# A Direct Approach to Arbitrage-Free Pricing of Credit Derivatives[1]

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## Abstract

This paper develops a framework for modelling risky debt and valuing credit derivatives that is flexible and simple to implement, and that is, to the maximum extent possible, based on observables. Our approach is based on expanding the Heath-Jarrow-Morton term-structure model to allow for defaultable debt. We do not follow the procedure of implying out the behavior of spreads from assumptions concerning the default process, instead working directly with the evolution of spreads. We show that risk-neutral drifts in the resulting model possess a recursive representation that particularly facilitates implementation and makes it possible to handle path-dependence and early exercise features without difficulty. The framework permits embedding a variety of specifications for default; we present an empirical example of a default structure which provides promising calibration results.

# 1 Introduction

Credit derivatives have attracted much attention in recent years. A number of models have been developed aimed at the pricing of these instruments. In this paper, we offer a framework for modelling risky debt and valuing credit derivatives that is flexible and simple to implement, and that is, to the maximum extent possible, based on observables. We utilise the risk-neutral pricing methodology, providing an arbitrage-free model for valuing credit derivatives.

Two distinct approaches are visible in the literature to the modelling of credit risk. One, following the lead of Merton [32], views debt as a contingent claim written on the assets on the firm. The typical model here posits a process for the evolution of firm value, and specifies the conditions leading to bankruptcy, as well as the payoffs to various parties in the event of bankruptcy. The value of debt is then derived as a consequence.[1]

An appealing feature of this approach is that once default regions are specified, the stochastic process driving default occurrence may be endogenously determined. The recovery rate in default may also be endogenized by assuming as in Merton [32], for instance, that the absolute-priority rule holds in the event of default; alternatively, it may be specified exogenously as in Longstaff and Schwartz [29]. However, there are also important practical weaknesses. First, since many of the firm's assets are typically not traded, the firm's value process is fundamentally unobservable; this makes these models difficult to implement. Second, in valuing a particular tranche of corporate debt in this approach, one also has to simultaneously value all debt senior to it, thus increasing computational complexity significantly.

An alternative that has gained in popularity over the past few years is to take a "reduced form" approach and directly model the default process of risky debt. Combining this with a term-structure model and assumptions concerning the recovery rate in default, the value of risky debt may be determined. One set of models along these lines employs a "credit-rating" based approach in which default is depicted through a gradual change in ratings driven by a Markov transition matrix (see, e.g., Das and Tufano [8], Jarrow, et al [21], or Lando [27]). Others, such as Duffie and Singleton [16], and Madan and Unal [30],[31], model the default process without reference to a credit-rating scheme.[2] The models differ in their assumptions concerning the recovery rate. For example, Jarrow, et al [21] use a "Recovery of Treasury" assumption (the terminology is from Duffie and Singleton [16]) that upon default a zero-coupon risky bond trades for the same price as $\delta$ units of a default-risk-free zero coupon bond with the same maturity, where $\delta$ is an exogenously given constant. Duffie and Singleton [16] employ instead what they term a "Recovery of Market Value" condition that upon default a zero-coupon risky bond trades for a fraction $a$ of its market value.

In this paper, we offer a discrete-time reduced-form model for valuing risky debt. Our model possesses the advantage of simple implementation mechanics and requires as inputs only easily available information. Our approach is based on the term-structure model of Heath, Jarrow, and Morton [19] (hereafter HJM). We extend the HJM model to include risky debt by adding a "forward spread" process

[1] Among the many papers that have built on and extended Merton's model in various directions are Bhattacharya and Mason [3], Black and Cox [4], Crosbie [6], Das [7], Delianedis and Geske [10], Geske [11], Huang [20], Kim, Ramaswamy and Sundaresan [25], KMV [26], Leland [28], Longstaff and Schwartz [29], Nielsen, Saa-Requejo and Santa-Clara [33], and Shimko, Tejima and VanDeventer [35].

[2] Other papers employing a reduced-form approach include Duffee [12], Duffie and Huang [14], Duffie, et al [15], Jarrow and Turnbull [22], Kijima and Komorobayashi [24], Kijima [23], and Ramaswamy and Sundaresan [34].

to the forward rate process for default risk-free bonds. No restrictions are placed on the correlation between the two processes. The probability of default at any point is allowed to depend on the entire history of the process to that point. Upon default, we assume that the RMV condition of Duffie and Singleton [16] applies.

The model takes as inputs information on (i) the term structures of default-free forward rates and credit spreads, and (ii) the term-structures of volatilities of these quantities. It then solves for the correct risk-neutral drifts of the stochastic processes to make all securities (including the credit-risky ones) martingales after discounting. Under the RMV condition, these drifts are shown to possess a particularly useful recursive structure.[3] By combining specific assumptions about the default process with this recursive structure, we can determine the evolution of the tree of forward rates and forward spreads; by construction, the default probabilities and recovery rates at each node on this tree are consistent with credit spreads at that node. We illustrate this procedure using a logit equation for the default probabilities, and provide several examples of the actual implementation process.

Our model has several distinguishing features. First, by taking existing spreads as an input into the model—rather than deriving it from implications on default probabilities and recovery rates—our model bypasses the potentially complex process of calibration in which credit spreads implied by the model are compared to observed credit spreads. Of course, this also guarantees that our model is consistent with any observed term structure of credit spreads. It also facilitates the pricing of credit derivatives whose payoffs depend directly on the spread. Second, rather than work with spot yield curves for default-free and risky debt, we work with forward rates and "forward spreads." When combined with the RMV assumption, this results in a third distinguishing feature of our approach: the recursive representations of the risk-neutral drifts of the forward rate and spread processes in the model. These recursive equations are central to facilitating implementation of the model. They render the path-dependence injected by default events computationally tractable, and make possible easy handling of American features in the pricing of credit derivatives is possible. They also enable keeping track of additional information such as the cumulative default probabilities needed for pricing certain types of credit derivatives.

The remainder of this paper is organized as follows. Section 2 describes the model and underlying assumptions. Section 3 describes the derivation of the recursive representation for the risk-neutral drifts, while Section 4 describes a recursive representation of risky bond prices in our model. Section 5 describes the additional structure required for actual implementation, and illustrates with one particular approach. Section 6 discusses the engineering details of the actual implementation process, and illustrates with several examples. Section 7 concludes.

## 2 The Model

The model is developed in discrete time, since a computer implementation for options with American features and path-dependence is envisaged. We consider an economy on a finite time interval $[0, T^*]$. Periods are taken to be of length $h > 0$; thus, a typical time-point $t$ has the form $kh$ for some integer $k$. It is assumed that at all times $t$, a full range of default-free zero-coupon bonds trades, as does a full range of risky zero-coupon bonds. It is also assumed that markets are free of arbitrage, so there

[3] A similar result has also been derived in the continuous version of the HJM model by Duffie and Singleton [16].

exists an equivalent martingale measure $Q$ for this economy;[4] all references to randomness below and all expectations are with respect to this measure.

For any given pair of time-points $(t, T)$ with $0 \leq t \leq T \leq T^* - h$, let $f(t, T)$ denote the forward rate on the default-free bonds applicable to the period $(T, T + h)$; in words, $f(t, T)$ is the rate as viewed from time $t$ for a default-free lending/investment transaction over the interval $(T, T + h)$. (All interest rates in the model are expressed in continuously-compounded terms.) When $t = T$, the rate $f(t, t)$ will be called the "short rate" and denoted by $r(t)$. The forward rate curve is assumed to evolve according to the process

$$f(t+h, T) \;=\; f(t, T) + \alpha(t, T)h + \sigma(t, T)X_1\sqrt{h}, \tag{2.1}$$

where $\alpha$ is the drift of the process and $\sigma$ its volatility; and $X_1$ is a random variable. Both $\alpha$ and $\sigma$ may depend on other information available at $t$, such as the time–$t$ forward rates. To keep notation simple, we have suppressed this possible dependence.

Analogously, for $0 \leq t \leq T \leq T^* - h$, let $\varphi(t, T)$ denote the "forward rate" on the risky bonds implied from the spot yield curve. The *forward spread* $s(t, T)$ on the risky bonds is then defined as

$$s(t, T) \;=\; \varphi(t, T) - f(t, T).$$

Our second assumption concerns the evolution of these forward spreads (and, thus, of the forward rates on the risky bonds). We take these spreads to follow the process

$$s(t+h, T) \;=\; s(t, T) + \beta(t, T)h + \eta(t, T)X_2\sqrt{h}, \tag{2.2}$$

where $\beta(t, T)$ and $\eta(t, T)$ are the drift and volatility coefficients, respectively, and $X_2$ is a random variable. Both $\beta$ and $\eta$ may depend on other information available at $t$. At this point, we place no restrictions on the joint distribution of $X_1$ and $X_2$. For specificity, we will later take both of them to be (arbitrarily correlated) binomial variables that take on the values $\pm 1$ with equal probability.

We will denote by $P(t, T)$ the time–$t$ price of a default-free zero-coupon bond of maturity $T \geq t$, and by $\Pi(t, T)$ its risky counterpart. Note that, by definition, we have

$$P(t, T) \;=\; \exp\left\{ -\sum_{k=t/h}^{T/h-1} f(t, kh) \cdot h \right\} \tag{2.3}$$

$$\Pi(t, T) \;=\; \exp\left\{ -\sum_{k=t/h}^{T/h-1} \varphi(t, kh) \cdot h \right\} \tag{2.4}$$

The spreads on the risky bonds represent the cost of default, and as such, depend on both the probability of default as well as the amount that bond holders expect to recover in the event of default.

[4] Specifically, we assume that $Q$ is an equivalent martingale measure with respect to the money-market account $B(t)$ defined in (3.1) below. See Harrison and Kreps [17] or Harrison and Pliska [18] for the role of equivalent martingale measures in securities modelling.

Given that default has not occurred up to $t$, we denote by $\lambda(t)$ the probability of default by time $t+h$; $\lambda$ may depend on the entire history of the model up to $t$. Concerning the recovery rate, we will use the "Recovery of Market Value" or RMV condition of Duffie and Singleton [16]. Let $\Phi^t$ denote the recovery amount in the event of default at $t$. The RMV condition then states that conditional on default occuring at time $t+h$, the time–$t$ expectation $E^t[\Phi^{t+h}]$ of the amount bondholders will receive is given by

$$E^t[\Phi^{t+h}] \;=\; \phi(t)E^t[\Pi(t+h,T)], \tag{2.5}$$

where $\phi(t)$ denotes the time–$t$ "recovery rate." As with $\lambda$, $\phi(t)$ may also depend on all information in the model up to and including period $t$.

The model's objective is to develop a risk-neutral lattice for pricing risky debt. This is undertaken in several steps. First, the default-free interest rates lattice is generated by solving for the risk-neutral drifts so that all default-free securities are martingales. Then, a lattice for credit spreads is superimposed on the first lattice, and risk-neutral drifts are computed for the forward spread process so as to make the discounted prices of risky debt martingales. Finally, the recursive structure of the model is used together with a specific assumption concerning the default process to illustrate implementation of the model. We begin with identification of the risk-neutral drifts.

## 3 Identifying the Risk-Neutral Drifts

In this section, we derive recursive expressions for the drifts $\alpha$ and $\beta$ of the forward-rate and spread processes, respectively, in terms of the volatilities $\sigma$ and $\eta$. To this end, we define $B(t)$ to be the time–$t$ value of a "money-market account" that uses an initial investment of \$1, and rolls the proceeds over at the default-free short rate:

$$B(t) \;=\; \exp\left\{\sum_{k=0}^{t/h-1} r(kh)\cdot h\right\}. \tag{3.1}$$

We assume without loss that the equivalent martingale measure $Q$ was defined with respect to $B(t)$ as numeraire; thus, under $Q$ all asset prices in the economy discounted by $B(t)$ will be martingales.

We will first identify the risk-neutral drifts $\alpha$ of the default-free forward rates in terms of the volatilities $\sigma$ of these rates. Let $Z(t,T)$ denote the price of the default-free bond discounted using $B(t)$:

$$Z(t,T) \;=\; \frac{P(t,T)}{B(t)}. \tag{3.2}$$

Since $Z$ is a martingale under $Q$, for any $t<T$ we must have $Z(t,T)=E^t[Z(t+h,T)]$, or, equivalently,

$$E^t\left[\frac{Z(t+h,T)}{Z(t,T)}\right] \;=\; 1. \tag{3.3}$$

Now, $Z(t+h,T)/Z(t,T) = (P(t+h,T)/P(t,T)) \cdot (B(t)/B(t+h))$. Using (2.3), some algebra reveals the first term to be

$$\frac{P(t+h,T)}{P(t,T)} \;=\; \exp\left\{ -\left( \sum_{k=t/h+1}^{T/h-1} [f(t+h,kh) - f(t,kh)] \cdot h \right) + f(t,t)h \right\}. \tag{3.4}$$

The second term $B(t)/B(t+h)$ is evidently just $\exp\{-f(t,t)h\}$. Combining these, we obtain

$$\frac{Z(t+h,T)}{Z(t,T)} \;=\; \exp\left\{ - \sum_{k=t/h+1}^{T/h-1} [f(t+h,kh) - f(t,kh)] \cdot h \right\}, \tag{3.5}$$

Using (3.5) in (3.3), the martingale condition becomes

$$E^t\left[ \exp\left\{ - \sum_{k=t/h+1}^{T/h-1} [f(t+h,kh) - f(t,kh)] \cdot h \right\} \right] \;=\; 1. \tag{3.6}$$

Substituting for $(f(t+h,kh) - f(t,kh))$ from (2.1), this is the same as

$$E^t\left[ \exp\left\{ - \sum_{k=t/h+1}^{T/h-1} [\alpha(t,kh)h^2 + \sigma(t,kh)X_1 h^{3/2}] \right\} \right] \;=\; 1. \tag{3.7}$$

Since $\alpha(t,\cdot)$ is known at $t$, it may be pulled out of the expectation. This gives us after some rearranging the promised recursive expression relating the risk-neutral drifts $\alpha$ to the volatilities $\sigma$ at each $t$:

$$\sum_{k=t/h+1}^{T/h-1} \alpha(t,kh) \;=\; \frac{1}{h^2} \ln\left( E^t\left[ \exp\left\{ - \sum_{k=t/h+1}^{T/h-1} \sigma(t,kh)X_1 h^{3/2} \right\} \right] \right). \tag{3.8}$$

We turn to the drifts $\beta(t,T)$. The following preliminary result relating short spreads to the default probabilities and recovery rates under $Q$ will come in handy here and in the rest of the paper:

$$s(t,t) \;=\; -\frac{1}{h} \ln[1 - \lambda(t) + \lambda(t)\phi(t)]. \tag{3.9}$$

To see (3.9), consider a risky bond at $t$ that matures at $(t+h)$. By definition, its time–$t$ price is given by

$$\Pi(t,t+h) \;=\; \exp\{-(f(t,t) + s(t,t)) \cdot h\}. \tag{3.10}$$

Now, a one period investment in this bond fetches a cash flow of \$1 at time $(t+h)$ if there is no default at $t+h$, and a cash flow of $\phi(t)$ if there is a default. When discounted at the short rate, the expected cash flow (in the risk-neutral world) must equal the initial price of the bond, so we obtain

$$\Pi(t,t+h) \;=\; \exp\{-f(t,t)h\}[1 - \lambda(t) + \lambda(t)\phi(t)] \tag{3.11}$$

Expression (3.9) is an immediate consequence of (3.10) and (3.11).

Now, pick any $t < T$ and consider a one-period investment in $\Pi(t,T)$ at $t$. Viewed from time $t$, there are two possibilities regarding expected cash flows at $t+h$ from this investment. If the bond has not defaulted by $t+h$, there is an expected cash flow of $E^t[\Pi(t+h,T)]$; if the bond has defaulted on the other hand, the expected cash flow is $\phi(t)E^t[\Pi(t+h,T)]$. Since the probability of default by $t+h$ is $\lambda(t)$, the expected cash flow at $t+h$ is

$$(1-\lambda(t))\, E^t[\Pi(t+h,T)] + \lambda(t)\phi(t)\, E^t[\Pi(t+h,T)] \tag{3.12}$$

which is the same as

$$[1-\lambda(t)+\lambda(t)\phi(t)]\, E^t[\Pi(t+h,T)]. \tag{3.13}$$

By definition of $Q$, when discounted at the short rate $r(t)$, this expected cash flow must equal $\Pi(t,T)$, so we have

$$E^t\left[\frac{[1-\lambda(t)+\lambda(t)\phi(t)]\,\Pi(t+h,T)}{\exp\{r(t)h\}\Pi(t,T)}\right] \;=\; 1. \tag{3.14}$$

Now, using (2.4) and the definitional relation $s(t,t) = \varphi(t,t) - f(t,t)$, some algebra reveals that

$$\frac{\Pi(t+h,T)}{\exp\{r(t)h\}\Pi(t,T)} \;=\; \exp\left\{-\left(\sum_{k=t/h+1}^{T/h-1}[\varphi(t+h,kh)-\varphi(t,kh)]\cdot h\right)+s(t,t)h\right\}. \tag{3.15}$$

Further, by (3.9), we know that $[1-\lambda(t)+\lambda(t)\phi(t)] = \exp\{-s(t,t)h\}$. Combining this with (3.14) and (3.15), we finally obtain

$$E^t\left[\exp\left\{-\sum_{k=t/h+1}^{T/h-1}[\varphi(t+h,kh)-\varphi(t,kh)]\cdot h\right\}\right] \;=\; 1. \tag{3.16}$$

Using the definition $\varphi(t,T) = f(t,T)+s(t,T)$, we can substitute for $(\varphi(t+h,kh)-\varphi(t,kh))$ from (2.1) and (2.2). Some rearranging now gives us the second recursive relation, this one defining $\alpha$ and $\beta$ in terms of $\sigma$ and $\eta$:

$$\exp\left\{\sum_{t/h+1}^{T/h-1}[\alpha(t,kh)+\beta(t,kh)]h^2\right\} \;=\; E^t\left[\exp\left\{-h^{3/2}\sum_{t/h+1}^{T/h-1}[\sigma(t,kh)X_1+\eta(t,kh)X_2]\right\}\right] \tag{3.17}$$

Since we have solved for $\alpha$ in terms of $\sigma$ using (3.8), we may now use (3.17) to solve for $\beta$ in terms of $\sigma$ and $\eta$. This completes the derivation of the risk-neutral drifts in terms of the volatilities. The recursive equations (3.8) and (3.17) play a central role in facilitating implementation of our model. We discuss this further in Section 6.

# 4 A Recursive Representation of Risky Bond Prices

Analogous to the risk-neutral drifts, the prices of a risky bond in our model also have a recursive representation, which leads, in turn, to a representation in terms of bond prices of short maturities (i.e., of the form $\Pi(\tau, \tau+h)$). We describe this representation here. We have seen that, under our assumptions, we must have

$$E^t \left[ \exp\left\{ - \sum_{k=t/h+1}^{T/h-1} [\varphi(t+h, kh) - \varphi(t, kh)] \cdot h \right\} \right] \;=\; 1. \tag{4.1}$$

Equivalently, this says

$$E^t \left[ \exp\{-\varphi(t,t)h\} \frac{\Pi(t+h, T)}{\Pi(t, T)} \right] \;=\; 1. \tag{4.2}$$

Rearranging terms and using the fact that $\exp\{-\varphi(t,t)h\} = \Pi(t, t+h)$, we now obtain

$$\Pi(t,T) \;=\; \Pi(t, t+h) \cdot E^t[\Pi(t+h, T)]. \tag{4.3}$$

Iterating on this expression, we finally obtain

$$\begin{aligned} \Pi(t,T) &= \Pi(t,t+h) \cdot E^t[\Pi(t+h,T)] \\ &= \Pi(t,t+h) \cdot E^t[\Pi(t+h,t+2h) \cdot E^{t+h}[\Pi(t+2h,T)]] \\ &= \Pi(t,t+h) \cdot E^t[\Pi(t+h,t+2h) \cdot E^{t+h}[\Pi(t+2h,t+3h) \cdot E^{t+2h}[\ldots] \cdots]] \end{aligned} \tag{4.4}$$

The recursive structure of the prices of risky bonds as embodied in (4.4) facilitates computation of these prices. Note that since all terms on the right-hand side have the form $F(\tau, \tau+h)$, we can make use of (3.11) to employ the forward spread components (i.e., the default and recovery rates) in this process.

# 5 Towards Implementation of the Model

To be able to implement the model, we must be more precise about quantities that have so far been left unspecified, viz., the random variables $X_1$ and $X_2$ and either the default probability $\lambda$ or the recovery rate $\phi$. In this section, we describe the assumptions that we will use in the rest of this paper. These assumptions were chosen with an eye towards simplicity both in exposition and in implementation, but they are primarily meant to be illustrative; alternative assumptions may, of course, be similarly handled.

Concerning the variables $X_1, X_2$, we make the standard discrete-time assumption that $X_1$ and $X_2$ are binomial random variables, specifically, that each variable takes on the values $\pm 1$ with probability $1/2$.

We will also allow for arbitrary correlation $\rho$ between the variables, so the assumed joint distribution of $(X_1, X_2)$ is

$$(X_1, X_2) \;=\; \begin{cases} (+1,+1), & \text{w.p. } (1+\rho)/4 \\ (+1,-1), & \text{w.p. } (1-\rho)/4 \\ (-1,+1), & \text{w.p. } (1-\rho)/4 \\ (-1,-1), & \text{w.p. } (1+\rho)/4 \end{cases} \tag{5.1}$$

We note that, in general, $\rho$ may not be equal to zero or even constant over the tree. Empirically, spreads and interest rates may be positively or negatively correlated depending on the economic regime. For example, a comprehensive study by Duffee [13], finds that yield spreads and treasury yields are negatively correlated over the period 1985-95. On the other when data spanning the period 1978–87 are used (see Adler and Altman [1]), the correlation between treasury rates and high-yield spreads is 0.20; for the period 1988-97 the correlation rises to 0.61, yet taken over the entire 20–year period it is slightly negative at $-0.07$. In addition, the degree of correlation tends to increase as the quality of risky debt declines (see Das and Tufano [8]). The specification we have adopted is flexible enough to accomodate these concerns. In particular, (5.1) can apply in each period with a different value for $\rho$. Thus, if we so desire, we may allow $\rho$ to change with time or depend on interest rate and spread levels, etc.

Turning to the components of the forward spreads, we have already identified a relationship (3.9) that links the short spread $s(t,t)$ to the default probability $\lambda(t)$ and the recovery rate $\phi(t)$. Without an additional equation, it is not possible to decompose the spread into these two fundamental components. In keeping with the spirit of this paper, a readily observable means of identification is desirable.

Many alternatives suggest themselves. One is to specify default probabilities or recovery rates as exogenously given, say, to correspond to average levels typically observed; such an approach has been used elsewhere in the literature.[5] More generally, we may assume that one of these variables evolves according to a specified stochastic process. The process driving the other variable then obtains by combining this with expression (3.9) and the process for the spreads (2.2).

We shall adopt the second approach in this paper for the purposes of illustration. For specificity, we shall assume that the default probability $\lambda(t)$ at time $t$ is a function of the level of interest rates in the model at time–$t$. Since $\lambda$ is a probability, we must choose a function whose range lies in $[0,1]$. Letting $F$ and $S$ denote the entire forward and spread curves, a simple specification that meets our requirements is the logit equation

$$\lambda(F,S) \;=\; \frac{1}{e^x + 1}, \qquad x \;=\; a + b \cdot F + c \cdot S \tag{5.2}$$

Of course, we could also use specifications other than (5.2) for default probabilities, even while assuming that these probabilities depend only on information embodied in the term-structure. For example, we may wish to allow default probabilities to depend on the slope of the yield curve as well: flat or inverted yield curves are frequently associated with recessionary trends in the economy, and to the extent that such a relationship holds, we would expect default probabilities to increase as the

[5] As mentioned earlier, for example, Jarrow and Turnbull [22] use a "Recovery of Treasury" condition in which upon default the risky bond is worth an exogenously given fraction $\delta$ of a corresponding default-free bond.

slope of the term-structure decreases. From the standpoint of illustration and expositional simplicity, however, (5.2) seems adequate, and we shall use it in the rest of the paper.[6] Of course, alternative formulations may also be used with the obvious changes in what follows.

One last, and non-trivial, issue remains before we can discuss the engineering details of model implementation. Estimates of the parameters of equation (5.2) based on the data cannot be directly used in our model, since our model (including the probability of default $\lambda$) is set in the risk-neutral world, while (5.2) taken to the data will yield the actual probability of default. Thus, a translation from the actual to the risk-neutral measure is required. To this end, suppose that $\lambda^P$ denotes the actual probability of default. We will make the natural assumption that the recovery rates are the same in the risk-neutral and actual worlds, so realized cash flows coincide in the two cases. Letting $\xi(t)$ be the time–$t$ premium for bearing default risk, the analog of (3.9) under the actual probabilities is easily derived:

$$\exp\{-s(t,t)h\} \;=\; \exp\{-\xi(t)h\}[1-\lambda^P(t)+\phi(t)\lambda^P(t)]. \tag{5.3}$$

Comparing (3.9) and (5.3), we may express $\lambda$ in terms of $\lambda^P$ and the risk-premium $\xi$:

$$\lambda(t) \;=\; \lambda^P(t)\left[\frac{1-\exp\{-s(t,t)h\}}{1-\exp\{-(s(t,t)-\xi(t))h\}}\right]. \tag{5.4}$$

Expression (5.4) implies the intuitive condition that $\lambda > \lambda^P$ whenever the risk-premium $\xi$ is positive.

These expressions may be used in the estimation process to extract not only the parameters of (5.2), but also the risk-premium $\xi$. Specifically, note that from (5.3), we have

$$\phi(t) \;=\; \frac{1}{\lambda^P(t)}[\exp\{-(s(t,t)-\xi(t))h\}-1+\lambda^P(t)]. \tag{5.5}$$

while (5.2) may be rewritten as

$$\ln\left(\frac{1}{\lambda^P(t)}-1\right) \;=\; a+b\cdot F+c\cdot S. \tag{5.6}$$

For notational ease, let $m(x)=\ln[(1/x)-1]$. Lastly, let $\bar{\phi}_{av}$ be the average recovery rate in the data. Consider the following estimation exercise:

$$\begin{array}{lrcl} \text{Minimize} & \sum_t \epsilon_t^2 & & \\ & & & \\ \text{subject to} & m(\lambda^P(t)) & = & a+b\cdot F+c\cdot S+\epsilon_t \\ & \phi(t) & = & [\exp\{-(s(t,t)-\xi(t))h\}-1+\lambda^P(t)]/\lambda^P(t) \end{array} \tag{5.7}$$

The first constraint in (5.7) is simply (5.6) with an error term. The second repeats the definition (5.5) of $\phi$ in terms of $\lambda^P$. Once we have values for the parameters $(a,b,c)$ and the risk-premium $\xi$, the

[6] We should note also that Wilson [37] provides strong support for a specification of this sort. He finds that a logit regression fits default rates with $R^2$ values in the range of 80–90% for many countries.

actual default probability at time $t$ may be obtained from (5.2); the recovery rate $\phi(t)$ by using (5.5); and the risk-neutral default probability $\lambda(t)$ by appealing to (5.4).

### An Empirical Illustration

We estimated (5.7) using data described in Adler and Altman [1] and Altman and Kishore [2]. The data consist of time series of treasury yields, high-yield spreads, and default rates for each year from 1978–1997. The default rate is measured as the number of issuers defaulting as a percentage of total issuers. Given this data, we used as explanatory variables only the short forward rate $f(t,t) = r(t)$ and the short spread (i.e., the difference in the high yield and treasury short rates). In this case, (5.2) specializes to

$$\lambda^P(t) \;=\; \frac{1}{1+e^x}, \qquad x \;=\; a + br(t) + cs(t,t). \tag{5.8}$$

Secondly, to pin down the values of $\phi(t)$, we assumed the the default risk-premium $\xi$ to be proportional to the short spread $s(t,t)$: $\xi(t) = \pi s(t,t)$. This is analogous to the risk-premium in the Cox-Ingersoll-Ross model of the term structure where the risk-premium is proportional to the short rate $r(t)$. The parameter values we obtained are described in Table 1 when $\pi$ is taken to be 0.50.[7] The model appears to fit the data well. The fitted time series of $\lambda^P$ and the actual data have a high correlation of 0.86. The regression $R^2$ is 0.58.[8]

## 6 Implementation of the Model

The model we have developed above may be easily implemented on a lattice. Its double-binomial structure as described in expression (5.1) results in a branching process with four branches emanating from each node. Specifically, once the risk-neutral drifts $\alpha(\cdot), \beta(\cdot)$ have been computed at any $t$, the possible values of the forward rates and forward spreads one period out are readily obtained using (2.1) and (2.2). Given, moreover, the forward and spread curves $F(\tau) \;=\; (f(\tau,\cdot))$ and $S(\tau) = (s(\tau,\cdot))$ at any $\tau$, we can also compute the one-period default probability $\lambda(\tau)$ using the estimated parameters of the logit equation (5.2); and the recovery rate $\phi(\tau)$ using (5.5). Therefore, at each node on the lattice we have information related to all three risks involved in the valuation of risky debt (interest rates, default probabilities, and recovery rates) and their possible one-period ahead values.

In addition to these variables, two other values of interest may be computed at each node. First, at each node, we can compute the state-prices at each of the four possible successor nodes using the following recursive equation:

$$w(t+h,\varpi) \;=\; w(t)\exp\{-r(t)h\}\pi(\varpi), \qquad w(0) = 1. \tag{6.1}$$

[7] The choice of the appropriate value of $\pi$ is not obvious. One way to proceed might be to use the value of $\pi$ under which the average recovery rate using the estimated parameters most closely matches the observed recovery rate. Unfortunately, the Adler–Altman and Altman–Kishore papers from which we took our data do not provide information on the market values of the defaulting bonds, so we could not carry out this step.

[8] In an earlier paragraph, we suggested that the slope of the term structure may be a plausible explanatory variable to include in the regression. We carried out this estimation, adding the term $d{\cdot}slope$ to the right-hand side of (5.8). The $R^2$ showed a significant increase to 0.68. The correlation between the fitted series of $\lambda^P$ and the actual values remained unchanged at its high value of 0.86.

Table 1: Parameters of the Default Probability

This table describes our estimates of the parameters of (5.7) using data from Adler and Altman [1]. The default probability function is specialized to (5.8), and estimation is undertaken by solving (5.7). The default risk-premium $\xi$ is taken to have the form $\xi(t) = \pi s(t,t)$, where $\pi$ is a constant and $s(t,t)$ the short spread. The estimation assuming $\pi$ to be 0.50. The expression "corr" in the table refers to the correlation between the fitted time-series of $\lambda^P$ based on our estimates and that in the actual data.

| Parameter | Estimate |
|---|---|
| $a$ | 5.44 |
| $b$ | −10.43 |
| $c$ | −27.24 |

| | |
|---|---|
| corr | 0.86 |
| $R^2$ | 0.58 |

Here, $\varpi$ represents the random choice of one of the four branches of the lattice at each node at time $t$, and $\pi(\varpi)$ the probability of $\varpi$. The advantage of computing state prices derives from the fact that it makes it easy to compute the price of a derivative claim by generating payoffs on the lattice and directly multiplying these by the state prices and aggregating the values. A second piece of information we could easily generate on our lattice are the cumulative probabilities of default. We denote these values by $\Lambda(t)$. Since $\lambda(t)$ denotes the one-period probability of default at $t$, the following recursive equation describes the evolution of cumulative default probabilities in the model:

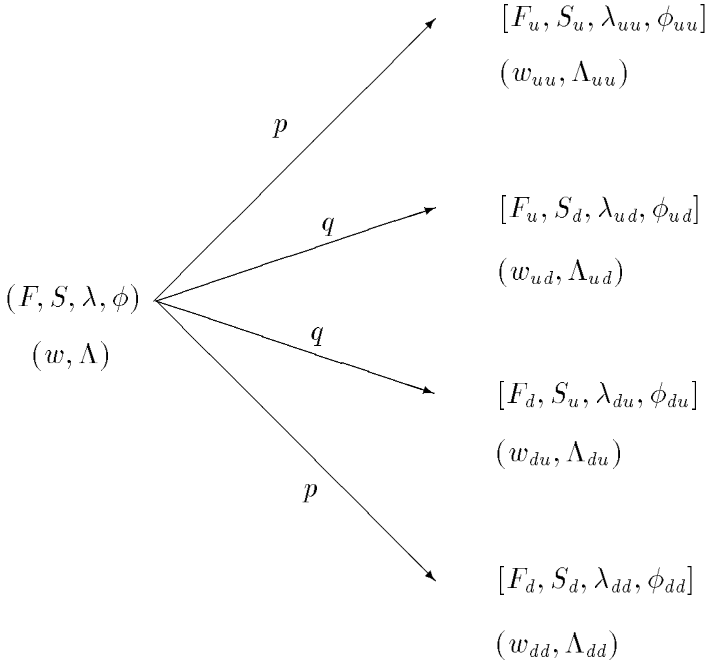
$$\Lambda(t+h,\varpi) \;=\; \Lambda(t) + [1-\Lambda(t)]\lambda(t,\varpi)h, \qquad \Lambda(0) = 0. \tag{6.2}$$

It is easy to see from the structure of (6.2) that $\Lambda(t)$ lies in the interval $[0,1]$ at all $t$.

With the computation of these additional values, the lattice is enhanced to include cumulative default probabilities and state prices. This provides all the necessary information for pricing a range of credit derivatives. The branching lattice together with the information available at each node now appears in Figure 1. In the figure, $p$ and $q$ are used as shorthand notation for $(1+\rho)/4$ and $(1-\rho)/4$, respectively. The remaining notation in the figure is largely self-explanatory:

1. $F_u$ and $F_d$ refer, respectively, to the forward rates that result from $F$ if $X_1 = +1$ and $X_1 = -1$.
2. $S_u$ and $S_d$ are the forward spreads that result from $S$ if $X_2 = +1$ and $X_2 = -1$, respectively.
3. $\lambda_{uu}$ refers to the default probability given $(F_u, S_u)$; $\lambda_{ud}$, $\lambda_{du}$, and $\lambda_{dd}$ are defined analogously.
4. $\phi_{uu}$ refers to the recovery rate given $(F_u, S_u)$; $\phi_{ud}$, $\phi_{du}$, and $\phi_{dd}$ are defined analogously.
5. $w_{uu}$ refers to the state prices that result given $(F_u, S_u)$; $w_{ud}$, $w_{du}$, and $w_{dd}$ are defined analogously.
6. $\Lambda_{uu}$ refers to the cumulative default probability at $(F_u, S_u)$; $\Lambda_{ud}$, $\Lambda_{du}$, and $\Lambda_{dd}$ are defined analogously.

Figure 1: Information Generated at Each Node in the Branching Process



## 6.1 Computing the Lattice via Recursion

The lattice described above may be generated using a recursive algorithm. A recursive algorithm is a function that calls itself at each step. Option pricing uses a recursion scheme quite naturally because on a lattice, the value at any node is a function of values at future nodes. Recursive algorithms provide two simple benefits: (i) the program code is parsimonious, and (ii) the recombination of branches on the lattice is not an issue. The recursive algorithm simply follows each sample path to its conclusion. However, when a satisfactory recombination scheme is available, a recursive algorithm may be inefficient, because it misses the opportunity to optimize the lattice for speed. We describe the recursive algorithm in some detail.

Recursion is available since the lattice satisfies no-arbitrage conditions at each node. Hence, each subtree on the lattice may be treated separately. Knowledge of the values $[F, S, \lambda, \phi, w, \Lambda](t)$ allows direct extension into the next period's nodes $[F, S, \lambda, \phi, w, \Lambda](t + h, \varpi)$. Since the recursion allows path dependent values to be carried on the tree by forward induction, the value of any path-dependent derivative asset is easy to compute.

In order to make the algorithm transparent, we describe the programs provided later in a simple fashion using pseudo-code. The algorithm has the following components:

1. Input the necessary raw material (i.e., program parameters). These are: the initial forward curve, the forward rate volatilities, initial spread curve, spread volatilities, the correlation between spread and forward rates, the time step $h$, and parameters of the default function. The exercise price of the credit derivative is also needed.

2. With this input, the recursion may begin. At the initial node are the current values of the forward rate curve and spread curve $[F(t), S(t)]$. From the initial node, four branches emanate. At the end of each of these branches is another set each of $[F(t + h), S(t + h)]$; and so on until the terminal nodes are reached. A recursive scheme contains two essential components: (i) the recursive calls, and (ii) the terminating conditions. In our model, the recursive function calls itself by passing the next period's values $[F(t + h, T), S(t + h, T)]$ to itself. In between, the code in the function generates the forward values ensuring that the conditions of no-arbitrage are met, i.e. the function solves for the risk-neutral drift parameters $\alpha(\cdot)$, $\beta(\cdot)]$. This comprises the bulk of the recursive function and is an implementation of the expressions in Section 3. The terminating conditions are the payoff values for the credit derivative at maturity, or in the case of an American style contract, the intermediate stopping rules.

3. The recursion handles generation of the lattice in forward rates and spreads. It also accounts for derivative payoffs. Unless additional complexity is called for in the contract, this is sufficient. However, for credit options that require specific information about recovery rates, or default probabilities, further work is required on the lattice. The spread values at each node may be easily decomposed into default probabilities and recovery rates using additional information, as described in Section 5. Since these may be handled as part of the recursion, it poses no further complexity in encoding the algorithm.

4. Finally, as explained at the top of this section, state prices and cumulative probabilities of default may be generated in the recursive procedure. The state prices allow for rapid aggregation of value

over intermediate payoffs, if any. The cumulative probabilities of default allow computation of forward default probabilities, if required.

The pseudo-code below embeds each of the above components and should help clarify the implementation:

```
PSEUDO-CODE for Recursive Credit Derivative Model
Step 1: Read in basic data: forward curves, spread curves, volatilities,
correlations, default model parameters, credit derivative payoffs.
Call this DATA1(t=0)
Step 2: Recursive function call
CRVAL[level, DATA1(t), DATA2(t)]:=
  {
      If stopping_rule OK,
      Then implement terminal/boundary/stopping rules;
      Else
      {
  Generate DATA1(t+1): obey no-arbitrage rules;
  Generate DATA2(t+1): use default function;
  Call CRVAL[level+1, DATA1(t+1), DATA2(t+1)]: recursive step;
      }
  }
```

The algorithm is economical. The data is read in first (Item 1 above). "DATA1" comprises forward rates and spreads, and their volatilities. This is basic data (Item 2). "DATA2" comprises additional data generated in the recursion such as state prices, default probabilities, cumulative default probabilities (Items 3 and 4 above).

In the next subsection, we price some popular credit derivatives as an illustration of our model. The examples of pseudo-code using *Mathematica* [38] demonstrate the programming technique.

## 6.2 Pricing Credit Derivatives: Examples

We consider two credit derivatives in this section: a credit-spread option and a credit default swap. Pricing each of these is facilitated by an aspect of our approach, the first since our model directly keeps track of the evolution of spreads over the tree, and the second because our model's recursive structure makes it possible to keep track of cumulative default probabilities. Following this discussion, we present numerical examples that price these and other derivatives using the codes presented below.

### 6.2.1 Credit Spread Options

A credit spread option is a credit derivative that is written on an underlying credit spread. A call option on a credit spread may be defined as a contract that pays off at some defined maturity if the spread is trading above a strike level $K$ (the exercise price). The payoff to this contract is

$$100 \times \max\left[0, s(T,T) - K\right] \tag{6.3}$$

where the payoff is per \$100 face value. These options allow investors to act on a view regarding the quality of a bond. They may evidently be used to insure a bond portfolio against declines in credit quality. They are also of value to option writers since since credit spreads tend to be more volatile than interest rates, resulting in large option premia. Furthermore, spread volatility declines rapidly towards bond maturity, making time decay an attractive feature for spread option writers.

Pricing this derivative is not difficult. Terminal payoffs from the option are generated on the lattice by comparing the spot spreads at that time with the exercise level of the option. Discounting these back appropriately delivers the option price. The code for this procedure follows. For simplicity, we have assumed in this pseudo-code that $\sigma(t,T)$ depends only on $T$.

```
(* Program to generate the HJM Tree with default risk recursively *)
CRD[f0_,fsig0_,s0_,ssig0_,rho_,h_,exprice_,a_,b_,c_,xi_]:=Module[
   {n,puu,pud,pdu,pdd},
   n=Length[f0];
   puu=(1+rho)/4; pdd=puu; pud=(1-rho)/4; pdu=pud;
   CRVAL[level_,f_,fsig_,s_,ssig_,cumdef_]:=
      CRVAL[level,f,fsig,s,ssig,cumdef]=
      Module[{i,m,j,alpha,beta,fuu,fud,fdu,fdd,suu,sud,sdu,
         sdd,fsigma,ssigma,pd,recov,cumd},
         If[level==n-1,    (*careful about n or n-1 *)
            result=Max[0,s[[1]]-exprice]*100;
         ];
         If[level<n-1,
            m=Length[f]-1;
            fuu=Take[f,-m]; fud=fuu; fdu=fud; fdd=fdu;
            suu=Take[s,-m]; sud=suu; sdu=sud; sdd=sdu;
            fsigma=Take[fsig,-m];
            ssigma=Take[ssig,-m];
            alpha=Table[0,{k,m}];
            beta=Table[0,{k,m}];
            For[j=1,j<=m,j++,
               If[j==1,
                     alpha[[j]]=Log[0.5*(Exp[-fsigma[[j]]*h*Sqrt[h]]+
                                Exp[fsigma[[j]]*h*Sqrt[h]])]/h^2;
                     beta[[j]]=Log[puu*Exp[(-fsigma[[j]]-ssigma[[j]])*h*Sqrt[h]]+
                        pud*Exp[(-fsigma[[j]]+ssigma[[j]])*h*Sqrt[h]]+
                        pdu*Exp[(fsigma[[j]]-ssigma[[j]])*h*Sqrt[h]]+
                        pdd*Exp[(fsigma[[j]]+ssigma[[j]])*h*Sqrt[h]]]/h^2-
                           alpha[[j]];
               ];
               If[j>1,
                    alpha[[j]]=Log[0.5*
                        (Exp[-Sum[fsigma[[k]],{k,j}]*h*Sqrt[h]]+
                         Exp[Sum[fsigma[[k]],{k,j}]*h*Sqrt[h]])]/h^2-
                           Sum[alpha[[k]],{k,j-1}];
```

```
36                  beta[[j]]=Log[
37                      puu*Exp[Sum[(-fsigma[[j]]-ssigma[[j]])*h*Sqrt[h],{k,j}]]+
38                      pud*Exp[Sum[(-fsigma[[j]]+ssigma[[j]])*h*Sqrt[h],{k,j}]]+
39                      pdu*Exp[Sum[(fsigma[[j]]-ssigma[[j]])*h*Sqrt[h],{k,j}]]+
40                      pdd*Exp[Sum[(fsigma[[j]]+ssigma[[j]])*h*Sqrt[h],{k,j}]]]/h^2-
41                        Sum[alpha[[k]],{k,j}]-Sum[beta[[k]],{k,j-1}];
42              ];
43           ];
44           fuu=fuu+alpha*h+fsigma*Sqrt[h];
45           fud=fud+alpha*h+fsigma*Sqrt[h];
46           fdu=fdu+alpha*h-fsigma*Sqrt[h];
47           fdd=fdd+alpha*h-fsigma*Sqrt[h];
48           suu=suu+beta*h+ssigma*Sqrt[h];
49           sud=sud+beta*h-ssigma*Sqrt[h];
50           sdu=sdu+beta*h+ssigma*Sqrt[h];
51           sdd=sdd+beta*h-ssigma*Sqrt[h];
52           cumd=(1-cumdef)/(1+Exp[a+b*f[[1]]+c*s[[1]]])*
53                (1-Exp[-s[[1]]*h])/(1-Exp[(xi*s[[1]]-s[[1]])*h]);
54           result=Exp[-(f[[1]])*h]*
55               (puu*CRVAL[level+1,fuu,fsigma,suu,ssigma,cumd]+
56                pud*CRVAL[level+1,fud,fsigma,sud,ssigma,cumd]+
57                pdu*CRVAL[level+1,fdu,fsigma,sdu,ssigma,cumd]+
58                pdd*CRVAL[level+1,fdd,fsigma,sdd,ssigma,cumd]];
59
60        ];  (* end IF Level < n-1 *)
61        Return[result];
62     ];
63  Return[CRVAL[0,f0,fsig0,s0,ssig0,0]];
64]
```

Two features of this code bear attention: (a) the use of recursive programming, and (b) the embedded boundary conditions. The boundary condition is provided in lines 10–11. The contract is assumed to be written on a notional value of $100 for this implementation. Lines 6 thru 62 contain the recursive code function CRVAL. This code segment performs two distinct operations. First, in lines 21–43, it computes the risk-neutral drifts using the closed form expressions for $\alpha(\cdot), \beta(\cdot)$ from the paper. Second, the function CRVAL calls itself recursively in lines 54–58. This recursive call exploits the feature that each subtree is arbitrage free in and of itself. Lines 44-52 of the program set up the forward induction segment of the program, while the lines 54-58 implement backward recursion. In line 63, the initial call of the recursive function is made with the starting parameters of the model.

The recursive code has the advantage that the entire lattice need not reside in memory. If non-recursive methods are used, the need to store the lattice in memory increases the computing hardware requirement. The number of terminal nodes on the tree grows rapidly as $4^n$, where $n$ is the number of time steps. For even a "small" value of $n$ such as 8, this results in over 65,000 end nodes.

### 6.2.2 Credit Default Swap

In a credit default swap, one party makes a steady stream of payments in exchange for a single contingent payment that is made only in the event of default of an underlying bond. The simplest form of this contract would be one where a single upfront payment is made to purchase default insurance. The payoff on default may be a predetermined lump-sum amount. It may also be defined as the loss in value on the bond.

Valuing this option on the recursive lattice requires information on cumulative default probabilities $[\Lambda(t)]$ along the sample path. This in turn requires the one-period probabilities of default $[\lambda(t)]$. As before, we define the probabilities using a logit function:

$$\lambda^P(t) \;=\; [1 + \exp\{a + bf(t,t) + cs(t,t)\}]^{-1}.$$

The values of the parameters $(a, b, c)$ are determined empirically; the risk-premium $\xi$ is once again presumed to have the form $\xi(t) = \pi s(t,t)$. If the underlying bond has $n$ periods to maturity, then default is possible in any of the periods. To value the credit default swap, we generate default payoffs based on the recovery rates $(\phi)$ at all points on the sample path. These payoffs are multiplied by the "first-passage" default probability, which is the probability of default conditional on no prior default. The first passage probability is given by the expression $[1 - \Lambda(t)]\lambda(t)$. The payoff of the credit default swap is the loss on default, i.e., $[1 - \phi(t)]$. Using equations (5.4) and (5.5) this may be rewritten as

$$1 - \phi(t) \;=\; \frac{1}{\lambda^P(t)}[1 - e^{-[s(t,t)-\xi(t)]h}] \;=\; \frac{1}{\lambda(t)}[1 - e^{-s(t,t)h}] \tag{6.4}$$

The expected cashflow to the derivative is given by the first-passage default probability times the loss on default, which is

$$[1 - \Lambda(t)]\lambda(t) \times [1 - \phi(t)] \;=\; [1 - \Lambda(t)][1 - e^{-s(t,t)h}] \tag{6.5}$$

This expression is embedded in the program below in lines 12 and 55 of the code.

```
1 (* Program to generate the HJM Tree with default risk recursively *)
2  CRD[f0_,fsig0_,s0_,ssig0_,rho_,h_,exprice_,a_,b_,c_,xi_]:=Module[
3    {n,puu,pud,pdu,pdd},
4    n=Length[f0];
5    puu=(1+rho)/4; pdd=puu; pud=(1-rho)/4; pdu=pud;
6    CRVAL[level_,f_,fsig_,s_,ssig_,cumdef_]:=
7       CRVAL[level,f,fsig,s,ssig,cumdef]=
8       Module[{i,m,j,alpha,beta,fuu,fud,fdu,fdd,suu,sud,sdu,
9              sdd,fsigma,ssigma,pd,recov,cumd},
10         If[level==n-1,pd=(1-cumdef)/(1+Exp[a+b*f[[1]]+c*s[[1]]])*
11            (1-Exp[-s[[1]]*h])/(1-Exp[(xi*s[[1]]-s[[1]])*h]);
12            result=(1-cumdef)*(1-Exp[-s[[1]]*h]);
13         ];
14         If[level<n-1,
15            m=Length[f]-1;
```

```
16          fuu=Take[f,-m]; fud=fuu; fdu=fud; fdd=fdu;
17          suu=Take[s,-m]; sud=suu; sdu=sud; sdd=sdu;
18          fsigma=Take[fsig,-m];
19          ssigma=Take[ssig,-m];
20          alpha=Table[0,{k,m}];
21          beta=Table[0,{k,m}];
22          For[j=1,j<=m,j++,
23             If[j==1,
24                alpha[[j]]=Log[0.5*(Exp[-fsigma[[j]]*h*Sqrt[h]]+
25                       Exp[fsigma[[j]]*h*Sqrt[h]])]/h^2;
26                beta[[j]]=Log[puu*Exp[(-fsigma[[j]]-ssigma[[j]])*h*Sqrt[h]]+
27                     pud*Exp[(-fsigma[[j]]+ssigma[[j]])*h*Sqrt[h]]+
28                     pdu*Exp[(fsigma[[j]]-ssigma[[j]])*h*Sqrt[h]]+
29                     pdd*Exp[(fsigma[[j]]+ssigma[[j]])*h*Sqrt[h]]]/h^2-
30                      alpha[[j]];
31            ];
32            If[j>1,
33                alpha[[j]]=Log[0.5*
34                    (Exp[-Sum[fsigma[[k]],{k,j}]*h*Sqrt[h]]+
35                     Exp[Sum[fsigma[[k]],{k,j}]*h*Sqrt[h]])]/h^2-
36                       Sum[alpha[[k]],{k,j-1}];
37                beta[[j]]=Log[
38                   puu*Exp[Sum[(-fsigma[[j]]-ssigma[[j]])*h*Sqrt[h],{k,j}]]+
39                   pud*Exp[Sum[(-fsigma[[j]]+ssigma[[j]])*h*Sqrt[h],{k,j}]]+
40                   pdu*Exp[Sum[(fsigma[[j]]-ssigma[[j]])*h*Sqrt[h],{k,j}]]+
41                   pdd*Exp[Sum[(fsigma[[j]]+ssigma[[j]])*h*Sqrt[h],{k,j}]]]/h^2-
42                       Sum[alpha[[k]],{k,j}]-Sum[beta[[k]],{k,j-1}];
43            ];
44         ];
45         fuu=fuu+alpha*h+fsigma*Sqrt[h];
46         fud=fud+alpha*h+fsigma*Sqrt[h];
47         fdu=fdu+alpha*h-fsigma*Sqrt[h];
48         fdd=fdd+alpha*h-fsigma*Sqrt[h];
49         suu=suu+beta*h+ssigma*Sqrt[h];
50         sud=sud+beta*h-ssigma*Sqrt[h];
51         sdu=sdu+beta*h+ssigma*Sqrt[h];
52         sdd=sdd+beta*h-ssigma*Sqrt[h];
53         cumd=cumdef+(1-cumdef)/(1+Exp[a+b*f[[1]]+c*s[[1]]])*
54             (1-Exp[-s[[1]]*h])/(1-Exp[(xi*s[[1]]-s[[1]])*h]);
55         result=(1-cumdef)*(1-Exp[-s[[1]]*h])+Exp[-(f[[1]])*h]*
56              (puu*CRVAL[level+1,fuu,fsigma,suu,ssigma,cumd]+
57               pud*CRVAL[level+1,fud,fsigma,sud,ssigma,cumd]+
58               pdu*CRVAL[level+1,fdu,fsigma,sdu,ssigma,cumd]+
59               pdd*CRVAL[level+1,fdd,fsigma,sdd,ssigma,cumd]);
60       ];  (* end IF Level < n-1 *)
61       Return[result];
62   ];
63   Return[CRVAL[0,f0,fsig0,s0,ssig0,0]];
64]
```

Some further aspects of the program code elicit interest. The code in lines 53-54 implements the recursive equation for forward induction of the cumulative default probabilities. The expression used is given before in equation (6.2) of the paper. Lines 55-59 implement backward recursion where the function CRVAL calls itself. Line 54 contains the expression for the one-period probability of default. In lines 56-59, the program passes on the current value of the cumulative default probability to the function next period, so that it can be carried ahead in time through forward induction. Since the cumulative probability of default has a trajectory that is path-dependent, the use of the recursive method offers efficient implementation.

### 6.3 Numerical Examples

We implement a simple example to demonstrate the model and the two segments of code provided above. The implementation is undertaken using *Mathematica* [38]. The data used is as follows:

| Period | $T$ | $(T-h,T)$ | $f(0,T)$ | $\sigma_f$ | $s(0,T)$ | $\sigma_s$ |
|---|---|---|---|---|---|---|
| 1 | 0.5 | (0,0.5) | 0.06 | 0.015 | 0.010 | 0.005 |
| 2 | 1.0 | (0.5,1) | 0.07 | 0.012 | 0.015 | 0.006 |
| 3 | 1.5 | (1,1.5) | 0.08 | 0.011 | 0.020 | 0.007 |
| 4 | 2.0 | (1.5,2) | 0.09 | 0.010 | 0.022 | 0.008 |

| | |
|---|---|
| $\rho$ | $-0.074$ |
| $h$ | 0.50 |
| $a$ | 5.44 |
| $b$ | $-10.43$ |
| $c$ | $-27.24$ |
| $\pi$ | 0.50 |

These parameters were used on the code above. The model is run over 4 periods, each half-year in length. These periods are not necessarily short for a credit model since changes in credit quality do not occur as rapidly as do other underlying variables in other markets; nonetheless, the length may be chosen to be smaller if so desired. The examples here are intentionally kept simple, so that the interested reader may quickly implement the provided code if so desired, and compare the results to those reported here. The examples are:

- Credit spread option: The option was struck at a strike spread of 0.015 with a maturity of 3 periods. The notional value of the contract is \$100. The price of the option amounts to \$0.75, i.e. a premium of 0.75%.
- Credit default swap: On a notional value of a dollar the default insurance premium computed is \$0.03, i.e. a premium of 3.0%. The forward default probabilities were generated using the logit function (5.2) with the parameters in Table 1 that were obtained empirically.

Since the recursive model results in parsimonious program code, the scheme in the paper allows for boundary conditions and other forms of payoff functions which only require minor program modifications.

### 6.4 Extensions

Two extensions of the codes provided above will further illustrate the use of our approach. The first considers path-dependence, the second early-exercise.

In order to demonstrate how easily the model can handle a path-dependent contract, we priced a call option on the average credit spread over the four periods of the model. This requires carrying an additional state variable for the average along the sample path on the lattice. The function call in the program is modified to be:

```
CRVAL[level_,f_,fsig_,s_,ssig_,cumdef_,sumspr_]:=
   CRVAL[level,f,fsig,s,ssig,cumdef,sumspr]=
   Module[{i,m,j,alpha,beta,fuu,fud,fdu,fdd,suu,sud,sdu,
          sdd,fsigma,ssigma,pd,recov,cumd,sumsp},
```

A new state variable "sumspr" has been added to the function call. This carries the sum of all the spreads from period to period. The next change required is to the payoff function which compares the average spread to the strike spread. The modified lines of code are below:

```
If[level==n-1,
   result=Max[0,(sumspr+s[[1]])/n-exprice]*100;
```

Here, the comparison variable is the average spread. At the last stage in the program, the previous sum of spreads is added to the last observed spread and the total is divided by the number of periods to get the average spread. Finally, a change is needed to accumulate and pass on the sum of spreads from period to period in a recursive way:

```
sumsp=sumspr+s[[1]];
result=Exp[-(f[[1]])*h]*
     (puu*CRVAL[level+1,fuu,fsigma,suu,ssigma,cumd,sumsp]+
      pud*CRVAL[level+1,fud,fsigma,sud,ssigma,cumd,sumsp]+
      pdu*CRVAL[level+1,fdu,fsigma,sdu,ssigma,cumd,sumsp]+
      pdd*CRVAL[level+1,fdd,fsigma,sdd,ssigma,cumd,sumsp]);
```

We retain the original set of parameters and strike price. Running the modified algorithm resulted in a price for the average spread option of \$0.27. In comparison, the simple call option we priced earlier had a price of \$0.75. Since the average option effectively reduces the volatlity of the underlying variable, the option price naturally becomes lower. This follows from the fact that the average of a variable has a lower variance than the variable itself.

Our second extension concerns early-exercise. We demonstrate this with an American style put option on the spread. The program requires a few minor changes. First the payoff function at terminal time changes for a put option:

```
If[level==n-1,
   result=Max[0,exprice-s[[1]]]*100;
```

Next, at each node on the lattice we check for whether it is optimal to exercise the option early. This is undertaken by a simple comparison of the continuation value versus the value from immediate exercise. The code changes to the following:

```
        result=Max[exprice-s[[1]], Exp[-(f[[1]])*h]*
            (puu*CRVAL[level+1,fuu,fsigma,suu,ssigma,cumd]+
             pud*CRVAL[level+1,fud,fsigma,sud,ssigma,cumd]+
             pdu*CRVAL[level+1,fdu,fsigma,sdu,ssigma,cumd]+
             pdd*CRVAL[level+1,fdd,fsigma,sdd,ssigma,cumd]);
```

With these two changes, the program provided an option value of \$0.11.

### 6.5 Comments

The program code may be extended to value several other instruments such as total return swaps, floating rate notes, auction-reset notes, and spread-adjusted notes, which are a small subset of the credit-linked securities that have been created over the past few years. Additionally, the code may be translated into programming languages that compile and run faster, such as C++.

The simple examples described above highlight the advantages of our framework.

- Our approach is consistent with arbitrage-free dynamics. Many lattice implementations require numerical solutions of equations in order to generate the correct tree. Our method provides analytic expressions for the drift terms $[\alpha, \beta]$ so that the lattice can be generated without resorting to numerical root finding procedures. Moreover, this is done in a bivariate setting, and the additional dimension for credit risk is added seamlessly without eliminating any of the desirable features of the Heath-Jarrow-Morton [19] approach.
- The implementation algorithm is simple because it makes use of recursive programming.
- The lattice contains a rich information set, and thus, a wide range of securities can be priced.
- Complex securities, requiring path-dependence, can be easily handled by the algorithm.
- Since the arbitrage-free dynamics for default risk are driven by observable spread curves, the user is allowed the freedom to choose any model for the process of default. Several such approaches exist, and our approach provides an encompassing framework within which to embed the preferred default model. For example, we have already described the logit method, discussed in Wilson [37]. Another approach, through the use of Markov rating transition matrices is also useable, and has been exploited in the papers of Jarrow, et al [21] and Das and Tufano [8]. Also, the hazard-rate models of Madan and Unal [30],[31] or the factor models of Duffie and Singleton [16] may be used. If the hazard rate is chosen to be path-dependent, it can still be handled in the model.
- American style derivatives may also be priced.

Of course, the model is computationally expensive, but this is necessary to capture path-dependence. As the number of time periods in the model grows, the computational time grows exponentially. However, since we are dealing with credit risk, and credit quality does not change immediately from instant to instant, working with time periods of a quarter is probably sufficient. Credit derivatives are usually priced over maturities out to two years, and this poses no problems for the model. Despite the computational intensity, the framework provided, including the pseudo-code, demonstrates that the model has the advantage of algorithmic simplicity, which makes it easy to adapt to new derivative products, as and when the need arises.

## 7 Concluding Comments

This paper develops a model for the pricing of credit derivatives using observables. The model is (i) arbitrage-free, (ii) accommodates path-dependence, and (iii) handles a range of securities, even with American features. The computer implementation uses a recursive scheme that is convenient and seamlessly processes forward induction and backward recursion, needed to compute more complicated derivative securities.

The model may be enhanced in many ways. It is possible to improve the recursive implementation using improved computer science methods. That has not been the focus of this paper. Here, we attempt to simply develop the arbitrage-free framework underlying the computer implementation. Once the bivariate lattice in default-free rates and spreads has been developed, the spread itself may be decomposed into default probabilities and recovery rates in many ways other than the one suggested in the paper. For example, information in rating transition matrices may be employed instead of the logit regression. This paper leaves these avenues of research for later work.

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