Rent Division among Trading Firms and Exchanges: A Bargaining Model and Evidence from the Japanese Stock Market

57181514-8 ZHAO, Jiaxi

C.E. Prof. YOTSUZUKA, Toshiki

D.E. Prof. UNO, Jun D.E. Prof. COAD, Alex

Summary

With the rise of technology these days, high frequency trading (HFT) has become a new norm in the financial world, especially in the most liquid markets such as equity. Previous study has shown that arbitrage opportunities are built into the current most popular CLOB market design, and there is rent devision between exchanges and trading firms. This thesis applies general results in the Nash-in-Nash bargaining model to a trading game setting, and derives conditions for existence of Nash-in-Nash equilibrium and the Nash-in-Nash prices. It can then be shown that the condition for there to exist Nash-in-Nash equilibrium is equivalent to the condition for there to exist Order Book Equilibrium (OBE) in the trading game setting in [3], and the implication on rent division by Nash prices are equivalent to the ESST prices in the OBE in [3]. This thesis also investigates five out of the seven stylized facts documented in [3] in the Japan equity market setting. Market shares of exchanges in Japan are stable over time, yet the market is tipping significantly. The per share trading fee was economically small before merger of OSE and TSE, yet it has not been economically small in JPX after the two exchanges merged. Exchanges in Japan do not earn significant revenue from technology and information service, and there is no economically significant upper trend in this part of revenue. The empirical validation of these stylized facts in Japan equity market, as inverse of what holds in the US, supports the necessity of an integrated market for (i) the market shares to be interior, (ii) the per share trading fee to be economically small, and (iii) the part of revenue from exchange specific speed technology to be economically significant and growing.

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Contents

1 Introduction

With the rise of technology these days, high frequency trading (HFT) has become a new norm in the financial world, especially in the most liquid markets such as equity. There has been discussions on the current market design against this background. As shown in Budish, Cramton, and Shim (2015, hereinafter referred to as [2]), arbitrage opportunity is built into the continuous limit order book (CLOB) market design. It has been further shown in Budish, Lee, and Shim (2019, [3]) that there is division of the latency arbitrage rents between trading firms and exchanges realized by the Exchange Specific Speed Technology (ESST) fee, which depends on the ESST user-provider relationship formed among them.

While these results are based upon strategic argument without being embedded into any specific game model, the Nash-in-Nash bargaining model seems to be a natural fit for this situation and potential to provide more understanding of it, as is also brought forward in [3]. In another previous study by Collard-Wexler, Gowrisankaran, and Lee (2019, [6]), it has been shown that Nash-in-Nash equilibrium for general bipartite multiple-player-to-multiple-player bargaining games exists under certain set of assumptions. The theoretical part of this thesis applies the general bilateral oligopoly Nash bargaining model in [6] to a setting in which multiple exchanges and multiple trading firms negotiate upon division of the arbitrage surplus resulted from HFT, investigates conditions for existence of solutions in the Nash-in-Nash bargaining game setting, and compares these to those in [3].

[3] has also documented and validated seven stylized facts in the US equity market, some of which are implied by the model. It is interesting to see whether this model can be generalized to situations in other markets, and how and why so (not). The second part of this thesis tries to validate or invalidate five out of the seven stylized facts in the Japan equity market.

In the theoretical part, we consider a model setting that is similar to that in [3]. The object is one security, whose fundamental value can be perfectly observed from a signal, which is a random variable that evolves with compound jump process. There are four types of strategic players in the game. Investors mainly get his/her payoff from satisfaction of buying/selling needs for the security itself. Trading Firms (TFs) and Informed Traders are arbitrageurs. Exchanges earn payoff from charging the other three agents for speed technology and order matching. The game consists of a one-time played pregame and two infinitely repeated sub-games. In the pregame, Exchanges posts per share trading fees. The first sub-game is a bargaining game, where bipartite negotiations happen between TFs and Exchanges on the Exchange Specific Speed Technology (ESST) fees and ESST user-provider relationships forms. The second is a trading game similar to the Stage Three game set in the Multiple Exchange Game in [3], consisting of two sessions per time period. In the first session TFs post their orders and make liquidity in Exchanges, and in the second session potential liquidity takers come and act. In each time period the bargaining game is first played, and then the trading game.

It can then be shown that the condition for there to exist an equilibrium in the Nash-in-Nash bargaining game is equivalent to the condition for there to exist OBE in the trading game setting in [3], and the Nash prices are equivalent to the ESST prices in the OBE in [3]. The bargaining game setting also shed a light on equilibrium where there are discrete and continuous exchanges in the market. We use the order book equilibrium (OBE) as solution concept in the trading game, and pure-strategy weak perfect Bayesian equilibrium concept with passive beliefs in the bargaining game. For different sets of relationships formed between TFs and Exchanges, there exists different equilibria for the trading game, resulting in different flow profit expectations. This gives the flow profit function for TFs and Exchanges. Given these flow profit functions, Nash-in-Nash equilibrium prices, specific form of the marginal contribution functions, and conditions for existence of Nash-in-Nash equilibrium can be derived.

We then go on to validate five out of the seven the stylized facts first documented by [3] in Japan equity market[1]. A summary of the seven stylized facts in the original paper can be found in Table 1.

The most significant difference between Japan and US equity market is that the Japan market is not integrated in the sense that with regulations such as the Unlisted Trading Privileges (UTP) and Regulation National Market System (Reg NMS) in the US, assets are accessible and fungible across exchanges. What is more, after the merger of OSE and TSE in 2012, although the market is fragmented

[1] Stylized facts 1 and 2 are not studied due to difficulty to access relevant data.

Table 1. Summary of the Seven Stylized Facts

Stylized Fact No.	Statement in [3]	In Japan Equity Market
1	"Many Exchanges Simultaneously at the Best Bid and Best Offer"	Not checked
2	"Linear Depth-Volume Relationship"	Not checked
3	"Exchange Market Shares are Interior and Relatively Stable, Both Aggregate and Within-Symbol"	Partly True (Stable, but not Interior)
4	"Average Trading Fees are Economically Small"	Partly True (Different among Exchanges)
5	"Money-Pump Constraint Binds"	True
6	"Exchanges Earn Significant Revenues from Data and Co-Location/Connectivity (i.e., ESST)"	Not True
7	"Exchange Revenue from Data and Co-Location/Connectivity has Grown Significantly in the Reg NMS Era"	Not True

in form, it lacks actual competition. The empirical validation of these stylized facts in Japan, as inverse of what is validated in the US, support the necessity of an integrated market for (i) the market shares to be interior, (ii) the per share trading fee to be economically small, and (iii) the part of revenue from exchange specific speed technology to be economically significant and growing.

More specifically, it is shown that market shares of exchanges in Japan is very stable over time, yet not interior. The market is tipping significantly in Japan, even more so on individual stock level than on the aggregate level. As for per share trading fee, the money pump restriction holds in Japan; the per share trading fee was economically small in OSE before merger, TSE before merger, and NSE, yet it is not economically small in JPX. Furthermore, there is a negative correlation between exchange groups' per share trading fees in absolute terms and their market shares. As for revenues from ESST, exchanges in Japan do not earn significant revenue from it; although there is a statistically significant upper trend in the percentage of revenues from ESST fee, the trend is not economically significant.

Equity markets witnesses fragmentation in the recent decade, especially in the US equity market. There have been studies that investigate interactions among exchanges. Chen and Duffie (2020, [5]) develops a model where assets can be traded in different exchanges and found that market fragmentation leads to lower depth but more informative prices. Baldauf and Mollner (2019, [1]) develops a model of imperfect competition and studies the tradeoff between competed down trading fees and increased arbitrage rent in fragmented equity market. They also estimates the model for an Australian security and found that competition does not increase investors' benefits in that case.

There has been not many theoretical studies built on the Nash-in-Nash bargaining model. Spulber (2017, [39]) extend the model for the situation where the downstream firms are complements. By considering a two-stage game where the players make supply schedule proposals in the first stage and engages in Nash-in-Nash bargaining in the second stage. Spulber found that there exists a unique equilibrium with weakly dominant strategy and the outcome turns out to be joint-profit maximizing. The theoretical part of this thesis tries to extend the Nash-in-Nash bargaining model by applying it to a more specific setting. The contribution of this part is two-fold. On the one hand, by applying the general Nash-in-Nash bargaining model to a more specific trading setting, the result from the bargaining model supports the equilibrium results in the trading game. On the other hand, the consistency between the condition for equilibrium existence of the Nash-in-Nash bargaining game and that in the trading game, which is independently derived with purely strategic arguments without being embedded in a bargaining game, supports and provides more intuitive understanding for the former.

Empirically, bargaining models are most often used to understand integration in typically oligopolistic markets, such as the health insurance market, the television market, etc. In Ho and Lee (2019, [13]), a bargaining model is estimated to investigate how horizontal integration affect equilibria in health care markets, with interactions among hospitals, insurer providers, employees (large insurance demanders), and individuals considered. It is argued that although concentration in insurance providers may cause increase in insurance premium and hospitals' income, the resulted stronger bargaining power of them may exert offsetting effect. In another work of Ho and Lee (2020, [14]), Nash-in-Nash bargaining solution is adapted to this insurer-hospital negotiation framework. Dafny, Ho and Lee[8] further investigate the bargaining game in health care market, looking into cross-market hospital mergers and their effect on price change. They found that price increase will arise when marginal contribution of new agreement among insurers and hospitals is decreasing. In particular, adjacent hospital mergers and mergers of hospitals who share common insurers generates more price effect. Effect of vertical integration is investigated in Crawford et. al.[7] in the US television broadcasting market, which also has its pros and cons towards social welfare, and it is found that the final result highly relies on program access rules. Both these two studies do simulation for an integrated market. In [3], equity market which is effectively integrated due to certain regulations is considered. The empirical part of this thesis contributes to the market integration literature in the sense that an actually fully integrated, competition-less market, i.e. the Japan equity market is studied. There are two significant mergers in Japan equity market, the merger of JASDAQ and OSE, and that of OSE with TSE. Effect of these two mergers are also implied by the results of the empirical part of this thesis.

The model in this thesis, which has been first built up by [2], has done well in depicting the essentials of US equity exchange competitions, as is shown in [3]. However, this market fragmentation and integration is not true in many other markets, including Japan equity market. There has long been studies that supports this market concentration equilibrium result. Pagano (1989, [33]) has proposed a model based on the idea that "trading volume and absorptive capacity of the market tend to feed positively on each other", and proved that there exists more than one equilibria depending on the belief about the other traders. When there are two markets and the trading cost is the same on both markets, then there is one unique equilibrium where all traders trade on one of the markets; when there are two markets and the transaction cost is not the same, then there are multiple equilibria where market is concentrated as well as fragmented. When the two-market fragmentation equilibrium does happen, traders form clusters on different markets by sizes of their intended transactions. The empirical part of this thesis suggests that the model in [33] better captures the Japan equity market. Another theoretical explanation can be found in Ellison and Fudenberg (2003, [9]), who has shown that in models where there are both increasing returns from concentration and the "market-impact effect" that results in preference to less crowded market, there exists a "plateau of equilibria" with two markets, and market will tip only when the market share of one of them falls beyond a certain threshold. This why concentration or fragmentation question was raised again in Cantillon and Yin (2010, [4]), which propose to endogenize market structure in further studies. They also put forward many other interesting questions for potential research, including considering the multi-sidedness of exchanges and how other businesses such as listing affect the competition among them.

The rest of this thesis is organized as follows. Section 2 deals with the theoretical model where all exchanges are continuous. Section 2.1 sets up the model and the game. After game set-up, the solution concept used is specified in Section 2.2, and useful definitions in the bargaining game equilibrium are given in Section 2.3. Section 2.4 first prepares some useful definitions, and then go on to describe the four equilibria corresponding to four equivalent classes of sets of relationships formed between TFs and Exchanges in the trading game, and give the flow profit to TFs and Exchanges under this set of relationship formed. Section 2.5 derives the marginal contribution functions, Nash prices, and conditions for equilibrium to exist. Section 3 then discusses the situation where there are discrete and continuous exchanges. Section 4 illustrates empirical study of five out of seven of the stylized facts documented by [3] in the Japan equity market, with each subsection discussing one fact. Section 5 concludes.

2 'Nash-in-Nash' Bargaining Approach to Rent Division in CLOB Market Design

2.1 Model Setup and Review of Results in Previous Works

2.1.1 Value of the Security

Notations in this thesis are mostly inherited from [2] and [3].

Let there be a security, x, that trades in the market, and a signal, y, that is perfectly correlated to the fundamental value of x. Further assume that x can always be liquidated at no cost at its fundamental value, and that x can be traded in continuous units and prices.

We consider discrete time setting with time periods indexed $t = 1, 2, \ldots$ being infinitely many and time between time periods being $\Lambda > 0$. Let y be a random variable that follows a compound jump process, with jump probability λ_{jump} per time period. Each time period is divided into two sessions, and jumps happen at the end of the first session. Jumps are distributed symmetrically with mean 0 and bounded support. The limit of this process when $\Lambda \to 0$ is consistent with the geometric Brownian Motion process, which is much more used for modeling equity value evolvement in financial literatures. The specific distribution of jumps is irrelevant here. Formally, let y_{t^-} be the value of y at the first session of time t, and let y_{t^+} be the value of y at the second session of time t, then y is a random variable such that

$$\Pr[y_{t^-} - y_{t^+} \neq 0] = \lambda_{jump}$$

with the probability density function $f_y(j) = \frac{d}{dj}F_y(j)$ symmetric around $j = 0$, where

$$F_y(j) \equiv \Pr[y_{t^-} - y_{t^+} \leqslant j | y_{t^-} - y_{t^+} \neq 0].$$

Let the size of jump that happens at time t be denoted by

$$J_t = |y_{t^-} - y_{t^+}|.$$

Then this J is also a random variable with a distribution that we call jump-size distribution.

Each jump in y, when it happens, is likely to be observed either privately (by only one player) or publicly. Let the probability of a jump happens and can only be observed privately per period be λ_{pri} and the probability of a publicly observable jump per period be λ_{pub}. The public jump and the private jump are two mutually exclusive events, i.e. $\lambda_{pri} + \lambda_{pub} = \lambda_{jump}$. Assume that private jumps and public jumps have the same jump size distribution.

2.1.2 Players, Orders, and Market Rules

There are four types of strategic players that we consider, the Exchanges, the Trading Firms (TFs), the Investors, and the Informed Traders.

An Investor is a player with inelastic need to to buy or sell one unit of x, with buying and selling needs equally probable. This need happens stochastically with the probability of λ_{inv} per time period. Investors seek to maximize their payoff, where the payoff function of an investor is $\pi_{invest}(p, y) = v + (y - p)$ when the need is to buy one unit of x, and $\pi_{invest}(p, y) = v - (y - p)$ when the need is to sell one unit of x. Here, p is the price at which the investor trade x, and y is the signal that represent fundamental value of x when investor trade; $v \gg 0$ represent the inelastic need such that $v > |y - p|$ whenever $|y - p|$ is not infinity[2]. This guarantees that it is always optimal for an investor when it arrives the market to trade immediately, as long as there is liquidity.

TFs and Informed Traders are arbitrageurs and do not have intrinsic needs to buy or sell x. They seek to maximize their expected payoff $y - p$ when they buy one unit of x, and $p - y$ when they sell one unit of x. An Informed Trader is the single player that observes a private jump, if one happens. Informed Trader has definite payoff given a private jump happens. However, TFs has indefinite payoff in a continuous limit order book (CLOB) exchange given public jump happens, which will be clearer after elaboration of the CLOB rules and the game setting.

TFs are classified into three categories based on the fastest speed at which they can trade, which has a one-to-one relationship to the set of speed technology that they possess. There are TFs with no cutting edge general-purpose speed technology (hereafter referred to as slow TFs), TFs with general-purpose speed technology but no exchange specific speed technology(ESST)[3] (hereafter referred to as fast TFs),

[2] $|y - p| = \infty$ represents the fact that there is no liquidity offered in the market.

[3] ESST started to appear in many exchanges with the arising of high frequency trading (HFT). The forms of ESST include co-location services, network services, proprietary high frequency data feed, etc. This will be discussed upon in the empirical validation section of this thesis.

and TFs with ESST (hereafter referred to as TFs with ESST on X_j). Note here that all TFs that possess ESST for some Exchange(s) must possess general speed technology in the first place. There is no such speed classification or any other classification for Investors and Informed Traders. Exchanges can operate under two kinds of market designs, the CLOB design, and the Frequent Batch Auction (FBA) design. We will set the game for CLOB design exchanges, and discuss what will happen for FBA design exchanges in Section 3.

TFs, Investors, and Informed Traders can send messages to an Exchange or Exchanges. These messages can be limit orders - a tuple consists of a specified unit of the security, a "buy/sell" action, and a specified highest(lowest) price to buy(sell) - or cancellations, to cancel previously placed limit orders[4]. Then, according to CLOB rules, each Exchange process these orders serially in the order of their arrival, construct limit order book with stack of "ask"s (limit sell order prices) and "bid"s (limit buy order prices), match marketable orders with orders on the book, or eliminate an order from the order book when processing cancellations. If there are multiple orders arriving at X_j at the same time period, orders sent by TFs with ESST on X_j will be processed with highest priority, orders by fast TFs and Informed Traders with second highest priority, and orders by slow TFs and Investors with lowest priority. If there are orders by agents within the same priority group arriving at X_j at the same time, they are processed serially in random order. It is also required in this thesis that "each Exchange sell ESST to at least two TFs or not sell ESST at all" in this thesis, realized in the same manner as in [3]. Thus when there is only one fast TF with ESST on some exchange X_k, then it is equivalent to the situation where there is no fast TF with ESST on this exchange. There is no place for slow TFs in equilibria in the CLOB Exchanges, according to a similar model in [2] Budish, Cramton, and Shim (2015) (hereafter [2]). For this reason, from now on we will refer to fast TFs simply as TFs, and specify when they possess ESST of a particular Exchange.

Exchanges earn profit from two sources, charging TFs for using its ESST, and charging trading agents per-share trading fee when orders are matched and completed. Let the ESST fee be denoted by F and the per-share trading fee be f. ESST fee take the form of a one-time payment, and per-share trading fee can have three different structures: the two-sided structure, the maker-taker structure, and the taker-maker structure. In a two-sided structure, the two counter-parties in a completed trade are both charged f per share of the transaction; in a maker-taker structure, liquidity maker is charged f, and liquidity taker is subsidized f; in a taker-maker structure, it is the other way around. In equilibrium, f will be zero in the two-sided and the taker-maker structures, and effectively zero in the maker-taker structure.

There are two assumptions that are important to this model, the accessibility and the fungibility of securities. These assumptions are satisfied in US by the Unites States securities regulation ([3, section 2, pp.7]), and when applying this model to other markets, it is necessary to check whether these two

[4] Other kinds of orders such as market orders, immediate or cancel(IOC) orders, etc. as proxies to place limit orders and cancellations(for example, a market buy order tells the Exchange to place a limit buy order with price equal to the lowest ask in the current order book, an IOC order tell the Exchange to place a limit order at specified price and cancel that order at the end of this time period if it still remains in the order book). We model these other types of orders using explicit combinations of the limit orders and cancellations. That is, no other type of order will be involved in the model in this thesis.

assumptions are met. By assumptions 2.2 and 2.1, all messages from players are effectively sent to all Exchanges simultaneously.

Assumption 2.1 (A.Fungibility). *Assume that x can always be traded on any exchange, i.e., the trading fee $f < \infty$ on any exchange, regardless of the trading history of x, where it is listed, etc. The fundamental value of x is also independent of where it is traded*[5].

Assumption 2.2 (A.Accessibility). *Assume that any order on any exchange can be access without friction at any time. The per share trading fee f must be economically small. Let f_{ij} be the per-share trading fee that X_j charges on entity i (can be an investor or a TF), then f_{ij} must be the same for any i, regardless of any other conditions (e.g. whether i has purchased ESST from X_j or not. Furthermore, all trading cost on an exchange can be incorporated into F and f, and no other cost will be incurred in the process (e.g. quoting fee, etc.)*[6].

2.1.3 The Pregame

All M Exchanges post their per-share trading fees $\mathbf{f} = (f_1, \ldots, f_M)$. This pregame is played one-time before the following games start.

2.1.4 The Bargaining Game

The bargaining game is an application of the Nash bargaining model in [6, Section II, pp. 170]. This model is suitable for the situation between the Exchanges and TFs for agreements among them are interdependent and have externalities.

Only TFs and Exchanges are involved in the bargaining game. Let $\mathcal{X}$ be the set of Exchanges on which x trades, indexed by $j = 1, 2, \ldots, M$ ($\mathcal{X} = \{X_1, X_2, \ldots X_M\}$), and $\mathcal{T}$ be the set of fast TFs that trades x, indexed by $j = 1, 2, \ldots, N$ ($\mathcal{T} = \{T_1, T_2, \ldots T_N\}$), where $M \geqslant 2$ and $N \geqslant 3$ [7].

Let $\mathcal{G} = \mathcal{T} \times \mathcal{X}$ be the set of potential relationship between $\mathcal{T}$ and $\mathcal{X}$, with elements being ordered pairs (i, j), representing the relationship that T_i has purchased ESST from X_j. Let $\mathcal{G}_{i,T}$ be the subset of relationships that involve T_i, and $\mathcal{G}_{-i,T}$ be the subset of relationships that does not involve T_i. Define $\mathcal{G}_{j,X}$ and $\mathcal{G}_{-jX}$ analogously. For any subset $\mathcal{A} \subseteq \mathcal{G}$, let $\mathcal{A}_{i,T} = \mathcal{A} \cap \mathcal{G}_{i,T}$, and $\mathcal{A}_{-i.T}, \mathcal{A}_{j,X}, \mathcal{A}_{-j,X}$ be analogously defined.

Let $\{\pi_{i,T}(\mathcal{A})\}_{i=1,\ldots,N;\mathcal{A}\subseteq\mathcal{G}}$ and $\{\pi_{j,X}(\mathcal{A})\}_{j=1,\ldots,M;\mathcal{A}\subseteq\mathcal{G}}$ be the expected profit that TFs realize from arbitrage in trading game per time period, and expected profit that Exchanges realize from transaction fees per time period, given that a set of ESST user-provider relations $\mathcal{A}$ has been formed. Note that ESST fees are not included in Exchanges' profit functions. The specific form that these profit functions take will be discussed in Section 2.4.

[5] The regulatory rule Unlisted Trading Privileges (UTP) guarantees that any securities traded in the US satisfy this assumption.

[6] This assumption is satisfied in the US based on a set of rules by Regulation National Market System (Reg NMS).

[7] With the assumptions 2.2 and 2.1, $\mathcal{X}$ should be all the exchanges in the system that governed by these institutional regulations. And by considering x as the value-weighted portfolio of all securities in the system, this is a model for the division of all rents from latency arbitrage in the system.

For immediate equilibrium to exist, discounting is adopt in this model. Assuming all TFs are faced with the same discount factor and all Exchanges are are faced with another, denote the discount factors for TF's by $\delta_T = e^{-r_T \Lambda}$, and the discount factors for Exchanges by $\delta_X = e^{-r_X \Lambda}$, where r_T and r_X are risk-free interest rates. Let $\mathcal{A}^t$ be the set of ESST user-provider relationship formed up to time period t (t included).

In the bargaining game, every pair of T_i and X_j negotiate independently over the lump-sum ESST fee F_{ij} that T_i pays to X_j to become a possessor of ESST on X_j. Assume that all trading firms are symmetric, and hence in equilibrium, F_{ij} will be the same for all i with fixed j. For this reason, we use F_j to represent the ESST fee that X_j charge for all T_i's in equilibrium.

The bargaining game then runs as a generalized Rubinstein (1982) bargaining game [34]. It begins in an odd period $t_0 \geqslant 1$ with $\mathcal{A}^{t_0-1} = \emptyset$. In odd periods t, each Exchange X_j proposes simultaneously $\{F_{ij}\}_{(i,j)\in\mathcal{G}_{j,X}\setminus\mathcal{A}^{t-1}}$ to each T_i with which X_j has a potential relationship but which has not been formed yet; each T_i received a proposed ESST price then simultaneously decide whether to purchase ESST at the proposed price or not. In even periods t, each T_i simultaneously offers $\{F_{ij}\}_{(i,j)\in\mathcal{G}_{i,T}\setminus\mathcal{A}^{t-1}}$ to Exchanges that T_i has a potential relationship with but not formed yet; each X_j received an offer then simultaneously decide whether to accept or reject this offer. In each period t, if an ESST price offer F_{ij} is accepted, then the ESST fee is paid from T_i to X_j , and the set of ESST user-provider relationship $\mathcal{A}^t$ is formed between the two immediately in this period t. During period t, all offer information is private. At the beginning of the next period $t+1$, each TF and Exchange observe the history of play, including what offers are made and whether they are accepted and rejected, and of course including $\mathcal{A}^t$, the formed set of relationships so far. The formed relationship remain formed throughout the rest of the time. There will be no meaningful strategic play in the bargaining game after any period T where $\mathcal{A}^T = \mathcal{G}$.

The trading game then started in the same time period. After the trading game, the expected profit $\pi_{i.T}(\mathcal{A}^t)$ and $\pi_{j,X}(\mathcal{A}^t)$ will be received by T_i and X_j at the end of time t.

2.1.5 The Trading Game

We now review the game setting and equilibrium results of the multi-exchange trading game as in [3, Section 3.2, pp. 19], with modifications to fit the setting in this thesis. The modifications do not affect the equilibrium results. All four types of players and nature are involved in the trading game[8].

There is a pre-game before the trading games. Then starting from time t_0, the trading game starts. There is one trading game per time period, which is divided into two sessions. The trading game is repeated infinitely. The discount factors are the same as what is specified in the bargaining game.

Session 1: At the beginning of the trading game at time t, a state vector $(y_{t^-}, \boldsymbol{\omega}_{t^-})$ is publicly observed by all players as a common knowledge, where y_{t^-} is the signal value, and $\boldsymbol{\omega}_{t^-} = (\omega_1, \dots, \omega_M)_{t^-}$ where ω_j represents the order book of X_j. All T_i have the chance to send limit orders to all exchanges

[8] This game model can also be viewed as an adaptation from the Multiple-Exchange Game in [3, Section 3.2, pp. 19] by merging the Stage One and Stage Two game into the bargaining Game, turning it from non-cooperative games to a cooperative game, and from a finite unrepeated game to an infinitely repeated game.

(recall that this is effectively the case assuming 2.2) to trade up to q_i units of x at price better than or equal to p_i (let this limit order be denoted by $\mathbf{o}_{ij} = (q_{ij}, p_{ij})$, or to cancel an order sent before. When $q_i > 0$, this is a limit buy order, and when $q_i < 0$, this is a limit sell order. Each exchange then serially process the orders they receive, if there are messages arriving at the same time, ties are broken by speed category of the TFs sending those messages, and randomly within the speed technology groups. The order book state then change to $\boldsymbol{\omega}_{t^+}$ and is observed publicly by all players.

Session 2: Nature moves and there are three events that may happen:

- Event 1: With probability λ_{inv}, an investor arrives with demand to trade one unit of x. The investor can send one message to all exchanges.
- Event 2: With probability λ_{pri}, a jump in y happens and y_{t^+} is observed by only a single Informed Trader. The Informed Trader can send one message to all exchanges.
- Event 3: With probability λ_{pub}, a jump in y happens and y_{t^+} is observed publicly by all players. Each TF can send one limit order or one cancellation message to all exchanges.

Here, Event 2 and Event 3 are mutually exclusive events, Event 1 and Event 2 are independent, and Event 1 and Event 3 are also independent. We model investor need and jump in y as independent events here, instead of mutually exclusive events as in [2]. Thus, the probability that there is no event in this session is $1 - \lambda_{inv} - \lambda_{pri} - \lambda_{pub} + \lambda_{inv}(\lambda_{pri} + \lambda_{pub}) \geqslant 0$.

Each exchange then serially processes the orders they receive. If there are messages arriving at the same time, ties are broken by the speed category to which the sender of the message belong, and randomly within the speed groups. After processing all orders, the order book state is observed by all players, and y_{t^+} is also observed by all players. All players then have a single opportunity to send cancellation messages to all Exchanges. Note that it is optimal for all players whose orders remain on the order book of any Exchange to send cancellation messages due to the Markov property of y. Processing of the cancellation orders does not affect the equilibrium[9]. The order book state is then changed to $\boldsymbol{\omega}_{(t+1)^-}$, and the next time period starts.

2.2 Equilibrium Concept

For the solution concept of the trading game, we adopt the pure-strategy order book equilibrium (OBE) concept, first defined in [3, Appendix A.1, pp.70]. In a word, an OBE is a set of orders such that there is "no profitable price improvements" and "no robust deviations". This solution concept take "the presence of other potential liquidity providers" into account for evaluating profitability of deviation, resulting in equilibrium where TFs provide liquidity at competitive prices. Explanation for non-existence of MPE with more detail and formal definition of OBE can be found in [2]. For the solution concept of the bargaining game, the pure-strategy weak perfect Bayesian equilibrium concept with passive beliefs as specified in [6, Section II.A, pp.172] is adopted.

[9] This is equivalent to the case where all players use immediate-or-cancel orders instead of limit orders.

2.3 Rubinstein and Nash-in-Nash Prices

We now recall definitions for Nash-in-Nash prices and the generalized Rubinstein prices in [6, Section II.B, pp.174].

To prepare for the necessary ingredients for the two set of prices, define the marginal contribution functions derived from the profit functions

$$\Delta\pi_{i,T}(\mathcal{A},\mathcal{B}) \equiv \pi_{i,T}(\mathcal{A}) - \pi_{i,T}(\mathcal{A}\setminus\mathcal{B}), \quad \mathcal{B}\subset\mathcal{A}\subset\mathcal{G},$$

and similarly

$$\Delta\pi_{j,X}(\mathcal{A},\mathcal{B}) \equiv \pi_{j,X}(\mathcal{A}) - \pi_{j,X}(\mathcal{A}\setminus\mathcal{B}), \quad \mathcal{B}\subset\mathcal{A}\subset\mathcal{G}.$$

Definition 2.3 (Rubinstein prices). *The Rubinstein prices form a vector* $\{F^R_{ij,T}, F^R_{ij,X}\}_{\{i,j\}\in\mathcal{G}}$ *where*

$$F^R_{ij,X} = \frac{(1-\delta_{i,T})\Delta\pi_{i,T}(\mathcal{G},\{(i,j)\}) - \delta_{i,T}(1-\delta_{j,X})\Delta\pi_{j,X}(\mathcal{G},\{(i,j)\})}{1-\delta_{i,T}\delta_{j,X}},$$

$$F^R_{ij,T} = \frac{\delta_{j,X}(1-\delta_{i,T})\Delta\pi_{i,T}(\mathcal{G},\{(i,j)\}) - (1-\delta_{j,X})\Delta\pi_{j,X}(\mathcal{G},\{(i,j)\})}{1-\delta_{i,T}\delta_{j,X}}.$$

Note that in this Rubinstein alternating offer bargaining setting, the strategy for each Exchange is a set of prices $\{F_{ij,X}\}_{(i,j)\in\mathcal{G}_{j,X}}$ that it will offer in odd periods; the strategy for each TF is the set of prices $\{F_{ij,T}\}_{(i,j)\in\mathcal{G}_{i,T}}$ that it will offer in even periods. In equilibrium, $F_{ij,X}$ should make T_i indifferent between accepting this offer and having its own offer accepted in the next period, vice versa for $F_{ij,T}$, given that (i,j) is the last unformed relationship in $\mathcal{G}$.

Definition 2.4 (Nash-in-Nash prices). *The Nash-in-Nash prices are a vector* $\{F^{Nash}_{ij}\}_{\{i,j\}\in\mathcal{G}}$ *where*

$$\begin{aligned} F^{Nash}_{ij} &\equiv \arg\max_p [\Delta\pi_{j,X}(\mathcal{G},\{(i,j)\}) + p]^b_{j,X} \times [\Delta\pi_{i,T}(\mathcal{G},\{(i,j)\}) - p]^{b_{i,T}} \\ &= \frac{b_{j,X}\Delta\pi_{i,T}(\mathcal{G},\{(i,j)\}) - b_{i,T}\Delta\pi_{j,X}(\mathcal{G},\{(i,j)\})}{b_{i,T}+b_{j,X}}. \end{aligned}$$

For any i,j such that $\{(i,j)\}\in\mathcal{G}$, the Nash-in-Nash price p^{Nash}_{ij} is the Nash bargaining solution between T_i and X_j given this relationship is the last one to form in $\mathcal{G}$. The sign of this definition is inverse of that in [6] because the payment direction is reversed in the TF and Exchange model (i.e., here the payment goes from TFs to Exchanges, rather than from downstream firm to upstream firms).

By [6, Lemma 2.1, pp.174], Rubinstein prices converge to Nash-in-Nash prices when the time interval $\Lambda \to 0$.

2.4 Equilibrium of the Trading Game and Latency Arbitrage, and Flow Profit Functions for the Bargaining Game

For disposition convenience, we make the following definitions:

Definition 2.5. *It is said that σ_j amount of liquidity of security x is provided on X_j at spread s around y, if and only if the order book ω_j of x on X_j contains limit orders $o_{ij} = (q_{ij}, p_{ij})$ such that (i) $\sum_{i:q_{ij}>0} q_{ij} = \sigma_j$, (ii) $\sum_{i:q_{ij}<0} q_{ij} = -\sigma_j$, (iii) $p_{ij} = y - \frac{s}{2}$, $\forall(i,j)$ s.t. $q_{ij} > 0$, and (iv) $p_{ij} = y + \frac{s}{2}$, $\forall(i,j)$ s.t. $q_{ij} < 0$.*

Definition 2.6. *s^*_{ctn} is the solution for s to the equation*

$$\lambda_{inv} \cdot \frac{s}{2} = (\lambda_{pri} + \lambda_{pub})L(s), \tag{1}$$

where

$$L(s) \equiv \Pr\left[J > \frac{s}{2}\right] \mathrm{E}\left[J - \frac{s}{2}\middle| J > \frac{s}{2}\right].$$

Note that there exists one unique strictly positive solution to this equation, as the left-hand-side of the equation is a monotone increasing function of s that ranges from 0 to $+\infty$, and the right-hand-side is a monotone decreasing function of s with an infimum of 0.

s^*_{ctn} is the bid-ask spread that leaves TFs indifferent between providing one unit of liquidity and sniping other's stale quotes on any single exchange where all the TFs are in the same speed group and where the per share trading fee is 0, first deducted in [3, equation (3.1)]. To snipe a stale quote is when a profitable public jump happens. to try to trade with the quotes at the price before jump. This is profitable due to the assumption that x can always be liquidated for its fundamental value with no cost.

Proof of this runs as follows. It is profitable for all TFs that are not the liquidity provider to snipe the stale quote when such a jump happens, and for the liquidity provider with the stale quote to submit cancellation message at the same time to avoid being sniped. Since all of their orders are with same processing priority, the conditional probability for one stale quote sniping TF to successfully snipe or for the liquidity provider to successfully avoid being sniped, is equal to the probability that a certain order wins over the other $N-1$ in random tie-breaking. This probability is $\frac{1}{N}$, and the conditional probability that the stale quote is sniped is $1 - \frac{1}{N} = \frac{N-1}{N}$. Therefore, the expected payoff of sniping other's quotes at spread s is $\lambda_{pub}L(s)$; and the expected payoff of providing liquidity at s is

$$\lambda_{inv} \cdot \frac{s}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s),$$

the expected gain from trade with investors at spread s, subtract the expected loss from jump. Equating this and the expected payoff of sniping stale quotes, we have the equation 1.

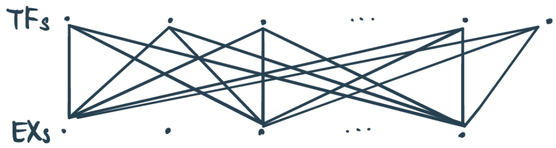
One important and interesting point to notice is that the equilibrium spread is independent of N, the number of TFs. That is, with an arbitrary greater than 1 number of TFs on an Exchange, no matter how many of them there are, there will be a strictly positive constant liquidity cost (bid-ask spread) and latency arbitrage prize, which results from the CLOB design. As is argued in [2], inefficiency is built into the CLOB design and cannot be competed away.

Moreover, the following condition for Exchanges is useful for illustration.

Definition 2.7. *Given $\mathcal{A}$ a set of relations formed, if for an Exchange j, either (i) all TFs have purchased ESST from X_j, or (ii) the number of TFs which have purchased ESST from X_j is less than or equal to 1, then X_j is referred to as a fair Exchange. The exchange is said to be unfair otherwise.*

If all TFs have purchased ESST on X_k or the number of TFs that have purchased ESST on X_k is no more than 1, then all TFs will be in the same speed group on this Exchange and the tie-breaking when processing orders arriving at the same time is fair among all TFs.

There are 4 equivalent classes of set of ESST use-provider relationships that are of use for solution of the bargaining aame, and there exists an equilibrium for each one of them in the trading tame. We describe the equilibria, the equilibria payoff for TFs and Exchanges, and illustrate proofs for them one by one.



Proposition 2.8. *Assume $M \geqslant 3$ and $N \geqslant 3$. For any time t such that all Exchanges are fair exchanges as defined in Definition 2.7, for any vector of market shares $\boldsymbol{\sigma}^* = (\sigma_1^*, \ldots, \sigma_M^*)$ where $\sum_j \sigma_j^* = 1$, there exists an equilibrium of the trading game specified as follows:*

***Pre-game**: X_j's simultaneously post per share trading fees $f_j^* = 0, \forall j$;*

***Session 1**: $\frac{\sigma_j^*}{\sum_{j \in \arg\min_k f_k} \sigma_k^*}$ of total liquidity is provided on X_j such that $j \in \arg\min_j f_j$ at spread s_{ctn}^* around y_{t^-}. There may be arbitrary units of x offered at arbitrary spread that is out of J's support.*

***Session 2**:*

- *When an investor arrives, he/she immediately sends orders for $\frac{\sigma_j^*}{\sum_{j \in \arg\min_k (\frac{s_k}{2} + f_k)} \sigma_k^*}$ units of x to X_j ($j \in \arg\min_j f_j$) at marketable price in those exchanges.*

- *If $J > \frac{s}{2} + f_j$ happens for some $s \geqslant 0$ and some j, and is observed by a single Informed Trader, the Informed Trader immediately sends orders to sell infinite units of x to X_j at $y_{t^-} - \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (-\infty, -\frac{s}{2} - f_j)$, or to buy infinite units of x to X_j at $y_{t^-} + \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (\frac{s}{2} + f_j, \infty)$.*

- *If $J > \frac{s}{2} + f_j$ happens for some $s \geqslant 0$ and some j, and is observed by all players, all TFs, regardless of speed group, immediately send orders to sell infinite units of x to X_j at $y_{t^-} - \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (-\infty, -\frac{s}{2} - f_j)$, or to buy infinite units of x to X_j at $y_{t^-} + \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (\frac{s}{2} + f_j, \infty)$; all TFs with outstanding orders on any Exchange with ask prices $< y_{t^+} - f_j$ or bid prices $> y_{t^+} + f_j$ immediately send cancellation messages for all such orders.*

After the above messages are processed, and y_{t^+} observed by all players, they simultaneously send cancellation messages for all orders that they have post earlier and still remain on the order book of any Exchange.

In this equilibrium, the profit to Exchanges is

$$\pi_{j,X}(\mathcal{A}^t) = 0, \quad \forall j \in \{1, \ldots, M\}; \tag{2}$$

and the expected profit to TFs is

$$\pi_{i,T}(\mathcal{A}^t) = \frac{1}{N} \lambda_{pub} L(s_{ctn}^*) \equiv \frac{\Pi_{ctn}^*}{N}, \quad \forall i \in \{1, \ldots, N\}. \tag{3}$$

The proof of this proposition can be done by rejecting all profitable deviation.

Proof. We first consider the deviation by some X_k to post per share trading fee $f_k \neq 0$. First, note that the payoff function for Exchanges given the trading game equilibrium is

$$u_k(f_k) = \begin{cases} \frac{\sigma_k^*}{\sum_{j \in \arg\min_j f_j} \sigma_j^*} \times f_j \times (\lambda_{inv} + \lambda_{jump} - \lambda_{inv}\lambda_{jump}), & \text{if } j \in \arg\min_j f_j \\ \\ 0, & \text{if } j \notin \arg\min_j f_j \end{cases}$$

and the payoff in proposed equilibrium is 0. Posting $f_k < 0$ is not profitable because it will render weakly negative payoff. Posting $f_k > 0$ is also not profitable because it will leave $k \notin \arg\min_j f_j = \{j : f_j = 0\}$, resulting in 0 payoff. This subgame is a Bertrand price competition, where in equilibrium prices are competed to the lowest possible value, which is 0 in this case due to the money-pump restriction [3, pp.26].

We then consider the possible deviations by some T_h in Session 1. (i) It is either not profitable or not a robust profitable deviation for T_h to offer liquidity on any X_j such that $j \notin \arg\min_j f_j$. Because of higher cost of trading fee, higher spread should be taken in such X_j's to render strictly higher profit. Then if there is already no less than one unit of x offered in the Exchanges with minimum trading fee, no investor will trade with the additional liquidity offered at wider spread. If the total unit of x offered in the Exchanges with minimum trading fee is less than one, then there will be safe price improvement by other TFs to provide the remaining unit in the Exchanges with minimum trading fee at the breakeven spread, which will make the deviation in question no longer profitable.

(ii) It is not a robust deviation for T_h to offer additional units of x on any Exchange X_j such that $j \in \arg\min_j f_j$ at spread s^*_{ctn}. Since then there will be more than one unit of x offered, probability that the liquidity will be taken by an investor will be strictly less than 1, but the expected loss from being sniped will stay the same, resulting in a strictly smaller expected payoff for T_h. (iii) It is not profitable for T_h to offer additional units of x on any Exchange X_j such that $j \in \arg\min_j f_j$ at a spread $s' > s^*_{ctn}$. The additional liquidity with wider spread will never be taken by investors but the expected loss from being sniped stays the same, resulting in strictly negative expected payoff. (iv) It may be profitable for T_h to offer additional $l \leqslant 1$ units of x on an Exchange X_j such that $j \in \arg\min_j f_j$ at a spread $s' < s^*_{ctn}$ but it is not a safe profitable price improvement. Note that there exists some $\varepsilon > 0$, such that

$$\left[\lambda_{inv}\frac{s^*_{ctn} - \varepsilon}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s^*_{ctn} - \varepsilon)\right] \cdot l + \frac{1}{N}\lambda_{pub}L(s^*_{ctn}) > \frac{1}{N}\lambda_{pub}L(s^*_{ctn}),$$

$$\text{i.e. } \lambda_{inv}\frac{s^*_{ctn} - \varepsilon}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s^*_{ctn} - \epsilon) > 0^{10}.$$

To provide $l \leqslant 1$ liquidity at spread $s^*_{ctn} - \varepsilon$ is profitable because narrower spread guarantees that the liquidity will be taken, and X_h still has the opportunity to snipe other TFs quotes at the spread s^*_{ctn}. But to respond to this price improvement, other TFs can withdraw l units of liquidity that they offer, resulting in expected payoff of X_h being reduced to

$$\left[\lambda_{inv}\frac{s^*_{ctn} - \varepsilon}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s^*_{ctn} - \varepsilon)\right] \cdot l < \frac{1}{N}\lambda_{pub}L(s^*_{ctn}),$$

[10] Since $\lambda_{pri} + \frac{N-1}{N}\lambda_{pub} < \lambda_{pri} + \lambda_{pub}$, the solution to $\lambda_{inv}\frac{s}{2} = \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s)$ is strictly smaller than the solution to $\lambda_{inv}\frac{s}{2} = (\lambda_{pri} + \lambda_{pub})L(s)$.

and this price improvement is no longer profitable.

(v) It is not a robust deviation for T_h to change the bid or ask prices in any of its order to a price at spread $s' > s^*_{ctn}$. This deviation is profitable because

$$\lambda_{inv}\frac{s'}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s') > \lambda_{inv}\frac{s^*_{ctn}}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s^*_{ctn}).$$

The order will be taken by investors at a probability 1 with strictly larger gain and the expected loss from being sniped will be strictly smaller. Yet if it does, other TFs can engage in safe profitable price improvement by offering the quantity of liquidity at spread s^*_{ctn} on exchanges where price is not at spread s^*_{ctn} in the deviation. This response is profitable because from this particular quantity of liquidity, the expected payoff by making this response

$$\lambda_{inv}\frac{s^*_{ctn}}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s^*_{ctn}) + \frac{1}{N}\lambda_{pub}L(s'),$$

is strictly greater than $\frac{1}{N}\lambda_{pub}L(s')$, the expected payoff by not making this response, and the expected payoff for the rest of the liquidity stays the same; it is safe because the expected payoff from this response on this particular quantity of liquidity even if the deviating TF(s) withdraw(s) that liquidity, as

$$\lambda_{inv}\frac{s^*_{ctn}}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s^*_{ctn}) = \frac{1}{N}\lambda_{pub}L(s^*_{ctn})$$

is still strictly greater than $\frac{1}{N}\lambda_{pub}L(s')$, i.e. the deviation still remains profitable.

(vi) It is not profitable for T_h to change the bid or ask price in any of his equilibrium order to a price at spread $s' < s^*_{ctn}$ because

$$\lambda_{inv}\frac{s'}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s') < \lambda_{inv}\frac{s^*_{ctn}}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s^*_{ctn}).$$

(vii) It is not profitable for T_h to offer less quantity than what it offers in the equilibrium because the profit from providing liquidity $\lambda_{inv}\frac{s^*_{ctn}}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(s^*_{ctn})$ is strictly positive.

Due to inelasticity of demand, investors, when they arrive, cannot be better off by waiting instead of trading immediately. It is also not profitable to take on any other unit vector because any $\sigma_j > \sigma^*_j$ units of x, if any, is at a much wider spread. As for Informed Traders, it is obvious that there is no profitable deviations since the strategy as described realizes the maximum possible expected payoff for the Informed Traders

$$\sum_j \lambda_{pri}\Pr\left[J > \frac{s^*_{ctn}}{2} + f_j\right] \mathrm{E}\left[J - \frac{s^*_{ctn}}{2} - f_j \middle| J > \frac{s^*_{ctn}}{2} + f_j\right],$$

given the other players' strategies in the trading game. The same goes for TFs in Session 2. It is always more profitable to try to snipe than not to, and to try to cancel the stale quotes than not to. At last it is optimal for all players to cancel all remaining orders on the order book at the end of the game because of the Markov property of y.

Therefore, there is neither safe profitable price improvements nor robust deviations for any players given this strategy profile, and such a strategy profile is an OBE of the trading game. In this equilibrium, profit to any Exchange is 0 because the only source of their profit would be f_j's. Profit to any TF is equal to expected payoff from sniping quotes or providing liquidity at spread s^*_{ctn}. □

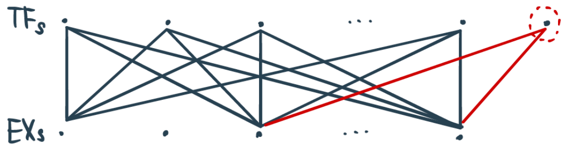
In particular, we have

$$\pi_{j,X}(\mathcal{G}) = 0, \quad \forall\, j \in \{1, \ldots, M\}; \tag{4}$$

and

$$\pi_{i,T}(\mathcal{G}) = \frac{1}{N}\lambda_{pub} L(s^*_{ctn}) \equiv \frac{\Pi^*_{ctn}}{N}, \quad \forall\, i \in \{1, \ldots, N\}. \tag{5}$$

We now consider any other time t when not all Exchanges are fair. This is equivalent to saying that there is some TF who has not purchased ESST from all Exchanges, inheriting the name in [3], such a TF T_k is called a lone-wolf. We first consider the situation where there is only one lone-wolf TF and there is only one unfair Exchange left (which is going to be fair if the lone-wolf TF purchase ESST from it).



Proposition 2.9. *Assume $M \geqslant 3$ and $N \geqslant 3$, for any time t such that there is exactly one X_{j_0} for which $|\mathcal{A}^t_{j_0}| = N-1$ and for all $j \neq j_0$, $|\mathcal{A}^t_j| = N$ or $|\mathcal{A}^t_j| \leqslant 1$, let $i \notin \mathcal{A}^t_{j_0}$ be i_0, for any vector of market shares $\boldsymbol{\sigma}^* = (\sigma^*_1, \ldots, \sigma^*_M)$ where $\sum_j \sigma^*_j = 1$, there exists an equilibrium of the trading game specified as follows:*

***Pre-game**: All X_j's simultaneously post per share trading fees $f^*_j = 0, \forall\, j$;*

***Session 1**: With $\overline{s}_N$, $\widetilde{s}_{i_0}(f_{j_0})$, and $\widetilde{s}_{-i_0}(f_{j_0})$ defined as below, assuming $\overline{s}_N < \widetilde{s}_{i_0}(f_{j_0}) < \widetilde{s}_{-i_0}(f_{j_0}) + 2f_{j_0}$, (i) when $\widetilde{s}_{i_0}(f_{j_0}) \leqslant s^*_{ctn}$, T_{i_0} provides $\frac{\sigma^*_j}{\sum_{j\in \arg\min_k f_k, j\neq j_0} \sigma^*_k}$ of liquidity on X_j $(j \neq j_0)$'s at spread $\widetilde{s}_{i_0}(f_{j_0})$ around y_{t^-}, and no liquidity is provided on X_{j_0}; (ii) when $\widetilde{s}_{i_0}(f_{j_0}) > s^*_{ctn}$, $\frac{\sigma^*_j}{\sum_{j\in \arg\min_k f_k, j\neq j_0} \sigma^*_k}$ of liquidity is provided on X_j $(j \neq j_0)$'s at spread s^*_{ctn} by arbitrary set of TFs, and no liquidity is provided on X_{j_0}. There may be arbitrary units of x offered at an arbitrary spread that is out of J's support on any exchange(s). The strategy given $\overline{s}_N < \widetilde{s}_{i_0}(f_{j_0}) < \widetilde{s}_{-i_0}(f_{j_0})$ does not hold is irrelevant.*

Session 2:

- *When an investor arrives, it immediately sends orders for $\frac{\sigma^*_j}{\sum_{j\in \arg\min_k \left(f_k + \frac{s_k}{2}\right)} \sigma^*_k}$ units of x to X_j at marketable price in that exchange, $\forall\, j$ such that $j \neq j_0$; or if liquidity is only provided on X_{j_0}, then investor sends order for 1 unit of x to X_{j_0} at marketable price.*

- *If $J > \frac{s}{2} + f_j$ happens for some $s \geqslant 0$ and some j, and is observed by a single Informed Trader, the Informed Trader immediately sends orders to sell infinite units of x to X_j at $y_{t^-} - \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (-\infty, -\frac{s}{2} - f_j)$, or to buy infinite units of x to X_j at $y_{t^-} + \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (\frac{s}{2} + f_j, \infty)$.*

- *If $J > \frac{s}{2} + f_j$ happens for some $s \geqslant 0$ and some j, and is observed by all players, all TFs, regardless of speed group, immediately send orders to sell infinite units of x to X_j at $y_{t^-} - \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (-\infty, -\frac{s}{2} - f_j)$, or to buy infinite units of x to X_j at $y_{t^-} + \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (\frac{s}{2} + f_j, \infty)$;*

all TFs with outstanding orders on any Exchange with ask prices $< y_{t^+} - f_j$ or bid prices $> y_{t^+} + f_j$ immediately send cancellation messages for all such orders.

After the above messages are processed, and y_{t^+} observed by all players, they simultaneously send cancellation messages for all orders that they have post earlier and still remain on the order book of any Exchange.

The spread $\overline{s}_N$ is the solution to the equation

$$\lambda_{inv}\frac{\overline{s}_N}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L\left(\overline{s}_N\right) = 0 \tag{6}$$

The spread $\widetilde{s}_{i_0}(f_{j_0})$ is the solution to the equation

$$\begin{aligned}&\frac{1}{N}\lambda_{pub}L(\widetilde{s}_{i_0}(f_{j_0}))\\ =&\lambda_{inv}\left(\frac{\widetilde{s}_{i_0}(f_{j_0})}{2} - 2f_{j_0}\right) - \left(\lambda_{pri} + \frac{N-2}{N-1}\lambda_{pub}\right)\Pr\left[J > \frac{\widetilde{s}_{i_0}(f_{j_0})}{2}\right]\mathrm{E}\left[J - \frac{\widetilde{s}_{i_0}(f_{j_0})}{2} + 2f_{j_0}\middle| J > \frac{\widetilde{s}_{i_0}(f_{j_0})}{2}\right].\end{aligned} \tag{7}$$

The spread $\widetilde{s}_{-i_0}(f_{j_0})$ is the solution to the equation

$$\begin{aligned}&\frac{1}{N-1}\lambda_{pub}\Pr\left[J > \frac{\widetilde{s}_{-i_0}(f_{j_0})}{2} + f_{j_0}\right]\mathrm{E}\left[J - \frac{\widetilde{s}_{-i_0}(f_{j_0})}{2} - f_{j_0}\middle| J > \frac{\widetilde{s}_{-i_0}(f_{j_0})}{2} + f_{j_0}\right]\\ =&\lambda_{inv}\left(\frac{\widetilde{s}_{-i_0}(f_{j_0})}{2} - f_{j_0}\right) - \left(\lambda_{pri} + \frac{N-2}{N-1}\lambda_{pub}\right)\Pr\left[J > \frac{\widetilde{s}_{-i_0}(f_{j_0})}{2} + f_{j_0}\right]\mathrm{E}\left[J - \frac{\widetilde{s}_{-i_0}(f_{j_0})}{2} + f_{j_0}\middle| J > \frac{\widetilde{s}_{-i_0}(f_{j_0})}{2} + f_{j_0}\right].\end{aligned} \tag{8}$$

In this equilibrium, the expected profit to Exchanges is

$$\pi_{j,X}(\mathcal{A}^t) = 0, \quad \forall j \in \{1, \ldots, M\}; \tag{9}$$

the expected profit to TFs is

$$\pi_{i_0,T}(\mathcal{A}^t) = \frac{1}{N}\frac{N-2}{N-1}\lambda_{pub}L(\widetilde{s}_N) \in \left(\frac{N-2}{N-1}\frac{\Pi^*_{ctn}}{N},\ \frac{\Pi^*_{ctn}}{N}\right) \tag{10}$$

for the lone-wolf TF i_0, and

$$\pi_{i,T}(\mathcal{A}^t) = \frac{1}{N}\lambda_{pub}L(\widetilde{s}_N) \in \left(\frac{\Pi^*_{ctn}}{N},\ \frac{N-1}{N-2}\frac{\Pi^*_{ctn}}{N}\right), \quad \forall i \in \{1, \ldots, N\} \setminus \{i_0\} \tag{11}$$

for all other TFs, where $\widetilde{s}_N$ is but a more convenient way to denote $\widetilde{s}_N(0)$.

Note that this situation is a special case of all cases where there are lone-wolf TF(s). The proof of this proposition is thus very similar to the proof of Lemma A.2 ("Lone-Wolf Lemma") in [3, Section A.2.2, pp.74]. The two propositions are different in that (i) the condition of $f = 0$ is given as preliminary in the said Lemma but it is a result of equilibrium here; (ii) the profit bound derived here is different from that in [3], and it will be shown that the one derived in this thesis is correct.

We first explain the meaning of the three kinds of spread that are important to this proposition.

Equation (6) is the breakeven condition for T_{i_0}. At any spread s greater than $\overline{s}_N$, lone-wolf TF T_{i_0} will be willing to provide liquidity, and it would prefer providing liquidity at this spread s and earn positive profit, if not providing liquidity leads to liquidity not provided on X_j $(j \neq j_0)$'s at all.

Solution $\widetilde{s}_{i_0}(f_{j_0})$ to equation (7) is the spread thatleave any non-lone-wolf TF indifferent between sniping T_{i_0} in fair exchanges at spread $\widetilde{s}_{i_0}(f_{j_0})$ and providing liquidity in the one unfair exchanges X_{j_0}

at spread $\widetilde{s}_{i_0}(f_{j_0}) - 2f_{j_0}$. To see this, note that the left hand side of equation (7) is the total profit that each T_i, $(i \neq i_0)$ can get from sniping T_{i_0} on X_j $(j \neq j_0)$, where the bid ask spread is $\widetilde{s}_{i_0}(f_{j_0})$ and all X_j $(j \neq j_0)$ has zero transaction fee. The right hand side is the expected net profit that some T_i $(i \neq i_0)$ can get being the sole liquidity provider on X_{j_0}, providing liquidity at spread $\widetilde{s}_{i_0}(f_{j_0}) - 2f_{j_0}$. The first term is the profit from investor, and the second is the loss from being sniped.

As for equation (8), it is the condition for T_i $(i \neq i_0)$'s to be indifferent between sniping and providing liquidity on X_{j_0}. The left hand side is the expected profit that each T_i, $(i \neq i_0)$ can get from sniping other liquidity providers on X_{j_0}, where the bid ask spread is $\widetilde{s}_{-i_0}(f_{j_0})$ and the transaction fee on X_{j_0} is f_{j_0}. On the right hand side, it is the expected net profit that each T_i, $(i \neq i_0)$ can get from being the sole liquidity provider on X_{j_0}, with trading fee f_{j_0} and bid-ask spread $\widetilde{s}_{-i_0}(f_{j_0})$. The solution to this equation thus leave T_i $(i \neq i_0)$'s indifferent between sniping and providing liquidity on X_{j_0}.

Proof. We first consider the deviation by some X_k, $k \neq j_0$ to post per share trading fee $f_k > 0$. This deviation is not profitable given the strategy of TFs and Investors. We then consider the deviation by X_{j_0} to post per share trading fee $f_{j_0} > 0$. First note that any $f_{j_0} > 0$ such that $\overline{s}_N < \widetilde{s}_{i_0}(f_{j_0}) < \widetilde{s}_{-i_0}(f_{j_0}) + 2f_{j_0}$ will not result in any profit for X_{j_0}, because in all circumstances liquidity will not be provided and traded on X_{j_0}. Now we show that this condition always holds and strategies under the condition that this relation does not hold will never be reached by changing $f_{j_0}$and thus does not affect the equilibrium.

Fix $f_{j_0} = f$, by (6), we have

$$\lambda_{inv}\frac{\overline{s}_N}{2} = \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L\left(\overline{s}_N\right).$$

By (7), we have

$$\begin{aligned}
\lambda_{inv}\frac{\widetilde{s}_{i_0}(f)}{2} =& 2\lambda_{inv} f + \frac{1}{N}\lambda_{pub}\Pr\left[J > \frac{\widetilde{s}_{i_0}(f)}{2}\right] \mathrm{E}\left[J - \frac{\widetilde{s}_{i_0}(f)}{2}\Big| J > \frac{\widetilde{s}_{i_0}(f)}{2}\right] \\
&+ \left(\lambda_{pri} + \frac{N-1}{N-2}\lambda_{pub}\right)\Pr\left[J > \frac{\widetilde{s}_{i_0}(f)}{2}\right]\left(\mathrm{E}\left[J - \frac{\widetilde{s}_{i_0}(f)}{2}\Big| J > \frac{\widetilde{s}_{i_0}(f)}{2}\right] + 2f\right) \\
&= \left(\lambda_{pri} + \frac{N(N-1)-1}{N(N-1)}\lambda_{pub}\right) L(\widetilde{s}_{i_0}(f)) + 2\left[\lambda_{inv} + \Pr\left[J > \frac{\widetilde{s}_{i_0}(f)}{2}\right]\left(\lambda_{pri} + \frac{N-2}{N-1}\lambda_{pub}\right)\right] f \\
&> \left(\lambda_{pri} + \frac{N(N-1)-1}{N(N-1)}\lambda_{pub}\right) L(\widetilde{s}_{i_0}(f)) \\
&> \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(\widetilde{s}_{i_0}(f)).
\end{aligned}$$

By (8), we have

$$\begin{aligned}
\lambda_{inv}\left(\frac{\widetilde{s}_{-i_0}(f)}{2} + f\right) &= 2\lambda_{inv} f + \frac{1}{N-1}\lambda_{pub} L(\widetilde{s}_{-i_0}(f) + 2f) \\
&+ \left(\lambda_{pri} + \frac{N-2}{N-1}\lambda_{pub}\right)\left(L(\widetilde{s}_{-i_0}(f) + 2f) + 2\Pr\left[J > \frac{\widetilde{s}_{-i_0}(f)}{2} + f\right] f\right) \\
&= (\lambda_{pri} + \lambda_{pub}) L(\widetilde{s}_{-i_0}(f) + 2f) + 2\left[\lambda_{inv} + \Pr\left[J > \frac{\widetilde{s}_{-i_0}(f)}{2} + f\right]\left(\lambda_{pri} + \frac{N-2}{N-1}\lambda_{pub}\right)\right] f \\
&> \left(\lambda_{pri} + \frac{N(N-1)-1}{N(N-1)}\lambda_{pub}\right) L(\widetilde{s}_{i_0}(f) + 2f) + 2\left[\lambda_{inv} + \Pr\left[J > \frac{\widetilde{s}_{i_0}(f)}{2} + f\right]\left(\lambda_{pri} + \frac{N-2}{N-1}\lambda_{pub}\right)\right] f
\end{aligned}$$

Since $\frac{\lambda_{inv}}{2}x$ is increasing with respect to x and $C \cdot L(x) + D$ is decreasing in x given exogenous variables C and D, the above equations determine unique $\overline{s}_N$, $\widetilde{s}_{i_0}(f_{j_0})$, $\widetilde{s}_{-i_0}(f_{j_0}) + 2f_{j_0}$ and by the pairwise comparison of the right hand side of the equations, we have

$$\overline{s}_N < \widetilde{s}_{i_0}(f) < \widetilde{s}_{-i_0}(f) + 2f, \quad \forall f.$$

Also, since s^*_{ctn} is just the solution to (8) when $f = 0$, $s^*_{ctn} < \widetilde{s}_{-i_0}(f),\ \forall f > 0$.

Now that the pre-game action is well justified, the rest of the proof of the equilibrium part of this proposition is the same as [3, pp.74].

We now proceed to the proof of profit in equilibrium.

By taking $f = 0$ in (7), and denote $\widetilde{s}_{i_0}(0)$ by $\widetilde{s}_N$ for illustration convenience, we have

$$\lambda_{inv}\frac{\widetilde{s}_N}{2} = \left(\lambda_{pri} + \frac{N(N-1)-1}{N(N-1)}\lambda_{pub}\right) L(\widetilde{s}_N). \tag{12}$$

Now, by providing liquidity at spread $\widetilde{s}_N$ on fair exchanges, the profit earned by the lone-wolf TF is

$$\pi_{i_0,T}(\mathcal{A}^t) = \lambda_{inv}\frac{\widetilde{s}_N}{2} - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(\widetilde{s}_N). \tag{13}$$

Plugging (12) in, we have

$$\pi_{i_0,T}(\mathcal{A}^t) = \left(\lambda_{pri} + \frac{N(N-1)-1}{N(N-1)}\lambda_{pub}\right) L(\widetilde{s}_N) - \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right) L(\widetilde{s}_N) \tag{14}$$

$$= \left(\frac{N(N-1)-1}{N(N-1)} - \frac{N-1}{N}\right) \lambda_{pub} L(\widetilde{s}_N) \tag{15}$$

$$= \frac{1}{N}\frac{N-2}{N-1}\lambda_{pub} L(\widetilde{s}_N). \tag{16}$$

Now since the function $L(\cdot)$ is decreasing, and $\widetilde{s}_N < s^*_{ctn}$, we have

$$\pi_{i_0,T}(\mathcal{A}^t) \in \left(\frac{N-2}{N-1}\frac{\Pi^*_{ctn}}{N},\ \frac{\Pi^*_{ctn}}{N}\right).$$

The lower bound is derived from (14) and the upper bound from (13). Now the profit that all the other $N-1$ none-lone-wolf T_i's profit from sniping T_{i_0} is

$$\pi_{-i_0,T}(\mathcal{A}^t) = \frac{1}{N}\lambda_{pub} L(\widetilde{s}_N).$$

Then,

$$\pi_{-i_0,T}(\mathcal{A}^t) \in \left(\frac{\Pi^*_{ctn}}{N},\ \frac{N-1}{N-2}\frac{\Pi^*_{ctn}}{N}\right).$$

□

Remark 2.10. *There are three points to notice for this proposition.*

1. *In particular, when $\mathcal{A}^t_{i,T} = \mathcal{G}_{i,T},\ \forall i \neq i_0$, for all i,*

$$\pi_{i,T}(\mathcal{G} \setminus \{i,j\}) = \frac{1}{N}\frac{N-2}{N-1}\lambda_{pub} L(\widetilde{s}_N) \in \left(\frac{N-2}{N-1}\frac{\Pi^*_{ctn}}{N},\ \frac{\Pi^*_{ctn}}{N}\right), \quad \forall j; \tag{17}$$

$$\pi_{k,T}(\mathcal{G} \setminus \{i,j\}) = \frac{1}{N}\lambda_{pub} L(\widetilde{s}_N) \in \left(\frac{\Pi^*_{ctn}}{N},\ \frac{N-1}{N-2}\frac{\Pi^*_{ctn}}{N}\right), \quad \forall k \neq i,\ \forall j; \tag{18}$$

and

$$\pi_{h,X}(\mathcal{G} \setminus \{i,j\}) = 0, \quad \forall h \in \{1, \dots, M\},\ \forall (i,j). \tag{19}$$

We denote

$$\pi^{LW}_N \equiv \pi_{i,T}(\mathcal{G} \setminus \{i,j\}) = \frac{1}{N}\frac{N-2}{N-1}\lambda_{pub} L(\widetilde{s}_N)$$

from now on.

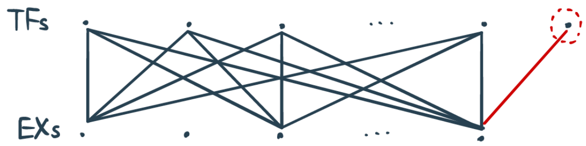
2. *The expected profit for the lone-wolf TF is strictly less than the expected profit it could have been collected if it made another purchase to make all Exchanges fair, while the expected profit for all other TFs is greater than the amount they can get if all Exchanges are fair. By exerting the threat to offer liquidity only on the unfair Exchange X_{j_0}, the none-lone-wolf TFs gain more power than the lone-wolf and manage to gain more from the game. The Exchange whose existence enables this threat gain profit only from selling ESST to all none-lone-wolf TFs through the bargaining game to share the surplus.*

3. *Now expected profit earned by all TFs is*

$$\begin{aligned}\pi_{i_0,T}(\mathcal{A}^t) + (N-1)\pi_{-i_0,T}(\mathcal{A}^t) &= \frac{1}{N}\frac{N-2}{N-1}\lambda_{pub}L(\widetilde{s}_N) + \frac{N-1}{N}\lambda_{pub}L(\widetilde{s}_N) \\ &= \frac{N(N-1)-1}{N(N-1)}\lambda_{pub}L(\widetilde{s}_N) \\ &< \lambda_{pub}L(\widetilde{s}_N).\end{aligned}$$

This fact is surprising because $\lambda_{pub}L(\widetilde{s}_N)$ is the gross latency arbitrage prize that is created by TFs, and no profit has gone anywhere else - exchanges realizing positive trading fee, for instance. Nevertheless, the total profit that all TFs gain is strictly less than the gross prize they create, in fact, exactly a portion of $\frac{1}{N(N-1)}$ of the prize is lost. It could be an interesting topic to investigate the reason for this loss, and the possibility of further reducing this total profit by further diversification in the portfolio of each TF's ESST providers, to allow for more lone-wolves and more room for check among them.

We now investigate the equilibrium where there is still only one lone-wolf TF and the number of exchanges with which it does not has ESST provider-purchaser relationship is more than 1.



Proposition 2.11. *Assume $M \geqslant 3$ and $N \geqslant 3$. For any time t such that $\mathcal{A}^t_{i,T}$ contains the same set of exchanges for all $i \neq i_0$, $\mathcal{A}^t_{i_0,T} \subset \mathcal{A}^t_{i,T}$ is the proper subset, and $|\mathcal{A}^t_{i_0,T}| \in [1, M-2]$, for any vector of market shares $\boldsymbol{\sigma}^* = (\sigma^*_1, \ldots, \sigma^*_M)$ where $\sum_j \sigma^*_j = 1$, there exists an equilibrium of the trading game exactly the same as the one above.*

Proof is similar to the one for proposition 2.9. It is actually much simpler because now the action by exchanges to post zero trading fee is justified by the price competition among themselves. From this we have that in particular (when $\mathcal{A}^t_{i,T} = \mathcal{G}_{i,T}$, $\forall\, i \neq i_0$), $\forall\, i$,

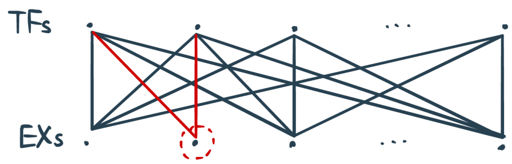
$$\pi_{i,T}(\mathcal{G}\setminus\mathcal{A}) = \frac{1}{N}\frac{N-2}{N-1}\lambda_{pub}L(\widetilde{s}_N) \in \left(\frac{N-2}{N-1}\frac{\Pi^*_{ctn}}{N},\ \frac{\Pi^*_{ctn}}{N}\right),\ \forall \mathcal{A} \subseteq \mathcal{G}_{i,T}. \tag{20}$$

$$\pi_{k,T}(\mathcal{G}\setminus\mathcal{A}) = \frac{1}{N}\lambda_{pub}L(\widetilde{s}_N) \in \left(\frac{\Pi^*_{ctn}}{N},\ \frac{N-1}{N-2}\frac{\Pi^*_{ctn}}{N}\right), \quad \forall k \neq i,\ \forall \mathcal{A} \subseteq \mathcal{G}_{i,T}; \tag{21}$$

and

$$\pi_{h,X}(\mathcal{G} \setminus \mathcal{A}) = 0, \quad \forall\, h \in \{1, \ldots, M\},\ \forall \mathcal{A} \subseteq \mathcal{G}_{i,T}. \tag{22}$$

Last, we consider a situation where there the lone-wolf is not TF but Exchange. Consider the set of relationship where there is one lone-wolf exchange for which the number of formed relationship with TFs is greater than or equal to 1 but less than the number connections all other exchanges.



Proposition 2.12. *Assume $M \geqslant 3$, $N \geqslant 3$. For any time t such that $\mathcal{A}^t_{j,X}$ contains the same set of TFs for all $j \neq j_0$, $\mathcal{A}^t_{j_0,X} \subset \mathcal{A}^t_{j,X}$ is a proper subset, and $|\mathcal{A}^t_{j_0,X}| \in [2, N-2]$, for any vector of market shares $\boldsymbol{\sigma}^* = (\sigma^*_1, \ldots, \sigma^*_M)$ where $\sum_j \sigma^*_j = 1$, there exists an equilibrium of the trading game specified as follows:*

***Pre-game**: All X_j's simultaneously post per share trading fees $f^*_j = 0$;*

***Session 1**: Define $N_{j_0} = |\mathcal{A}^t_{j_0,X}|$. With $\widetilde{s}_{N,N_{j_0}}(f_{j_0})$ defined as below, T_i's which do not purchase ESST from X_{j_0} collectively provides $\frac{\sigma^*_j}{\sum_{j \in \arg\min_k f_k, j \neq j_0} \sigma^*_k}$ of liquidity on X_j $(j \neq j_0)$'s at spread $\widetilde{s}_{N,N_{j_0}}(f_{j_0})$ around y_{t^-}, and no liquidity is provided on X_{j_0}. There may be arbitrary units of x offered at an arbitrary spread that is out of J's support on any arbitrary Exchange(s).*

***Session 2**:*

- *When an investor arrives, it immediately sends orders for $\frac{\sigma^*_j}{\sum_{j \in \arg\min_k \left(f_k + \frac{s_k}{2}\right)} \sigma^*_k}$ units of x to X_j at marketable price in that exchange, $\forall\, j$ such that $j \neq j_0$; or if liquidity is only provided on X_{j_0}, then investor sends order for 1 unit of x to X_{j_0} at marketable price.*

- *If $J > \frac{s}{2} + f_j$ happens for some $s \geqslant 0$ and some j, and is observed by a single Informed Trader, the Informed Trader immediately sends orders to sell infinite units of x to X_j at $y_{t^-} - \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (-\infty, -\frac{s}{2} - f_j)$, or to buy infinite units of x to X_j at $y_{t^-} + \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (\frac{s}{2} + f_j, \infty)$.*

- *If $J > \frac{s}{2} + f_j$ happens for some $s \geqslant 0$ and some j, and is observed by all players, all TFs, regardless of speed group, immediately send orders to sell infinite units of x to X_j at $y_{t^-} - \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (-\infty, -\frac{s}{2} - f_j)$, or to buy infinite units of x to X_j at $y_{t^-} + \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (\frac{s}{2} + f_j, \infty)$; all TFs with outstanding orders on any Exchange with ask prices $< y_{t^+} - f_j$ or bid prices $> y_{t^+} + f_j$ immediately send cancellation messages for all such orders.*

After the above messages are processed, and y_{t^+} observed by all players, they simultaneously send cancellation messages for all orders that they have post earlier and still remain on the order book of any Exchange.

The spread $\widetilde{s}_{N,N_{j_0}}(f_{j_0})$ *is the solution to the equation*

$$\begin{aligned}&\frac{1}{N}\lambda_{pub}L(\widetilde{s}_{N,N_{j_0}}(f_{j_0}))\\=&\lambda_{inv}\left(\frac{\widetilde{s}_{N,N_{j_0}}(f_{j_0})}{2}-2f_{j_0}\right)-\left(\lambda_{pri}+\frac{N_{j_0}-1}{N_{j_0}}\lambda_{pub}\right)\Pr\left[J>\frac{\widetilde{s}_{N,N_{j_0}}(f_{j_0})}{2}\right]\mathrm{E}\left[J-\frac{\widetilde{s}_{i_0}(f_{j_0})}{2}+2f_{j_0}\Big|J>\frac{\widetilde{s}_{N,N_{j_0}}(f_{j_0})}{2}\right].\end{aligned} \tag{23}$$

In this equilibrium, the expected profit to Exchanges is

$$\pi_{j,X}(\mathcal{A}^t)=0,\quad \forall j\in\{1,\ldots,M\}; \tag{24}$$

the expected profit to TFs is

$$\pi_{i,T}(\mathcal{A}^t)=\begin{cases}\frac{2N_{j_0}-N}{N_{j_0}}\lambda_{pub}L(\widetilde{s}_{N,N_{j_0}}), & T_i \text{ does not purchase ESST from } X_{j_0};\\ \frac{1}{N}\lambda_{pub}L(\widetilde{s}_{N,N_{j_0}}), & \text{otherwise},\end{cases} \tag{25}$$

where $\widetilde{s}_{N,N_{j_0}}\equiv\widetilde{s}_{N,N_{j_0}}(0)$.

Proof of this is again similar to proof of proposition 2.9. The only point that needs more justification is that despite that in this situation providing liquidity is actually less profitable than sniping on the same set of exchanges, given the formed equilibrium, none of the TFs which do not purchase ESST from X_{j_0} would withdraw their liquidity provision. This is justified by the fact that $\widetilde{s}_{N,N_{j_0}}>\overline{s}_N$.

Now note that for any set of relationships formed, at equilibrium, all Exchanges will post zero trading fee. This is true because positive trading fee would not only increase the cost to provide liquidity, but also the cost to snipe. For any j, either it is fair, or it is unfair. If it is fair and other exchanges are also fair, then it has to post zero trading fee because of Bertrand competition; if it is fair and other exchanges are unfair, then trading on X_j already costs more than trading on other exchanges, and posting positive trading fee would not encourage any TF to provide liquidity on it. If X_j is unfair and all other exchanges are fair, then according to proposition 2.9 it will post zero trading fee. In fact, in this case, the advantage of the unfairness is taken by the lone-wolf TF by offering narrower bid-ask spread, and there is no surplus left for the exchange. This can also generalize to the situation where X_j is unfair, there are other unfair exchanges, but trading on them are more expensive than trading on X_j. If X_j is unfair, there are other unfair exchanges, and trading on them are the same or less expensive than trading on X_j, it is also not profitable to charge positive trading fee because again this will only increase the cost to trade on it.

2.5 Equilibrium of the Bargaining Game

Now we are ready to derive all the marginal contribution functions that is needed in the bargaining model.

$$\Delta\pi_{i,T}(\mathcal{G},\mathcal{A})=\frac{1}{N}\lambda_{pub}\left[L(s^*_{ctn})-\frac{N-2}{N-1}L(\widetilde{s}_N)\right]=\frac{\Pi^*_{ctn}}{N}-\pi_N^{LW},\quad \forall\mathcal{A}\subseteq\mathcal{G}_{i,T},\ \mathcal{A}\neq\mathcal{G}_{i,T};$$

$$\Delta\pi_{i,T}(\mathcal{G},\mathcal{G}_{i,T})=\frac{1}{N}\lambda_{pub}L(s^*_{ctn})=\frac{\Pi^*_{ctn}}{N};$$

$$\Delta\pi_{j,X}(\mathcal{G},\mathcal{A})=0,\quad\forall\mathcal{A}\subseteq\mathcal{G}_{j,X}.$$

With these flow profit functions well defined, we have the Nash-in-Nash price in the particular bargaining game we defined in this thesis being

$$\begin{aligned} F_{ij}^{Nash} =& \frac{b_{j,X}\Delta\pi_{i,T}(\mathcal{G},\{i,j\}) - b_{i,T}\Delta\pi_{j,X}(\mathcal{G},\{i,j\})}{b_{i,T}+b_{j,X}} \\ =& \frac{b_{j,X}}{b_{i,T}+b_{j,X}}\frac{1}{N}\lambda_{pub}\left[L(s^*_{ctn}) - \frac{N-2}{N-1}L(\widetilde{s}_N)\right]. \end{aligned}$$

Assuming that the TFs and the Exchanges are facing same risk-free rate, we have $b_{j,X} = b_{i,T}, \forall\ i,\ j$. Denoting this Nash-in-Nash price by F_j^{Nash}, then

$$F_j^{Nash} = \frac{1}{2}\frac{1}{N}\lambda_{pub}\left[L(s^*_{ctn}) - \frac{N-2}{N-1}L(\widetilde{s}_N)\right].$$

We go on to check whether the assumptions raised in [6] are satisfied in this specific setting, which generally are useful when combined in particular ways for existence of an equilibrium in the bargaining game .

i. The assumption A.GFT (Gains from trade) is satisfied.

$$\Delta\pi_{i,T}(\mathcal{G},\{i,j\}) + \Delta\pi_{j,X}(\mathcal{G},\{i,j\}) = \frac{1}{N}\lambda_{pub}\left[L(s^*_{ctn}) - \frac{N-2}{N-1}L(\widetilde{s}_N)\right] > 0, \quad \forall\{i,j\}\in\mathcal{G}.$$

ii. For Exchanges, the assumption A.SDCMC (strong conditional decreasing marginal contribution) is satisfied. And thus by Lemma 3.3 [6], A.WCDMC (weak conditional decreasing marginal contribution) and A.FEAS (feasibility) are also satisfied.

A.SCDMC is satisfied as for all $\{i,j\}\in\mathcal{G},\ \mathcal{B}\subseteq\mathcal{G}_{-i,T}$ and $\mathcal{A},\mathcal{A}'\subseteq\mathcal{G}_{i,T}\setminus\{i,j\}$,

$$\pi_{j,X}(\mathcal{A}\cup\mathcal{B}\cup\{i,j\}) - \pi_{j,X}(\mathcal{A}'\cup\mathcal{B}) = 0 = \Delta\pi_{j,X}(\mathcal{G},\{i,j\}.$$

iii. For TFs, A.WCDMC is not satisfied, and thus by Lemma 3.3 [6], A.SCDMC is not satisfied either. Fix i, note that for $\mathcal{A} = \{\{i,j\},\{i,k\}\}\subseteq\mathcal{G}_{i,T}$,

$$\Delta\pi_{i,T}(\mathcal{G},\mathcal{A}) = \frac{1}{N}\lambda_{pub}\left[L(s^*_{ctn}) - \frac{N-2}{N-1}L(\widetilde{s}_N)\right],$$

$$\Delta\pi_{i,T}(\mathcal{G},\{i,j\}) + \Delta\pi_{i,T}(\mathcal{G},\{i,k\}) = \frac{2}{N}\lambda_{pub}\left[L(s^*_{ctn}) - \frac{N-2}{N-1}L(\widetilde{s}_N)\right] > \Delta\pi_{i,T}(\mathcal{G},\mathcal{A}).$$

iv. For TFs, A.FEAS is satisfied[11] if and only if

$$\sum_{j:\ \sigma_j^*>0} F_j^{Nash} \leqslant \frac{\Pi^*_{ctn}}{N} - \max\{0,\ \pi_N^{LW} - \min_j F_j^{Nash}\}. \tag{26}$$

Proof. ((26)⇒A.FEAS for TFs) For any i, $\forall\mathcal{A}\subset\mathcal{G}_{i,T}$, assume (26), then

$$\frac{\Pi^*_{ctn}}{N} - \sum_{j:\ \sigma_j^*>0} F_j^{Nash} \geqslant \max\{0,\ \pi_N^{LW} - \min_j F_j^{Nash}\}.$$

[11] The sign for Nash-in-Nash price in A.FEAS condition for our setting is inverse of the original A.FEAS condition, since the payment direction is reversed heret.

That is,

$$\frac{\Pi^*_{ctn}}{N} - \sum_{j:\ \sigma^*_j>0} F_j^{Nash} \geqslant 0, \quad \text{and} \quad \frac{\Pi^*_{ctn}}{N} - \sum_{j:\ \sigma^*_j>0} F_j^{Nash} \geqslant \pi_N^{LW} - \min_j F_j^{Nash}.$$

Now from the first inequality

$$\frac{\Pi^*_{ctn}}{N} - \sum_{j:\ \sigma^*_j>0} F_j^{Nash} \geqslant 0 \ \Leftrightarrow\ \Delta\pi_{i,T}(\mathcal{G}, \mathcal{G}_{i,T}) \equiv \frac{\Pi^*_{ctn}}{N} - 0 \geqslant \sum_{\{i,j\}\in\mathcal{G}_{i,T}} F_j^{Nash};$$

and from the second inequality we have

$$\frac{\Pi^*_{ctn}}{N} - \pi_N^{LW} \geqslant \sum_{j:\ \sigma^*_j>0} F_j^{Nash} - \min_j F_j^{Nash},$$

where the right-hand-side is the sum of the largest $M-1$ items in $\{F_j\}_{j\in\mathcal{G}_{i,T}}$, and thus

$$\Delta\pi_{i,T}(\mathcal{G}, \mathcal{A}) \equiv \frac{\Pi^*_{ctn}}{N} - \pi_N^{LW} \geqslant \sum_{\{i,j\}\in\mathcal{A}} F_j^{Nash},\ \forall\, \mathcal{A} \subset \mathcal{G}_{i,T},\ \mathcal{A} \neq \mathcal{G}_{i,T}.$$

Combine the two derived inequalities, we have that for any i and any $\mathcal{A} \subset \mathcal{G}_{i,T}$

$$\Delta\pi_{i,T}(\mathcal{G}, \mathcal{A}) \geqslant \sum_{\{i,j\}\in\mathcal{A}} F_j^{Nash}.$$

(A.FEAS for TFs $\Rightarrow$ (26)) Fix i. Since $\mathcal{G}_{i,T} \subseteq \mathcal{G}_{i,T}$, by A.FEAS we have

$$\frac{\Pi^*_{ctn}}{N} = \Delta\pi_{i,T}(\mathcal{G}, \mathcal{G}_{i,T}) \geqslant \sum_{\{i,j\}\in\mathcal{G}_{i,T}} F_j^{Nash} = \sum_{j:\ \sigma^*_j>0} F_j^{Nash}.$$

Also, take $j_0 = \arg\min_j F_j^{Nash}$ (this in fact can be any j in our setting), since $\mathcal{G}_{i,T} \setminus \{i, j_0\} \subseteq \mathcal{G}_{i,T}$, by A.FEAS we have

$$\begin{aligned}
\frac{\Pi^*_{ctn}}{N} - \pi_N^{LW} =& \frac{1}{N}\lambda_{pub}\left[L(s^*_{ctn}) - \frac{N-2}{N-1}L(\widetilde{s}_N)\right] \\
=& \Delta\pi_{i,T}(\mathcal{G}_{i,T}, \mathcal{G}_{i,T} \setminus \{i, j_0\}) \\
\geqslant& \sum_{j\neq j_0} F_j^{Nash} \\
=& \sum_{j:\ \sigma^*_j>0} F_j^{Nash} - \min_j F_j^{Nash}.
\end{aligned}$$

Therefore,

$$\frac{\Pi^*_{ctn}}{N} - \sum_{j:\ \sigma^*_j>0} F_j^{Nash} \geqslant 0, \quad \text{and} \quad \frac{\Pi^*_{ctn}}{N} - \sum_{j:\ \sigma^*_j>0} F_j^{Nash} \geqslant \pi_N^{LW} - \min_j F_j^{Nash},$$

and hence (26). □

Recall that equilibrium existence result in [6] says that (i) when A.GFT holds, A.FEAS holds for one side of the market, and A.SCDMC holds for the other side of the market, then "there exists a Nash-in-Nash limit equilibrium in which all agreements in $\mathcal{G}$ immediately form" ([6, theorem 3.4, pp.180]); and (ii) if A.FEAS does not hold, then "a Nash-in-Nash limit equilibrium in which all agreements in $\mathcal{G}$ immediately form does not exist" ([6, theorem 3.2, pp.180]).

Then according to the what we have checked above, it can be argued that given the setting of the Trading Game, there exists a Nash-in-Nash limit equilibrium in which all TFs purchase ESST from all Exchanges immediately, if and only if (26) holds. Note that (26) is also the sufficient condition for there to exist an equilibrium of the multi-exchange game in [3, proposition 3.2, pp.23]. Then in the sense of equilibrium existence condition, the two game settings are equivalent.

Now note that if plugging in the Nash-in-Nash price, the first condition among the two implied by 26 (and thus by A.FEAS) translates into

$$\frac{L(\widetilde{s}_N)}{L(s^*_{ctn})} \geqslant \frac{M-2}{M}\frac{N-1}{N-2},$$

which always holds by definition of $L(\cdot)$ and its decreasing property; the second condition translates into $\frac{M-1}{2} \leqslant 1 \Leftrightarrow M \leqslant 3$. The necessary condition of existence of an immediate all-relationship-formed equilibrium is satisfied only when the number of exchanges is no greater than three. That is, if there are more than three exchanges in the market, at least under the bargaining game setting, an equilibrium where all TFs purchase ESST from all Exchanges immediately does not exist.

3 'Nash-in-Nash' Bargaining with Existence of Discrete Exchange(s)

With everything else stay the same as the model described above, a discrete exchange is one that takes on the frequent batch auction (FBA) rules instead of the CLOB rules. The exact description for FBA rules can be found in [3, Section 5.1.1, pp.48]. The major difference in setting is that in the FBA exchange, orders are not processed serially (continuously), but in batches (discretely) every small interval. This interval is small enough to distinguish TFs in slow, fast, and ESST speed categories, but large enough to ensure that all messages sent at the same time point by fast TFs with ESST of this exchange can be processed in the same batch. This ensures that when a public jump happens, liquidity provider attempts to cancel its stale quote while other TFs attempts to snipe, the cancellation can always succeed (jumps that can be observed by only one TF is not relevant). Therefore, in an exchange with such rules, there will be no gain from public jump and arbitrage.

With the bargaining approach, the rent devision can be viewed in a more clear-cut manner with the trading game itself. It then follows naturally that with existence of one or more discrete exchange(s), the ESST fee, which is the slice of the arbitrage rent pie that exchanges get, will be zero. To show this, we still need several profit functions for TFs and Exchanges.

Definition 3.1. *Let $s^*_{dis}(f)$ be the solution to the following equation*

$$\lambda_{inv}\left(\frac{s}{2}-f\right) - \lambda_{pri}\Pr\left[J > \frac{s}{2}+f\right]\mathrm{E}\left[J-\frac{s}{2}+f\middle|J>\frac{s}{2}+f\right] = 0,$$

and in particular, denote $s^*_{dis}(0)$ by s^*_{dis}.

Proposition 3.2. *Consider a situation where there is only one discrete exchange, let this exchange be X_{j_0}. Let there be at least one continuous exchange. Then at any time point t, with arbitrary set of rela-*

tionship formed by X_j and the continuous excahgnes, for any vector of market shares $\boldsymbol{\sigma}^* = (\sigma_1^*, \ldots, \sigma_M^*)$ where $\sum_j \sigma_j^* = 1$, there exists an equilibrium of the trading game specified as follows:

***Pre-game**: X_j('s) simultaneously post(s) per share trading fee $f_j^* = 0, \forall j \neq j_0$; X_{j_0} posts per share trading fee $f_{j_0,dis}^* > 0$ such that*

$$s_{dis}^* < s_{dis}^*(f_{j_0,dis}^*) + 2f_{j_0,dis}^* < \overline{s}_N, \tag{27}$$

where $\overline{s}_N$ is defined as in 6.

***Session 1**: 1 unit of liquidity is provided on X_{j_0} at spread $s_{dis}^*(f_{j_0,dis}^*)$ around y_{t^-}. There may be arbitrary units of x offered at arbitrary spread that is out of J's support on arbitrary exchange(s).*

***Session 2**:*

- *When an investor arrives, it immediately sends orders for $\frac{\sigma_j^*}{\sum_{j \in \arg\min_h(\frac{s_h}{2}+f_h)} \sigma_j^*}$ unit of x to X_j at marketable price in that exchange, $\forall j$ such that $j \in \arg\min_h(\frac{s_h}{2} + f_h)$.*

- *If $J > \frac{s}{2} + f_k$ happens for some $s \geqslant 0$ and some k, and is observed by a single Informed Trader, the Informed Trader immediately sends orders to sell infinite units of x to X_k at $y_{t^-} - \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (-\infty, -\frac{s}{2} - f_k)$, or to buy infinite units of x to X_k at $y_{t^-} + \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (\frac{s}{2} + f_k, \infty)$.*

- *If $J > \frac{s}{2} + f_k$ happens for some $s \geqslant 0$ and some k, and is observed by all players, all TFs, regardless of speed group, immediately send orders to sell infinite units of x to X_k at $y_{t^-} - \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (-\infty, -\frac{s}{2} - f_k)$, or to buy infinite units of x to X_k at $y_{t^-} + \frac{s}{2}$ when $y_{t^+} - y_{t^-} \in (\frac{s}{2} + f_k, \infty)$; all TFs with outstanding orders on any Exchange with ask prices $< y_{t^+} - f_k$ or bid prices $> y_{t^+} + f_k$ immediately send cancellation messages for all such orders.*

After the above messages are processed, and y_{t^+} observed by all players, they simultaneously send cancellation messages for all orders that they have post earlier and still remain on the order book of any Exchange.

In this equilibrium, the expected profit to Exchanges is

$$\pi_{j_0,X}(\mathcal{A}^t) = 2f_{j_0,dis}^* \times (\lambda_{inv} + \lambda_{pri}) > 0; \tag{28}$$

$$\pi_{j,X}(\mathcal{A}^t) = 0, \quad \forall j \in \{1, \ldots, M\} \setminus \{j\}; \tag{29}$$

and the profit to TFs is

$$\pi_{i,T}(\mathcal{A}^t) = 0, \quad \forall i \in \{1, \ldots, N\}. \tag{30}$$

We show proof that there exist some $f_{j,dis}^*$ such that the key inequality (27) for this proposition holds.

Proof. By 3.1, we have that

$$\lambda_{inv} \frac{s_{dis}^*(0)}{2} = \lambda_{pri} L(s_{dis}^*(0)),$$

and that for any $f > 0$,

$$\lambda_{inv} \left(\frac{s_{dis}^*(f)}{2} - f \right) = \lambda_{pri} \Pr\left[J > \frac{s_{dis}^*(f)}{2} + f \right] \mathrm{E}\left[J - \frac{s_{dis}^*(f)}{2} + f \Big| J > \frac{s_{dis}^*(f)}{2} + f \right]$$

$$\Rightarrow \frac{\lambda_{inv}}{2}\left(s^*_{dis}(f)+2f\right) = \lambda_{pri}\Pr\left[J > \frac{s^*_{dis}(f)}{2}+f\right]\mathrm{E}\left[J - \frac{s^*_{dis}(f)}{2} - f \Big| J > \frac{s^*_{dis}(f)}{2}+f\right]$$
$$+ \lambda_{pri}\Pr\left[J > \frac{s^*_{dis}(f)}{2}+f\right]\mathrm{E}\left[2f \Big| J > \frac{s^*_{dis}(f)}{2}+f\right] + 2\lambda_{inv}f$$
$$= \lambda_{pri}L\left(s^*_{dis}(f)+2f\right) + 2\left(\lambda_{pri}\Pr\left[J > \frac{s^*_{dis}(f)}{2}+f\right] + \lambda_{inv}\right)f$$
$$> \lambda_{pri}L\left(s^*_{dis}(f)+2f\right)$$

Compare these two equations, by decreasing property of the function $L(\cdot)$, we have that

$$s^*_{dis}(0) < s^*_{dis}(f) + 2f, \quad \forall f > 0.$$

Similarly, since

$$\lambda_{inv}\frac{\overline{s}_N}{2} = \left(\lambda_{pri} + \frac{N-1}{N}\lambda_{pub}\right)L\left(\overline{s}_N\right) > \lambda_{pri}L\left(\overline{s}_N\right),$$

we have

$$s^*_{dis}(0) < \overline{s}_N.$$

Now by continuity of the function $s^*_{dis}(f) + 2f$ with respect to f, there exists some $f > 0$, such that

$$s^*_{dis}(0) < s^*_{dis}(f) + 2f < \overline{s}_N.$$

□

Note that this is the equilibrium for any t such that $\mathcal{A}^t = \mathcal{G} \setminus \mathcal{A}$, $\forall \mathcal{A} \subseteq \mathcal{G}_{j,X}$, and that f^*_{dis} is not dependent on $\mathcal{A}^t$. That is, for X_{j_0} being the discrete exchange,

$$\Delta\pi_{j_0,X}(\mathcal{G},\mathcal{A}) = 0, \quad \forall \mathcal{A} \subseteq \mathcal{G}_{j_0,X}.$$

and it is more obvious for X_j the continuous exchanges,

$$\Delta\pi_{j,X}(\mathcal{G},\mathcal{A}) = 0, \quad \forall \mathcal{A} \subseteq \mathcal{G}_{j,X};$$

and for TFs,

$$\Delta\pi_{i,T}(\mathcal{G},\mathcal{A}) = 0, \quad \forall i \in \{1,\dots,N\},\ \forall \mathcal{A} \subseteq \mathcal{G}_{i,T}$$

since no TF can get any positive expected profit from any continuous exchange (no liquidity will be taken from these exchanges) and neither can they get any positive expected profit from the discrete exchange (by definition 3.1).

Now it is easy to check that A.WCDMC holds for both TFs and Exchanges. For all $i \in \{1,\dots,N\}$, for all $\mathcal{A} \subseteq \mathcal{G}_{i,T}$,

$$\sum_{\{i,k\}\in\mathcal{A}} \Delta\pi_{i,T}(\mathcal{G},\{i,k\}) = 0 = \Delta\pi_{i,T}(\mathcal{G},\mathcal{A});$$

For all $j \in \{1,\dots,M\}$, for all $\mathcal{A} \subseteq \mathcal{G}_{j,X}$

$$\sum_{\{h,j\}\in\mathcal{A}} \Delta\pi_{j,X}(\mathcal{G},\{h,j\}) = 0 = \Delta\pi_{j,X}(\mathcal{G},\mathcal{A}).$$

Therefore, in the Bargaining Game, there exists an equilibrium where all TFs purchase ESST from all Exchanges at the Nash-in-Nash price

$$F_k = \frac{b_{j,X}\Delta\pi_{i,T}(\mathcal{G},\{i,j\}) - b_{i,T}\Delta\pi_{j,X}(\mathcal{G},\{i,j\})}{b_{i,T} + b_{j,X}} = 0.$$

By being effectively a monopoly, the discrete Exchange is able to extract all profit from the private information arbitrage by charging per share trading fee, which rooted from the bid-ask spread charged by TFs (the bid-ask spread is still strictly greater than zero as the private jumps continue to exist).

The situation with more than one discrete exchanges will have an equilibrium very similar to the above, with the per share trading fee post by discrete exchanges also zero, driven by Bertrand competition among the discrete exchanges, and no party earning positive expected profit. The ESST fee as a result of the Bargaining Game is obviously still zero.

4 Empirical Validation in Japanese Stock Market

In this section, we validate some of the seven stylized facts documented in [3, Section 4, pp.29] for Japanese equity market. We first note that it is after January 2010 that millisecond-level HFT has been made possible on TSE. However, our analysis will not only be focused on the period after that because we are interested in how these facts hold in different situations. We now point out some important differences and similarities between the US and Japanese equity market.

First, the Japanese stock market is not as well fragmented as the US stock market. After JASDAQ being acquired by Osaka Stock Exchange (OSE) in 2010 and the merger of OSE and Tokyo Stock Exchange (TSE) in 2013, there is effectively four equity exchange groups in Japan. Apart from Japan Exchange Group (JPX), there are Sapporo Stock Exchange (SSE), Nagoya Stock Exchange (NSE), and Fukuoka Stock Exchange (FSE). The equity markets that has existed are listed in Table 2, where the indents show the subsidiary relationships among them (for OSE and JASDAQ, the subsidiary relationship before OSE acquired JASDAQ is shown).

Second, there is no regulation in the Japanese equity market that ensures satisfaction of A. Accessibility and A. Fungibility. Many companies are only listed on a single exchange and can only be traded there. To put this in the context of the model, for almost all securities, the number of exchanges where they can be traded (i.e. M) equals to 1, and for some, $M = 2$. This means there will not be competition in the sense described in the model in Japan equity market.

Third, all exchanges use CLOB design for normal operation hours and auction rules for hour before market open and some other extreme situations. Trading rules in all four exchange groups treat orders with price priority first, and time priority when there are ties. However, the method of breaking ties when there are orders made at same time with same price is different from what is specified in the model (breaking ties randomly). In all four exchange groups, it is specified that when there are simultaneous orders at the same price, the orders will be put in sequence, grouped by trading firms, according to the total number of orders placed by the trading firm, from orders by the trading firm which placed the most orders, to those by trading firms which placed the least orders. If there is still a tie (trading firms place same number of orders), then orders that are received by the exchange system earlier will be put further in the queue. And then, counter-party orders will be allocate to these trading firms in that sequence, one unit at a time, until all counter-party orders are allocated [11, 19, 29, 30, 37]. This rule is effectively equivalent with the random tie-breaking rule in the model, though. With this allocation rule,

Table 2. Exchanges and Boards in Japan

Exchange (group)	Abbreviated
Japan Exchange Group	JPX
1st Section	TS1
2nd Section	TS2
Mothers	TMO
TOKYO PRO Market	TKP
JASDAQ Standard	JQS
JASDAQ Growth	JQG
Fukuoka Stock Exchange	FSE
Nagoya Stock Exchange	NSE
1st Section	NS1
2nd Section	NS2
Centrix	NCT
Sapporo Stock Exchange	SSE
Sapporo Stock Exchange Main Market	SSM
Ambitious	SSA
Osaka Stock Exchange	OSE
1st Section	OS1
2nd Section	OS2
Hercules Standard	OHS
Hercules Growth	OHG
NEO	OSN
JASDAQ	JDQ

TFs, when trying to snipe, will place the largest possible number of orders to get the number-of-order priority, resulting in all TFs place same number of orders. Then it comes back to the situation where ties are broken almost randomly for all TFs with the same speed technology. Lastly, the allocation method ensures that each TF can snipe approximately $\frac{1}{N}$ of the stale quote with error within ± 1 unit, where N is the total number of TFs. This $\frac{1}{N}$ of the stale quote is an expectation in the model, but a definite number under the Japan trading rules. Therefore, the trading rules in Japan equity market is effectively equivalent with the CLOB design specified in the model.

4.1 Exchange Market Shares in the Whole Market

We will now validate the aggregate part of Stylized Fact #3 ("Exchange Market Shares are Interior and Relatively Stable, Both Aggregate and Within-Symbol) documented in [3, Section 4.1]. Note that even in [3], it is argued that this fact is not a result of the model, but a result of the "stationary routing table strategies". In Japan, however, there is no accessibility nor fungibility for the strategy to work. It is interesting to validate them to find out whether the assumptions that are satisfied in US market, for instance assumptions 2.2, 2.1, and an integrated market are the necessary conditions for those facts to hold.

We use both cash equity transaction volume and transaction value data in validation of interior and relatively stable exchange market shares. The reason is also that because neither assumption Accessibility nor Fungibility is satisfied in Japan, transaction volumes (or transaction values) by itself could not represent market shares accurately.

The data are accessed through market statistics published on official websites of exchanges. We use monthly data for all trading days from 2009, since when all exchanges have historical data on transaction volume and value [10, 15, 21, 22, 23, 24, 25, 27, 35]. Treatment of the two merger events that happened during this period is as follows:

1. Due to the merger of JASDAQ, NEO, and Hercules on October 12, 2010, transaction volume and value on JASDAQ Standard before October 2010 is calculated by summing up those of JASDAQ and Hercules Standard; transaction volume and value on JASDAQ Growth before October 2010 is calculated by summing up those of NEO and Hercules Growth.

2. Due to merger of TSE and OSE in 2013 and the actual market integration in July 16, 2013, TSE 1st section (old) and TSE 2nc section (old) data before July 2013 are regarded equal to the historical data of the same month of TSE 1st section (new) and TSE 2nd section (new); TSE 1st Section (new) data before July 2013 are calculated by summing up those of TSE 1st Section (old) and OSE 1st Section, similar to TSE 2nd Section (new) data before July 2013.

A summary of data descriptives can be found in Table 3.

Table 3. Summary of Aggregate Transaction Value and Transaction Volume Data

	No. of obs.	Mean	Median	Range	SD (% of mean)	Skewness	Kurtosis
Panel A: Transaction value (mn JPY)							
JPX	133	48,012,858	52,105,541	70,593,383	34.345%	-0.116	-0.991
FSE	133	1,905	1,320	14,500	107.384%	3.400	14.925
NSE	133	11,877	7,758	100,227	109.901%	4.323	23.588
SSE	133	5,716	1,430	82,622	195.542%	4.346	24.070
TS1	133	44,860,751	47,594,437	63,052,790	33.018%	-0.071	-0.951
TS2	133	474,218	415,930	1,453,746	79.070%	0.784	-0.297
TMO	133	1,575,970	1,738,090	4,647,780	72.023%	0.501	-0.312
TKP	79	130	1	2,603	380.937%	4.134	16.617
JQS	133	982,413	861,161	3,315,881	63.603%	0.916	0.856
JQG	133	119,427	99,619	520,982	84.080%	1.374	2.085
FSE	133	1,905	1,320	14,500	107.384%	3.400	14.925
NS1	133	2,990	2,096	21,154	108.185%	3.754	17.095
NS2	133	3,320	2,739	9,637	55.109%	1.669	3.388
NCT	133	5,565	1,787	94,026	220.701%	4.971	28.593
SSM	133	244	187	1,092	71.166%	2.186	6.401
SSA	133	5,471	1,208	81,948	203.426%	4.339	23.975
Total	133	48,032,357	52,118,961	70,610,823	34.356%	-0.116	-0.991
Panel B: Transaction volume ('000 shs)							
JPX	133	50,344,531	48,204,232	84,426,686	27.514%	1.368	3.648
FSE	133	2,646	2,294	7,998	55.947%	1.802	4.752

Table 3.(Continued) Summary of Aggregate Transaction Value and Transaction Volume Dat

	No. of obs.	Mean	Median	Range	SD (% of mean)	Skewness	Kurtosis
NSE	133	20,014	12,636	126,411	103.333%	3.096	10.719
SSE	133	7,269	2,620	85,279	160.329%	3.488	16.666
TS1	133	44,860,751	47,594,437	63,052,790	33.018%	-0.071	-0.951
TS2	133	474,218	415,930	1,453,746	79.070%	0.784	-0.297
TMO	133	1,575,970	1,738,090	4,647,780	72.023%	0.501	-0.312
TKP	79	64	2	1,220	305.573%	4.146	18.503
JQS	133	982,413	861,161	3,315,881	63.603%	0.916	0.856
JQG	133	119,427	99,619	520,982	84.080%	1.374	2.085
FSE	133	1,905	1,320	14,500	107.384%	3.400	14.925
NS1	133	2,990	2,096	21,154	108.185%	3.754	17.095
NS2	133	3,320	2,739	9,637	55.109%	1.669	3.388
NCT	133	5,565	1,787	94,026	220.701%	4.971	28.593
SSM	133	244	187	1,092	71.166%	2.186	6.401
SSA	133	5,471	1,208	81,948	203.426%	4.339	23.975
Total	133	48,032,357	52,118,961	70,610,823	34.356%	-0.116	-0.991
Panel C: Market shares by transaction value							
JPX	133	99.963%	99.972%	0.175%	0.028%	-2.749	9.227
FSE	133	0.004%	0.003%	0.026%	88.093%	3.895	21.156
NSE	133	0.024%	0.019%	0.146%	80.494%	3.900	20.065
SSE	133	0.010%	0.003%	0.106%	169.864%	3.754	18.208
TS1	133	94.038%	93.971%	11.291%	2.946%	-0.505	-0.382
TS2	133	0.863%	0.757%	2.090%	61.484%	0.870	-0.156
TMO	133	2.919%	2.821%	7.312%	61.071%	0.899	0.504
TKP	79	0.000%	0.000%	0.005%	383.042%	4.256	18.280
JQS	133	1.919%	1.651%	4.143%	43.975%	1.277	1.288
JQG	133	0.225%	0.199%	0.704%	66.543%	1.184	0.951
FSE	133	0.004%	0.003%	0.026%	88.093%	3.895	21.156
NS1	133	0.007%	0.005%	0.034%	88.416%	2.171	6.178
NS2	133	0.007%	0.006%	0.015%	40.590%	1.260	2.138
NCT	133	0.010%	0.004%	0.138%	186.341%	4.584	24.471
SSM	133	0.001%	0.000%	0.002%	69.234%	2.758	9.964
SSA	133	0.009%	0.002%	0.106%	179.706%	3.736	18.056
Panel D: Market shares by transaction volume							
JPX	133	99.939%	99.958%	0.254%	0.047%	-2.003	4.526
FSE	133	0.005%	0.005%	0.021%	53.695%	2.702	11.234
NSE	133	0.039%	0.028%	0.230%	87.481%	3.078	11.930
SSE	133	0.017%	0.005%	0.213%	176.466%	3.462	15.066
TS1	133	90.018%	89.628%	20.519%	5.854%	-0.318	-0.955
TS2	133	4.110%	3.312%	12.461%	57.622%	1.204	1.635
TMO	133	1.985%	2.095%	5.600%	76.293%	0.494	-0.724
TKP	79	0.000%	0.000%	0.004%	340.706%	5.324	32.270
JQS	133	3.488%	3.326%	6.960%	49.950%	0.311	-0.993
JQG	133	0.338%	0.263%	1.661%	97.160%	1.263	1.787
FSE	133	0.005%	0.005%	0.021%	53.695%	2.702	11.234
NS1	133	0.007%	0.005%	0.029%	74.719%	1.712	3.905

Table 3.(Continued) Summary of Aggregate Transaction Value and Transaction Volume Dat

	No. of obs.	Mean	Median	Range	SD (% of mean)	Skewness	Kurtosis
NS2	133	0.011%	0.010%	0.025%	38.233%	1.311	2.525
NCT	133	0.021%	0.008%	0.231%	157.440%	3.320	14.430
SSM	133	0.002%	0.001%	0.029%	188.914%	6.344	45.691
SSA	133	0.016%	0.001%	0.213%	199.410%	3.429	14.693

For a visualization of the data, see Figure below.

For statistical analysis, we first calculate market shares vector by both transaction volume and transaction value data. Let transaction volume on exchange X of the mmth month of year yy be denoted Vol_X^{yymm}, and in relative terms $vol_{X,\mathcal{X}}^{yymm}$. That is, $vol_{X,\mathcal{X}}^{yymm} = \frac{Vol_X^{yymm}}{\sum_{X\in\mathcal{X}} Vol_X^{yymm}} \times 100\%$. Similarly denote the absolute transaction value and relative transaction value on exchange X of the mmth month of year yy be denoted Val_X^{yymm} and $val_{X,\mathcal{X}}^{yymm}$. We may calculate different relative transaction data by taking $\mathcal{X}_i$ as different subsets of the set of all markets. For example, $vol_{\text{FSE},\{\text{JPX, FSE, NSE, SSE}\}}^{1809}$ means the percentage of total cash equity transaction volume on FSE in the total cash equity transaction volume on all major Japan exchanges in Sep 2018. In this thesis market shares on exchange group level and exchange level are considered, i.e. for the following $\mathcal{X}$'s:

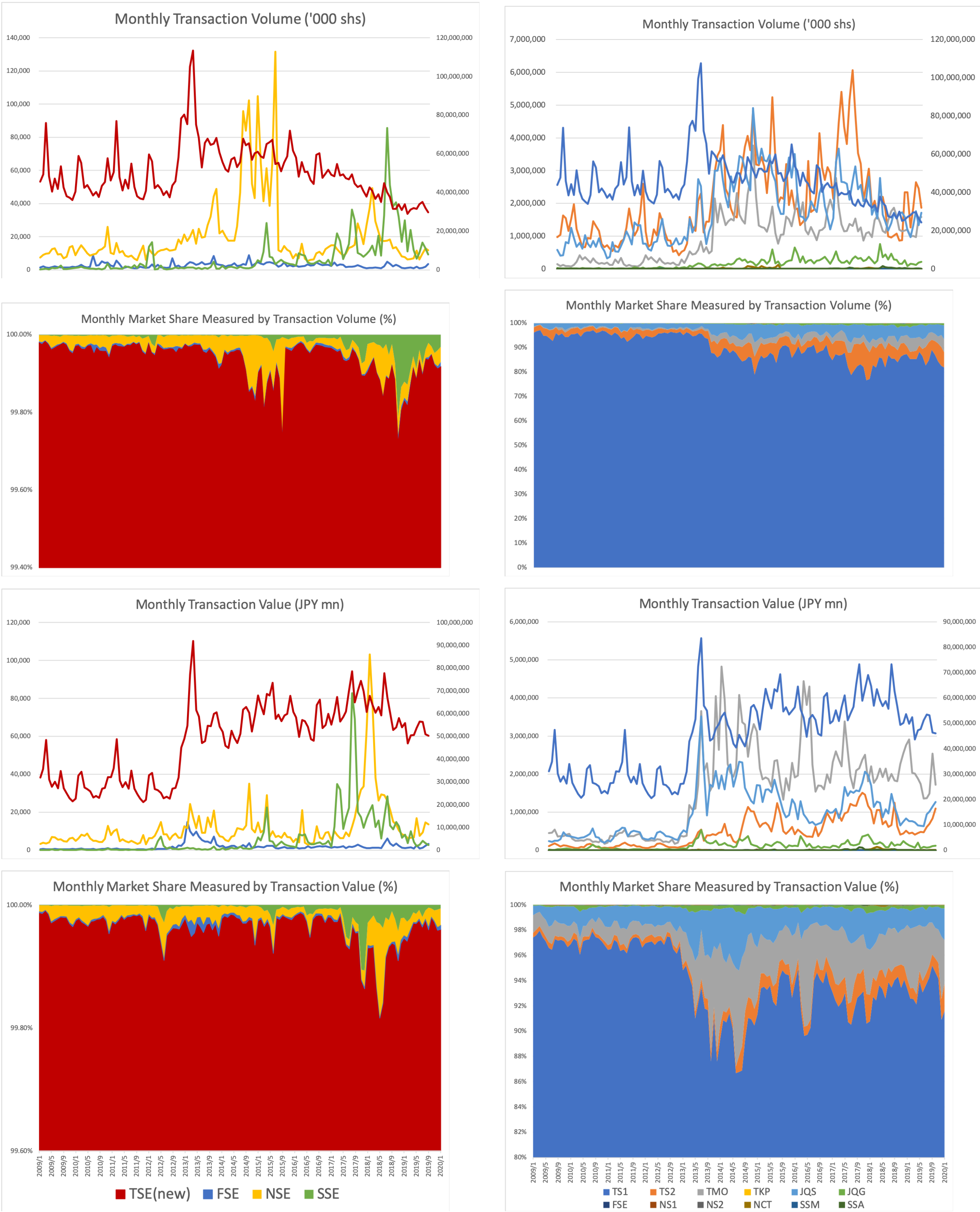
$$\mathcal{X}_1 = \{\text{JPX, FSE, NSE, SSE}\},$$
$$\mathcal{X}_2 = \{\text{TS1, TS2, TMO, TKP, JQS, JQG, FSE, NS1, NS2, NCT, SSM, SSA}\}.$$

We then estimate the following trend models with fixed effects.

$$\begin{aligned}
vol_{X,\mathcal{X}_1}^{yymm} =& \alpha + \beta_0 t + \beta_1\text{JPX} + \beta_2\text{FSE} + \beta_3\text{NSE} + \beta_4\text{SSE} + \sum_{i=1}^{6}\gamma_i P_i + \epsilon;\\
vol_{X,\mathcal{X}_2}^{yymm} =& \alpha + \beta_0 t + \beta_1\text{TS1} + \beta_2\text{TS2} + \beta_3\text{TMO} + \beta_4\text{TKP} + \beta_5\text{JQS} + \beta_6\text{JQG} + \beta_7\text{FSE} + \beta_8\text{NS1} + \beta_9\text{NS2}\\
&+ \beta_{10}\text{NCT} + \beta_{11}\text{SSM} + \sum_{i=1}^{6}\gamma_i P_i + \epsilon;\\
val_{X,\mathcal{X}_1}^{yymm} =& \alpha + \beta_0 t + \beta_1\text{JPX} + \beta_2\text{FSE} + \beta_3\text{NSE} + \beta_4\text{SSE} + \sum_{i=1}^{6}\gamma_i P_i + \epsilon;\\
val_{X,\mathcal{X}_2}^{yymm} =& \alpha + \beta_0 t + \beta_1\text{TS1} + \beta_2\text{TS2} + \beta_3\text{TMO} + \beta_4\text{TKP} + \beta_5\text{JQS} + \beta_6\text{JQG} + \beta_7\text{FSE} + \beta_8\text{NS1} + \beta_9\text{NS2}\\
&+ \beta_{10}\text{NCT} + \beta_{11}\text{SSM} + \sum_{i=1}^{6}\gamma_i P_i + \epsilon,
\end{aligned}$$

where the explaining variables with abbreviated exchange names are dummy variables capturing the fixed effects, P_i's are dummy variables capturing major merger events, and thus β_0 captures the trend over time without effects from exchanges and merger events. More specifically, P_1 equals 1 if $t <$ Oct

Figure 1. Market Shares by Transaction Volume and Value



2010, 0 otherwise; P_2 equals 1 if $t =$ Oct 2010 , 0 otherwise; P_3 equals 1 if Oct 2010 $< t <$ Jan 2012, 0 otherwise; P_4 equals 1 if $t =$ Jan 2012 , 0 otherwise ; P_5 equals 1 if Jan 2012 $< t <$ Jul 2013, 0 otherwise; P_6 equals 1 if $t =$ Jul 2013 , 0 otherwise.

The results of estimation are presented in Table 4. In the estimated model for transaction volume market shares on exchange group level (for $\mathcal{X}_1$), all three exchange group variables are significant, which means that the effect of being in different exchange groups is significant in market shares, while none of the merger events variables is significant. In the estimated model for transaction volume market shares on exchange group level (for $\mathcal{X}_1$), all three exchange group variables are significant, while none of the merger events variables is significant. In the estimated model for transaction volume market shares on exchange level (for $\mathcal{X}_2$), dummy variables indicating exchanges TS1, TS2, TMO, and JQS are significant, while none of the merger events variables is significant. In the estimated model for transaction volume market shares on exchange level (for $\mathcal{X}_2$), dummy variables indicating exchanges TS1, TS2, TMO, and JQS are significant, while none of the merger events variables is significant. Note that these are also the four largest exchanges in the Japan equity market in terms of transaction value.

It can be concluded from the monthly market share measured by both transaction volume and value that the market shares are stable. However, they are obviously not interior, but with market tipping. That is, JPX in the exchange group level, and TSE Section 1 in a single exchange level, is controlling virtually the whole equity market in Japan.

Therefore, we can say that neither an integrated market where assumptions 2.2 and 2.1 are satisfied, nor the stationary routing table strategies, is a necessary condition for stable market shares. However, they may be necessary for interior market shares to form.

Another pronounced result can be observed from the data that deserves further study is that the integration of TSE and OSE results in a discrete increase in monthly transaction value in the newly formed TSE Section 1 (compared with the sum of its two predecessors), and a gradual decrease in transaction volume; which is also true for the newly formed TSE as a whole.

4.2 Exchange Market Shares for Individual Symbols

Now we validate Stylized Fact #3 ("Exchange Market Shares are Interior and Relatively Stable, Both Aggregate and Within-Symbol) documented in [3, Section 4.1] on individual stock level.

We use cross sectional data of the month February 2020. First, a list of stocks that are listed on both regional exchanges and JPX are collected from the regional exchanges' official websites [12, 31, 38]. Then, information of trading volume and value in monthly reports are retrieved for FSE, NSE, and SSE [12, 32, 38]. SSE does not disclose trading value information. We aggregate for monthly trading volume and trading value for each individual stock in each Exchange for the month February in 2020. Then, we access JPX monthly trading volume and trading value information for the corresponding individual stocks. The market shares of Exchanges for each stock are then calculated. A summary of data descriptives can be found in Table 5.

Table 4. Exchange Market Shares Over Time Aggregate

	Group MS by Vol	Board MS by Vol	Group MS by Val	Board MS by Val
Intercept	0.0002*	0.0002	9.522e-05*	8.994e-05
	(2.094)	(0.054)	(1.979)	(0.056)
JPX	0.9992***	-	0.9995***	-
	(2.47e+04)	-	(4.26e+04)	-
FSE	-0.0001***	-	-5.816e-05**	-
	(-2.993)	-	(-2.482)	-
NSE	0.0002***	-	0.0001***	-
	(5.227)	-	(6.194)	-
TS1	-	0.9000***	-	0.9403***
	-	(407.193)	-	(769.228)
TS2	-	0.0409***	-	0.0085***
	-	(18.523)	-	(6.984)
TMO	-	0.0197***	-	0.0291***
	-	(8.912)	-	(23.804)
TKP	-	-0.0002	-	-8.862e-05
	-	(-0.070)	-	(-0.072)
JQS	-	0.0347***	-	0.0191***
	-	(15.708)	-	(15.625)
JQG	-	0.0032	-	0.0022
	-	(1.461)	-	(1.764)
FSE	-	-0.0001	-	-5.288e-05
	-	(-0.047)	-	(-0.043)
NS1	-	-8.983e-05	-	-2.171e-05
	-	(-0.041)	-	(-0.018)
NS2	-	-4.558e-05	-	-1.954e-05
	-	(-0.021)	-	(-0.016)
NCT	-	5.274e-05	-	1.182e-05
	-	(0.024)	-	(0.010)
SSM	-	-0.0001	-	-8.467e-05
	-	(-0.063)	-	(-0.069)
P1	2.257e-15	3.458e-15	2.235e-15	3.648e-15
	(2.85e-11)	(1.38e-12)	(4.87e-11)	(2.64e-12)
P2	-3.678e-15	1.922e-15	-3.858e-15	1.915e-15
	(-2.09e-11)	(3.46e-13)	(-3.78e-11)	(6.23e-13)
P3	-5.378e-16	4.061e-15	-5.169e-16	4.286e-15
	(-7.52e-12)	(1.8e-12)	(-1.25e-11)	(3.44e-12)
P4	-3.039e-15	5.163e-15	-2.942e-15	5.371e-15
	(-1.76e-11)	(9.49e-13)	(-2.94e-11)	(1.78e-12)
P5	2.426e-15	2.175e-15	2.414e-15	2.272e-15
	(4.09e-11)	(1.16e-12)	(7.03e-11)	(2.2e-12)
P6	-1.471e-15	1.728e-15	-1.443e-15	1.846e-15
	(-8.69e-12)	(3.24e-13)	(-1.47e-11)	(6.26e-13)
t	-3.95e-16	3.581e-18	-3.958e-16	7.805e-18
	(-4.84e-10)	(1.39e-13)	(-8.38e-10)	(5.49e-13)
R^2	99.999943%	99.4753%	99.999981%	99.8534%

*: 95% level, **: 97.5% level, ***: 99% level.

Table 5. Summary of Transaction Value and Transaction Volume Data for Individual Symbols

	No. of obs.	Mean	Median	Range	SD (% of mean)	Skewness	Kurtosis
Panel A: Transaction value of repetitively listed stocks (JPY)							
Symbol total	351	21,493,881	2,671,400	1,030,492,800	321.509%	10.016	132.927
JPX	351	21,676,661	2,735,600	1,030,470,100	320.014%	9.979	131.921
FSE	87	242	0	4,100	267.336%	4.205	20.355
NSE	226	1,315	0	93,700	540.463%	10.810	132.288
SSE	41	182	0	2,000	236.683%	3.093	10.031
Panel B: Transaction volume of repetitively listed stocks (shs)							
Symbol total	351	41,530,724,653	4,202,816,459	884,191,945,981	225.971%	4.270	24.860
JPX	351	41,884,423,857	4,321,505,916	884,114,418,281	224.824%	4.252	24.653
FSE	87	582,985	0	11,214,800	298.646%	4.437	21.690
NSE	227	1,731,426	0	58,903,500	326.458%	6.349	52.036
SSE	0	N.A.	N.A.	N.A.	N.A.	N.A.	N.A.

The average market shares over symbols of FSE, NSE, and SSE for February 2020 are respectively 0.002%, 0.014%, and 0.001% in trading volume measure, those of FSE and NSE in trading value measure are 0.002% and 0.013%. Although they are all significantly not zero statistically, they are indeed economically zero. It can be safely concluded that the market is even more tipped on individual stock level than on the aggregate level. This is contradictory with the case in US [3, Figure 4.4], where market shares are interior on individual symbol level and ranging from $0-30\%$ for both NYSE-Listed and non-NYSE-Listed symbols.

The reason may be the fact that although companies listed on both regional exchanges and JPX in Japan can be traded on both exchanges, the bids and asks on one exchange cannot be accessed on another. So that for repetitively listed companies, trading them on JPX is always cost friendlier than trading on regional exchanges. This result in Japan equity market may be a strong evidence that the assumptions (2.2 and 2.1) are necessary for interior market shares to form.

4.3 Per Share Trading Fees (f)

We go on to validate Stylized Fact #4 ("Average Trading Fees are Economically Small") and Stylized Fact #5 ("Money-Pump Constraint Binds") that are documented in [3, Section 4.2]. The latter is more of a general fact for all exchanges anywhere. For the former, however, its truth in the US is largely based on the effect of price competition in an integrated market. If we think that the model and the price competition can explain the economically small transaction fees in US perfectly, this fact will not hold in Japan equity market, since there is neither integration nor proper competition in Japan.

Two approaches are used to investigate per share trading fees in Japanese exchanges. First, we refer to the fee schedule published by exchanges themselves. Only JPX publishes its fee schedule online [26], and the fee structure can be described in Table 6.

Since the fee structure is quite complicated, it is not obvious enough from the official fee schedule

Table 6. JPX fee schedule

	JPX (after September 1 2016)		
Basic Fee	*one-off payment for general trading participant (JPY)*		
	500,000		
Trading Fees	*based on monthly trading value (JPY / 10,000 JPY)*		
	auction trading (excl. JQS, JQG, TMO, TKP)	auction trading in JQS, JQG, TMO, TKP	off-auction single-issue trading and basket trading
	0 - 0.30	0 - 0.90	0.06
Access Fees	*based on number of orders per month (JPY / order)*		
	auction trading basic access fee	auction trading (JPY / order)	ToSTNeT trading (single-issue trading and basket trading)
	200,000	0.08 - 3.00	0 - 67
Trading System Facility Usage Fees	*based on the number of servers and/or terminals for the trading system (JPY / server)*		
	auction trading		ToSTNeT trading
	server for order: 5 orders /sec	4,000 (from 5th server)	8,000 (from 3rd server)
	server for order: 60 orders /sec	30,000 (from 3rd server)	
	server for order: 200 orders /sec	90,000	
	server for inquiries	15,000 (from 3rd server)	
	server for drop copy	30,000	

whether the per share trading fee is economically small or not. So a second approach to per share trading fee is also used.

We use exchanges' earning reports to estimate revenues from per share trading fee and ESST for JPX, OSE (before merger) and NSE. These are the only exchange groups that reports revenue breakdown to the level of ESST and transaction fees. SSE and FSE do not report their revenue in any form. For JPX, we use quarterly data from fourth quarter of financial year 2012 (ended March 31, 2013) to the third quarter of financial year 2019 (ended December 31, 2019); for OSE, we use quarterly data from first quarter of financial year 2008 (ended June 30, 2008) to the second quarter of financial year 2012 (ended September 30, 2012); for NSE, we use semi-annual data from the second half of financial year 2007 (ended March 31, 2008) to the first half of financial year 2019 (ended September 30, 2019).

JPX breaks down its operating revenue into five categories [17]: Trading Services Revenue, Clearing Services Revenue, Listing Services Revenue, Information Services Revenue, and Others. The "Trading Services Revenue" category is further broken down to five subcategories, Transaction Fees, Basic Fees, Access Fees, Trading System Facilities Usage Fees, and others. The "Others" category includes four items, usage fees for Arrownet (a network service), usage fees related to co-location services, revenue from provision of trading system and other services, and revenue from system development and operations, among which JPX disclose exact number for the first two items in its quarterly earnings reports.

According to the charging basis for trading services revenue (Table 2) by JPX, we regard the whole Trading Service Revenue category as f in the model, since all fees are charged on general trading participants. The trading system facility usage fees is the only one that may be in question, yet as it only takes up less than 10% of the whole category revenue, and the fee is charged on both general users (for low frequency servers) and high frequency users, categorizing it as part of f will not cause significant problem. We also regard clearing services revenue as f, as is similarly done in [3, Footnote 63] for CME's clearing revenue. We do not include listing services revenue in the revenue decomposition analysis since it is not related to trading activities. We take the sum of revenue from Information Services, usage fees for Arrownet, and usage fees for co-location services as the ESST F in the model.

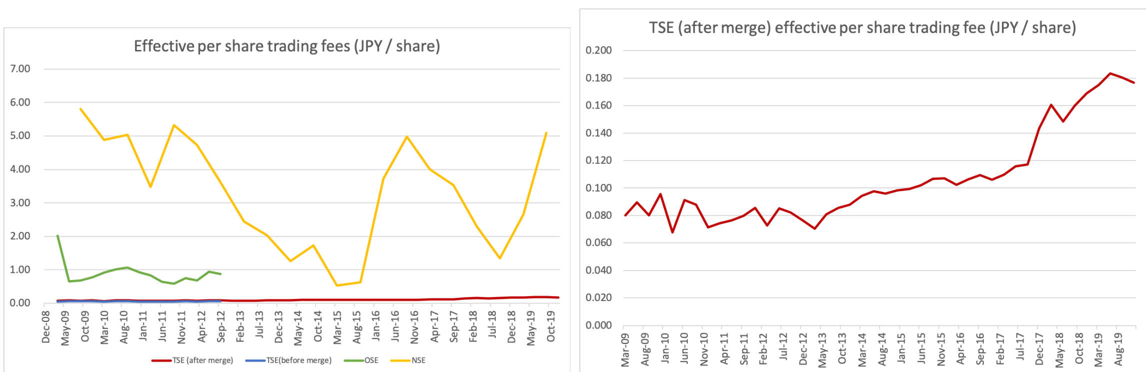
OSE disclose decomposition of its revenue into four categories, Participant Fees, Listing Fees, Equipment and Information Services Fees, and Others. The first category further consists of Trading Fees, Clearing Fees, Access Fees, Basic Fees, and others. We take this category as f in the model. The third category further consists of Market Information Fees, Network Line Fees, Co-location Service Fees, and Others. We take this category as F in the model. Listing Fees and Others are not taken into account since they are irrelevant to trading activities.

NSE's operating revenue is broken down to four categories, 取引参加料金[trading participation fee], 上場関係収入[listing related revenue], 情報関係収入[information related revenue], and その他の営業利益[other operating revenue]. We take trading participation fee as f, and information related revenue as F in the model. We do not consider listing related revenue and other operating revenue.

A summary of data descriptives can be found in Table 7.

With these data, we take quarterly revenue from participation fees reported by exchanges as the total revenue from f (as specified in the last section), divide them by quarterly transaction volume of the corresponding period to get the effective per share trading fee on a quarterly basis for JPX as a whole, TSE before merge, and OSE before merge. We do the same thing with NSE on a semiannual basis. Note that before January 2012, TSE revenue and OSE revenue are summed up and the sum is regarded as JPX revenue for the corresponding quarter; the same method is used to get transaction volume. This is for the purpose of better investigation of the effect of merger event on effective trading fee. Since no exchange group report revenue for the exchanges it operates, analysis on that level cannot be done. Visualization of effective per share trading fee from the quarter ended December 2008 to that ended December 2019 can be found in Figure 2.

Figure 2. Effective per share trading fees in four exchanges



Averaging over time, the effective per share trading fee on JPX is JPY0.106 per share; that on TSE before merge is JPY0.053 per share; OSE before merge, JPY1.756 per share; NSE, JPY3.289 per share.

From the effective per share trading fee data, the following observations can be made.

Table 7. Summary of Revenue Decomposition Data

	No. of obs.	Mean	Median	Range	SD (% of mean)	Skewness	Kurtosis
Panel A: Revenue from per share trading fee (mn JPY)							
JPX	44	15,475	16,709	14,211	23.915%	-0.124	-0.989
TSE	15	6,977	6,909	2,050	7.684%	0.350	0.615
OSE	15	4,111	3,859	3,325	20.180%	2.354	6.509
NSE	21	267	251	119	13.248%	1.647	1.713
Panel B: Trading volume (mn shs)							
JPX	44	151,500	147,585	199,877	25.397%	1.220	3.165
TSE	15	132,716	129,949	60,889	14.913%	0.487	-0.609
OSE	15	4,924	4,696	4,122	25.324%	0.469	-0.811
NSE	21	123	71	434	92.588%	2.409	5.438
Panel C: Effective Trading Fee (JPY / share)							
JPX	44	0.106	0.097	0.116	31.562%	1.150	0.155
TSE	15	0.053	0.054	0.023	12.167%	-0.001	-0.690
OSE	15	0.889	0.828	1.438	38.699%	2.768	9.145
NSE	21	3.289	3.532	5.277	49.884%	-0.198	-1.220
Panel D: Revenue from ESST (mn JPY)							
JPX	28	6,156	5,971	2,016	11.123%	0.328	-1.470
OSE	18	4,958	4,890	2,010	9.766%	0.148	0.964
NSE	25	217	227	667	48.220%	-3.741	18.634
Panel E: JPX revenue further decomposition (mn JPY)							
Trading Participant Fees	28	12,386	11,881	6,473	11.248%	1.812	4.532
Securities settlement	28	5,589	5,653	2,839	12.693%	-0.269	-0.493
Information Services	28	4,606	4,478	1,602	11.385%	0.119	-1.334
Arrownet	28	746	774	384	14.312%	-0.289	-0.825
Colocation	28	803	785	424	16.219%	0.137	-1.249

1. The larger market share the exchange has, the smaller its effective per share trading fee. In particular, effective per share trading fee on JPX is significantly smaller than other exchanges.
2. Effective trading fees on all exchanges at any time point between September 2008 to December 2020 are greater than zero.
3. There is no trend over time in effective per share trading fees on OSE or NSE. However, there is a significant increasing trend in effective per share trading fees in JPX starting from January 2012, when TSE and OSE merged.

From observation 2, it can be concluded that Stylized Fact #5 is validated in Japan equity market.

As for Stylized Fact #4 [3, Section 4.2] to be validated or invalidated, we compare this revenue from participation fees to the operational expense of the corresponding exchange for the most recent reported period to try to argue what is economically small in our context. Similar argument is also used in [3]. By taking the coverage ratio of revenue from per share trading fee (including settlement fee for JPX) over operating expense, we have the coverage ratio for JPX for the three months ended December 2019 being 1.23, the coverage ratio for TSE for the three months ended September 2012 being 0.53, the coverage ratio for OSE for the three months ended September 2012 being 0.89, and that of NSE for the six months ended September 2019 being 0.49.

	JPX three months ended Dec-19	OSE three months ended Sep-12	TSE three months ended Sep-12	NSE six months ended Sep-19
Revenue from per share trading fee (JPY mn)(A)	$17,851$	$3,345$	$6,255$	240
Operating expense (JPY mn)(B)	$14,536$	$3,758$	$11,759$	486
A/B	1.23	0.89	0.53	0.49

From these comparisons, the conclusion can be made that per share trading fees in the previous TSE before merger, OSE, and NSE are economically small, and Sylized Fact #4 (”Average Trading Fees are Economically Small”) [3, Section 4.2] holds for these three exchange groups.

In the current JPX group, however, although the effective per share trading fee is still smaller than that in any other exchange, the revenue it earns from per share trading fee is not small in the sense that it covers 1.23 of its operating expense. That is to say, JPX is significantly profitable with participant fee revenue alone. This is consistent with the conclusion in the next section.

Moreover, note that the economically small per share trading fees in TSE before merger, OSE before merger, and NSE are not an effect of price competition as predicted by the model for US equity market. Since neither assumption 2.2 nor 2.1 is satisfied, there could not be price competition in Japan equity market. A correlated item is market shares. To take a closer look into the relationship between the two, we calculate average semiannual market share and average semiannual effective per share trading fee for JPX (averaging over the period from January 2013 to September 2019, when the exchange group actually operate as one group), TSE (averaging over the period from March April 2019 to September 2012), OSE(averaging over the period from March April 2019 to September 2012), and NSE (averaging over the period from March April 2019 to September 2019); after that, we plot the four pairs of data on a graph. The details data points and the graph are shown in Figure 3. Recall, that market shares in Japan equity market is not interior as in the US. This negative correlation between market share and effective per share trading fee of an exchange is an area that deserves more study.

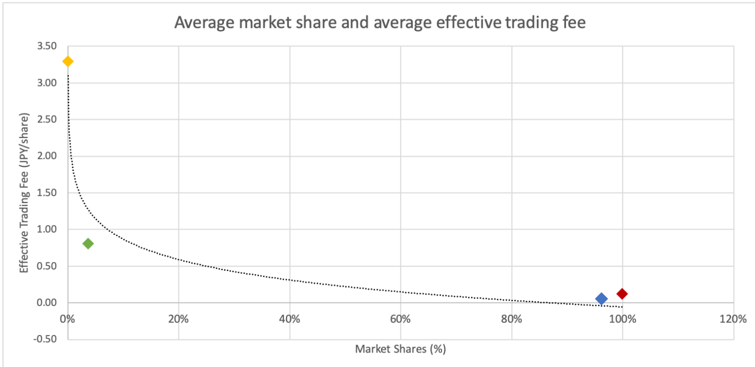
4.4 Exchange-Specific Speed Technology (F)

Now we go on to investigate the validity of stylized fact #6 documented in [3, Section 4.3, pp. 40], “Exchanges earn significant revenues from data and co-location/connectivity (i.e. ESST)” in Japan equity market. Note that although the two assumptions 2.2 and 2.1 are not satisfied, this facts only increase the market power of exchanges over ESST on itself. Therefore, exchanges should still earn significant revenues from ESST in Japan, and they may even get a fraction that is not bounded by (26).

Data used are the same as what are used in the the second approach in the last section.

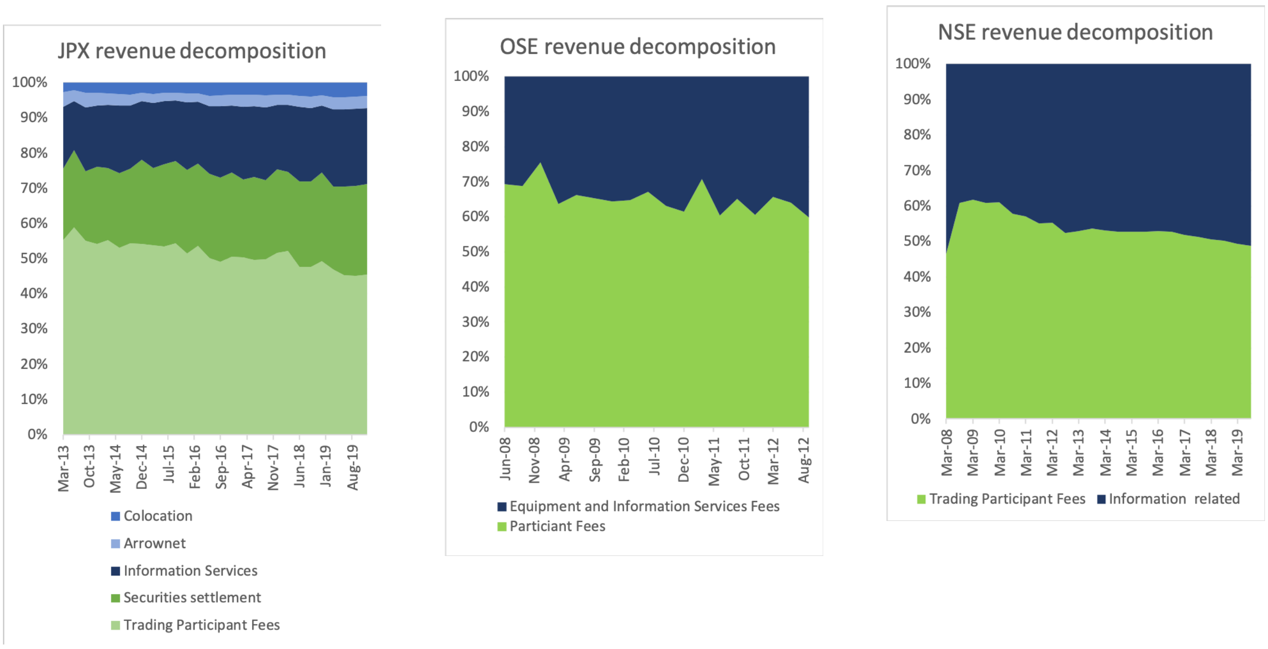
Figure 3. Average market share and average effective trading fee

	Average market share (%)	Average effective trading fee (JPY/share)
JPX	99.95	0.12
TSE	96.25	0.05
OSE	3.72	0.80
NSE	0.04	3.29



A visualization of trading-related revenue decomposition for JPX, NSE, and OSE can be found in figure 4.

Figure 4. Exchange revenue decomposition



The following observations can be made from the results:

1. Around 70% to 80% of trading-related revenue of JPX comes from f, which is similar to the revenue decomposition situation of CME [3, footnote 63]. Also similar to CME, JPX is able to earn more from trading participation fees because of lack of integration and competition in the Japan equity market. JPX also earn from clearing fees, which adds to the source of revenue it can earn from per share trading fees.

2. NSE earns 50% to 60% of trading-related revenue from trading participant fees. This may be a result from the tipping market (JPX owns significant markets share) and lack of market power of

NSE. However, this percentage is still significantly larger than the percentage of per share trading fee revenue in other US equity exchanges, such as BATS group, which earns only 31.2% of operating revenue on transaction fees [3, Figure 4.5].

3. Approximately 50% to 60% of trading-related revenue of OSE came from per share trading fee f. This is less than JPX but still more than NSE. This is consistent with one of our observations in the last section that the more market share an exchange has, the more it can earn from per share trading fee.

From these observations, we can conclude that in a market where there is no accessibility and fungibility, ESST cannot be a large source for revenue for exchanges compared with per share trading fee.

4.5 Growth of exchange revenue from ESST (F)

Lastly, we investigate the time series property of exchange revenue from ESST. Results of linear regressions of the percentage of revenue from ESST of JPX on time can be found in Table 8. Although the trends are significant, their magnitudes are economically small. Revenue from ESST has been growing 0.25% per quarter for JPX, 0.21% per six months for NSE, and 0.47% per quarter for OSE before it was acquired. This is nothing compared with what the US exchanges have been growing at in the Reg NMS era, ranging from 5.1% to 11.7% annually [3, Section 4.3, pp. 44].

Table 8. Percentage of Exchange Revenue from ESST Overtime

	JPX	OSE	NSE
Intercept	-0.826***	-1.567*	-0.426*
	(-4.618)	(-2.316)	(-2.189)
t	2.54E-05***	4.74E-05**	2.13E-05***
	(6.048)	(2.829)	(4.570)
R^2	58.456%	33.339%	47.593%

*: 95% level, **: 97.5% level, ***: 99% level.

Therefore, as an inverse of Stylized Fact #7 ("Exchange revenue from data and co-location/connectivity has grown significantly in the Reg NMS era") documented in [3, Section 4.3], revenues from ESST of exchanges in Japan are not growing as significantly without regulations equivalent to Reg NMS. Together with Section 4.4, this could serve as an evidence supporting the necessity of an integrated market for a significant and growing part of revenue to be earned from ESST by exchanges.

5 Concluding Remarks

The model proposed by [2] and [3] pointed out the built-in arbitrage opportunities in a CLOB market. The Nash-in-Nash bargaining model offers more understanding of the rent division and the two models support each other reciprocally. The sufficient conditions for there to exist Nash-in-Nash equilibrium are equivalent to those for there to exist OBE in the trading game, and the implication on equilibrium ESST fee paid by TFs to Exchanges given by Nash-in-Nash prices are the same with that by the boundary solution in the OBE.

The seven stylized facts documented in [3] well captures the competition among exchanges in a fragmented but integrated market where there are accessibility and fungibility, such as the US equity market. Nevertheless, the situation in the Japan equity market is quite different. Market shares of exchanges in Japan is stable over time, and the market is tipping significantly. The per share trading fee was economically small before merger of OSE and TSE, yet it is not economically small in JPX after the merger. Exchanges in Japan do not earn significant revenue from technology and information service, and there is no economically significant upper trend in this part of revenue. The empirical validation of these stylized facts in Japan, as inverse of what holds in the US, supports the necessity of an integrated market for (i) the market shares to be interior, (ii) the per share trading fee to be economically small, and (iii) the part of revenue from exchange specific speed technology to be economically significant and growing.

One interesting result along the way of deriving the theoretical model is that when there are lone-wolf TFs and Exchanges, the total arbitrage rent that can be exploited by these two participants diminishes. How this effect can be enlarged and help eliminating arbitrage and adverse selection in the HFT era can be further studied.

The more participants considered, the more complex but more complete the picture will be. In the equity market case, more stakeholders remain to be explored. A most vital one would be the listed companies, whose stock are really what is being traded on the exchange, and who has the choice of which exchange to be listed on. To incorporate listed companies into the model will be another interesting topic for future research.

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