;=\; \omega(t) - x(t)S(t) + y(t)f(t, \theta^{x,y}(t)), \quad 0 \leq t \leq T. \tag{2.7}$$

Therefore, a trading strategy $(x,y)$ is said to be admissible if the induced consumption rate function is nonnegative. We shall denote by $\mathcal{A}$ the set of all admissible trading strategies, i.e.

$$\mathcal{A} \;=\; \left\{(x,y) \in L^1_+ \times L^1_+ \;:\; \int_0^\cdot x \geq \int_0^\cdot y \;\text{ hold and } c^{x,y} \geq 0\right\}. \tag{2.8}$$

The agent's preferences are represented by a time-additive utility function from consumption $U(t,c)$. We assume throughout the paper that $U$ is $C^{1,2}([0,T], I\!R_+)$, decreasing in $t$, concave nondecreasing in $c$ and

$$\sup_{0 \leq t \leq T} U_c(t, 0+) \;<\; \infty \tag{2.9}$$

$$U_c(t, +\infty) \;:=\; \lim_{c \to +\infty} U_c(t,c) \;=\; 0 \quad \text{for all }\; t \in [0,T] \tag{2.10}$$

Notice that the latter conditions are not assumed in Jouini, Koehl and Touzi (1999). The agent's optimal control problem is :

$$\sup_{(x,y)\in\mathcal{A}} \int_0^T U\left(t, c^{x,y}(t)\right) dt, \tag{2.11}$$

i.e. maximize utility from consumption over all admissible trading strategies. In the sequel we shall denote

$$\phi(x,y) \ := \ \int_0^T U(t, c^{x,y}(t))dt \quad \text{for all} \ (x,y) \ \in \ \mathcal{A}.$$

**Remark 2.2** From condition (2.9), the utility function $U$ is bounded from below by some constant and, therefore, function $\phi(x,y)$ is well-defined for all $(x,y) \in \mathcal{A}$ and takes values in $I\!R \cup \{+\infty\}$.

We now recall the basic result of Jouini, Koehl and Touzi (1999) which allows to simplify the nonnegativity constraint on consumption.

**Theorem 2.1** *Let $(x,y)$ be some admissible strategy in $\mathcal{A}$. Then, there exists an admissible strategy $(\tilde{x},\tilde{y}) \in \mathcal{A}$ such that $c^{x,y} \le c^{\tilde{x},\tilde{y}}$ and :*

$$\tilde{x}(t)\tilde{y}(t) \ = \ 0 \quad 0 \le t \le T \ \text{ a.e.} \tag{2.12}$$

An important consequence of the last Theorem is that the set of admissible strategies in the optimization problem (2.11) can be restricted to the set $\mathcal{A}_0$ defined by :

$$\mathcal{A}_0 \ = \ \{(x,y) \in \mathcal{A} \ : \ x(t)y(t) = 0, \ \ 0 \le t \le T \ \text{ a.e.}\}$$

in the sense that

$$\sup_{(x,y)\in\mathcal{A}} \phi(x,y) \ = \ \sup_{(x,y)\in\mathcal{A}_0} \phi(x,y)$$

It is then easily seen that all strategies $(x,y) \in \mathcal{A}_0$ satisfy $xS \le \omega$. Conversely, if $(x,y) \in L^1_+ \times L^1_+$ satisfies $xy = 0$ a.e. and $xS \le \omega$, then we have $c^{x,y} \ge 0$ a.e. It follows that the set $\mathcal{A}_0$ may be rewritten as :

$$\mathcal{A}_0 \ = \ \left\{(x,y) \in L^1_+ \times L^1_+ \ : \ xS \le \omega, \ \ xy = 0 \ \text{ a.e. and } \ \int_0^{\cdot} x \ge \int_0^{\cdot} y\right\}. \tag{2.13}$$

The last condition together with the no short sales condition (2.3) provide the following result.

**Lemma 2.3** $\sup_{(x,y)\in\mathcal{A}} \phi(x,y) = \sup_{(x,y)\in\mathcal{A}_0} \phi(x,y) < \infty$.

**Proof.** Since $U$ is nonincreasing in $t$ and concave in $c$, we see that

$$\begin{aligned}\phi(x,y) &\leq TU\left(0, \frac{1}{T}\int_0^T [\omega(t) - x(t)S(t) + y(t)f(t,\theta^{x,y}(t))]dt\right)\\ &\leq TU\left(0, \frac{1}{T}\|\omega\|_1(1+\|f\|_\infty)\right),\end{aligned}$$

where we used (2.3) and the fact that $xS \leq \omega$. □

The set $\mathcal{A}_0$ introduced above is not convex; however, its convex hull is included in the set of admissible strategies :

**Lemma 2.4** *$conv(\mathcal{A}_0) \subset \mathcal{A}$ where $conv(\mathcal{A}_0)$ is the smallest convex set containing $\mathcal{A}_0$.*

**Proof.** Consider some $(x_1, y_1)$, $(x_2, y_2) \in \mathcal{A}_0$ and $\lambda \in (0,1)$. Define $(\hat{x}, \hat{y}) := \lambda(x_1, y_1) + (1-\lambda)(x_2, y_2)$. Then it is clear that $(\hat{x}, \hat{y}) \in L^1_+ \times L^1_+$ and satisfies the no short sales condition (2.3). As for the consumption induced by $(\hat{x}, \hat{y})$, we have :

$$c^{\hat{x},\hat{y}}(t) = \omega(t) - \hat{x}(t)S(t) + \hat{y}(t)f(t,\theta^{\hat{x},\hat{y}}(t)) \geq 0$$

since $\hat{x}S = \lambda x_1 S + (1-\lambda)x_2 S \leq \omega$ and $\hat{y} = \lambda y_1 + (1-\lambda)y_2 \geq 0$.

□

# 3 An existence result

In this section we prove the following result.

**Theorem 3.1** *The optimal control problem* (2.11) *has a solution, i.e. there exists a strategy $(x^*, y^*) \in \mathcal{A}$ such that*

$$\phi(x^*, y^*) = \sup_{(x,y)\in\mathcal{A}} \phi(x,y).$$

**Remark 3.1** By Theorem 2.1, the solution $(x^*, y^*)$ may be chosen in $\mathcal{A}_0$. We have then an existence result in $\mathcal{A}_0$.

Let $(x_n, y_n)_{n\in I\!N}$ a maximizing sequence of trading strategies in $\mathcal{A}_0$, i.e.

$$(x_n, y_n) \in \mathcal{A}_0 \quad \text{and} \quad \lim_{n\to\infty} \phi(x_n, y_n) = \sup_{(x,y)\in\mathcal{A}} \phi(x,y).$$

**Lemma 3.1** *Let $p$ be an arbitrary value in $(1,\infty)$. For all $n$, there exist coefficients $(\lambda_k^n)_{k\geq n}$, with $\lambda_k^n \geq 0$ and $\sum_{k\geq n}\lambda_k^n = 1$, such that*

$$\hat{x}_n := \sum_{k\geq n}\lambda_k^n x_k \longrightarrow x^* \quad in \ L^p$$
$$\hat{y}_n := \sum_{k\geq n}\lambda_k^n y_k \longrightarrow y^* \quad a.e.$$

*where $x^*S \leq \omega$, $y^* \in L^1$ and $(\hat{x}_n,\hat{y}_n) \in \mathcal{A}$ for all $n$.*

**Proof.** the fact that the sequence $(\hat{x}_n,\hat{y}_n) \in \mathcal{A}$ follows from Lemma 2.4. We now prove the convergence result. Since $(x_n,y_n) \in \mathcal{A}_0$ for all $n$, the sequence $(x_n)_n$ is bounded in $L^\infty$ and therefore in $L^p$ for all $p > 1$ with $x^*S \leq \omega$. Then there exists a subsequence of $(x_n)_n$ which converges weakly in $L^p$ to some $x^* \in L^p$. By Mazur's Lemma there exists a convex combination of $\{x_k,\ k \geq n\}$ converging towards $x^*$ in $L^p$; see e.g. Dellacherie and Meyer (1975). Hence there exist coefficients $(\mu_k^n)$ with $\mu_k^n \geq 0$ and $\sum_{k\geq n}\mu_k^n = 1$ such that

$$\sum_{k\geq n}\mu_k^n x_k \longrightarrow x^* \quad \text{in } L^p.$$

Notice that, since $x_n S \leq \omega$ for each $n$, we have $x^*S \leq \omega$. Next notice that, from the no short sales condition (2.3), we have :

$$\int_0^T \sum_{k\geq n}\mu_k^n y_k(t)dt \leq \int_0^T \sum_{k\geq n}\mu_k^n x_k(t)dt \leq \int_0^T \omega(t)dt;$$

recall that $S \geq 1$. Then, from Komlòs Theorem, there exists a subsequence of $(\sum_{k\geq n}\mu_k^n y_k)_n$ named $(\sum_{k\geq\pi(n)}\mu_k^{\pi(n)} y_k)$ such that

$$\frac{1}{n}\sum_{p=1}^{n}\sum_{k\geq\pi(p)}\mu_k^{\pi(p)} y_k \longrightarrow y^* \quad \text{a.e.}$$

for some $y^* \in L^1$; see Hall and Heyde (1980). This proves that there exist coefficients $(\lambda_k^n)$ with $\lambda_k^n \geq 0$ and $\sum_{k\geq n}\lambda_k^n = 1$ satisfying the requirements of the lemma.

□

We now recall an important notion of convergence which will be used for the sequence of delay functions. Let $f$ and $g$ be two nondecreasing right-continuous functions defined on $[0,T]$. The Levy distance $\delta(f,g)$ is defined by :

$$\delta(f,g) := \inf\{\varepsilon > 0 : f(t-\varepsilon)-\varepsilon \leq g(t) \leq f(t+\varepsilon)+\varepsilon \text{ for } t\in[0,T]\}.$$

In words, $\delta(f,g)$ is the shortest distance between the graph of $f$ and the graph of $g$ along lines in the direction of the second diagonal (spanned by $(-1,1)$).

**Lemma 3.2** *There exists a nondecreasing right-continuous function $\theta^*$ such that*

$$\theta_n \ := \ \theta^{x_n,y_n} \ \longrightarrow \ \theta^* \quad \text{and} \quad \hat{\theta}_n \ := \ \theta^{\hat{x}_n,\hat{y}_n} \ \longrightarrow \ \theta^*$$

*in the sense of the Levy metric, possibly along some subsequence.*

**Proof.** We refer to Lemma 3.7 of Jouini, Koehl and Touzi (1999) for the existence of $\theta^*$ as the limit in the sense of the Levy metric of the sequence $(\theta_n)_n$ ; this result is in fact a consequence of the Prohorov Theorem, see Billingsley (1968) p37. To see that $(\hat{\theta}_n)_n$ also converges to $\theta^*$ in the sense of the Levy metric, notice that :

$$\int_0^{\hat{\theta}_n(t)} \sum_{k\geq n} \lambda_k^n x_k(s)ds \ = \ \int_0^t \sum_{k\geq n} \lambda_k^n y_k(s)ds \ = \ \sum_{k\geq n} \lambda_k^n \int_0^{\theta_k(t)} x_k(s)ds$$

so that :

$$\inf_{k\geq n} \theta_k(t) \ \leq \ \hat{\theta}_n(t) \ \leq \ \sup_{k\geq n} \theta_k(t); \quad 0 \leq t \leq T.$$

Using the last inequality, it is easily checked that $(\hat{\theta}_n)_n$ converges to $\theta^*$ in the sense of the Levy metric.

□

In our problem, we do not know wether function $\phi(x, y)$ (to be maximized) is convex. Therefore, we can not deduce immediately that $(\hat{x}_n, \hat{y}_n)$ is a maximizing sequence. We now establish the latter result without appealing to convexity of $\phi$.

**Lemma 3.3** *The sequence $(\hat{x}_n, \hat{y}_n)_{n\in I\!N}$ is a maximizing sequence, i.e.*

$$\lim_{n\to\infty} \phi(\hat{x}_n, \hat{y}_n) \ = \ \sup_{(x,y)\in\mathcal{A}} \phi(x, y).$$

**Proof.** Fix some $\varepsilon > 0$. From the uniform continuity of the function $f(t, u)$, there exists some $\alpha_\varepsilon > 0$ such that

$$|t - t'| + |u - u'| \ \leq \ \alpha_\varepsilon \ \Longrightarrow \ |f(t, u) - f(t', u')| \ \leq \ \varepsilon \tag{3.1}$$

for all $(t, u)$ and $(t', u')$ in $\Delta$. Define $\alpha := \min(\varepsilon, \alpha_\varepsilon)$. From the Levy convergence of $(\theta_n)_n$ and $(\hat{\theta}_n)_n$ towards $\theta^*$ (see Lemma 3.2), there exists some $N \in I\!N$ such that

$$\hat{\theta}_n(t) \ \geq \ \theta^*(t - (\alpha/2)) - (\alpha/2) \ \geq \ \theta_k(t - \alpha) - \alpha; \quad \text{for all } \ k \geq n \geq N \ \text{ and } \ t \in [0, T].$$

From the last inequality and the increase of function $f(t,.)$ (see (2.2)), it follows that :

$$\begin{aligned}\phi(\hat{x}_n,\hat{y}_n) &= \int_0^T U\left[t,\omega(t)-\hat{x}_n(t)S(t)+\hat{y}_n(t)f(t,\hat{\theta}_n(t))\right]dt \\ &\geq \int_0^T U\left[t,\sum_{k\geq n}\lambda_k^n\left(\omega(t)-x_k(t)S(t)+y_k(t)f(t,\theta_k(t-\alpha)-\alpha)\right)\right]dt \\ &\geq \int_0^T U\left[t,\sum_{k\geq n}\lambda_k^n\left(\omega(t)-x_k(t)S(t)+y_k(t)f(t,\theta_k(t-\alpha))-\varepsilon y_k(t)\right)\right]dt,\end{aligned}$$

where the last inequality follows from (3.1). Using the concavity of $U$, this provides :

$$\phi(\hat{x}_n,\hat{y}_n) \geq \sum_{k\geq n}\lambda_k^n\int_0^T U\left[t,c^{x_k,y_k}(t)+y_k(t)\left(f(t,\theta_k(t-\alpha))-f(t,\theta_k(t))-\varepsilon\right)\right]dt.$$

Now, from condition (2.9)), function $U(t,c)$ is Lipschitz in $c$ uniformly in $t$, and by the no short-sales condition (2.3), $\int_0^T y_k \leq \int_0^T x_k \leq \|\omega/S\|_1$ since $(x_k,y_k)\in\mathcal{A}_0$. Then

$$\begin{aligned}\phi(\hat{x}_n,\hat{y}_n) &\geq \sum_{k\geq n}\lambda_k^n\phi(x_k,y_k)-\varepsilon A\|\omega/S\|_1 \\ &\quad +A\sum_{k\geq n}\lambda_k^n\int_0^T y_k(t)\left(f(t,\theta_k(t-\alpha))-f(t,\theta_k(t))\right)dt.\end{aligned}$$

where $A:=\sup_{t\in[0,T]}U_c(t,0)$. We now claim that

$$\int_0^T y_n(t)\left(f(t,\theta_n(t-\alpha))-f(t,\theta_n(t))\right)dt \leq C\varepsilon\|\omega\|_1 \tag{3.2}$$

for some constant $C$ defined below. To see this, recall that the function $(t,u)\longmapsto f(t,u)$ is $C^1$ on the compact set $\Delta$ and therefore :

$$\begin{aligned}&\left|\int_0^T y_n(t)\left(f(t,\theta_n(t-\alpha))-f(t,\theta_n(t))\right)dt\right| \\ &\leq C\int_0^T y_n(t)\left|\theta_n(t-\alpha)-\theta_n(t)\right|dt\end{aligned}$$

for some constant $C$. Since $\int_0^t y(s)ds=\int_0^{\theta(t)}x(s)ds$ by Lemma 2.1 and by virtue of Lemma 2.2, the change of variable formula provides :

$$\begin{aligned}&\left|\int_0^T y_n(t)\left(f(t,\theta_n(t-\alpha))-f(t,\theta_n(t))\right)dt\right| \\ &\leq C\int_0^{\theta_n(T)} x_n(t)\alpha dt \\ &\leq \varepsilon C\|\omega/S\|_1,\end{aligned}$$

which proves (3.2). We then get :

$$\phi(\hat{x}_n,\hat{y}_n) \quad\geq\quad \sum_{k\geq n}\lambda_k^n\phi(x_k,y_k)-\varepsilon A\|\omega/S\|_1(1+C)$$

which provides :

$$\liminf_{n\to\infty}\phi(\hat{x}_n,\hat{y}_n) \quad\geq\quad \lim_{n\to\infty}\phi(x_n,y_n) \quad=\quad \sup_{(x,y)\in\mathcal{A}}\phi(x,y).$$

Since $(\hat{x}_n,\hat{y}_n)\in\mathcal{A}$ (see Lemma (3.1)), this proves that $(\hat{x}_n,\hat{y}_n)_n$ is a maximizing sequence.

□

**Remark 3.2** This proof is the only place where we need condition (2.9) ensuring that $U(t,c)$ is Lipschitz in $c$ uniformly in $t$.

So far, we only have an $L^1$ bound on the sequence $(y_n)_n$. This allowed to construct the sequence $(\hat{y}_n)_n$ of convex combinations, which converges only in the a.s. sense. For reason to be clear in the proof of the main theorem, we need an $L^1$ convergence. This is obtained by the following result.

**Lemma 3.4** *Let $(\bar{x}_n,\bar{y}_n)\in\mathcal{A}_0^{I\!N}$ be a maximizing sequence, i.e.*

$$\lim_{n\to\infty}\phi(\bar{x}_n,\bar{y}_n) \quad=\quad \sup_{(x,y)\in\mathcal{A}_0}\phi(x,y).$$

*Then the sequence $(\bar{y}_n)_n$ is uniformly integrable, i.e.*

$$\lim_{M\to\infty}\sup_{n\geq 0}\int_0^T \bar{y}_n 1_{\{\bar{y}_n\geq M\}} \quad=\quad 0.$$

**Proof.** Fix some $M>0$ and define the sequence $(\tilde{x}_n,\tilde{y}_n)_n$ by :

$$\tilde{y}_n := \begin{cases}\frac{1}{2}\bar{y}_n & \text{on } A:=\{\bar{y}_n\geq M\}\\ \bar{y}_n & \text{on } [0,T]\setminus A\end{cases} \quad\text{and}\quad \tilde{x}_n := \begin{cases}\frac{1}{2}\bar{x}_n & \text{on } \bar{\theta}_n(A)\\ \bar{x}_n & \text{on } [0,T]\setminus\bar{\theta}_n(A).\end{cases}$$

where $\bar{\theta}_n := \theta^{\bar{x}_n,\bar{y}_n}$. By Lemma 2.2, it is clear that we have $\tilde{\theta}_n := \theta^{\tilde{x}_n,\tilde{y}_n} = \bar{\theta}_n$. Now, recall that $(\bar{x}_n,\bar{y}_n)\in\mathcal{A}_0$ and therefore $\bar{x}_n\bar{y}_n=0$ a.e. and $A\cap[\bar{\theta}_n(A)\cap\{x>0\}]=\emptyset$. Therefore :

$$\begin{aligned}
& \phi(\tilde{x}_n,\tilde{y}_n)-\phi(\bar{x}_n,\bar{y}_n)\\
= & \int_A\left\{U\left[t,\omega(t)+\frac{1}{2}\bar{y}_n(t)f(t,\bar{\theta}_n(t))\right]-U\left[t,\omega(t)+\bar{y}_n(t)f(t,\bar{\theta}_n(t))\right]\right\}dt\\
& +\int_{\bar{\theta}_n(A)\cap\{x>0\}}\left\{U\left[t,\omega(t)-\frac{1}{2}\bar{x}_n(t)S(t)\right]-U\left[t,\omega(t)-\bar{x}_n(t)S(t)\right]\right\}dt\\
= & \int_A\left\{U\left[t,\omega(t)+\frac{1}{2}\bar{y}_n(t)f(t,\bar{\theta}_n(t))\right]-U\left[t,\omega(t)+\bar{y}_n(t)f(t,\bar{\theta}_n(t))\right]\right\}dt\\
& +\int_{\bar{\theta}_n(A)}\left\{U\left[t,\omega(t)-\frac{1}{2}\bar{x}_n(t)S(t)\right]-U\left[t,\omega(t)-\bar{x}_n(t)S(t)\right]\right\}dt.
\end{aligned}$$

Using the concavity of $U(t,.)$ and the fact that $\omega \geq 0$ and $S \geq 1$, we see that :

$$
\begin{aligned}
& \phi(\tilde{x}_n, \tilde{y}_n) - \phi(\bar{x}_n, \bar{y}_n) && (3.3)\\
\geq\ & \|f\|_\infty \int_A -\frac{1}{2}\bar{y}_n(t) U_c\left(t, \frac{1}{2}M\right) dt + \int_{\bar{\theta}_n(A)} \frac{1}{2}\bar{x}_n(t) U_c\left(t, \|\omega\|_\infty\right) dt \\
\geq\ & \frac{1}{2}\int_A \bar{y}_n(t)\left[U_c\left(t, \|\omega\|_\infty\right) - \|f\|_\infty U_c\left(t, \frac{1}{2}M\right)\right] dt && (3.4)
\end{aligned}
$$

where the last inequality (which is in fact an equality) follows from the change of variable formula for Lebesgue integrals, see Lemma 2.2. Now, fix some $\varepsilon > 0$. Since $(\bar{x}_n, \bar{y}_n)_n$ is a maximizing sequence, we must have for $n$ sufficiently large

$$
\phi(\tilde{x}_n, \tilde{y}_n) - \phi(\bar{x}_n, \bar{y}_n) \ \leq\ \varepsilon C \quad \text{where} \quad C \ :=\ \frac{1}{4}\sup_{t\in[0,T]} U_c(t, \|\omega\|_\infty). \tag{3.5}
$$

Moreover since $U_c(t, +\infty) = 0$, we have

$$
\|f\|_\infty U_c\left(t, \frac{1}{2}M\right) \ \leq\ \frac{1}{2}U_c\left(t, \|\omega\|_\infty\right); \qquad t \in [0,T] \tag{3.6}
$$

for $M \geq M^*$ with $M^*$ independent of $(\bar{x}_n, \bar{y}_n)$ and $t \in [0,T]$; recall that $U_c(.,c)$ is continuous on $[0,T]$. Combining (3.4), (3.5) and (3.6), we see that :

$$
\int_{\{\bar{y}_n \geq M\}} \bar{y}_n \ \leq\ \varepsilon \quad \text{for } M \geq M^*,
$$

which proves the required result since $M^*$ does not depend on $n$. □

**Lemma 3.5** *Let $(c_n)_n$ be a uniformly integrable sequence with $\int_0^T U(t, c_n(t))dt < \infty$. Then, the sequence $(U(., c_n(.)))_n$ is uniformly integrable.*

**Proof.** If the utility function is bounded, then the result is trivial. We then consider the case $U$ unbounded. Moreover, from condition (2.9), we have $U(0,0) > -\infty$; we can assume without loss of generality (by possibly by adding a constant) that $U(0,0) = 0$. Since $U(.,c)$ is increasing, we have :

$$
\int_0^T U(t, c_n(t)) 1_{\{U(t,c_n(t)) \geq M\}} dt \ \leq\ \int_0^T U(0, c_n(t)) 1_{\{U(0,c_n(t)) \geq M\}} dt.
$$

We denote by $U^0$ the function $U(0,.)$. Let $M' = (U^0)^{-1}(M)$. Then

$$
\int_0^T U(t, c_n(t)) 1_{\{U(t,c_n(t)) \geq M\}} dt \ \leq\ \int_0^T U^0(c_n(t)) 1_{\{c_n(t) \geq M'\}} dt.
$$

Notice that since $U^0$ is strictly increasing and unbounded we have that $M' \longrightarrow \infty$ as $M \longrightarrow \infty$. Next, using the concavity of $U^0$, we get :

$$
\int_0^T U(t, c_n(t)) 1_{\{U(t,c_n(t)) \geq M\}} dt \ \leq\ \lambda(c_n \geq M') U^0\left(\frac{1}{\lambda(c_n \geq M')}\int_0^T c_n(t) 1_{\{c_n(t) \geq M'\}} dt\right),
$$

where $\lambda$ is the Lebesgue measure and we set by convention $(1/\lambda(A)) \int_0^T f(t) 1_A(t) dt = 0$ whenever $\lambda(A) = 0$. Now, define the function :

$$V(x) \quad := \quad \sup_{y \in (0,T]} y U^0 \left( \frac{x}{y} \right).$$

Then, it is clear that $V$ is nondecreasing and therefore :

$$0 \leq \lim_{M \to \infty} \sup_n \int_0^T U(t, c_n(t)) 1_{\{U(t,c_n(t)) \geq M\}} dt \quad \leq \quad \lim_{M \to \infty} V \left( \sup_n \int_0^T c_n(t) 1_{c_n(t) \geq M'\}} dt \right) \tag{3.7}$$

Next, notice that the supremum in the definition of function $V$ is either attained in an interir point (satisfying the first order conditions) or on the boundaries. Direct computation leads to :

$$V(x) \quad = \quad \max \left\{ \frac{x}{K} U^0(K) \; , \; x U^{0\prime}(+\infty) \; , \; T U^0(x/T) \right\}$$

where $K$ solves (if it exists) $U^0(K) - K U^{0\prime}(K) = 0$ on the interval $[(x/T), \infty)$. If such a $K$ does not exist the latter maximum is taken over the two last arguments. Notice that, from the strict concavity of $U^0$, it follows that the equation $U^0(K) - K U^{0\prime}(K) = 0$ admits at most one solution. From this expression of $V$ and recalling that $U^0(0) = 0$, we see that :

$$\lim_{x \to 0^+} V(x) \quad = \quad \max \left\{ \lim_{x \to 0^+} \frac{x}{K} U^0(K) \; , \; \lim_{x \to 0^+} x U^{0\prime}(+\infty) \; ; \; T U^0(0) \right\} \; = \; 0.$$

The required result is therefore obtained from (3.7).

□

**Remark 3.3** The previous proof can be considerably simplified if we use the Lipschitz property of $U$ in the $c$ variable. Notice that condition (2.9) is only used to ensure that $U(0,0) > -\infty$.

**Proof of Theorem 3.1.** By Lemma 3.4, the sequence $(y_n)_n$ is uniformly integrable. It follows that the sequence $(\hat{y}_n)_n$ introduced in Lemma 3.1 is also uniformly integrable, see e.g. Theorem 20 p35 of Delacherie and Meyer (1975). We then have :

$$\hat{x}_n \longrightarrow x^* \text{ in } L^p \quad \text{and} \quad \hat{y}_n \longrightarrow y^* \text{ in } L^1, \tag{3.8}$$

where $p > 1$.[2] From Lemma 2.1, we have

$$\int_0^{\theta^{\hat{x}_n, \hat{y}_n}(t)} \hat{x}_n(u) du \quad = \quad \int_0^t \hat{y}_n(u) du \quad \text{for all } t \in [0,T]$$

[2]The $L^1$ convergence result of $(\hat{y}_n)_n$ can also be obtained as follows. From the uniform integrability of the maximizing sequence $(y_n)_n$, it follows that, after passing to a subsequence, $(y_n)_n$ converges to some $y^* \in L^1$ in the sense of the weak topology $\sigma(L^1, L^\infty)$, see e.g. Theorem 25 (Dunford-Pettis compactness criterion) p43 of Dellacherie and Meyer (1975). Then Mazur's Lemma ensures the existence of a sequence $\hat{y}_n \in \text{conv}(y_k, k \geq n)$ which converges in $L^1$ to $y^*$.

with $(\theta^{\hat{x}_n,\hat{y}_n})_n$ converging towards $\theta^*$ in the sense of the Levy metric, see Lemma 3.2. Fix some $\varepsilon > 0$, then, by definition of the Levy convergence, we have for sufficiently large $n$ :

$$\int_0^{\theta^*(t-\varepsilon)-\varepsilon} \hat{x}_n(u)du \;\leq\; \int_0^t \hat{y}_n(u)du \;\leq\; \int_0^{\theta^*(t+\varepsilon)+\varepsilon} \hat{x}_n(u)du$$

for al $t \in [0,T]$. From (3.8), this provides, by sending $\varepsilon$ to zero,

$$\int_0^{\theta^*(t-)} \hat{x}^*(u)du \;\leq\; \int_0^t \hat{y}^*(u)du \;\leq\; \int_0^{\theta^*(t+)} \hat{x}^*(u)du;$$

recall that $x^* \in L^1$ and therefore $t \longmapsto \int_0^t x^*(u)du$ is (absolutely) continuous. Since $\theta^*$ is a right-continuous nondecreasing function, it is continuous a.e. on $[0,T]$ and therefore :

$$\int_0^{\theta^*(t)} x^*(u)du \;=\; \int_0^t y^*(u)du \quad \text{a.e. on } [0,T].$$

Since $t \longmapsto \int_0^t y^*$ is nondecreasing and (absolutely) continuous as integral of an $L^1_+$ function, we get :

$$\int_0^{\theta^*(t)} x^*(u)du \;=\; \int_0^t y^*(u)du \quad \text{for all } t \in [0,T].$$

and therefore

$$\theta^{x^*,y^*} \;=\; \theta^*. \tag{3.9}$$

Now, from Lemma 3.3,

$$\begin{aligned} \sup_{(x,y)\in\mathcal{A}} \phi(x,y) &= \lim_{n\to\infty} \phi(\hat{x}_n,\hat{y}_n) \\ &= \lim_{n\to\infty} \int_0^T U(t, c^{\hat{x}_n,\hat{y}_n}(t))dt \end{aligned} \tag{3.10}$$

where $U(., c^{\hat{x}_n,\hat{y}_n}(.))$ converges a.e. towards $U(., c^{x^*,y^*}(.))$ by (3.8) and (3.9). Notice that

$$c^{\hat{x}_n,\hat{y}_n}(t) \;\leq\; \omega(t) + \hat{y}_n(t)\|f\|_\infty.$$

The last inequality proves that the sequence $c^{\hat{x}_n,\hat{y}_n}$ inherits the uniform integrability property from $(\hat{y}_n)_n$ and therefore the sequence $\left(U(., c^{\hat{x}_n,\hat{y}_n}(.))\right)_n$ is uniformly integrable by Lemma 3.5. This implies that $U(., c^{\hat{x}_n,\hat{y}_n}(.))$ converges to $U(., c^{x^*,y^*}(.))$ in the sense of $L^1$ and therefore we obtain from (3.10) :

$$\sup_{(x,y)\in\mathcal{A}} \phi(x,y) \;=\; \phi(x^*,y^*),$$

which ends the proof of the existence theorem.

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