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The Greed and Regret Problem

Anshul Sheopuri
Department of Information, Operations and Management Science, Leonard N. Stern School of Business, New York University. 8-154, 44^{th} West, 4^{th} Street New York, NY- 10012. asheopur@stern.nyu.edu

Eitan Zemel
Department of Information, Operations and Management Science, Leonard N. Stern School of Business, New York University. 8-65, 44^{th} West, 4^{th} Street New York, NY- 10012. zemel@stern.nyu.edu

In this paper, we study a model, called the Greed and Regret (GR) problem that provides a unified approach for analyzing numerous situations in a wide range of areas including studies of crime, fraud*, aggression and threats, "take it or leave it" offers and ultimatums, optimal bidding, service levels, buffers, and inventory, sales force compensation, and more. We model these situations as continuous decision problems, where the decision maker seeks to set the level of his activity as high as possible without crossing the line and going overboard. A special case of this problem, which we call the Market Price Newsvendor (MNV), involves a Newsvendor whose selling price depends on whether the supply or demand is tight. The (MNV) problem bears an interesting and surprising relation to the "Selling to the Newsvendor" problem studied by Lariviere and Porteus (2001).

In general, the (GR) problem need not be concave. Our main results are to prove several monotonicity properties for the (GR) problem and to identify a sufficient condition for uniqueness of the optimal solution, which we call Increasing Composite Failure Rate (ICFR), and study its properties. The condition is a relaxation of the IGFR property introduced by Lariviere and Porteus (2001) but is stronger than the IFR property. We extend the analysis to a larger class of incentive and disincentive functions and to the finite horizon version of the problem.

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1. Introduction

In this paper, we study a model, called the Greed and Regret (GR) problem that provides a unified approach for analyzing numerous situations in a wide range of areas including studies of crime, corruption, aggression and threats, "take it or leave it" offers and ultimatums, optimal bidding, service levels, buffers, and inventory, sales force compensation, and more. We model these

* We also briefly summarize an implementation (work undertaken as part of summer internship at IBM T.J. Watson Research Center with IBM Managed Business Process Services (MBPS)) of some ideas included in this article in the context of fraud detection for the Travel and Entertainment Expense application.

situations as continuous decision problems, where the decision maker seeks to set the level of his activity as high as possible without crossing the line and going overboard.

The (GR) problem is a single period decision problem with uncertainty, involving a risk neutral decision maker (DM), and a single activity, p. For small levels of p, (DM)'s objective function increases with p, and so he wishes to set p as high as possible ("be greedy"). However, if (DM) sets p too high, his objective may suffer ("regret"). We model the "greed" and the "regret" aspect of this problem using linear functions, but generalizations for the non-linear are readily available. In most cases the (GR) problem is not concave.

We motivate the development of our model with an illustrative application, which we call the Market Price Newsvendor, and which is a (non-concave) generalization of the classic Newsvendor problem. In this problem, a risk-neutral Newsvendor decides how much inventory, y, to stock in the face of stochastic demand, D. Let c be the per-unit cost price, and assume the salvage value and back-order cost are zero. Assume that the selling price depends on the supply / demand conditions. If the supply is tight ($y \leq D$), the market price is $r_1 > c$. If the demand is tight ($y > D$), the market price is $r_2 : 0 \leq r_2 \leq r_1$. Then, the pay-off function is given by:

$$\begin{cases} (r_1 - c)y & y \leq D; \\ -cy + r_2 D & y > D. \end{cases}$$

For the case of $r_2 = r_1$, this reduces to the classic Newsvendor problem. As we will see in section 3, the case $r_2 = 0$ bears an interesting and surprising relation to the "Selling to the Newsvendor" problem studied by Lariviere and Porteus (2001).

To demonstrate the range of applications of the (GR) problem, we outline below a few additional examples that motivate our interest.

(GR1) **Fraud and Aggression**. Consider an individual (or company) who is planning to "cook the books". Let the level of this activity be denoted p and the reward per one unit of activity p be denoted α. Let the probability of being caught be denoted $F(p)$, increasing with p, and the penalty of being caught be $B + \beta p$. Then, the payoff from the activity can be modeled as

$$\begin{cases} \alpha p & \text{if not caught (with probability} 1 - F(p)); \\ -B - \beta p & \text{if caught (with probability} F(p)). \end{cases}$$

In some cases, B or β would be set to 0 to reflect the extreme cases of pure fixed or pure variable penalties. In this formulation, the activity p could represent tax evasion, money laundering, backdating options, stealing from one's employer and so on. Note that problem looks at fraud as a continuous activity, rather than discrete (see Tsebelis (1990)). That is, the issue is how much to cheat, rather than whether or not to cheat.

A related application is that of modelling aggression. For example, in many contact sport competitions, a participant may gain advantage by being overly aggressive. However, if he overdoes it, he may be penalized by the referee. For example, Cachon (private communication, 2006) has pointed out that in soccer competitions, as the game moves from a non-critical to a critical phase, the levels of aggression typically rise. We will show that such behavior is predicted by the model.

Another application, using the ideas outlined above is the problem of fraud detection with imperfect audit proxy, in the context of Travel and Entertainment Expense claim reimbursement studied by Sheopuri *et.* al (2006)[1]. In their setting, they note the "unavailability of a random subset of the data for which claims have been partitioned into fraudulent and non-fraudulent after a complete audit. However, the dataset contains an imperfect proxy of fraud, an indicator audit variable that is flagged by certain adhoc business rules." They approach the problem by using a classical statistical model for fraud detection (for example, the Logit model) and update the estimates of fraud obtained by evaluating the equilibrium of a game between the auditor and the claimant. They note that "embedded as a sub-problem of this game is a class of optimization problems studied by Sheopuri and Zemel (2007)."

(GR2) **Take it or leave it offers and Ultimatums.** An individual makes an offer to sell an object to another individual who has a reservation value V, unknown to the maker of the offer. If we denote the offer by $p+r$, where r is the value of the object to the seller, then the seller's objective function is given by

$$\begin{cases} p & p \leq V - r; \\ 0 & \text{otherwise.} \end{cases}$$

Lambert-Mogiliansky *et.* al (2004) begin their analysis of petty corruption within the framework of this model (with $r = 0$). In their model, the decision maker is a corrupt bureaucrat who is deciding how much bribe to request.

The problem of issuing an ultimatum is another example, the first extensive experiments on which were done by Guth *et.* al (1982). In the ultimatum game, "A sum of USD 10 is available for distribution on just this one occasion to two people, A and B, strangers to each other, provided they can reach an agreement on how to share it. A can only make an offer to B of any amount between 0 and USD 10. If B rejects the offer, they can each share the USD 10 according to their agreement. If B rejects the offer, they each get zero. The claim is that accepted economic theory asserts that it is optimal for A to offer B as little as possible, say one penny, and that B should

[1] Work undertaken as part of summer internship at IBM T.J. Watson Research Center with inputs from IBM Managed Business Process Services (MBPS).

accept this offer because even one penny is better than nothing." (see Tesler (1995)). However, most experiments with this game show that B rejects the offer below a certain threshold (Tesler (1995) and Frank (1992)). Given the prior on the cut-off of B, $A's$ problem may be modelled is a "Take-it-or-leave-it" offer, as discussed above.

(GR3) **Sales Compensation.** Gonik (1978) suggests the following compensation scheme for a salesperson which depends upon his forecast, F, of the actual sales, A, which is stochastic:

$$\begin{cases} k\frac{A+F}{2} & F \leq A; \\ k\frac{3A-F}{2} & \text{otherwise,} \end{cases}$$

with $k > 0$.

(GR4) **Multiple unit Auctions.** Consider an auction in which the seller has to decide how many units to put on the market. For simplicity, assume there are 2 buyers, whose per-unit bid prices and quantities are the ordered pairs (r_1, q_1) and (r_2, q_2) respectively (assume that $r_1 \geq r_2 \geq 0$). Let c be the reservation value of the seller. Then, if the auction is first-price, the seller's pay-off function (for $0 \leq q \leq q_1 + q_2$) is given by

$$\begin{cases} (r_1 - c)q & q \leq q_1; \\ (r_2 - c)q & \text{otherwise.} \end{cases}$$

(GR5) **Service Level.** Sheopuri and Zemel (2005) study a supplier's quality problem, in which a supplier provides a shipment with a fraction of defects denoted by p. The buyer will accept the shipment as long as $p \leq T$, where the cut-off level T is unknown to the seller. Let α be the marginal benefit from producing poor quality in the case of $p \leq T$. Let β be the marginal penalty and B the fixed penalty of poor quality in the case of $p > T$. The problem may be modelled as:

$$\begin{cases} \alpha p & p \leq T; \\ -(B + \beta p) & \text{otherwise.} \end{cases}$$

(GR6) **Service Level with Noise.** In this problem, the service level cut-off T is known to the supplier. However, if the supplier sets the service level at p, than the actual service level produced is $p - \epsilon$, where ϵ is independent of p. The problem may be modelled as:

$$\begin{cases} \alpha p & p - \epsilon \geq T; \\ -B - \beta(p - \epsilon) & \text{otherwise.} \end{cases}$$

In this paper, we provide a unified approach for studying problems of this type. In section 2, we formulate the linear model of the (GR) problem and establish equivalence of two versions, which we denote the "canonical" and "reduced" formulations. Section 3 describes how the well-known "Selling to the Newsvendor" problem studied by Lariviere and Portues (2001) and the classical Newsvendor problem relate to our problem, and in particular to the Market Price Newsvendor, which is a special case of the (GR) problem with zero fixed penalty. We show that the profit function of the Market Price Newsvendor can be expressed as a convex combination of the profit functions of the Newsvendor and Selling to the Newsvendor problems.

The expected profit function of the (GR) problem need not be concave. Thus, in section 4, we study properties of the optimal set. In particular, we study monotonicity of the optimal set with the various penalty and reward parameters, and demonstrate some counter-intuitive properties. We also show the equivalence of linear transformation of support of the underlying random variable and a shift in the penalty for the (GR) problem.

In section 5, we identify a class of distributions, which we call "Increasing Composite Failure Rate (ICFR)", which provide sufficient conditions for uniqueness of the optimal solution of the (GR) problem. This class retains several properties of IGFR distributions (see Lariviere (2003) and Paul (2004)). Interestingly, we observe that the ICFR property is preserved only under "scale-up" and not under "scale-down", unlike the IGFR and IFR distributions. In section 6, we extend our results to the non-linear case and the finite-horizon version of the (GR) problem. Section 7 summarizes the results of the paper.

2. The Linear Model

We now describe the linear model. Let A, B, α, β, u and v be constants with $A+B \geq 0$, $\alpha > 0$, $\alpha + \beta \geq 0$. Let X be a random variable (r.v.) with support $[a,b]$, $a \geq 0$, and with cumulative distribution function (c.d.f.) and probability density function (p.d.f.) denoted by $F(.)$ and $f(.)$ respectively. Assume that $f(.)$ is continuous over $[a,b]$ and that $f(x) > 0$ for $x \in [a,b]$. Let

$$\Pi(p,X) = \begin{cases} A + \alpha p - uX & p \leq X; \\ -(B + \beta p - vX) & p > X, \end{cases} \tag{1}$$

and

$$\pi(p) = E_X[\Pi(p,X)].$$

We denote the problem

$$\pi^* = \max_{a \leq p \leq b} \pi(p),$$

where p is the decision on the activity, as the Greed and Regret (GR) problem.

We denote (1) as the "canonical" formulation of the (GR) problem. We refer to the case $p \leq X$ as being on the "greed line" and to the case $p > X$ as being on the"regret line". To gain intuition into the tradeoffs involved, it is sometimes convenient to transform the formulation (1) into a more condensed, equivalent, formulation, involving four coefficients, α, C, β, and w, with $\alpha > 0$, $C \geq 0$, $\alpha + \beta \geq 0$:

$$\Pi^R(p, X) = \begin{cases} \alpha p & p \leq X; \\ -(C + \beta p - wX) & p > X. \end{cases} \tag{2}$$

We refer to the formulation (2) as the "reduced" formulation. With the proper selection of coefficients, the two formulations are equivalent (see Lemma 4 in Appendix).

In the remainder of the paper, we drop the superscript R while referring to the profit function and expected profit functions of the reduced formulation (2).

In the reduced formulation, the term αp represents the profits levels gained by the forbidden activity, p, as long as we stay on the greed line. However, in case X turns out to be larger than p, the decision maker moves to the regret line. In this case, he is subject to four types of penalties:

1. disgorge the "profits" αp,
2. pay a fixed penalty C,
3. pay a proportional penalty βp,
4. receive a random adjustment wX.

The first penalty ("disgorgement") is always non-negative, as is the second, fixed penalty. The third is positive in most applications, but some exceptions are also possible. The fourth, term, received rather than paid by the decision maker in terms of regret, will typically (but not necessarily) be positive (an adjustment to the penalties). Figure 1 depicts the case $w = 0$, with each of the first three penalties being positive. The significance of the random adjustment wX will be examined in section 3, where we study the Market Price Newsvendor and its two special cases, the Newsvendor and the "Selling to the Newsvendor" problem.

To examine the relation between the models (1) and (2), and the applications GR1-GR6 consider the following table:

Table 2.1: Values of parameters for the canonical and reduced formulation of the (GR) problem.

Problem	A	B	α	β	u	v	C	w	X
(GR1)	0	B	α	β	0	0	B	$\alpha+\beta$	An artificial r.v. with c.d.f. $F(.)$
(GR2)	0	0	1	0	0	0	0	1	$V - r$
(GR3)	0	0	$\frac{k}{2}$	$\frac{k}{2}$	$-\frac{k}{2}$	$\frac{3k}{2}$	0	k	A
(GR4)	0	0	r_1	$-r_2$	0	0	0	$r_1 - r_2$	X
(GR5)	0	B	α	β	0	0	B	$\alpha+\beta$	T
(GR6)	0	B	α	β	α	β	B	$\alpha+\beta$	$T+\epsilon$

3. The Market-Price Newsvendor

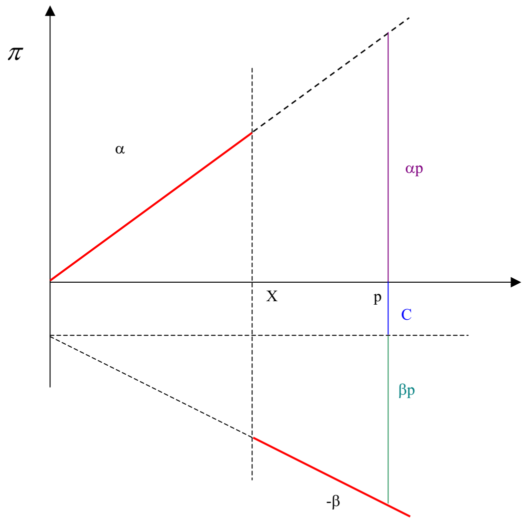

Figure 1 **(GR) problem with** $w = 0$.

Recall the Market Price Newsvendor (MNV) problem introduced in Section 1:

$$\Pi(y, D) = \begin{cases} (r_1 - c)y & y \leq D; \\ -cy + r_2 D & y > D, \end{cases}$$

Clearly, (MNV) is an equivalent formulation of the (GR) problem with $C = 0$ via the relations

$$\alpha = r_1 - c, \beta = c, w = r_2.$$

If $r_1 = r_2$, the (MNV) reduces to the well-known Newsvendor problem.

We now examine the relation between the case $r_2 = 0$ and the "Selling to the Newsvedor" (SNV) problem: In the (SNV) problem, a manufacturer (first mover) produces a good at a marginal cost c and sells it to a retailer at price $w(y)$, where y is the stocking level. The salvage value and the

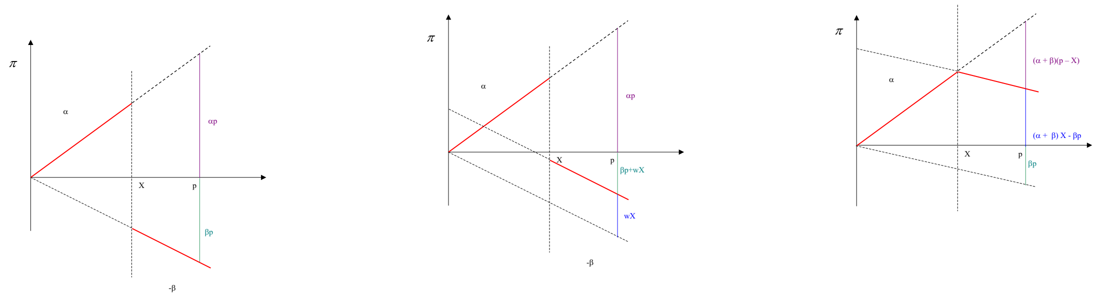

Figure 2 **(MNV) problem for $w = 0$, $w = \alpha + \beta$ and $w = \lambda(\alpha + \beta)$ for $0 < \lambda < 1$.**

backordering cost are zero. The retail price is fixed at $r > c$. The manufacturer faces the inverse demand curve

$$w(y) = r\bar{F}(y),$$

where $F(.)$ is the c.d.f. of the demand faced by the retailer. The manufacturer's expected profit function is given by:

$$\pi^M(y) = y(w(y) - c) = ry\bar{F}(y) - cy.$$

Note that the problems (GR) and (SNV) are not the same: while (SNV) involves two decision makers and the source of variability is market demand, which is external to both, (GR) involves one decision maker, and the source of uncertainty is in the trigger level X. Also, the basic structure of (SNV) is naturally continuous, while the main building block of (GR) is inherently discontinuous (in general). However, for a risk-neutral decision-maker, the (SNV) problem is mathematically equivalent to the following special case of the (GR) problem:

$$\Pi(y, D) = \begin{cases} (r_1 - c)y & y \le D; \\ -cy & y > D, \end{cases}$$

i.e., to the case $r_1 = r$ and $r_2 = 0$. Figures 2 depicts the objective function of the (MNV) problem for the cases $w = 0$, $w = \alpha + \beta$ and $w = \lambda(\alpha + \beta)$ for $0 < \lambda < 1$, respectively. Further, since $r_2 \le r_1$, the profit function for the (MNV) problem may be written as a convex combination of the profit functions of the (NV) and (SNV) problems.

Another related problem of interest is the the Selling to the "Newsvendor" Problem with a forecast update studied by Ozer *et.* al (2005). The manufacturer's expected profit, in this case, is again equivalent to the (GR) problem, but with non-zero fixed penalty, i.e., $C > 0$. We will discuss in more detail the implication of this observation on the assumption on the forecast update distribution in Section 5.

4. Monotonicity Properties of the Optimal Set

While the Newsvendor problem is known to be concave, the (GR) problem in general, is not. Thus, it may admit several local optima, and the global optimum may not be expressed in a closed form. In consequence, one can not study the connection between parameters such as α, β or C and the optimal behavior of the decision maker using standard analysis of the first order conditions. Nevertheless, we show below that the (GR) objective function is submodular with respect to some of its variables, and thus can demonstrate that the optimal solution(s) behave monotonically with the right parameters. We first examine the profit function, $\pi(p)$, in its reduced formulation, by taking the expectation of the expression (2);

$$\pi(p) = \alpha p(1 - F(p)) - CF(p) - \beta pF(p) + w\int_a^p xf(x)dx.$$

The first order condition (F.O.C.) is

$$\alpha - (\alpha + \beta)(pf(p) + F(p)) - Cf(p) + wpf(p) = 0.$$

Using Lemma 5 (see Appendix), we can now show the monotonicity of the optimal set:

Proposition 1. *(Monotonicity of P^*)*
Consider a (GR) problem P_1, defined by $[a_1, b_1], F_1, \alpha_1, \beta_1, w_1$ and C_1. Let P_2 be derived from P_1 by

$$\beta_2 \geq \beta_1,$$

$$C_2 \geq C_1,$$

$$w_2 \leq w_1.$$

with at least one of the inequalities being strict. Let all other parameters of P_2 be the same as that of P_1. Let p_1^* and p_2^* be any pair of optimal solutions for P_1 and P_2 respectively. Then,
(i) $p_2^* \leq p_1^*$.
(ii) Either $p_2^* < p_1^*$ or $p_2^* = p_1^* = a$ or $p_2^* = p_1^* = b$.

Remark 1. Suppose y^{NV}, y^{SNV} and y^{MNV} be any optimal solution to the (NV), (SNV) and (MNV) problems respectively. Then, from Proposition 1,

$$y^{NV} \geq y^{MN} \geq y^{SN}.$$

Note that Proposition 1 does not specify the impact of α. In fact, $\pi(p)$ is, in general, not submodular with respect to p and $\pm\alpha$. However, we now state a result similar to proposition 1 by considering a (GR) problem defined by $[a,b], F, \alpha, \beta$, C and λ.

Proposition 2. *(Monotonicity of P^*)*
Consider a (GR) problem P_1, defined by $[a_1,b_1], F_1, \alpha_1, \beta_1, \lambda_1$ and C_1. Let P_2 be derived from P_1 by

$$\lambda_2 \le \lambda_1$$

$$C_2 \ge C_1,$$

$$\beta_2 \ge \beta_1 (\text{if } \lambda_1 \le 1),$$

$$\alpha_2 \le \alpha_1 (\text{if } \lambda_2 \ge 1),$$

with at least one of the inequalities being strict. Let all other parameters of P_2 be the same as that of P_1. Let p_1^* and p_2^* be any pair of optimal solutions for P_1 and P_2 respectively. Then,
(i) $p_2^* \le p_1^*$.
(ii) Either $p_2^* < p_1^*$ or $p_2^* = p_1^* = a$ or $p_2^* = p_1^* = b$.

Proof. Lemma 6 (see Appendix). □

We note that condition $\lambda \le 1$ does not hold in any of the examples (GR1-GR6) that we examined.

We discuss briefly the implication of the result in proposition 2 for the (NV), (SNV) and (MNV) problems respectively. We first note that an increase in α is equivalent to a decrease in β, C and w. Thus, for the (NV) and (SNV) problem, the (DM) is more “risky” with higher α. In both these cases, the greed parameter, α is the profit margins. However, the result need not be true for the (MNV) problem (Recall that the profit function of the (MNV) problem is a convex combination of the (SNV) and (NV) problems). To see this, consider the (MNV) problem with zero marginal cost of production. In this case, (DM) is more “conservative” with higher α.

Proposition 3. *(Equivalence of Linear Transformation and penalty shifts (see also (Lariviere and Porteus (2001)))*
Consider a (GR) problem P_0, defined by $[a_0,b_0], F_0, \alpha_0, \beta_0$, C_0 and w_0.
Let P_1 be derived from P_0 by the linear transformation:
$[a_1,b_1] = [\rho + \gamma a_0, \rho + \gamma b_0]$,
$F_1(\rho + \gamma p) = F_0(p)$, *with all other parameters of P_1 being the same as that of P_0.*
Let P_2 be derived from P_0 by the penalty shifts:
$C_2 = \frac{C_0 + (\alpha+\beta)\rho}{\gamma} - w_0 \rho$,

$w_2 = \gamma w_0$, *with all other parameters of* P_2 *being the same as that of* P_0.
Let $P_1^* = \{p_{1i}^*, i = 1, 2, ..., k\}$ *and* $P_2^* = \{p_{2i}^*, i = 1, 2, ..., r\}$ *(Wlog, assume that* $p_{11}^* \le p_{12}^* \le ... \le p_{1k}^*$ *and* $p_{21}^* \le p_{22}^* \le ... \le p_{2r}^*$*). Then,*
$k = r$, *and*
$p_{1i}^* = \rho + \gamma p_{2i}^*, i = 1, 2, ..., k$.

Proof. Let $\pi_i(p)$ be the objective function for the (GR) problem P_i, $i = 1, 2$. Then,

$$\begin{aligned}\pi_1(p) &= \alpha_1(\rho+\gamma p) - [C_1 + (\alpha_1+\beta_1)(\rho+\gamma p)]F_1(\rho+\gamma p) + w_1 \int_a^{(\rho+\gamma p)} (\rho+\gamma x) f_1(x)dx \\ &= \alpha\rho + \gamma\pi_2(p),\end{aligned}$$

which gives the result. □

Another way of expressing the equivalence of p_1^* and p_2^* is in terms of the penalty probabilities:

$$F(p_{1i}^*) = F(p_{2i}^*), i = 1, 2, 3, ..., k.$$

Combining propositions 1 and 3, we note that a linear transformation of $[a, b]$ would result in a reduction of the probability of punishment iff

$$(\gamma - 1)C < \rho(\alpha + \beta - w\gamma).$$

For the case of $\alpha + \beta \ge w$, the inequality holds, for instance, if $\gamma \le 1$, $\rho \ge 0$ (with at least one inequality strict). This is satisfied if the intervals $[a_1, b_1]$ and $[a_2, b_2]$ are such that $b_2 \le a_2 + (b_1 - a_1)$ and $a_2 \ge a_1$. Thus,

Proposition 4. *(Monotonicity of* P^* *with support shifts)*
Consider a (GR) problem P_1*, defined by* $[a_1, b_1], F_1, \alpha_1, \beta_1$, C_1 *and* w_1.
Let P_2 *be derived from* P_1 *by the linear transformation:*
$[a_2, b_2] = [\rho + \gamma a_1, \rho + \gamma b_1]$,
$F_2(\rho + \gamma p) = F_1(p)$, *with all other parameters of* P_2 *being the same as that of* P_1.
If $\alpha_1 + \beta_1 \ge w_1$, *(DM) is more conservative if*
(i) $[a_2, b_2] = x + [a_1, b_1]$, $x > 0$,
(ii) $[a_2, b_2] \subseteq [a_1, b_1]$.

Proof. The result follows from that fact that $b_2 \le a_2 + (b_1 - a_1)$ and $a_2 \ge a_1$ are satisfied if either (i) or (ii) hold. □

Note that (i) and (ii) correspond to dominance in the sense of first and second order respectively (see Shaked and Shanthikumar (1994)).

5. Composite Failure Rates

We start with the case $\alpha+\beta>0$. Let

$$\delta=\frac{C}{\alpha+\beta}$$

and

$$\lambda=\frac{w}{\alpha+\beta}.$$

Note that $\delta\geq 0$, though λ may be negative.

We now discuss conditions for the optimal solution to be unique. If $\hat{p}$ is an upper bound on p^*, it is sufficient to require monotonicity[2] of (for $p<b$)

$$\frac{(\delta+\bar{\lambda}p)f(p)}{1-F(p)}$$

over $[a,\hat{p}]$. (To see this, note that, for $p<b$:

$$\pi'(p)=-\beta+(\alpha+\beta)\bar{F}(p)[1-\frac{(\delta+\bar{\lambda}p)f(p)}{1-F(p)}]).$$

For the case of $\delta=0,\lambda=0$, $\alpha+\beta>0$, Lariviere and Porteus (2001) provide a special case of this condition, which they denote the Increasing Generalized Failure Rate (IGFR), namely that

$$\frac{pf(p)}{1-F(P)}$$

be monotone increasing over $[a,\hat{p}]$.

Remark 2. For the (NV) problem, $\delta=0$ and $\lambda=1$, hence the monotonicity holds and the optimal solution is always unique.

Below, we study the class of distributions that satisfies the sufficient condition for uniqueness, which we call Increasing Composite Failure Rate (ICFR).

Let X be a r.v. with p.d.f. $f(.)$ and c.d.f. $F(.)$. Let $0\leq\theta\leq 1$. Let

$$g_X(x|\theta)=\frac{(\bar{\theta}+\theta x)f(x)}{1-F(x)}.$$

Definition 1. X is said to be increasing in Composite Failure Rate with parameter $\theta\in[0,1]$ (denoted by $ICFR(\theta)$) if

$$g_X(x|\theta)$$

is monotone increasing in x.

[2] In this section, we use monotonicity in the weak sense, unless otherwise stated.

REMARK 3. We use the term "Composite Failure Rate" as the function, $g_X(x|\theta)$, is a convex combination of two terms, the failure rate, $g_X(x|0)$ and the generalized failure rate, $g_X(x|1)$, i.e.,

$$g_X(x|\theta) = \bar{\theta} g_X(x|0) + \theta g_X(x|1).$$

From the definition, it follows that:

(1) X is IFR if $g_X(x|0)$ is increasing in x.

(2) X is IGFR if $g_X(x|1)$ is increasing in x.

LEMMA 1. *(a) (Monotonicity with θ) Let X be $ICFR(\theta_0)$. Then,*
X is ICFR(θ), $\theta \geq \theta_0$,
(b) (Implication relations) Let $\theta \in [0,1]$. Then,
(i) $IFR \Rightarrow ICFR(\theta) \Rightarrow IGFR$,
(ii) $IGFR \not\Rightarrow ICFR(\theta)$.

PROOF. (a)

$$g_X(x|\theta) = \frac{\bar{\theta} + \theta x}{\bar{\theta}_0 + \theta_0 x} g_X(x|\theta_0).$$

The result follows by noting that

$$\frac{\bar{\theta} + \theta x}{\bar{\theta}_0 + \theta_0 x}$$

is increasing if $\theta \geq \theta_0$.

(b) (i) The results follows from (a).

(ii) We provide a counterexample to show the result. Let X be a Pareto r.v. Then,

$$g_X(x|0) = \frac{\sigma}{x},$$

$\sigma > 0$, $x \geq a$, $a > 0$. Clearly, X is IGFR. The result follows by observing that

$$g_X(x|\theta) = (\bar{\theta} + \theta x)\frac{\sigma}{x} = \bar{\theta}\frac{\sigma}{x} + \theta\sigma,$$

is decreasing in x for $\theta \in [0,1)$. □

LEMMA 2. *(Preservation under shifting) Let X be ICFR(θ). Then,*
kX is ICFR($\frac{\theta}{\theta + k\bar{\theta}}$).

PROOF. By definition $g_X(x|\omega)$,

$$g_{kX}(x|\omega) = \frac{\bar{\omega} + \omega x}{\bar{\theta} + \theta x} g_X(x/k|\theta).$$

The sufficient condition for $g_X(x|\omega)$ to be increasing is that $\frac{\bar{\omega}+\omega x}{\bar{\theta}+\theta x}$ be increasing or that $\omega \geq \frac{\theta}{\theta + k\bar{\theta}}$.
□

Remark 4. Let X be $ICFR(\theta)$. It follows from Lemma 6 that kX is ICFR(θ), $\forall k \geq 1$ (Preservation under scale-up). However, note that kX need not be ICFR(θ), $k < 1$ (Non-preservation under scale-down). To see this, consider the following counter example: Let X be a Log-logistic random variable with $a = 1$ and parameter θ. Then,

$$g_X(x|0) = \frac{\theta}{1+\theta x}, x \geq 0.$$

Note that X is ICFR($\frac{\theta}{1+\theta}$). The result follows by observing that

$$g_{kX}(x|\frac{\theta}{1+\theta}) = \frac{1+\theta x}{1+\theta}\frac{1}{k}\frac{\theta}{1+\frac{\theta x}{k}}$$

is decreasing in x if

$$\frac{1+\theta x}{k+\theta x}$$

is decreasing.

Finally, we review some results in Paul (2005) for IGFR distributions which hold for ICFR also:
(1) ICFR(θ) distributions are not closed under convolutions,
(2) ICFR(θ) distributions are not closed under shifting,
(3) ICFR(θ) distributions are not closed under the operation of taking arbitrary powers of the underlying variable.

Example 1. Let p_{unv} be the optimal solution for the (NV) problem in the case of the uniform distribution. In the (GR) problem for the uniform distribution, the monotonicity condition is satisfied and the optimal solution, p_u^*, is given by:

$$p_u^* = max\{\frac{p_{unv} - \delta}{2 - \lambda}, a\}.$$

In the special case of $\delta = 0, \lambda = 0$ (the equivalent (SNV) problem), this reduces to

$$p_u^* = max\{\frac{p_{unv}}{2}, a\}$$

(see Lariviere and Porteus(2001)). Further, if $a = 0$, then

$$p_u^* = \frac{p_{unv}}{2}.$$

Remark 5. Recall the observation (Section 3) on the equivalence of the Selling to the "Newsvendor" with forecast update studied by Ozer *et.* al (2005) to the (GR) problem with $\delta > 0$ and $\lambda = 0$. They assume that the the distribution of the forecast update is IFR to obtain uniqueness results. Our analysis allows us to relax this assumption to ICFR.

Remark 6. If $\alpha+\beta=0$, recall that

$$\pi'(p)=\alpha-Cf(p)+wpf(p).$$

In this case, the montonicity condition is that $(wp-C)f(p)$ is decreasing. Further, the objective function is, indeed, submodular with p and $-\alpha$.

6. Extensions

6.1. The Non-Linear Model

6.1.1. Equivalence of "canonical" and "reduced" form

Analogous to section 2, we formulate the non-linear model in "canonical" form (3) and "reduced" form (4). We provide a generalization for the (GR) problem with piece-wise non-linear functions. Let

$$\Pi(p,X) = \begin{cases} A+\alpha(p)-u(X) & p\le X; \\ -(B+\beta(p)-v(X)) & p>X, \end{cases} \tag{3}$$

with $A+B\ge 0$, $\alpha(0)=\beta(0)=u(0)=v(0)=0$. α, β, u and v are continuous and differentiable. Further, $\alpha'(p)>0$ and $\alpha'(p)+\beta'(p)\ge 0$.

$$\Pi^R(p,X) = \begin{cases} \alpha(p) & p\le X; \\ -(C+\beta(p)-w(X)) & p>X, \end{cases} \tag{4}$$

Remark 7. The "canonical" and "reduced" forms are equivalent under the following transformations:

$$C=A+B, w(x)=u(x)+v(x).$$

We work with the "reduced" formulation and restrict our attention to the case when $\alpha'(p)+\beta'(p)>0$.

Define

$$\Delta(p)=\frac{C}{\alpha'(p)+\beta'(p)}$$

and

$$\Lambda(p) = \frac{(\alpha(p)+\beta(p))-w(p)}{(\alpha'(p)+\beta'(p))}.$$

6.1.2. Uniqueness condition Next, we identify uniqueness conditions on similar lines developed for the linear model. Assume that $\beta'(p) > 0$. First, note that

$$\pi'(p) = \beta'(p)\{[1 + \frac{\alpha'(p)}{\beta'(p)}]\bar{F}(p)[1 - [\Delta(p) + \Lambda(p)]\frac{f(p)}{\bar{F}(p)}] - 1\}.$$

To derive the sufficient condition for uniqueness of optimal solution, we restrict our attention to the class of problems where
(A1) $\beta(p)$ is concave, and,
(A2) $\frac{\alpha'(p)}{\beta'(p)}$ is decreasing.

We thus identify a sufficient condition for uniqueness of the solution to be that

$$[\Delta(p) + \Lambda(p)]\frac{f(p)}{\bar{F}(p)}$$

is monotone increasing.

From a practical perspective, and for tractability, we are interested in analyzing the case when $\alpha(p), \beta(p)$ and $w(p)$ belong to the same family, i.e.,

$$\alpha(p) = \alpha P(p),$$

$$\beta(p) = \beta P(p),$$

$$w(p) = wP(p),$$

where $P(p)$ is continuous and differentiable and satisfies $P(0) = 0$, $P'(p) > 0$. We note that (A2) is satisfied trivially for this case and (A1) is satisfied if $P(p)$ is concave. In this case, the sufficient condition reduces to requiring the monotonicity of

$$\frac{\delta + \bar{\lambda}P(p)}{P'(p)}\frac{f(p)}{\bar{F}(p)}$$

and the FOC reduces to

$$F(p) = \frac{\alpha}{\alpha + \beta} - \frac{\delta + \bar{\lambda}P(p)}{P'(p)}f(p).$$

Further, we note that

$$\frac{\delta + \bar{\lambda}P(p)}{P'(p)}$$

is increasing, since $P(p)$ is concave.

EXAMPLE 2. Consider the case when $P(p)$ is a pure polynomial, i.e,

$$P(p) = p^n, 0 < n \leq 1.$$

Then, the FOC reduces to

$$F(p) = \frac{\alpha}{\alpha+\beta} - \frac{\delta p^{1-n} + \bar{\lambda} p}{n} f(p).$$

Interestingly, when $\delta = 0$, the monotonicity condition is IGFR, similar to the case of $P(p) = p$.

EXAMPLE 3. Consider the case when $P(p)$ is exponential, i.e,

$$P(p) = 1 - e^{-np}, n > 0.$$

Then, the F.O.C. reduces to

$$F(p) = \frac{\alpha}{\alpha+\beta} - \frac{\delta e^{np} + \bar{\lambda}(e^{np} - 1)}{n} f(p).$$

We are now ready to generalize the ICFR(θ) property. Let X be a r.v. with p.d.f. $f(.)$ and c.d.f. $F(.)$. Let $0 \leq \theta \leq 1$. Let

$$G_X(x|\theta, P(x)) = \frac{(\bar{\theta} + \theta P(x))}{P'(x)} \frac{f(x)}{1 - F(x)}.$$

Recall that:

(1) X is IFR if $G_X(x|0, P(x) = x)$ is increasing in x.

(2) X is IGFR if $G_X(x|1, P(x) = x)$ is increasing in x.

Recall that $P(x)$ is continuous and differentiable and satisfies $P(x) = 0$, $P'(x) > 0$. Further, $P(x)$ is concave.

DEFINITION 2. X is said to be increasing in Generalized Composite Failure Rate with parameter $\theta \in [0,1]$ and function $P(x)$ (denoted by $IGCFR(\theta, P(x))$) if

$$G_X(x|\theta, P(x))$$

is monotone increasing in x.

LEMMA 3. *(a) (Monotonicity with θ) Let X be $IGCFR(\theta_0, P(x))$. Then,*
X is IGCFR(θ, $P(x)$), $\theta \geq \theta_0$.

(b) (Implication relations) Let $\theta \in [0,1]$. Then,
$IFR \Rightarrow IGCFR(\theta, P(x))$

PROOF. (a)

$$G_X(x|\theta, P(x)) = \frac{\bar{\theta} + \theta P(x)}{\bar{\theta}_0 + \theta_0 P(x)} G_X(x|\theta_0, P(x)).$$

The result follows by noting that

$$\frac{\bar{\theta} + \theta P(x)}{\bar{\theta}_0 + \theta_0 P(x)}$$

is increasing if $\theta \geq \theta_0$.

(b) The results follows from (a). □

EXAMPLE 4. Let X be a Pareto r.v. with $g_X(x|1) = \frac{\eta}{x}, x \geq 1, \eta > 0$. Then, X is $ICGFR(0, 1 - e^{-nx})$, $n \geq 0$. Recall that X is IGFR but not $ICFR(\theta)$ $\forall\ \theta \in (0,1)$.

6.2. The Multi-period (GR) Problem

In the finite-horizon (GR) problem, a risk-neutral decision-maker chooses p_t in period t, $t = 1, 2, ..., T$. However, due to a shock ϵ_t, the realized decision is $q_t = Z_t(p_t, \epsilon_t)$. Assume that the ϵ_t are independent and that p_t and ϵ_t are independent.

Let $L_t(p_t)$ be the one-period expected cost incurred in period t, i.e., $L_t(p_t) = -\pi_t(p_t)$. Define the penalty incurred from period to period (the switching cost function):

$$\delta(p_t, q_{t-1}) = \delta_1(q_{t-1} - p_t)^+ + \delta_2(p_t - q_{t-1})^+.$$

This formulation captures many of the applications that we studied in section 1. For example, in the (MN) problem, we permit salvage in every period at δ_1 and set the marginal cost of buying one unit at δ_2 ($\delta_2 > \delta_1$), with

$$\Pi(y) = \begin{cases} ry & y \leq D; \\ r_1 D & y > D. \end{cases}$$

In this case, $\epsilon_t = D_t$, the demand in period t and $Z_t(p_t, \epsilon_t) = p_t - \epsilon_t$. For the case of petty crime, p_t represents the bribe level demanded in period t and $Z_t(p_t, \epsilon_t) = p_t$. Another interesting application is the service level with noise example. Here, p_t is the inverse service level decision in period t and $Z_t(p_t, \epsilon_t) = p_t + \epsilon_t$.

Let $f_t(q_{t-1})$ be the optimal cost-to-go function. Then,

$$f_t(q_{t-1}) = min\{min_{p_t \geq q_{t-1}}\{G_t(p_t) - \delta_2 q_{t-1}\}, \{min_{p_t \leq q_{t-1}}\{g_t(p_t) + \delta_1 q_{t-1}\}\}$$

where

$$G_t(p_t) = \delta_2 p_t + L_t(p_t) + \alpha E[f_{t+1}(q_t)],$$

and

$$g_t(p_t) = -\delta_1 p_t + L_t(p_t) + \alpha E[f_{t+1}(q_t)].$$

Suppose that X is ICFR(θ) for $\theta = \frac{\bar{\lambda}}{\bar{\lambda} + \delta}$, whenever θ is defined. Let $Z_t(x_t, y_t)$ be an additive function of x_t and y_t.

PROPOSITION 5. *A target interval base-stock policy is optimal for the multi-period (GR) problem.*

PROOF. $L_t(p_t)$ is convex as X is ICFR(θ). Since $Z_t(.)$ is additive, $f_{t+1}(Z_t(p_t,\epsilon_t))$ is convex for every realization of ϵ_t whenever $f_{t+1}(.)$ is convex. The remainder of the proof follows a standard technique outlined in Porteus (Pp 116). □

7. Conclusions

We now summarize the results of the article. We develop a framework for analyzing commonly researched problems in the Operations Management Literature, which we call the Greed and Regret (GR) problem. The (GR) problem has applications in sales force compensation, petty crime, corruption, service levels, buffer sizes, inventory decisions, bidding and so on. We identify the Market-price Newsvendor (whose selling price depends on whether the supply or demand is tight) as an equivalent formulation of the (GR) problem with zero fixed penalty and show that its profit function is a convex combination of the Newsvendor and "Selling to the Newsvendor" problem (Lariviere and Portues (2001)).

Our main results are to prove several monotonicity properties for the (GR) problem and to identify a sufficient condition for uniqueness of the optimal solution, which we call Increasing Composite Failure Rate (ICFR). This property is weaker than the IGFR property but stronger than the IFR property. This class retains several properties of IGFR distributions (see Lariviere (2003) and Paul (2005)). Interestingly, we observe that the ICFR property is preserved only under "scale-up" and not under "scale-down", unlike the IGFR and IFR distributions. The IGFR property has received a lot of attention recently in the contract theory literature. We believe that our contribution here would help researchers working on various OM problems (for example, Wright *et.* al's (2006) observation on the (GR) problem in Revenue Management in the context of airline alliances).

Finally, we extend the analysis to a larger class of incentive and disincentive functions. We extend the ICFR property to the non-linear case and study some interesting results. This model captures many of the scale-effects which the linear model does not. Our final extension is to study the multi-period version of the (GR) problem.

Appendix A:

LEMMA 4. *(Equivalence of canonical and reduced form)*
The canonical formulation (1) *is equivalent to the reduced formulation* (2) *with:*

$$C = A + B,$$

and

$$w = u + v.$$

Proof. The result follows from the observation that when $C = A + B$ and $w = u + v$, $E[\Pi(p, X)]$ and $E[\Pi^R(p, X)]$ differ by a constant, $A - uE[X]$. The optimal p is identical for both problems. □

Lemma 5. *(Submodularity over the parameter space)*
Consider the (GR) problem in reduced form. Then, for $p > a$, $\pi(p)$ is (strictly) submodular with p and C, $-w$ and β.

Proof.

$$\pi'(p) = \alpha - Cf(p) - (\alpha + \beta - w)pf(p) - (\alpha + \beta)F(p).$$

Note that for $p > a$, $f(p)$, $pf(p)$, and $F(p)$ are strictly positive. Thus, $\pi'(p)$ is strictly decreasing with C, w and β (see Topkis (1998) and Sundaram (1999)). □

Lemma 6. *(Submodularity over the transformed parameter space)*
Let $\alpha > 0, \alpha + \beta > 0, w = \lambda(\alpha + \beta)$. Then, for $p > a$, $\pi(p)$ is (strictly) submodular with p and C, $-\lambda$ and
(i) $-\alpha$ if $\lambda \geq 1$
(ii) β if $\lambda \leq 1$.

Proof. The result follows by noting that $\pi'(p)$ may be re-written as:

$$\pi'(p) = \alpha[(1 - F(p)) - (1 - \lambda)pf(p)] - \beta[(1 - \lambda)pf(p) + F(p)] - Cf(p).$$

(see Topkis (1998) and Sundaram (1999)). □

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