The Role of Outside Options in Auction Design*

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Abstract

This paper studies revenue maximizing auctions when buyers' outside options depend on their private information. The set-up is very general and encompasses a large number of potential applications. The main novel message of our analysis is that with type-dependent non-participation payoffs, the revenue maximizing assignment of objects can crucially depend on the outside options that buyers face. Outside options can therefore affect the degree of efficiency of revenue maximizing auctions. We show that depending on the shape of outside options, sometimes an optimal mechanism will allocate the objects in an ex-post efficient way, and other times, buyers will obtain objects more often than it is efficient. Our characterization rings a bell of caution. Modeling buyers' outside options as being independent of their private information, *is* with loss of generality and can lead to quite misleading intuitions. Our solution procedure can be useful also in other models where type-dependent outside options arise endogenously, because, for instance, buyers can collude or because there are competing sellers. Keywords: *Optimal Multi Unit Auctions, Type Dependent Outside Options, Externalities, Mechanism Design, Type-Dependent Outside Options: JEL D44, C7, C72.*

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1. Introduction

This paper studies revenue maximizing allocation mechanisms for multiple objects in a very general model that allows buyers' outside payoffs to depend on their types. Objects can be heterogeneous, and they can be simultaneously complements for some buyers and substitutes for others. Buyers' payoffs may depend on the *entire* allocation of the objects, not merely on the ones they obtain, on their costs, which are private information, and on the costs of their competitors. Therefore the auction outcome may affect buyers irrespectively of whether they win any objects or not, and irrespectively of whether they participate in the auction or not. Non-participation payoffs may then very well depend on their cost, (type). Applications of this problem range from the allocation of positions in teams, to the allocation of airport take-off and landing slots, privatization, advertising and many more.

We show that with type-dependent non-participation payoffs, a revenue maximizing assignment of the objects can crucially depend on the outside options that buyers face. Therefore, outside options can affect the *degree of efficiency* of revenue maximizing auctions. Depending on the shape of outside payoffs, sometimes an optimal mechanism will allocate the objects in an *ex-post efficient* way. An important insight of monopoly theory is that a monopolist faces a trade-off between revenue maximization and efficiency, and sacrifices efficiency to increase revenue by selling less than it is socially desirable. The monopolist in this paper[1] does not always face this trade-off since a revenue maximizing allocation of the goods can be ex-post efficient. However, our analysis also shows that sometimes a revenue maximizing seller will sell "too much"compared to the socially desirable level. The second lesson is that with type-dependent outside options a revenue maximizing monopolist may induce inefficiencies of a different nature compared to the classical monopoly theory.

We now illustrate with a simple example how outside options can increase both revenue and efficiency of revenue maximizing mechanisms when outside payoffs are type-dependent. Suppose that a small company in Silicon Valley develops a valuable new technology. This company does not have the necessary infrastructure to reap its benefits, so it is essentially worthless for it. There is, however, a large firm, (say company A), that is willing to purchase it. The value of the new technology to company A is given by $500,000 - 500,000c$, where c is private information and uniformly distributed on [0,1]. We assume that irrespective of its cost realization, giving the technology to A maximizes the sum of consumer and producer surplus. If company A does not get the technology and no-one else does either, A's payoff is zero. From Myerson (1981) or from Riley and Samuelson (1981), we know that the best that the developer can do is to make a take-it-or leave-it offer to company A of $\$250,000$. Then, company A will get the invention only if its cost parameter is below $\frac{1}{2}$. This maximizes ex-ante expected revenue, which is $\$125,000$, but it is inefficient, because the developer is stuck half the time with a worthless (for it) invention, whereas company A would generate

[1] Such a comparison is legitimate, since the seller in our model is a multiproduct monopolist who instead of choosing revenue maximizing prices, is choosing revenue maximizing mechanisms.

non-negative payoff for all cost realizations.

Now suppose that the developer can make the invention publicly available by making it open source. This possibility changes A's outside options. The payoff of company A in case of open-sourcing is given by $100,000 - 1,000,000c$. If the developer considers threatening company A, in case it drops out of the sale, which threat should it use? The answer is not obvious since the developer does not know company A's cost parameter, so it does not know which alternative "hurts more."[2] If A is very efficient, ($c < \frac{1}{10}$), it would prefer the invention to become open-source, instead of the seller keeping it, since $100,000 - 1,000,000c > 0$, whereas the reverse is true if $c > \frac{1}{10}$. In this paper we show that the optimal threat is to tell A that in the event it does not participate, the seller keeps the invention with probability $\frac{1}{2}$, and makes it open source with probability $\frac{1}{2}$. Faced with this lottery, then company A's expected outside payoff is $50,000 - 500,000c$. Then, as we show,[3] the best that the seller can do is ask a price of \$450,000. Firm A *always(!)* agrees to buy the invention at the asking price of \$450,000, since $500,000 - 500,000c - 450,000 = 50,000 - 500,000c$ and hence its payoff is (weakly) greater than its outside option. Thus, the open source option, even though is never implemented, has an extraordinary effect on the revenue maximizing allocation. It guarantees a higher expected revenue (\$450,000), and makes the mechanism efficient. This is one of the main economic messages of this paper: when outside options depend on the buyers' private information, the seller can increase *both* revenue and efficiency by designing appropriate outside options. If the payoff from open sourcing did not depend on $A's$ cost parameter, then the allocation of the invention at the optimal mechanism would have been identical, and as inefficient, as in the case where open sourcing were not an option.[4] This example highlights the crucial role of outside options on the degree of efficiency of revenue maximizing mechanisms when outside payoffs are type-dependent.

Myerson (1981) studies revenue maximizing mechanisms of a single unit in an independent private value environment, where each buyer's outside option is a constant that is independent from the outcome of the auction. This seminal contribution establishes that at a revenue maximizing auction the seller gives the good to the buyer with the highest virtual surplus, whenever this virtual surplus is above the seller's valuation. Because a buyer's virtual surplus is equal to his valuation minus information rents, optimal auctions are inef-

[2] Both of these threats are credible. In case firm A does not participate in the sale, the seller is indifferent between keeping the invention and making it open source, and since there is nothing else the seller can do in that case, both these options are optimal.

[3] This example is, essentially, the one that is formally analyzed in section 5.2.

[4] If $A's$ payoff from open sourcing were independent of its type, say it were $-\$100,000$, then, from the work of Jehiel, Moldovanu and Stacchetti (1996) we know that the developer by threatening company A to make the invention open source, can extract payments even if company A does not get the technology, so long as it does not become open source. In this case the optimal auction will have an entry fee of \$100,000 and a take-it-or-leave-it offer of \$250,000. Company A will get the new technology when its cost is below $\frac{1}{2}$. Now the expected revenue for the developer will be higher and it will be \$225,000, but the optimal auction is inefficient, since there is trade only half the time, exactly as in the case without the open sourcing option.

ficient, even when buyers are ex-ante symmetric.[5] Jehiel, Moldovanu and Stachetti (1996), JMS'96, examine revenue maximizing mechanisms of a single object, where as in Myerson (1981), each buyer's outside option is a constant, but with the important difference that the outside option depends on the allocation of the object. The new insight of JMS'96 is that the seller increases *revenue* by choosing the appropriate outside options. In JMS'96 because outside options are type-independent the revenue maximizing *allocation* of the good is *never* affected by the outside options that buyers face. Only payments are affected. Therefore, the kind of inefficiencies that appear in Myerson (1981) are still present. A more recent paper with type-independent externalities is Aseff and Chade (2006).

In this paper we study revenue maximizing auctions when outside options can depend on buyers' types, and show that the revenue maximizing *allocation* of the goods crucially depends on the shape of the outside options that buyers face. The reason for this is that with type-dependent outside options the virtual surplus of an allocation is *"modified"* to account for the shape of the outside options. The shape of the outside options, together with the allocation mechanism determine the critical types, that is the types where the participation constraints bind. The "modified virtual surplus" of an allocation can be *equal* or *strictly greater* than its *actual* surplus. Depending on how the modified virtual surplus of an allocation compares to its actual surplus, a revenue maximizing mechanism can be ex-post efficient, as is the example in Section 5.2, or it may be "overselling" compared to the ex-post efficient level, as in the example presented in Section 5.1.

The dependence of the "modified virtual surplus" on the allocation through the vector of critical types makes the problem sometimes non-linear. Hence, a general, analytical solution seems intractable. Because of the possible nonlinearities, this problem is similar to Maskin and Riley (1984), who study revenue maximizing auctions with risk averse buyers. Fortunately, here we are able to identify a large class of environments where the problem becomes linear, as it is in Myerson (1981). In these cases, the vector of critical types does not depend on the allocation that the seller chooses, because, (roughly), buyers' outside payoffs have extreme slopes. The analytical solutions of these cases show the possibilities of efficiency and "overselling." We choose to state these results as possibilities, rather then to describe the complete list of cases where they would be true, because this seems like a very long and tedious task. Whether efficiency, "overselling" or "underselling" occurs depends on the vector of critical types. These features will be present also when revenue depends non-linearly in the assignment rule.

It is very important to stress that the virtual surplus is modified only when outside-payoffs are type-dependent. Thus, overselling *cannot* occur when there are externalities, (positive or negative), but the outside options are flat,[6] as is the case in JMS'96. Also the presence of externalities is just one instance where outside options may be type-dependent,

[5] This is because it is possible that the highest virtual surplus, (valuation minus information rents), is below the seller's valuation, whereas the highest valuation is above.

[6] This point is elaborated at the end of Section 6.1.

there can be many more, think for instance, a procurement setting where bidders have to give up the possibility of undertaking other projects in order to participate in the current auction.

Our model allows for an elegant description of a large number of allocation problems because it allows for multiple heterogeneous goods, type-dependent outside options and externalities; as well as for the goods to be simultaneously complements for some buyers, and substitutes for others. We now list a few of the potential applications of our model.

• *Allocation of rights to a new technology.* Our analysis could offer useful insights on the debate about how new technologies or ideas should be sold. In the example just discussed, we saw the crucial role of the presence of the open source option on the efficiency properties of the revenue maximizing mechanisms: it increased both revenue and efficiency. This is an important area, since the way property rights are assigned on new ideas and technologies does not only affect the way the particular ideas will be implemented in practice, but also the incentives to produce new ones.

• *Auctioning of advertisement slots on the internet, TV or radio.* Airtime for advertisements on TV and radio is often priced using conventional mechanisms. However, exploiting the presence of externalities is not far from what we already observe in reality. In Germany during the soccer world cup, advertisement slots were sold by category. For instance, a slot was allocated only to brewing companies. Then a potential buyer knew a priori that if it did not buy the slot, it will go to a competitor. Nowadays, companies like Yahoo! and Google auction-off their advertising slots and are thinking of optimal ways to do so. Our model fits very well many aspects of the problem these companies face: they are selling many advertising slots that can be heterogeneous, some slots may be substitutes and some complements of one another, and clearly buyers care about the slots that their competitors obtain.

• *Team formation.* Our model can be used to study a type of procurement auction where the buyer is an organization, (consulting firm, sports team), that wants to hire individuals to perform a task as a team. The compensation that an individual requires depends on who else will join the team. For instance, if individuals joining consist of gurus in the field, someone may consider the experience of working with such people so important, that he may be willing to participate with minimal compensation. On the other hand, if team members are of very poor quality the compensation that he requires may be higher.[7]

• *Optimal auction design with endogenous market structure.* Our model captures scenarios of auctions with endogenous market structure and generalizes previous work by Dana and Spier (1994), and Milgrom (1996).[8]

[7] These insights can be useful when one thinks about academic hiring. Clearly academics care a lot about the quality of their colleagues in absolute sense, and also relatively, meaning how good is the match.

[8] Gale (1990) also considers a variation of this problem but because he imposes a very strong super-additivity condition to the profit function, he shows that an optimal mechanism always gives all the "permits" to at most one buyer, so the market structure is always a monopoly.

Other applications include firm take-overs,[9] allocation of airport take-off and landing slots, and optimal bundling. We finish with a historical application.

• *A historical example.* The praetorian guard realized the additional benefits of running an auction when negative externalities are present. In the year 196 A.D. they killed the emperor Pertinax and, making a break with "tradition," decided not to hand over the title to someone else for a fixed price, but to run an auction. Historians[10] cite the fact that there was heavy overbidding, since participants were afraid that in case of not winning the auction, they would be killed by the next emperor, since they would be potential conspirators. This is an example of extreme negative externalities! The experiment was successful from the point of view of the guard, since the auction generated very high revenue, but was not repeated, probably since Didius Iulianus (the winner) lasted only 65 days as emperor and was killed after that, making next bidders reluctant to participate in another auction of this sort.

To summarize, our model is tractable, despite its generality, and has a very large number of potential applications. Our main message is that when outside options are type-dependent the revenue maximizing assignment of the objects will depend on them. The seller can then increase both revenue and efficiency by choosing the appropriate outside options. This issue seems to be known to practitioners, as it is suggested by the design of the UK spectrum auctions,[11] see for instance Klemperer (2004). Moreover, our solution technique of analyzing type-dependent outside options can be useful in models where type-dependent outside options arise endogenously, because buyers can collude or because there are competing sellers.

Other papers that study optimal multi-unit auctions when private information is single dimensional are Maskin and Riley (1989), who analyze the case of unit demands and continuously divisible goods, Gale (1990) who analyzes the case of discrete goods and super-additive valuations and, finally, Levin (1997) the case of complements. As in these papers, uncertainty in our model is single dimensional and buyers are risk neutral, but we allow for many goods, (that can be bundled any way the seller likes), multi-unit demands and payoff functions that allow for complements, substitutes and externalities. A number of papers on optimal multi-unit auctions model types as being multidimensional. With multi-dimensional types the characterization of the optimum is extremely difficult. Significant progress has been made, but no analytical solution, nor general algorithm is known. Important contributions there are Armstrong (2000), Avery and Hendershott (2000) and Jehiel and Moldovanu (2001). This paper is less general in the dimensionality of the types, but

[9] Externalities are of huge importance in firm take-overs: Recently (February 2004), Cingular bought AT&T wireless for $41 billion after a bidding war with Vodafone. Some perceive that the big winner of this sale will be Verizon even though it was not a participant in the auction (NY Times February 17, 2004 "Verizon Wireless May Benefit From Results of Auction").

[10] This is stated by Edward Gibbon, (1737-94), English historian, in his book *"The History of the Decline and Fall of the Roman Empire."*

[11] We are grateful to Sushil Bikhchandani for pointing this out.

much more general in all other dimensions.

This paper is also related to the literature on mechanism design with type-dependent outside options and most notably to the paper by Krishna and Perry (2000) who examine efficient mechanisms, whereas our focus is revenue maximization. Jehiel-Moldovanu (2001b) are also concerned with the design of efficient mechanisms. Lewis and Sappington (1989) study an agency problem where the outside option of the agent is type-dependent. Among other things, the fact that the critical type is not necessarily the "worst" one mitigates the inefficiencies that arise from contracting under private information. This feature also appears sometimes in our analysis, but we also show that sometimes inefficiencies are not reduced, but they change in nature, and the monopolist instead of selling too little, she sells too much. Jullien (2000) uses a dual approach to characterize properties of the optimal incentive scheme such as the possibility of separation, non-stochasticity, etc. In this paper we do not rely on dual methods. Other differences from Jullien are that we allow for multiple agents and for the principle to choose the outside options that agents face.

Externalities, and hence outside options, are also type dependent in Jehiel-Moldovanu and Stacchetti (1999), JMS'99, who consider the design of optimal auctions of a single unit in the presence of type-dependent externalities and *multi-dimensional types.* A buyer's type is a vector, where each component indicates his/her utility as a function of who gets the object. In JMS'99 the multi-dimensionality of types makes the solution of the general problem intractable.[12]

We conclude with a brief outline of our paper. In Section 2 we introduce the model. Our analysis starts in Section 3 by establishing properties of feasible mechanisms, that is mechanisms that satisfy incentive, voluntary participation, and resource constraints. Section 4 characterizes revenue maximizing mechanisms. In Section 5 we present two largely self contained examples. A reader can get a flavor of our findings by looking directly at these examples.

2. The model

A risk neutral seller owns N indivisible, possibly heterogeneous, objects that are of 0 value to her and faces I risk-neutral buyers. Both N and I are finite natural numbers. The seller (indexed by zero) can bundle these N objects in any way she sees fit. An allocation z is an assignment of objects to the buyers and to the seller. It is a vector with N components, where each component stands for an object and it specifies who gets it, therefore the set of possible allocations is finite and given by $Z \subseteq [I \cup \{0\}]^N$. Buyer i's valuation from

[12] JMS (1999) restrict attention auctions where (1) the buyers submit scalar bids and (2) the seller transfers the object to one of the buyers for sure, and show that a second-price auction is an optimal mechanism among this class. They also slightly relax the "for sure sale" assumption, by allowing for reserve prices and show that with two buyers, this auction remains optimal.

allocation z is denoted by $\pi_i^z(c_i, c_{-i})$ and it depends on buyer $i's$ cost parameter c_i and on the cost parameters of all the other buyers c_{-i}. Values are therefore *interdependent.* Buyer i's cost parameter c_i is private information and is distributed on $C_i = [\underline{c_i}, \overline{c_i}]$, with $0 \leq \underline{c_i} \leq \overline{c_i} < \infty$, according to a distribution F_i that has a strictly positive and continuous density f_i. All buyers' types are *independently* distributed. We use $f(c) = \times_{i\in I} f_i(c_i)$, where $c \in C = \times_{i\in I} C_i$ and $f_{-i}(c_{-i}) = \times_{\substack{j\in I \\ j\neq i}} f_j(c_j)$.

We assume that, for all $i \in I$, $\pi_i^z(\cdot, c_{-i})$ is *decreasing*, *convex* and *differentiable* for all z and c_{-i}. We impose no restrictions on how π_i depends on z nor c_{-i}. This formulation allows for buyers to be demanding many objects, which may be complements or substitutes, and for externalities, that can be type and identity dependent. It is very well possible that $\pi_i^z(c_i, c_{-i}) \neq 0$ even when the allocation z does not include any objects for i. An instance of that, is a situation where buyers are firms competing in different markets, and whatever happens in the current sale will affect their positioning and interaction relative to the other buyers in other markets. More importantly, an allocation may affect buyer i even if he is not taking part in the auction, which implies that non-participation payoffs may depend on $i's$ type. Type-dependent non-participation payoffs are the key force behind our new insights.

The objective of the seller is to design a mechanism that maximizes expected revenue, and buyers aim to maximize expected surplus.

Mechanisms

By the revelation principle it is without loss of generality to restrict attention to truth-telling equilibria of direct revelation games where *all* buyers participate. To see this, note that the set of possible allocations is $Z = \{I \cup \{0\}\}^N$, which is larger, the more buyers participate. The seller can then replicate an equilibrium outcome of some auction, where a subset of the buyers for some realizations of their private information do not participate, with a mechanism where all these buyers participate, and by mapping their corresponding reports to the allocation that would have prevailed at the equilibrium of the original auction game.

A *direct revelation mechanism,(DRM)*, $M = (p, x)$ consists of an *assignment rule* $p : C \longrightarrow \Delta(Z)$ and a *payment rule* $x : C \longrightarrow \mathbb{R}^I$.

The assignment rule specifies the probability of each allocation for a given vector of reports. We denote by $p^z(c)$ the probability that allocation z is implemented when the vector of reports is c. Observe that the assignment rule has as many components as the number of possible allocations. The payment rule x specifies, for each vector of reports c, a vector of payments, one for each buyer.

Non-Participation Assignment Rules

In the event that a buyer i does not participate in the mechanism, then his payoff is determined by the allocation that prevails when he is not around, which we denote by p^{-i}. A *non-participation assignment rule* specifies a p^{-i} for each $i \in I$. We are assuming that the seller has the commitment power to choose the non-participation assignment rule, in such a way, as to maximize ex-ante expected revenue.[13] The seller chooses p^{-i} out of $\mathcal{P}^{-i} = \{p^{-i} : C_{-i} \to \Delta(Z^{-i})\}$, where $Z^{-i} \subset Z$ is the set of allocations that are feasible without i. If the seller does not have such commitment power, then $\mathcal{P}^{-i}$ contains all the assignment rules that are feasible *and* optimal when i is not around (therefore $\mathcal{P}^{-i} \subset \{p^{-i} : C_{-i} \to \Delta(Z^{-i})\}$). It is worth stressing, that the qualitative features of our results depend on the fact that outside payoffs are type dependent, and not on whether the seller has the power to choose p^{-i} or not.

We now proceed to describe the seller's and the buyers' payoffs.

Payoffs from Participation

The interim expected utility of a buyer of type c_i when he participates and declares c_i' is

$$U_i(c_i, c_i'; (p, x)) = E_{c_{-i}} \left[\sum_{z \in Z} (p^z(c_i', c_{-i}) \pi_i^z(c_i, c_{-i})) - x_i(c_i', c_{-i}) \right].$$

Let also $V_i(c_i) \equiv U_i(c_i, c_i; (p, x))$.

Payoffs from Non-Participation

The payoff that accrues to buyer i from non participation depends on what allocations will prevail in that case, which are determined by p^{-i}, and on his type c_i, and it is given by

$$\underline{U}_i(c_i, p^{-i}) = E_{c_{-i}} \left[\sum_{z \in Z^{-i}} (p^{-i})^z (c_{-i}) \pi_i^z (c_i, c_{-i}) \right],$$

where $(p^{-i})^z$ denotes the probability assigned to allocation z by p^{-i}. The fact that $i's$ non-participation payoffs depend on his type is arguably the most crucial feature of our model.

[13] This is also the assumption in JMS'96. In that paper there is a single object for sale, and constant with respect to type, outside options.

For an illustration of participation and non-participation payoffs, see **Figure 1.**

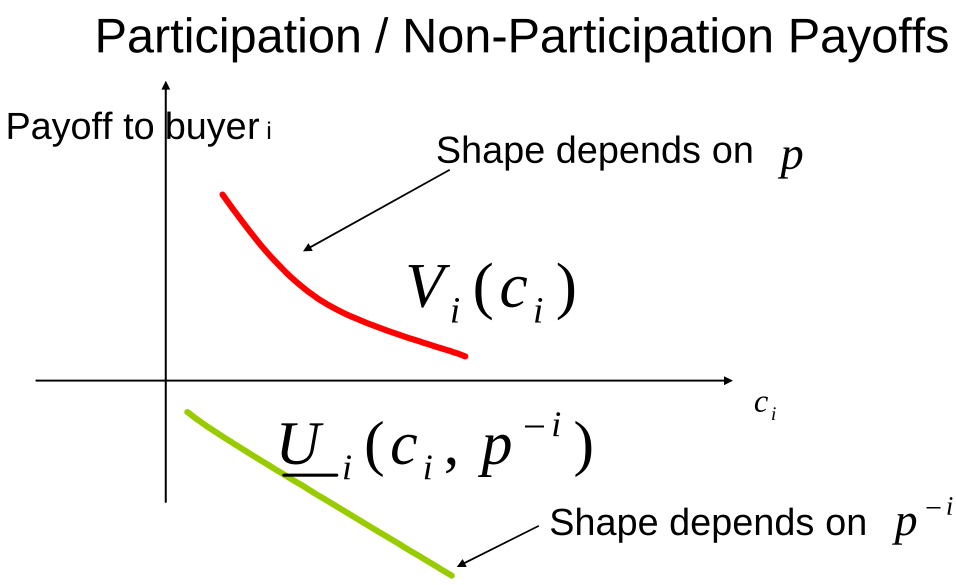

Figure 1

We proceed to describe the timing

Timing

Stage 0: The seller chooses a mechanism (p, x) and p^{-i}, for all i.

Stage 1: Buyers decide whether to participate or not, and which report to make. If all make a report, the mechanism determines the assignment of objects and the payments. If buyer i decides not to participate, the objects are assigned according to $\{p^{-i}\}$. If more than one buyers fail to participate, we assume that the seller keeps the objects.

In order for a mechanism to be feasible it must be the case that all buyers choose to participate and to report their true type. We are capturing a one-shot scenario. Given that others participate and tell the truth about their types, is it a best response for buyer i to participate and tell the truth about his type? In such a one-shot scenario, buyers are not making inferences about the types of buyer i in the event that buyer i does not participate.

We now provide a formal definition of what it entails for a direct revelation mechanism to be feasible.

Feasible Mechanisms

Definition 1. (Feasible Mechanisms) For a given non-participation assignment rule, $(p^{-i})_{i\in I}$, we say that a mechanism (p,x) is *feasible* iff it satisfies

(IC) "incentive constraints," a buyer's strategy is such that
$U_i(c_i, c_i; (p,x)) \geq U_i(c_i, c_i'; (p,x))$ for all $c_i, c_i' \in C_i$, and $i \in I$
(PC) "voluntary participation constraints,"
$U_i(c_i, c_i; (p,x)) \geq \underline{U}_i(c_i, p^{-i})$ for all $c_i \in C_i$, and $i \in I$
(RES) "resource constraints" $\sum\limits_{z\in Z} p^z(c) = 1$, $p^z(c) \geq 0$ for all $c \in C$

Summarizing, feasibility requires that p and x are such that buyers (1) prefer to tell the truth about their cost parameter, (2) buyers choose voluntarily to participate in the mechanism and (3) p is a probability distribution over Z.[14] We now state the seller's problem.

The Seller's Problem

With the help of the revelation principle the seller's problem can be written as

$$\max \int\limits_C \sum_{i=1}^{I} x_i(c) f(c) dc \tag{1}$$

subject to (p,x) being "feasible."

This completes the description of our model and the seller's problem. We proceed with the analysis of it. Proofs of the results not presented in the main text can be found in the Appendix A.

3. Implications of Incentive and Participation Constraints

The seller's objective is to maximize expected revenue subject to incentive, participation and resource constraints. This section studies implications of these constraints.

Implications of Incentive Compatibility

Given a DRM (p,x) buyer $i's$ maximized payoff,

$$V_i(c_i) = \max_{c_i'} \int\limits_{C_{-i}} \left(\sum_{z\in Z} p^z(c_i', c_{-i}) \pi_i^z(c_i, c_{-i}) - x_i(c_i', c_{-i}) \right) f_{-i}(c_{-i}) dc_{-i}, \tag{2}$$

is convex, since it is a maximum of convex functions. In the next Lemma we show that the incentive constraints translate into the requirement that the derivative of V_i

$$P_i(c_i) \equiv \int\limits_{C_{-i}} \sum_{z\in Z} p^z(c_i, c_{-i}) \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i}, \tag{3}$$

[14] Notice that Z contains the allocation where the seller keeps all the objects, thus $\sum\limits_{z\in Z} p^z(c) = 1$.

(more precisely a selection from its subgradient, which is single valued almost surely), evaluated at the true type is weakly increasing.[15]

Lemma 1 *A mechanism (p, x) is incentive compatible iff*

$$P_i(c_i') \geq P_i(c_i) \qquad \text{for all } c_i' > c_i \tag{4}$$

$$V_i(c_i) = V_i(\overline{c}_i) - \int_{c_i}^{\overline{c_i}} P_i(s)ds \quad \text{for all } c_i \in C_i. \tag{5}$$

With the help of Lemma 1 and using standard arguments, we can write buyer i's expected payment as a function of the assignment rule p, and the payoff that accrues to his worst type,[16] $V_i(\overline{c}_i)$

$$\int_C x_i(c)f(c)dc = \int_C \sum_{z\in Z} p^z(c_i, c_{-i}) \left(\pi_i^z(c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i} \right) f(c)dc - V_i(\overline{c}_i).$$

Let

$$J_z(c) \equiv \sum_{i=1}^{I} [\pi_i^z(c_i, c_{-i}) + \frac{F_i(c_i)}{f_i(c_i)} \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i}]$$

denote the virtual surplus of allocation z. Notice that we are summing over all buyers because an allocation may affect all of them, and not just the ones that obtain objects. Therefore the virtual surplus of allocation z may depend on the whole vector of types.[17]

Using this definition, the seller's objective function can be rewritten as

$$\sum_{i=1}^{I} \int_C x_i(c)f(c)dc = \int_C \sum_{z\in Z} p^z(c)J_z(c)f(c)dc - \sum_{i=1}^{I} V_i(\overline{c}_i). \tag{6}$$

Now we turn to examine the implications of the participation constraints.

Implications of Participation Constraints

Since the seller's revenue is decreasing in $V_i(\overline{c}_i)$, at a solution this term must be as small as possible subject to the participation constraint $V_i(c_i) \geq \underline{U}_i(c_i, p^{-i})$ for all $c_i \in C_i$. This observation implies that there will be at least one type c_i where $V_i(c_i) = \underline{U}_i(c_i, p^{-i})$. We call this *the critical type of* i and denote it by $c_i^*(p, p^{-i})$. In the event that there is more

[15] In the classical case, where there is only one object and $i's$ payoff from obtaining the object is v_i, (see Myerson (1981)), the analog of P_i is $P_i(v_i) = \int_{V_{-i}} p(v_i, v_{-i})f_{-i}(v_{-i})dv_{-i}$.

[16] For more details see Appendix.

[17] In Myerson (1981) virtual valuations are buyer-specific. For buyer i we have $J_i(v_i) = v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$, ($v_i$ is i's valuation for the object).

than one type where $V_i(c_i) = \underline{U}_i(c_i, p^{-i})$, then any one of them will do. From (5) we have that $V_i(c_i)$ =constant$-\int\limits_{c_i}^{\overline{c_i}} P_i(s)ds)$, so c_i^* must be such that

$$c_i^*(p, p^{-i}) \in \arg\min_{c_i} \left[-\int_{c_i}^{\overline{c}_i} P_i(s)ds - \underline{U}_i(c_i, p^{-i}) \right]. \tag{7}$$

See **Figure 2.**

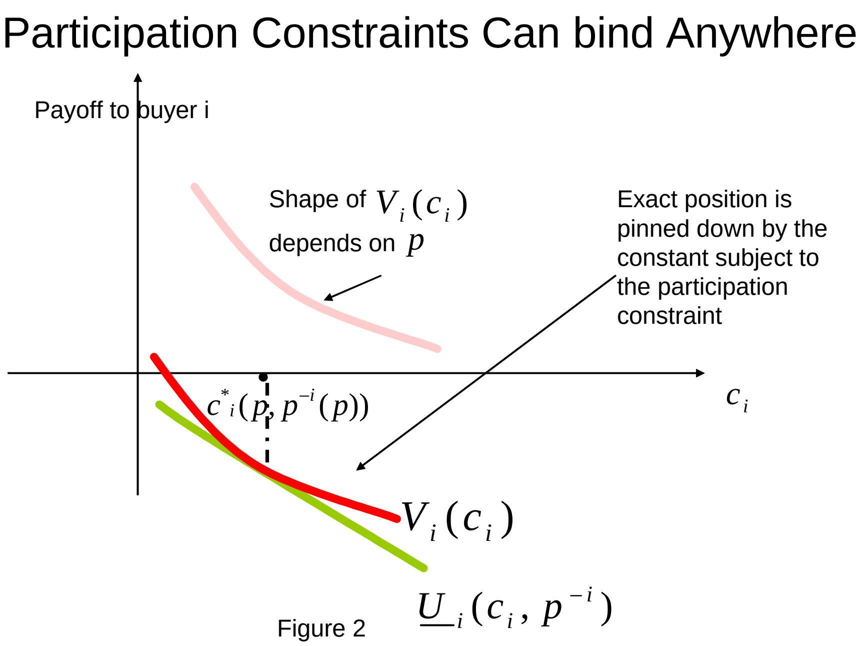

Figure 2

Note that (7) implies that if c_i^* is interior, V_i and $\underline{U}_i$ must be tangent at c_i^*, namely it must be the case that

$$\frac{\partial \underline{U}_i(c_i^*, p^{-i})}{\partial c_i} \in \partial V_i(c_i^*). \tag{8}$$

If we are at a corner, that is $c_i^* = \overline{c}_i$ then it must be the case that $\frac{\partial \underline{U}_i(\overline{c}_i, p^{-i})}{\partial c_i} \geq \frac{dV_i(\overline{c}_i)}{dc_i}$, and if we are at $c_i^* = \underline{c}_i$ then it must hold that $\frac{\partial \underline{U}_i(\underline{c}_i, p^{-i})}{\partial c_i} \leq \frac{dV_i(\underline{c}_i)}{dc_i}$. Moreover, (7) implies that at c_i^* we have that

$$V_i(c_i^*) = \underline{U}_i(c_i^*, p^{-i}) \tag{9}$$

and from a generalization of the Fundamental Theorem of Calculus (see Krishna and Maenner (2001)), and incentive compatibility, it follows that

$$V_i(\overline{c}_i) = \underline{U}_i(c_i^*(p, p^{-i}), p^{-i}) + \int\limits_{c_i^*(p, p^{-i})}^{\overline{c_i}} P_i(s)ds. \tag{10}$$

From (10) we see that $V_i(\overline{c}_i)$ depends on p through two channels: P_i and $c_i^*(p, p^{-i})$.

Moreover, as already discussed, p^{-i} is often chosen by the seller in order to minimize $V_i(\overline{c}_i)$, namely

$$p^{-i}(p) \in \arg \min_{\rho^{-i} \in \mathcal{P}^{-i}} \underline{U}_i(c_i^*(p, \rho^{-i}), \rho^{-i}) + \int\limits_{c_i^*(p,\rho^{-i})}^{\overline{c_i}} P_i(s)ds. \tag{11}$$

For each assignment of the objects, p, there is a potentially different optimal "threat" $p^{-i}(p)$, which can be random. The dependence of p^{-i} on p adds an additional level of complication.

By substituting a solution of the program described in (11) into (10), we have that at an optimum it must be the case that

$$V_i(\overline{c}_i; p, p^{-i}(p)) = \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}) + \int\limits_{c_i^*(p,p^{-i}(p))}^{\overline{c_i}} P_i(s)ds. \tag{12}$$

Modified Virtual Surpluses

We now proceed to demonstrate how the presence of type-dependent outside options modifies the virtual surpluses of allocations. By substituting (12) into (6), the objective function of the seller's problem can be rewritten as

$$\int\limits_C \sum_{z \in Z} p^z(c) J_z(c) f(c) dc - \sum_{i=1}^{I} \left[\underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}) + \int\limits_{c_i^*(p,p^{-i}(p))}^{\overline{c_i}} P_i(s)ds \right]. \tag{13}$$

Recalling that $P_i(c_i) = \int\limits_{C_{-i}} \sum\limits_{z \in Z} p^z(c) \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i}$, and by rearranging the terms in (13), we can rewrite it as

$$\int\limits_C \sum_{z \in Z} p^z(c) \left[J_z(c) - \sum_{i=1}^{I} 1_{c_i \geq c_i^*(p,p^{-i}(p))} \frac{\partial \pi_i^z(c)}{\partial c_i} \frac{1}{f_i(c_i)} \right] f(c) dc - \sum_{i=1}^{I} \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}).$$

We define the *"modified virtual surplus"* of allocation z by

$$\hat{J}_z(c) \equiv J_z(c) - \sum_{i=1}^{I} 1_{c_i \geq c_i^*(p,p^{-i}(p))} \frac{\partial \pi_i^z(c)}{\partial c_i} \frac{1}{f_i(c_i)}. \tag{14}$$

Observe that the modified virtual surplus depends on p and on p^{-i} through $c_i^*(p, p^{-i}(p))$, which depends on the shape of the participation payoffs, which are determined by p, and on the shape of non-participation payoffs, which are determined by $\{p^{-i}\}_{i \in I}$.

It is useful to compare the modified virtual surplus of an allocation z, $\hat{J}_z$, with the virtual surplus of that allocation, J_z, and with the actual surplus of that allocation S_z, which is given by $S_z(c) = \sum_{i=1}^{I} \pi_i^z(c_i, c_{-i})$. This is interesting because the degree of efficiency of a revenue maximizing mechanism depends on these comparisons.

If $c_i^* = \bar{c}_i$ for all i the modified virtual surplus coincides with the virtual surplus hence

$$\hat{J}_z(c) = J_z(c). \tag{15}$$

This is because the virtual surplus is modified only for $c_i \geq c_i^*$. The condition $c_i^* = \bar{c}_i$ for all i holds when outside options are type-independent as is the case in in Myerson (1981) and in JMS'96. To see this, note that if outside options are type-independent and equal to $\underline{U}_i(p^{-i})$ for all c_i, then because $V_i(c_i)$ is decreasing in c_i it is follows immediately that the participation constraint will be binding at the highest cost type, namely $c_i^* = \bar{c}_i$, irrespectively of the exact shape of V_i, which depends on p. For an illustration see **Figure 3.**

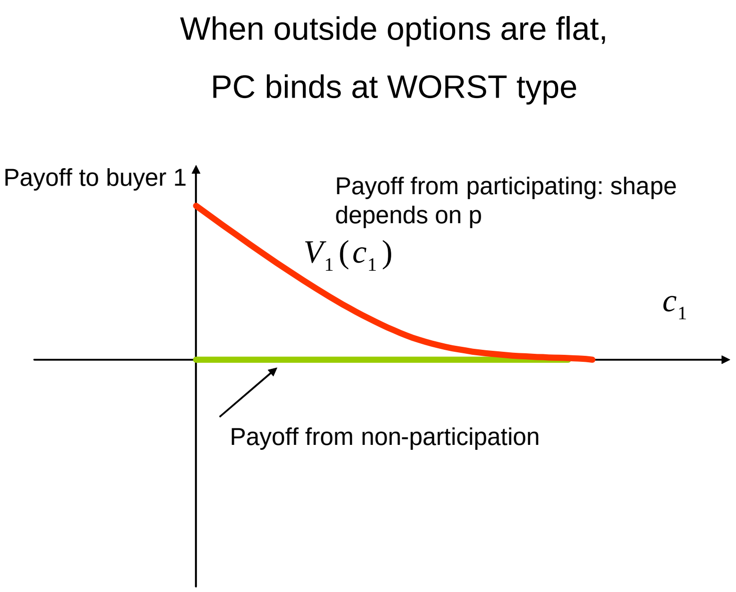

Figure 3

If on the other hand, $c_i^* = \underline{c}_i$ for all i, then[18]

$$\hat{J}_z(c) = J_z(c) - \sum_{i=1}^{I} \frac{\partial \pi_i^z(c)}{\partial c_i} \frac{1}{f_i(c_i)}, \tag{16}$$

which can be rewritten as

$$\hat{J}_z(c) = \sum_{i=1}^{I} \left[\pi_i^z(c_i, c_{-i}) + \frac{F_i(c_i) - 1}{f_i(c_i)} \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i} \right]. \tag{17}$$

[18] This is the case in the example we study in subsection 5.1.

In this case $\hat{J}_z(c) > J_z(c)$ because $\sum_{i=1}^{I} \frac{\partial \pi_i^z(c)}{\partial c_i} \frac{1}{f_i(c_i)}$ is negative, which follows from the fact that π_i^z is decreasing in c_i. Moreover, since the amount $\left(\frac{F_i(c_i)-1}{f_i(c_i)} \frac{\partial \pi_i^z(c_i,c_{-i})}{\partial c_i} \right)$ is positive, we also have that the "modified virtual surplus" of allocation z, is actually larger than the actual surplus of allocation z, that is $\hat{J}_z(c) \geq S_z(c)$.

Finally, when c_i^* is interior[19] for all i, namely $c_i^* \in (\underline{c}_i, \bar{c}_i)$, then depending on how a vector (c_i, c_{-i}) compares to (c_i^*, c_{-i}^*), $\hat{J}_z$ differs. Take for instance a $(\tilde{c}_i, \tilde{c}_{-i})$, where for all i we have that $\tilde{c}_i < c_i^*$, then it holds that $\hat{J}_z(\tilde{c}) = J_z(\tilde{c})$, as in (15), and at that $\tilde{c}$ the modified virtual surplus is less than $S_z(\hat{c})$. Now take a $(\hat{c}_i, \hat{c}_{-i})$, where for all i we have that $\hat{c}_i \geq c_i^*$, then it holds that $\hat{J}_z(\hat{c}) = J_z(\hat{c}) - \sum_{i=1}^{I} \frac{\partial \pi_i^z(\hat{c})}{\partial c_i} \frac{1}{f_i(\hat{c}_i)}$, as in (16), and at $\hat{c}$ we have that $\hat{J}_z(\hat{c}) > J_z(\hat{c})$ and $\hat{J}_z(\hat{c}) \geq S_z(\hat{c})$. For a vector (c_i, c_{-i}) where $c_i > c_i^*$ for some i, and $c_j \leq c_j^*$ for some j, we can see from (14), there is not modification to J_z for j, but there is for i. Then we can still conclude that $\hat{J}_z(c) \geq J_z(c)$, but depending on the exact comparison of (c_i, c_{-i}) with (c_i^*, c_{-i}^*) both $\hat{J}_z(c) \geq S_z(c)$ and $\hat{J}_z(c) < S_z(c)$ are possible.

How the modified virtual surplus of an allocation, (the $\hat{J}_z$), with the actual virtual surplus of that allocation, (the S_z), is important because, as we will see later, it affects the degree of efficiency of the revenue maximizing mechanisms, which are studied in the following section.

4. Optimal Mechanisms

Here we put together all the implications we have derived in the previous section, and describe the conditions revenue maximizing mechanisms satisfy.

Using (14) the seller's objective function given by (13) can be rewritten as

$$\int_C \sum_{z \in Z} p^z(c) \hat{J}_z(c) f(c) dc - \sum_{i=1}^{I} \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}). \tag{18}$$

The following Proposition characterizes the problem solved by revenue maximizing mechanisms.

Proposition 2 *If in a mechanism (p, x) the assignment function p satisfies resource constraints, (4), and maximizes (18), with $c_i^*(p, p^{-i})$ given by (7), and the payment function x for all i is given by:*

$$x_i(c) = \sum_{z \in Z} p^z(c) \pi_i^z(c) + \int_{c_i}^{\overline{c_i}} \sum_{z \in Z} p^z(s, c_{-i}) \frac{\partial \pi_i^z(s, c_{-i})}{\partial s} ds - V_i(\bar{c}_i; p, p^{-i}(p)), \tag{19}$$

with $V_i(\bar{c}_i; p, p^{-i}(p))$ given by (12), then the mechanism is optimal.

[19] This is the case in the example we study in subsection 5.2.

Proof. We have already argued why at an optimal mechanism, there must exists at list one type for each buyer where the participation constraint binds. We have called this type $c_i^*(p, p^{-i})$, and it satisfies (7). With the help of (7) we got (9). These two equations are implications of the participation constraints on the solutions.

The implications of the incentive constraints are that revenue can be expressed as in (6). Combining this, with the implications of the participation constraints, namely (7) and (9) we showed how we can express revenue by (18).

Now in order for a mechanism to be a valid solution it must have an allocation rule $\hat{p}$ that satisfies (4), and resource constraints.

Finally, if in a mechanism the payment rule is given (19), then for all $i \in I$, $i's$ payoff given p and $p^{-i}(p)$ subject to the participation constraints is minimized, since the type c_i^* is indifferent between participation or not. To see this, note that by substituting (12) into (19), and taking expectations with respect to c_{-i}, we obtain that

$$\int_{C_{-i}} x_i(c) f_{-i}(c_{-i}) dc_{-i} = \int_{C_{-i}} \left[\sum_{z \in Z} p^z(c) \pi_i^z(c) + \int_{c_i}^{\overline{c_i}} \sum_{z \in Z} p^z(s, c_{-i}) \frac{\partial \pi_i^z(s, c_{-i})}{\partial s} ds \right] f_{-i}(c_{-i}) dc_{-i}$$
$$- \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}) - \int_{c_i^*(p, p^{-i}(p))}^{\overline{c_i}} P_i(s) ds. \tag{20}$$

By recalling (3), (20) implies that

$$V_i(c_i) = \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}) - \int_{c_i}^{c_i^*(p, p^{-i}(p))} P_i(s) ds,$$

from which we immediately get that

$$V_i(c_i^*) = \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}).$$

From these considerations it follows that a mechanism (p, x) that satisfies all these conditions is optimal. ■

Note that Proposition 2 is analogous to Lemma 3 in Myerson (1981). As in that paper, we have revenue equivalence. Any two mechanisms that allocate the objects in the same way and give the same expected payoff to the worst type, generate the same revenue. There are however, important differences. The most important one is that in our problem the objective function can depend non-linearly on p. The reason is that $c_i^{*\prime}s$ may depend on non-linearly on p directly and through $p^{-i}(p)$, and revenue depends on c_i^* through $\hat{J}_z$, and the term $\sum_{i=1}^{I} \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i})$.

In Myerson (1981) and JMS (1996) c_i^* is *always* equal to $\overline{c}_i$ for all i because non-participation payoffs are independent of types. This implies that the modified virtual surplus is equal to the virtual surplus and independent of the assignment rule p. In this case, a revenue maximizing p is independent of the outside options that buyers face, and it has a simple characterization, because revenue is always linear in p. This is true even if, as in this paper and in JMS (1996), the seller can choose p^{-i}. The reason is that, when outside options give a type-independent payoff, they are essentially just a number. All the seller needs to do is to choose the option that guarantees the lowest number for i. In that case optimal threats p^{-i} are independent of p and deterministic. In contrast, with type-dependent outside options p_{-i} can depend on p, can be random and cannot be chosen by simple inspection, as we illustrate in Example 5.2.

From the previous discussion, it is clear the impossibility of finding an analytical expression for $p's$ that maximizes (13) for all cases because the seller's objective function is non-linear[20] in p. Fortunately, the problem has enough structure to allow the use of variational methods. In particular, if the functions $\pi_i^z(\cdot, c_{-i})$ are smooth enough, then $c_i^*(p, p^{-i}(p))$ is a differentiable function of p, thus guaranteeing that the objective function is differentiable, and hence continuous. It is not hard to show that the feasible set is sequentially compact. A continuous function over a sequentially compact set has a maximum. The solution will depend on the particular shapes of π_i^z and of the distributions F_i. This environment is more complicated than the ones considered by Jullien (2000), because there are multiple agents and the seller can choose the outside options. However in Figueroa and Skreta (2005) we show that the problem often, (but not always), reduces to one with essentially exogenous non-participation assignment rules. Unfortunately, the difficulties arising from having a non-linear objective function remain. In that respect, Proposition 2 is analogous to Theorem 8 in Maskin and Riley (1984), who characterize revenue maximizing auctions with risk averse buyers. As here, in that paper too, the non-linear nature of the program prohibits an analytical expression in general.

Fortunately, we are able to identify interesting sub-classes of problems where the problem becomes linear and hence analytical solutions can be obtained through a procedure similar to the one used in Myerson (1981). As we later argue, these classes of problems are by themselves economically relevant, and allow us to analyze the qualitative effects that type-dependent outside options have on revenue maximizing mechanisms, and to compare them to the particular case of type-independent ones. This by no means implies that they are the only relevant ones, but their analytical tractability is used to illuminate the more general role of outside options on the shape of revenue maximizing mechanisms.

4.1 Optimal Mechanisms when Revenue is Linear in p

Whether the problem turns out to be linear or not, depends on how sensitive the outside payoff of a buyer is with respect to his own type, relative to the sensitiveness of the payoffs

[20] The objective function is non-linear in p when $c_i^*(p, p^{-i}(p))$ depends on p. For an example see Appendix B.

received if the buyer actually participates. Many cases with interesting economic insights turn out to be linear, as we illustrate in our examples in Section 5. They include the case where outside options can depend on p and on the type of competitors, but not on the buyer's type. They also include the somewhat opposite polar case, where the outside option depends very strongly on the buyer's type, and an intermediate case where both options are present: the buyer can be threatened with an allocation that yields him a type independent payoff, and with an allocation where the payoff is very sensitive to type.

We start by describing under what circumstances revenue will be linear in p. In one sentence, revenue is linear in p in cases where c_i^* does not depend on p, when p^{-i} is chosen optimally. This can occur in many cases, like the three ones we just described. We analyze them in detail in what follows, since they suffice to illustrate the main economic insights of the influence of outside options in the shape of revenue maximizing mechanisms. It is important to stress that all the conditions that we will be discussing are imposed only on the shape of $\pi_z^i(\cdot, c_{-i})$ with $z \in Z^{-i}$.

4.1.1 Environments where Revenue is Linear in p

We now present the three environments described before. A more detailed description can be found in Appendix C.

In what follows we use the notation:

$$\bar{\pi}_i^z(c_i) \equiv \int_{C_{-i}} \pi_i^z(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}.$$

Case 1: Flat Payoff from Worst Allocation for i

Suppose that there is an allocation in $z_i^F \in Z^{-i}$, that gives i a type-independent payoff and satisfies (with some abuse of notation)

$$\pi_i^{z_i^F}(c_{-i}) \leq \pi_i^z(c_i, c_{-i}) \text{ for all } z \in Z^{-i} \text{ and } c_i \in C_i.$$

Then, an optimal outside option from the seller's perspective, is $(p^{-i})^{z_i^F} = 1$, since it solves, for all p

$$p^{-i}(p) \in \arg \min_{\rho^{-i} \in \mathcal{P}^{-i}} \underline{U}_i(c_i^*(p, \rho^{-i}), \rho^{-i}) + \int_{c_i^*(p,\rho^{-i})}^{\overline{c_i}} P_i(s) ds \tag{21}$$

In that case we have (see Figure 3)

$$\begin{aligned} c_i^*(p, p^{-i}(p)) &= \bar{c}_i \text{ and} \\ \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}(p)) &= \bar{\pi}_i^{z_i^F}(\bar{c}_i). \end{aligned} \tag{22}$$

Environments that fall in this category are in Myerson (1981) and in JMS (1996).

Case 2: Very Steep Payoff from Worst Allocation for i

Another case, is the polar opposite of the previous one. Here the worst allocation for buyer i is type dependent, and very sharply so. More precisely, there exists an allocation $z_i^S \in Z^{-i}$, at which $i's$ payoff is very sensitive to type, and guarantees the lowest payoff at $\underline{c}_i$:[21]

$$\begin{aligned} \frac{d\overline{\pi}_i^{z_i^S}(c_i)}{dc_i} &\leq \frac{d\overline{\pi}_i^z(c_i)}{dc_i} \text{ for all } z \in Z \\ \overline{\pi}_i^{z_i^S}(\underline{c}_i) &\leq \overline{\pi}_i^z(\underline{c}_i) \text{ for all } z \in Z. \end{aligned}$$

It is easy to see (the details are in Appendix C) that the optimal outside option from the seller's perspective is $(p^{-i})^{z_i^S} = 1$ for all p, since it solves, for all p,

$$p^{-i}(p) \in \arg\min_{\rho^{-i} \in \mathcal{P}^{-i}} \underline{U}_i(c_i^*(p, \rho^{-i}), \rho^{-i}) + \int_{c_i^*(p,\rho^{-i})}^{\overline{c_i}} P_i(s)ds, \tag{23}$$

In that case we have (see **Figure 4**)

$$\begin{aligned} c_i^*(p, p^{-i}(p)) &= \underline{c}_i \text{ and} \\ \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}(p)) &= \bar{\pi}_i^{z_i^S}(\underline{c}_i). \end{aligned} \tag{24}$$

[21] Such a case is illustrated in Scenario 2 in Section 5.1.

When outside options are steep,

PC binds at BEST type

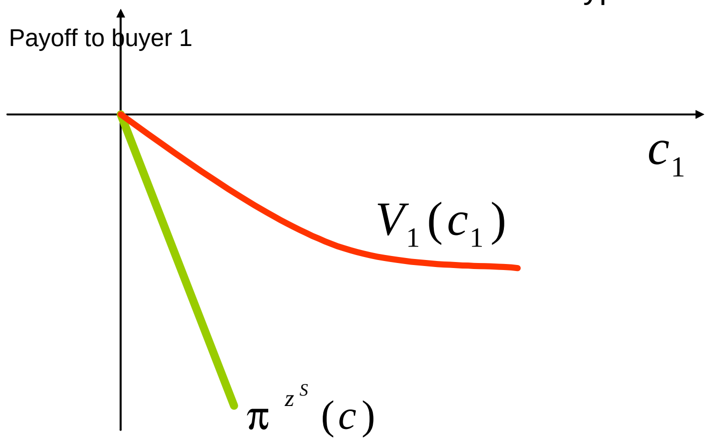

Figure 4

Case 3: Coexistence of Flat and Very Steep Worst Allocations for i

Another interesting case is the one where options like z_i^S and z_i^F coexist, and it is not obvious which one should be used by the seller, because

$$\begin{aligned} \frac{d\bar{\pi}_i^{z_i^S}(c_i)}{dc_i} &\leq \frac{d\bar{\pi}_i^{z}(c_i)}{dc_i} \leq \frac{d\bar{\pi}_i^{z_i^F}(c_i)}{dc_i} \text{ for all } z \in Z, c_i \in C_i \\ \bar{\pi}_i^{z_i^S}(\underline{c}_i) &\geq \bar{\pi}_i^{z_i^F}(\underline{c}_i) \end{aligned}$$

As one can see from **Figure 5**, for some types, z_i^F hurts more, and for others z_i^S.

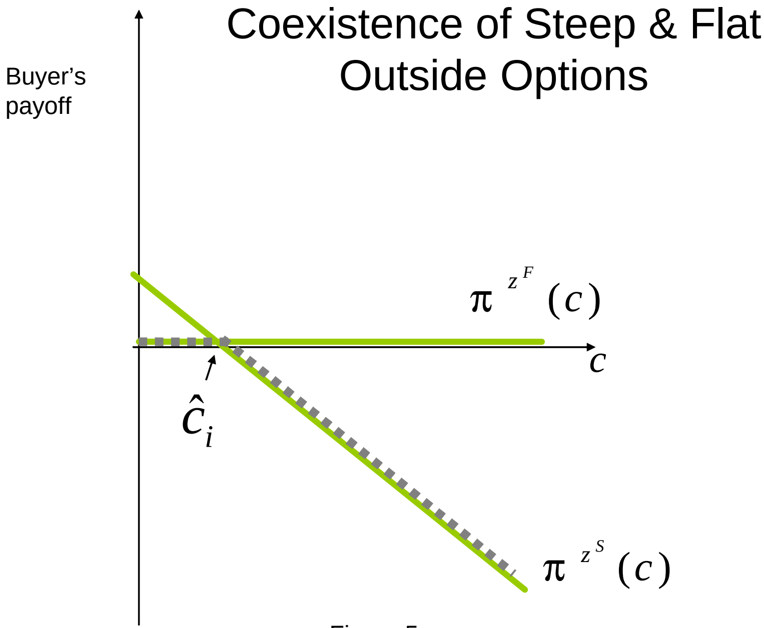

Figure 5

In this case[22], the solution to

$$p^{-i}(p) \in \arg \min_{\rho^{-i} \in \mathcal{P}^{-i}} \rho^{-i} \bar{\pi}_i^{z_i^F}(c_i^*) + (1-\rho^{-i}) \bar{\pi}_i^{z_i^S}(c_i^*) + \int_{c_i^*}^{\overline{c_i}} P_i(s)ds, \tag{25}$$

is such that

$$\begin{aligned} c_i^*(p, p^{-i}(p)) &= \hat{c}_i \text{ and} \\ \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}(p)) &= \bar{\pi}_i^{z_i^F}(\hat{c}_i) = \bar{\pi}_i^{z_i^S}(\hat{c}_i) \text{ for all } p \text{ and } p^{-i}(p) \end{aligned} \tag{26}$$

where $\hat{c}_i$ is the type where the payoffs cross, that is

$$\bar{\pi}_i^{z_i^F}(\hat{c}_i) = \bar{\pi}_i^{z_i^S}(\hat{c}_i). \tag{27}$$

In this case, the critical type is always the same, but the optimal p^{-i} depends on the assignment rule that the seller wishes to implement.

[22] Such a scenario is illustrated in Section 5.2.

Summing up, in all the cases[23] described so far neither $c_i^*(p, p^{-i}(p))$ nor the level of $\underline{U}_i(., p^{-i}(p))$ evaluated at the critical type c_i^*, depend on p. This is despite the fact that p^{-i} can depend on p.[24] This means that for every possible assignment rule p, when the seller chooses $p^{-i} \in \mathcal{P}^{-i}$ optimally, that is according to (11), the following are true[25]

$$\begin{aligned} c_i^* &\equiv c_i^*(p, p^{-i}(p)) \text{ and} \\ \underline{U}_i(c_i^*) &\equiv \underline{U}_i(c_i^*, p^{-i}(p)). \end{aligned} \tag{28}$$

Proposition 3 *If (28) is satisfied, the seller's expected revenue can be expressed as a linear function of the assignment rule,*

$$\int_C \sum_{z \in Z} p^z(c) \hat{J}_z(c) f(c) dc - \sum_{i=1}^{I} \underline{U}_i(c_i^*),$$

where $\hat{J}_z$ is the modified virtual surplus of allocation z defined in (14).

4.1.2 Analysis of the Problem when Revenue is Linear in p

When Proposition 3 holds, we can break the characterization of revenue maximizing mechanisms into two steps: first find an optimal non-participation assignment rule $\{p^{-i}(p)\}_{i \in I}$, as we have done in (21), (23), or (25), and then find an optimal assignment rule p that solves:

$$\begin{aligned} &\max_{p \in \Delta(Z)} \int_C \sum_{z \in Z} p^z(c) \hat{J}_z(c) f(c) dc \\ &s.t.\ P_i \text{ increasing.} \end{aligned} \tag{29}$$

This problem has a similar structure to the classical one in Myerson (1981), but with modified virtual surpluses, and can be solved using relatively conventional methods. Despite this, the solution will often exhibit stark differences from the solution to the classical one.

The solution is straightforward if the assignment rule that solves the relaxed program

$$\max_{p \in \Delta(Z)} \int_C \sum_{z \in Z} p^z(c) \hat{J}_z(c) f(c) dc$$

also satisfies the requirement of P_i being increasing, since in that case, the relaxed program can be solved by pointwise maximization. Following Myerson (1981) we will refer to this as

[23] These are not the only cases where revenue will be linear in p, but they are suggestive on what classes of environments are likely to exhibit this property.

[24] This occurs in the cases where c_i^* is interior and p^{-i} must also satisfy (8).

[25] Notice that if p^{-i} is exogenous ($\mathcal{P}^{-i}$ is a singleton) the second requirement is trivially satisfied.

the *regular* case. On the other hand, in the *general* case, pointwise optimization will lead to a mechanism that may not be feasible.

In the classical problem, a sufficient condition for the problem to be regular is that the virtual surpluses are increasing. A mild condition on the distribution function F_i (MHR) guarantees that. Unfortunately, in our more general environment the problem fails to be regular even if virtual surpluses, (or modified virtual surpluses), are monotonic, so Myerson's technique of obtaining 'ironed' virtual valuations will not work. In Figueroa and Skreta (2007) we illustrate this phenomenon in a concrete example and show a way to solve the general case, which does not impose additional assumptions, such as differentiability, on the mechanism. There we argue that in the general case an optimal mechanism will involve randomizations between allocations. Such lotteries are quite surprising given that buyers are risk neutral and types are single dimensional.

We now state a condition which guarantees that pointwise optimization will lead to a feasible solution. This condition generalizes the one in Myerson (1981), since with independent private values and linear utility functions our condition is satisfied whenever MHR is satisfied.

Before stating the Assumption, let us provide some explanation. Recall that IC requires P_i, to be increasing in c_i. Pointwise optimization assigns probability one to the allocation with the highest virtual surplus at each vector of types. Along a region where there is no switch, one allocation, say z_1, is selected throughout and $P_i(c_i) = \int\limits_{C_{-i}} \frac{\partial \pi_i^{z_1}(c_i,c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i}$, which is increasing by the convexity of π_i. Incentive compatibility can be violated though, when the seller wishes to switch, say, from allocation z_1 to z_2. At such a point c we have that $\hat{J}_{z_2}(c) \geq \hat{J}_{z_1}(c)$ and IC requires that P_i does not decrease, namely $\int\limits_{C_{-i}} \frac{\partial \pi_i^{z_2}(c_i,c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i} \geq \int\limits_{C_{-i}} \frac{\partial \pi_i^{z_1}(c_i,c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i}$. Our condition guarantees precisely this.

Assumption 4 [26] *Let* $z_1, z_2 \in Z$ *be any two allocations. For a given cost realization* (c_i, c_{-i}) *if*[27] $z_1 \in arg\max\limits_{z\in Z} \hat{J}_z(c_i^-, c_{-i})$ *and* $z_2 \in arg\max\limits_{z\in Z} \hat{J}_z(c_i^+, c_{-i})$ *, then* $\frac{\partial \bar{\pi}_i^{z_2}(c_i)}{\partial c_i} \geq \frac{\partial \bar{\pi}_i^{z_1}(c_i)}{\partial c_i}$.

We now state another condition, which is more stringent, but often easier to verify than Assumption 4.

Assumption 5 *For all* i *and for all* c_{-i}*, when* $\frac{\partial J_{z_2}(c_i,c_{-i})}{\partial c_i} \geq \frac{\partial J_{z_1}(c_i,c_{-i})}{\partial c_i}$ *then* $\frac{\partial \bar{\pi}_i^{z_2}(c_i)}{\partial c_i} \geq \frac{\partial \bar{\pi}_i^{z_1}(c_i)}{\partial c_i}$.

Lemma 6 *Assumption 5 is sufficient for Assumption 4.*

[26] This condition has similar flavor to condition 5.1 in the environment of Jehiel and Moldovanu (2001b). We are grateful to Benny Moldovanu for bringing to our attention this connection.

[27] The notation c_i^- means limit from the left to c_i and c_i^+ means limit from the right to c_i.

For the special class where payoffs are linear in own type, there is an even simpler condition that is sufficient for Assumption 4. This is the well known monotone hazard rate condition.

Lemma 7 *If the payoff functions are of the form $\bar{\pi}_i^z(c_i) \equiv A_i^z + B_i^z c_i$, and $\frac{F_i(c_i)}{f_i(c_i)}$ is increasing in c_i for all i, then Assumption 4 is satisfied.*

With the help of Assumption 4, it is straightforward to find an optimal assignment rule which is described in the following result.

Proposition 8 *Suppose that (28) holds.*[28] *If Assumption 4 is satisfied, then an optimal allocation p is given by:*[29]

$$p^{z^*}(c) = \begin{cases} 1 & \text{if } z^* \in arg\max_z \hat{J}_z(c) \\ 0 & \text{otherwise} \end{cases}.$$

The qualitative features of the solution depend on whether the conditions in (28) are satisfied for $c_i^* = \bar{c}_i$, $c_i^* = \underline{c}_i$, or $c_i^* \in (\underline{c}_i, \bar{c}_i)$. If $c_i^* = \bar{c}_i$, then $\hat{J}_z(c) < S_z(c)$ and the seller sells less often than it is efficient. When the conditions in (28) are satisfied for $c_i^* = \underline{c}_i$, $\hat{J}_z(c) \geq S_z(c)$ and overselling occurs, as stated in the next corollary:

Corollary 9 *Suppose that $c_i^*(p, p^{-i}(p)) = \underline{c}_i$ for all i. Suppose also that when the seller keeps all objects, every buyer gets a payoff independent of his type, for example zero. Then, at a revenue maximizing assignment rule the seller keeps all the objects **less** often than what is ex-post efficient.*

The situation in Example 5.1 exhibits this feature. As noted in the introduction, "overselling" is in contrast with a standard intuition from monopoly theory, where the monopolist restricts supply in order to generate higher revenue.

When $c_i^* \in (\underline{c}_i, \bar{c}_i)$, then $\hat{J}_z(c) < S_z(c)$ for some type profiles, and $\hat{J}_z(c) \geq S_z(c)$ for others. Here, underselling and overselling can occur simultaneously, (the seller keeps the objects in some cases where he should sell, and sells them in cases where he should keep them), or even ex-post efficiency can occur. Example 5.2 illustrates a scenario where the critical type is interior and the revenue maximizing mechanism is ex-post efficient.

Now we move on to see how much of what we learnt, by examining revenue maximizing mechanisms in linear cases, applies when revenue is non-linear in p. As discussed, this occurs when $c_i^*(p, p^{-i}(p))$ depends on p. The intuitions discussed above remain: the relation between modified virtual surpluses, and real surpluses will depend on the actual values of $c_i^*(p, p^{-i}(p))$, for $i \in I$, at the optimal p. We proceed to examples.

[28] In Appendix C we describe a couple of specific environments where (28) holds. The list is not, nor it is meant to be exhaustive.

[29] Ties can be broken arbitrarily.

5. Illustration of the Solution

The purpose of this section is to illustrate the solution in simple but economically insightful examples.

5.1 The Role of Steep Outside Options

Consider 2 firms fighting for a single slot to advertise their products. There are three feasible allocations. The seller keeps the slot, z_0; firm 1 gets the slot, z_1 or firm 2 gets the slot, z_2.

The value of airing a spot depends on the actual cost parameter c_i of firm i, which is private information and is uniformly and independently distributed in $[0,1]$ for both firms. The value of not airing a spot depends on the allocation implemented: a firm suffers an externality, (which depends on its cost parameter c_i), if its competitor gets the spot, while it gets a payoff of 0 in case nobody gets it. Let $\pi_i^{z_j}(c_i)$ denote the payoff of firm i if allocation z_j is implemented and its type is c_i. The payoffs that accrue to each firm from each of these alternatives are

$$\begin{array}{ll} \pi_1^{z_0}(c_1) = 0 & \pi_2^{z_0}(c_2) = 0 \\ \pi_1^{z_1}(c_1) = 1 - c_1 & \pi_2^{z_1}(c_2) = -2c_2 \\ \pi_1^{z_2}(c_1) = -2c_1 & \pi_2^{z_2}(c_2) = 1 - c_2 \end{array} .$$

An assignment rule here is $p(c) = (p^{z_0}(c), p^{z_1}(c), p^{z_2}(c))$, where $c = (c_1, c_2)$.

The virtual surpluses of allocations z_0, z_1 and z_2 are given by

$$\begin{aligned} J_{z_0}(c) &= 0 \\ J_{z_1}(c) &= 1 - 2c_1 - 4c_2 \\ J_{z_2}(c) &= 1 - 2c_2 - 4c_1. \end{aligned}$$

Using (6) we can write the seller's problem as:

$$\max_p \int_{[0,1]} \int_{[0,1]} [p^{z_0}(c)J_{z_0}(c) + p^{z_1}(c)J_{z_1}(c) + p^{z_2}(c)J_{z_2}(c)]dc_1 dc_2 - V_1(1) - V_2(1) \tag{30}$$

subject to:

$$\begin{array}{l} P_1(c_1) \equiv -\int [p^{z_1}(c) + 2p^{z_2}(c)]dc_2 \text{ be increasing} \\ P_2(c_2) \equiv -\int [2p^{z_1}(c) + p^{z_2}(c)]dc_1 \text{ be increasing} \\ 0 \le p^{z_i}(c) \le 1,\ i = 0,1,2 \text{ and } \sum_{i=0}^{2} p^{z_i}(c) = 1 \end{array}$$

The solution of this problem crucially depends on the allocations that prevail if a buyer refuses to participate in the mechanism, since these determine $V_i(1, p, p^{-i}(p))$, $i = 1, 2$. We demonstrate this point by solving for the optimal mechanism under two different scenaria regarding the outside options that buyers face.

Scenario 1: Flat Outside Options

In this case if a buyer does not participate the seller must keep the slot. Then

$$p^{-1} = p^{-2} = (p^{z_0}(c), p^{z_1}(c), p^{z_2}(c)) = (1, 0, 0).$$

Given this non-participation assignment rule, the payoff to buyer i from not participating is $\pi_i^{z_0}(c_i) = 0$, which is independent of $i's$ type. Then participation constraint binds at the "worst" type $\bar{c}_1 = \bar{c}_2 = 1$, because at an incentive compatible assignment rule V_i is decreasing in c_i.[30] This implies immediately that

$$V_1(1) = V_2(1) = 0,$$

and the objective function in (30), after substituting for the $J_z's$, it becomes

$$\max_p \int_{[0,1]} \int_{[0,1]} [p^{z_1}(c)(1 - 2c_1 - 4c_2) + p^{z_2}(c)(1 - 2c_2 - 4c_1)] dc_1 dc_2. \tag{31}$$

Pointwise maximization gives us

$$p(c) = \begin{cases} (0,1,0) & \text{if } c_2 \leq c_1 \text{ and } 1 \geq 2c_1 + 4c_2 \\ (0,0,1) & \text{if } c_1 \leq c_2 \text{ and } 1 \geq 2c_2 + 4c_1 \\ (1,0,0) & \text{if } 2c_1 + 4c_2 > 1 \text{ and } 2c_2 + 4c_1 > 1 \end{cases},$$

which is feasible, and hence optimal. Feasibility follows from Lemma 7, since we have linear payoffs and the uniform distribution satisfies MHR. We graph the revenue assignment rule

[30] Recall Figure 3.

in **Figure 6.**

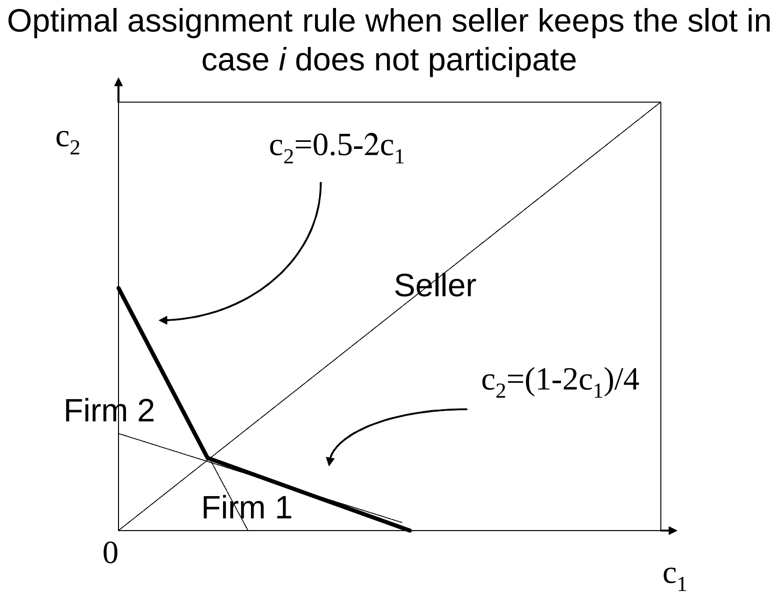

Figure 6

Scenario 2: Steep Outside Options

In this case, if a firm fails to participate the seller gives the slot to its competitor, the other firm, that is

$$p^{-1} = (p^{z_0}(c), p^{z_1}(c), p^{z_2}(c)) = (0, 0, 1) \text{ and } p^{-2} = (p^{z_0}(c), p^{z_1}(c), p^{z_2}(c)) = (0, 1, 0).$$

It is not hard to see that this is the optimal way for the seller to threaten buyers, since giving the slot to the competitor has the lowest payoff for buyer i. We now argue that in this case the critical type will be $c_i^* = \underline{c}_i = 0$, for all i and p. This is because allocation z_j gives the lowest payoff to i when his cost is the smallest possible, that is

$$\pi_i^{z_j}(0) \leq \pi_i^z(0) \text{ for all } z \in \{z_0, z_1, z_2\} \tag{32}$$

and it gives the steepest payoff to buyer i, for $i, j = 1, 2$ since we have that

$$\frac{d\pi_i^{z_j}(c_i)}{dc_i} \leq \frac{d\pi_i^z}{dc_i} \text{ for all } z \in \{z_0, z_1, z_2\}. \tag{33}$$

From (32) and (33) it follows that the participation constraint will always bind at $c_i^* = \underline{c}_i = 0$. Such a situation is depicted in Figure 4.

Then for $i, j = 1, 2$ we have

$$
\begin{aligned}
V_i(1) &= \bar{\pi}_i^{z_j}(c_i^*) + \int_{c_i^*}^{1} P_i(c_i)dc_i \\
&= \bar{\pi}_i^{z_j}(0) + \int_{0}^{1} P_i(c_i)dc_i \\
&= 0 + \int_0^1 \int_0^1 [-p^{z_i}(c) - 2p^{z_j}(c)]dc.
\end{aligned} \tag{34}
$$

Substituting these expressions in the objective function, we get "modified virtual surpluses",

$$
\begin{aligned}
\hat{J}_{z_0}(c) &= 0 \\
\hat{J}_{z_1}(c) &= 4 - 2c_1 - 4c_2 \\
\hat{J}_{z_2}(c) &= 4 - 2c_2 - 4c_1
\end{aligned}
$$

and the seller's problem can be rewritten us

$$
\begin{array}{ll}
\max\limits_{p} & \int\limits_{[0,1]} \int\limits_{[0,1]} [p^{z_1}(c)\,(4 - 2c_1 - 4c_2) + p^{z_2}(c)\,(4 - 2c_2 - 4c_1)]dc_1dc_2 \\
s.t. & P_1(c_1) \equiv -\int [p^{z_1}(c) + 2p^{z_2}(c)]dc_2 \text{ is increasing} \\
& P_2(c_2) \equiv -\int [2p^{z_1}(c) + p^{z_2}(c)]dc_1 \text{ is increasing} \\
& 0 \le p^{z_i}(c) \le 1,\ i = 0, 1, 2 \text{ and } \sum\limits_{i=0}^{2} p^{z_i}(c) = 1.
\end{array} . \tag{35}
$$

By comparing (31) and (35), we see that how the terms $V_1(1)$ and $V_2(1)$ can affect the objective function.

The assignment rule corresponding to pointwise maximization is given by

$$
p(c) = \begin{cases}
(0, 1, 0) & \text{if } c_2 \le c_1 \text{ and } 4 \ge 2c_1 + 4c_2 \\
(0, 0, 1) & \text{if } c_1 \le c_2 \text{ and } 4 \ge 2c_2 + 4c_1 \\
(1, 0, 0) & \text{if } 2c_1 + 4c_2 > 4 \text{ and } 2c_2 + 4c_1 > 4
\end{cases},
$$

which by Lemma 7 is feasible, and hence optimal. This assignment rule is shown in **Figure**

7.

Optimal assignment rule when seller gives the slot to *j* in case *i* does not participate

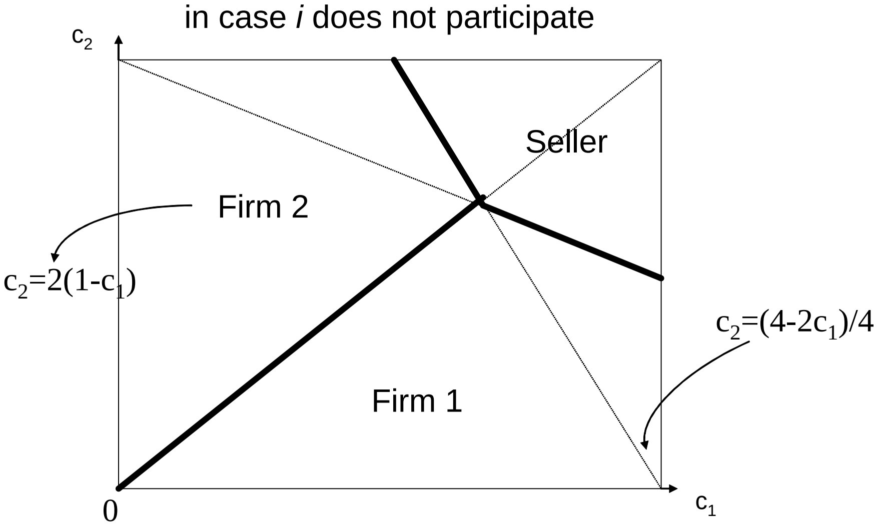

Figure 7

As discussed earlier, when participation constraints bind at the smallest cost "overselling" occurs, compared to what is efficient. In this example, the "ex-post efficient allocation" is given by

$$p^e(c) = \begin{cases} (0,1,0) & \text{if } c_2 \leq c_1 \text{ and } c_1 + 2c_2 \leq 1 \\ (0,0,1) & \text{if } c_1 \leq c_2 \text{ and } 2c_1 + c_2 \leq 1 \\ (1,0,0) & \text{otherwise} \end{cases} .$$

We illustrate it in **Figure 8.**

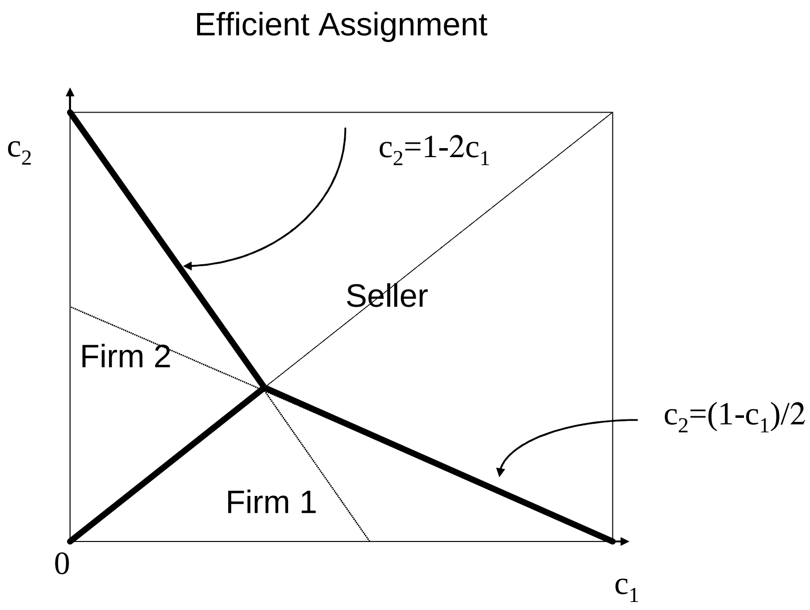

Figure 8

Comparing $p(c)$ and $p^e(c)$, depicted in Figures 7 and 8 respectively, we see that at the revenue maximizing assignment rule firms 1 and 2 obtain the slot for cost realizations where efficiency dictates that the seller should keep it.

This example illustrates that the optimal assignment rule critically depends on the outside options that each buyer faces. When the seller can only keep the object if a buyer fails to participate, the optimal assignment rule assigns the slot less often than it is efficient. In contrast, in the case where if a buyer fails to participate, the seller gives the slot to the other firm, the revenue maximizing assignment rule allocates the spot more often then it is efficient. The reason why the solution in these two scenaria differs, is that in the second one when firm i fails to participate its payoff depends on its cost, $(\pi_i^{z_j}(c_i) = -2c_i)$.

Before closing, we would like to stress that the mere presence of externalities, (regardless of whether they are positive or negative), will *not* lead to an optimal mechanism where overselling occurs compared to the ex-post efficient level. This is illustrated in the following small modification of the current example:

$$\begin{array}{ll} \pi_1^{z_0}(c_1) = 0 & \pi_2^{z_0}(c_2) = 0 \\ \pi_1^{z_1}(c_1) = 1 - c_1 & \pi_2^{z_1}(c_2) = -0.5 \\ \pi_1^{z_2}(c_1) = -0.5 & \pi_2^{z_2}(c_2) = 1 - c_2 \end{array} .$$

The virtual surpluses of the various allocations in this case are given by

$$\begin{aligned}J_{z_0}(c) &= 0\\ J_{z_1}(c) &= 1-2c_1-0.5\\ J_{z_2}(c) &= 1-2c_2-0.5.\end{aligned}$$

Observe that as in the original example there are negative externalities. If a firm's competitor gets the slot it gets a negative payoff of -0.5. The important difference is that now this payoff does not depend on the firm's cost. Consequently, when the seller is threatening to assign the slot to a firms competitor, that firm faces a payoff that is independent from its type. A consequence of that is that irrespective of whether the seller keeps the slot if a firm does not participate, or she gives it to a competitor, the virtual surpluses are unaffected, since a firm's outside payoff is a straight line, (0 in the case where the seller keeps the good and -0.5 in the case that its gives the slot to the other firm), which implies that in both cases the critical type is $c_i^* = \bar{c}_i = 1$. Also notice that the virtual surpluses of z_0, z_1 and z_2 are smaller, (possibly weakly smaller), than the actual surpluses of these allocations, which are given by

$$\begin{aligned}S_{z_0}(c) &= 0\\ S_{z_1}(c) &= 1-c_1-0.5\\ S_{z_2}(c) &= 1-c_2-0.5,\end{aligned}$$

hence "overselling" cannot occur.

Summarizing, outside options affect the optimal assignment rule only if the payoffs from non-participation are type-dependent. In the next example we show how in the case of type-dependent outside options the seller can increase both revenue and efficiency by appropriately choosing the right outside options.

5.2 An Example with Coexistence of Steep & Flat Outside Options[31]

A seller has an invention which is of potential interest to firm A. The firm has a cost parameter c distributed uniformly in $[0, 1]$. In case firm A gets the exclusive rights, its valuation is given by $\pi^{z_A}(c) = 5-5c$. In case that there is no sale, the seller can either keep the invention or open source it. A very efficient firm is not afraid of competition and prefers open sourcing to no sale at all, whereas a more inefficient firm prefers the opposite.[32] In particular, in the case of no sale firm A gets $\pi^{z_1}(c) = 0$, and in the case of open sourcing

[31] This is essentially the example described in the introduction.

[32] This is different from the previous example, where the allocation that hurts buyers the most is always the same. Irrespective of the cost parameter, a firm prefers that the seller keeps the slot, to its competitor obtaining it.

it gets $\pi^{z_2}(c) = 1 - 10c$. So if firm A is very efficient with $c \leq \frac{1}{10}$, the option of no one obtaining the invention is worse than open sourcing. The opposite is true when $c \geq \frac{1}{10}$. An assignment rule here is $p(c) = (p^{z_A}(c), p^{z_1}(c), p^{z_2}(c))$. In case firm A does not participate in the sale, the seller is indifferent between keeping the invention and making it open source. In fact, since there is nothing else the seller can do in that case, any randomization between these options is optimal from her perspective and hence credible.

The seller solves:

$$\begin{array}{ll} \max\limits_{p} & \int\limits_{0}^{1} [p^{z_A}(c)\,(5-10c) + p^{z_2}(c)\,(1-20c)]dc - V(1, p, p^{-i}(p)) \\ s.t. & -[5p^{z_A}(c) + 10p^{z_2}(c)] \text{ is increasing} \\ & 0 \leq p^{z}(c) \leq 1 \text{ for all } z \in \{z^A, z^1, z^2\} \text{ and } \Sigma_{z\in\{z^A,z^1,z^2\}} p^{z}(c) = 1 \end{array} \tag{36}$$

This example belongs to the class of problems which satisfy (28) for an interior c_i^*. As already discussed, in this case an optimal non-participation rule, which we call p^{-A}, depends on the assignment rule p that the seller wants to implement. We therefore start by specifying the optimal p^{-A} as a function of p and then solve for an optimal p.

1. Finding an optimal $p^{-A}(p)$

With a slight abuse of notation, let p^{-A} denote the probability that allocation z_2 is chosen if A fails to participate, and let $(1 - p^{-A})$ denote the probability that allocation z_1 will be chosen. Associated with this non-participation assignment rule is the payoff that will accrue to A if it fails to participate

$$\begin{aligned} \underline{U}_A(c, p^{-A}) &= (1 - p^{-A}) \cdot 0 + p^{-A}(1 - 10c) \\ &= p^{-A} - 10p^{-A}c. \end{aligned} \tag{37}$$

We know that the optimal non-participation assignment rule must minimize

$$V(1) = \underline{U}_A(c^*(p, \rho^{-A}), p^{-A}) + \int\limits_{c^*(p,\rho^{-A})}^{1} \frac{dV(c)}{dc} dc,$$

which by using (37) can be rewritten as

$$V(1) = \rho^{-A} - 10\rho^{-A}c^*(p, \rho^{-A}) + \int\limits_{c^*(p,\rho^{-A})}^{1} \frac{dV(c)}{dc} dc. \tag{38}$$

Now at a solution[33] $p^{-A}(p)$ the total derivative of $V(1)$ with respect to ρ^{-A} is equal to the partial, and it is given by

$$\left.\frac{dV(1)}{d\rho^{-A}}\right|_{\rho^{-A}=p^{-A}(p)} = 1 - 10c^*(p, p^{-A}(p)).$$

[33] This property is an envelope condition. We state it formally in Lemma A in Appendix C.

Moreover, at an interior minimum it must be the case that

$$\left.\frac{dV(1)}{d\rho^{-A}}\right|_{\rho^{-A}=p^{-A}(p)} = 1 - 10c^*(p, p^{-A}(p)) = 0,$$

which implies that

$$c^*(p, p^{-A}(p)) = \frac{1}{10} \text{ for all } p \text{ and } p^{-A}. \tag{39}$$

We have therefore verified that this example satisfies (28) for $c_i^* = \frac{1}{10}$, and hence the critical type is independent of p and p^{-A}.

We proceed to find an optimal p^{-A} as a function of an assignment rule p. The slope of the payoff from non-participation is

$$\frac{\partial \underline{U}_A(c, p^{-A})}{\partial c} = -10p^{-A}.$$

At an optimal p^{-A} this has to be equal to the slope of the participation payoff V at $c^*(p, p^{-A}(p))$ which in our case it is $\frac{1}{10}$. In other words

$$\left.\frac{dV(c)}{dc}\right|_{c^*=\frac{1}{10}} = -10p^{-A}, \tag{40}$$

now given an assignment rule $p(c) = (p^{z_A}(c), p^{z_1}(c), p^{z_2}(c))$, $V(c)$ is given by

$$V(c) = p^{z_A}(c)(5 - 5c) + p^{z_1}(c) \cdot 0 + p^{z_2}(c)(1 - 10c),$$

and its slope is given by

$$\frac{dV(c)}{dc} = -5p^{z_A}(c) - 10p^{z_2}(c). \tag{41}$$

With the help of (41), (40) can be rewritten as

$$-10p^{-A} = -5p^{z_A}(\frac{1}{10}) - 10p^{z_2}(\frac{1}{10}),$$

which reduces to

$$p^{-A}(p) = \frac{1}{2}p^{z_A}(\frac{1}{10}) + p^{z_2}(\frac{1}{10}). \tag{42}$$

Equation (42) gives us an optimal p^{-A} as a function of the assignment rule p.[34]

[34] This example illustrates the interdependence of optimal non-participation assignment rules with the assignment rules, that is how p^{-A} can depend on p. This feature is novel and does not appear in the earlier work, (see for instance JMS (1996)), where optimal threats are independent from the way the seller wants to allocate the goods. Here equation (42) tells us that for different assignment rules, the optimal, from the seller's point of view, non-participation assignment rule, is different. For example, for

$$(\tilde{p}^{z_A}(c), \tilde{p}^{z_1}(c), \tilde{p}^{z_2}(c)) = \begin{cases} (1,0,0) & \text{if } c \in [0, \frac{1}{2}] \\ (0,1,0) & \text{if } c \in [\frac{1}{2}, 1] \end{cases}$$

2. Finding an optimal p

With the help of (39) $V(1)$, given by (38), can be rewritten as

$$V(1) = -\int_{\frac{1}{10}}^{1} [5p^{z_A}(c) + 10p^{z_2}(c)]dc. \tag{43}$$

Now by substituting (43) into (36), the seller's problem can be rewritten as

$$\begin{array}{ll} \max\limits_{p} & \int_{0}^{\frac{1}{10}} [p^{z_A}(c)\,(5-10c) + p^{z_2}(c)\,(1-20c)]dc + \int_{\frac{1}{10}}^{1} p^{z_A}(c)\,(10-10c) + p^{z_2}(c)\,(11-20c)]dc \\ s.t. & -[5p^{z_A}(c) + 10p^{z_2}(c)] \text{ is increasing} \\ & 0 \le p^z(c) \le 1 \text{ for all } z \in \{z^A, z^1, z^2\} \text{ and } \Sigma_{z\in\{z^A,z^1,z^2\}} p^z(c) = 1. \end{array}$$

Pointwise maximization gives us that $p^{z_A}(c) = 1$ for all c, and the optimal assignment rule is

$$p(c) = (p^{z_A}(c) = 1, p^{z_1}(c) = 0, p^{z_2}(c) = 0) \tag{44}$$

which is feasible, since irrespective of report, firm A obtains the object with probability 1 and pays the same price, which is equal to 4.5. By substituting (44) into (42), we get that the optimal non-participation assignment rule is given by

$$\begin{aligned} p^{-A}(p) &= \frac{1}{2} \text{ or more precisely} \\ p^{-A}(p) &= (p^{z_A}(c), p^{z_1}(c), p^{z_2}(c)) = (0, \frac{1}{2}, \frac{1}{2}). \end{aligned}$$

In this example the revenue maximizing assignment rule is ex-post efficient. To see this, notice that $\pi^{z_A}(c) \ge \pi^{z_1}(c)$ and $\pi^{z_A}(c) \ge \pi^{z_2}(c)$ for all $c \in [0,1]$, so it's always efficient to sell the invention to the firm. This efficiency property is rather surprising given the presence of private information that is statistically independent. The seller's expected revenue is 4.5. A's payoff is the $5 - 5c - 4.5$ which is exactly equal to its outside option which is

the optimal non-participation assignment rule is $p^{-A}(\tilde{p}) = \frac{1}{2}$. If the assignment rule is instead

$$(\hat{p}^{z_A}(c), \hat{p}^{z_1}(c), \hat{p}^{z_2}(c)) = \begin{cases} (\frac{1}{2}, 0, \frac{1}{2}) & \text{if } c \in [0, \frac{1}{2}] \\ (0,1,0) & \text{if } c \in [\frac{1}{2}, 1] \end{cases}$$

the optimal non-participation assignment rule is $p^{-A}(\hat{p}) = \frac{3}{4}$.

$0.5 \cdot 0 + \cdot 0.5 \cdot (1 - 10c) = 0.5 - 5c$. These payoffs are graphed in **Figure 9.**

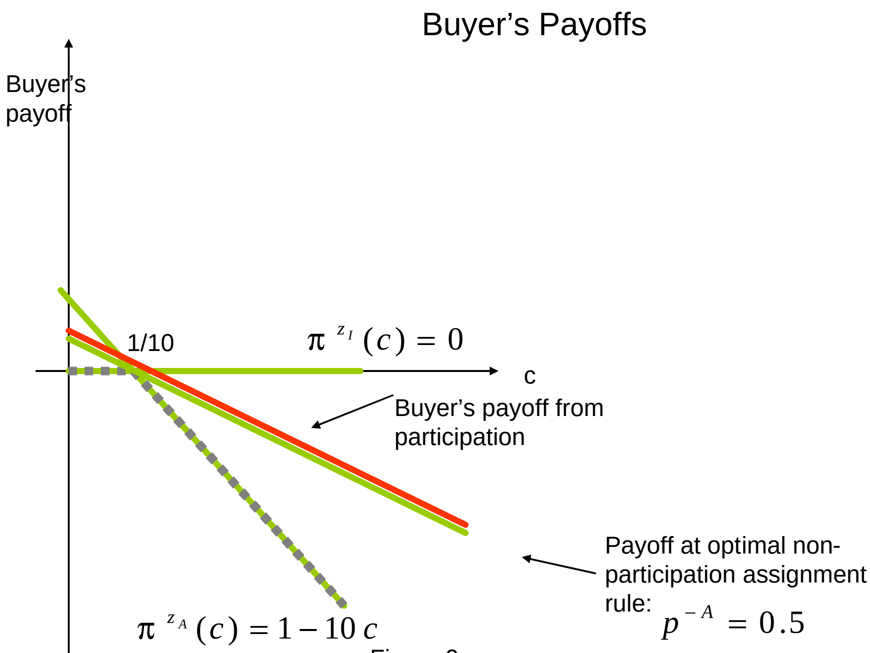

Figure 9

It is interesting to compare this solution to the one when open sourcing (allocation z_2) is not an available option. In this case the optimal assignment rule is

$$p(c) = (p^{z_A}(c), p^{z_1}(c)) = \begin{cases} (1,0) & \text{if } c \in [0, \frac{1}{2}] \\ (0,1) & \text{if } c \in [\frac{1}{2}, 1] \end{cases},$$

and trivially the non-participation assignment rule is $p^{-A}(p) = (0,1)$. Then the seller's expected revenue is 1.25. This assignment rule is inefficient, since half of the time Firm A does not obtain the invention, whereas it is always efficient that it does. Comparing to the previous case, we see that the option of open sourcing increases both the seller's revenue, (it more than triples), and efficiency. This is despite the fact that open sourcing is never implemented.

This example highlights an important new insight. When the payoff from non-participation depends on a buyer's type, even allocations that are never implemented can crucially affect the revenue maximizing assignment of the objects. The introduction of the option of open sourcing increased the revenue of the seller, and made the revenue maximizing assignment rule ex-post efficient, even though it is never implemented. This example also shows that optimal non-participation assignment rules can be random.

6. Concluding Remarks

In this paper we study revenue maximizing auctions when buyers' outside payoffs depend on their type. Our analysis shows that key intuitions from earlier work on optimal auctions fail to generalize. Very often efficiency and revenue maximization are conflicting objectives. However, here we show that a revenue maximizing mechanism sometimes will allocate the objects in an ex-post efficient way, and sometimes it will sell "too often". The broad message is that type-dependent non-participation payoffs change the nature of the distortions that arise from the presence of asymmetric information. The designer by creating the "appropriate" outside options can increase both revenue, and the overall efficiency of the mechanism. This paper also encompasses a large number of important allocation problems as a special case. Potential applications range from the allocation of airport take-off and landing slots, to the allocation of positions in teams.

7. Appendix A

Proof of Lemma 1[35]

By the convexity of $\pi_i^z(\cdot, c_{-i})$ we have that V_i is a maximum of convex functions, so it is convex, and therefore differentiable a.e. It is also easy to check that the following are equivalent:

(a) (p, x) is incentive compatible

(b) $P_i(c_i) \in \partial V_i(c_i)$

(c) $U_i(c_i, c_i; (p, x)) = V_i(c_i)$

We now use these equivalent statements to prove necessity and sufficiency in our Lemma.

($\Longrightarrow$) Here we use the fact that incentive compatibility implies (b). A result in Krishna and Maenner (2001) then implies (5). By the convexity of V_i, we know that ∂V_i is monotone, so:

$$(P_i(c_i) - P_i(c_i'))(c_i - c_i') \geq 0.$$

This immediately implies (4).

($\Longleftarrow$) To prove that (4) implies incentive compatibility it's enough to show that $P_i(c_i) \in \partial V_i(c_i)$. By (4) and (5),

$$\begin{aligned} V_i(c_i') - V_i(c_i) &= \int_{c_i}^{c_i'} P_i(s)ds \\ &\geq P_i(c_i)(c_i' - c_i) \end{aligned}$$

which shows $P_i(c_i) \in \partial V_i(c_i)$. ■

[35] This proof is relatively standard, see for instance, Jehiel, Moldovanu and Stacchetti (1999) and is included for completeness.

Expected Payment at an Incentive Compatible Mechanism[36]

Recall that

$$V_i(c_i) = \int_{C_{-i}} \left[\sum_{z\in Z} p^z(c)\pi_i^z(c) - x_i(c) \right] f_{-i}(c_{-i})dc_{-i}. \tag{45}$$

By integrating (45) with respect to c_i, and by rearranging we get that

$$\int_C x_i(c)f(c)dc = \int_C \sum_{z\in Z} p^z(c)\pi_i^z(c)f(c)dc - \int_{C_i} V_i(c_i)f_i(c_i)dc_i. \tag{46}$$

Integrating the second condition in (5) over C_{-i} and by changing the order of integration we get:

$$\begin{aligned}
\int_{C_i} V_i(c_i)dc_i &= \int_{C_i} [V_i(\bar{c}_i) - \int_{c_i}^{\overline{c_i}} P_i(s_i)ds_i] f_i(c_i)dc_i \\
&= V_i(\bar{c}_i) - \int_{C_i} P_i(s_i) \int_{\underline{c_i}}^{s_i} f_i(c_i)dc_i ds_i \\
&= V_i(\bar{c}_i) - \int_{C_i} P_i(c_i)F_i(c_i)dc_i \\
&= V_i(\bar{c}_i) - \int_{C_i}\int_{C_{-i}} \sum_{z\in Z} p^z(c_i,c_{-i}) \frac{\partial \pi_i^z(c_i,c_{-i})}{\partial c_i} f_{-i}(c_{-i})dc_{-i}F_i(c_i)dc_i \\
&= V_i(\bar{c}_i) - \int_C \sum_{z\in Z} p^z(c_i,c_{-i}) \frac{\partial \pi_i^z(c_i,c_{-i})}{\partial c_i} \frac{F_i(c_i)}{f_i(c_i)} f(c)dc.
\end{aligned}$$

Combining (46) with last expression, the result follows. ■

Proof of Lemma 6

If there exists a point (c_i, c_{-i}) such that $z_1 \in arg \max_{z\in Z} \hat{J}_z(c_i^-, c_{-i})$ and $z_2 \in arg \max_{z\in Z} \hat{J}_z(c_i^+, c_{-i})$, then it must be the case that $\frac{\partial J_{z_2}(c_i,c_{-i})}{\partial c_i} \geq \frac{\partial J_{z_1}(c_i,c_{-i})}{\partial c_i}$. If Assumption 5 is satisfied, then we have that $\frac{d\bar{\pi}_i^{z_2}(c_i)}{dc_i} \geq \frac{d\bar{\pi}_i^{z_1}(c_i)}{dc_i}$, which implies that Assumption 4 is also satisfied. ■

Proof of Lemma 7

We just need to prove that Assumption 5 is satisfied. For that, suppose that $\frac{\partial J_{z_1}(c_i,c_{-i})}{\partial c_i} \geq \frac{\partial J_{z_2}(c_i,c_{-i})}{\partial c_i}$. By the linearity assumption, we have that $B_i^{z_1}\left[1 + \left(\frac{F_i(c_i)}{f_i(c_i)}\right)'\right] \geq B_i^{z_2}\left[1 + \left(\frac{F_i(c_i)}{f_i(c_i)}\right)'\right]$. Then, since $\left(\frac{F_i(c_i)}{f_i(c_i)}\right)' \geq 0$ by assumption, we get $B_i^{z_1} \geq B_i^{z_2}$, which is equivalent to $\frac{d\bar{\pi}_i^{z_1}(c_i)}{dc_i} \geq \frac{d\bar{\pi}_i^{z_2}(c_i)}{dc_i}$ under the linearity assumption. ■

[36] This proof is very standard and is included for completeness.

Proof of Theorem 8

The solution proposed corresponds to pointwise maximization, so the only possibility that is not optimal is that is not feasible. To check that feasibility is satisfied remember that

$$P_i(c_i) = \int\limits_{C_{-i}} \sum_{z \in Z} p^z(c_i, c_{-i}) \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i}$$

and consider a fixed c_{-i}. In a region of cost realizations where $\bar{z} \in arg \max_{z \in Z} \hat{J}_z(c)$, the allocation rule $p(c)$ does not change since along this region $p^{\bar{z}}(c) = 1$. Then, $P_i(c_i)$ is increasing by the convexity of $\pi_i^{\bar{z}}(\cdot, c_{-i})$. For a c_i^* where $z_1 \in arg \max_{z \in Z} \hat{J}_z(c_i^{*-}, c_{-i})$ and $z_2 \in arg \max_{z \in Z} \hat{J}_z(c_i^{*+}, c_{-i})$, $p^{z_1}(c_i^{*-}, c_{-i}) = 1$ and $p^{z_2}(c_i^{*+}, c_{-i}) = 1$, $P_i(c_i)$ is increasing by Assumption 4. ■

Proof of Corollary 9

Let's denote by z_0 the allocation where the seller keeps all the objects and consider a fixed realization of types c. Since $\pi_i^{z_0}(c)$ is constant for all i, its derivative vanishes, and we have that $J_{z_0}(c) = \sum_{i=1}^{N} \pi_i^{z_0}(c) = S_{z_0}(c)$. On the other hand, for every allocation z, its virtual surplus is given by

$$J_z(c) = \sum_{i=1}^{N} \left[\pi_i^z(c) + \frac{\partial \pi_i^z(c)}{\partial c_i} \frac{F_i(c_i) - 1}{f_i(c_i)} \right] > S_z(c) \equiv \sum_{i=1}^{N} \pi_i^z(c).$$

Then it is easy to see that the set where the seller keeps the objects, $\left\{ c | z_0 \in arg \max_z S_z(c) \right\}$, is a subset of the set where it would be efficient that she keeps them, $\left\{ c | z_0 \in arg \max_z J_z(c) \right\}$.
■

8. Appendix B: An Example where Revenue Depends Non-Linearly in p.

Suppose there is one buyer and three possible allocations z_1, z_2, z_3 and that c is uniformly distributed on $[0, 1]$. The payoffs of the allocations are $\pi^{z_1}(c) = 10 - 10c$, $\pi^{z_2}(c) = 0$ and $\pi^{z_3}(c) = -5c$, where $c \in [\underline{c}, \bar{c}]$. Then it is easy to that irrespective of p an optimal non-participation assignment rule is $(p^{-1})^{z_3} = 1$, so the non-participation assignment rule assigns probability one to allocation z_3. An assignment rule $p(c) = (p^{z_1}(c), p^{z_2}(c), p^{z_3}(c))$ induces a surplus

$$V(c) = V(\bar{c}; p, p^{-1}) - \int\limits_{c}^{\bar{c}} P(s) ds$$

which, in the points where it is differentiable satisfies $\frac{dV(c)}{dc} = P(c) = -10p^{z_1}(c) - 5p^{z_3}(c)$. The type where the participation constraint binds depends on how $P(c)$, which is the slope of the payoff from participating in the mechanism, compares to the slope of the payoff from non-participating, which is given by -5. The critical type c^* depends non-linearly on p, and it is given by

$$c^*(p, p^{-1}) = \begin{cases} \underline{c} & \text{if } -5 \leq -10p^{z_1}(0) - 5p^{z_3}(0) \\ \bar{c} & \text{if } -5 \geq -10p^{z_1}(1) - 5p^{z_3}(1) \\ c^* & \text{otherwise} \end{cases},$$

where c^* satisfies that $-10p^{z_1}(c^{*^-}) - 5p^{z_3}(c^{*^-}) \leq -5 \leq -10p^{z_1}(c^{*^+}) - 5p^{z_3}(c^{*^+})$. Since

$$V(\bar{c}, p, p^{-1}) = -5c^*(p, p^{-1}) + \int_{c^*(p,p^{-1})}^{\bar{c}} [-10p^{z_1}(c) - 5p^{z_3}(c)]dc,$$

we have that the objective function is non-linear in the assignment rule p.

9. Appendix C: Two Specific Environments where Critical Types are Independent of p.

I. Steep Outside Options: Participation Constraints bind at the best type $c_i^* = \underline{c}_i$.

We now provide the precise conditions for the case of "very responsive" outside options, and argue that under those conditions (28) are satisfied at $c_i^* = \underline{c}_i$.

Recall that we use $\overline{\pi}_i^z(c_i) = \int_{C_{-i}} \pi_i^z(c_i, c_{-i}) f_{-i}(c_{-i}) dc_{-i}$ to denote the expected payoff to agent i if allocation z is implemented.

Assumption 10 *Suppose that outside options are steep, in the sense that for all $i \in I$, there exists an allocation $z_i^S \in Z^{-i}$ such that*

$$\frac{d\overline{\pi}_i^{z_i^S}(c_i)}{dc_i} \leq \frac{d\overline{\pi}_i^z(c_i)}{dc_i} \text{ for all } z \in Z \tag{47}$$

and

$$\overline{\pi}_i^{z_i^S}(\underline{c}_i) \leq \overline{\pi}_i^z(\underline{c}_i) \text{ for all } z \in Z. \tag{48}$$

Proposition 11 *Under Assumption 10 it follows that for all p (a) $(\hat{p}^{-i})^z \equiv \begin{cases} 1 & \text{if } z = z_i^S \\ 0 & \text{if not} \end{cases}$, for all i is an optimal non-participation assignment rule, (b) $c_i^S = \underline{c}_i$, for all i, and (c) $\underline{U}_i(c_i^S) \equiv \overline{\pi}_i^{z_i^S}(\underline{c}_i)$.*

Proof. (a)The optimality of $\hat{p}^{-i}$ follows immediately from (47) and (48).

(b) Now we show that $c_i^S = \underline{c}_i$, by establishing that if the participation constraint is satisfied at $c_i = \underline{c}_i$, then it is satisfied for all $c_i \in C_i$. This follows from three observations.

(i) $P_i(c_i) \in \partial V_i(c_i)$,

(ii)

$$\begin{aligned} P_i(c_i) &= \int\limits_{C_{-i}} \sum_{z\in Z} p^z(c) \frac{\partial \pi_i^z(c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i} \\ &\geq \int\limits_{C_{-i}} \sum_{z\in Z} p^z(c) \frac{\partial \pi_i^{z_i^S}(c_i, c_{-i})}{\partial c_i} f_{-i}(c_{-i}) dc_{-i} \\ &= \frac{d\bar{\pi}_i^{z_i^S}(c_i)}{dc_i} \end{aligned}$$

(iii) $V_i(\underline{c}_i) \geq \bar{\pi}_i^{z_i^S}(\underline{c}_i)$.

Observations (i) and (ii) imply that the derivative of V_i is always greater than the derivative of $\bar{\pi}_i^{z_i^S}$. These two, together with (iii) imply that $V(c_i) \geq \bar{\pi}_i^{z_i^S}(c_i)$ for all $c_i \in C_i$.

(c) Finally, it follows immediately that $\underline{U}_i(c_i^*) \equiv \bar{\pi}_i^{z_i^S}(\underline{c}_i)$. ■

II. Coexistence of Steep and Flat Outside Options: Participation Constraints bind at interior types $c_i^* \in (\underline{c}_i, \bar{c}_i)$.

Suppose that there are two extreme allocations for each buyer, one that gives the flattest payoff z_i^S, and one that gives the steepest, z_i^F. If the flattest option were to be used then $c_i^* = \bar{c}_i$ and if the steepest option were to be used, then $c_i^* = \underline{c}_i$. When neither of these two options is clearly worse, it turns out that an optimal $p^{-i}(p)$ randomizes between the two options and the participation constraint always binds at the type who is indifferent between z_i^S and z_i^F. We now describe the precise conditions and establish the claim.

Assumption 12 *Suppose that $Z^{-i} = \{z_i^S, z_i^F\}$ and that $\bar{\pi}_i^{z_i^S}(c_i), \bar{\pi}_i^{z_i^F}(c_i)$ satisfy $\frac{d\bar{\pi}_i^{z_i^S}(c_i)}{dc_i} \leq \frac{d\bar{\pi}_i^{z}(c_i)}{dc_i} \leq \frac{d\bar{\pi}_i^{z_i^F}(c_i)}{dc_i}$ for all $z \in Z$ and $c_i \in C_i$ and $\bar{\pi}_i^{z_i^S}(\underline{c}_i) \geq \bar{\pi}_i^{z_i^F}(\underline{c}_i)$. Suppose also that either (i) values are private or (ii) the seller can only use non-participation assignment rules that do not depend on the types of other players (that is $p^{-i} \in \mathcal{P}^{-i} \Longrightarrow p^{-i}(c_{-i}) \equiv p^{-i}$).*

Proposition 13 *Under Assumption 12 it follows that (a) for all p the critical type is $c_i^* = \hat{c}_i$ where $\hat{c}_i$ satisfies*

$$\bar{\pi}_i^{z_i^S}(\hat{c}_i) = \bar{\pi}_i^{z_i^F}(\hat{c}_i) \tag{49}$$

(b) an optimal p^{-i} given p is determined by the condition $(p^{-i}(p))^{z_i^S} \frac{d\bar{\pi}_i^{z_i^S}(\hat{c}_i)}{dc_i} + (1-(p^{-i}(p))^{z_i^F}) \frac{d\bar{\pi}_i^{z_i^F}(\hat{c}_i)}{dc_i} \in \partial V_i(\hat{c}_i)$, and (c) for all p and $p^{-i}(p)$ we have $\underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}(p)) = \bar{\pi}_i^{z_i^F}(\hat{c}_i) = \bar{\pi}_i^{z_i^S}(\hat{c}_i)$.

Proof: To prove this Proposition, we first prove the following Lemma:
Lemma A.

$$\left.\frac{dV_i(\bar{c}_i)}{d(\rho^{-i})^z}\right|_{\rho^{-i}=p^{-i}(p)} = \left.\frac{\partial V_i(\bar{c}_i)}{\partial(\rho^{-i})^z}\right|_{\rho^{-i}=p^{-i}(p)} = \bar{\pi}_i^z(c_i^*(p, p^{-i}(p))), \text{ for all } z \in Z^{-i}. \tag{50}$$

Proof. We suppose for simplicity that the derivative $\frac{\partial c_i^*(p,\rho^{-i})}{\partial \rho^{-i}}$ is well defined, (otherwise we can do all the analysis with subgradients).

Then, differentiating $V_i(c_i) = \underline{U}_i(c_i^*(p, \rho^{-i}(p)), \rho^{-i}) - \int_{c_i}^{c_i^*(p,\rho^{-i}(p))} P_i(s)ds$ with respect to $(\rho^{-i})^z$ we obtain that

$$\frac{dV_i(\bar{c}_i)}{d(\rho^{-i})^z} = \frac{\partial \underline{U}_i(c_i^*(p, \rho^{-i}), \rho^{-i})}{\partial(\rho^{-i})^z} + \left[\frac{\partial \underline{U}_i(c_i^*(p, \rho^{-i}), \rho^{-i})}{\partial c_i} - P_i(c_i^*(p, \rho^{-i}))\right] \frac{\partial c_i^*(p, \rho^{-i})}{\partial(\rho^{-i})^z}. \tag{51}$$

Given an assignment rule p and a non-participation assignment rule ρ^{-i}, we know that at an optimal mechanism $c_i^*(p, \rho^{-i})$ satisfies $c_i^*(p, \rho^{-i}) \in \arg\min_{c_i} \left[-\int_{c_i}^{\bar{c}_i} P_i(s)ds - \underline{U}_i(c_i, \rho^{-i})\right]$. Depending on whether $c_i^*(p, \rho^{-i}) \in (\underline{c_i}, \bar{c}_i)$, or $c_i^*(p, \rho^{-i}) = \underline{c_i}$ or $c_i^*(p, \rho^{-i}) = \bar{c}_i$, there are three cases to consider.

Case 1: $c_i^*(p, \rho^{-i}) \in (\underline{c}_i, \bar{c}_i)$

Since $c_i^*(p, \rho^{-i}) \in \arg\min_{c_i} \left[-\int_{c_i}^{\bar{c}_i} P_i(s)ds - \underline{U}_i(c_i, \rho^{-i})\right]$, an interior solution (which is precisely the case under investigation), must satisfy

$$\left.\frac{dV_i(c_i)}{dc_i}\right|_{c_i=c_i^*(p,\rho^{-i})} = \left.\frac{\partial \underline{U}_i(c_i(p, \rho^{-i}), \rho^{-i})}{\partial c_i}\right|_{c_i=c_i^*(p,\rho^{-i})}. \tag{52}$$

Then recall that $V_i(c_i) = V_i(\bar{c}_i) - \int_{c_i}^{\overline{c_i}} P_i(s)ds$, which implies that

$$\left.\frac{dV_i(c_i)}{dc_i}\right|_{c_i=c_i^*(p,\rho^{-i})} = P_i(c_i^*(p, \rho^{-i})). \tag{53}$$

Then, substituting (52) and (53) into (51), we obtain that

$$\left.\frac{dV_i(\bar{c}_i)}{d(\rho^{-i})^z}\right|_{\rho^{-i}=p^{-i}(p)} = \frac{\partial \underline{U}_i(c_i^*(p, p^{-i}(p)), p^{-i}(p))}{\partial(\rho^{-i})^z} = \bar{\pi}_i^z(c_i^*(p, p^{-i}(p))), \text{ for all } z \in Z^{-i},$$

which is what we wanted to show.

Case 2: $c_i^*(p, \rho^{-i}) = \underline{c_i}$

If p and ρ^{-i}such that $c_i^*(p, \rho^{-i}) = \underline{c_i}$ and we change z^{th} component of the non-participation assignment rule p^{-i} then two things can happen. One possibility is that

$$\frac{\partial c_i^*(p, \rho^{-i})}{\partial(\rho^{-i})^z} = 0,$$

in that case (51), reduces to (50). Another possibility is that we move to a c_i^* in the interior, in which case we are back to Case 1.[37]

Case 3: $c_i^*(p,\rho^{-i}) = \bar{c}_i$

This case is identical to the previous one. ■

Now, we prove the Proposition.

(a) Because there are only z_i^S and z_i^F in Z^{-i}, we can write

$$V_i(\bar{c}_i) = \rho^{-i}\bar{\pi}_i^{z_i^S}(c_i^*(p,\rho^{-i})) + (1-\rho^{-i})\bar{\pi}_i^{z_i^F}(c_i^*(p,\rho^{-i})) + \int_{c_i^*(p,\rho^{-i})}^{\bar{c}_i} P_i(s)ds.$$

Because of the envelope condition proved before, that is (50), we can write

$$\begin{aligned} \left.\frac{dV_i(\bar{c}_i)}{d\rho^{-i}}\right|_{\rho^{-i}=p^{-i}(p)} &= \left.\frac{\partial V_i(\bar{c}_i)}{\partial\rho^{-i}}\right|_{\rho^{-i}=p^{-i}(p)} \\ &= \bar{\pi}_i^{z_i^S}(c_i^*(p,\rho^{-i})) - \bar{\pi}_i^{z_i^F}(c_i^*(p,\rho^{-i})). \end{aligned} \tag{54}$$

When ρ^{-i} is in a neighborhood of 0 then the outside option is flat and $c_i^* = \bar{c}_i$. When ρ^{-i} is in a neighborhood of 1 then the outside option is very steep and $c_i^* = \underline{c}_i$. This means that $\left.\frac{\partial c_i^*(p,\rho^{-i})}{\partial\rho^{-i}}\right|_{\rho^{-i}=0} = \left.\frac{\partial c_i^*(p,\rho^{-i})}{\partial\rho^{-i}}\right|_{\rho^{-i}=1} = 0$, and also we get that

$$\begin{aligned} \left.\frac{dV_i(\bar{c}_i)}{d\rho^{-i}}\right|_{\rho^{-i}=0} &= \bar{\pi}_i^{z_i^S}(c_i^*(p,0)) - \bar{\pi}_i^{z_i^F}(c_i^*(p,0)) \\ &= \bar{\pi}_i^{z_i^S}(\bar{c}_i) - \bar{\pi}_i^{z_i^F}(\bar{c}_i) < 0 \end{aligned}$$

$$\begin{aligned} \left.\frac{dV_i(\bar{c}_i)}{d\rho^{-i}}\right|_{\rho^{-i}=1} &= \bar{\pi}_i^{z_i^S}(c_i^*(p,1)) - \bar{\pi}_i^{z_i^F}(c_i^*(p,1)) \\ &= \bar{\pi}_i^{z_i^S}(\underline{c}_i) - \bar{\pi}_i^{z_i^F}(\underline{c}_i) > 0. \end{aligned}$$

These two inequalities imply that the optimally chosen ρ^{-i}, that is $p^{-i}(p)$, is interior, so it satisfies the FONC $\left.\frac{dV_i(\bar{c}_i)}{d\rho^{-i}}\right|_{\rho^{-i}=p^{-i}(p)} = 0$, because of (54) it implies $\bar{\pi}_i^{z_i^S}(c_i^*(p,\rho^{-i})) = \bar{\pi}_i^{z_i^F}(c_i^*(p,\rho^{-i}))$, from which we get that irrespective of p we have that

$$c_i^* = \hat{c}_i,$$

where $\hat{c}_i$ satisfies (49). Moreover, because of the assumptions, the functions $\bar{\pi}_i^{z_i^S}$ and $\bar{\pi}_i^{z_i^F}$ cross at most once, so c_i^* is uniquely determined.

[37] Note that since both V_i and $\underline{U}_i$ are decreasing and convex in c_i, so changing $(p^{-i})^z$ slightly cannot result in c_i^* moving from $\underline{c}_i$ to $\bar{c}_i$.

(b) By (8) it follows immediately that an optimal p^{-i} given p must satisfy that $p^{-i}(p)\frac{d\bar{\pi}_i^{z_i^S}(\hat{c}_i)}{dc_i}+(1-p^{-i}(p))\frac{d\bar{\pi}_i^{z_i^F}(\hat{c}_i)}{dc_i}\in\partial V_i(\hat{c}_i)$.

(c) Is immediate. ■

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