Strategic Defense and Attack for Series and Parallel Reliability Systems: Simultaneous Moves by Defender and Attacker

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Abstract

A system of independent components is defended by a strategic defender and attacked by a strategic attacker. The reliability of each component depends on how strongly it is defended and attacked, and on the intensity of the contest. In a series system the attacker benefits from a substitution effect since attacker benefits flow from attacking any of the components, while the defender needs to defend all components. Even for a series system, when the attacker is sufficiently disadvantaged with high attack inefficiencies, and the intensity of the contest is sufficiently high, the defender earns maximum utility and the attacker earns zero utility. The results for the defender (attacker) in a parallel system are equivalent to the results for the attacker (defender) in a series system. Hence the defender benefits from the substitution effect in parallel systems. With budget constraints the ratio of the investments for each component, and the contest success function for each component, are the same as without budget constraints when replacing the system values for the defender and attacker with their respective budget constraints. For a series system of parallel subsystems, inifinitely many components in parallel benefits the defender maximally regardless of the finite number of parallel subsystems in series. Ceteris paribus, and with equivalence requirements, the defender prefers the series system of parallel subsystems rather than the parallel system of series subsystems, and conversely for the attacker.

1 Introduction

This is the first article in a sequence of two that analyzes strategic defense and attack for series and parallel reliability systems. This article considers simultaneous moves by the defender and attacker. The second article considers sequential moves by the defender and attacker.

A large literature on security and safety applies reliability analysis to determine the most cost-effective risk reduction strategies. Examples are Levitin (2002, 2003a, 2003b), Levitin and Lisnianski (2000, 2001, 2003), and Levitin et al. (2003). The external threat is usually assumed to be static. Within cyber security Gordon and Loeb (2002) and Gordon et al. (2003) determine the optimal investment for information protection, and Gal-Or and Ghose (2005) analyze how market characteristics affect security investment. Again the threat is assumed fixed and immutable.

Some research applying game theory considers components in isolation (Major 2002; Woo, 2002, 2003; O'Hanlon et al, 2002). For multiple components one strand of literature associates one defender with each component. Hausken (2002) lets each agent dichotomously choose a strategy which for its component causes either reliability zero with no cost of effort or reliability one for a fixed cost of effort. He finds that the series, parallel, and summation systems frequently correspond to the coordination game, the battle of the sexes and the chicken game, and prisoner's dilemma, respectively.

Kunreuther and Heal (2003) analyze interdependent systems where one target's defense benefits all targets.[1] Bier and Gupta (2006) explore the effects of heterogeneous discount rates on the optimal defensive strategy in such systems. Hausken (2006) finds that with increasing interdependence, each defending agent free rides by investing less, suffers lower profit, while the attacker enjoys higher profit. Enders and Sandler (2003) and Hausken (2006) analyze the substitution effect which causes a strategic attacker to substitute into the most optimal attack allocation across multiple targets, and the income effect which eliminates parts of the attacker's resource base.

[1] Examples occur within the airline industry, computer networks, fire protection, theft protection, bankruptcy protection, vaccinations.

Another strand of literature lets one defender defend entire systems. For series and parallel systems with independent components Bier and Abhichandani (2002) and Bier et al. (2005) assume that the defender minimizes the success probability, and expected damage, respectively, of an attack. The success probability depends on the resources expended by the defender to strengthen each component.[2] Although the approach implicitly accounts for a strategic attacker (for series systems the defender equalizes the expected damage of attacks against multiple components), a more general approach would assume that the success probability of an attack depends on resource investments by both the defender and the attacker for each component. The attack September 11, 2001 illustrated that major threats today involve strategic attackers. Threats emerge from nature, technology, and humans, but increasing complexity and human involvement suggest that the strategic factor needs to be given increased emphasis in future research.

Azaiez and Bier (2006) discuss various simplifications to this general approach. For example, the level of effort expended by the attacker on each component could be a constant. They choose the simplification that the success probability of an attack on each component is constant, and assume that the defender attempts to deter attacks by making them as costly as possible to the attacker. This enables them to find closed-form results for systems with moderately general structures with both parallel and series subsystems.

The objective of this article is to extend the research where an entire system of independent components is defended by a fully strategic defender and attacked by a fully strategic attacker. The external threat is neither static, fixed, nor immutable. Series and parallel systems are considered, as well as series systems of parallel subsystems, and parallel systems of series subsystems. The defender and attacker adapt to each other optimally choosing continuous strategic variables for each component under defense and attack. The reliability of each component depends on the relative investments the defender and attacker direct into defending versus attacking that component. The defender seeks to maximize the reliability of the system, accounting for its assessment of the value of the system, while the attacker seeks to minimize the

[2] Further results are by Bier et al. (2005: 322) who show that "if one component is more valuable than another, but has a lower probability of being attacked, then the more vulnerable but less valuable component may be more likely to be attacked, and hence merit greater investment." Also, Bier et al. (2006) analyze the optimal allocation of defensive resources in the face of uncertainty about attacker goals, motivations, and valuations of potential targets.

reliability, accounting for its often different assessment of the system value. One paramount consideration is how each agent substitutes investments across multiple components.

One defense inefficiency and one attack inefficiency are associated with each component, which specify unit costs of defense and attack. Such inefficiencies vary considerably across components. A component such as the US Gold Reserve stored at Ft. Knox has high defense inefficiency and even higher attack inefficiency. It is located for optimal defense, and is very hard to attack. Another component such as the U.S. Statue of Liberty has a more vulnerable location which increases the defense inefficiency and decreases the attack inefficiency. A component such as an underground transport system has high defense inefficiency since it is geographically dispersed, and low attack inefficiency. In contrast, a component buried deep within a mountain has low defense inefficiency and high attack inefficiency.

The article assumes variation in the intensity of the contest between the defender and attacker for each component. The intensity can vary greatly across components. Low intensity occurs for components and systems that are defendable, and where the individual components are dispersed. In such cases neither the defender nor the attacker can easily get a significant upper hand. High intensity occurs for systems that are easier to attack, and where the individual components are concentrated. This may cause “winner-take-all” battles and dictatorship by the strongest agent.

A crucial issue for defense and attack is how the various components, such as in these examples, are interlinked in series and parallel. As is conventional in the literature, components are assumed independent. If they are not, the analysis presumes application of Simon’s (1969:217) principle of “near decomposability”. Complex or hierarchic systems are frequently nearly decomposable, and intracomponent linkages are generally stronger than intercomponent linkages. Multiple subcomponents that are sufficiently interdependent are joined together to form one larger aggregate component which thus has a more complex internal structure from which the relevant parameters such as the unit costs of defense and attack, and the intensity of the contest, are determined. This process continues until each component is sufficiently independent from the other components so that the analysis can justifiably assume independent components as an approximation.

Section 2 introduces component and system reliabilities, utilities, and contest success functions. Sections 3 and 4 analyze the series system without and with budget constraints. Sections 5 and 6 analyze the parallel system without and with budget constraints. Section 7 considers the series system of parallel subsystems, and section 8 considers the parallel system of series subsystems. Section 9 concludes.

2 Component and system reliabilities, utilities, and contest success functions

A system of components configured in some manner is under attack. A defender of the system invests in security technology and safety measures to ensure that the system is secure and safe, which is needed for it to function reliably. The defender incurs an effort t_i, hereafter referred to as an investment, at unit cost c_i to defend component i. Higher c_i means greater inefficiency of investment, and $1/c_i$ is the efficiency. The security and safety investment expenditure is f^i, i=1,…,n, which can be capital and/or labor, where $\partial f^i/\partial t_i$>0. We consider the simple case $f^i = c_i t_i$. If the system is a cyber security system, the defender hires security experts, installs firewalls, applies encryption techniques, access control mechanisms, develops intrusion detection systems, and designs the optimal defense for the system. If the system consists of serially linked components in a societal infrastructure, for example production of goods and services such as water and food, communication, transport, finance, governmental functions, and health services, the investments consist in safeguarding the components with human inspection and patrolling, development of procedures, technology investments, surveillance of potential sources of threats, elimination of threats, and deterrence.

Conversely, the attacker seeks to attack the system to ensure that it does not function reliably. Analogously, it incurs an efforts T_i at unit cost C_i to attack component i. C_i is the inefficiency of investment, and $1/C_i$ is the efficiency. Its investment expenditure is F^i, capital and/or labor, i=1,…,n, where $\partial F^i/\partial T_i$>0. We assume $F^i = C_i T_i$. If the system is a cyber security system, the attacker seeks to break through the security defense, circumvent the work of the security experts, penetrate the firewalls, decipher the encryption, and bypass the access control mechanisms and intrusion detection systems. A successful attack reduces the reliability of the system through appropriating, getting access to, or confiscating, something of value within or related to the system, or securing information which can be used as means of reducing system reliability. If the system is a part of the societal infrastructure, the attacker seeks to reduce the reliability through destruction, distortion, theft, and interfering with production, human inspection and patrolling, avoidance of surveillance, covert action to avoid detection, manipulation of information, and public revelation of system weaknesses. Both the defender and the attacker are assumed to be risk neutral.[3]

We formulate the reliability p_i of component i, i=1,…,n, as a contest success function p_i between the defender and attacker. The reliability in our context corresponds to the asset in the conflict literature (Hausken 2005). There is conflict over reliability between the defender and the attacker, just as there is conflict over an asset between contending agents. The defender enjoys contest success p_i, while the attacker enjoys contest success $1-p_i$. Skaperdas (1996) has presented three axioms for contest success functions between two agents. First, $0 \leq p_i \leq 1$, and the contest success for the defender and attacker sum to one. Second, $\partial p_i / \partial t_i > 0$ and $\partial p_i / \partial T_i$ which means that the reliability increases in the defender's investment, and decreases in the attacker's investment. Third, each agent's contest success depends on its investment, t_i or T_i, and not on the identity of the agent or opponent (anonymity property).

The reliability p_i can be given two interpretations. The first is that component i is 100% reliable with probability p_i, and 100% unreliable with probability $1- p_i$. This means 100% functionality or 100% incapacitation. The second is that component i functions deterministically to a fixed degree p_i. That is, with 100% certainty damage is caused to component i, but the component functions with guaranteed reliability p_i nevertheless. For the second interpretation consider a defensive force and an offensive force fighting to keep a road open versus blocked. Neither side is 100% successful. Assume that p_i=0.65 so that the defense has 65% success keeping the road open while the attack has 35% success keeping the road blocked. This means that 65% of all traffic passes through the road.

The reliability p of a system is determined from the reliabilities p_i of the individual components dependent on how these are configured in series and parallel. The two interpretations of reliability also apply for the system as a whole. The defender and attacker are concerned about the damage to the system. The defender seeks to minimize the damage, while the attacker seeks

[3] An alternative analysis may assume that the defender is risk averse and the attacker risk seeking. Assuming risk neutrality simplifies the analysis. Much of the economic conflict literature related to production, appropriation, defense, and rent seeking assumes risk neutrality. See Skaperdas (1991) for an exception.

to maximize it. We express the linkage from reliability to damage for the defender as d(t,T)=r(1-p), where r is the damage as perceived by the defender if the system is 100% disabled. We also refer to r as the value of the system for the defender when it is not under attack, and rp as the system value when it is under attack. The defender's utility is

$$u = rp - \sum_{i=1}^{n} c_i t_i \tag{1}$$

which the defender seeks to maximize, and which can be measured e.g. in dollar, and where the expenditures of defending the n components are subtracted. The defender seeks to increase the system reliability, but not at any expenditure. When the expenditures exceed the benefit, the defender chooses zero effort $t_i=0$ for all components, and earns zero utility. The defender and attacker have subjective utilities and often assess damage differently. We express the linkage from reliability to damage for the attacker as D(t,T)=R(1-p), where R is the damage as perceived by the attacker if the system is 100% disabled. We also refer to R as the value of the system for the attacker. The attacker's utility is

$$U = R\left(1-p\right) - \sum_{i=1}^{n} C_i T_i \tag{2}$$

which the attacker seeks to maximize, and where the expenditures of attacking the n components are subtracted. Analogously, the attacker seeks to decrease the system reliability, but not at any expenditure.

The most common functional form for the contest success function is the ratio form

$$p_i = \frac{t_i^{m_i}}{t_i^{m_i} + T_i^{m_i}} \tag{3}$$

where $m_i \geq 0$ is a parameter for component i. The second most common form is the difference or logit form

$$p_i = \frac{Exp(a_i t_i)}{Exp(a_i t_i) + Exp(a_i T_i)} \tag{4}$$

where $a_i \geq 0$ is a mass effect parameter for component i. Both these two forms satisfy the three axioms above. Choosing between these functional forms is analogous to choosing e.g. between Cobb-Douglas and CES production and utility functions in economics. The results are usually qualitatively similar, but there can be differences e.g. for investments close to zero or infinity. Determining the appropriate functional form is an important empirical issue, as is also determining the appropriate values of m_i and a_i, the unit costs of investments, and the values the defender and attacker assess for the value of the system. Hybrids of (3) and (4) can also be considered, and different m_i's and a's for the defender and attacker.

Fig. 1 illustrates for the ratio form how, for T_i held fixed at T_i=1 for the attacker, the reliability p_i responds to changes in the investment t_i for the defender. At the limit, with infinitely much defensive investment, and finite offensive investment, component i is 100% reliable. The same result follows with finite defensive investment and zero offensive investment. At the other limit, with infinitely much offensive investment, and finite defensive investment, component i is 0% reliable. The same result follows with finite offensive investment and zero defensive investment. The sensitivity of p_i to t_i increases as m_i increases. When m_i=0, the investments t_i and T_i have equal impact on the reliability regardless of their size which gives 50% reliability, p_i=1/2.[4] $0<m_i<1$ gives a disproportional advantage of investing less than one's opponent. When m_i=1, the investments have proportional impact on the reliability. $m_i>1$ gives a disproportional advantage of investment more effort than one's opponent. This is often realistic in praxis, as evidenced by benefits from economies of scale. Finally, $m_i=\infty$ gives a step function where "winner-takes-all". That is, the defender earns zero reliability when t_i is marginally smaller than T_i, and earns maximum reliability when t_i is marginally larger than T_i. This illustrates that the contest intensifies as m increases.

The parameter m_i is a characteristic of the contest over component i. It can be illustrated by the history of warfare. Low intensity occurs for components that are defendable, predictable, and where the individual ingredients of each components are dispersed, i.e. physically distant or separated by barriers of various kinds. Neither the defender nor the attacker can get a significant upper hand. An example is the time prior to the emergence of cannons and modern fortifications in the fifteenth century. Another example is entrenchment combined with the machine gun, in multiply dispersed locations, in World War I (Hirshleifer 1995:32-33). High m_i occurs for components that are less predictable, easier to attack, and where the individual ingredients of each component are concentrated, i.e. close to each other or not separated by particular barriers. This may cause "winner-take-all" battles and dictatorship by the strongest. Either the defender or

[4] In the conflict literature this is referred to as egalitarian distribution of an asset independent of effort, so that each agent receives 50%. In our context m_i=0 gives "egalitarianism" between the defender and the attacker in the sense that the defender obtains half as much reliability as it maximally hopes for for component i. We ignore $m_i<0$ which corresponds in one sense to altruism and in another sense to punishing individual investments and

the attacker may get the upper hand. The combination of airplanes, tanks, and mechanized infantry in World War II allowed both the offense and defense to concentrate firepower more rapidly, which intensified the effect of force superiority.[5]

Fig. 2 illustrates for the difference form how, for T_i held fixed at T_i=1 for the attacker, the reliability p_i responds to changes in the investment t_i for the defender. The curves have large similarities with Fig. 1. A main difference is that the reliability is strictly positive p_i>0 also when the defender invests zero, t_i =0. If the defender invests zero, it is not always realistic that the defender suffers zero reliability when the attacker invests a finite, and possibly arbitrarily small, amount. This is possible for components that are technologically designed in a hardened manner, or when the attacker is less than fully alert and determined. Hirshleifer (1989:104) argues that "in a military context we might expect the ratio form of the Contest Success Function to be applicable when clashes take place under close to 'idealized' conditions such as: an undifferentiated battlefield, full information, and unflagging weapons effectiveness. In contrast, the difference form tends to apply where there are sanctuaries and refuges, where information is imperfect, and where the victorious player is subject to fatigue and distraction." Hence, applying the difference form, in struggles between nations, one side may surrender rather than resist against an unappeasable opponent, with the expectation of not losing everything, realizing the cost to the victor of locating and extracting all the spoils.

3 Series system

The reliability of a series system is a benefit to the defender and equals the product of the component reliabilities p_i, i=1,…,n. This benefit expresses the value of system functionality, scaled between zero and one. The benefit is scaled relative to the expenditures, by adjusting the c_i's. Inserting into (1), and applying the contest success function in (3), we model the defender's utility as[6]

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placing a premium on laziness.

[5] We could assume different m_i's for the offense and defense, e.g. high m_i for the offense if enabled to concentrate firepower and low m_i for the defense if not enabled to concentrate firepower. But, this has not been common in the conflict literature, and makes the analysis intractable. Furthermore, different unit costs c_i and C_i of effort for the defender and offender gives all the differentiating characteristics we need since one contestant less enabled to concentrate firepower can be modeled as incurring a higher unit cost of generating firepower.

[6] Equation (5) corresponds to Bier et al.'s (2005) analysis in that they also consider the value of system functionality, but differs in that they also consider the value of each component, i.e. the loss if one component is disabled irrespective of whether the system is disabled.

$$u = r\prod_{i=1}^{n}\frac{t_i^{m_i}}{t_i^{m_i}+T_i^{m_i}} - \sum_{i=1}^{n} c_i t_i \tag{5}$$

In contrast, the attacker seeks to decrease the system reliability. Since $0 \le \prod_{i=1}^{n} p_i \le 1$, we formulate this such that the attacker seeks to increase $1-\prod_{i=1}^{n} p_i$. Inserting into (2), the attacker's utility is

$$U = R\left(1-\prod_{i=1}^{n}\frac{t_i^{m_i}}{t_i^{m_i}+T_i^{m_i}}\right) - \sum_{i=1}^{n} C_i T_i \tag{6}$$

The defender has n free choice variables t_i. The attacker has n free choice variables T_i. Applying the ratio form in (3), the 2n FOCs are

$$\frac{\partial u}{\partial t_i} = r\frac{m_i t_i^{m_i-1}T_i^{m_i}}{(t_i^{m_i}+T_i^{m_i})^2}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{t_j^{m_j}}{t_j^{m_j}+T_j^{m_j}} - c_i = 0, \quad \frac{\partial U}{\partial T_i} = R\frac{m_i T_i^{m_i-1}t_i^{m_i}}{(t_i^{m_i}+T_i^{m_i})^2}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{t_j^{m_j}}{t_j^{m_j}+T_j^{m_j}} - C_i = 0 \tag{7}$$

The FOCs give $t_i = T_i rC_i / Rc_i$. In the case $c_i/r=C_i/R$ zero investment $T_i=t_i=0$ is not possible since one agent investing arbitrarily little earns a jump from 50% to 100% of the reliability r. Applying the difference form in (4), the 2n FOCs are

$$\frac{\partial u}{\partial t_i} = r\frac{a_i Exp[a_i(t_i+T_i)]}{(Exp(a_i t_i)+Exp(a_i T_i))^2}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{Exp(a_j t_j)}{Exp(a_j t_j)+Exp(a_j T_j)} - c_i = 0,$$

$$\frac{\partial U}{\partial T_i} = R\frac{a_i Exp[a_i(t_i+T_i)]}{(Exp(a_i t_i)+Exp(a_i T_i))^2}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{Exp(a_j t_j)}{Exp(a_j t_j)+Exp(a_j T_j)} - C_i = 0 \tag{8}$$

$$\Rightarrow \frac{1}{r}\left(\frac{\partial u}{\partial t_i}+c_i\right) = \frac{1}{R}\left(\frac{\partial U}{\partial T_i}+C_i\right)$$

A special case of (8) has been analyzed by Hirshleifer (1989). In the case $c_i/r=C_i/R$, zero investment $T_i=t_i=0$ is possible since one agent investing arbitrarily little does not earn a jump from 50% to 100% of the reliability r. As an example, assume $T_i=0$. Inserting into (8) implies that $t_i=0$ is a solution when $c_i / r = a_i / 2^{n+1}$. When $c_i/r \neq C_i/R$, there is no interior solution to (8). As Hirshleifer (1989:107) points out, the reason is that for the difference form the marginal cost of contest effort is constant. The implication for our purpose is that the defender enjoys 100% reliability for component i when $c_i/r>C_i/R$, that the attacker enjoys 0% reliability for component i when $c_i/r<C_i/R$, and that they both earn 50% reliability when $c_i/r=C_i/R$. In this latter case (8) gives one equation with two unknown t_i and T_i. The characteristics of (8) are such that the ratio

form which allows for general and often realistic interior solutions is more interesting to analyze, and is the main focus in this article.

Proposition 1. *The ratio of investments for each component is inverse proportional to the ratio of the unit costs to the system value,* $t_i / T_i = (C_i / R)/(c_i / r)$*, unrelated to that of the other components. The investment expenditures relative to the system value for the defender and the attacker are equal for each component, that is* $t_i c_i / r = T_i C_i / R$*, regardless of the parameter* m_i.

Proposition 1 follows since components are assumed independent. It states that neither the defender nor the attacker has an incentive to increase or decrease his/her investment expenditure relative to the system value above or below that of the opponent. An increase for the defender would increase the reliability, but the additional cost is not worth it. Conversely, a decrease for the defender would decrease the reliability below the optimum, and incurring additional cost would be appropriate. Deviation from equality is equally suboptimal for the attacker. For a given ratio of attack expenditure to system value, $T_i C_i / R$, increasing (decreasing) the defense inefficiency c_i for component i causes the defender to decrease (increase) its investment t_i such that $t_i c_i / r = T_i C_i / R$ is preserved. The logic is analogous for the attacker.

Solving (7) gives

$$\begin{aligned} t_i &= \frac{m_i (c_i/r)^{m_i-1}(C_i/R)^{m_i}}{((c_i/r)^{m_i}+(C_i/R)^{m_i})^2}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{(C_j/R)^{m_j}}{(c_j/r)^{m_j}+(C_j/R)^{m_j}}, \quad \frac{t_i}{T_i}=\frac{C_i/R}{c_i/r} \\ T_i &= \frac{m_i (C_i/R)^{m_i-1}(c_i/r)^{m_i}}{((c_i/r)^{m_i}+(C_i/R)^{m_i})^2}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{(C_j/R)^{m_j}}{(c_j/r)^{m_j}+(C_j/R)^{m_j}}, \quad \frac{t_i}{t_i+T_i}=\frac{C_i/R}{C_i/R+c_i/r} \end{aligned} \tag{9}$$

Proposition 2. *(a) The investments decrease in n. (b) Assume* $m_i = m_j = 1$*. For the defender,* $\partial t_i / \partial(c_i / r) < 0$*,* $\partial t_i / \partial(c_j / r) < 0$*,* $\partial t_i / \partial(C_i / R) < 0$ *when* $c_i / r < C_i / R$*,* $\partial t_i / \partial(C_j / R) > 0$*. For the attacker,* $\partial T_i / \partial(C_i / R) < 0$*,* $\partial T_i / \partial(C_j / R) > 0$*,* $\partial T_i / \partial(c_i / r) > 0$ *when* $C_i / R > c_i / r$*,* $\partial T_i / \partial(c_j / r) < 0$*,* $i,j = 1,\dots,n$*,* $i \neq j$*.*

Proof. (a) Since $(C_j/R)^{m_j}/((c_j/r)^{m_j}+(C_j/R)^{m_j})<1$ in (9), adding an additional component causes multiplication with yet another ratio less than one, which causes lower t_i and lower T_i. (b) See Appendix 1.

Both agents invest less when the number of components increases. It is more difficult for the defender to defend many components in a series system. And, the attacker benefits without having to invest much. When m_i=m_j=1, the defender invests less to defend component i when any defense inefficiency increases. Conversely, it invests more in component i when the attack inefficiency of component i increases given that $c_i/r > C_i/R$, and it invests more in component i when the attack inefficiency of any component j increases. Higher defense inefficiency for any component has impact throughout causing lower investment. In contrast, higher attack inefficiency for a given component causes higher defense investment for all other components, and for the given component if high ratio of unit cost to system value c_i/r.

Analogously, the attacker invests less to attack component i when the attack inefficiency of component i increases. However, in contrast to the defender, the attacker invests more to attack component i when the attack inefficiency of component j increases. The substitution effect applies strongly for the attacker, but less so for the defender. When the attacker suffers attack inefficiency for component j, it substitutes to attacking component i. The reason is that the attacker benefits from attacking any of the components in a series system, and choosing the most cost effective attack is beneficial. The defender needs to defend all components for the series system to function. It cannot substitute strongly from component j to component i when the inefficiency of component j increases (since $\partial t_i/\partial(c_j/r)$<0) since that would make component j vulnerable for a strong attack due to substitution by the attacker. Mathematically, observe in (9) that c_j/r is not present in the numerator of t_i, while C_j/R is indeed present in the numerator of T_i.

However, when choosing t_i and assuming m_i=m_j=1, the defender is more negatively influenced by a high c_i/r than by a high c_j/r. This can be interpreted as a moderate substitution effect also for the defense. This can be seen from the four denominators in (9) which have one square bracket and one bracket that is not squared. For component i the investments t_i and T_i have squared brackets in the denominators for the inefficiencies c_i / r and C_i / R, which causes reduced investment if these inefficiencies are large. We formulate this as a follows.

Proposition 3. *In a series system the attacker benefits more from the substitution effect than the defender since attacker benefits flow from attacking any of the components, while defender benefits flow from defending all components. The attacker is highly sensitive to the difference in the attack efficiencies for the components. The defender is less sensitive to the difference in the defense efficiencies for the components.*

Proposition 2 also states that when $m_i = m_j = 1$, the defender increases the investment t_i when C_j / R increases ($\partial t_i / \partial(C_j / R) > 0$). This is a natural consequence of the attacker's substitution effect. When the attack inefficiency C_j / R increases, the attacker substitutes to attacking component i, causing the defender to increase t_i to defend component i. Conversely, the attacker decreases the investment T_i when c_j increases ($\partial T_i / \partial(c_j / r) < 0$). The reason is that increasing c_j / r causes reduced investment also of t_i for the defender. To preserve $t_i c_i / r = T_i C_i / R$ in Proposition 1, T_i decreases too.

To further illustrate the substitution effect, (9) implies

$$\frac{t_i}{t_j} = \frac{m_i (c_i / r)^{m_i - 1} \left((c_j / r)^{m_j} + (C_j / R)^{m_j} \right)}{m_j (c_j / r)^{m_j - 1} \left((c_i / r)^{m_i} + (C_i / R)^{m_i} \right)}, \quad \frac{T_i}{T_j} = \frac{m_i (c_i / r)^{m_i} (C_j / R) \left((c_j / r)^{m_j} + (C_j / R)^{m_j} \right)}{m_j (c_j / r)^{m_j} (C_i / R) \left((c_i / r)^{m_i} + (C_i / R)^{m_i} \right)} \quad (10)$$

for any two components i,j=1,…,n, $i \neq j$. Both agents' investment ratios increases in m_i / m_j, so larger intensity causes larger investment for any given component. When $m_i = m_j = 1$, the ratio t_i / t_j of the defender's investments simply increases in the sum of the inefficiencies of component j, and decreases in the sum of the inefficiencies of component i. The defender is investment sensitive in a straightforward manner. However, when $m_i = m_j = 1$, the attacker's investment ratio T_i / T_j is impacted by two additional ratios. First, c_i / c_j partly counterbalances the ratio of the sums of the inefficiencies, tilting the balance toward attacking component i if the defense inefficiency of component i is large. Secondly, and more importantly, C_j / C_i

demonstrates the substitution effect. Component i is attacked more fiercely if the attack inefficiency of component j is large.

Inserting (9) into (5) and (6) gives the utilities

$$u = r\left(1-\sum_{i=1}^{n}\frac{m_i(c_i/r)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\right)\prod_{i=1}^{n}\frac{(C_i/R)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}},$$
$$U = R\left(1-\left(1+\sum_{i=1}^{n}\frac{m_i(c_i/r)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\right)\prod_{i=1}^{n}\frac{(C_i/R)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\right) \tag{11}$$

which depend on the ratio $(c_i/r)/(C_i/R)$, and not on the absolute values of c_i/r and C_i/R, When (11) gives negative utilities, a corner solution emerges with zero utility and zero investment for that agent (either the defender or the attacker) which according to (11) would otherwise get negative utility. Using (5) or (6), the other agent gets a utility equal to the value of the system by investing arbitrarily small but positive amounts into defending or attacking the components. The product $\prod_{i=1}^{n}(\cdot)$ causes high utility to the defender of high attack inefficiencies. Subtraction of this product causes low utility to the attacker. Let us consider three benchmarks. First, inserting m_i=0 for the "egalitarian" contest into (9) and (11) gives

$$t_i = T_i = 0,\quad u = \frac{r}{2^n},\quad U = R\left(1-\frac{1}{2^n}\right) \tag{12}$$

Egality has four implications. First, the agents incur zero investments since utilities are independent of investments. Second, for each component viewed in isolation the reliability is 50% which benefits the defender and attacker equally much when R=r. When R=r they meet half way between their preferred utility of the system value and the non-preferred utility zero. Third, the defender's utility equals ½ raised to the number of components for the series system, multiplied with r, which decreases convexly in n. Fourth, there is no waste in the sense that the two utilities sum to r when R=r, so the attacker receives r minus the utility of the defender, which increases concavely in n. For all positive values of m investments are positive which causes the sum of the utilities to the defender and attacker to be less than r when R=r. Second, inserting C_i/R=c_i/r into (9) and (11) gives

$$t_i = T_i = \frac{m_i}{2^{n+1}c_i/r},\quad u = r\left(1-\sum_{i=1}^{n}\frac{m_i}{2}\right)\frac{1}{2^n},\quad U = R\left(1-\left(1+\sum_{i=1}^{n}\frac{m_i}{2}\right)\frac{1}{2^n}\right) \tag{13}$$

Although the agents have equal unit costs of investment, the defender is strongly disadvantaged by both high m_i and high n. The defender's utility reaches zero when $\sum_{i=1}^{n}\frac{m_i}{2}\geq 1$, which makes the n components 100% insecure. The attacker also suffers from high m_i, but benefits from high n as shown in (A2). The sum of the utilities for the agents is $r\left(1-\frac{1}{2^{n-1}}\sum_{i=1}^{n}\frac{m_i}{2}\right)$ when R=r, which is less than r because of the wastage of investment. This sum decreases in m_i, but increases in n since the attacker benefits more than the defender suffers as the number of components increases. As our third benchmark, inserting $c_i=c$, $C_i=C$, $m_i=m$ into (9) and (11) gives

$$t_i=\frac{m(c/r)^{m-1}(C/R)^{mn}}{\left((c/r)^m+(C/R)^m\right)^{n+1}},\quad T_i=\frac{m(c/r)^m(C/R)^{mn-1}}{\left((c/r)^m+(C/R)^m\right)^{n+1}},$$

$$u=r\left(1-\frac{mn\left((c/r)/(C/R)\right)^m}{\left((c/r)/(C/R)\right)+1}\right)\frac{1}{\left(\left((c/r)/(C/R)\right)+1\right)^n},\tag{14}$$

$$U=R\left(1-\left(1+\frac{mn\left((c/r)/(C/R)\right)^m}{\left((c/r)/(C/R)\right)+1}\right)\frac{1}{\left(\left((c/r)/(C/R)\right)+1\right)^n}\right)$$

Proposition 4. *(a) When $C_i/R=c_i/r$, both agents' utilities decrease linearly in m_i. (b) When the defender is advantaged with a sufficiently small $(c_i/r)/(C_i/R)$ for all i, u increases in $m_i=m$. (c) When the attacker is advantaged with a sufficiently large $(c_i/r)/(C_i/R)$ for all i, U increases in $m_i=m$. (d) The defender's utility decreases in n. (e) The attacker's utility increases in n. (f) When $m_i=m,n,c_i/r,C_i/R$ in (11) are such that u<0 and U>0, which implies u=0 and U=R, the series system is 100% insecure. When $m,n,c_i/r,C_i/R$ in (11) are such that u>0 and U<0, which implies u=r and U=0, the series system is 100% secure.*

Proof. (a) Differentiating (13) gives $\partial u/\partial m_i=-r/2^{n+1}$, $\partial U/\partial m_i=-R/2^{n+1}$, $\partial^2 u/\partial m_i^2=\partial^2 U/\partial m_i^2=0$. (b) The bracket before $Ln\left((c/r)/(C/R)\right)$ in (A3) is negative when $(c/r)/(C/R)$ is arbitrarily small. $Ln\left((c/r)/(C/R)\right)$ is then also negative with arbitrarily high absolute value. Hence $\partial u/\partial m>0$. (c) The bracket before $Ln\left((c/r)/(C/R)\right)$ in (A4) is positive when $(c/r)/(C/R)$ is arbitrarily large, causing arbitrarily large $Ln\left((c/r)/(C/R)\right)$ and $\partial U/\partial m>0$. (d) and (e) See Appendix 1. (f) Follows from (11).

Although the agents' utilities decrease in the intensity of the contest for a broad set of parameter values, increased intensity can benefit the attacker, and even the defender, when advantaged with a sufficiently low unit cost of investment compared with that of the opponent. But ceteris paribus, adding several components to the series system benefits the attacker, and not the defender. Let us consider two examples.

Example 1. Inserting m_i=1 and n=2 into (11) gives

$$u = r\frac{\frac{C_1}{R}\frac{C_2}{R}\left(\frac{C_1}{R}\frac{C_2}{R}-\frac{c_1}{r}\frac{c_2}{r}\right)}{\left(\frac{c_1}{r}+\frac{C_1}{R}\right)^2\left(\frac{c_2}{r}+\frac{C_2}{R}\right)^2},\quad U = R\frac{\frac{C_1^2}{R}\frac{c_2^2}{r}+\frac{c_1^2}{r}\left(\frac{c_2}{r}+\frac{C_2}{R}\right)^2+\frac{c_1}{r}\frac{C_1}{R}\frac{c_2}{r}\left(\frac{C_2}{R}+2\frac{c_2}{r}\right)}{\left(\frac{c_1}{r}+\frac{C_1}{R}\right)^2\left(\frac{c_2}{r}+\frac{C_2}{R}\right)^2} \tag{15}$$

The defender's utility is positive when $C_1 C_2 / R^2 > c_1 c_2 / r^2$. Otherwise the defender gives up, does not invest, and earns zero utility, while the attacker earns a utility equal to the value of the system and invests arbitrarily little. The attacker's utility is always positive, but making the attacker maximally disadvantaged gives $\underset{\substack{C_1\to\infty\\ C_2\to\infty}}{Lim}\ u = r$, $\underset{\substack{C_1\to\infty\\ C_2\to\infty}}{Lim}\ U = 0$, using L'Hopital's rule. For a series system of two components where m_i=1, the product of the defense inefficiencies must be less than the product of the attack inefficiencies in order for the defender to earn positive utility. Series systems are hard to defend, and the defender needs superior defense efficiencies in order to ensure overall system reliability. The numerator in the attacker's utility in (15) contains the product of the defense inefficiencies raised to both the third and the fourth power, while the product of the attack inefficiencies are not raised to more than the second power.

Example 2. To place the attacker in a more disadvantaged situation, inserting $c_1/r=c_2/r=1$, $C_1/R=C_2/R=n=2$, $m_i=m$ into (9) and (11) gives

$$t_1 = t_2 = \frac{4^m m}{(1+2^m)^3},\quad T_1 = T_2 = \frac{2^{2m-1} m}{(1+2^m)^3},\quad u = r\frac{4^m(1+2^m-2m)}{(1+2^m)^3},\quad U = R\frac{1+3\times 2^m+2^{2m+1}(1-m)}{(1+2^m)^3} \tag{16}$$

For m=1, (16) gives $t_1=t_2=4/27$, $T_1=T_2=2/27$, u=4r/27, U=7R/27, with a utility advantage when R=r for the attacker for the series system despite the double attack inefficiencies. For m=2, (16) gives $t_1=t_2=32/125$, $T_1=T_2=16/125$, u=16r/125 and U=-19R/125, which implies a corner solution with u=r and U=0. High attack inefficiencies and high m induce a high toll on the attacker in series systems. For this series system, where the attacker is sufficiently disadvantaged with high attack inefficiencies, and the intensity of the contest is sufficiently high, a corner solution emerges with maximum utility r to the defender and minimum utility zero to the attacker.

Fig. 3 shows the four investments and two utilities (divided by system value) as functions of the defense inefficiency c_1/r when n=2 and $c_2/r=C_1/R=C_2/R=m_i=1$. Both the defender's investments t_1 and t_2 decrease in c_1/r, as stated in Proposition 2. The attacker's investment T_2 equals t_2 for these parameters, which decreases, while T_1 increases in c_1/r when $c_1/r < C_1/R$ (Proposition 2). The defender's utility decreases to zero when $c_1/r=1$, and the attacker's utility increases in c_1/r. When $c_1/r>1$, the defender earns zero, and the attacker earns one. Division with 2 in U/2R is for scaling purposes.

Fig. 4 keeps the same parameters except doubles C_2/R to $C_2/R=2$. The defender benefits since the range of positive utility increases to $0<c_1/r<2$. The attacker reduces T_2 as c_1/r increases, substituting to attacking component 1 (Proposition 2), causing the defender to increase t_1. Observe the inverse U form for T_1. When c_1/r is small, the defense t_1 is substantial, and the attacker chooses low T_1. As c_1/r increases toward 2, the defender decreases t_1 so substantially that the attacker can cash in on its investment. The defender faces a substantial threat, gives gradually up, and the attacker benefits. Inverse U form for investments has also been reported by Hausken (2006). The insight is that a contestant invests little when the threat is negligible or overwhelming, and invests maximum when the threat is intermediate.

4 Series system with budget constraints

Assume that the defender and attacker have budget constraints b and B,

$$\sum_{i=1}^{n} c_i t_i \le b, \quad \sum_{i=1}^{n} C_i T_i \le B \tag{17}$$

If b and B are large, the constraints are not binding, and the previous analysis applies. This section assumes that both the constraints are binding. (The two cases when one constraint is binding and the other not binding, are cumbersome to analyze.) Assume without loss of generality that the investments for the n'th component are

$$t_n = \frac{b - \sum_{i=1}^{n-1} c_i t_i}{c_n}, \quad T_n = \frac{B - \sum_{i=1}^{n-1} C_i T_i}{C_n} \tag{18}$$

The investments for components 1,…,n-1 are determined by optimization. The analysis is analytically intractable for general m_i, so we set m_i=1. In order to differentiate with respect to t_1 and T_1 for the first component, we write the utilities in (5) and (6) as

$$\begin{aligned}
u &= r\frac{t_1}{t_1+T_1}s_{11}\frac{\dfrac{b-c_1t_1-\sum_{i=2}^{n-1}c_it_i}{c_n}}{\dfrac{b-c_1t_1-\sum_{i=2}^{n-1}c_it_i}{c_n}+\dfrac{B-C_1T_1-\sum_{i=2}^{n-1}C_iT_i}{C_n}}-b,\quad s_{11}=\prod_{i=2}^{n-1}\frac{t_i}{t_i+T_i},\quad s_1=\sum_{i=2}^{n-1}c_it_i,\\
U &= R\left(1-\frac{t_1}{t_1+T_1}s_{11}\frac{\dfrac{b-c_1t_1-\sum_{i=2}^{n-1}c_it_i}{c_n}}{\dfrac{b-c_1t_1-\sum_{i=2}^{n-1}c_it_i}{c_n}+\dfrac{B-C_1T_1-\sum_{i=2}^{n-1}C_iT_i}{C_n}}\right)-B,\quad S_1=\sum_{i=2}^{n-1}C_iT_i
\end{aligned}\tag{19}$$

where s_{11},s_1,S_1 are not functions of t_1 and T_1. Setting the derivatives with respect to t_1 and T_1 equal to zero, $\partial u/\partial t_1$=0 and $\partial U/\partial T_1$=0, and solving gives

$$\begin{aligned}
t_1 &= \frac{(b-s_1)[c_n(B-S_1)+C_n(b-s_1)]}{2c_1c_n(B-S_1)+(c_1C_n+C_1c_n)(b-s_1)},\\
T_1 &= \frac{c_1(B-S_1)[c_n(B-S_1)+C_n(b-s_1)]}{C_1[2c_1c_n(B-S_1)+(c_1C_n+C_1c_n)(b-s_1)]},\quad \frac{t_1}{T_1}=\frac{C_1(b-s_1)}{c_1(B-S_1)}
\end{aligned}\tag{20}$$

In order to differentiate with respect to t_2 and T_2 for the second component, we write the utilities in (5) and (6) as

$$\begin{aligned}
u &= r\frac{t_2}{t_2+T_2}s_{22}\frac{\dfrac{b-c_2t_2-\sum_{i=1,i\neq2}^{n-1}c_it_i}{c_n}}{\dfrac{b-c_2t_2-\sum_{i=1,i\neq2}^{n-1}c_it_i}{c_n}+\dfrac{B-C_2T_2-\sum_{i=1,i\neq2}^{n-1}C_iT_i}{C_n}}-b, s_{22}=\prod_{i=1,i\neq2}^{n-1}\frac{t_i}{t_i+T_i}, s_2=\sum_{i=1,i\neq2}^{n-1}c_it_i,\\
U &= R\left(1-\frac{t_2}{t_2+T_2}s_{22}\frac{\dfrac{b-c_2t_2-\sum_{i=1,i\neq2}^{n-1}c_it_i}{c_n}}{\dfrac{b-c_2t_2-\sum_{i=1,i\neq2}^{n-1}c_it_i}{c_n}+\dfrac{B-C_2T_2-\sum_{i=1,i\neq2}^{n-1}C_iT_i}{C_n}}\right)-B,\quad S_2=\sum_{i=1,i\neq2}^{n-1}C_iT_i
\end{aligned}\tag{21}$$

where s_{22},s_2,S_2 are not functions of t_2 and T_2. Setting the derivatives with respect to t_2 and T_2 equal to zero, $\partial u / \partial t_2$=0 and $\partial U / \partial T_2$=0, and solving gives

$$t_2 = \frac{(b-s_2)[c_n(B-S_2)+C_n(b-s_2)]}{2c_2c_n(B-S_2)+(c_2C_n+C_2c_n)(b-s_2)},$$
$$T_2 = \frac{c_2(B-S_2)[c_n(B-S_2)+C_n(b-s_2)]}{C_2[2c_2c_n(B-S_2)+(c_2C_n+C_2c_n)(b-s_2)]}, \quad \frac{t_2}{T_2} = \frac{C_2(b-s_2)}{c_2(B-S_2)} \tag{22}$$

The analysis for n=2 components gives $t_1c_1/b = T_1C_1/B$ and $t_2c_2/b = T_2C_2/B$, and the analysis for n=3 components additionally gives $t_3c_3/b = T_3C_3/B$. We thus generally set $t_ic_i/b = T_iC_i/B$ and verify the solution below. Observe the difference to $t_ic_i/r = T_iC_i/R$ in Proposition 1 where the system values are replaced by the budget constraints. Solving $t_ic_i/b = T_iC_i/B$ together with the ratios in (20) and (22) gives $S_1=s_1B/b$ and $S_2=s_2B/b$, which are inserted into (20) and (22) to yield

$$t_1 = \frac{(b-s_1)(c_nB+C_nb)}{2c_1c_nB+(c_1C_n+C_1c_n)b}, \quad T_1 = \frac{c_1B(b-s_1)(c_nB+C_nb)}{C_1b[2c_1c_nB+(c_1C_n+C_1c_n)b]}, \quad \frac{t_1}{T_1} = \frac{C_1/B}{c_1/b},$$
$$t_2 = \frac{(b-s_2)(c_nB+C_nb)}{2c_2c_nB+(c_2C_n+C_2c_n)b}, \quad T_2 = \frac{c_2B(b-s_2)(c_nB+C_nb)}{C_2b[2c_2c_nB+(c_2C_n+C_2c_n)b]}, \quad \frac{t_2}{T_2} = \frac{C_2/B}{c_2/b} \tag{23}$$

Solving (23) with respect to s_1 and s_2 gives

$$s_1 = b - c_1t_1 - c_n\frac{c_1/b+C_1/B}{c_n/b+C_n/B}t_1, \quad s_2 = b - c_2t_2 - c_n\frac{c_2/b+C_2/B}{c_n/b+C_n/B}t_2 \tag{24}$$

Inserting s_1, $S_1=s_1B/b$, and $t_1 = T_1bC_1/Bc_1$ into the first order conditions for t_1 and T_1 confirm that these equal to zero. The analysis for components 2,…,n-1 is analogous. The definitions of s_1 and s_2 in (19) and (21) imply $s_1 + c_1t_1 = s_2 + c_2t_2$. Hence (24) implies

$$t_2 = \frac{c_1/b+C_1/B}{c_2/b+C_2/B}t_1 \tag{25}$$

which gives the ratio of the investments in components 1 and 2 for the defender. The defender's investment in component 2 is inverse proportional to the sum of the unit costs, divided by the respective constraints, of investing in component 1. Generalizing to components i and j gives

$$t_j = \frac{c_i/b+C_i/B}{c_j/b+C_j/B}t_i \tag{26}$$

Inserting (26) when i=1 into (24) and applying the definition of s_1 from (19) gives

$$s_1 = \sum_{j=2}^{n-1} c_j t_j = \sum_{j=2}^{n-1} c_j \frac{c_1/b + C_1/B}{c_j/b + C_j/B} t_1 = b - c_1 t_1 - c_n \frac{c_1/b + C_1/B}{c_n/b + C_n/B} t_1$$

$$\Rightarrow t_1 = \frac{1}{c_1/b + (c_1/b + C_1/B)\sum_{j=2}^{n} \frac{c_j/b}{c_j/b + C_j/B}} \tag{27}$$

Generalizing to component i gives

$$t_i = \frac{1}{c_i/b + (c_i/b + C_i/B)\sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{c_j/b}{c_j/b + C_j/B}}, \quad T_i = \frac{c_i/b}{C_i/B} t_i, \quad \frac{t_i}{t_i + T_i} = \frac{C_i/B}{c_i/b + C_i/B},$$

$$u = r\prod_{i=1}^{n} \frac{C_i/B}{c_i/b + C_i/B} - b, \; U = R\left(1 - \prod_{i=1}^{n} \frac{C_i/B}{c_i/b + C_i/B}\right) - B \tag{28}$$

Proposition 5. *(a) With budget constraints the ratio of the investments for each component, and the contest success function for each component, are the same as without budget constraints when replacing the system values r and R with the budget constraints b and B,* $t_i/T_i = (C_i/B)/(c_i/b)$. *(b) With budget constraints the utilities are the same as without budget constraints replacing r with b and R with B in the contest success function, and replacing the expenditures with the budget constraints. (c) The investments decrease in n. (d) For the defender,* $\partial t_i/\partial(c_i/b) < 0$, $\partial t_i/\partial(c_j/b) < 0$, $\partial t_i/\partial(C_i/B) < 0$, $\partial t_i/\partial(C_j/B) > 0$. *For the attacker,* $\partial T_i/\partial(C_i/B) < 0$, $\partial T_i/\partial(C_j/B) > 0$, $\partial T_i/\partial(c_i/b) > 0$, $\partial T_i/\partial(c_j/b) < 0$, *i,j=1,…,n,* $i \neq j$. *(e) The defender's utility decreases in n. (f) The attacker's utility increases in n. (g) When* $n, c_i/b, C_i/B, b, B$ *in (28) are such that u<0 and U>0, which implies u=0 and U=R, the series system is 100% insecure. When* $n, c_i/b, C_i/B, b, B$ *in (28) are such that u>0 and U<0, which implies u=r and U=0, the series system is 100% secure.*

Proof. (a) Follows from comparing (28) and (9). (b) Follows from comparing (28) with (5), (6), (11). (c) Since $(c_j/b)/(c_j/b + C_j/B) > 0$ in (28), adding an additional component adds yet another ratio in the denominator, which causes lower t_i and lower T_i. (d) Follows from differentiating (28). (The signs of the first derivatives are observed straightforwardly from (28), and are not set up due to space constraints.) (e) Since $(C_i/B)/(c_i/b + C_i/B) < 1$ in (28), adding an additional component causes multiplication with yet another ratio less than one, which causes lower u. (f) Since the product decreases in n, one minus the product increases in n. (g) Follows from (28).

The results in Proposition 5 are mostly similar to the results in Propositions 1-4 without budget constraints. The budget constraints can indeed have a significant impact, especially for an agent who would otherwise choose a much higher investment. An agent with a large system value, but a low budget, is especially constrained. The signs of the derivatives for the investments in Proposition 5c are equivalent to the signs in Proposition 2b, with two exceptions. First, $\partial t_i / \partial(C_i / B) < 0$ with budget constraints, while $\partial t_i / \partial(C_i / R) < 0$ when $c_i / r < C_i / R$ without budget constraints. Second, $\partial T_i / \partial(c_i / b) > 0$ with budget constraints, while $\partial T_i / \partial(c_i / r) > 0$ when $C_i / R > c_i / r$ without budget constraints. Both these exceptions concern the impact of the other agent's unit cost for the same component, which has more indirect impact than an agent's own unit cost on own investment.

Let us consider the same special cases as above. First, inserting $C_i/B=c_i/b$ and $b=B$ into (28) gives

$$t_i = T_i = \frac{1}{nc_i / b}, \quad u = \frac{r}{2^n} - b, \; U = R\left(1 - \frac{1}{2^n}\right) - b, \quad \sum_{i=1}^{n} c_i t_i = \sum_{i=1}^{n} C_i T_i = b \tag{29}$$

Comparing (29) with (13), both agents would invest more with budget constraints when $b > rn / 2^{n+1}$. However, (13) implies $\sum_{i=1}^{n} c_i t_i = rn / 2^{n+1}$, so such a budget constraint would not be binding, and the agents always invest less with budget constraints when equipped with equal unit costs and budgets. Second, inserting $c_i=c$ and $C_i=C$ into (28) gives

$$t_i = \frac{1}{nc / b}, \quad T_i = \frac{1}{nC / B}, \quad u = r\left(\frac{C / B}{c / b + C / B}\right)^n - b, \; U = R\left(1 - \left(\frac{C / B}{c / b + C / B}\right)^n\right) - B \tag{30}$$

which can be compared with (14). Third, inserting n=2 into (28) gives

$$t_i = \frac{1}{c_i / b + c_j / b \dfrac{c_i / b + C_i / B}{c_j / b + C_j / B}}, \quad T_i = \frac{c_i / b}{C_i / B} t_i,$$

$$u = r \frac{\dfrac{C_1}{B} \dfrac{C_2}{B}}{\left(\dfrac{c_1}{b} + \dfrac{C_1}{B}\right)\left(\dfrac{c_2}{b} + \dfrac{C_2}{B}\right)} - b, \; U = R\left(1 - \frac{\dfrac{C_1}{B} \dfrac{C_2}{B}}{\left(\dfrac{c_1}{b} + \dfrac{C_1}{B}\right)\left(\dfrac{c_2}{b} + \dfrac{C_2}{B}\right)}\right) - B \tag{31}$$

which can be compared with (15). $\underset{b \to \infty}{Lim} \; u = r - b$ and $\underset{B \to \infty}{Lim} \; U = R - B$, but unlimited budgets cause the solution in (15). Fourth, inserting $c_1/b=c_2/b=1$, $C_1/B=C_2/B=n=2$ into (28) gives

$$t_i = \frac{1}{n} = \frac{1}{2}, \quad T_i = \frac{1}{2n} = \frac{1}{4}, \quad u = r\left(\frac{2}{3}\right)^n - b = \frac{4r}{9} - b, \; U = R\left(1 - \left(\frac{2}{3}\right)^n\right) - B = \frac{5R}{9} - B \tag{32}$$

which can be compared with (16). The attacker enjoys a utility advantage when R=r and B=b for the series system despite the double attack inefficiencies, as without budget constraints.

5 Parallel system

The reliability of a parallel system equals one minus the product of the component unreliabilities, that is $1 - \prod_{i=1}^{n} \left(1 - p_i\right)$. Applying (3), and analogously to (5), we model the defender's utility as

$$u = r\left(1 - \prod_{i=1}^{n}\left(1 - \frac{t_i^{m_i}}{t_i^{m_i} + T_i^{m_i}}\right)\right) - \sum_{i=1}^{n} c_i t_i \tag{33}$$

Analogously to (6), the attacker's utility is

$$U = R \prod_{i=1}^{n}\left(1 - \frac{t_i^{m_i}}{t_i^{m_i} + T_i^{m_i}}\right) - \sum_{i=1}^{n} C_i T_i \tag{34}$$

The 2n FOCs are

$$\frac{\partial u}{\partial t_i} = r \frac{m_i t_i^{m_i - 1} T_i^{m_i}}{(t_i^{m_i} + T_i^{m_i})^2} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{T_j^{m_j}}{t_j^{m_j} + T_j^{m_j}} - c_i = 0, \quad \frac{\partial U}{\partial T_i} = R \frac{m_i T_i^{m_i - 1} t_i^{m_i}}{(t_i^{m_i} + T_i^{m_i})^2} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{T_j^{m_j}}{t_j^{m_j} + T_j^{m_j}} - C_i = 0 \tag{35}$$

Compared with (7) for the series system, only one variable in the numerator gets changed for each FOC in (35). For the FOCs for t_i and T_i, the change is from $t_j^{m_j}$ to $T_j^{m_j}$ inside the product sign for j=1,…,n, $j \neq$ i. The impact of this apparently minor change is substantial. Also for the parallel system the FOCs give $t_i = T_i r C_i / R c_i$, which we formulate as follows.

Proposition 1P. *The ratio of investments for each component is inverse proportional to the ratio of the unit costs to the system value,* $t_i / T_i = (C_i / R)/(c_i / r)$*, unrelated to that of the other components. The investment expenditures relative to the system value for the defender and the attacker are equal for each component, that is* $t_i c_i / r = T_i C_i / R$*, regardless of the parameter* m_i*.*

Solving (35) gives

$$\begin{aligned} t_i &= \frac{m_i (c_i / r)^{m_i - 1} (C_i / R)^{m_i}}{((c_i / r)^{m_i} + (C_i / R)^{m_i})^2} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(c_j / r)^{m_j}}{(c_j / r)^{m_j} + (C_j / R)^{m_j}}, \quad \frac{t_i}{T_i} = \frac{C_i / R}{c_i / r} \\ T_i &= \frac{m_i (C_i / R)^{m_i - 1} (c_i / r)^{m_i}}{((c_i / r)^{m_i} + (C_i / R)^{m_i})^2} \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{(c_j / r)^{m_j}}{(c_j / r)^{m_j} + (C_j / R)^{m_j}}, \quad \frac{t_i}{t_i + T_i} = \frac{C_i / R}{C_i / R + c_i / r} \end{aligned} \tag{36}$$

Just as for the FOCs, only one variable in the numerator gets changed for each variable in (36), compared with the series system in (9). For t_i and T_i, the change is from $(C_j / R)^{m_j}$ to $(c_j / r)^{m_j}$ inside the product sign for j=1,…,n, $j \neq$ i.

Proposition 2P. *(a) The investments decrease in n. (b) Assume* $m_i=m_j=1$*. For the defender,* $\partial t_i / \partial (c_i / r) < 0$*,* $\partial t_i / \partial (c_j / r) > 0$*,* $\partial t_i / \partial (C_i / R) > 0$ *when* $c_i / r > C_i / R$*,* $\partial t_i / \partial (C_j / R) < 0$*. For the attacker,* $\partial T_i / \partial (C_i / R) < 0$*,* $\partial T_i / \partial (C_j / R) < 0$*,* $\partial T_i / \partial (c_i / r) < 0$ *when* $C_i / R < c_i / r$*,* $\partial T_i / \partial (c_j / r) > 0$*, i,j=1,…,n,* $i \neq j$*.*

Proof. (a) Since $(c_j / r)^{m_j} / ((c_j / r)^{m_j} + (C_j / R)^{m_j}) < 1$ in (36), adding an additional component causes multiplication with yet another ratio less than one, which causes lower t_i and lower T_i. (b) See Appendix 2.

Both agents invest less when the number of components increases. It is more difficult for the attacker to attack many components in a parallel system. And, the defender benefits without having to invest much. The four inequalities for the defender in Proposition 2P for the parallel system correspond to the four inequalities for the attacker in Proposition 2 for the series system, obtained by permuting capital and regular letters. That is, T_i is replaced with t_i, C_i is replaced with c_i, C_j is replaced with c_j, c_i is replaced with C_i, and c_j is replaced with C_j. Analogously, the four inequalities for the attacker in Proposition 2P correspond to the four inequalities for the defender in Proposition 2. This means that the advantage for the attacker in a series system gets replaced by an analogous advantage for the defender in a parallel system.

Analogously, the defender invests less to defend component i when the defense inefficiency of component i increases. However, in contrast to the attacker, the defender invests more to defend component i when the defense inefficiency of component j increases. In contrast to series systems, the substitution effect applies strongly for the defender, but less so for the attacker. When the defender suffers defense inefficiency for component j, it substitutes to defending component i. The defender benefits from defending any of the components in a parallel system, and choosing the most cost effective defense is beneficial. The attacker needs to attack both components to reduce the reliability of the parallel system. It cannot substitute strongly from component j to component i when the inefficiency of component j increases (since $\partial T_i / \partial(C_j / R)$<0) since that would increase the reliability of component j substantially due to substitution by the defender. Mathematically, observe in (36) that C_j / R is not present in the numerator of T_i, while c_j / r is indeed present in the numerator of t_i.

However, when choosing T_i and assuming m_i=m_j=1, the attacker is more negatively influenced by a high C_i / R than by a high C_j / R. This can be interpreted as a moderate substitution effect also for the attack, for parallel systems. This can be seen from the four denominators in (36) which have one square bracket and one bracket that is not squared. For component i the investments t_i and T_i have squared brackets in the denominators for the inefficiencies c_i / r and C_i / R, which cause reduced investment if these inefficiencies are large. We formulate this as a follows.

Proposition 3P. *In a parallel system the defender benefits more from the substitution effect than the attacker since defender benefits flow from defending any of the components, while attacker benefits flow from attacking all components. The defender is highly sensitive to the difference in the defense efficiencies for the components. The attacker is less sensitive to the difference in the attack efficiencies for the components.*

Proposition 2P also states that when $m_i=m_j=1$, the attacker increases the investment T_i when c_j/r increases ($\partial T_i/\partial(c_j/r)>0$). This is a natural consequence of the defender's substitution effect. When the defense inefficiency c_j/r increases, the defender substitutes to defending component i, causing the attacker to increase T_i to attack component i, in order to reduce the overall reliability of the parallel system. Conversely, the defender decreases the investment t_i when C_j/R increases ($\partial t_i/\partial(C_j/R)<0$). The reason is that increasing C_j/R causes reduced investment also of T_i for the attacker. To preserve $t_ic_i/r=T_iC_i/R$ in Proposition 1P, t_i decreases too.

To further illustrate the substitution effect, (36) implies

$$\frac{t_i}{t_j}=\frac{m_i(C_i/R)^{m_i}(c_j/r)\left((c_j/r)^{m_j}+(C_j/R)^{m_j}\right)}{m_j(C_j/R)^{m_j}(c_i/r)\left((c_i/r)^{m_i}+(C_i/R)^{m_i}\right)}\quad \frac{T_i}{T_j}=\frac{m_i(C_i/R)^{m_i-1}\left((c_j/r)^{m_j}+(C_j/R)^{m_j}\right)}{m_j(C_j/R)^{m_j-1}\left((c_i/r)^{m_i}+(C_i/R)^{m_i}\right)} \tag{37}$$

for any two components i,j=1,…,n, $i\neq j$. Equation (37) follows from (10) by permuting capital and regular letters. Both agents' investment ratios increases in m_i/m_j. When $m_i=m_j=1$, the ratio T_i/T_j of the attacker's investments increases in the sum of the inefficiencies of component j, and decreases in the sum of the inefficiencies of component i. The attacker is investment sensitive in a straightforward manner. However, when $m_i=m_j=1$, the defender's investment ratio t_i/t_j is impacted by two additional ratios. First, C_i/C_j partly counterbalances the ratio of the sums of the inefficiencies, tilting the balance toward defending component i if the attack inefficiency of component i is large. Secondly, and more importantly, c_j/c_i demonstrates the substitution effect where component i is defended more fiercely if the defense inefficiency of component j is large.

Inserting (36) into (33) and (34) gives the utilities

$$\begin{aligned} u&=r\left(1-\left(1+\sum_{i=1}^{n}\frac{m_i(C_i/R)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\right)\prod_{i=1}^{n}\frac{(c_i/r)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\right),\\ U&=R\left(1-\sum_{i=1}^{n}\frac{m_i(C_i/R)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\right)\prod_{i=1}^{n}\frac{(c_i/r)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\end{aligned} \tag{38}$$

The defender utility u in (38) follows from the attacker utility U in (11) by permuting capital and regular letters. Analogously, the attacker utility U in (38) follows from the defender utility u in (11) by permuting capital and regular letters. The product $\prod_{i=1}^{n}(\cdot)$ causes high utility to the attacker of high defense inefficiencies. Subtraction of this product causes low utility to the defender. For the three benchmarks, inserting m_i=0 into (36) and (38) gives

$$t_i = T_i = 0, \quad u = r\left(1-\frac{1}{2^n}\right), \quad U = \frac{R}{2^n} \tag{39}$$

In contrast to the series system, the defender's utility increases concavely in the number of components for the parallel system, while the attacker's utility decreases convexly in n. Second, inserting $C_i/R=c_i/r$ into (36) and (38) gives

$$t_i = T_i = \frac{m_i}{2^{n+1}c_i/r}, \quad u = r\left(1-\left(1+\sum_{i=1}^{n}\frac{m_i}{2}\right)\frac{1}{2^n}\right), \quad U = R\left(1-\sum_{i=1}^{n}\frac{m_i}{2}\right)\frac{1}{2^n} \tag{40}$$

The attacker is strongly disadvantaged by both high m_i and high n. The attacker's utility reaches zero when $\sum_{i=1}^{n}\frac{m_i}{2} \geq 1$, which makes the n components 100% secure. The defender also suffers from high m_i, but benefits from high n as shown in (A14). The sum $r\left(1-\frac{1}{2^{n-1}}\sum_{i=1}^{n}\frac{m_i}{2}\right)$, when R=r, of the utilities decreases in m_i, but increases in n since the defender benefits more than the attacker suffers as the number of components increases. For the third benchmark, inserting c_i=c, C_i=C, m_i=m into (36) and (38) gives

$$\begin{gathered}
t_i = \frac{m(C/R)^m(c/r)^{mn-1}}{\left((c/r)^m+(C/R)^m\right)^{n+1}}, \quad T_i = \frac{m(C/R)^{m-1}(c/r)^{mn}}{\left((c/r)^m+(C/R)^m\right)^{n+1}}, \\
u = r\left(1-\left(1+\frac{mn\left((C/R)/(c/r)\right)^m}{\left((C/R)/(c/r)\right)+1}\right)\frac{1}{\left(\left((C/R)/(c/r)\right)+1\right)^n}\right), \\
U = R\left(1-\frac{mn\left((C/R)/(c/r)\right)^m}{\left((C/R)/(c/r)\right)+1}\right)\frac{1}{\left(\left((C/R)/(c/r)\right)+1\right)^n}
\end{gathered} \tag{41}$$

Proposition 4P. *(a) When $C_i/R=c_i/r$, both agents' utilities decrease linearly in m_i. (b) When the attacker is advantaged with a sufficiently small $(C_i/R)/(c_i/r)$ for all i, U increases in $m_i=m$. (c) When the defender is advantaged with a sufficiently large $(C_i/R)/(c_i/r)$ for all i, u increases in $m_i=m$. (d) The attacker's utility decreases in n. (e) The defender's utility increases* *in n. (f) When m_i=m,n,c_i/r,C_i/R in (38) are such that u<0 and U>0, which implies u=0 and U=R, the parallel system is 100% insecure. When m_i=m,n,c_i/r,C_i/R in (38) are such that u>0 and U<0, which implies u=r and U=0, the parallel system is 100% secure.*

Proof. (a) Differentiating (40) gives $\partial u/\partial m_i = -r/2^{n+1}$, $\partial U/\partial m_i = -R/2^{n+1}$, $\partial^2 u/\partial m_i^2 = \partial^2 U/\partial m_i^2 = 0$. (b) The bracket before $Ln\big((C/R)/(c/r)\big)$ in (A15) is negative when $(C/R)/(c/r)$ is arbitrarily small. $Ln\big((C/R)/(c/r)\big)$ is then also negative with arbitrarily high absolute value. Hence $\partial U/\partial m > 0$. (c) The bracket before $Ln\big((C/R)/(c/r)\big)$ in (A16) is positive when $(C/R)/(c/r)$ is arbitrarily large, causing arbitrarily large $Ln\big((C/R)/(c/r)\big)$ and $\partial u/\partial m > 0$. (d) and (e) Since the defender utility u in (38) follows from the attacker utility U in (11) by permuting capital and regular letters, the proof is equivalent with Appendix 1 permuting capital and regular letters. (f) Follows from (38).

Although the agents' utilities decrease in the intensity of the contest for a broad set of parameter values, increased intensity can benefit the defender, and even the attacker, when advantaged with a sufficiently low unit cost of investment compared with that of the opponent. But ceteris paribus, adding several components to the parallel system benefits the defender, and not the attacker. Let us consider two examples.

Example 3. Inserting m_i=1 and n=2 into (38) gives

$$u = r\frac{\frac{c_1^2}{r}\frac{C_2^2}{R} + \frac{C_1^2}{R}\left(\frac{c_2}{r}+\frac{C_2}{R}\right)^2 + \frac{C_1}{R}\frac{c_1}{r}\frac{C_2}{R}\left(\frac{c_2}{r}+2\frac{C_2}{R}\right)}{\left(\frac{c_1}{r}+\frac{C_1}{R}\right)^2\left(\frac{c_2}{r}+\frac{C_2}{R}\right)^2},\quad U = R\frac{c_1 c_2\left(\frac{c_1}{r}\frac{c_2}{r}-\frac{C_1}{R}\frac{C_2}{R}\right)}{\left(\frac{c_1}{r}+\frac{C_1}{R}\right)^2\left(\frac{c_2}{r}+\frac{C_2}{R}\right)^2} \qquad (42)$$

which also follows from permuting capital and regular letters in (15). The attacker's utility is positive when $c_1 c_2/r^2 > C_1 C_2/R^2$. Otherwise the attacker earns zero utility and does not invest, while the defender earns a utility equal to the value of the system and invests arbitrarily little. The defender's utility is always positive, but making the defender maximally disadvantaged gives $\underset{\substack{c_1\to\infty\\ c_2\to\infty}}{Lim} U = R$, $\underset{\substack{c_1\to\infty\\ c_2\to\infty}}{Lim} u = 0$ (L'Hopital's rule). For a parallel system of two components where m_i=1, the product of the attack inefficiencies must be less than the product of the defense inefficiencies in order for the attacker to earn positive utility. Parallel systems are hard to attack, and the attacker needs superior attack efficiencies in order to ensure overall system reliability. The numerator in the defender's utility in (42) contains the product of the attack inefficiencies raised to both the third and the fourth power, while the product of the defense inefficiencies are not raised to more than the second power.

Example 4. To place the defender in a more disadvantaged situation, inserting $c_1/r=c_2/r=n=2$ and $C_1/R=C_2/R=1$, $m_i=m$ into (36) and (38) gives

$$t_1=t_2=\frac{2^{2m-1}m}{(1+2^m)^3},\ T_1=T_2=\frac{4^m m}{(1+2^m)^3},\ u=r\frac{1+3\times2^m+2^{2m+1}(1-m)}{(1+2^m)^3},\ U=R\frac{4^m(1+2^m-2m)}{(1+2^m)^3} \quad (43)$$

For m=1, (43) gives $t_1=t_2=2/27$, $T_1=T_2=4/27$, $u=7r/27$, $U=4R/27$, with a utility advantage when R=r for the defender for the parallel system despite the double defense inefficiencies. For m=2, (43) gives $t_1=t_2=16/125$, $T_1=T_2=32/125$, $u=-19r/125$ and $U=16R/125$, which implies a corner solution with u=0 and U=R. High defense inefficiencies and high m induce a high toll on the defender in parallel systems. For this parallel system, where the defender is sufficiently disadvantaged with high defense inefficiencies, and the intensity of the contest is sufficiently high, a corner solution emerges with maximum utility R to the attacker and minimum utility zero to the defender.

Of course, permuting capital and regular letters for the six variables in the series systems simulations in section 4, gives equivalent simulations for the parallel system. The results for the defender (attacker) in the series system are equivalent to the results for the attacker (defender) in the parallel system with such permutation. To generate new interesting results, we keep the c_1/r dependency in section 4, and the same parameters, without permutation.

Fig. 5 shows the four investments and two utilities (divided by system value) as functions of the defense inefficiency c_1/r with the same parameters as in Fig. 3, i.e. n=2 and $c_2/r=C_1/R=C_2/R=m_i=1$. Whereas Fig. 3 for the series system gives zero utility to the defender and a utility equal to the value of the system to the attacker when $c_1/r>1$ (due to negative u in (15) where $C_1C_2/R^2-c_1c_2/r^2$ is present), Fig. 5 for the parallel system gives zero utility to the attacker and a utility equal to the value of the system to the defender when $c_1/r<1$ (due to negative U in (42) where $c_1c_2-C_1C_2$ is present). The defender is very vulnerable to high defense inefficiencies in series systems, since it has to defend both components. Conversely, the attacker is very vulnerable to low defense inefficiencies in parallel systems, since it has to attack both components, and the defender substitutes the defense optimally across the two components. The defender's investment t_1 naturally decreases c_1/r, but t_2 increases in c_1/r as the defender optimally substitutes to defending component 2 in a parallel system (Proposition 3P). The attacker's investment T_2 equals t_2 for these parameters, which increases, while T_1 decreases in c_1/r when $c_1/r < C_1/R$ (Proposition 2P). The defender's utility increases to one, and the attacker's utility decreases to zero, as c_1/r decreases to $c_1/r=1$. Division with 2 in u/2r is for scaling purposes.

Analogously to Fig. 4, Fig. 6 keeps the same parameters except doubles C_2/R to $C_2/R=2$. Also in Fig. 6 the defender benefits, and especially so since the range of maximum utility increases to $0<c_1/r<2$. Let us consider $c_1/r>2$. Despite the high defense inefficiency for component 1, the defender retains a substantial utility which decreases in c_1/r. Conversely, the attacker's utility increases in c_1/r. The defender reduces t_1 as c_1/r increases, substituting to defending component 2. This causes the attacker to increase T_2, and in fact to decrease T_1. The reason is that high c_1/r causes component 2 to be the main source of high reliability for the defender, and the attacker must attack component 2 to reduce that high reliability. This illustrates how the parallel system benefits the defender, and not the attacker.

6 Parallel system with budget constraints

Assume that the defender and attacker have budget constraints b and B defined in (17) and that the investments for the n'th component are as in (18). In order to differentiate with respect to t_1 and T_1 for the first component, we write the utilities in (33) and (34) as

$$u = r\left(1 - \frac{T_1}{t_1 + T_1} q_{11} \frac{\dfrac{B - C_1T_1 - \sum_{i=2}^{n-1} C_iT_i}{C_n}}{\dfrac{b - c_1t_1 - \sum_{i=2}^{n-1} c_it_i}{c_n} + \dfrac{B - C_1T_1 - \sum_{i=2}^{n-1} C_iT_i}{C_n}}\right) - b,\ q_{11} = \prod_{i=2}^{n-1} \frac{T_i}{t_i + T_i},$$

$$U = R\frac{T_1}{t_1 + T_1} q_{11} \frac{\dfrac{B - C_1T_1 - \sum_{i=2}^{n-1} C_iT_i}{C_n}}{\dfrac{b - c_1t_1 - \sum_{i=2}^{n-1} c_it_i}{c_n} + \dfrac{B - C_1T_1 - \sum_{i=2}^{n-1} C_iT_i}{C_n}} - B,\ q_1 = \sum_{i=2}^{n-1} c_it_i,\ Q_1 = \sum_{i=2}^{n-1} C_iT_i \tag{44}$$

where q_{11},q_1,Q_1 are not functions of t_1 and T_1. Setting the derivatives with respect to t_1 and T_1 equal to zero, $\partial u / \partial t_1$=0 and $\partial U / \partial T_1$=0, and solving gives

$$t_1 = \frac{C_1(b - q_1)[c_n(B - Q_1) + C_n(b - q_1)]}{c_1[2C_1C_n(b - q_1) + (c_1C_n + C_1c_n)(B - Q_1)]},$$
$$T_1 = \frac{(B - Q_1)[c_n(B - Q_1) + C_n(b - q_1)]}{2C_1C_n(b - q_1) + (c_1C_n + C_1c_n)(B - Q_1)},\quad \frac{t_1}{T_1} = \frac{C_1(b - q_1)}{c_1(B - Q_1)} \tag{45}$$

In order to differentiate with respect to t_2 and T_2 for the second component, we write the utilities in (5) and (6) as

$$u = r\left(1 - \frac{T_2}{t_2 + T_2} q_{22} \frac{\dfrac{B - C_2T_2 - \sum_{i=1, i\neq 2}^{n-1} C_iT_i}{C_n}}{\dfrac{b - c_2t_2 - \sum_{i=1, i\neq 2}^{n-1} c_it_i}{c_n} + \dfrac{B - C_2T_2 - \sum_{i=1, i\neq 2}^{n-1} C_iT_i}{C_n}}\right) - b,\ q_{22} = \prod_{i=1, i\neq 2}^{n-1} \frac{T_i}{t_i + T_i},$$

$$U = R\frac{T_2}{t_2 + T_2} q_{22} \frac{\dfrac{B - C_2T_2 - \sum_{i=1, i\neq 2}^{n-1} C_iT_i}{C_n}}{\dfrac{b - c_2t_2 - \sum_{i=1, i\neq 2}^{n-1} c_it_i}{c_n} + \dfrac{B - C_2T_2 - \sum_{i=1, i\neq 2}^{n-1} C_iT_i}{C_n}} - B,\ q_2 = \sum_{i=1, i\neq 2}^{n-1} c_it_i,\ Q_2 = \sum_{i=1, i\neq 2}^{n-1} C_iT_i \tag{46}$$

where q_{22},q_2,Q_2 are not functions of t_2 and T_2. Setting the derivatives with respect to t_2 and T_2 equal to zero, $\partial u / \partial t_2$=0 and $\partial U / \partial T_2$=0, and solving gives

$$t_2 = \frac{C_2(b-q_2)[c_n(B-Q_2)+C_n(b-q_2)]}{c_2[2C_2C_n(b-q_2)+(c_2C_n+C_2c_n)(B-Q_2)]},$$
$$T_2 = \frac{(B-Q_2)[c_n(B-Q_2)+C_n(b-q_2)]}{2C_2C_n(b-q_2)+(c_2C_n+C_2c_n)(B-Q_2)}, \quad \frac{t_2}{T_2} = \frac{C_2(b-q_2)}{c_2(B-Q_2)} \tag{47}$$

As in section 4b for the series system, and analogously to the case without budget constraints, we set $t_ic_i/b = T_iC_i/B$ and verify the solution below. Observe the difference to $t_ic_i/r = T_iC_i/R$ in Proposition 1P where the system values are replaced by the budget constraints. Solving $t_ic_i/b = T_iC_i/B$ together with the ratios in (45) and (47) gives $q_1=Q_1b/B$ and $q_2=Q_2b/B$, which are inserted into (45) and (47) to yield

$$t_1 = \frac{C_1b(B-Q_1)(c_nB+C_nb)}{c_1B[2C_1C_nb+(c_1C_n+C_1c_n)B]}, \quad T_1 = \frac{(B-Q_1)(c_nB+C_nb)}{2C_1C_nb+(c_1C_n+C_1c_n)B}, \quad \frac{t_1}{T_1} = \frac{C_1/B}{c_1/b},$$
$$t_2 = \frac{C_2b(B-Q_2)(c_nB+C_nb)}{c_2B[2C_2C_nb+(c_2C_n+C_2c_n)B]}, \quad T_2 = \frac{(B-Q_2)(c_nB+C_nb)}{2C_2C_nb+(c_2C_n+C_2c_n)B}, \quad \frac{t_2}{T_2} = \frac{C_2/B}{c_2/b} \tag{48}$$

Solving (48) with respect to Q_1 and Q_2 gives

$$Q_1 = B - C_1T_1 - C_n\frac{c_1/b+C_1/B}{c_n/b+C_n/B}T_1, \quad Q_2 = B - C_2T_2 - C_n\frac{c_2/b+C_2/B}{c_n/b+C_n/B}T_2 \tag{49}$$

Inserting Q_1, $q_1=Q_1b/B$, and $t_1 = T_1bC_1/Bc_1$ into the first order conditions for t_1 and T_1 confirm that these equal to zero. The analysis for components 2,…,n-1 is analogous. The definitions of Q_1 and Q_2 in (44) and (46) imply $Q_1 + C_1T_1 = Q_2 + C_2T_2$. Hence (49) implies

$$T_2 = \frac{c_1/b+C_1/B}{c_2/b+C_2/B}T_1 \tag{50}$$

which gives the ratio of the investments in components 1 and 2 for the attacker. The attacker's investment in component 2 is inverse proportional to the sum of the unit costs, divided by the respective constraints, of investing in component 1. Generalizing to components i and j gives

$$T_j = \frac{c_i/b+C_i/B}{c_j/b+C_j/B}T_i \tag{51}$$

Inserting (51) when i=1 into (49) and applying the definition of s_1 from (44) gives

$$Q_1 = \sum_{i=2}^{n-1} C_iT_i = \sum_{j=2}^{n-1} C_j\frac{c_1/b+C_1/B}{c_j/b+C_j/B}T_1 = B - C_1T_1 - C_n\frac{c_1/b+C_1/B}{c_n/b+C_n/B}T$$
$$\Rightarrow T_1 = \frac{1}{C_1/b+(c_1/b+C_1/B)\sum_{j=2}^{n}\frac{C_j/b}{c_j/b+C_j/B}} \tag{52}$$

Generalizing to component i gives

$$T_i=\frac{1}{C_i/B+(c_i/b+C_i/B)\sum_{\substack{j=1\\j\neq i}}^{n}\frac{C_j/b}{c_j/b+C_j/B}},\quad t_i=\frac{C_i/B}{c_i/b}T_i,\quad \frac{T_i}{t_i+T_i}=\frac{c_i/b}{C_i/B+c_i/b},$$
$$u=r\left(1-\prod_{i=1}^{n}\frac{c_i/b}{c_i/b+C_i/B}\right)-b,\quad U=R\prod_{i=1}^{n}\frac{c_i/b}{c_i/b+C_i/B}-B \tag{53}$$

Observe how the analysis and results for the parallel system in this sections follows from the analysis and results for the series system in section 4b by permuting capital and regular letters.

Proposition 5P. *(a) With budget constraints the ratio of the investments for each component, and the contest success function for each component, are the same as without budget constraints when replacing the system values r and R with the budget constraints b and B,* $t_i/T_i=(C_i/B)/(c_i/b)$. *(b) With budget constraints the utilities are the same as without budget constraints replacing r with b and R with B in the contest success function, and replacing the expenditures with the budget constraints. (c) The investments decrease in n. (d) For the defender,* $\partial t_i/\partial(c_i/b)<0$, $\partial t_i/\partial(c_j/b)>0$, $\partial t_i/\partial(C_i/B)>0$, $\partial t_i/\partial(C_j/B)<0$. *For the attacker,* $\partial T_i/\partial(C_i/B)<0$, $\partial T_i/\partial(C_j/B)<0$, $\partial T_i/\partial(c_i/b)<0$, $\partial T_i/\partial(c_j/b)>0$, *i,j=1,…,n,* $i\neq j$. *(e) The defender's utility increases in n. (f) The attacker's utility decreases in n. (g) When* $n,c_i/b,C_i/B,b,B$ *in (53) are such that u<0 and U>0, which implies u=0 and U=R, the parallel system is 100% insecure. When* $n,c_i/b,C_i/B,b,B$ *in (53) are such that u>0 and U<0, which implies u=r and U=0, the parallel system is 100% secure.*

Proof. (a) Follows from comparing (53) and (36). (b) Follows from comparing (53) with (33), (34), (38). (c) Since $(C_j/B)/(c_j/b+C_j/B)>0$ in (53), adding an additional component adds yet another ratio in the denominator, which causes lower t_i and lower T_i. (d) Follows from differentiating (53). (The signs of the first derivatives are observed straightforwardly from (53), and are not set up due to space constraints.) (e) Since $(c_i/b)/(c_i/b+C_i/B)<1$ in (53), adding an additional component causes multiplication with yet another ratio less than one, which causes higher u. (f) Follows since the product decreases in n. (g) Follows from (53).

The results in Proposition 5P are mostly similar to the results in Propositions 1P-4P without budget constraints, and follow from Proposition 5P when permuting capital with regular letters. The budget constraints can indeed have a significant impact, especially for an agent who would otherwise choose a much higher investment. An agent with a large system value, but a low budget, is especially constrained. The signs of the derivatives for the investments in Proposition 5Pc are equivalent to the signs in Proposition 2Pb, with two exceptions. First, $\partial t_i / \partial(C_i / B) > 0$ with budget constraints, while $\partial t_i / \partial(C_i / R) > 0$ when $c_i / r > C_i / R$ without budget constraints. Second, $\partial T_i / \partial(c_i / b) < 0$ with budget constraints, while $\partial T_i / \partial(c_i / r) < 0$ when $C_i / R < c_i / r$ without budget constraints. Both these exceptions concern the impact of the other agent's unit cost for the same component, which has more indirect impact than an agent's own unit cost on own investment.

Let us consider the same special cases as above. First, inserting $C_i/B=c_i/b$ and $b=B$ into (53) gives

$$t_i = T_i = \frac{1}{nc_i / b}, \quad u = r\left(1 - \frac{1}{2^n}\right) - b, \quad U = \frac{R}{2^n} - B, \quad \sum_{i=1}^{n} c_i t_i = \sum_{i=1}^{n} C_i T_i = b \tag{54}$$

Comparing (54) with (40), both agents would invest more with budget constraints when $b > rn / 2^{n+1}$. However, (40) implies $\sum_{i=1}^{n} c_i t_i = rn / 2^{n+1}$, so such a budget constraint would not be binding, and the agents always invest less with budget constraints when equipped with equal unit costs and budgets. Second, inserting $c_i=c$ and $C_i=C$ into (53) gives

$$t_i = \frac{1}{nc / b}, \quad T_i = \frac{1}{nC / B}, \quad u = r\left(1 - \left(\frac{c / b}{c / b + C / B}\right)^n\right) - b, \quad U = R\left(\frac{c / b}{c / b + C / B}\right)^n - B \tag{55}$$

which can be compared with (41). Third, inserting n=2 into (53) gives

$$T_i = \frac{1}{C_i / B + C_j / b \dfrac{c_i / b + C_i / B}{c_j / b + C_j / B}}, \quad t_i = \frac{C_i / B}{c_i / b} T_i,$$

$$u = r\left(1 - \frac{\frac{c_1}{b}\frac{c_2}{b}}{\left(\frac{c_1}{b} + \frac{C_1}{B}\right)\left(\frac{c_2}{b} + \frac{C_2}{B}\right)}\right) - b, \quad U = R\frac{\frac{c_1}{b}\frac{c_2}{b}}{\left(\frac{c_1}{b} + \frac{C_1}{B}\right)\left(\frac{c_2}{b} + \frac{C_2}{B}\right)} - B \tag{56}$$

which can be compared with (42). $\lim_{b \to \infty} u = r - b$ and $\lim_{B \to \infty} U = R - B$, but unlimited budgets cause the solution in (42). Fourth, inserting $c_1/b=c_2/b=n=2$, $C_1/B=C_2/B=1$ into (53) gives

$$t_i = \frac{1}{2n} = \frac{1}{4}, \quad T_i = \frac{1}{n} = \frac{1}{2}, \quad u = r\left(1 - \left(\frac{2}{3}\right)^n\right) - b = \frac{5r}{9} - b, \quad U = \frac{4R}{9} - B \tag{57}$$

which can be compared with (43). The defender enjoys a utility advantage when R=r and B=b for the parallel system despite the double defense inefficiencies, as without budget constraints.

7 Series system of parallel subsystems

Consider the n parallel subsystems s times in series in Fig. 7. The system reliability is

$$p=\prod_{i=1}^{s}\left(1-\prod_{j=1}^{n}\left(1-\frac{t_{ij}^{m_{ij}}}{t_{ij}^{m_{ij}}+T_{ij}^{m_{ij}}}\right)\right) \tag{58}$$

and inserting into the utilities in (1) and (2) gives

$$\begin{aligned}
u&=r\prod_{i=1}^{s}\left(1-\prod_{j=1}^{n}\left(1-\frac{t_{ij}^{m_{ij}}}{t_{ij}^{m_{ij}}+T_{ij}^{m_{ij}}}\right)\right)-\sum_{i=1}^{s}\sum_{j=1}^{n}c_{ij}t_{ij},\\
U&=R\left(1-\prod_{i=1}^{s}\left(1-\prod_{j=1}^{n}\left(1-\frac{t_{ij}^{m_{ij}}}{t_{ij}^{m_{ij}}+T_{ij}^{m_{ij}}}\right)\right)\right)-\sum_{i=1}^{s}\sum_{j=1}^{n}C_{ij}T_{ij}
\end{aligned} \tag{59}$$

The defender has sn free choice variables t_{ij}. The attacker has sn free choice variables T_{ij}. There are 2sn FOCs. To determine the FOCs for t_{11} and T_{11} we write (59) as

$$\begin{aligned}
u&=r\left(1-\left(1-\frac{t_{11}^{m_{11}}}{t_{11}^{m_{11}}+T_{11}^{m_{11}}}\right)S_3\right)S_4-\sum_{i=1}^{s}\sum_{j=1}^{n}c_{ij}t_{ij},\\
U&=R\left(1-\left(1-\left(1-\frac{t_{11}^{m_{11}}}{t_{11}^{m_{11}}+T_{11}^{m_{11}}}\right)S_3\right)S_4\right)-\sum_{i=1}^{s}\sum_{j=1}^{n}C_{ij}T_{ij},\\
S_3&=\prod_{k=2}^{n}\left(1-\frac{t_{1k}^{m_{1k}}}{t_{1k}^{m_{1k}}+T_{1k}^{m_{1k}}}\right),\quad S_4=\prod_{q=2}^{s}\left(1-\prod_{k=1}^{n}\left(1-\frac{t_{qk}^{m_{qk}}}{t_{qk}^{m_{qk}}+T_{qk}^{m_{qk}}}\right)\right)
\end{aligned} \tag{60}$$

where S_3 and S_4 are not functions of t_{11} and T_{11}. Differentiating with respect to t_{11} and T_{11} gives

$$\frac{\partial u}{\partial t_{11}}=r\frac{m_{11}t_{11}^{m_{11}-1}T_{11}^{m_{11}}}{(t_{11}^{m_{11}}+T_{11}^{m_{11}})^2}S_3S_4-c_{11}=0,\quad \frac{\partial U}{\partial T_{11}}=R\frac{m_{11}T_{11}^{m_{11}-1}t_{11}^{m_{11}}}{(t_{11}^{m_{11}}+T_{11}^{m_{11}})^2}S_3S_4-C_{11}=0 \tag{61}$$

Solving gives $t_{11}=T_{11}rC_{11}/Rc_{11}$ and

$$\begin{aligned}
t_{11}&=\frac{m_{11}(c_{11}/r)^{m_{11}-1}(C_{11}/R)^{m_{11}}}{((c_{11}/r)^{m_{11}}+(C_{11}/R)^{m_{11}})^2}S_3S_4,\quad T_{11}=\frac{m_{11}(C_{11}/R)^{m_{11}-1}(c_{11}/r)^{m_{11}}}{((c_{11}/r)^{m_{11}}+(C_{11}/R)^{m_{11}})^2}S_3S_4,\\
\frac{t_{11}^{m_{11}}}{t_{11}^{m_{11}}+T_{11}^{m_{11}}}&=\frac{(C_{11}/R)^{m_{11}}}{(c_{11}/r)^{m_{11}}+(C_{11}/R)^{m_{11}}}\Rightarrow\frac{t_{ij}^{m_{ij}}}{t_{ij}^{m_{ij}}+T_{ij}^{m_{ij}}}=\frac{(C_{ij}/R)^{m_{ij}}}{(c_{ij}/r)^{m_{ij}}+(C_{ij}/R)^{m_{ij}}}
\end{aligned} \tag{62}$$

where the general ratio follows since FOCs analogous to (61) exist for any pair of t_{ij} and T_{ij}. Inserting the general ratio in (62) into (60), and S_3 and S_4 into (62), gives

$$S_3 = \prod_{k=2}^{n} \frac{(c_{1k}/r)^{m_{1k}}}{(c_{1k}/r)^{m_{1k}} + (C_{1k}/R)^{m_{1k}}}, \quad S_4 = \prod_{q=2}^{s} \left(1 - \prod_{k=1}^{n} \frac{(c_{qk}/r)^{m_{qk}}}{(c_{qk}/r)^{m_{qk}} + (C_{qk}/R)^{m_{qk}}} \right),$$

$$t_{11} = \frac{m_{11}(c_{11}/r)^{m_{11}-1}(C_{11}/R)^{m_{11}}}{((c_{11}/r)^{m_{11}} + (C_{11}/R)^{m_{11}})^2} \prod_{k=2}^{n} \frac{(c_{1k}/r)^{m_{1k}}}{(c_{1k}/r)^{m_{1k}} + (C_{1k}/R)^{m_{1k}}} \prod_{q=2}^{s} \left(1 - \prod_{k=1}^{n} \frac{(c_{qk}/r)^{m_{qk}}}{(c_{qk}/r)^{m_{qk}} + (C_{qk}/R)^{m_{qk}}} \right),$$

$$T_{11} = \frac{m_{11}(C_{11}/R)^{m_{11}-1}(c_{11}/r)^{m_{11}}}{((c_{11}/r)^{m_{11}} + (C_{11}/R)^{m_{11}})^2} \prod_{k=2}^{n} \frac{(c_{1k}/r)^{m_{1k}}}{(c_{1k}/r)^{m_{1k}} + (C_{1k}/R)^{m_{1k}}} \prod_{q=2}^{s} \left(1 - \prod_{k=1}^{n} \frac{(c_{qk}/r)^{m_{qk}}}{(c_{qk}/r)^{m_{qk}} + (C_{qk}/R)^{m_{qk}}} \right) \tag{63}$$

Generalizing to component ij gives

$$t_{ij} = \frac{m_{ij}(c_{ij}/r)^{m_{ij}-1}(C_{ij}/R)^{m_{ij}}}{((c_{ij}/r)^{m_{ij}} + (C_{ij}/R)^{m_{ij}})^2} \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{(c_{ik}/r)^{m_{ik}}}{(c_{ik}/r)^{m_{ik}} + (C_{ik}/R)^{m_{ik}}} \prod_{\substack{q=1 \\ q \neq i}}^{s} \left(1 - \prod_{k=1}^{n} \frac{(c_{qk}/r)^{m_{qk}}}{(c_{qk}/r)^{m_{qk}} + (C_{qk}/R)^{m_{qk}}} \right),$$

$$T_{ij} = \frac{m_{ij}(C_{ij}/R)^{m_{ij}-1}(c_{ij}/r)^{m_{ij}}}{((c_{ij}/r)^{m_{ij}} + (C_{ij}/R)^{m_{ij}})^2} \prod_{\substack{k=1 \\ k \neq j}}^{n} \frac{(c_{ik}/r)^{m_{ik}}}{(c_{ik}/r)^{m_{ik}} + (C_{ik}/R)^{m_{ik}}} \prod_{\substack{q=1 \\ q \neq i}}^{s} \left(1 - \prod_{k=1}^{n} \frac{(c_{qk}/r)^{m_{qk}}}{(c_{qk}/r)^{m_{qk}} + (C_{qk}/R)^{m_{qk}}} \right) \tag{64}$$

and inserting into (59) gives the utilities. Taking the limits as n and s approach infinity, (64) and (59) imply

$$\begin{aligned} &\lim_{n\to\infty} t_{ij} = \lim_{n\to\infty} T_{ij} = \lim_{n\to\infty} U = 0, \quad \lim_{n\to\infty} u = r \\ &\lim_{s\to\infty} t_{ij} = \lim_{s\to\infty} T_{ij} = \lim_{s\to\infty} u = 0, \quad \lim_{s\to\infty} U = R \end{aligned} \tag{65}$$

Proposition 6. *Infinitely many components n in parallel causes maximum utility u=r to the defender and minimum utility U=0 to the attacker regardless of the finite number s of parallel subsystems in series. Conversely, infinitely many components s in series causes minimum utility u=0 to the defender and maximum utility U=R to the attacker regardless of the finite number n of components in parallel in each subsystem. The investments t_i and T_i approach zero asymptotically as n or s approaches infinity.*

Proof. Follows from (59), (64), and (65).

Inserting $c_{ij}=c_j$, $C_{ij}=C_j$, $m_{ij}=m_j$ for all i=1,…,s into (64), which means that component j in each of the s parallel subsystems has equal unit costs of defense and attack, and equal intensity, gives

$$t_{ij}=\frac{m_j(c_j/r)^{m_j-1}(C_j/R)^{m_j}}{((c_j/r)^{m_j}+(C_j/R)^{m_j})^2}\prod_{\substack{k=1\\k\neq j}}^{n}\frac{(c_k/r)^{m_j}}{(c_k/r)^{m_j}+(C_k/R)^{m_j}}\left(1-\prod_{k=1}^{n}\frac{(c_k/r)^{m_j}}{(c_k/r)^{m_j}+(C_k/R)^{m_j}}\right)^{s-1},$$
$$T_{ij}=\frac{m_j(C_j/R)^{m_j-1}(c_j/r)^{m_j}}{((c_j/r)^{m_j}+(C_j/R)^{m_j})^2}\prod_{\substack{k=1\\k\neq j}}^{n}\frac{(c_k/r)^{m_j}}{(c_k/r)^{m_j}+(C_k/R)^{m_j}}\left(1-\prod_{k=1}^{n}\frac{(c_k/r)^{m_j}}{(c_k/r)^{m_j}+(C_k/R)^{m_j}}\right)^{s-1} \quad (66)$$

Conversely, inserting $c_{ij}=c_i$, $C_{ij}=C_i$, $m_{ij}=m_i$ for all j=1,…,n into (64), which means that all the n components within each parallel subsystem have equal unit costs of defense and attack, and equal intensity, gives

$$t_{ij}=\frac{m_i(c_i/r)^{m_i-1}(C_i/R)^{m_i}}{((c_i/r)^{m_i}+(C_i/R)^{m_i})^2}\left(\frac{(c_i/r)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\right)^{n-1}\prod_{\substack{q=1\\q\neq i}}^{s}\left(1-\left(\frac{(c_q/r)^{m_i}}{(c_q/r)^{m_i}+(C_q/R)^{m_i}}\right)^{n}\right),$$
$$T_{ij}=\frac{m_i(C_i/R)^{m_i-1}(c_i/r)^{m_i}}{((c_i/r)^{m_i}+(C_i/R)^{m_i})^2}\left(\frac{(c_i/r)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\right)^{n-1}\prod_{\substack{q=1\\q\neq i}}^{s}\left(1-\left(\frac{(c_q/r)^{m_i}}{(c_q/r)^{m_i}+(C_q/R)^{m_i}}\right)^{n}\right) \quad (67)$$

The joint assumption of (66) and (67) where $c_{ij}=c$, $C_{ij}=C$, $m_{ij}=m$ for all i=1,…,s, j=1,…,n, which means that all the sn components have the same unit costs of defense and attack, and equal intensity, gives

$$t_{ij}=\frac{m(c/r)^{m-1}(C/R)^{m}}{((c/r)^{m}+(C/R)^{m})^2}\left(\frac{(c/r)^{m}}{(c/r)^{m}+(C/R)^{m}}\right)^{n-1}\left(1-\left(\frac{(c/r)^{m}}{(c/r)^{m}+(C/R)^{m}}\right)^{n-1}\right)^{s-1},$$
$$T_{ij}=\frac{m(C/R)^{m-1}(c/r)^{m}}{((c/r)^{m}+(C/R)^{m})^2}\left(\frac{(c/r)^{m}}{(c/r)^{m}+(C/R)^{m}}\right)^{n-1}\left(1-\left(\frac{(c/r)^{m}}{(c/r)^{m}+(C/R)^{m}}\right)^{n-1}\right)^{s-1} \quad (68)$$

If C/R=c/r, which means that the attacker's ratio of unit cost to system value equals the defender's ratio of unit cost to system value, (68) and the utilities in (59) become

$$t_{ij}=T_{ij}=\frac{m}{2^{n+1}c/r}\left(1-\frac{1}{2^{n-1}}\right)^{s-1},$$
$$u=r\left(1-\frac{1}{2^{n}}\right)^{s}-rmsn\frac{1}{2^{n+1}}\left(1-\frac{1}{2^{n-1}}\right)^{s-1}, \quad (69)$$
$$U=R\left(1-\left(1-\frac{1}{2^{n}}\right)^{s}\right)-Rmsn\frac{1}{2^{n+1}}\left(1-\frac{1}{2^{n-1}}\right)^{s-1}$$

Assume that C/R=c/wr and m=1. If w>1, the defender is disadvantaged with a c/r that is w times as high as C/R for the attacker. Conversely, if w<1, the attacker is disadvantaged. Then (68) and (59) become

$$t_{ij} = \frac{w}{(w+1)^2 c/r}\left(\frac{w}{w+1}\right)^{n-1}\left(1-\left(\frac{w}{w+1}\right)^{n-1}\right)^{s-1},$$

$$T_{ij} = \frac{w^2}{(w+1)^2 c/r}\left(\frac{w}{w+1}\right)^{n-1}\left(1-\left(\frac{w}{w+1}\right)^{n-1}\right)^{s-1}, \quad \frac{t_{ij}}{T_{ij}} = \frac{1}{w},$$

$$u = r\left(1-\left(\frac{w}{w+1}\right)^{n}\right)^{s} - rsn\frac{w}{(w+1)^2}\left(\frac{w}{w+1}\right)^{n-1}\left(1-\left(\frac{w}{w+1}\right)^{n-1}\right)^{s-1}, \tag{70}$$

$$U = R\left(1-\left(1-\left(\frac{w}{w+1}\right)^{n}\right)^{s}\right) - Rsn\frac{w}{(w+1)^2}\left(\frac{w}{w+1}\right)^{n-1}\left(1-\left(\frac{w}{w+1}\right)^{n-1}\right)^{s-1}$$

8 Parallel system of series subsystems

Consider the s series subsystems n times in parallel in Fig. 8. The system reliability is

$$p = 1-\prod_{i=1}^{s}\left(1-\prod_{j=1}^{n}\frac{t_{ij}^{m_{ij}}}{t_{ij}^{m_{ij}}+T_{ij}^{m_{ij}}}\right) \tag{71}$$

and inserting into the utilities in (1) and (2) gives

$$u = r\left(1-\prod_{i=1}^{s}\left(1-\prod_{j=1}^{n}\frac{t_{ij}^{m_{ij}}}{t_{ij}^{m_{ij}}+T_{ij}^{m_{ij}}}\right)\right) - \sum_{i=1}^{s}\sum_{j=1}^{n}c_{ij}t_{ij},$$

$$U = R\prod_{i=1}^{s}\left(1-\prod_{j=1}^{n}\frac{t_{ij}^{m_{ij}}}{t_{ij}^{m_{ij}}+T_{ij}^{m_{ij}}}\right) - \sum_{i=1}^{s}\sum_{j=1}^{n}C_{ij}T_{ij} \tag{72}$$

To determine the FOCs for t_{11} and T_{11} we write (72) as

$$u = r\left(1-\left(1-\frac{t_{11}^{m_{11}}}{t_{11}^{m_{11}}+T_{11}^{m_{11}}}Q_3\right)Q_4\right) - \sum_{i=1}^{s}\sum_{j=1}^{n}c_{ij}t_{ij},$$

$$U = R\left(1-\left(1-\left(1-\frac{t_{11}^{m_{11}}}{t_{11}^{m_{11}}+T_{11}^{m_{11}}}Q_3\right)Q_4\right)\right) - \sum_{i=1}^{s}\sum_{j=1}^{n}C_{ij}T_{ij}, \tag{73}$$

$$Q_3 = \prod_{k=2}^{n}\frac{t_{1k}^{m_{1k}}}{t_{1k}^{m_{1k}}+T_{1k}^{m_{1k}}}, \quad Q_4 = \prod_{q=2}^{s}\left(1-\prod_{k=1}^{n}\frac{t_{qk}^{m_{qk}}}{t_{qk}^{m_{qk}}+T_{qk}^{m_{qk}}}\right)$$

where Q_3 and Q_4 are not functions of t_{11} and T_{11}. Differentiating with respect to t_{11} and T_{11} gives

$$\frac{\partial u}{\partial t_{11}} = r\frac{m_{11}t_{11}^{m_{11}-1}T_{11}^{m_{11}}}{(t_{11}^{m_{11}}+T_{11}^{m_{11}})^2}Q_3Q_4 - c_{11} = 0, \quad \frac{\partial U}{\partial T_{11}} = R\frac{m_{11}T_{11}^{m_{11}-1}t_{11}^{m_{11}}}{(t_{11}^{m_{11}}+T_{11}^{m_{11}})^2}Q_3Q_4 - C_{11} = 0 \tag{74}$$

Solving gives $t_{11} = T_{11}rC_{11}/Rc_{11}$ and

$$t_{11} = \frac{m_{11}(c_{11}/r)^{m_{11}-1}(C_{11}/R)^{m_{11}}}{((c_{11}/r)^{m_{11}} + (C_{11}/R)^{m_{11}})^2} Q_3Q_4, \quad T_{11} = \frac{m_{11}(C_{11}/R)^{m_{11}-1}(c_{11}/r)^{m_{11}}}{((c_{11}/r)^{m_{11}} + (C_{11}/R)^{m_{11}})^2} Q_3Q_4,$$
$$\frac{t_{11}^{m_{11}}}{t_{11}^{m_{11}} + T_{11}^{m_{11}}} = \frac{(C_{11}/R)^{m_{11}}}{(c_{11}/r)^{m_{11}} + (C_{11}/R)^{m_{11}}} \Rightarrow \frac{t_{ij}^{m_{ij}}}{t_{ij}^{m_{ij}} + T_{ij}^{m_{ij}}} = \frac{(C_{ij}/R)^{m_{ij}}}{(c_{ij}/r)^{m_{ij}} + (C_{ij}/R)^{m_{ij}}} \tag{75}$$

where the general ratio follows since FOCs analogous to (74) exist for any pair of t_{ij} and T_{ij}. Equations (74) and (75) are equivalent to (61) and (62) substituting S_3S_4 with Q_3Q_4. Inserting the general ratio in (75) into (73), and Q_3 and Q_4 into (75), gives

$$Q_3 = \prod_{k=2}^{n} \frac{(C_{1k}/R)^{m_{1k}}}{(c_{1k}/r)^{m_{1k}} + (C_{1k}/R)^{m_{1k}}}, \quad Q_4 = \prod_{q=2}^{s}\left(1 - \prod_{k=1}^{n} \frac{(C_{qk}/R)^{m_{qk}}}{(c_{qk}/r)^{m_{qk}} + (C_{qk}/R)^{m_{qk}}}\right)$$
$$t_{11} = \frac{m_{11}(c_{11}/r)^{m_{11}-1}(C_{11}/R)^{m_{11}}}{((c_{11}/r)^{m_{11}} + (C_{11}/R)^{m_{11}})^2} \prod_{k=2}^{n} \frac{(C_{1k}/R)^{m_{1k}}}{(c_{1k}/r)^{m_{1k}} + (C_{1k}/R)^{m_{1k}}} \prod_{q=2}^{s}\left(1 - \prod_{k=1}^{n} \frac{(C_{qk}/R)^{m_{qk}}}{(c_{qk}/r)^{m_{qk}} + (C_{qk}/R)^{m_{qk}}}\right),$$
$$T_{11} = \frac{m_{11}(C_{11}/R)^{m_{11}-1}(c_{11}/r)^{m_{11}}}{((c_{11}/r)^{m_{11}} + (C_{11}/R)^{m_{11}})^2} \prod_{k=2}^{n} \frac{(C_{1k}/R)^{m_{1k}}}{(c_{1k}/r)^{m_{1k}} + (C_{1k}/R)^{m_{1k}}} \prod_{q=2}^{s}\left(1 - \prod_{k=1}^{n} \frac{(C_{qk}/R)^{m_{qk}}}{(c_{qk}/r)^{m_{qk}} + (C_{qk}/R)^{m_{qk}}}\right) \tag{76}$$

Permuting capital and regular letters, Q_3 follows from S_3, and Q_4 follows from S_4. Generalizing to component ij gives

$$t_{ij} = \frac{m_{ij}(c_{ij}/r)^{m_{ij}-1}(C_{ij}/R)^{m_{ij}}}{((c_{ij}/r)^{m_{ij}} + (C_{ij}/R)^{m_{ij}})^2} \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{(C_{ik}/R)^{m_{ik}}}{(c_{ik}/r)^{m_{ik}} + (C_{ik}/R)^{m_{ik}}} \prod_{\substack{q=1 \\ q\neq i}}^{s}\left(1 - \prod_{k=1}^{n} \frac{(C_{qk}/R)^{m_{qk}}}{(c_{qk}/r)^{m_{qk}} + (C_{qk}/R)^{m_{qk}}}\right),$$
$$T_{ij} = \frac{m_{ij}(C_{ij}/R)^{m_{ij}-1}(c_{ij}/r)^{m_{ij}}}{((c_{ij}/r)^{m_{ij}} + (C_{ij}/R)^{m_{ij}})^2} \prod_{\substack{k=1 \\ k\neq j}}^{n} \frac{(C_{ik}/R)^{m_{ik}}}{(c_{ik}/r)^{m_{ik}} + (C_{ik}/R)^{m_{ik}}} \prod_{\substack{q=1 \\ q\neq i}}^{s}\left(1 - \prod_{k=1}^{n} \frac{(C_{qk}/R)^{m_{qk}}}{(c_{qk}/r)^{m_{qk}} + (C_{qk}/R)^{m_{qk}}}\right) \tag{77}$$

and inserting into (72) gives the utilities. Taking the limits as n and s approach infinity, (77) and (72) imply

$$\begin{aligned} &\lim_{n\to\infty} t_{ij} = \lim_{n\to\infty} T_{ij} = \lim_{n\to\infty} u = 0, \quad \lim_{n\to\infty} U = R \\ &\lim_{s\to\infty} t_{ij} = \lim_{s\to\infty} T_{ij} = \lim_{s\to\infty} U = 0, \quad \lim_{s\to\infty} u = r \end{aligned} \tag{78}$$

Proposition 7. *Infinitely many components n in series causes minimum utility u=r to the defender and maximum utility U=R to the attacker regardless of the finite number s of series subsystems in parallel. Conversely, infinitely many components s in parallel causes maximum utility u=r to the defender and minimum utility U=0 to the attacker regardless of the finite number n of components in series in each subsystem. The investments t_i and T_i approach zero asymptotically as n or s approaches infinity.*

Proof. Follows from (72), (77), and (78).

Inserting $c_{ij}=c_j$, $C_{ij}=C_j$, $m_{ij}=m_j$ for all i=1,…,s into (77), which means that component j in each of the s series subsystems has equal unit costs of defense and attack, and equal intensity, gives

$$t_{ij}=\frac{m_j(c_j/r)^{m_j-1}(C_j/R)^{m_j}}{((c_j/r)^{m_j}+(C_j/R)^{m_j})^2}\prod_{\substack{k=1\\k\neq j}}^{n}\frac{(C_k/R)^{m_j}}{(c_k/r)^{m_j}+(C_k/R)^{m_j}}\left(1-\prod_{k=1}^{n}\frac{(C_k/R)^{m_j}}{(c_k/r)^{m_j}+(C_k/R)^{m_j}}\right)^{s-1},$$
$$T_{ij}=\frac{m_j(C_j/R)^{m_j-1}(c_j/r)^{m_j}}{((c_j/r)^{m_j}+(C_j/R)^{m_j})^2}\prod_{\substack{k=1\\k\neq j}}^{n}\frac{(C_k/R)^{m_j}}{(c_k/r)^{m_j}+(C_k/R)^{m_j}}\left(1-\prod_{k=1}^{n}\frac{(C_k/R)^{m_j}}{(c_k/r)^{m_j}+(C_k/R)^{m_j}}\right)^{s-1} \quad (79)$$

Conversely, inserting $c_{ij}=c_i$, $C_{ij}=C_i$, $m_{ij}=m_i$ for all j=1,…,n into (77), which means that all the n components within each series subsystem have equal unit costs of defense and attack, and equal intensity, gives

$$t_{ij}=\frac{m_i(c_i/r)^{m_i-1}(C_i/R)^{m_i}}{((c_i/r)^{m_i}+(C_i/R)^{m_i})^2}\left(\frac{(C_i/R)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\right)^{n-1}\prod_{\substack{q=1\\q\neq i}}^{s}\left(1-\left(\frac{(C_q/R)^{m_i}}{(c_q/r)^{m_i}+(C_q/R)^{m_i}}\right)^{n}\right),$$
$$T_{ij}=\frac{m_i(C_i/R)^{m_i-1}(c_i/r)^{m_i}}{((c_i/r)^{m_i}+(C_i/R)^{m_i})^2}\left(\frac{(C_i/R)^{m_i}}{(c_i/r)^{m_i}+(C_i/R)^{m_i}}\right)^{n-1}\prod_{\substack{q=1\\q\neq i}}^{s}\left(1-\left(\frac{(C_q/R)^{m_i}}{(c_q/r)^{m_i}+(C_q/R)^{m_i}}\right)^{n}\right) \quad (80)$$

The joint assumption of (79) and (80) where $c_{ij}=c$, $C_{ij}=C$, $m_{ij}=m$ for all i=1,…,s, j=1,…,n, which means that all the sn components have the same unit costs of defense and attack, and equal intensity, gives

$$t_{ij}=\frac{m(c/r)^{m-1}(C/R)^{m}}{((c/r)^{m}+(C/R)^{m})^2}\left(\frac{(C/R)^{m}}{(c/r)^{m}+(C/R)^{m}}\right)^{n-1}\left(1-\left(\frac{(C/R)^{m}}{(c/r)^{m}+(C/R)^{m}}\right)^{n-1}\right)^{s-1},$$
$$T_{ij}=\frac{m(C/R)^{m-1}(c/r)^{m}}{((c/r)^{m}+(C/R)^{m})^2}\left(\frac{(C/R)^{m}}{(c/r)^{m}+(C/R)^{m}}\right)^{n-1}\left(1-\left(\frac{(C/R)^{m}}{(c/r)^{m}+(C/R)^{m}}\right)^{n-1}\right)^{s-1} \quad (81)$$

If C/R=c/r, which means that the attacker's ratio of unit cost to system value equals the defender's ratio of unit cost to system value, (81) and the utilities in (72) become

$$t_{ij}=T_{ij}=\frac{m}{2^{n+1}c/r}\left(1-\frac{1}{2^{n-1}}\right)^{s-1},$$
$$u=r\left(1-\left(1-\frac{1}{2^n}\right)^s\right)-rmsn\frac{1}{2^{n+1}}\left(1-\frac{1}{2^{n-1}}\right)^{s-1}, \tag{82}$$
$$U=R\left(1-\frac{1}{2^n}\right)^s-Rmsn\frac{1}{2^{n+1}}\left(1-\frac{1}{2^{n-1}}\right)^{s-1}$$

The investments in (82) and (69) are equivalent since (81) is equivalent to (68) when C/R=c/r. The utilities are different since (72) differ from (59). Observe how u in (82) follows from U in (69) replacing R with r, and conversely, how U in (82) follows from u in (69) replacing r with R. The defender prefers to increase n in (69) and to increase s in (82). The next proposition specifies the defender's preference when s=n.

Proposition 8. *With equivalent components, equal unit costs for defender and attacker, i.e. C/R=c/r, equal intensity m for all components, and equally many components in series and parallel, i.e. s=n, the defender always prefers n parallel subsystems n times in series rather than n series subsystems n times in parallel, and conversely for the attacker.*

Proof. Requiring that the defender's utility in (69) is larger than the defender's utility in (82) implies $2^{-n}+2^{-1/n}<1$ which is always satisfied when n>1. (When s=n=1 there is one component and the systems are equivalent.)

This means that if the defender can control system configurations, series subsystems should be avoided even when multiple of these in parallel are present. If possible, each component should be supplemented with multiple components in parallel, and these parallel subsystems should then be linked in series.

Assume that C/R=c/wr and m=1. If w>1, the defender is disadvantaged with a c/r that is w times as high as C/R for the attacker. Conversely, if w<1, the attacker is disadvantaged. Then (81) and (72) become

$$t_{ij} = \frac{w}{(w+1)^2 c/r}\left(\frac{1}{w+1}\right)^{n-1}\left(1-\left(\frac{1}{w+1}\right)^{n-1}\right)^{s-1},$$

$$T_{ij} = \frac{w^2}{(w+1)^2 c/r}\left(\frac{1}{w+1}\right)^{n-1}\left(1-\left(\frac{1}{w+1}\right)^{n-1}\right)^{s-1}, \quad \frac{t_{ij}}{T_{ij}} = \frac{1}{w},$$

$$u = r\left(1-\left(1-\left(\frac{1}{w+1}\right)^{n}\right)^{s}\right) - rsn\frac{w}{(w+1)^2}\left(\frac{1}{w+1}\right)^{n-1}\left(1-\left(\frac{1}{w+1}\right)^{n-1}\right)^{s-1}, \tag{83}$$

$$U = R\left(1-\left(\frac{1}{w+1}\right)^{n}\right)^{s} - Rsn\frac{w}{(w+1)^2}\left(\frac{1}{w+1}\right)^{n-1}\left(1-\left(\frac{1}{w+1}\right)^{n-1}\right)^{s-1}$$

Both the investments and utilities in (83) differ from (70).

9 Conclusion

The article considers a system of independent components defended by a strategic defender and attacked by a strategic attacker. The reliability of each component depends on how strongly it is defended and attacked, and on the intensity of the contest. High intensity causes higher investment, but ceteris paribus depresses utilities. In a series system the attacker benefits from a substitution effect since attacker benefits flow from attacking any of the components. The attacker is highly sensitive to differences in attack efficiencies for the components. Conversely, the defender is less sensitive to differences in defense efficiencies for the components, since defender benefits flow from defending all components. With two components in series and intermediate intensity (with value one), the defender's utility is positive when the product of the attack inefficiencies exceeds the product of the defense inefficiencies. Otherwise a corner solution emerges with zero utility to the defender and maximum utility to the attacker. The series system is then 100% insecure. However, even for a series system which usually benefits the attacker, when the attacker is sufficiently disadvantaged with high attack inefficiencies, and the intensity of the contest is sufficiently high, a corner solution emerges with maximum utility to the defender and zero utility to the attacker. The series system is then 100% secure.

The results for the defender in a parallel system are equivalent to the results for the attacker in a series system, determined by permutation. Analogously, the results for the attacker in a parallel system are equivalent to the results for the defender in a series system. This means that whereas the attacker benefits from the substitution effect in series systems, the defender benefits from the substitution effect in parallel systems. Hence whereas the attacker usually benefits in series systems, the defender usually benefits in parallel systems.

With budget constraints the ratio of the investments for each component, and the contest success function for each component, are the same as without budget constraints when replacing the system values for the defender and attacker with their respective budget constraints. The results with budget constraints largely confirm the results without budget constraints.

For a series system of parallel subsystems, inifinitely many components in parallel benefits the defender maximally regardless of the finite number of parallel subsystems in series. Conversely, infinitely many components in series benefits the attacker maximally regardless of the finite number of components in parallel in each subsystem. For a parallel system of series subsystems, the results are opposite. With equivalent components, equal unit costs for defender and attacker, equal intensity for all components, and equally many components in series and parallel, the defender always prefers the series system of parallel subsystems rather than the parallel system of series subsystems, and conversely for the attacker.

Appendix 1

Proof of Proposition 2b. Differentiating (9) when $m_i = m_j = 1$ gives

$$\frac{\partial t_i}{\partial \frac{c_i}{r}} = -\frac{2\frac{C_i}{R}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^3}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\frac{C_j}{R}}{\frac{c_j}{r}+\frac{C_j}{R}}, \quad \frac{\partial t_i}{\partial \frac{C_i}{R}} = \frac{\frac{c_i}{r}-\frac{C_i}{R}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^3}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\frac{C_j}{R}}{\frac{c_j}{r}+\frac{C_j}{R}},$$

$$\frac{\partial t_i}{\partial \frac{c_j}{r}} = -\frac{\frac{C_i}{R}\frac{C_j}{R}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^2\left(\frac{c_j}{r}+\frac{C_j}{R}\right)^2}\prod_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\frac{\frac{C_k}{R}}{\frac{c_k}{r}+\frac{C_k}{R}}, \quad \frac{\partial t_i}{\partial \frac{C_j}{R}} = \frac{\frac{C_i}{R}\frac{c_j}{r}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^2\left(\frac{c_j}{r}+\frac{C_j}{R}\right)^2}\prod_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\frac{\frac{C_k}{R}}{\frac{c_k}{r}+\frac{C_k}{R}},$$

$$\frac{\partial T_i}{\partial \frac{C_i}{R}} = -\frac{2\frac{c_i}{r}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^3}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\frac{C_j}{R}}{\frac{c_j}{r}+\frac{C_j}{R}}, \quad \frac{\partial T_i}{\partial \frac{c_i}{r}} = \frac{\frac{C_i}{R}-\frac{c_i}{r}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^3}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\frac{C_j}{R}}{\frac{c_j}{r}+\frac{C_j}{R}},$$

$$\frac{\partial T_i}{\partial \frac{C_j}{R}} = \frac{\frac{c_i}{r}\frac{c_j}{r}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^2\left(\frac{c_j}{r}+\frac{C_j}{R}\right)^2}\prod_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\frac{\frac{C_k}{R}}{\frac{c_k}{r}+\frac{C_k}{R}}, \quad \frac{\partial T_i}{\partial \frac{c_j}{r}} = -\frac{\frac{c_i}{r}\frac{C_j}{R}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^2\left(\frac{c_j}{r}+\frac{C_j}{R}\right)^2}\prod_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\frac{\frac{C_k}{R}}{\frac{c_k}{r}+\frac{C_k}{R}}$$

(A1)

Differentiating (13), and considering the range of m_i=m where $u \geq 0$, gives

$$\begin{aligned}
\frac{\partial u}{\partial n} &= r\frac{[m(nLn(2)-1)-Ln(4)]}{2^{n+1}} < 0 \ when\ m \leq \frac{2}{n}, \\
\frac{\partial^2 u}{\partial n^2} &= r\frac{[m(-nLn(2)+2)+Ln(4)]Ln(2)}{2^{n+1}} > 0 \ when\ m \leq \frac{2}{n}, \\
\frac{\partial U}{\partial n} &= R\frac{[m(nLn(2)-1)+Ln(4)]}{2^{n+1}} > 0 \ when\ m \leq \frac{2}{n}, \\
\frac{\partial^2 U}{\partial n^2} &= -R\frac{[m(nLn(2)-2)+Ln(4)]Ln(2)}{2^{n+1}} < 0 \ when\ m \leq \frac{2}{n}
\end{aligned} \qquad \text{(A2)}$$

Differentiating (14) gives

$$\frac{\partial u}{\partial m} = \frac{rn\left((c/r)/(C/R)\right)^m}{\left(\left((c/r)/(C/R)\right)^m+1\right)^{n+2}}\left[\left((mn-1)\left(\frac{c/r}{C/R}\right)^m - m - 1\right)Ln\left(\frac{c/r}{C/R}\right) - \left(\frac{c/r}{C/R}\right)^m - 1\right] \qquad \text{(A3)}$$

$$\frac{\partial U}{\partial m} = \frac{Rn\left((c/r)/(C/R)\right)^m}{\left(\left((c/r)/(C/R)\right)^m+1\right)^{n+2}}\left[\left((mn+1)\left(\frac{c/r}{C/R}\right)^m - m + 1\right)Ln\left(\frac{c/r}{C/R}\right) - \left(\frac{c/r}{C/R}\right)^m - 1\right] \qquad \text{(A4)}$$

Proof of Proposition 4d. Differentiating (14) gives

$$\frac{\partial u}{\partial n} = r\left(1+\left(\frac{c/r}{C/R}\right)^m\right)^{-n-1}\left[m\left(\frac{c/r}{C/R}\right)^m\left[nLn\left(1+\left(\frac{c/r}{C/R}\right)^m\right)-1\right]-\left[1+\left(\frac{c/r}{C/R}\right)^m\right]Ln\left(1+\left(\frac{c/r}{C/R}\right)^m\right)\right]$$

(A5)

When $(c/r)/(C/R)$ is low, the bracket after $m\left((c/r)/(C/R)\right)^m$ in (A5) can be negative when n is low, causing $\partial u/\partial n$<0. As $(c/r)/(C/R)$ increases, the bracket becomes positive. When C_i/R=c_i/r, $\partial u/\partial n$<0 as shown in (A2). As c/r increases above C/R, the bracket can become quite large, and even larger when the RHS replaces the LHS in the inequality

$$m\left(\frac{c/r}{C/R}\right)^m < \frac{1}{n}\left[1+\left(\frac{c/r}{C/R}\right)^m\right] \tag{A6}$$

which follows from requiring u>0 in (14). Inserting the RHS in (A6) into (A5) gives

$$\frac{\partial u}{\partial n} < -\frac{r}{n}\left(1+\left(\frac{c/r}{C/R}\right)^m\right)^{-n} < 0 \tag{A7}$$

Proof of Proposition 4e. Differentiating (14) gives

$$\frac{\partial U}{\partial n} = R\left(1+\left(\frac{c/r}{C/R}\right)^m\right)^{-n-1}\left[m\left(\frac{c/r}{C/R}\right)^m\left[nLn\left(1+\left(\frac{c/r}{C/R}\right)^m\right)-1\right]+\left[1+\left(\frac{c/r}{C/R}\right)^m\right]Ln\left(1+\left(\frac{c/r}{C/R}\right)^m\right)\right] \tag{A8}$$

This expression is evidently positive when $(c/r)/(C/R)$ is large. When $(c/r)/(C/R)$ is small, the bracket after $m\left((c/r)/(C/R)\right)^m$ in (A8) is negative, and it is even more negative when the RHS replaces the LHS in the inequality

$$m\left(\frac{c/r}{C/R}\right)^m < \frac{1}{n}\left[1+\left(\frac{c/r}{C/R}\right)^m\right]\left[\left(1+\left(\frac{c/r}{C/R}\right)^m\right)^n-1\right] \tag{A9}$$

which follows from requiring U>0 in (14). Inserting the RHS in (A9) into (A8) gives

$$\frac{\partial U}{\partial n} > R\left(1+\left(\frac{c/r}{C/R}\right)^m\right)^{-n}\left[\left(1+\left(\frac{c/r}{C/R}\right)^m\right)^n\left[Ln\left(1+\left(\frac{c/r}{C/R}\right)^m\right)-\frac{1}{n}\right]+\frac{1}{n}\right] \tag{A10}$$

Proving that (A10) is positive amounts to proving that

$$f(x) = x^n(Ln(x)-1/n)+1/n \tag{A11}$$

is positive when x>1, which is the case since f(1)=0 and

$$\frac{\partial f(x)}{\partial x} = nx^{n-1}Ln(x) > 0 \quad when \quad x > 1 \tag{A12}$$

which implies $\partial U/\partial n$>0.

Appendix 2

Proof of Proposition 2Pb. Differentiating (36) when m_i=m_j=1 gives

$$\frac{\partial t_i}{\partial \frac{c_i}{r}} = -\frac{2\frac{C_i}{R}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^3}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\frac{c_j}{r}}{\frac{c_j}{r}+\frac{C_j}{R}}, \quad \frac{\partial t_i}{\partial \frac{C_i}{R}} = \frac{\frac{c_i}{r}-\frac{C_i}{R}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^3}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\frac{c_j}{r}}{\frac{c_j}{r}+\frac{C_j}{R}},$$

$$\frac{\partial t_i}{\partial \frac{c_j}{r}} = \frac{\frac{C_i}{R}\frac{C_j}{R}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^2\left(\frac{c_j}{r}+\frac{C_j}{R}\right)^2}\prod_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\frac{\frac{c_k}{r}}{\frac{c_k}{r}+\frac{C_k}{R}}, \quad \frac{\partial t_i}{\partial \frac{C_j}{R}} = -\frac{\frac{C_i}{R}\frac{c_j}{r}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^2\left(\frac{c_j}{r}+\frac{C_j}{R}\right)^2}\prod_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\frac{\frac{c_k}{r}}{\frac{c_k}{r}+\frac{C_k}{R}},$$

$$\frac{\partial T_i}{\partial \frac{C_i}{R}} = -\frac{2\frac{c_i}{r}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^3}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\frac{c_j}{r}}{\frac{c_j}{r}+\frac{C_j}{R}}, \quad \frac{\partial T_i}{\partial \frac{c_i}{r}} = \frac{\frac{C_i}{R}-\frac{c_i}{r}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^3}\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\frac{c_j}{r}}{\frac{c_j}{r}+\frac{C_j}{R}},$$

$$\frac{\partial T_i}{\partial \frac{C_j}{R}} = -\frac{\frac{c_i}{r}\frac{c_j}{r}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^2\left(\frac{c_j}{r}+\frac{C_j}{R}\right)^2}\prod_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\frac{\frac{c_k}{r}}{\frac{c_k}{r}+\frac{C_k}{R}}, \quad \frac{\partial T_i}{\partial \frac{c_j}{r}} = \frac{\frac{c_i}{r}\frac{C_j}{R}}{\left(\frac{c_i}{r}+\frac{C_i}{R}\right)^2\left(\frac{c_j}{r}+\frac{C_j}{R}\right)^2}\prod_{\substack{k=1\\ k\neq i\\ k\neq j}}^{n}\frac{\frac{c_k}{r}}{\frac{c_k}{r}+\frac{C_k}{R}}$$

(A13)

Differentiating (40), and considering the range of m_i=m where $U \geq 0$, gives

$$\begin{aligned}
\frac{\partial U}{\partial n} &= R\frac{[m(nLn(2)-1)-Ln(4)]}{2^{n+1}} < 0 \; when \; m \leq \frac{2}{n}, \\
\frac{\partial^2 U}{\partial n^2} &= R\frac{[m(-nLn(2)+2)+Ln(4)]Ln(2)}{2^{n+1}} > 0 \; when \; m \leq \frac{2}{n}, \\
\frac{\partial u}{\partial n} &= r\frac{[m(nLn(2)-1)+Ln(4)]}{2^{n+1}} > 0 \; when \; m \leq \frac{2}{n}, \\
\frac{\partial^2 u}{\partial n^2} &= -r\frac{[m(nLn(2)-2)+Ln(4)]Ln(2)}{2^{n+1}} < 0 \; when \; m \leq \frac{2}{n}
\end{aligned} \tag{A14}$$

Differentiating (41) gives

$$\frac{\partial U}{\partial m} = \frac{Rn\left((C/R)/(c/r)\right)^m}{(\left((C/R)/(c/r)\right)^m+1)^{n+2}}\left[\left((mn-1)\left(\frac{C/R}{c/r}\right)^m - m - 1\right)Ln\left(\frac{C/R}{c/r}\right) - \left(\frac{C/R}{c/r}\right)^m - 1\right] \tag{A15}$$

$$\frac{\partial u}{\partial m} = \frac{rn\left((C/R)/(c/r)\right)^m}{(\left((C/R)/(c/r)\right)^m+1)^{n+2}}\left[\left((mn+1)\left(\frac{C/R}{c/r}\right)^m - m + 1\right)Ln\left(\frac{C/R}{c/r}\right) - \left(\frac{C/R}{c/r}\right)^m - 1\right] \tag{A16}$$

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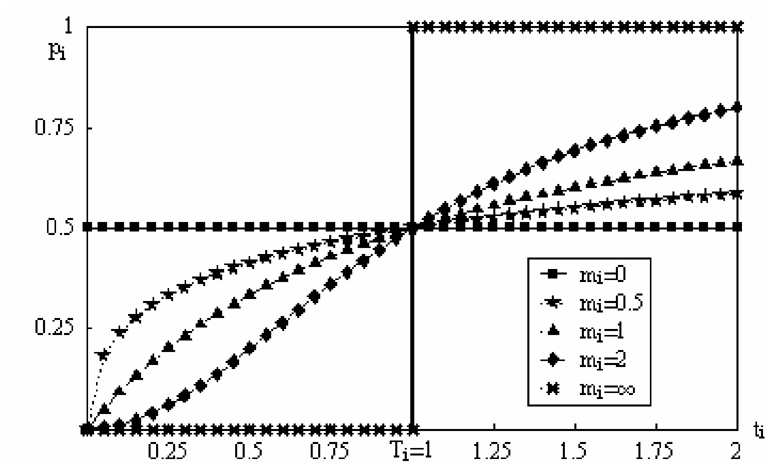

Fig. 1. Ratio form reliability p_i as a function of the investment t_i for various m_i when T_i=1.

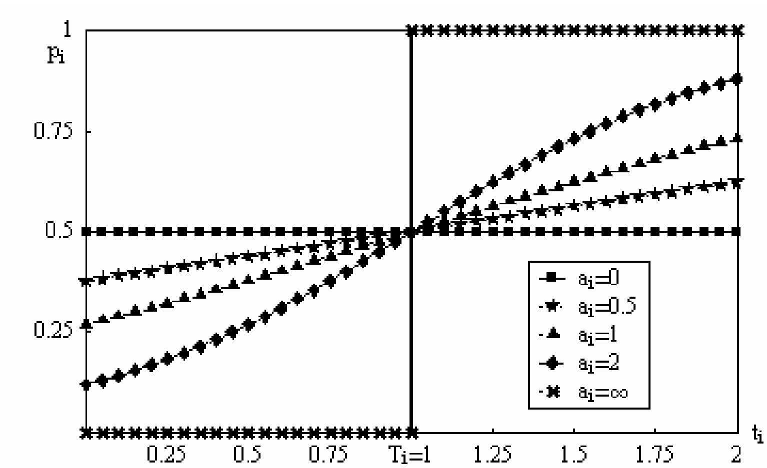

Fig. 2. Difference form reliability p_i as a function of the investment t_i for various a_i, T_i=1.

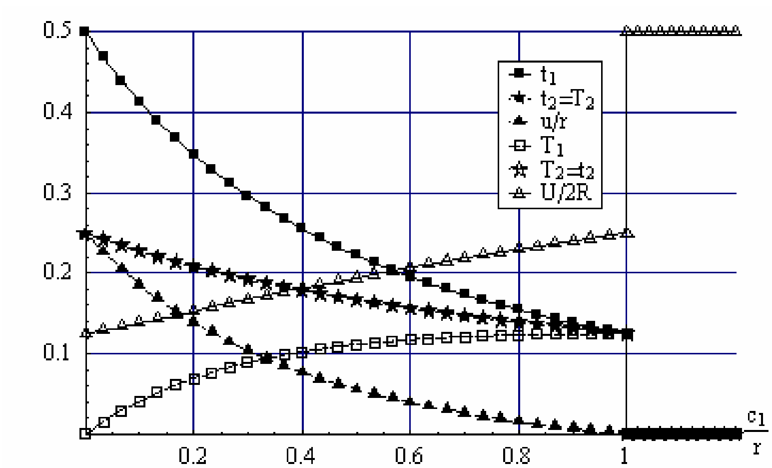

Fig. 3. Series system: The six variables as functions of c_1/r when $c_2/r=C_1/R=C_2/R=m_i=1$.

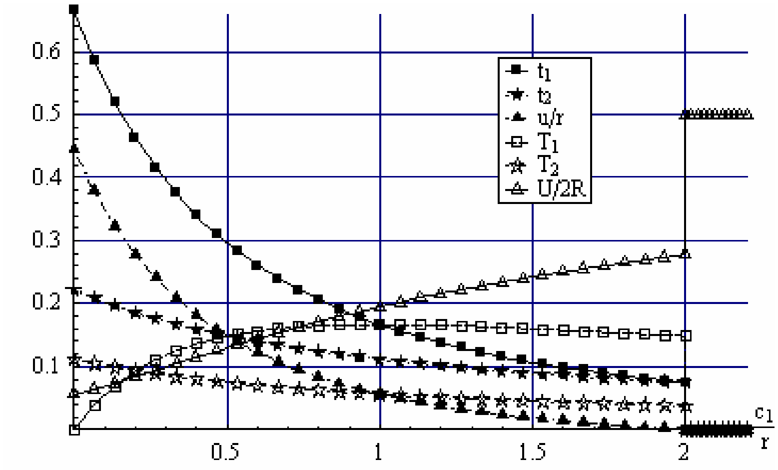

Fig. 4. Series system: The six variables as functions of c_1/r when $c_2/r=C_1/R=m_i=1$, $C_2/R=2$.

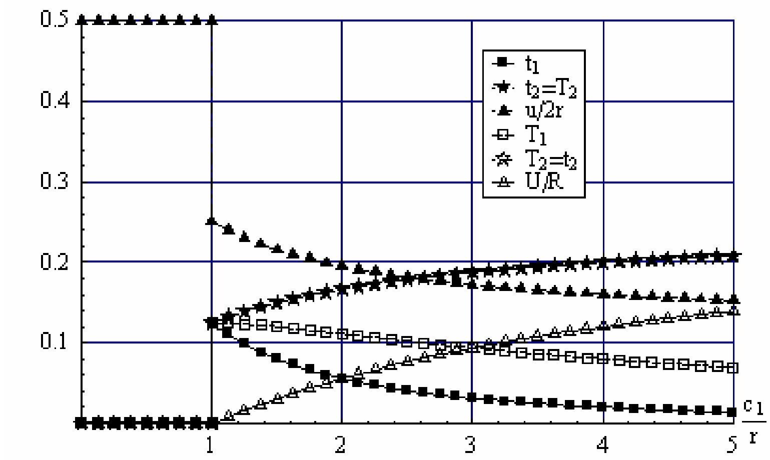

Fig. 5. Parallel system: Same parameters as in Fig. 3.

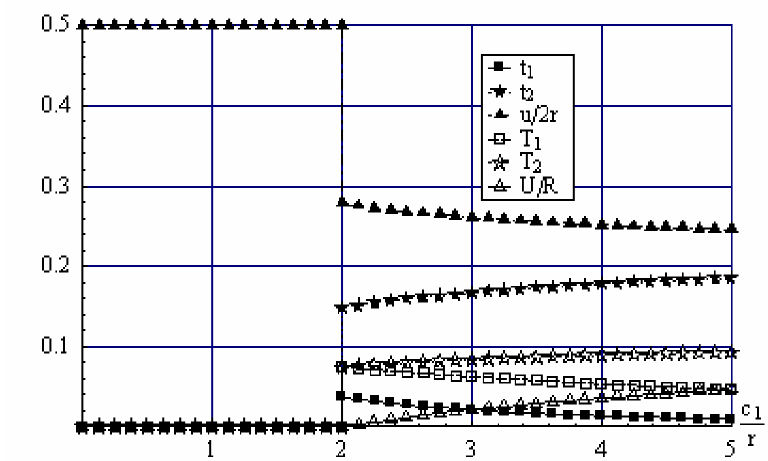

Fig. 6. Parallel system: Same parameters as in Fig. 4.

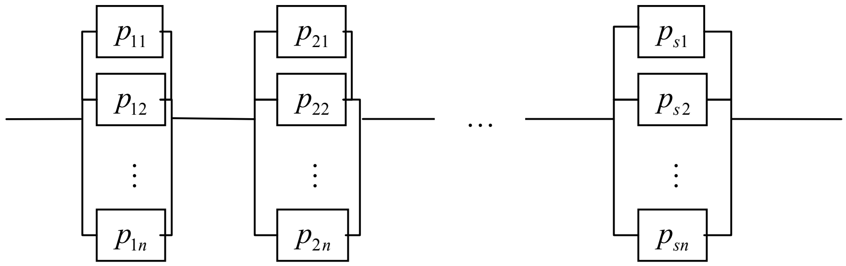

Fig. 7 n parallel subsystems s times in series.

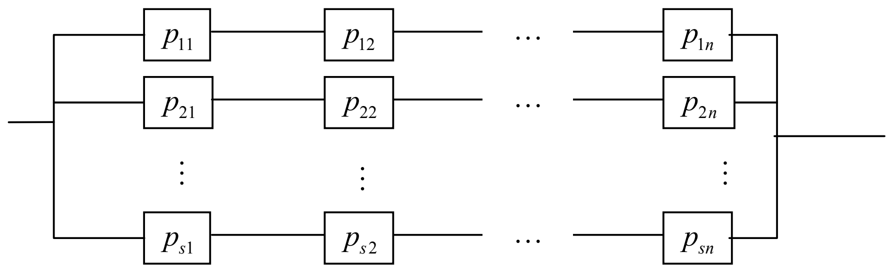

Fig. 8 s series subsystems n times in parallel.

Strategic Defense and Attack for Series and Parallel Reliability Systems: Sequential Moves by Defender and Attacker

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Abstract

For systems built over time where defense investments are known, the defender moves first. When the defender gains attacker intelligence, or the attacker announces forthcoming attacks, the attacker moves first. When the defender moves first in a series system, and is sufficiently advantaged with a low unit cost of investment, the attacker is 100% deterred. Such deterrence is not possible in the simultaneous move game. With two equivalent components in series, the defender always prefers the two period game over the one period simultaneous move game. The attacker prefers the one period game, but prefers the two period game when the defense inefficiency is between 50% and 100% of the attack inefficiency. When the attacker moves first in a series system, for a broad set of parameter values, simulations illustrate that the attacker mostly prefers the first mover advantage, and prefers to attack two components rather than one component. When the attacker moves first in a parallel system, and is sufficiently advantaged with a low unit cost of investment, the defender is 100% deterred. Such deterrence is not possible in the simultaneous move game. With two equivalent components in parallel, the attacker always prefers the two period game over the one period simultaneous move game. The defender prefers the one period game, but prefers the two period game when the attack inefficiency is between 50% and 100% of the defense inefficiency. When the defender moves first in a parallel system, for a broad set of parameter values, simulations illustrate that the defender mostly prefers the first mover advantage, and prefers to defend two components rather than one component.

10 Introduction

This is the second article in a sequence of two that analyzes strategic defense and attack for series and parallel reliability systems. The first article considered simultaneous moves by the defender and attacker. This second article considers sequential moves by the defender and attacker.

That the defender and attacker choose their strategies simultaneously is often realistic, but not always. Sometimes one agent chooses its strategy first, which means that the other agent chooses its strategy thereafter, taking the first agent's strategy as given. The optimization strategies are quite different. Sometimes there may be a first mover advantage, and sometimes a second mover advantage.

For systems and infrastructures that are built over time, and where the defender is open and candid about its defense investment and installations, it is often realistic that the defender moves first. The attacker then takes the defender's defense level as given when choosing the optimal attack. If the defender prefers to, or is able to, keep its defense investment secret, and when the attacker's investment is also unknown, the two agents optimize in a simultaneous game. The opposite case means that the attacker announces its investment strategy up front, or the defender gets to know the attacker's investment capacity or strategy through intelligence operations. In this case the attacker moves first, and the defender takes the attacker's strategy as given when choosing the optimal defense. To determine which agent moves first, the essential issue is to determine what each agent knows about the opponent's strategy when choosing its own strategy.

As an example, the September 11, 2001 attack is probably best interpreted as a two period game where the defender moves first with an extremely weak defense. The defender had weak intelligence on the attacker's strategy, and the intelligence that existed was not taken into account since the notion of flying planes into buildings lay outside the defender's imagination. The attacker observed the weak defense, took it as given in its optimization, and launched an overwhelming unexpected attack. No replication of 9/11 has occurred at the time of writing this paper. Potential attackers reasonably expect the defense of most targets to be much improved, which causes an alternative solution to the attacker's second period optimization problem.

In the cases where the defender moves first by preparations, no contest occurs before the attacker moves second. The attacker can often observe these preparations, but not always. Conversely, the defender frequently works to understand the attacker's strategy. One example is President Bill Clinton's attempts to observe Osama Bin Laden's strategy in the late 1990s. Clearly, determining which decision is made first can be difficult. Was the bad defense decided first, with an overwhelming attack thereafter? Or was an overwhelming attack planned and decided first, with a bad defense decided thereafter? This may become a chicken and egg problem. Which comes first? But, we have the following difference between the two options: 1. Defender maximizes first, expecting a future decision by the attacker. Attacker maximizes second taking the defender's decision as given. 2. Attacker maximizes first, expecting a future decision by the defender. Defender maximizes second taking the attacker's decision as given.

The key determinator is to assess who takes what as given when making a decision, and whether one decision is given in a more fundamental sense than the other decision. The defense sometimes goes first when it is designed without explicit knowledge about the nature and scope of the attack. The attacker thereafter reacts to that with a second decision. Conversely, a clear cut case when the attacker moves first is when the attacker announces up front that a new attack for example on some more or less specified target in the Western World will occur at some point in the future, and that the credibility is supported by demonstrated force investments for the attacker. This can happen even without an announced attack. For example, the defender can gain intelligence that estimates force investments by the attacker, and then design a new defensive strategy thereafter. If the time between the intelligence and the attack is short, the defender is constrained in its ability to revise its strategy. It may for example merely be able to adjust alertness levels, and revise and slightly upgrade patrolling by guards and security personnel. However, with sufficient time between the intelligence and the attack, the defender can design an appropriate an entirely new strategy, and at best move second with full knowledge of the attacker's strategy in the sense that it can be taken as given in the defender's optimization problem. This may involve substantial capital investment into hardening targets and infrastructures, and training of security personnel with various kinds of competence.

There are often grey zones with intermediate knowledge and uncertain estimates. However, the three cases simultaneous game, defender moves first, attacker moves first, are clear cut and archetypical. For intermediate cases, solutions intermediate between the three archetypical can be assessed.

Assessing whether the defense is superior to attack, or the attack is superior to defense, is essential. Clausewitz (1832) presented the classical argument for the "superiority of defense over attack," i.e. that "the defensive form of warfare is intrinsically stronger than the offensive": He writes (Clausewitz 1832:6.1.2): "The defender enjoys optimum lines of communication and retreat, and can choose the place for battle. By standing on the defensive and awaiting "the blow," the defender does not expose himself, and can respond to the enemy's attack according to its nature and character. The weaker commander thus often adopts the defensive because its inherent strength tends to offset his weakness. Hence the weak may, through delay, wear out the strong. The attacking army, on the other hand (Clausewitz 1832:6.3), "cuts itself off from its own theater of operations, ...suffers by having to leave its fortresses and depots behind," and wastes strength by moving forward. The attacker's main advantages are due to the attack's initial surprise, and the benefit of choosing the time, the nature, and the form of the attack. This, however, if one is weak, does not outweigh the benefits of the defense, unless one has a political initiative or expects to grow even weaker as time proceeds" (Clausewitz 1832:8.5).

Hausken (2004:574) argues that the superiority of the defense over the attack appears to be even larger for production facilities and produced goods than for Clausewitz's mobile army. The location of production facilities can be chosen for optimum defense. Although retreat is not especially easy for most production facilities, production is often deeply entrenched in an agent's way of life, and its resources and goods can be camouflaged, hidden, and moved.

Examples of features improving the defense are the use of trenches in World War I, castles and fortresses with canon fire from higher altitudes, and the use of checks and guards in broad senses of those terms. In WWI, the machine gun and trench warfare allowed for superiority of defense over attack. In WWII, tanks and better airplanes tilted some advantage over toward the attack.

In the internet cyber era, the attacker has an advantage. Anderson (2001) argues as follows: "So information warfare looks rather like air warfare looked in the 1920s and 1930s. Attack is simply easier than defense. Defending a modern information system could also be likened to defending a large, thinly-populated territory like the nineteenth century Wild West: the men in black hats can strike anywhere, while the men in white hats have to defend everywhere. Another possible relevant analogy is the use of piracy on the high seas as an instrument of state policy by many European powers in the sixteenth and seventeenth centuries. Until the great powers agreed to deny pirates safe haven, piracy was just too easy. The technical bias in favor of attack is made even worse by asymmetric information." Levitin (2007) argues more bluntly that “choosing the time, place, and means of attacks, the attacker has always an advantage over the defender.” Levitin’s argument (personal communication) applies especially with the presence of mobile offensive means (especially for cyber attack) and a huge number of possible targets (not limited now by military facilities), in contrast to conventional warfare where much resources should be moved to provide the offensive and where the only target is the defender’s position.

Section 11 analyzes the two period series system where the defender chooses investment in the first period, while the attacker chooses investment in the second period. Section 12 considers the series system when the attacker moves first. Section 13 analyzes the two period parallel system where the attacker moves first, and the defender moves second. Section 14 considers the parallel system when the defender moves first. Section 15 concludes.

11 Two period game for series system when defender moves first

It is sometimes realistic that the defender chooses investment in the first period, while the attacker chooses investment in the second period. The second period is solved first. The n attacker FOCs for the second period are as in (7) and can for components i and j be expressed as

$$\begin{aligned}\frac{\partial U}{\partial T_i} &= R\frac{m_i T_i^{m_i-1} t_i^{m_i}}{(t_i^{m_i}+T_i^{m_i})^2}\frac{t_j^{m_j}}{t_j^{m_j}+T_j^{m_j}}\prod_{\substack{k=1\\k\neq i\\k\neq j}}^{n}\frac{t_k^{m_k}}{t_k^{m_k}+T_k^{m_k}} - C_i = 0,\\ \frac{\partial U}{\partial T_j} &= R\frac{m_j T_j^{m_j-1} t_j^{m_j}}{(t_j^{m_j}+T_j^{m_j})^2}\frac{t_i^{m_j}}{t_i^{m_j}+T_i^{m_j}}\prod_{\substack{k=1\\k\neq i\\k\neq j}}^{n}\frac{t_k^{m_k}}{t_k^{m_k}+T_k^{m_k}} - C_j = 0\end{aligned} \tag{84}$$

The two product signs are equal, so this is solved to yield

$$t_j^{m_j}+T_j^{m_j} = \frac{C_i T_j^{m_j-1}(t_i^{m_j}+T_i^{m_j})}{C_j T_i^{m_j-1}} \tag{85}$$

Inserting (85) into (7) gives

$$\frac{\partial U}{\partial T_i} = R\frac{m_i (T_i^{m_i-1})^{n+1}}{C_i^n (t_i^{m_i} + T_i^{m_i})^{n+1}} \prod_{i=1}^{n} \frac{C_i t_i^{m_i}}{T_i^{m_i-1}} - C_i = 0 \tag{86}$$

which for m_i=1 is solved to yield

$$T_i = \frac{1}{C_i / R}\left(\prod_{i=1}^{n} t_i C_i / R\right)^{1/(n+1)} - t_i \tag{87}$$

which implies

$$C_i (T_i + t_i) = C_j (T_j + t_j) \tag{88}$$

for i,j=1,…,n, $i \neq j$, which is a consequence of the n attacker FOCs. Inserting (87) into the denominator in (5) and simplifying gives the defender's first period utility

$$u = r\left(\prod_{i=1}^{n} t_i C_i / R\right)^{1/(n+1)} - \sum_{i=1}^{n} c_i t_i \tag{89}$$

Two of the n defender FOCs, for components i and j, for the first period are

$$\frac{\partial u}{\partial t_i} = \frac{r}{(n+1)t_i}\left(\prod_{i=1}^{n} t_i C_i / R\right)^{1/(n+1)} - c_i = 0, \quad \frac{\partial u}{\partial t_j} = \frac{r}{(n+1)t_j}\left(\prod_{i=1}^{n} t_i C_i / R\right)^{1/(n+1)} - c_j = 0 \tag{90}$$

which are solved to yield

$$t_j = c_i t_i / c_j \tag{91}$$

Inserting (91) into (90) and solving for t_i, and inserting into (87), gives

$$t_i = \frac{1}{(n+1)^{(n+1)} c_i / r} \prod_{i=1}^{n} \frac{C_i / R}{c_i / r}, \quad T_i = \frac{(n+1)c_i / r - C_i / R}{(n+1)^{n+1} (C_i / R) c_i / r} \prod_{i=1}^{n} \frac{C_i / R}{c_i / r} \tag{92}$$

which implies $t_i / (t_i + T_i) = (C_i / R)/((n+1)c_i / r)$. Inserting into the utilities in (5) and (6) gives

$$u = \frac{r}{(n+1)^{n+1}} \prod_{i=1}^{n} \frac{C_i / R}{c_i / r}, \quad U = R\left(1 - \frac{1}{(n+1)^n}\left(1 + \frac{1}{(n+1)}\sum_{i=1}^{n} \frac{(n+1)c_i / r - C_i / R}{c_i / r}\right)\prod_{i=1}^{n} \frac{C_i / R}{c_i / r}\right) \tag{93}$$

When $c_i / r < (C_i / R)/(n+1)$, which is satisfied when the unit cost of defense is low when n is large, a corner solution exists where T_i=0, which deters the attacker successfully from component i. Assume without loss of generality that $c_1 / r < (C_1 / R)/(n+1)$ for component 1, which causes T_1=0. Assume that the defender invests an arbitrarily small but positive amount $t_1 = 0^+$ into defending component 1, causing component 1 to be 100% secure. Assuming that the k components 1,2,…,k are secure, replacing n with n-k in (92) and (93), and using subscript j instead of i, gives the investments and utilities for components j=k+1,…,n,

$$t_j = \frac{1}{(n-k+1)^{(n-k+1)} c_j / r} \prod_{j=k+1}^{n} \frac{C_j / R}{c_j / r}, \quad T_j = \frac{(n-k+1)c_j / r - C_j / R}{(n-k+1)^{n-k+1} (C_j / R) c_j / r} \prod_{j=k+1}^{n} \frac{C_j / R}{c_j / r}$$

$$u = \frac{r}{(n-k+1)^{n-k+1}} \prod_{j=k+1}^{n} \frac{C_j / R}{c_j / r}, \tag{94}$$

$$U = R\left(1 - \frac{1}{(n-k+1)^{n-k}} \left(1 + \frac{1}{(n-k+1)} \sum_{j=k+1}^{n} \frac{(n-k+1)c_j / r - C_j / R}{c_j / r}\right) \prod_{j=k+1}^{n} \frac{C_j / R}{c_j / r}\right)$$

Proposition 9. *Two period series system when defender moves first: (a) When the defense inefficiencies for the k components 1, 2,…,k are sufficiently low,* $c_1 / r < (C_1 / R)/(n+1)$, $c_2 / r < (C_2 / R)/n$,…, $c_k / r < (C_k / R)/(n-k+2)$, *then these components are not subject to attack and are 100% secure,* T_i*=0. When additionally* $c_j / r < (C_j / R)/(n-k+1)$ *for any j=k+1,…,n, a (k+1)th corner solution exists where* T_j*=0, which deters the attacker successfully also from component j. (b) The series system is never 100% insecure except at the limit when* $n \to \infty$, *or* $c_i / r \to \infty$ *or* $C_i / R = 0$ *for at least one component which causes u=0 and U=R.*

Proof. (a) Follows from (93) and from requiring T_i>0 in (92) and T_j>0 in (94). (b) Follows from (93).

Whereas the one period series and parallel systems can be both 100% secure and 100% insecure for finite nonzero parameter values, the two period series system can be 100% secure, but not 100% insecure, except for three extreme and unlikely cases. First, infinitely many components in series reduces the defense of each component towards zero, and reduces the defender's utility towards zero, since $(n+1)^{(n+1)}$ in the denominators in (92) and (93) becomes overwhelmingly large. Second, if the defender has infinite unit cost of defending at least one component, that causes 100% insecurity for that component and thus for the series system. Third, if the attacker has zero unit cost of attacking at least one component, that causes 100% insecurity for that component and thus for the series system. This illustrates the defender's first mover advantage. The defender prefers to precommit to its optimal investment level in the first period in a two period game, rather than moving simultaneously with the attacker in a one period game.

Inserting n=2 into (92) and (93) gives

$$t_i = \frac{\frac{C_1}{R}\frac{C_2}{R}}{27\frac{c_i^2}{r}\frac{c_j}{r}}, \quad T_i = \frac{\left(3\frac{c_i}{r} - \frac{C_i}{R}\right)\frac{C_j}{R}}{27\frac{c_i^2}{r}\frac{c_j}{r}},$$
$$u = r\frac{\frac{C_1}{R}\frac{C_2}{R}}{27\frac{c_1}{r}\frac{c_2}{r}}, \quad U = R\left(1 - \frac{9\frac{c_1}{r}\frac{c_2}{r}\frac{C_1}{R}\frac{C_2}{R} - \frac{c_1}{r}\frac{C_1}{R}\frac{C_2^2}{R} - \frac{c_2}{r}\frac{C_2}{R}\frac{C_1^2}{R}}{27\frac{c_1^2}{r}\frac{c_2^2}{r}}\right) \tag{95}$$

When $(c_1/r)/(C_1/R) < 1/3$ so that the attacker is deterred from component 1, (94) gives

$$t_1 = T_1 = 0,\ t_2 = \frac{C_2/R}{4(c_2/r)^2},\ T_2 = \frac{2c_2/r - C_2/R}{4(c_2/r)^2},\ u = r\frac{C_2/R}{4c_2/r},\ U = R\frac{(2c_2/r - C_2/R)^2}{4(c_2/r)^2} \tag{96}$$

When additionally $(c_2/r)/(C_2/R) < 1/2$, the attacker is deterred from both components causing $t_1=t_2=T_1=T_2=U=0$ and $u=r$.

Fig. 9 illustrates as functions of the defense inefficiency c_1/r for the same parameters as in Fig. 3, $n=2$, $c_2/r=C_1/R=C_2/R=m_i=1$. Although Figs. 9 and 3 show qualitatively similar trends for increases and decreases, the two most noteworthy differences are that in Fig. 9 the defender's utility does not equal zero when $c_1/r>1$, and the attacker does not attack component 1 when $c_1/r<1/3$.

Assessing whether the agents prefer the two period game or the one period game is analytically challenging except when $n=2$, $c_i=c$, $C_i=C$, $m_i=1$. We let subscript "t" denote two period game, and subscript "o" denote one period game. The interior solutions follow from inserting $c_i=c$ into (95), (14), (15), which gives

$$t_{it} = \frac{1}{27\frac{c}{r}\left(\frac{c/r}{C/R}\right)^2}, \quad T_{it} = \frac{3\frac{c/r}{C/R} - 1}{27\frac{c}{r}\left(\frac{c/r}{C/R}\right)^2},$$
$$u_t = \frac{r}{27\left(\frac{c/r}{C/R}\right)^2}, \quad U_t = R\frac{\left(3\frac{c/r}{C/R} - 1\right)^2\left(3\frac{c/r}{C/R} + 2\right)}{27\left(\frac{c/r}{C/R}\right)^3} \tag{97}$$

$$t_{io}=\frac{\frac{c/r}{C/R}}{\frac{c}{r}\left(1+\frac{c/r}{C/R}\right)^3},\quad T_{io}=\frac{\frac{c/r}{C/R}}{\frac{C}{R}\left(1+\frac{c/r}{C/R}\right)^3},$$
$$u_o=r\frac{1-\frac{c/r}{C/R}}{\left(1+\frac{c/r}{C/R}\right)^3},\quad U_o=R\frac{\left(3+\frac{c/r}{C/R}\right)\left(\frac{c/r}{C/R}\right)^2}{\left(1+\frac{c/r}{C/R}\right)^3} \tag{98}$$

Three ranges for the ratio $\frac{c/r}{C/R}$ apply. When $0<\frac{c/r}{C/R}<1/3$, the two period game has a corner solution where the attacker is deterred from both components, U=0 and u=r, and the one period game has an interior solution. When $1/3<\frac{c/r}{C/R}<1$, both games have interior solutions. When $\frac{c/r}{C/R}>1$, the two period game has an interior solution and the one period game has a corner solution where the defender gives up, $u=t_1=t_2=0$, U=R. The ratio of the utilities are

$$\frac{u_t}{u_o}=\begin{cases}\frac{\left(1+\frac{c/r}{C/R}\right)^3}{\left(1-\frac{c/r}{C/R}\right)} \; when\; 0<\frac{c/r}{C/R}<\frac{1}{3}\\ \frac{\left(1+\frac{c/r}{C/R}\right)^3}{27\left(\frac{c/r}{C/R}\right)^2\left(1-\frac{c/r}{C/R}\right)} \; when\; \frac{1}{3}<\frac{c/r}{C/R}<1,\\ \infty \; when\; \frac{c/r}{C/R}>1\end{cases}\quad \frac{U_t}{U_o}=\begin{cases}0 \; when\; 0<\frac{c/r}{C/R}<\frac{1}{3}\\ \frac{\left(1-3\frac{c/r}{C/R}\right)^2\left(2+3\frac{c/r}{C/R}\right)\left(1+\frac{c/r}{C/R}\right)^3}{27\left(\frac{c/r}{C/R}\right)^5\left(3+\frac{c/r}{C/R}\right)} \; when\; \frac{1}{3}<\frac{c/r}{C/R}<1\\ \frac{\left(1-3\frac{c/r}{C/R}\right)^2\left(2+3\frac{c/r}{C/R}\right)}{27\left(\frac{c/r}{C/R}\right)^3} \; when\; \frac{c/r}{C/R}>1\end{cases} \tag{99}$$

Equation (99) is plotted in Fig. 10 as a function of $\frac{c/r}{C/R}$.

Proposition 10. *Two period series system when defender moves first: Assume two equivalent components, $c_i=c$, $C_i=C$, $m_i=1$. The defender always prefers the two period game over the one period simultaneous move game and is indifferent between the two games when $\frac{c/r}{C/R}=1/2$. The attacker prefers the two period game when $1/2<\frac{c/r}{C/R}<1$, and otherwise (when $\frac{c/r}{C/R}<1/2$ and $\frac{c/r}{C/R}>1$) prefers the one period game.*

Proof. Follows from inspecting (99) and Fig. 10.

The defender has a first mover's advantage and always prefers the two period game over the one period game. The reason is that the defender can invest based on its expectation about the attacker's strategic move in the second period. Since the attacker is deterred when $0<\frac{c/r}{C/R}<1/3$ causing constant utilities U=0 and u=r for the two period game, that u_o in (98) decreases in $\frac{c/r}{C/R}$ causes u_t/u_o to increase in $\frac{c/r}{C/R}$. When $\frac{c/r}{C/R}$ increases above 1/3, the attacker is no longer deterred and an interior solution applies for both games. Both u_t and u_o decrease convexly in $\frac{c/r}{C/R}$, but u_o eventually reaches zero when $\frac{c/r}{C/R}=1$. This causes u_t/u_o to be U shaped when $1/3<\frac{c/r}{C/R}<1$, reaching infinity as $\frac{c/r}{C/R}$ reaches one from below. The attacker's two period utility gradually increases from zero when $\frac{c/r}{C/R}=1/3$ as shown in (97), which causes U_t/U_o to increase. Interestingly, it increases above one when $\frac{c/r}{C/R}\geq 1/2$. To understand that, (97) and (98) imply the investment ratios

$$\frac{t_{it}}{t_{io}}=\left(\frac{1+\frac{c/r}{C/R}}{3\frac{c/r}{C/R}}\right)^3=\begin{cases}\frac{64}{27}\ when\ \frac{c/r}{C/R}=\frac{1}{3}\\ 1\ \ when\ \frac{c/r}{C/R}=\frac{1}{2}\\ \frac{8}{27}\ when\ \frac{c/r}{C/R}=1\end{cases},$$

$$\frac{T_{it}}{T_{io}}=\frac{\left(3\frac{c/r}{C/R}-1\right)\left(1+\frac{c/r}{C/R}\right)^3}{27\left(\frac{c/r}{C/R}\right)^4}=\begin{cases}0\ \ when\ \frac{c/r}{C/R}=\frac{1}{3}\\ 1\ \ when\ \frac{c/r}{C/R}=\frac{1}{2}\\ \frac{16}{27}\ when\ \frac{c/r}{C/R}=1\end{cases} \tag{100}$$

The defender's investment ratio decreases from a deterrence objective with large t_{it}/t_{io} when $\frac{c/r}{C/R}=1/3$, to a low t_{it}/t_{io} when deterrence is not possible when $\frac{c/r}{C/R}=1$. In contrast, the attacker's investment ratio is inverse U shaped with a maximum when $\frac{c/r}{C/R}$=1/2. There is high predictability in the two period game. Both agents invest less and prefer the two period game when 1/2<$\frac{c/r}{C/R}$<1. For the one period game both agents are more uncertain, especially when $\frac{c/r}{C/R}$ is below one since the defender's utility decreases and reaches zero when $\frac{c/r}{C/R} \geq 1$. This means that the defender despite a first mover advantage is in a weak position when 1/2<$\frac{c/r}{C/R}$<1, from which the attacker benefits. The reason U_t/U_o drops below one when $\frac{c/r}{C/R}$ increases above one is that the attacker in the one period game earns maximum utility R when the defender earns zero in the one period game.

There has been a discussion in the literature whether the defender should publicly announce defense investments. Infrastructures are built over time. Attackers sometimes take these as given when choosing components for attack. This suggests a two period game where the defender invests first. This section shows that the defender of a series system of equivalent components always prefers the two period game over the one period game. Even the attacker prefers the two period game when the defense inefficiency is between 50% and 100% of the attack inefficiency, but otherwise prefers the one period game.

12 Two period game for series system when attacker moves first

It is also possible, though often less common, that the attacker chooses investment in the first period, while the defender chooses investment in the second period. This sometimes happens in emergency response, or when the attacker has the possibility of announcing publicly and up front what the attack investment will be. The second period is solved first. The n defender FOCs for the second period are as in (7) and can for components i and j be expressed as

$$\frac{\partial u}{\partial t_i} = r\frac{m_i t_i^{m_i-1}T_i^{m_i}}{(t_i^{m_i}+T_i^{m_i})^2}\frac{t_j^{m_j}}{t_j^{m_j}+T_j^{m_j}}\prod_{\substack{k=1\\k\neq i\\k\neq j}}^{n}\frac{t_k^{m_k}}{t_k^{m_k}+T_k^{m_k}} - c_i = 0,$$

$$\frac{\partial u}{\partial t_j} = r\frac{m_j t_j^{m_j-1}T_j^{m_j}}{(t_j^{m_j}+T_j^{m_j})^2}\frac{t_i^{m_j}}{t_i^{m_j}+T_i^{m_j}}\prod_{\substack{k=1\\k\neq i\\k\neq j}}^{n}\frac{t_k^{m_k}}{t_k^{m_k}+T_k^{m_k}} - c_j = 0 \tag{101}$$

The two product signs are equal, so this is solved to yield

$$t_j^{m_j}+T_j^{m_j} = \frac{c_i T_j^{m_j} t_i (t_i^{m_j}+T_i^{m_j})}{c_j T_i^{m_j} t_j} \tag{102}$$

Inserting (102) into (7) gives

$$\frac{\partial u}{\partial t_i} = r\frac{m_i (T_i^{m_i})^{n+1}}{c_i^n t_i^{n+1}(t_i^{m_i}+T_i^{m_i})^{n+1}}\prod_{i=1}^{n}\frac{c_i t_i^{m_i+1}}{T_i^{m_i}} - c_i = 0 \tag{103}$$

which for m_i=1 is solved to yield

$$\frac{t_i(t_i+T_i)c_i/r}{T_i} = \left(\prod_{i=1}^{n}\frac{t_i^2 c_i/r}{T_i}\right)^{1/(n+1)} \tag{104}$$

which implies

$$\frac{T_i}{c_i t_i(t_i+T_i)} = \frac{T_j}{c_j t_j(t_j+T_j)} \tag{105}$$

for i,j=1,…,n, $i\neq j$. Hence a consequence of the n defender FOCs is that the contest success of the attacker divided by the defense expenditure of the defender is equal across all components. Equation (105) is of the second order in t_i. Inserting the solution of t_i into (101) for m_i=1 gives an equation of fourth order in t_j. Hence t_i and t_j are of the fourth order as functions of T_i and T_j. These expressions for t_i and t_j can be inserted into the attacker's first period utility, which can be differentiated with respect to T_i and T_j to determine the attacker's FOCs for the first period. The analytical solutions are too complex for straightforward interpretation. Furthermore, corner solutions with zero investment for the attacker have to scrutinized differently for the following reason. When the defender moves first, the defender has to defend each of the n components in series since any component not defended is attacked with arbitrarily small investment by the attacker causing utility zero to the defender and utility R to the attacker. In contrast, when the attacker moves first, it may attack one component, a subset of components, or all components.

In order to gain insight despite these analytical challenges into the case when the attacker moves first, let us consider the special case of equivalent components, c_i=c, C_i=C, m_i=1. If the attacker attacks more than one component, these are attacked equally much since the attacker has no reason to distinguish between them. The attacker has three options. First, to attack any one component, and not the others. Second, to attack a subset k of components equally much, k=2,…,n-1. Third, to attack all n components equally much. Starting with the first option, assume that the attacker attacks component i with T_i and component j with $T_j=0$, j=1,…,i-1,i+1,…,n. Solving the second period for the defender first, the solution of (7) for $m_i=1$ (general m_i gives no analytical solution) gives

$$\frac{\partial u}{\partial t_i}=r\frac{T_i}{(t_i+T_i)^2}-c_i=0 \quad\Rightarrow\quad t_i=\sqrt{\frac{T_i}{c_i/r}}-T_i \tag{106}$$

Inserting into (6) for n=1 gives the attacker's first period utility

$$U=R\sqrt{T_i c_i/r}-C_i T_i \tag{107}$$

Differentiating with respect to the attacker's free choice variable T_i for the first period gives

$$\frac{\partial U}{\partial T_i}=\frac{R}{2}\sqrt{\frac{c_i/r}{T_i}}-C_i=0 \quad\Rightarrow\quad T_i=\frac{c_i/r}{4(C_i/R)^2} \tag{108}$$

Inserting into (106), (5), (6) gives

$$t_i=\frac{2C_i/R-c_i/r}{4(C_i/R)^2},\quad u=\frac{(2C_i/R-c_i/r)^2}{4(C_i/R)^2},\quad U=\frac{c_i/r}{4C_i/R} \tag{109}$$

When the ratio of the defender's unit cost to system value is more than twice as large as that of the attacker, $c_i/r>2C_i/R$, the defender refrains from investing, component i is 100% insecure, and the defender earns zero utility u=0, while the attacker earns U=R. This gives a first mover advantage to the attacker.

Proceeding with option 2, inserting $t_i=t$, $T_i=T$, and $m_i=1$ into (7), and replacing n with k gives

$$r\frac{Tt^{k-1}}{(t+T)^{k+1}}=c \tag{110}$$

For k=2 this equation also follows from, in accordance with (5), assuming defender utility

$$u=r\left(\frac{t}{t+T}\right)^2-2ct,\quad \frac{\partial u}{\partial t}=r\frac{2tT}{(t+T)^3}-2c=0 \quad\Rightarrow\quad \frac{tT}{(t+T)^3}=c/r \tag{111}$$

This is a third order equation in t with three solutions. Testing reveals that the following solution applies for our purpose

$$t=\frac{\left(3\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{27\frac{c}{r}T-4}-27\left(\frac{c}{r}\right)^2T^2\right)^{1/3}}{3\,2^{1/3}\frac{c}{r}}+T\left(\frac{2^{1/3}}{\left(3\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{27\frac{c}{r}T-4}-27\left(\frac{c}{r}\right)^2T^2\right)^{1/3}}-1\right)$$

(112)

Using (6), for k=2 the attacker's utility is

$$U = R\left(1-\left(\frac{t}{t+T}\right)^2\right) - 2CT \tag{113}$$

Inserting (112) into (113) gives the attacker's first period utility as a function of the variable T and parameters c,r,C,R,

$$\mathbf{U} = \mathbf{R}\left(1 - \left(\left(6\sqrt{3}\left(\frac{c}{r}\right)^{3/2} T^{3/2}\sqrt{27\frac{c}{r}T-4} - 54\left(\frac{c}{r}\right)^2 T^2\right)^{2/3} + 6\frac{c}{r}T\left(2^{1/3} - \left(3\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{27\frac{c}{r}T-4} - 27\left(\frac{c}{r}\right)^2 T^2\right)^{1/3}\right)\right)^{\wedge}2 \Big/ \left(6\, 2^{1/3}\frac{c}{r}T + \left(6\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{27\frac{c}{r}T-4} - 54\left(\frac{c}{r}\right)^2 T^2\right)^{2/3}\right)^2\right) - 2\,\mathbf{C}\,\mathbf{T} \tag{114}$$

Differentiating gives the attacker's FOC

$$
\begin{aligned}
\frac{\partial U}{\partial T} = -R\Bigg(\Bigg(12\,\frac{c}{r}\Bigg(&2^{1/3}+\frac{6\left(\frac{c}{r}\right)^{3/2}T^{3/2}\left(\sqrt{3}-9\sqrt{3}\,\frac{c}{r}\,T+3\sqrt{\frac{c}{r}}\,\sqrt{T}\,\sqrt{-4+27\,\frac{c}{r}\,T}\right)}{\sqrt{-4+27\,\frac{c}{r}\,T}\left(-27\left(\frac{c}{r}\right)^{2}T^{2}+3\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{-4+27\,\frac{c}{r}\,T}\right)^{2/3}}\\
&-\left(-27\left(\frac{c}{r}\right)^{2}T^{2}+3\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{-4+27\,\frac{c}{r}\,T}\right)^{1/3}\\
&-\frac{4\sqrt{\frac{c}{r}}\,\sqrt{T}\left(\sqrt{3}-9\sqrt{3}\,\frac{c}{r}\,T+3\sqrt{\frac{c}{r}}\,\sqrt{T}\,\sqrt{-4+27\,\frac{c}{r}\,T}\right)}{\sqrt{-4+27\,\frac{c}{r}\,T}\left(-54\left(\frac{c}{r}\right)^{2}T^{2}+6\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{-4+27\,\frac{c}{r}\,T}\right)^{1/3}}\Bigg)\\
&\Bigg(\left(-54\left(\frac{c}{r}\right)^{2}T^{2}+6\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{-4+27\,\frac{c}{r}\,T}\right)^{2/3}+\\
&\quad 6\,\frac{c}{r}\,T\left(2^{1/3}-\left(-27\left(\frac{c}{r}\right)^{2}T^{2}+3\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{-4+27\,\frac{c}{r}\,T}\right)^{1/3}\right)\Bigg)\Bigg)\Bigg/\\
&\left(6\,2^{1/3}\,\frac{c}{r}\,T+\left(-54\left(\frac{c}{r}\right)^{2}T^{2}+6\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{-4+27\,\frac{c}{r}\,T}\right)^{2/3}\right)^{2}+\\
&\Bigg(12\,\frac{c}{r}\left(2^{1/3}-\frac{4\sqrt{\frac{c}{r}}\,\sqrt{T}\left(\sqrt{3}-9\sqrt{3}\,\frac{c}{r}\,T+3\sqrt{\frac{c}{r}}\,\sqrt{T}\,\sqrt{-4+27\,\frac{c}{r}\,T}\right)}{\sqrt{-4+27\,\frac{c}{r}\,T}\left(-54\left(\frac{c}{r}\right)^{2}T^{2}+6\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{-4+27\,\frac{c}{r}\,T}\right)^{1/3}}\right)\\
&\Bigg(\left(-54\left(\frac{c}{r}\right)^{2}T^{2}+6\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{-4+27\,\frac{c}{r}\,T}\right)^{2/3}+\\
&\quad 6\,\frac{c}{r}\,T\left(2^{1/3}-\left(-27\left(\frac{c}{r}\right)^{2}T^{2}+3\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{-4+27\,\frac{c}{r}\,T}\right)^{1/3}\right)\Bigg)\hat{\ }2\Bigg)\Bigg/\\
&\left(6\,2^{1/3}\,\frac{c}{r}\,T+\left(-54\left(\frac{c}{r}\right)^{2}T^{2}+6\sqrt{3}\left(\frac{c}{r}\right)^{3/2}T^{3/2}\sqrt{-4+27\,\frac{c}{r}\,T}\right)^{2/3}\right)^{3}\Bigg)-2C=0
\end{aligned}
\tag{115}
$$

which determines T numerically for specific values of c,r,C,R. Inserting T into (111), (112), (114) gives numerical values for u, t, U. Fig. 11 plots t,u/r,T as a function of c/r when C/R=1 when the attacker attacks k=1 component (subscript I) and k=2 components (subscript II). The attacker attacks each component slightly more when attacking two components rather than one component, $T_{II}>T_I$ when $c_i<0.6$. This has a detrimental impact on the defender. Whereas t_I for one component decreases linearly reaching zero when c/r=2, as determined by (109), t_{II} for two components decreases more sharply and drops to zero when c/r=0.6 which causes zero utility u/r=0. The attacker's utilities are shown in Fig. 12. For one component U_I/R increases linearly to 0.5 when c/r=2, and jumps to one as determined by (109). For two components U_{II}/R increases to 0.32 when c/r=0.6, and jumps to one. The last two curves show the two period game with two components when the defender moves first, and the one period game with two components. Aside from the small range 0.5<c/r<0.6 when the attacker prefers the defender to move first, the attacker prefers the first mover advantage, and prefers to attack two components rather than one component. Attacking two components forces the defender to defend both, which is an overwhelming task as the defender's unit cost of investment increases.

13 Two period game for parallel system when attacker moves first

The two period parallel system where the attacker invests in the first period, while the defender invests in the second period, corresponds to the two period series system where the defender invests in the first period, while the attacker invests in the second period. The solution is found by permuting capital and regular letters. The n defender FOCs for the second period are as in (35) and can for components i and j be expressed as

$$\frac{\partial u}{\partial t_i}=r\frac{m_i t_i^{m_i-1}T_i^{m_i}}{(t_i^{m_i}+T_i^{m_i})^2}\frac{T_j^{m_j}}{t_j^{m_j}+T_j^{m_j}}\prod_{\substack{k=1\\k\neq i\\k\neq j}}^{n}\frac{T_k^{m_k}}{t_k^{m_k}+T_k^{m_k}}-c_i=0,$$
$$\frac{\partial u}{\partial t_j}=r\frac{m_j t_j^{m_j-1}T_j^{m_j}}{(t_j^{m_j}+T_j^{m_j})^2}\frac{T_i^{m_j}}{t_i^{m_j}+T_i^{m_j}}\prod_{\substack{k=1\\k\neq i\\k\neq j}}^{n}\frac{T_k^{m_k}}{t_k^{m_k}+T_k^{m_k}}-c_j=0 \tag{116}$$

The two product signs are equal, so this is solved to yield

$$t_j^{m_j}+T_j^{m_j}=\frac{c_i t_j^{m_j-1}(t_i^{m_j}+T_i^{m_j})}{c_j t_i^{m_j-1}} \tag{117}$$

Inserting (117) into (35) gives

$$\frac{\partial u}{\partial t_i}=r\frac{m_i(t_i^{m_i-1})^{n+1}}{c_i^n(t_i^{m_i}+T_i^{m_i})^{n+1}}\prod_{i=1}^{n}\frac{c_iT_i^{m_i}}{t_i^{m_i-1}}-c_i=0 \tag{118}$$

which for $m_i=1$ is solved to yield

$$t_i=\frac{1}{c_i/r}\left(\prod_{i=1}^{n}T_ic_i/r\right)^{1/(n+1)}-T_i \tag{119}$$

which implies

$$c_i(T_i+t_i)=c_j(T_j+t_j) \tag{120}$$

for i,j=1,…,n, $i\neq j$, which is a consequence of the n defender FOCs. Inserting (119) into the denominator in (34) and simplifying gives the attacker's first period utility

$$U=R\left(\prod_{i=1}^{n}T_ic_i/r\right)^{1/(n+1)}-\sum_{i=1}^{n}C_iT_i \tag{121}$$

Two of the n attacker FOCs, for components i and j, for the first period are

$$\frac{\partial U}{\partial T_i}=\frac{R}{(n+1)T_i}\left(\prod_{i=1}^{n}T_ic_i/r\right)^{1/(n+1)}-C_i=0,\quad \frac{\partial U}{\partial T_j}=\frac{R}{(n+1)T_j}\left(\prod_{i=1}^{n}T_ic_i/r_i\right)^{1/(n+1)}-C_j=0 \tag{122}$$

which are solved to yield

$$T_j=C_iT_i/C_j \tag{123}$$

Inserting (123) into (122) and solving for T_i, and inserting into (119), gives

$$T_i=\frac{1}{(n+1)^{(n+1)}C_i/R}\prod_{i=1}^{n}\frac{c_i/r}{C_i/R},\quad t_i=\frac{(n+1)C_i/R-c_i/r}{(n+1)^{(n+1)}(C_i/R)c_i/r}\prod_{i=1}^{n}\frac{c_i/r}{C_i/R} \tag{124}$$

which implies $T_i/(t_i+T_i)=(c_i/r)/((n+1)C_i/R)$. Inserting into the utilities in (33) and (34) gives

$$u=r\left(1-\frac{1}{(n+1)^n}\left(1+\frac{1}{(n+1)}\sum_{i=1}^{n}\frac{(n+1)C_i/R-c_i/r}{C_i/R}\right)\prod_{i=1}^{n}\frac{c_i/r}{C_i/R}\right),\quad U=\frac{R}{(n+1)^{n+1}}\prod_{i=1}^{n}\frac{c_i/r}{C_i/R} \tag{125}$$

When $C_i/R<(c_i/r)/(n+1)$, which is satisfied when the unit cost of attack is low when n is large, a corner solution exists where $t_i=0$, which deters the defender successfully from defending component i. Assume without loss of generality that $C_1/R<(c_1/r)/(n+1)$ for component 1, which causes $t_1=0$. Assume that the attacker invests an arbitrarily small but positive amount $T_1=0^+$ into attacking component 1, causing component 1 to be 100% insecure. Assuming that the k components 1,2,…,k are insecure, replacing n with n-k in (124) and (125), and using subscript j instead of i, gives the investments and utilities for components j=k+1,…,n,

$$\begin{gathered}t_j=\frac{(n-k+1)C_j/R-c_j/r}{(n-k+1)^{n-k+1}(C_j/R)c_j/r}\prod_{j=k+1}^{n}\frac{c_j/r}{C_j/R},\quad T_j=\frac{1}{(n-k+1)^{(n-k+1)}C_j/R}\prod_{j=k+1}^{n}\frac{c_j/r}{C_j/R},\\ u=r\left(1-\frac{1}{(n-k+1)^{n-k}}\left(1+\frac{1}{(n-k+1)}\sum_{j=k+1}^{n}\frac{(n-k+1)C_j/R-c_j/r}{C_j/R}\right)\prod_{j=k+1}^{n}\frac{c_j/r}{C_j/R}\right),\\ U=\frac{R}{(n-k+1)^{n-k+1}}\prod_{j=k+1}^{n}\frac{c_j/r}{C_j/R}\end{gathered} \tag{126}$$

Proposition 9P. *Two period parallel system when attacker moves first: (a) When the attack inefficiencies for the k components 1, 2,…,k are sufficiently low,* $C_1/R<(c_1/r)/(n+1)$, $C_2/R<(c_2/r)/n$*,…,* $C_k/R<(c_k/r)/(n-k+2)$*, then these components are not defended and are 100% insecure,* $t_i=0$*. When additionally* $C_j/R<(c_j/r)/(n-k+1)$ *for any j=k+1,…,n, a (k+1)th corner solution exists where* $t_j=0$*, which causes the defender not to defend component j.*

(b) The parallel system is never 100% secure except at the limit when $n \to \infty$, *or* $C_i / R \to \infty$ *or* $c_i / r = 0$ *for at least one component which causes u=r and U=0.*

Proof. (a) Follows from (125) and from requiring $T_i>0$ in (124) and $T_j>0$ in (126). (b) Follows from (125).

Whereas the one period series and parallel systems can be both 100% secure and 100% insecure for finite nonzero parameter values, the two period parallel system can be 100% insecure, but not 100% secure, except for three extreme and unlikely cases. First, infinitely many components in parallel reduces the attack on each component towards zero, and reduces the attacker's utility towards zero, since $(n+1)^{(n+1)}$ in the denominators in (124) and (125) becomes overwhelmingly large. Second, if the attacker has infinite unit cost of attacking at least one component, that causes 100% security for that component and thus for the parallel system. Third, if the defender has zero unit cost of defending at least one component, that causes 100% security for that component and thus for the parallel system. This illustrates the attacker's first mover advantage. The attacker prefers to precommit to its optimal investment level in the first period in a two period game, rather than moving simultaneously with the defender in one period game.

Inserting n=2 into (124) and (125) gives

$$T_i = \frac{\frac{c_1}{r}\frac{c_2}{r}}{27\frac{C_i^2}{R}\frac{C_j}{R}}, \quad t_i = \frac{\left(3\frac{C_i}{R} - \frac{c_i}{r}\right)\frac{c_j}{r}}{27\frac{C_i^2}{R}\frac{C_j}{R}},$$
$$U = R\frac{\frac{c_1}{r}\frac{c_2}{r}}{27\frac{C_1}{R}\frac{C_2}{R}}, \quad u = r\left(1 - \frac{9\frac{C_1}{R}\frac{C_2}{R}\frac{c_1}{r}\frac{c_2}{r} - \frac{C_1}{R}\frac{c_1}{r}\frac{c_2^2}{r} - \frac{C_2}{R}\frac{c_2}{r}\frac{c_1^2}{r}}{27\frac{C_1^2}{R}\frac{C_2^2}{R}}\right) \tag{127}$$

When $(C_1 / R)/(c_1 / r)<1/3$ so that the defender is deterred from defending component 1, (126) gives

$$T_1 = t_1 = 0,\ t_2 = \frac{2C_2 / R - c_2 / r}{4(C_2 / R)^2},\ T_2 = \frac{c_2 / r}{4(C_2 / R)^2},\ u = \frac{(2C_2 / R - c_2 / r)^2}{4(C_2 / R)^2},\ U = \frac{c_2 / r}{4(C_2 / R)^2} \tag{128}$$

When additionally $(C_2 / R)/(c_2 / r)<1/2$, the defender is deterred from defending both components causing $T_1=T_2=t_1=t_2=u=0$ and U=R.

Fig. 13 illustrates as functions of the defense inefficiency c_1/r for the same parameters as in Fig. 5, n=2, $c_2/r=C_1/R=C_2/R=m_i=1$. The two most noteworthy differences between Figs. 10 and 5 are that in Fig. 13 the attacker's utility does not equal zero when $c_1/r<1$, and the defender does not defend component 1 when $c_1/r>3C_1/R$.

Assessing whether the agents prefer the n period game or the one period game is analytically challenging except when n=2, $c_i=c$, $C_i=C$, $m_i=1$. We let subscript "t" denote two period game, and subscript "o" denote one period game. The interior solutions follow from inserting $c_i=c$ into (127), (41), (42), which gives

$$t_{it}=\frac{3\frac{C/R}{c/r}-1}{27\frac{C}{R}\left(\frac{C/R}{c/r}\right)^2},\quad T_{it}=\frac{1}{27\frac{C}{R}\left(\frac{C/R}{c/r}\right)^2},$$
$$u_t=r\frac{\left(3\frac{C/R}{c/r}-1\right)^2\left(3\frac{C/R}{c/r}+2\right)}{27\left(\frac{C/R}{c/r}\right)^3},\quad U_t=\frac{R}{27\left(\frac{C/R}{c/r}\right)^2} \tag{129}$$

$$t_{io}=\frac{\frac{C/R}{c/r}}{\frac{c}{r}\left(1+\frac{C/R}{c/r}\right)^3},\quad T_{io}=\frac{\frac{C/R}{c/r}}{\frac{C}{R}\left(1+\frac{C/R}{c/r}\right)^3},$$
$$u_o=r\frac{\left(3+\frac{C/R}{c/r}\right)\left(\frac{C/R}{c/r}\right)^2}{\left(1+\frac{C/R}{c/r}\right)^3},\quad U_o=R\frac{1-\frac{C/R}{c/r}}{\left(1+\frac{C/R}{c/r}\right)^3} \tag{130}$$

Three ranges for the ratio $\frac{C/R}{c/r}$ apply. When $0<\frac{C/R}{c/r}<1/3$, the two period game has a corner solution where the defender is deterred from defending both components, u=0 and U=R, and the one period game has an interior solution. When $1/3<\frac{C/R}{c/r}<1$, both games have interior solutions. When $\frac{C/R}{c/r}>1$, the two period game has an interior solution and the one period game has a corner solution where the attacker is deterred, $U=T_1=T_2=0$, u=r. The ratio of the utilities are

$$\frac{u_t}{u_o}=\begin{cases}0 \textit{ when } 0<\frac{C/R}{c/r}<\frac{1}{3}\\ \frac{\left(1-3\frac{C/R}{c/r}\right)^2\left(2+3\frac{C/R}{c/r}\right)\left(1+\frac{C/R}{c/r}\right)^3}{27\left(\frac{C/R}{c/r}\right)^5\left(3+\frac{C/R}{c/r}\right)} \textit{ when } \frac{1}{3}<\frac{C/R}{c/r}<1,\\ \frac{\left(1-3\frac{C/R}{c/r}\right)^2\left(2+3\frac{C/R}{c/r}\right)}{27\left(\frac{C/R}{c/r}\right)^3} \textit{ when } \frac{C/R}{c/r}>1\end{cases} \quad \frac{U_t}{U_o}=\begin{cases}\frac{\left(1+\frac{C/R}{c/r}\right)^3}{\left(1-\frac{C/R}{c/r}\right)} \textit{ when } 0<\frac{C/R}{c/r}<\frac{1}{3}\\ \frac{\left(1+\frac{C/R}{c/r}\right)^3}{27\left(\frac{C/R}{c/r}\right)^2\left(1-\frac{C/R}{c/r}\right)} \textit{ when } \frac{1}{3}<\frac{C/R}{c/r}<1\\ \infty \textit{ when } \frac{C/R}{c/r}>1\end{cases} \quad (131)$$

Equation (131) is plotted in Fig. 14 as a function of $\frac{c/r}{C/R}$. (Plotting as a function of $\frac{C/R}{c/r}$ gives the same curves as in Fig. 10, permuting capital and regular letters for the defender and attacker.)

Proposition 10P. *Two period parallel system when attacker moves first: Assume two equivalent components, $c_i=c$, $C_i=C$, $m_i=1$. The attacker always prefers the two period game over the one period simultaneous move game and is indifferent between the two games when $\frac{C/R}{c/r}=1/2$. The defender prefers the two period game when $1/2<\frac{C/R}{c/r}<1$, and otherwise (when $\frac{C/R}{c/r}<1/2$ and $\frac{C/R}{c/r}>1$) prefers the one period game.*

Proof. Follows from inspecting (131) and Fig. 14.

The attacker has a first mover's advantage and always prefers the two period game over the one period game. The reason is that the attacker can invest based on its expectation about the defender's strategic move in the second period. Since the defender is deterred from defending when $0<\frac{C/R}{c/r}<1/3$ causing constant utilities u=0 and U=R for the two period game, that U_o in (130) decreases in $\frac{C/R}{c/r}$ causes U_t/U_o to increase in $\frac{C/R}{c/r}$. When $\frac{C/R}{c/r}$ increases above 1/3, the defender is no longer deterred from defending and an interior solution applies for both games. Both U_t and U_o decrease convexly in $\frac{C/R}{c/r}$, but U_o eventually reaches zero when

$\frac{C/R}{c/r}$=1. This causes U_t/U_o to be U shaped when 1/3<$\frac{C/R}{c/r}$<1, reaching infinity as $\frac{C/R}{c/r}$ reaches one from below, which means that $\frac{c/r}{C/R}$ reaches one from above in Fig. 14. The defender's two period utility gradually increases from zero when $\frac{C/R}{c/r}$=1/3 as shown in (129), which causes u_t/u_o to increase in $\frac{C/R}{c/r}$. Interestingly, it increases above one when $\frac{C/R}{c/r} \geq 1/2$. To understand that, (129) and (130) imply the investment ratios

$$\frac{T_{it}}{T_{io}} = \left(\frac{1+\frac{C/R}{c/r}}{3\frac{C/R}{c/r}}\right)^3 = \begin{cases} \frac{64}{27} \text{ when } \frac{C/R}{c/r} = \frac{1}{3} \\ 1 \text{ when } \frac{C/R}{c/r} = \frac{1}{2} \\ \frac{8}{27} \text{ when } \frac{C/R}{c/r} = 1 \end{cases},$$

$$\frac{t_{it}}{t_{io}} = \frac{\left(3\frac{C/R}{c/r}-1\right)\left(1+\frac{C/R}{c/r}\right)^3}{27\left(\frac{C/R}{c/r}\right)^4} = \begin{cases} 0 \text{ when } \frac{C/R}{c/r} = \frac{1}{3} \\ 1 \text{ when } \frac{C/R}{c/r} = \frac{1}{2} \\ \frac{16}{27} \text{ when } \frac{C/R}{c/r} = 1 \end{cases} \tag{132}$$

The attacker's investment ratio decreases from a deterrence objective with large T_{it}/T_{io} when $\frac{C/R}{c/r}$=1/3, to a low T_{it}/T_{io} when deterrence is not possible when $\frac{C/R}{c/r}$=1. In contrast, the defender's investment ratio is inverse U shaped with a maximum when $\frac{C/R}{c/r}$=1/2. There is high predictability in the two period game. Both agents invest less and prefer the two period game when 1/2<$\frac{C/R}{c/r}$<1.

For the one period game both agents are more uncertain, especially when $\frac{C/R}{c/r}$ is below one since the attacker's utility decreases and reaches zero when $\frac{C/R}{c/r} \geq 1$. This means that the attacker despite a first mover advantage is in a weak position when 1/2<$\frac{C/R}{c/r}$<1, from which the defender benefits. The reason u_t/u_o drops below one when $\frac{C/R}{c/r}$ increases above one, which means that $\frac{c/r}{C/R}$ decreases below one in Fig. 14, is that the defender in the one period game earns maximum utility r when the attacker earns zero in the one period game.

An interesting question is whether the attacker should publicly announce attack investments. This section shows that the attacker of a parallel system of equivalent components always prefers the two period game over the one period game. Even the defender prefers the two period game when the attack inefficiency is between 50% and 100% of the defense inefficiency, but otherwise prefers the one period game.

14 Two period game for parallel system when defender moves first

When the defender moves first in a parallel system, the n attacker FOCs for the second period are as in (35) and can for components i and j be expressed as

$$\begin{aligned}\frac{\partial U}{\partial T_i} &= R\frac{m_i T_i^{m_i-1} t_i^{m_i}}{(t_i^{m_i}+T_i^{m_i})^2}\frac{T_j^{m_j}}{t_j^{m_j}+T_j^{m_j}}\prod_{\substack{k=1\\k\neq i\\k\neq j}}^{n}\frac{T_k^{m_k}}{t_k^{m_k}+T_k^{m_k}}-C_i=0,\\ \frac{\partial U}{\partial T_j} &= R\frac{m_j T_j^{m_j-1} t_j^{m_j}}{(t_j^{m_j}+T_j^{m_j})^2}\frac{T_i^{m_j}}{t_i^{m_j}+T_i^{m_j}}\prod_{\substack{k=1\\k\neq i\\k\neq j}}^{n}\frac{T_k^{m_k}}{t_k^{m_k}+T_k^{m_k}}-C_j=0\end{aligned} \tag{133}$$

The two product signs are equal, so this is solved to yield

$$t_j^{m_j}+T_j^{m_j}=\frac{C_i t_j^{m_j} T_i (t_i^{m_j}+T_i^{m_j})}{C_j t_i^{m_j} T_j} \tag{134}$$

Inserting (134) into (35) gives

$$\frac{\partial U}{\partial T_i}=R\frac{m_i (t_i^{m_i})^{n+1}}{C_i^n T_i^{n+1}(t_i^{m_i}+T_i^{m_i})^{n+1}}\prod_{i=1}^{n}\frac{C_i T_i^{m_i+1}}{t_i^{m_i}}-C_i=0 \tag{135}$$

which for m_i=1 is solved to yield

$$\frac{T_i(t_i+T_i)C_i/R}{t_i}=\left(\prod_{i=1}^{n}\frac{T_i^2 C_i/R}{t_i}\right)^{1/(n+1)} \tag{136}$$

which implies

$$\frac{t_i}{C_i T_i (t_i+T_i)}=\frac{t_j}{C_j T_j (t_j+T_j)} \tag{137}$$

for i,j=1,…,n, $i \neq j$. Hence a consequence of the n attacker FOCs is that the contest success of the defender divided by the attack expenditure of the attacker is equal across all components. Equation (137) is of the second order in T_i. Inserting the solution of T_i into (133) for m_i=1 gives an equation of fourth order in T_j. Hence T_i and T_j are of the fourth order as functions of t_i and t_j. These expressions for T_i and T_j can be inserted into the defender's first period utility, which can be differentiated with respect to t_i and t_j to determine the defender's FOCs for the first period. The analytical solutions are too complex for straightforward interpretation. Furthermore, corner solutions with zero investment for the defender have to scrutinized differently for the following reason. When the attacker moves first, the attacker has to attack each of the n components in parallel since any component not attacked is defended with arbitrarily small investment by the defender causing utility zero to the attacker and utility r to the defender. In contrast, when the defender moves first, it may defend one component, a subset of components, or all components.

In order to gain insight despite these analytical challenges into the case when the defender moves first, let us consider the special case of equivalent components, c_i=c, C_i=C, m_i=1. If the defender defends more than one component, these are defended equally much since the defender has no reason to distinguish between them. The defender has three options. First, to defend any one component, and not the others. Second, to defend a subset k of components equally much, k=2,…,n-1. Third, to defend all n components equally much. Starting with the first option, assume that the defender defends component i with t_i and component j with t_j=0, j=1,…,i-1,i+1,…,n. Solving the second period for the attacker first, the solution of (35) for m_i=1 (general m_i gives no analytical solution) gives

$$\frac{\partial U}{\partial T_i} = R\frac{t_i}{(t_i + T_i)^2} - C_i = 0 \quad \Rightarrow \quad T_i = \sqrt{\frac{t_i}{C_i / R}} - t_i \tag{138}$$

Inserting into (33) gives the defender's first period utility

$$u = r\sqrt{t_i C_i / R} - c_i t_i \tag{139}$$

Differentiating with respect to the defender's free choice variable t_i for the first period gives

$$\frac{\partial u}{\partial t_i} = \frac{r}{2}\sqrt{\frac{C_i / R}{t_i}} - c_i = 0 \quad \Rightarrow \quad t_i = \frac{C_i / R}{4(c_i / r)^2} \tag{140}$$

Inserting into (138), (33), (34) gives

$$T_i = \frac{2c_i / r - C_i / R}{4(c_i / r)^2}, \quad u = \frac{C_i / R}{4c_i / r}, \quad U = \frac{(2c_i / r - C_i / R)^2}{4(c_i / r)^2} \tag{141}$$

When the attacker's unit cost is more than twice as large as the defender's unit cost, $C_i/R>2c_i/r$ the attacker refrains from investing, component i is 100% secure, and the attacker earns zero utility U=0, while the defender earns u=r. This gives a first mover advantage to the defender.

Proceeding with option 2, inserting t_i=t, T_i=t, and m_i=1 into (35), and replacing n with k gives

$$R\frac{tT^{k-1}}{(t+T)^{k+1}}=C \tag{142}$$

For k=2 this equation also follows from, in accordance with (34), assuming attacker utility

$$U=R\left(\frac{T}{t+T}\right)^2-2CT,\quad \frac{\partial U}{\partial T}=R\frac{2tT}{(t+T)^3}-2C=0 \quad\Rightarrow\quad \frac{tT}{(t+T)^3}=C/R \tag{143}$$

This is a third order equation in T with three solutions. Testing reveals that the following solution applies for our purpose

$$T=\frac{\left(3\sqrt{3}\left(\frac{C}{R}\right)^{3/2}t^{3/2}\sqrt{27\frac{C}{R}t-4}-27\left(\frac{C}{R}\right)^2t^2\right)^{1/3}}{3\,2^{1/3}\,\frac{C}{R}}+t\left(\frac{2^{1/3}}{\left(3\sqrt{3}\left(\frac{C}{R}\right)^{3/2}t^{3/2}\sqrt{27\frac{C}{R}t-4}-27\left(\frac{C}{R}\right)^2t^2\right)^{1/3}}-1\right) \tag{144}$$

Using (34), for k=2 the defender's utility is

$$u=r\left(1-\left(\frac{T}{t+T}\right)^2\right)-2ct \tag{145}$$

Inserting (144) into (145) gives the defender's first period utility as a function of the variable t and parameters c,r,C,R,

$$\begin{aligned}u=r\Bigg(1-\Bigg(\Bigg(\left(6\sqrt{3}\left(\frac{C}{R}\right)^{3/2}t^{3/2}\sqrt{27\frac{C}{R}t-4}-54\left(\frac{C}{R}\right)^2t^2\right)^{2/3}\\+6\frac{C}{R}t\left(2^{1/3}-\left(3\sqrt{3}\left(\frac{C}{R}\right)^{3/2}t^{3/2}\sqrt{27\frac{C}{R}t-4}-27\left(\frac{C}{R}\right)^2t^2\right)^{1/3}\right)\Bigg)\Bigg)\hat{}\,2\Big/\\\left(6\,2^{1/3}\frac{C}{R}t+\left(6\sqrt{3}\left(\frac{C}{R}\right)^{3/2}t^{3/2}\sqrt{27\frac{C}{R}t-4}-54\left(\frac{C}{R}\right)^2t^2\right)^{2/3}\right)^2\Bigg)-2ct\end{aligned} \tag{146}$$

Differentiating gives the defender's FOC

$$
\begin{aligned}
\frac{\partial u}{\partial t} = -r \Bigg(\Bigg(\Bigg(12\, \frac{C}{R} \Bigg(2^{1/3} + {}& \frac{6 \left(\frac{C}{R}\right)^{3/2} t^{3/2} \left(\sqrt{3} - 9\sqrt{3}\, \frac{C}{R}\, t + 3 \sqrt{\frac{C}{R}}\, \sqrt{t}\, \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)}{\sqrt{-4 + 27\, \frac{C}{R}\, t}\, \left(-27 \left(\frac{C}{R}\right)^2 t^2 + 3\sqrt{3} \left(\frac{C}{R}\right)^{3/2} t^{3/2} \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)^{2/3}} \\
& - \left(-27 \left(\frac{C}{R}\right)^2 t^2 + 3\sqrt{3} \left(\frac{C}{R}\right)^{3/2} t^{3/2} \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)^{1/3} \\
& - \frac{4 \sqrt{\frac{C}{R}}\, \sqrt{t} \left(\sqrt{3} - 9\sqrt{3}\, \frac{C}{R}\, t + 3 \sqrt{\frac{C}{R}}\, \sqrt{t}\, \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)}{\sqrt{-4 + 27\, \frac{C}{R}\, t}\, \left(-54 \left(\frac{C}{R}\right)^2 t^2 + 6\sqrt{3} \left(\frac{C}{R}\right)^{3/2} t^{3/2} \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)^{1/3}} \Bigg) \Bigg) \\
& \Bigg(\Bigg(\left(-54 \left(\frac{C}{R}\right)^2 t^2 + 6\sqrt{3} \left(\frac{C}{R}\right)^{3/2} t^{3/2} \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)^{2/3} + \\
& \quad 6\, \frac{C}{R}\, t \left(2^{1/3} - \left(-27 \left(\frac{C}{R}\right)^2 t^2 + 3\sqrt{3} \left(\frac{C}{R}\right)^{3/2} t^{3/2} \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)^{1/3} \right) \Bigg) \Bigg) \Bigg/ \\
& \left(6\, 2^{1/3}\, \frac{C}{R}\, t + \left(-54 \left(\frac{C}{R}\right)^2 t^2 + 6\sqrt{3} \left(\frac{C}{R}\right)^{3/2} t^{3/2} \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)^{2/3} \right)^2 + \\
& \Bigg(12\, \frac{C}{R} \Bigg(2^{1/3} - \frac{4 \sqrt{\frac{C}{R}}\, \sqrt{t} \left(\sqrt{3} - 9\sqrt{3}\, \frac{C}{R}\, t + 3 \sqrt{\frac{C}{R}}\, \sqrt{t}\, \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)}{\sqrt{-4 + 27\, \frac{C}{R}\, t}\, \left(-54 \left(\frac{C}{R}\right)^2 t^2 + 6\sqrt{3} \left(\frac{C}{R}\right)^{3/2} t^{3/2} \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)^{1/3}} \Bigg) \Bigg) \\
& \Bigg(\Bigg(\left(-54 \left(\frac{C}{R}\right)^2 t^2 + 6\sqrt{3} \left(\frac{C}{R}\right)^{3/2} t^{3/2} \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)^{2/3} + \\
& \quad 6\, \frac{C}{R}\, t \left(2^{1/3} - \left(-27 \left(\frac{C}{R}\right)^2 t^2 + 3\sqrt{3} \left(\frac{C}{R}\right)^{3/2} t^{3/2} \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)^{1/3} \right) \Bigg) {}^\wedge 2 \Bigg) \Bigg/ \\
& \left(6\, 2^{1/3}\, \frac{C}{R}\, t + \left(-54 \left(\frac{C}{R}\right)^2 t^2 + 6\sqrt{3} \left(\frac{C}{R}\right)^{3/2} t^{3/2} \sqrt{-4 + 27\, \frac{C}{R}\, t} \right)^{2/3} \right)^3 \Bigg) - 2c = 0
\end{aligned}
\tag{147}
$$

which determines t numerically for specific values of c,r,C,R. Inserting t into (143), (144), (146) gives numerical values for U, T, u.

Fig. 15 plots t,T,U/R as a function of c/r when C/R=1 when the attacker attacks k=1 component (subscript I) and k=2 components (subscript II). (Plotting as a function of C/R when c/r=1 gives the same curves as in Fig. 11, permuting capital and regular letters for the defender and attacker.) The defender defends each component slightly more when defending two components rather than one component, $t_{II}>t_I$ when c/r>1.8. This has a detrimental impact on the attacker. The attacker's investment is higher for one than for two components, $T_I>T_{II}$, and the utility is significantly higher, $U_I/R>U_{II}/R$. For one component the attacker ceases investment when c/r<0.5, as determined by $T_i<0$ in (141), while for two components the attacker ceases investment when c/r<1.8. Despite significantly higher unit cost, 1.8>C/R=1, the defender defending two components deters the attacker. The defender's utilities are shown in Fig. 16. For one component u_I/r equals one when c/r<0.5, and thereafter decreases convexly as determined by (141). For two components the defender enjoys utility r when c/r<1.8, and the utility thereafter decreases, $u_I/r>u_{II}/r$. The last two curves show the two period game with two components when the attacker moves first, and the one period game with two components. Aside from the small range 1.8<c/r<2 when the defender prefers the attacker to move first, the defender prefers the first mover advantage, and prefers to defend two components rather than one component. Defending two components forces the attacker to attack both, which is an overwhelming task as the defender's unit cost of investment decreases.

15 Conclusion

Contesting agents preferring high versus low reliability for a system are often in the time asymmetric situation where one agent (the defender or attacker) chooses its optimal strategy first, expecting a future decision by the other agent. The other agent chooses its optimal strategy thereafter taking the first agent's strategy as given. For systems and infrastructures that are built over time, where the defender is open and candid about its defense investment and installations, and lacks knowledge about potential attackers, the defender moves first. Conversely, an attacker may announce up front the specifics of a forthcoming attack, or the defender can gain intelligence that estimates force investments by the attacker. In this case the attacker moves first.

When the defender moves first in a series system, and the defense inefficiency divided by the system value is less than the attack inefficiency divided by the system value and divided by the number of components plus one, which is a strict requirement, then these components are not subject to attack and are 100% secure. A less strict requirement applies for the remaining components, and the attacker is deterred. Such 100% deterrence of the attacker is not possible in a simultaneous move game. The defender thus has a first mover advantage, but this requires substantially less unit costs of defense than unit costs of attack given the ceteris paribus advantage of the attacker in series systems. The series system is 100% insecure at the limit with infinitely many components, or when the unit cost of defense for any component divided by the system value approaches infinity, or when the unit cost of attack for any component divided by the system value approaches zero.

With two equivalent components in series, the defender always prefers the two period game over the one period simultaneous move game. The reason is that the defender can invest based on its expectation about the attacker's strategic move in the second period. In the two period game there is higher predictability, and both agents invest less. The attacker prefers the one period game, but prefers the two period game when the defense inefficiency is between 50% and 100% of the attack inefficiency. Over this range the defender is maximally vulnerable. Recall that in the one period game the defender withdraws from investment when the defense inefficiency exceeds the attack inefficiency.

When the defender moves first in a series system, the defender has to defend each component since any component not defended is attacked with arbitrarily small investment causing a 100% insecure system. This situation is analytically tractable. In contrast, when the attacker moves first in a series system, it may attack one component, a subset of components, or all components. This situation is quite complex to react to for the defender. In general, fourth order equations follow which are illustrated with simulations. With the special assumption that the attacker attacks only one component, this component is 100% insecure when the ratio of the defender's unit cost to system value is more than twice as large as that of the attacker. This means that the defender is twice as disadvantaged as the attacker. The consequence is that the defender refrains from investing, earns zero utility, while the attacker earns maximum utility and enjoys a first mover advantage. For a broad set of parameter values, the simulations illustrate that the attacker mostly prefers the first mover advantage, and prefers to attack two components rather than one component. Attacking two components forces the defender to defend both, which is an overwhelming task as the defender's unit cost of investment increases.

When the attacker moves first in a parallel system, and the attack inefficiency divided by the system value is less than the defense inefficiency divided by the system value and divided by the number of components plus one, which is a strict requirement, then these components are not defended and are 100% insecure. A less strict requirement applies for the remaining components, and these are then not defended. Such 100% insecurity for the defender is not possible in a simultaneous move game. The attacker thus has a first mover advantage, but this requires substantially less unit costs of attack than unit costs of defense given the ceteris paribus advantage of the defender in parallel systems. The parallel system is 100% secure at the limit with infinitely many components, or when the unit cost of attack for any component divided by the system value approaches infinity, or when the unit cost of defense for any component divided by the system value approaches zero.

With two equivalent components in parallel, the attacker always prefers the two period game over the one period simultaneous move game. The reason is that the attacker can invest based on its expectation about the defender's strategic move in the second period. In the two period game there is higher predictability, and both agents invest less. The defender prefers the one period game, but prefers the two period game when the attack inefficiency is between 50% and 100% of the defense inefficiency. Over this range the attacker is maximally vulnerable. Recall that in the one period game the attacker withdraws from investment when the attack inefficiency exceeds the defense inefficiency.

When the attacker moves first in a parallel system, the attacker has to attack each component since any component not attacked is defended with arbitrarily small investment causing a 100% secure system. This situation is analytically tractable. In contrast, when the defender moves first in a parallel system, it may defend one component, a subset of components, or all components. This situation is quite complex to react to for the attacker. In general, fourth order equations follow which are illustrated with simulations. With the special assumption that the defender defends only one component, this component is 100% secure when the ratio of the attacker's unit cost to system value is more than twice as large as that of the defender. This means that the attacker is twice as disadvantaged as the defender. The consequence is that the attacker refrains from investing, earns zero utility, while the defender earns maximum utility and enjoys a first mover advantage. For a broad set of parameter values, the simulations illustrate that the defender mostly prefers the first mover advantage, and prefers to defend two components rather than one component. Defending two components forces the attacker to attack both, which is an overwhelming task as the attacker's unit cost of investment increases.

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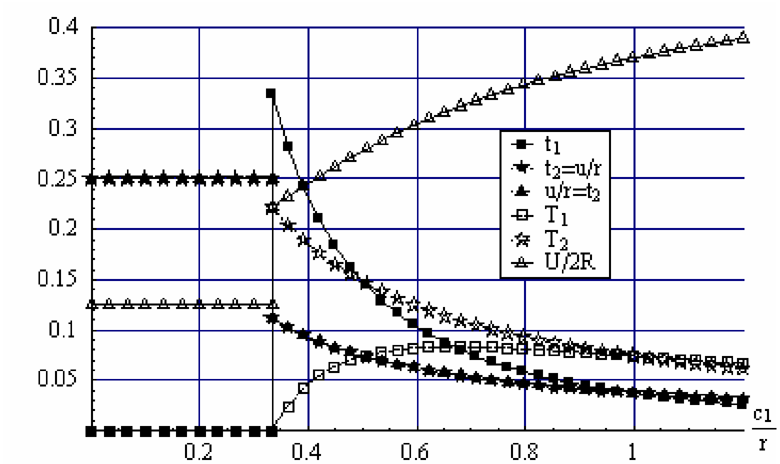

Fig. 9. Series system, defender moves first. The six variables as functions of c_1/r when $c_2/r=C_1/R=C_2/R=m_i=1$.

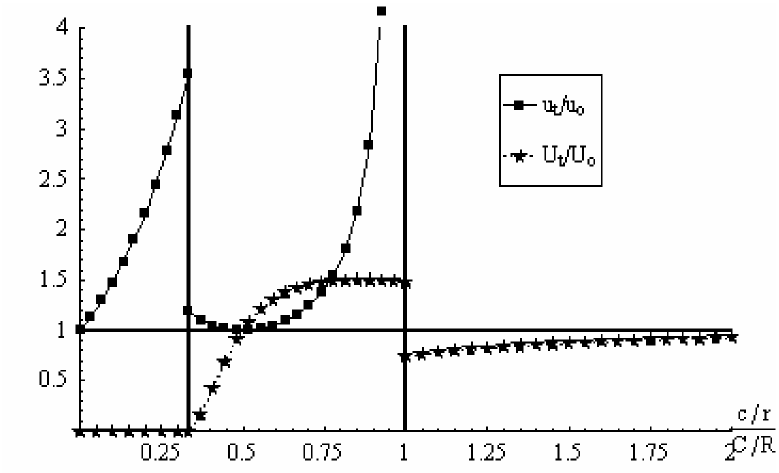

Fig. 10. Series system, defender moves first: Ratio of utilities for two period game versus one period game.

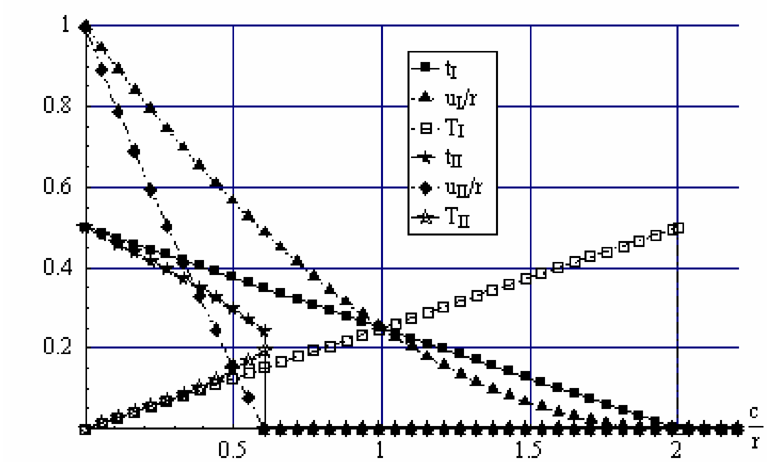

Fig. 11. Series system: Attacker moves first and attacks one component (subscript I) or two components (subscript II), C/R=1.

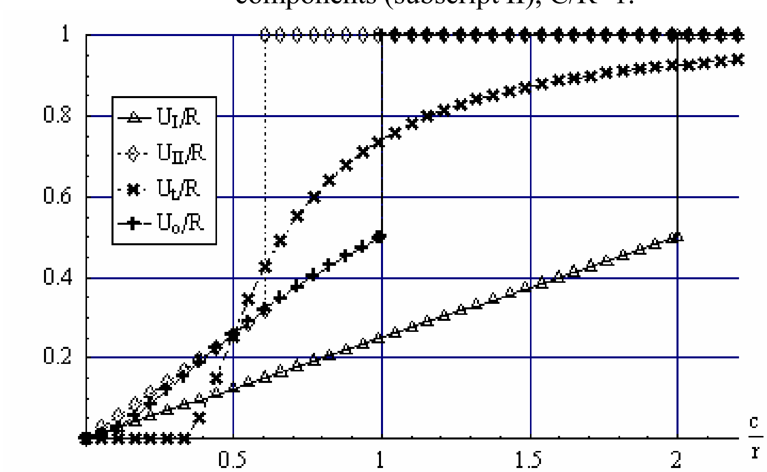

Fig. 12. Series system. Attacker moves first and attacks one component (subscript I) or two components (subscript II), C/R=1.

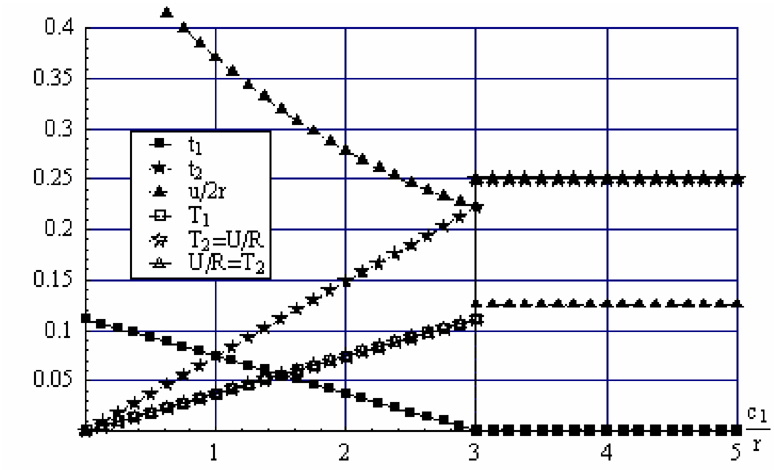

Fig. 13. Parallel system, attacker moves first. Same parameters as in Fig. 9.

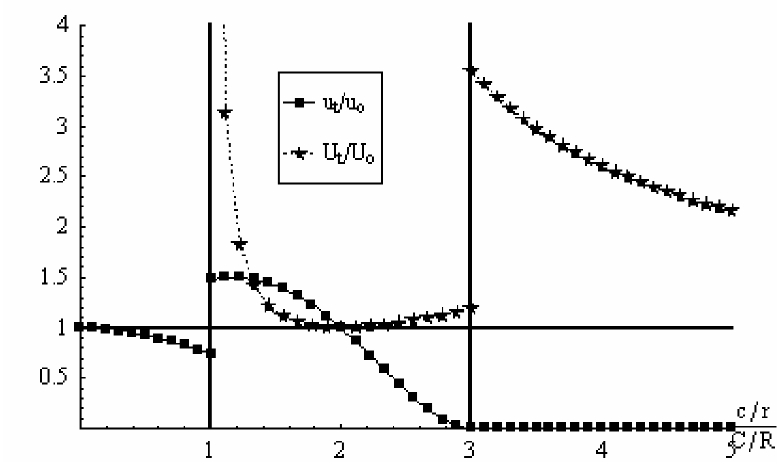

Fig. 14. Parallel system, attacker moves first: Ratio of utilities for two period game versus one period game.

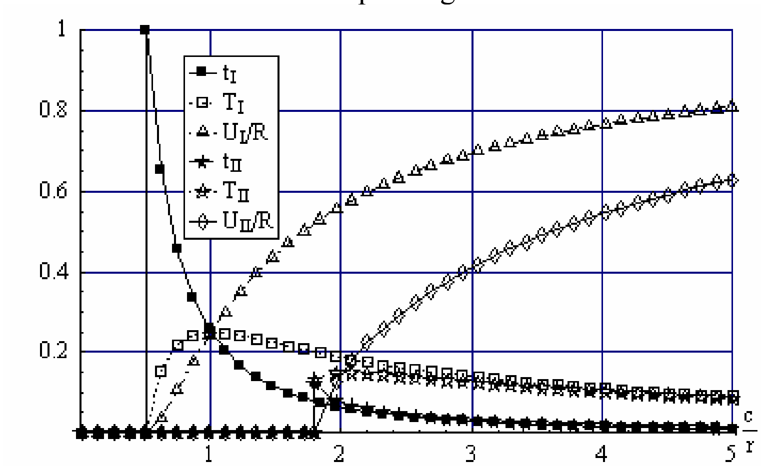

Fig. 15. Parallel system. Defender moves first and defends one component (subscript I) or two components (subscript II), C/R=1.

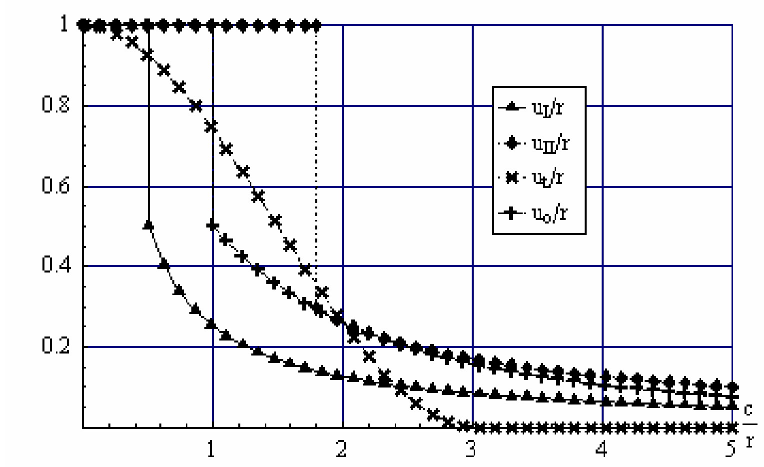

Fig. 16. Parallel system. Defender moves first and defends one component (subscript I) or two components (subscript II), C/R=1.