

A Multiple Indicators Model For Volatility Using Intra-Daily Data

Robert F. Engle* and Giampiero M. Gallo**

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Abstract

Many ways exist to measure and model financial asset volatility. In principle, as the frequency of the data increases, the quality of forecasts should improve. Yet, there is no consensus about a “true” or “best” measure of volatility. In this paper we propose to jointly consider absolute daily returns, daily high-low range and daily realized volatility to develop a forecasting model based on their conditional dynamics. As all are non-negative series, we develop a multiplicative error model that is consistent and asymptotically normal under a wide range of specifications for the error density function. The estimation results show significant interactions between the indicators. We also show that one-month-ahead forecasts match well (both in and out of sample) the market-based volatility measure provided by an average of implied volatilities of index options as measured by VIX.

Keywords: volatility modeling, volatility forecasting, GARCH, VIX, high-low range, realized volatility.

JEL Codes: C22, C32, C53.

* Department of Finance, Stern School of Business, New York, rengle@stern.nyu.edu .

** Dipartimento di Statistica “G.Parenti”. Università di Firenze, gallog@ds.unifi.it .

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1. Introduction

Models to describe and predict financial asset volatility abound. In practice, in addition to a model's capability to reproduce stylized facts in observed time series and exhibit desirable statistical properties, the ultimate way to evaluate a model is its usefulness as a tool in many areas such as derivative products pricing, risk evaluation and hedging, portfolio allocation, and the derivation of Value at Risk measures.

Yet, the concept of volatility itself is somewhat elusive, as many ways exist to measure it and hence to model it (cf. the survey by Andersen *et al.*, 2002). In recent times, the availability of ultra-high frequency data and the work done on them has shed new light on the concept of volatility: as a matter of fact, data sampled at regular intra-daily intervals can be summarized into a measure called realized volatility which under some assumptions is a consistent estimator of the quadratic variation of the underlying diffusion process. Such a measure was widely adopted as a target of forecast accuracy, but the dependence of the measure upon the frequency of observation of the data makes it difficult to come to clear conclusions. In principle, the volatility measures derived from ultra-high frequency data should prove to be more accurate, hence allowing for forecast efficiency gains. Nevertheless, the measures that make more intensive use of such data are prone to all sorts of microstructure problems. Bid-ask bounce (Roll, 1984), screen fighting (Zhou, 1996), price discreteness, irregular spacing of quotes and transactions can all bias volatility estimates.

Even if the growing literature on realized volatility has delivered promising results (Andersen *et al.*, 2000; Barndorff-Nielsen and Shephard, 2002), what is of interest is the appropriate way to provide accurate forecasts in the medium-to-long run, and the problem remains open as to whether daily or intra-daily models deliver the most successful answer. In the first category one can mention those models favoring the existence of long memory or high persistence in the process of volatility, such as the daily component model by Engle and Lee (1999), or the Fractionally Integrated GARCH (FIGARCH – Baillie *et al.*, 1996; or FIEGARCH, Bollerslev and Mikkelsen, 1996) or Long Memory Stochastic Volatility Models (LMSV – Breidt *et al.*, 1998; Deo and Hurvich, 1999). Intra-daily models are more recent: Ghose and Kroner (1996) adopt a signal plus noise model to estimate a persistent component as in Engle and Lee (1999); Andersen and Bollerslev (1997) show how accuracy of volatility forecasts can be improved if

one moves to analyzing five-minute returns; Engle (2000) derives a measure of volatility from transaction data.

The approach which we will pursue here is one in which several measures of volatility can be jointly used to see whether different features of observed time series can deliver an enrichment of volatility forecasting for the medium run. As a matter of fact, next to the traditional volatility modelling from daily returns measured as the log-difference of closing prices, we can consider absolute returns on which considerable modeling effort is present in the literature (Taylor, 1986; Ding, Granger and Engle, 1993; Granger and Sin, 2000) and, with the already mentioned provisos, realized volatility as the standard deviation of intra-daily returns observed at regular intervals. Furthermore, it has long been recognized that the spread between the highest recorded daily price and the lowest recorded daily price is a function of the volatility during the day and, as such, can lead to an improvement of the volatility estimates. Many authors (Taylor, 1987, Gallant *et al.*, 1999, Chou, 2001, Alizadeh *et al.*, 2002, Brandt and Diebold, 2002) have devoted considerable attention to the informational content of range data extending the relationship it has to the volatility parameter in a geometric Brownian motion context (cf. the early papers by Parkinson, 1980, Garman and Klass, 1980, and Beckers, 1983), and comparing its persistence characteristics with the ones of daily returns (Brunetti and Lildholdt, 2003).

It should be stressed that the three variables have different features relative to one another: the main difference is that the daily return uses information about the closing price of the previous trading day, while the high-low spread and the realized volatility are measured on the basis of what is observed during the day, the former taking all trade information into account, the latter being built on the basis of quotes sampled at discrete intervals. Thus, a zero return is not necessarily informative about what happened during the day, and, by the same token, a high return may signal high volatility during the day while it may just be due to an opening price much different from the closing price the previous day but very close to the closing price of the same trading day, with a small high-low spread. Also, note that the same value of realized volatility may correspond to different values of the range since we could both have recorded both high and low values fairly far apart during the day with a smooth transition of price movements between the two, or some vivacious price swings concentrated in a short period of time. For these reasons, they are all potentially useful.

In the present paper we exploit the fact that absolute returns, daily range and realized volatility are variables evolving as functions of the underlying time-varying volatility and all exhibit the usual conditional persistence of financial time series. They can each be considered as indicators of volatility and since they are all non-negative-valued they can be modelled with a multivariate extension of the Multiplicative Error Model suggested by Engle (2002). Each indicator is modeled as a GARCH-type process possibly augmented with weakly exogenous variables. This estimator is robust to a range of error distribution assumptions. We show that this three-variable model possesses interesting properties in that the forecasts for each indicator are augmented by the presence of the other indicators lagged and by asymmetric effects from the direction of price movements. The model can be solved dynamically for multi-period forecasts and the dynamic interdependence accounts for a substantial departure from the standard GARCH profile of dynamic forecasts. We can use our three equation model to predict multi-step volatility.

The model is estimated with daily index data over a relatively long sample period. A careful specification search selects models for each equation. We calculate 22-step volatility forecasts and compare these with one month option-implied volatilities as measured by the VIX. We examine the forecast performance both in and out of sample and estimate a combination of forecast equation to see whether the forecasts derived in our framework have a significant impact in predicting the VIX. The approach is quite different from that of Blair *et al.* (2001) who also analyze volatility forecasting and VIX but inserting it as a weakly exogenous variable in the conditional variance equation for the S&P100.

The reader should expect the following: in Section 2 we discuss the Multiplicative Error Model, with some general considerations about the estimator and its properties, and about the specifications of the three models adopted. Data issues and model selection procedures are described in detail in Section 3. Model performance is analyzed in Section 4 focusing on the characteristics of the model in multi-step-ahead forecasting. In order to show the usefulness of our forecasts when compared to a measure based on implied volatilities representing market evaluation of volatility, in Section 5 we build multi-step-ahead volatility forecasts and we use them as regressors in a combination of forecast regression where the (log of the) volatility index VIX is the forecast target. Concluding remarks follow.

2. The Model

We assume that the evolution of a non-negative valued process x_t , can be described by a Multiplicative Error Model (MEM, as discussed by Engle, 2002). That is, x_t is the product of a time varying scale factor (which depends upon the recent past of the series) and a standard positive valued random variable. Such a specification was adopted in Engle and Russell (1998) for durations, Manganelli (2000) for volume transaction data and Chou (2002) for high-low range. The scale factor is identified as the conditional mean if the error distribution is assumed to have unit mean. In general, therefore,

$$x_t = \mu_t \varepsilon_t, \quad \varepsilon_t | \mathfrak{F}_{t-1} \sim i.i.d. D(1, \varphi) \quad (1)$$

where μ_t can be rather flexibly specified as

$$\mu_t = \omega + \sum_{i=1}^p \alpha_i x_{t-i} + \sum_{j=1}^q \beta_j \mu_{t-j} + \gamma' z_t \quad (2).$$

Further terms can signal the dependence of the series on weakly exogenous variables (summarized in the vector z_t) included in the information set available at time $t-1$. The conditions to ensure stationarity and positive means for all possible realizations have been discussed in Engle (2002).

The density of the error term ε_t was left unspecified thus far. While in general one should specify the true DGP, it may be possible to find robust specifications. We now consider the family of gamma densities that have been used for ACD models.

$$f(\varepsilon_t | \mathfrak{F}_{t-1}) = \frac{1}{\Gamma(\alpha) \beta^\alpha} \varepsilon_t^{\alpha-1} \exp\left(-\frac{\varepsilon_t}{\beta}\right).$$

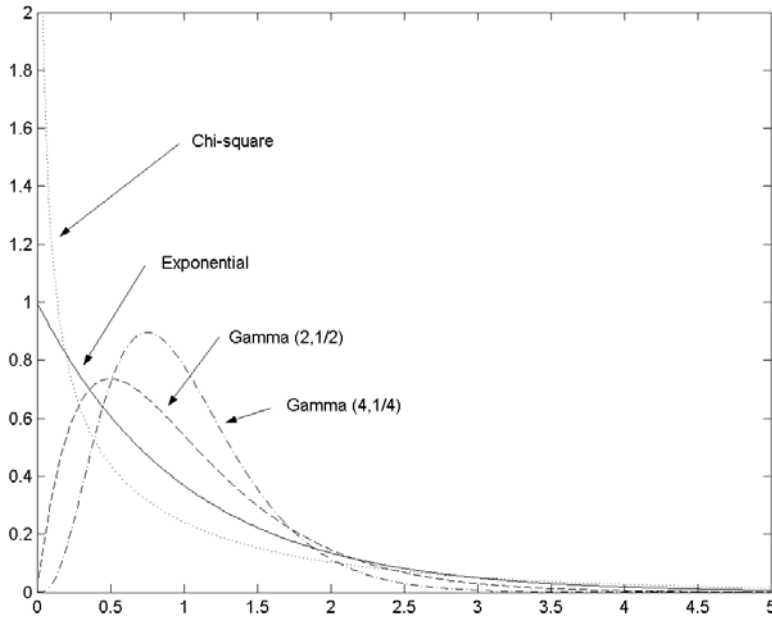
The property of unit mean¹ implies that $\beta = \frac{1}{\alpha}$, namely,

$$f(\varepsilon_t | \mathfrak{F}_{t-1}) = \frac{1}{\Gamma(\alpha)} \alpha^\alpha \varepsilon_t^{\alpha-1} \exp(-\alpha \varepsilon_t) \Rightarrow f(x_t | \mathfrak{F}_{t-1}) = \frac{1}{\Gamma(\alpha)} \alpha^\alpha x_t^{\alpha-1} \mu_t^{-\alpha} \exp\left(-\alpha \frac{x_t}{\mu_t}\right) \quad (3)$$

¹ Such a condition is not restrictive at all: consider that if ε_t were to be such that $E(\varepsilon_t) = \alpha\beta \neq 1$, it could be written as $\beta^* \varepsilon_t^*$ with $E(\varepsilon_t^*) = 1$ and β^* would be a multiplicative constant to be absorbed by μ_t .

The process would have conditional expectation $E(x_t | \mathfrak{F}_{t-1}) = \mu_t$ and conditional variance $Var(x_t | \mathfrak{F}_{t-1}) = \frac{\mu_t^2}{\alpha}$. A comparison between various densities corresponding to different choices of α subject to the constraint is reported in Figure 1: as well known, the choice of $\alpha = 1/2$, gives the pdf of a chi-square with one degree of freedom, while $\alpha = 1$ yields the unit exponential. In general, values of $\alpha < 1$ amount to attributing more weight to extremely small or large values of the random variable while values of $\alpha > 1$ generate a hump shaped density which approaches the normal distribution for large values of α .

Figure 1. Comparison among different gamma densities subject to the constraint $\alpha\beta = 1$



Let us suppose for the moment that the only parameters of interest are the ones defining μ_t (let us call them θ), and let us look at the corresponding log-likelihood function: the relevant object for estimation is given by

$$\log L = \text{constant} - \alpha \sum_{t=1}^T \left(\log \mu_t + \frac{x_t}{\mu_t} \right) \quad (4)$$

where the constant depends only on the x_t 's and α . It is clear that in maximizing this function with respect to θ , the value assumed by α is irrelevant, as the first order conditions must obey

$$\sum_{t=1}^T \left(\frac{x_t - \mu_t}{\mu_t^2} \right) \frac{\partial \mu_t}{\partial \theta} = 0. \quad (5)$$

Even from a numerical point of view, an iterative procedure is unaffected (in a Newton-type method the α terms present in the inverse Hessian and gradient would cancel each other) resulting in exactly the same estimates (apart from rounding off errors when numerical derivatives are used). Furthermore, these first order conditions coincide with the ones derived from an auxiliary model that specifies the square root of the variable of interest, $\sqrt{x_t}$, as the product of the square root of the scale factor $\sqrt{\mu_t}$ and a half Gaussian error term v_t , namely,

$$\sqrt{x_t} = \sqrt{\mu_t} v_t \quad v_t | \mathfrak{F}_{t-1} \sim \text{half Gaussian (standard)}. \quad (6)$$

This model has the clear advantage of being able to exploit any GARCH software which can estimate the parameters θ in μ_t from a model for $\sqrt{x_t}$ with no equation for the mean (as done for the ACD model by Engle and Russell, 1998; cf. also Engle, 2002).

Clearly, the second order conditions would differ, since the Hessian will be proportional to α (e.g. for the exponential case it would be twice the Hessian for the chi-square), and, correspondingly, the estimated parameter variance-covariance matrices. Note, however, that (exploiting the results of Bollerslev and Wooldridge, 1992 and Lee and Hansen, 1994) the robust variance covariance matrix, computed as the product

$$\widehat{Var}(\widehat{\mathfrak{S}}) = \widehat{H}^{-1} \widehat{OPG} \widehat{H}^{-1}$$

(with H the Hessian matrix and OPG the matrix of the outer products of the gradients) provides the obvious benefit of making this discrepancy irrelevant since the α 's cancel out:

$$\widehat{Var}(\widehat{\mathfrak{S}}) = \frac{1}{\alpha} \widehat{H}_{\alpha=1}^{-1} \alpha^2 \widehat{OPG}_{\alpha=1} \frac{1}{\alpha} \widehat{H}_{\alpha=1}^{-1} = \widehat{H}_{\text{exp}}^{-1} \widehat{OPG}_{\text{exp}} \widehat{H}_{\text{exp}}^{-1} \quad (7)$$

The strong lesson we learn from this discussion is that in the absence of justifications on the most appropriate distribution to adopt for the error term in an MEM, the avenue of deriving its parameter values through the estimation of an auxiliary variance equation for the positive square

root of the variable of interest² with a GARCH specification and normally distributed errors is straightforward and provides Quasi Maximum Likelihood estimators, hence consistent and asymptotically normal as discussed by Engle (2002), building on results by Lee and Hansen (1994). However, if we know that a Gamma distribution assumption for ε_t is appropriate, then the same procedure delivers consistency and efficiency for the estimators if α is known. Even if α is not known, using robust standard errors shields against the specific shape of the Gamma distribution. Introducing α among the parameters to be estimated would provide information on the shape of the distribution of the error term ε_t (some empirical results are discussed by Engle, 2002) which would be useful for simulating future values or scenario analysis but would not have any impact on the values of the estimates of θ , nor on their standard errors as $Cov(\hat{\theta}, \hat{\alpha}) = 0$.

More specifically for the case at hand here, let us indicate the daily closing price as C_t , and calculate the daily returns as $r_t = \log(C_t / C_{t-1})$. Let us thus consider squared returns r_t^2 , modeled as an MEM

$$r_t^2 = h_t^r \varepsilon_t, \quad \varepsilon_t | \mathfrak{F}_{t-1} \sim i.i.d. D(1, \varphi^r); \quad (8)$$

where h_t^r is the conditional mean of r_t^2 . The square, hl_t^2 , of the high-low range is defined as $hl_t = \log(H_t / L_t)$, where H_t and L_t are the highest, respectively, the lowest recorded prices during the day, and is modeled as:

$$hl_t^2 = h_t^h \eta_t, \quad \eta_t | \mathfrak{F}_{t-1} \sim i.i.d. D(1, \varphi^h); \quad (9)$$

and the square of the realized volatility, dv_t , defined as the standard deviation of observed returns over J subperiods within the day, between market opening time and market closing time³,

$$v_t^2 = h_t^v \zeta_t, \quad \zeta_t | \mathfrak{F}_{t-1} \sim i.i.d. D(1, \varphi^v) \quad (10)$$

² As established by Ding, Granger and Engle (1993), absolute returns exhibit high levels of serial correlation and can themselves be adopted as the first indicator of volatility. We prefer to think of the square root of r_t^2 as absolute returns even if the choice between writing (6) for absolute returns or for returns (with Gaussian innovations) is irrelevant from a numerical point of view, since the first-order conditions would not change.

³ The merits of the choice of the appropriate J to balance estimation accuracy and microstructure pitfalls are discussed in Andersen *et al.* (2000): empirically, 5 minutes intervals “work” in a foreign exchange framework (Andersen *et al.*, 2001), 15 minutes intervals are adopted by Schwert (1998), but we still lack a theory of what is the optimal length to choose. In this context we choose $J=78$ and we adopted the method discussed by Zhou (1996) on raw data to purge bid-ask bounce effects.

Put together, we can write this trivariate MEM as a system of equations

$$\begin{pmatrix} r_t^2 \\ hl_t^2 \\ v_t^2 \end{pmatrix} = \begin{pmatrix} h_t^r \\ h_t^h \\ h_t^v \end{pmatrix} \odot \begin{pmatrix} \varepsilon_t \\ \eta_t \\ \zeta_t \end{pmatrix} \quad \begin{pmatrix} \varepsilon_t \\ \eta_t \\ \zeta_t \end{pmatrix} \Big| \mathfrak{I}_{t-1} \sim i.i.d.D(\mathbf{1}, \Phi) \quad (11)$$

where \odot indicates the Hadamard product, $\mathbf{1}$ is a unit (3x1) vector, and Φ is a 3x3 matrix representing, respectively, the mean vector and the variance-covariance matrix of the innovation terms. The multivariate estimation problem becomes a series of univariate problems when it is assumed that the covariance matrix is diagonal. There is no loss of consistency if this assumption is false but there can be a loss of efficiency.

We can now turn to the details of the MEM specification. Let us consider the first MEM and write the basic form of the model for the scale factor h_t^r as

$$h_t^r = \omega + \alpha r_{t-1}^2 + \beta h_{t-1}^r \quad (12)$$

In order to take asymmetric reactions to shocks, this base model can be expanded, either by writing a threshold-type model involving the customary dummy variable for negative returns $d_t = I(r_t < 0)$,

$$h_t^r = \omega + \alpha r_{t-1}^2 + \beta h_{t-1}^r + \gamma r_{t-1}^2 d_{t-1} \quad (13)$$

or even adding lagged returns r_{t-1} as a different way of accounting for asymmetry (as in the APARCH model, Ding, Granger, and Engle, 1992) given that they maintain their sign,

$$h_t^r = \omega + \alpha r_{t-1}^2 + \beta h_{t-1}^r + \gamma r_{t-1}^2 d_{t-1} + \delta r_{t-1} \quad (14)$$

The main question we want to address at this stage is whether the inclusion of the lagged variables hl_{t-1}^2 and v_{t-1}^2 , which are part of the information set \mathfrak{I}_{t-1} , adds significant explanatory power to the specification for h_t^r . If so, then some of the information contained in these indicators relates to a variability in returns which cannot be accounted for by considering just squared returns and possible asymmetric effects. By the same token, a way to account for further asymmetric effects when returns in the previous day are negative is to include also the product of the indicators by the same dummy variable d_{t-1} . A model which would take all these effects into account can be written as

$$h_t^r = (\omega_r + \alpha_r r_{t-1}^2 + \beta_r h_{t-1}^r) + \gamma_r r_{t-1}^2 d_{t-1} + \delta_r r_{t-1} + \phi_r hl_{t-1}^2 + \vartheta_r hl_{t-1}^2 d_{t-1} + \psi_r v_{t-1}^2 + \lambda_r v_{t-1}^2 d_{t-1}, \quad (15)$$

where a subscript “ r ” was added to the coefficients indicating that they refer to the specification for returns. The augmentation of the GARCH(1,1) model in this case includes the six variables r_{t-1} , $r_{t-1}^2 d_{t-1}$, hl_{t-1}^2 , $hl_{t-1}^2 d_{t-1}$, v_{t-1}^2 , $v_{t-1}^2 d_{t-1}$ which are all known at time $t-1$.

A similar approach can be followed for the other two indicators. Considering again $hl_t^2 = h_t^h \eta_t$, and picking six relevant variables from \mathfrak{F}_{t-1} , namely, r_{t-1} , $hl_{t-1}^2 d_{t-1}$, r_{t-1}^2 , $r_{t-1}^2 d_{t-1}$, v_{t-1}^2 , $v_{t-1}^2 d_{t-1}$, one can write a full specification for h_t^h as

$$h_t^h = (\omega_h + \alpha_h hl_{t-1}^2 + \beta_h h_{t-1}^h) + \delta_h r_{t-1} + \gamma_h hl_{t-1}^2 d_{t-1} + \phi_h r_{t-1}^2 + \theta_h r_{t-1}^2 d_{t-1} + \psi_h v_{t-1}^2 + \lambda_h v_{t-1}^2 d_{t-1}. \quad (16)$$

Analogously, for the realized daily volatility indicator we start from $v_t^v = h_t^v \zeta_t$ and add the corresponding six variables, $v_{t-1}^2 d_{t-1}$, r_{t-1} , r_{t-1}^2 , $r_{t-1}^2 d_{t-1}$, hl_{t-1}^2 , $hl_{t-1}^2 d_{t-1}$, to the base specification of a GARCH(1,1) to get

$$h_t^v = (\omega_v + \alpha_v v_{t-1}^2 + \beta_v h_{t-1}^v) + \delta_v r_{t-1} + \gamma_v v_{t-1}^2 d_{t-1} + \phi_d r_{t-1}^2 + \theta_v r_{t-1}^2 d_{t-1} + \psi_v hl_{t-1}^2 + \lambda_d hl_{t-1}^2 d_{t-1}. \quad (17)$$

The question that we will discuss in the following section is based on the characteristics of the data that one analyzes, namely, whether adding the lagged indicators adds substantial explanatory power to the basic GARCH specification. In fact, it may happen that the MEM for one indicator does not require the information set to be augmented, while there may be significant effects for another.

3. The Data, Parameter Estimation and Model Selection

For our empirical models, we will use data on the Standard and Poor 500 index from January 4, 1988 to July 14, 1999 (2730 observations); we leave the last 217 observations for out-of-sample forecast comparison purposes. We build series for absolute returns, high-low range and realized volatility for this period: since the constructed variables have quite different scales relative to one another, we rescale all three variables so that they are all expressed in annual percentage terms and they share the in-sample quadratic mean of observed returns. Some descriptive statistics are reported in Table 1 and suggest that the absolute returns and the daily range have more characteristics in common with one other than each of them does with daily volatility. However,

a more complete picture is obtained if one looks also at the scatter-plots in Figure 2 and the correlation coefficients between the three series reported in the last lines of Table 1.

Figure 2. The Standard and Poor 500 Index. Scatter-plots between absolute returns, daily range and realized volatility. Sample Period: Jan.4, 1988 to Dec. 31, 1997.

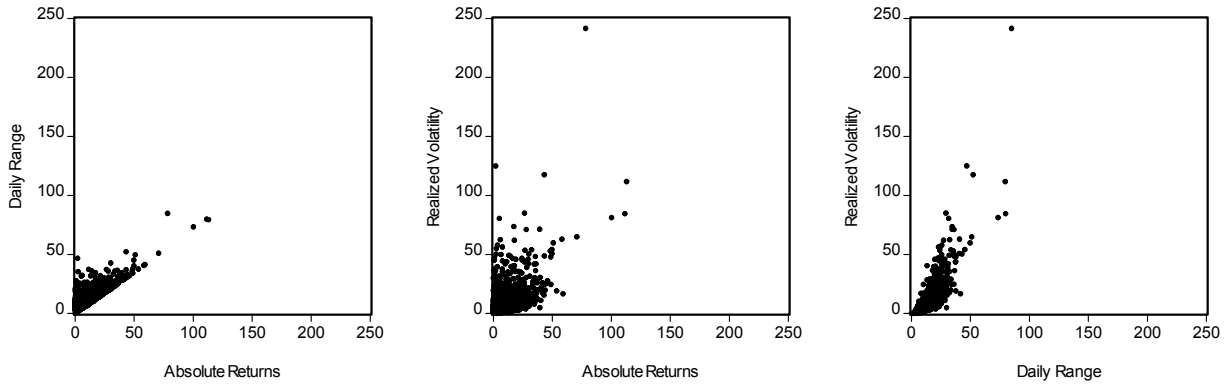


Table 1. S&P500 Absolute returns, daily range and realized volatility. Summary statistics.

	Absolute Returns	Daily Range	Realized Volatility
Mean	9.35	11.27	7.74
Median	6.72	9.67	4.89
Maximum	112.92	84.83	241.53
Minimum	0.000	0.098	0.004
Std. Dev.	9.29	6.83	10.67
Quadratic Mean	13.18	13.18	13.18
Skewness	2.93	2.90	7.60
Kurtosis	21.97	21.48	115.80
Correlations			
with absolute returns		0.81	0.51
with daily range			0.80

All variables are expressed in terms of percentage annual terms and they share the same quadratic mean. Sample period: Jan. 4, 1988 – Dec.30, 1997.

The absolute returns lie almost invariably below the daily range, signaling that the days in which the previous day closing price is lower (higher) than the lowest (highest) intradaily price are infrequent. A more dispersed pattern is shown between the absolute returns and the realized volatility as reflected also by the correlation coefficient of 0.51. The fact that the correlations are

relatively high but not nearly unity signifies that the indicators are different from one another. These preliminary stylized facts coupled with the characteristics of persistence exhibited by the absolute return, the daily range and the realized volatility series (cf. Figure 3), mean that an effort aimed at modeling the interactions between these three variables in a conditional context seems promising.

FIGURE 3 (at the end of the paper) about here

For each of these MEMs, we envisage the introduction of any of up to six weakly exogenous variables in addition to a GARCH(1,1)-type model which is kept as the base specification in order to avoid coefficient identification problems. For each indicator, therefore, we are estimating $2^6=64$ models ranging from the most general forms (15), (16) and (17) down to the base specification kept as a benchmark.

The two model selection strategies that we will adopt and compare are:

1. a general-to-specific strategy whereby we start pruning the coefficients that appear to be statistically insignificant (using Bollerslev and Wooldridge robust standard errors) in the most general expressions and go on to search down to the level where all coefficients are significant and
2. the smallest value of the Schwartz Information Criteria (BIC) among the 64 models.

The chosen models from the general-to-specific selection procedure are reported in Table 2.

Table 2. S&P500 - General-to-specific Model Selection.

Equations for the time-varying component in the equations for the squares of absolute returns, high-low range, and daily realized volatility. Sample period Jan 4, 1988 – Dec. 30, 1997 (Robust t-statistics in small font under the parameter value).

$$\begin{aligned}
 h^r_t &= 3.389 - 0.031 r_{t-1}^2 + 0.901 h^r_{t-1} - 0.747 r_{t-1} + 0.11 hl_{t-1}^2 - 0.010 v_{t-1}^2 \\
 &\quad \begin{matrix} 2.123 & -1.121 & 45.537 & -3.552 & 2.535 & 2.700 \end{matrix} \\
 h^h_t &= 7.622 + 0.109 hl_{t-1}^2 + 0.850 h^h_{t-1} - 0.878 r_{t-1} \\
 &\quad \begin{matrix} 4.885 & 5.407 & 32.713 & 3.608 \end{matrix} \\
 h^v_t &= 1.654 - 0.021 v_{t-1}^2 + 0.728 h^v_{t-1} - 1.596 r_{t-1} + 0.211 v_{t-1} d_{t-1} + 0.094 r_{t-1}^2 + \\
 &\quad 0.098 hl_{t-1}^2 + 0.114 hl_{t-1}^2 d_{t-1}. \\
 &\quad \begin{matrix} 4.005 & 1.292 & 80.909 & 14.932 & 7.278 & 9.801 & 4.286 & 4.127 \end{matrix}
 \end{aligned}$$

The results show that, when evaluated in terms of coefficient significance, the inclusion of other variables in the information set appears to add explanatory power in each of the expressions. The model for absolute returns includes an asymmetric effect captured by r_{t-1} (the term $r_{t-1}^2 d_{t-1}$ seems to be less important), and both the squared daily range and the squared daily volatility. The model for the high-low range seems to be the most parsimonious with the presence of just an asymmetric response of h_t^h to lagged values of the returns. Interestingly, the one model which attracts the highest number of significant variables is the model for the daily realized volatility in which there appear to be asymmetric effects from all variables, as well as lagged squared returns and lagged squared daily range. Some diagnostics (reported in the top panel of Table 3) show that there are no major specification problems: for reference we give the values of the BIC and the estimated log-likelihood, as well as the results of an ARCH(2) test (5% critical value = 5.99) and the Ljung-Box test Q(12) for the squared residuals (5% critical value = 21.03).

Table 3: S&P500 - Diagnostics on Selected Models.
Sample period Jan 4, 1988 – Dec. 30, 1997.

	BIC	ARCH(2)	Q(12)	LOGLIK
General-to-Specific Model Selection				
<i>Absolute returns</i>	7.8693	1.770	4.016	-9868.28
<i>Daily Range</i>	7.8622	2.25	5.86	-9867.127
<i>Realized Volatility</i>	7.2581	0.425	15.248	-9092.116
Smallest BIC Model Selection				
<i>Absolute returns</i>	7.8682	1.272	4.174	-9870.81
<i>Daily Range</i>	7.8622	2.25	5.86	-9867.127
<i>Realized Volatility</i>	7.2561	0.470	12.136	-9097.375

The specification search guided by the lowest value of the BIC gives the results we present in Table 4. The model selected for the daily range is the same as before. For absolute returns, the model selected here does not contain lagged square daily volatility; the values of the coefficients do not change much, so the profile of forecasts between the two models will differ mainly because of the impact of this variable. The model for the daily volatility here

does not include the terms involving the daily range, and the values of the coefficients are fairly different. Again (bottom panel of Table 3) no major problems are signaled by the residual diagnostics.

Table 4. S&P500 - Smallest BIC Model Selection.
Equations for the square of the time-varying component in absolute returns, high-low range, and daily realized volatility. Sample period Jan 4, 1988 – Dec. 30, 1997. (Robust t-statistics in small font under the parameter value)

$$\begin{aligned}
 h^r_t &= \underset{2.805}{5.026} - \underset{1.068}{0.030} r_{t-1}^2 + \underset{43.432}{0.901} h^r_{t-1} - \underset{3.293}{0.745} r_{t-1} + \underset{2.328}{0.101} hl_{t-1}^2 \\
 h^h_t &= \underset{4.885}{7.622} + \underset{5.407}{0.109} hl_{t-1}^2 + \underset{32.713}{0.850} h^h_{t-1} - \underset{3.608}{0.878} r_{t-1} \\
 h^v_t &= \underset{8.061}{2.123} + \underset{2.366}{0.035} v_{t-1}^2 + \underset{91.479}{0.736} h^v_{t-1} - \underset{28.350}{1.183} r_{t-1} + \underset{6.688}{0.122} v_{t-1} d_{t-1} + \underset{23.911}{0.123} r_{t-1}^2
 \end{aligned}$$

4. Model Performance

The three models thus estimated could be used separately for one-step-ahead predictions using the estimated coefficients and the actual value of the right-hand side variables: the performance of these single-equation models could be individually evaluated in relationship to the performance of the corresponding GARCH models. The interest of what is being done here, though, lies in the fact that the three equations together can be seen as a system which can be used as a tool for multi-step forecasting for medium horizons. Let us consider the left-hand side as a three dimensional vector h_t , and consider that at time T+1 we have

$$h_{T+1|T} = \begin{pmatrix} h^r_{T+1|T} \\ h^h_{T+1|T} \\ h^v_{T+1|T} \end{pmatrix} = \begin{pmatrix} \omega_r \\ \omega_h \\ \omega_v \end{pmatrix} + \mathbf{A}^* \left(r_T, r_T^2, hl_T^2, hl_T^2 d_T, v_T^2, v_T^2 d_T, h^r_T, h^h_T, h^v_T \right)' \quad (18)$$

where \mathbf{A}^* is a 3 by 9 matrix which includes the coefficients on the variables the value of which is known at time T. To forecast the various future second-order moments conditional on information at time T for maturities greater than 1, we need to substitute the right-hand side variables with their conditional expectation as of time T. For a generic horizon k, we will have

$$\begin{aligned}
E_T(r_{T+k-1}) &= 0, \\
E_T(r_{T+k-1}^2) &= h^r_{T+k-1|T}, \\
E_T(hl_{T+k-1}^2) &= h^h_{T+k-1|T} \\
E_T(hl_{T+k-1}^2 d_{T+k-1}) &= \frac{1}{2} h^h_{T+k-1|T} \\
E_T(v_{T+k-1}^2) &= h^v_{T+k-1|T} \\
E_T(v_{T+k-1}^2 d_{T+k-1}) &= \frac{1}{2} h^v_{T+k-1|T}
\end{aligned}$$

and, therefore the expression (18) is substituted by

$$h_{T+k|T} = \begin{pmatrix} h^r_{T+k|T} \\ h^h_{T+k|T} \\ h^v_{T+k|T} \end{pmatrix} = \begin{pmatrix} \omega r \\ \omega h \\ \omega v \end{pmatrix} + \mathbf{A} \begin{pmatrix} h^r_{T+k-1|T} \\ h^h_{T+k-1|T} \\ h^v_{T+k-1|T} \end{pmatrix} = \omega + \mathbf{A} h_{T+k-1|T} \quad (19)$$

The dynamic properties of the estimated system can therefore be evaluated by examining the characteristic roots of the matrix \mathbf{A} . In principle, there could be stable complex conjugate roots that would give the system some dampened cyclicity in the forecasting. For the case at hand the roots for the two sets of estimates are given as in Table 5.

Table 5. Characteristic Roots of the Matrix A

Model selection			
General-to-specific	0.958	0.860	0.833
Smallest BIC	0.958	0.870	0.832

The roots are fairly similar across the two selection procedures: one can therefore expect that the two multi-equation models will provide different forecasting profiles for the short horizon, while they will tend to be very similar in the medium to long run. For this reason we will report the graphs just for the models selected according to the smallest BIC criterion.

To evaluate the performance of the models let us first consider the profile of the out-of-sample forecasts generated by the system starting on Jan. 2, 1998, the first day after the estimation sample, and varying the starting date. We take the observed values as starting values for day T (corresponding to Dec. 30, 1997 and forecasting from Jan 2, 1998 to July 13, 1998), $T+4$ (corresponding to Jan 8, 1998), and so on until the last starting value which corresponds to Mar. 6, 1998 for a forecasting horizon going from Mar.9, 1998 to Sep. 14, 1998. In Figure 4 we

superimpose various profiles of the forecasts for different starting periods and for a horizon of 132 periods ahead. The forecasts for the daily range indicator exhibit the typical monotonic profile of a GARCH(1,1) model. This is not the case for the curves corresponding to forecasts obtained from the other two indicators: it is interesting to note that the forecasts produced by our model may have a non-monotonic behavior with over- (or under-) shooting of their long term (unconditional) values at intermediate horizons.

FIGURE 4 (at the end of the paper) about here

Let us now consider the term structure of volatility for an asset (or an index) at a given maturity $T+k$ as the square root of the cumulated sum of j -step-ahead forecasts generated by expression (18), when $j=1$, or (19), when j is between 1 and k (cf. Engle and Patton, 2001). This corresponds to evaluating the square root of the cumulated sum of the expected values of each volatility indicator at any time between 1 and k . As k increases the terms of the sum will tend to repeat themselves. We thus have

$$\begin{pmatrix} v_{T+k|T}^a \\ v_{T+k|T}^h \\ v_{T+k|T}^v \end{pmatrix} = \begin{pmatrix} \sqrt{\sum_{j=1}^k h_{T+j|T}^r} \\ \sqrt{\sum_{j=1}^k h_{T+j|T}^h} \\ \sqrt{\sum_{j=1}^k h_{T+j|T}^v} \end{pmatrix} = \begin{pmatrix} \sqrt{\sum_{j=1}^k E_T r_{T+j}^2} \\ \sqrt{\sum_{j=1}^k E_T h_{T+j}^2} \\ \sqrt{\sum_{j=1}^k E_T v_{T+j}^2} \end{pmatrix} \quad (20)$$

For reasons that will be clearer in the next section when we discuss the comparison of these forecasts with the VIX volatility index, we choose a horizon k equal to 22, that is, a one-month ahead forecasts. In Figure 5 we report the values of the cumulative volatility forecast as the (square root of the) average of 1-step, 2-steps, ..., 22-steps ahead out-of-sample forecasts obtained by the general-to-specific three-equation system (18) and (19) relative to the standard GARCH(1,1) specification. As one would expect, for the h^h , equation the differences between the values obtained with our model and with the standard base specification are quite minimal and can be ascribed to the presence of lagged returns in the former specification. For the other two indicators the results show that the estimates obtained with our model are generally higher and more persistent than the estimates obtained with the base specification when the absolute

return indicator is used whereas the reverse is true for the daily realized volatility equation. A remarkable difference is observed for estimates on or about August 31, 1998 for which the base specification provides much higher estimates. Whether this signals excessive volatility forecasts by the latter set of models is an issue that requires a type of more specific evaluation. What we need is an overall comparison of the two sets of forecasts gauged in reference to a market-based volatility measure such as the volatility index VIX. We turn to this in the next section.

FIGURE 5 (at the end of the paper) about here

5. A Volatility Index as a Forecast Target

The Chicago Board of Options Exchange (CBOE) is the world's largest options exchange where standardized stock and index options are traded.⁴ Among these options, CBOE offers an (American-style) index option on the S&P 100 index called OEX, the value of which is established as 100 times the current value of the index (e.g. on July 25, 2001 the index was 613.95 and hence the dollar value of the index was \$61,395). OEX options are the most actively traded on the CBOE and are very liquid. One interesting feature of OEX is that in 1993 the CBOE created a volatility index called VIX, which is calculated as a weighted average of the implied volatilities from eight at-the-money call and put options on the OEX which have an average time to maturity of 30 days. The VIX index therefore measures the implied volatility of a hypothetical option that is at-the-money and has 30 days to expiration.⁵ The behavior of the series from Jan. 4, 1988 to Dec. 14, 1998 is shown in Figure 6. Standard Augmented Dickey Fuller tests show that the unit root hypothesis is rejected, although the degree of persistence in the series is very high.

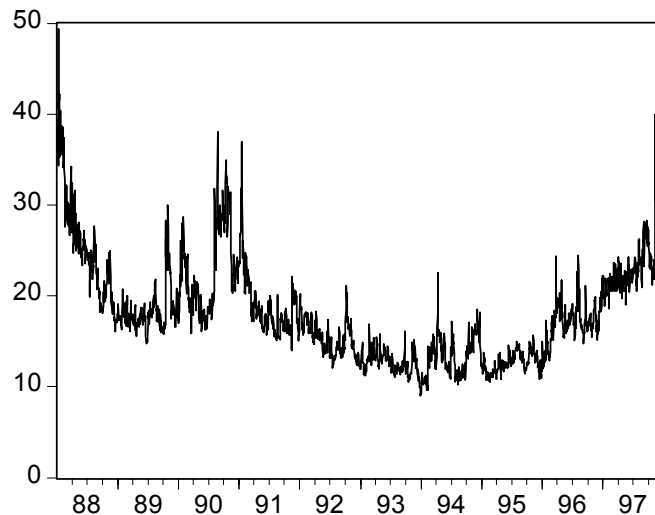
For the purposes of this paper, the analysis of VIX in reference to volatility estimates with a multiple indicator conditional model is relevant because we can use the value of VIX as a target for our volatility estimates. Under several auxiliary assumptions, the optimal forecast of volatility from past prices should match the Black-Scholes at-the-money implied volatility. However, information such as forthcoming announcements known to traders but not to the

⁴ For more detailed information cf. the CBOE web site www.cboe.com.

⁵ The value of the index is reckoned to measure investors' fears. A very high value signals bearishness as downward risk is perceived to be higher than upward risk. Chartists look at VIX for possible trend reversals.

econometricians may lead to episodes of positive or negative discrepancies. The 22-period-ahead horizon for the term structure of volatilities was chosen to ensure compatibility with the horizon considered when VIX is constructed.⁶

Figure 6. The CBOE VIX Index: Jan. 4, 1988 to Dec. 14, 1998.



We will therefore examine two main issues:

- whether the volatility forecasts provided by each set of MEMs (base and system specifications) and grouped according to the indicator they refer to contribute to the explanation of the behavior of the VIX index both in and out of sample. We will thus test whether the coefficients of the forecasts (grouped by indicator) are jointly equal to zero in a regression framework;
- whether the volatility forecasts provided by the MEMs for each indicator with a base specification and with a system specification help in explaining the behavior of the VIX index both in and out of sample. We will thus test whether the coefficients of each group of forecasts (base and system) are jointly equal to zero in a regression framework.

In order to answer these questions we ran two sets of regressions each over two separate periods (estimation sample and out-of-sample). The slope coefficient and intercept may adjust for the differences between the S&P500 and the S&P100 options and for potential, un-modeled

⁶ We are not overly concerned with the fact that VIX is calculated on the S&P100 index while we are calculating volatility measures on the S&P500. The two indices have a high correlation (in the sample period of interest here

volatility risk premia. In this context, we use both the forecasts provided by the models selected according to the smallest BIC criterion and to general-to-specific model selection procedure, since they provide partially different outcomes. Two models were estimated, both using the log of VIX as the dependent variable and the log of the one month volatility forecasts according to the base specification, and the system MEMs as independent variables. The difference between the two is that the second includes an AR(1) correction. It is not surprising that there would be serial correlation in these estimates. Differences between the optimal econometric forecast of volatility and implied volatilities would naturally arise from the reduced information set used by the econometrician. For example, an upcoming election would be incorporated in VIX but not GARCH and this would persist for many days.

The results are shown in Table 7 where we report the results of the in-sample and out-of-sample analysis with the top panel referring to the smallest BIC selection criterion and the bottom one to the results obtained with the general-to specific approach. The results are show that, with a few exceptions, all coefficients are individually statistically significant. One notices that in all specifications the constant term is highly significant and positive, signaling that the volatility estimates together somewhat underestimate the volatility level. This could just be due to the fact that our volatility estimates are referred to a wider-based index taking the average over a larger number of stocks or it could be due to the American option premium in the OEX contract or it could be due to a volatility risk premium. The specification which seems to have better properties is one in which an AR(1) correction is introduced which interestingly, keeps the joint explanatory power of the variables we derived. Beside an obvious improvement in the R^2 and in the value of the likelihood function, the diagnostics show the expected improvement since the residual autocorrelation and ARCH effects are reduced. The contribution of various sets of forecasts is assessed by means of HAC Wald F-tests and reported in the lower part of each panel. We test whether each forecast has an impact by type of indicator (rows labeled absolute returns, daily range or realized volatility) or by type of method (base specification or system specification) by means of Wald-type tests. There is no clear indication of a better in-sample model performance as a consequence of the selection criterion adopted. In both sets of results one notices that some negative coefficients appear: concentrating on the high-low range, for

about 98% for the returns and 99% for the returns squared, with high cross-correlations). One should keep in mind that the S&P500 is an average over a larger number of stocks and hence has a potentially lower volatility.

example, other things being equal, an increase in the forecast of the volatility value for this type of indicator would imply a reduction in the value of VIX. However, it is unlikely that other things are equal.

The same kind of analysis may be carried over to an out-of-sample combination of forecasts exercised over the period January 2, 1998 to November 10, 1998, in which the same variable ($\log(\text{VIX})$) is regressed on a constant and on the (logs of) 22-day-ahead volatility forecasts obtained from the same models, keeping the coefficients fixed at their estimated in-sample values. The results, again broken down between lowest BIC models and general-to-specific models, are presented in the last columns under the heading Out of Sample. Once again, the AR(1) correction achieves a substantial reduction in the values of the residual diagnostics tests and, again, the significance of each set of forecasts is maintained judging from the F-tests. The lowest BIC provides a slightly better performance: all parameters are significant. Some features of this combination of forecasts can also be assessed graphically (Figure 7) since it seems that the highest variance appears to correspond to periods in which the VIX index is higher and more volatile.

Figure 7. The CBOE VIX Index: Out of sample combination of forecasts – Jan. 2, 1998 to Nov. 10, 1998 – Smallest BIC Models with AR(1) correction (Coefficients in Table 7, last column).

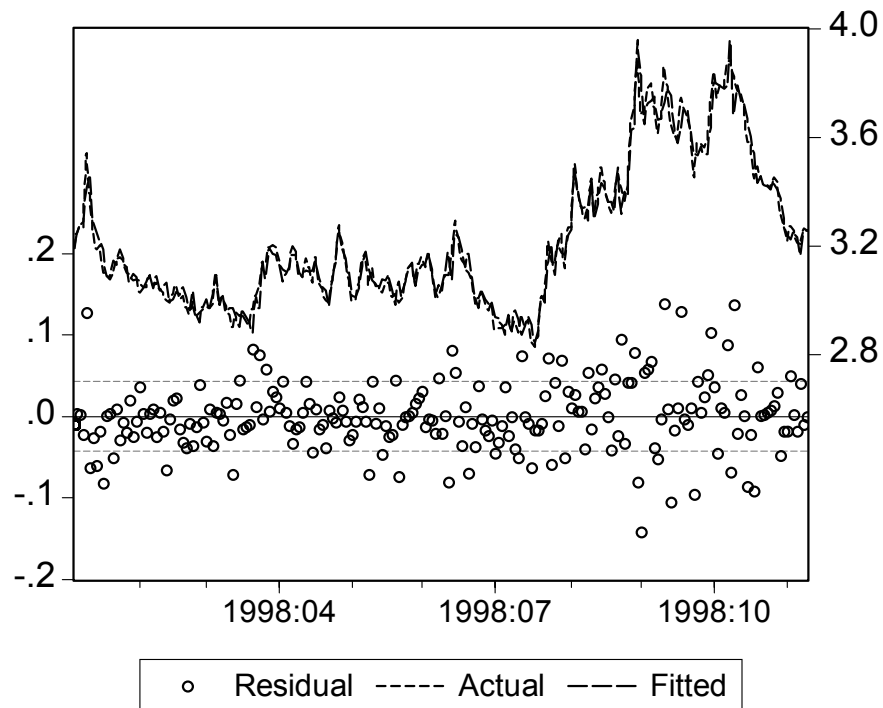


Table 7. Regression results log(VIX) regressed on a constant and the 22-day-ahead forecasts of volatility obtained using the base and the system specifications with and without an AR(1) correction.

		In sample		Out of sample	
Lowest BIC		w/ AR(1)		w/ AR(1)	
	Constant	0.778 (2.15)	0.412 (1.88)	1.393 (3.42)	0.973 (2.02)
Multi-step Average Volatility (Base specification)	Absolute Return	0.725 (10.57)	0.767 (9.44)	0.264 (2.56)	0.471 (2.49)
	Daily Range	0.723 (1.60)	1.328 (4.30)	1.682 (3.83)	2.513 (10.87)
	Realized Volatility	0.100 (0.52)	-1.216 (-12.25)	-0.961 (-3.01)	-1.565 (-7.61)
Multi-step Average Volatility (System specification)	Absolute Return	-1.480 (-2.53)	-0.071 (-0.21)	-0.898 (-4.09)	-0.790 (-7.84)
	Daily Range	-0.216 (-3.01)	-0.071 (-3.04)	0.049 (0.74)	-0.023 (-0.52)
	Realized Volatility	1.032 (3.72)	0.234 (2.69)	0.537 (3.84)	0.238 (2.03)
	AR(1)		0.950 (146.36)		0.908 (29.14)
Diagnostics	R-squared	0.83	0.97	0.91	0.97
	TR ² for AR(4)	2877.82**	9.12**	91.68**	4.17**
	TR ² for ARCH(4)	995.62**	11.91**	48.50**	3.21*
Robust Wald tests of joint zero coefficients	Abs. returns terms	56.33**	49.62**	7.47**	59.36**
	Daily Range terms	3.28*	102.40**	8.41**	57.59**
	Realized Volatility terms	7.37**	4.92**	9.31**	2.11
	Base terms	109.00**	77.56**	3.22*	29.59**
	System terms	15.56**	181.46**	10.05**	66.41**
General to Specific					
	Constant	1.178 (3.82)	0.581 (4.32)	1.234 (3.52)	1.203 (2.98)
Multi-step Average Volatility (base specification)	Absolute Return	0.679 (9.94)	0.837 (10.29)	0.194 (1.88)	0.557 (2.67)
	Daily Range	1.200 (4.16)	1.260 (5.52)	0.683 (2.61)	1.741 (7.40)
	Realized Volatility	0.056 (0.31)	-1.282 (-13.61)	-0.238 (-0.98)	-1.294 (-5.09)
Multi-step Average Volatility (System specification)	Absolute Return	-2.360 (-5.52)	-0.240 (-0.74)	-0.511 (-2.99)	-0.802 (-6.93)
	Daily Range	-0.068 (-1.06)	-0.023 (-1.30)	0.165 (2.48)	0.127 (2.62)
	Realized Volatility	1.238 (5.00)	0.362 (4.87)	0.430 (7.44)	0.431 (9.19)
	AR(1)		0.947 (137.97)		0.945 (37.27)
Diagnostics	R-squared	0.84	0.97	0.91	0.97
	TR ² for AR(4)	2542.08**	8.47**	74.73**	1.90
	TR ² for ARCH(4)	957.18**	11.92**	44.33**	6.87**
Robust Wald tests of joint zero coefficients	Abs. returns terms	55.28**	61.19**	3.54*	27.57**
	Daily Range terms	16.47**	104.32**	4.87**	26.08**
	Realized Volatility terms	13.37**	11.89**	37.81**	54.76**
	Base terms	81.79**	72.62**	7.13**	10.28**
	System terms	28.78**	182.80**	20.45**	34.22**

Robust t-values under the estimated coefficient values. * = significant at 5%; ** = significant at 1%.

6. Conclusions

The motivation for this paper stems from the consideration that various volatility indicators used in the literature (absolute daily returns, daily high-low range and intra-daily realized volatility) may present features which should be jointly combined in a dynamic model to enhance the information content of individual measures. We have adopted a novel approach, called Multiplicative Error Model (Engle, 2002), which is suited to model the conditional behavior of positively valued variables choosing a convenient GARCH-type structure to model persistence. It turns out that a wide variety of error assumptions can be accommodated with a standard easy-to-use procedure and inference conducted on the basis of the robust variance covariance matrix. For the problem at hand the chosen specification for the MEM is multivariate and is suitable to be dynamically solved for short to medium range forecasting horizons. For the data at hand (Standard and Poor 500), we show that the approach is rewarding in that the retained specifications for each indicator are augmented by the presence of lagged values of other indicators. In particular, we obtain the result that daily range and returns have explanatory power for realized volatility, and daily range is the indicator with the most parsimonious model.

We evaluate the performance of this model by producing 22-step-ahead volatility forecasts for each of the three indicators and by using them in a regression framework to detect their explanatory power for an index of market volatility such as the VIX index. The results show that the model performs well, with significant persistence of the VIX index being captured by this forecasts both individually and grouped either by type of indicator and by type of specification adopted.

The estimator thus derived is efficient when the choice of error distribution is within the class of a Gamma distribution: it retains consistency when other more flexible and time-varying density specifications may be more appropriate and research is under way to identify such densities. By the same token, in the present context we adopted a diagonal structure for the variance covariance matrix of the error terms, and further efficiency gains could be achieved by considering more complex structures of correlations among the error terms, paralleling the literature on multivariate GARCH models.

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OTHER FIGURES FROM THE TEXT

Figure 3. The S&P500 Index: Absolute returns, Daily Range, and Realized Volatility
Sample Period: 1/4/1988 – 12/30/1997.

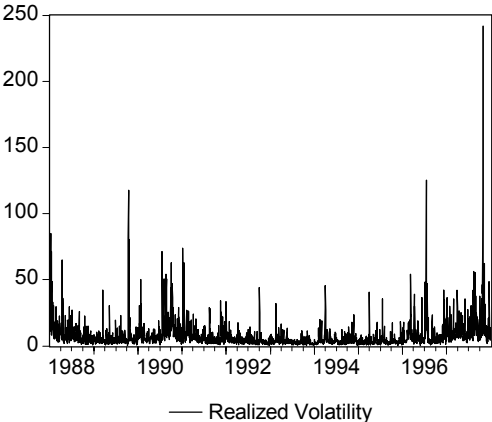
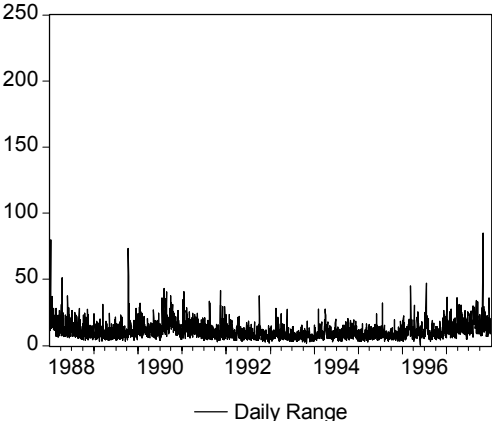
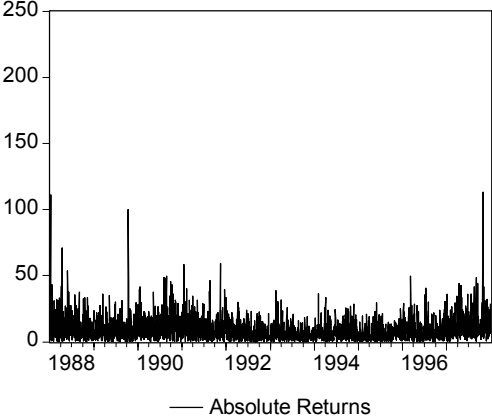
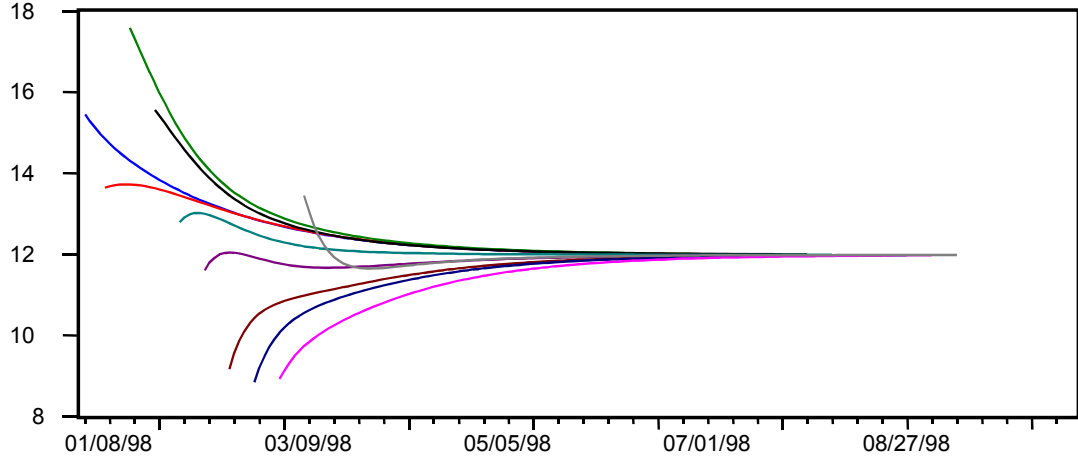
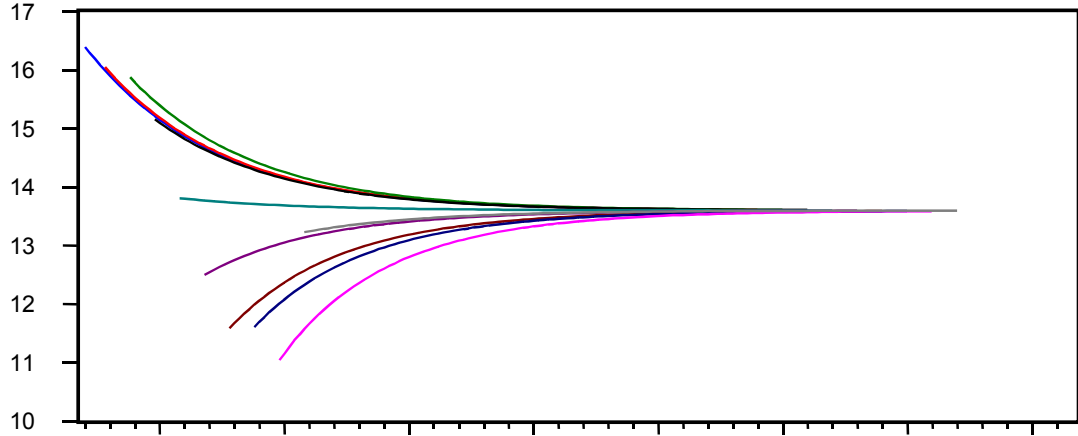
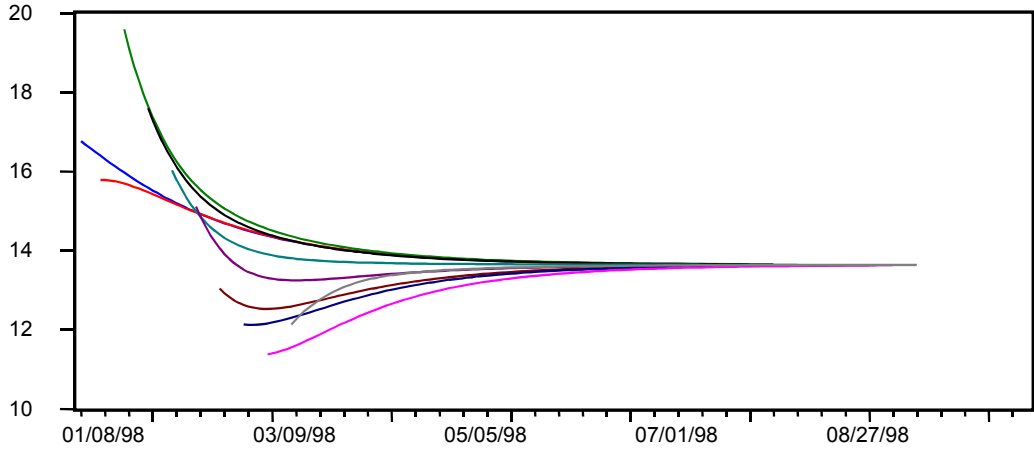


Figure 4. The S&P500 Index: Out-of-sample Volatility Forecasts: Abs. Returns, High-Low, Daily Volatility. Various starting dates.



**Figure 5. The S&P500 Index: One month-ahead term structure of volatility:
Abs. Returns, High-Low, Daily Volatility. Out of Sample - Jan. 2, 1998 to Nov. 10, 1998.**

