DE GRUYTER J. Inverse Ill-Posed Probl. 2019; 27(2): 171–184

Research Article

Igor Polovinkin, Elina Shishkina* and Sergei Sitnik

Space of images of the mixed Riesz hyperbolic B-potential and analytic continuation

https://doi.org/10.1515/jiip-2017-0112
Received November 30, 2017; revised May 3, 2018; accepted September 7, 2018

Abstract: In this paper we prove semigroup properties for the mixed Riesz hyperbolic B-potential, find its analytic continuation and describe a space of images of mixed hyperbolic Riesz B-potentials. This problem is closely related to the problem of inversion of the weighted Radon transform on Lorentzian manifolds.

Keywords: Bessel operator, transmutation operators, mixed Riesz hyperbolic B-potential, weighted Radon transform, Lorentzian manifolds, Lorentzian weighted spherical mean

MSC 2010: Primary 35L81, 35L20, 46E30, 31B99; secondary 47G40

1 Introduction

In this paper we deal with mixed hyperbolic Riesz B-potentials which are fractional powers of the operator

$$\frac{\partial^2}{\partial t^2} - \Delta_\nu, \quad \text{where } \Delta_\nu = \sum_{i=1}^{n} (B_{\nu_i})_{x_i}, \tag{1.1}$$

with

$$(B_{\nu_i})_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{\nu_i}{x_i}\frac{\partial}{\partial x_i}$$

being the singular differential Bessel operator, and $\nu_1 > 0, \dots, \nu_n > 0$.

The potential theory comes from mathematical physics. The most well-known areas of its applications are electrostatic and gravitational theory, probability theory, scattering theory, biological systems, among others. The Newton potential can be interpreted as the negative power of the Laplace operator. Marcell Riesz was the first mathematician who considered fractional negative powers of the Laplace operator, which are now called *Riesz potentials* (see [8, 9]). Riesz also introduced potentials with Lorentz distances, which are the fractional negative powers of the D'Alembert operator. Further studies, properties, and applications of classical Riesz potentials can be found in the books [11, pp. 49, 263], [16, p. 117], [1, p. 131], [10, pp. 483, 554], among others.

In this paper we prove semigroup properties for the mixed Riesz hyperbolic B-potential, find its analytic continuation and describe the space of images of mixed hyperbolic Riesz B-potentials. This potential is the negative real power of the hyperbolic operator (1.1). Such operator is closely related to the the problem of the inversion of the weighted Radon transform over Lorentzian manifolds; for the non-weighted case, see [2].

***Corresponding author: Elina Shishkina,** Voronezh State University, Universitetskaya pl. 1, 394006 Voronezh, Russia, e-mail: ilina_dico@mail.ru
Igor Polovinkin, Voronezh State University, Universitetskaya pl. 1, 394006 Voronezh, Russia, e-mail: polovinkin@yandex.ru
Sergei Sitnik, Belgorod State National Research University, Pobedy str. 85, 308015 Belgorod, Russia, e-mail: sitnik@bsu.edu.ru

1.1 Basic definitions

In this subsection we give a summary of the basic notation, the terminology and some results which will be used in this article.

Suppose that $\mathbb{R}^n$ is the n-dimensional Euclidean space,

$$\mathbb{R}^n_+ = \{(x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ x_1 > 0, \dots, x_n > 0\},$$

$\nu = (\nu_1, \dots, \nu_n)$ is a multiindex consisting of positive real numbers ν_i, $i = 1, \dots, n$, and $|\nu| = \nu_1 + \dots + \nu_n$. Let Ω be a finite or infinite open set in $\mathbb{R}^{n+1}$, symmetric with respect to each hyperplane $x_i = 0$, $i = 1, \dots, n$, and let $\Omega_+ = \Omega \cap (\mathbb{R} \times \mathbb{R}^n_+)$ and $\overline{\Omega}_+ = \Omega \cap (\mathbb{R} \times \overline{\mathbb{R}}^n_+)$, where $\overline{\mathbb{R}}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n,\ x_1 \geq 0, \dots, x_n \geq 0\}$. We deal with the class $C^m(\Omega_+)$, consisting of m times differentiable functions on Ω_+, and denote by $C^m(\overline{\Omega}_+)$ the subset of functions from $C^m(\Omega_+)$ such that all existing derivatives of these functions with respect to x_i, for any $i = 1, \dots, n$, are continuous up to $x_i = 0$, and all existing derivatives with respect to t are continuous for $t \in R$. The class $C^m_{\mathrm{ev}}(\overline{\Omega}_+)$ consists of all functions from $C^m(\overline{\Omega}_+)$ such that

$$\left.\frac{\partial^{2k+1} f}{\partial x_i^{2k+1}}\right|_{x=0} = 0$$

for all nonnegative integers $k \leq \frac{m-1}{2}$ and for $i = 1, \dots, n$ (see [17] and [3, p. 21]). In the following, we will denote $C^m_{\mathrm{ev}}(\mathbb{R} \times \overline{\mathbb{R}}^n_+)$ by C^m_{ev}. We set

$$C^\infty_{\mathrm{ev}}(\overline{\Omega}_+) = \bigcap C^m_{\mathrm{ev}}(\overline{\Omega}_+),$$

with intersection taken for all finite m. Let $C^\infty_{\mathrm{ev}}(\mathbb{R} \times \overline{\mathbb{R}}^n_+) = C^\infty_{\mathrm{ev}}$. Let $\overset{\circ}{C}{}^\infty_{\mathrm{ev}}(\overline{\Omega}_+)$ be the space of all functions $f \in C^\infty_{\mathrm{ev}}(\overline{\Omega}_+)$ with compact support. We will use the notations $\overset{\circ}{C}{}^\infty_{\mathrm{ev}}(\overline{\Omega}_+) = \mathcal{D}_+(\overline{\Omega}_+)$ and $\overset{\circ}{C}{}^\infty_{\mathrm{ev}}(\mathbb{R} \times \overline{\mathbb{R}}^n_+) = \mathcal{D}_+$.

Let $L^\nu_p(\Omega_+)$, $1 \leq p < \infty$, be the space of all measurable functions in Ω_+ such that

$$\int\limits_{\Omega_+} |f(t,x)|^p x^\nu \, dt\, dx < \infty, \quad \text{where } x^\nu = \prod_{i=1}^n x_i^{\nu_i}.$$

For a real number $p \geq 1$, the $L^\nu_p(\Omega_+)$-norm of $f = f(t,x)$ is defined by

$$\|f\|_{L^\nu_p(\Omega_+)} = \Big(\int\limits_{\Omega_+} |f(t,x)|^p x^\nu \, dt\, dx \Big)^{1/p}.$$

The weighted measure of Ω_+ is denoted by $\mathrm{mes}_\nu(\Omega_+)$ and is defined by the formula

$$\mathrm{mes}_\nu(\Omega_+) = \int\limits_{\Omega_+} x^\nu \, dt\, dx.$$

For every measurable function $f(t,x)$, defined for $t \in \mathbb{R}$, $x \in \mathbb{R}^n_+$, we consider

$$\mu_\nu(f,\sigma) = \mathrm{mes}_\nu\{t \in \mathbb{R}, x \in \mathbb{R}^n_+ : |f(t,x)| > \sigma\} = \int\limits_{\{(t,x):|f(t,x)|>\sigma\}^+} x^\nu \, dt\, dx,$$

where $\{(t,x) : |f(t,x)| > \sigma\}^+ = \{t \in \mathbb{R}, x \in \mathbb{R}^n_+ : |f(t,x)| > \sigma\}$. We will call the function $\mu_\nu = \mu_\nu(f,\sigma)$ a *weighted distribution function* $|f(t,x)|$.

Let $L^\nu_\infty(\Omega_+)$ be the set of measurable functions $f(t,x)$ on Ω_+ such that

$$\|f\|_{L^\nu_\infty(\Omega_+)} = \operatorname*{ess\,sup_\nu}_{(t,x)\in\Omega_+} |f(t,x)| = \inf_{\sigma\in\Omega_+} \{\mu_\nu(f,\sigma) = 0\} < \infty.$$

For $1 \leq p \leq \infty$, the $L^\nu_{p,\mathrm{loc}}(\Omega_+)$ is the set of functions u, defined almost everywhere in Ω_+, such that $uf \in L^\nu_p(\Omega_+)$ for any $f \in \mathcal{D}_+(\overline{\Omega}_+)$.

Let $\mathcal{D}'_+(\overline{\Omega}_+)$ be the set of continuous linear functionals on $\overline{\Omega}_+$. Each function $u \in L^{\nu}_{1,\mathrm{loc}}(\Omega_+)$ will be identified with the functional $u \in \mathcal{D}'_+(\overline{\Omega}_+)$ acting according to the formula

$$(u, f)_\nu = \int\limits_{\Omega_+} u(t, x) f(t, x) x^\nu \, dt \, dx, \quad f \in \mathcal{D}_+(\overline{\Omega}_+). \tag{1.2}$$

The functionals $u \in \mathcal{D}'_+(\overline{\Omega}_+)$ acting as in formula (1.2) will be called *regular weighted functionals*. All other continuous linear functionals $u \in \mathcal{D}'_+(\overline{\Omega}_+)$ will be called *singular weighted functionals*. We will use the notation $D'_+ = \mathcal{D}'_+(\mathbb{R} \times \overline{\mathbb{R}}^n_+)$.

The generalized function δ_ν is defined, analogously as in [3, p. 12], by

$$(\delta_\nu, \varphi)_\nu = \varphi(0), \quad \varphi \in \mathcal{D}_+(\overline{\Omega}_+).$$

We will use the generalized convolution product defined by the formula

$$(f * g)_\nu = \int\limits_{\mathbb{R}^{n+1}_+} f(\tau, y)({}^\nu\mathbf{T}^y_x g)(t - \tau, x) y^\nu \, d\tau \, dy, \tag{1.3}$$

where ${}^\nu\mathbf{T}^y_x$ is the multidimensional generalized translation

$$({}^\nu\mathbf{T}^y_x f)(t, x) = ({}^{\nu_1}T^{y_1}_{x_1} \cdots {}^{\nu_n}T^{y_n}_{x_n} f)(t, x).$$

Each of the one-dimensional generalized translations ${}^{\nu_i}T^{y_i}_{x_i}$ is defined, for $i = 1, \ldots, n$, by (see [4, p. 122, formula (5.19)])

$$({}^{\nu_i}T^{y_i}_{x_i} f)(t, x) = \frac{\Gamma(\frac{\nu_i+1}{2})}{\Gamma(\frac{\nu_i}{2})\Gamma(\frac{1}{2})} \int\limits_0^\pi \sin^{\nu_i - 1} \varphi_i \times f\Big(t, x_1, \ldots, x_{i-1}, \sqrt{x_i^2 + y_i^2 - 2x_i y_i \cos \varphi_i}, x_{i+1}, \ldots, x_n\Big) \, d\varphi_i.$$

Based on the multidimensional generalized translation ${}^\nu\mathbf{T}^y_x$, the weighted spherical mean $M^\nu_r[f(t, x)]$ of a suitable function, acting only by the variables $x_1, \ldots, x_n$, is constructed by the formula

$$(M^\nu_r)_x[f(t, x)] = \frac{1}{|S^+_1(n)|_\nu} \int\limits_{S^+_1(n)}^{\nu} \mathbf{T}^{r\theta}_x f(t, x) \theta^\nu \, dS, \tag{1.4}$$

where

$$\theta^\nu = \prod_{i=1}^n \theta_i^{\nu_i}, \quad S^+_1(n) = \{\theta : |\theta| = 1, \theta \in \mathbb{R}^n_+\} \quad \text{and} \quad |S^+_1(n)|_\nu = \frac{\prod_{i=1}^n \Gamma(\frac{\nu_i+1}{2})}{2^{n-1}\Gamma(\frac{n+|\nu|}{2})}.$$

The weighted spherical mean $M^\nu_r[f(t, x)]$ is the transmutation operator intertwining $(\Delta_\nu)_x$ and $(B_{n+|\nu|-1})_t$ for $f \in C^2_{\mathrm{ev}}$ (see [15]):

$$(B_{n+|\nu|-1})_r M^\nu_r[f(t, x)] = M^\nu_r[(\Delta_\nu)_x f(t, x)].$$

As the space of basic functions, we will use the subspace of rapidly decreasing functions

$$S_{\mathrm{ev}}(\mathbb{R} \times \overline{\mathbb{R}}^n_+) = S_{\mathrm{ev}} = \Big\{f \in C^\infty_{\mathrm{ev}} : \sup_{t \in \mathbb{R}, x \in \mathbb{R}^n_+} |t^{\alpha_0} x^\alpha D^\beta f(t, x)| < \infty\Big\},$$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_0, \beta_1, \ldots, \beta_n)$, with $\alpha_0, \alpha_1, \ldots, \alpha_n, \beta_0, \beta_1, \ldots, \beta_n$ being arbitrary integer non-negative numbers, and

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}, \qquad D^\beta = D_t^{\beta_0} D_{x_1}^{\beta_1} \cdots D_{x_n}^{\beta_n}, \qquad D_t = \frac{\partial}{\partial t}, \quad D_{x_j} = \frac{\partial}{\partial x_j}, \quad j = 1, \ldots, n.$$

In the same way as for $\mathcal{D}'_+$, we introduce the space S'_{ev}. In fact, we identify S'_{ev} with a subspace of $\mathcal{D}'_+$, since $\mathcal{D}_+$ is dense in S_{ev}.

Definition 1.1. The *Fourier–Bessel transform* of a function $f \in L_1^\nu(\mathbb{R} \times \mathbb{R}_+^n)$, $f = f(t, x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}_+^n$, is expressed by

$$\mathcal{F}_\nu[f](\tau, \xi) = \mathcal{F}_\nu[f(t, x)](\tau, \xi) = \widehat{f}(\tau, \xi) = \int_{-\infty}^{\infty} e^{it\tau}\, dt \int_{\mathbb{R}_+^n} f(t, x)\, \mathbf{j}_\nu(x; \xi) x^\nu\, dx,$$

with

$$\mathbf{j}_\nu(x; \xi) = \prod_{i=1}^{n} j_{\frac{\nu_i - 1}{2}}(x_i \xi_i), \quad \nu_1 > 0, \dots, \nu_n > 0,$$

where the symbol j_ν is used for the normalized Bessel function

$$j_\nu(r) = \frac{2^\nu \Gamma(\nu + 1)}{r^\nu} J_\nu(r)$$

and $J_\nu(r)$ is the Bessel function of the first kind of order ν.

For $f \in S_{\mathrm{ev}}$, the inverse Hankel transform is defined by

$$\mathcal{F}_\nu^{-1}[\widehat{f}(\tau, \xi)](t, x) = f(t, x) = \frac{2^{n-|\nu|-1}}{\pi \prod_{j=1}^{n} \Gamma^2(\frac{\nu_j + 1}{2})} \int_{-\infty}^{\infty} e^{-it\tau}\, d\tau \int_{\mathbb{R}_+^n} \mathbf{j}_\nu(x, \xi) \widehat{f}(\tau, \xi) \xi^\nu\, d\xi.$$

Let

$$\Psi_{\nu,V} = \{\psi \in S_{\mathrm{ev}} : (D^k \psi)(x) = 0,\ x \in V,\ |k| = 0, 1, 2, \dots\}$$

and

$$\Phi_{\nu,V} = \{\varphi : \mathcal{F}_\nu \varphi \in \Psi_{\nu,V}\}.$$

2 Mixed hyperbolic Riesz B-potential and its properties

2.1 Preliminary information on mixed hyperbolic Riesz B-potential

In this subsection we define a mixed hyperbolic Riesz B-potential, prove a theorem on its absolute convergence, obtain its representation using the transmutation operator (1.4), and give some examples and properties concerning boundedness and the application of the Fourier–Bessel transform to a mixed hyperbolic Riesz B-potential.

Let

$$|x| = \sqrt{x_1^2 + \dots + x_n^2}.$$

First, for $t \in \mathbb{R}$, $x \in \mathbb{R}_+^n$, $\lambda \in C$, we define

$$s^\lambda(t, x) = \begin{cases} \frac{(t^2 - |x|^2)^\lambda}{N(\alpha,\nu,n)} & \text{when } |x|^2 \le t^2 \text{ and } t \ge 0, \\ 0 & \text{when } t^2 < |x|^2 \text{ or } t < 0, \end{cases} \tag{2.1}$$

where

$$N(\alpha, \nu, n) = \frac{2^{\alpha-n-1}}{\sqrt{\pi}} \prod_{i=1}^{n} \Gamma\Big(\frac{\nu_i + 1}{2}\Big) \Gamma\Big(\frac{\alpha - n - |\nu| + 1}{2}\Big) \Gamma\Big(\frac{\alpha}{2}\Big).$$

The regular weighted generalized function corresponding to (2.1) will be denoted by s_+^λ.

We introduce the *mixed hyperbolic Riesz B-potential* $I_{s,\nu}^\alpha$ *of order* $\alpha > 0$ as a generalized convolution product (1.3) with a weighted generalized function $s_+^{(\alpha-n-|\nu|-1)/2}$ and $f \in S_{\mathrm{ev}}$:

$$(I_{s,\nu}^\alpha f)(t, x) = (s_+^{\frac{\alpha-n-|\nu|-1}{2}} * f)_\nu(t, x). \tag{2.2}$$

The explicit definition of the constant $N(\alpha, \nu, n)$ allows to obtain the semigroup property or index low of the potential (2.2).

We can rewrite formula (2.2) as

$$(I^{\alpha}_{s,\nu}f)(t,x)=\int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mathbb{R}^n_+}s_+^{(\alpha-n-|\nu|-1)/2}(\tau,y)({}^{\nu}\mathbf{T}^y_x)f(t-\tau,x)y^{\nu}\,dy\Big)\,d\tau. \tag{2.3}$$

Theorem 2.1. *Let $f\in S_{\mathrm{ev}}$ and $n+|\nu|-1<\alpha$. Then $(I^{\alpha}_{s,\nu}f)(t,x)$ converges absolutely for $t\in\mathbb{R}$, $x\in\mathbb{R}^n_+$.*

Proof. Passing to spherical coordinates, $(\tau,y)=\rho\sigma$, $\rho=\sqrt{\tau^2+|y|^2}$, $\sigma=(\sigma_1,\dots,\sigma_{n+1})$, in (2.3), we get

$$(I^{\alpha}_{s,\nu}f)(t,x)=\frac{1}{N(\alpha,\nu,n)}\int\limits_0^{\infty}\rho^{\alpha-1}\,d\rho\int\limits_{S_1^+(n),|\sigma'|<\sigma_1}(\sigma_1^2-|\sigma'|^2)^{\frac{\alpha-n-|\nu|-1}{2}}({}^{\nu}\mathbf{T}^{\rho\sigma'}_x f)(t-\rho\sigma_1,x)(\sigma')^{\nu}\,dS,$$

where $\sigma'=(\sigma_2,\dots,\sigma_{n+1})$, $(\sigma')^{\nu}=\prod_{i=1}^n\sigma_{i+1}^{\nu_i}$.

Using the fact that ${}^{\nu}\mathbf{T}^y_x f(t,x)={}^{\nu}\mathbf{T}^x_y f(t,y)$, property 6 of generalized translation from [4, p. 124], in the form

$$|{}^{\nu}\mathbf{T}^y_x f(t,x)|\le\sup_{x\in\mathbb{R}^n_+}|f(t,x)|,$$

and taking into account that $f\in S_{\mathrm{ev}}$, we obtain

$$(I^{\alpha}_{s,\nu}f)(t,x)=\frac{1}{N(\alpha,\nu,n)}\int\limits_0^{\infty}\frac{\rho^{\alpha-1}}{(1+\rho^2)^{\frac{\alpha+1}{2}}}\,d\rho\int\limits_{S_1^+(n),|\sigma'|<\sigma_1}(\sigma_1^2-|\sigma'|^2)^{\frac{\alpha-n-|\nu|-1}{2}}(\sigma')^{\nu}\,dS.$$

It is easy to see that for $\alpha>n+|\nu|-1$, the integral $(I^{\alpha}_{s,\nu}f)(t,x)$ converges absolutely. □

Lemma 2.2. *The following representation of the mixed hyperbolic Riesz B-potential is valid:*

$$(I^{\alpha}_{s,\nu}f)(t,x)=\frac{2^{2-\alpha}\sqrt{\pi}}{\Gamma(\frac{n+|\nu|}{2})\Gamma(\frac{\alpha-n-|\nu|+1}{2})\Gamma(\frac{\alpha}{2})}\int\limits_0^{\infty}\Big(\int\limits_0^{\tau}(M^{\nu}_r)_x f(t-\tau,x)(\tau^2-r^2)^{\frac{\alpha-n-|\nu|-1}{2}}r^{n+|\nu|-1}\,dr\Big)\,d\tau, \tag{2.4}$$

where M^{ν}_r is the transmutation operator (1.4).

Proof. We have

$$\begin{aligned}(I^{\alpha}_{s,\nu}f)(t,x)&=\int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mathbb{R}^n_+}s_+^{\frac{\alpha-n-|\nu|-1}{2}}(\tau,y)({}^{\nu}\mathbf{T}^y_x)f(t-\tau,x)y^{\nu}\,dy\Big)\,d\tau\\&=\frac{1}{N(\alpha,\nu,n)}\int\limits_0^{\infty}d\tau\int\limits_{|y|^2<\tau^2}(\tau^2-|y|^2)^{\frac{\alpha-n-|\nu|-1}{2}}({}^{\nu}\mathbf{T}^y_x)f(t-\tau,x)y^{\nu}\,dy\\&=\{y=r\theta\}=\frac{1}{N(\alpha,\nu,n)}\int\limits_0^{\infty}d\tau\int\limits_0^{\tau}(\tau^2-r^2)^{\frac{\alpha-n-|\nu|-1}{2}}r^{n+|\nu|-1}\,dr\int\limits_{S_1^+(n)}({}^{\nu}\mathbf{T}^{r\theta}_x)f(t-\tau,x)\theta^{\nu}\,dS.\end{aligned}$$

Using the transmutation operator (1.4), we obtain (2.4). □

Example. Let $f(t,x)=\lambda(t)\mathbf{j}_{\nu}(x;b)$, $b\in\mathbb{R}^n_+$. Using the formula (see [5])

$$M^{\nu}_r\mathbf{j}_{\nu}(x;b)=\mathbf{j}_{\nu}(x;b)j_{\frac{n+|\nu|}{2}-1}(r|b|),$$

we obtain

$$\begin{aligned}(I^{\alpha}_{s,\nu}f)(t,x)&=\frac{2^{2-\alpha}\sqrt{\pi}\,\mathbf{j}_{\nu}(x;b)}{\Gamma(\frac{n+|\nu|}{2})\Gamma(\frac{\alpha-n-|\nu|+1}{2})\Gamma(\frac{\alpha}{2})}\int\limits_0^{\infty}\lambda(t-\tau)\,d\tau\Big(\int\limits_0^{\tau}j_{\frac{n+|\nu|}{2}-1}(r|b|)(\tau^2-r^2)^{\frac{\alpha-n-|\nu|-1}{2}}r^{n+|\nu|-1}\,dr\Big)\\&=\frac{2^{\frac{n+|\nu|}{2}+1-\alpha}\sqrt{\pi}\,\mathbf{j}_{\nu}(x;b)}{|b|^{\frac{n+|\nu|}{2}-1}\Gamma(\frac{\alpha-n-|\nu|+1}{2})\Gamma(\frac{\alpha}{2})}\int\limits_0^{\infty}\lambda(t-\tau)\,d\tau\Big(\int\limits_0^{\tau}J_{\frac{n+|\nu|}{2}-1}(r|b|)(\tau^2-r^2)^{\frac{\alpha-n-|\nu|-1}{2}}r^{\frac{n+|\nu|}{2}}\,dr\Big).\end{aligned}$$

Using [7, formula (2.12.4.6)] in the form

$$\int_0^a x^{\nu+1}(a^2-x^2)^{\beta-1}J_\nu(cx)dx = \frac{2^{\beta-1}a^{\beta+\nu}}{c^\beta}\Gamma(\beta)J_{\beta+\nu}(ac), \quad a, \operatorname{Re}\beta > 0, \operatorname{Re}\nu > -1,$$

we get

$$\int_0^\tau J_{\frac{n+|\nu|}{2}-1}(r|b|)(\tau^2-r^2)^{\frac{\alpha-n-|\nu|-1}{2}} r^{\frac{n+|\nu|}{2}}\,dr = \frac{2^{\frac{\alpha-n-|\nu|-1}{2}}\tau^{\frac{\alpha-1}{2}}}{|b|^{\frac{\alpha-n-|\nu|+1}{2}}}\Gamma\Big(\frac{\alpha-n-|\nu|+1}{2}\Big)J_{\frac{\alpha-1}{2}}(\tau|b|)$$

and

$$I^\alpha_{s,\nu}\lambda(t)\mathbf{j}_\nu(x;b) = \frac{\sqrt{\pi}\,\mathbf{j}_\nu(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma(\frac{\alpha}{2})}\int_0^\infty \lambda(t-\tau)\tau^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|b|\tau)\,d\tau.$$

If we take $\lambda(t)=\theta(t)$, where θ is the Heaviside step function, for $t>0$, we get

$$I^\alpha_{s,\nu}\theta(t)\mathbf{j}_\nu(x;b) = \frac{\sqrt{\pi}\,\mathbf{j}_\nu(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma(\frac{\alpha}{2})}\int_0^t \tau^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|b|\tau)\,d\tau = \frac{\sqrt{\pi}t^\alpha\mathbf{j}_\nu(x;b)}{2^\alpha}{}_1F_2\Big(\frac{\alpha}{2};\frac{\alpha+1}{2},\frac{\alpha+2}{2};-\frac{|b|^2t^2}{4}\Big).$$

Assuming $\lambda(t)=\theta(t)t^{\beta-1}$, $\beta>0$, and using [7, formula (2.12.3.1)] in the form

$$\begin{aligned}&\int_0^a x^{\alpha-1}(a-x)^{\beta-1}J_\nu(cx)\,dx\\ &\quad= a^{\alpha+\beta+\nu-1}\Big(\frac{c}{2}\Big)^\nu\frac{\Gamma(\beta)\Gamma(\alpha+\nu)}{\Gamma(\nu+1)\Gamma(\alpha+\beta+\nu)}{}_2F_3\Big(\frac{\alpha+\nu}{2},\frac{\alpha+\nu+1}{2};\nu+1,\frac{\alpha+\beta+\nu}{2},\frac{\alpha+\beta+\nu+1}{2};-\frac{a^2c^2}{4}\Big),\end{aligned}$$

where $a, \operatorname{Re}\beta, \operatorname{Re}(\alpha+\nu)>0$, we get

$$\begin{aligned}I^\alpha_{s,\nu}\theta(t)t^{\beta-1}\mathbf{j}_\nu(x;b) &= \frac{\sqrt{\pi}\,\mathbf{j}_\nu(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma(\frac{\alpha}{2})}\int_0^t (t-\tau)^{\beta-1}\tau^{\frac{\alpha-1}{2}}J_{\frac{\alpha-1}{2}}(|b|\tau)\,d\tau\\ &= \frac{\sqrt{\pi}\,\mathbf{j}_\nu(x;b)}{(2|b|)^{\frac{\alpha-1}{2}}\Gamma(\frac{\alpha}{2})}\cdot t^{\alpha+\beta-1}\Big(\frac{|b|}{2}\Big)^{\frac{\alpha-1}{2}}\frac{\Gamma(\beta)\Gamma(\alpha)}{\Gamma(\frac{\alpha+1}{2})\Gamma(\alpha+\beta)}\\ &\quad\times{}_2F_3\Big(\frac{\alpha}{2},\frac{\alpha+1}{2};\frac{\alpha+1}{2},\frac{\alpha+\beta}{2},\frac{\alpha+\beta+1}{2};-\frac{t^2|b|^2}{4}\Big)\\ &= \frac{\sqrt{\pi}\Gamma(\beta)\Gamma(\alpha)t^{\alpha+\beta-1}\mathbf{j}_\nu(x;b)}{2^{\alpha-1}\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha+1}{2})\Gamma(\alpha+\beta)}{}_1F_2\Big(\frac{\alpha}{2};\frac{\alpha+\beta}{2},\frac{\alpha+\beta+1}{2};-\frac{t^2|b|^2}{4}\Big).\end{aligned}$$

Theorem 2.3. *Let $n+|\nu|-1<\alpha<n+|\nu|+1$, $1\le p<\frac{n+|\nu|+1}{\alpha}$. For the estimate*

$$\|I^\alpha_{s,\nu}f\|_{q,\nu}\le M\|f\|_{p,\nu},\quad f\in S_{\mathrm{ev}}, \tag{2.5}$$

to be valid, it is necessary and sufficient that $q=\frac{(n+|\nu|+1)p}{n+|\nu|+1-\alpha p}$. The constant M does not depend on f.

Remark. By virtue of (2.5), there is a unique extension of $I^\alpha_{s,\nu}$, to all L^ν_p for $1<p<\frac{n+|\nu|+1}{\alpha}$, preserving boundedness when $n+|\nu|-1<\alpha<n+|\nu|$. It follows that this extension is represented by the integral in (2.3), due to its absolute convergence. The boundedness of $I^\alpha_{s,\nu}$ was proved in [14].

Theorem 2.4. *For $f\in S_{\mathrm{ev}}$, the Fourier–Bessel transform of the mixed hyperbolic Riesz potential $I^\alpha_{s,\nu}f$ is defined by*

$$\mathcal{F}_\nu[I^\alpha_{s,\nu}f](\tau,\xi) = q|\tau^2-|\xi|^2|^{-\frac{\alpha}{2}}\cdot\mathcal{F}_\nu[f(t,x)](\tau,\xi), \tag{2.6}$$

where

$$q=\begin{cases}1, & |\xi|^2\ge\tau^2,\\ e^{-\frac{\alpha\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau\ge 0,\\ e^{\frac{\alpha\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau<0.\end{cases}$$

Theorem 2.4 was proved in [14].

2.2 Semigroup properties of the mixed hyperbolic Riesz B-potential

In this subsection we prove semigroup properties of the mixed hyperbolic Riesz B-potential and obtain formulas for $(\frac{\partial^2}{\partial t^2}-\Delta_\nu)^k I^{\alpha+2k}_{s,\nu}f$ and $I^{\alpha+2k}_{s,\nu}(\frac{\partial^2}{\partial t^2}-\Delta_\nu)^k$, where $k\in\mathbb{N}$.

Theorem 2.5. *The mixed hyperbolic Riesz potential for $f\in S_{\mathrm{ev}}$ satisfies*

$$I^\beta_{s,\nu}I^\alpha_{s,\nu}f=I^{\alpha+\beta}_{s,\nu}f,\quad n+|\nu|-1<\alpha,\ n+|\nu|-1<\beta. \tag{2.7}$$

Proof. Consider first Φ_V, where $V=\{\tau\in\mathbb{R},\ \xi\in\mathbb{R}^n_+:\tau^2-|\xi|^2=0\}$. Using (2.6), we obtain

$$\mathcal{F}_\nu[I^\beta_{s,\nu}I^\alpha_{s,\nu}f](\tau,\xi)=q_\beta|\tau^2-|\xi|^2|^{-\frac{\beta}{2}}\cdot\mathcal{F}_\nu[I^\alpha_{s,\nu}f(t,x)](\tau,\xi)=q_\alpha q_\beta|\tau^2-|\xi|^2|^{-\frac{\alpha+\beta}{2}}\cdot\mathcal{F}_\nu[f(t,x)](\tau,\xi),$$

where

$$q_\alpha=\begin{cases}1, & |\xi|^2\ge\tau^2,\\ e^{-\frac{\alpha\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau\ge0,\\ e^{\frac{\alpha\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau<0,\end{cases}\qquad q_\beta=\begin{cases}1, & |\xi|^2\ge\tau^2,\\ e^{-\frac{\beta\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau\ge0,\\ e^{\frac{\beta\pi}{2}i}, & |\xi|^2<\tau^2,\ \tau<0.\end{cases} \tag{2.8}$$

We have

$$\mathcal{F}_\nu[I^\beta_{s,\nu}I^\alpha_{s,\nu}f](\tau,\xi)=q_{\alpha+\beta}|\tau^2-|\xi|^2|^{-\frac{\alpha+\beta}{2}}\cdot\mathcal{F}_\nu[f(t,x)](\tau,\xi)=\mathcal{F}_\nu[I^{\alpha+\beta}_{s,\nu}f](\tau,\xi),$$

since from (2.8) it is clear that $q_\alpha q_\beta=q_{\alpha+\beta}$. Applying the inverse Fourier–Bessel transform, we get (2.7). Since $\Phi_{\nu,V}$ is dense in S_{ev}, we get the statement of the theorem. □

Theorem 2.6. *For $f\in S_{\mathrm{ev}}$, $1<p<\frac{n+|\nu|+1}{\alpha}$, $n+|\nu|-1<\alpha$ and $k\in\mathbb{N}$, the following formula is valid:*

$$\Big(\frac{\partial^2}{\partial t^2}-\Delta_\nu\Big)^k I^{\alpha+2k}_{s,\nu}f=I^\alpha_{s,\nu}f. \tag{2.9}$$

Proof. Using the property of generalized translation in the form (see [4])

$$\int\limits_{\mathbb{R}^n_+}{}^\nu\mathbf{T}^y_x f(x)g(y)y^\nu\,dy=\int\limits_{\mathbb{R}^n_+}f(y)\,{}^\nu\mathbf{T}^y_x g(x)y^\nu\,dy$$

and [3, formula (1.8.3)] in the form ${}^{\nu_i}T^{y_i}_{x_i}(B_{\nu_i})_{x_i}=(B_{\nu_i})^{\nu_i}_{x_i}T^{y_i}_{x_i}$, we obtain

$$\Big(\frac{\partial^2}{\partial t^2}-\Delta_\nu\Big)^k(I^{\alpha+2k}_{s,\nu}f)(t,x)=\int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mathbb{R}^n_+}({}^\nu\mathbf{T}^y_x)\Big(\frac{\partial^2}{\partial t^2}-(\Delta_\nu)_x\Big)^k s_+^{\frac{\alpha-n-|\nu|-1}{2}+k}(t-\tau,x)f(\tau,y)y^\nu\,dy\Big)\,d\tau.$$

Since

$$\Big(\frac{\partial^2}{\partial t^2}-(\Delta_\nu)_x\Big)^k(t^2-|x^2)^{\frac{\alpha-n-|\nu|-1}{2}+k}=2^{2k}\frac{\Gamma(\frac{\alpha-n-|\nu|+1}{2}+k)}{\Gamma(\frac{\alpha-n-|\nu|+1}{2})}\frac{\Gamma(\frac{\alpha}{2}+k)}{\Gamma(\frac{\alpha}{2})}(t^2-|x|^2)^{\frac{\alpha-n-|\nu|-1}{2}}$$

and

$$2^{2k}\frac{\Gamma(\frac{\alpha-n-|\nu|+1}{2}+k)}{\Gamma(\frac{\alpha-n-|\nu|+1}{2})}\frac{\Gamma(\frac{\alpha}{2}+k)}{\Gamma(\frac{\alpha}{2})}\cdot\frac{1}{N(\alpha+2k,\nu,n)}=\frac{1}{N(\alpha,\nu,n)},$$

we get

$$\begin{aligned}\Big(\frac{\partial^2}{\partial t^2}-\Delta_\nu\Big)^k(I^{\alpha+2k}_{s,\nu}f)(t,x)&=\int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mathbb{R}^n_+}({}^\nu\mathbf{T}^y_x)s_+^{\frac{\alpha-n-|\nu|-1}{2}}(t-\tau,x)f(\tau,y)y^\nu\,dy\Big)\,d\tau\\&=\int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mathbb{R}^n_+}s_+^{\frac{\alpha-n-|\nu|-1}{2}}(\tau,x)({}^\nu\mathbf{T}^y_x)f(t-\tau,y)y^\nu\,dy\Big)\,d\tau\\&=(I^\alpha_{s,\nu}f)(t,x).\end{aligned}$$

This completes the proof □

Theorem 2.7. *For $f \in S_{\mathrm{ev}}$, $1 < p < \frac{n+|\nu|+1}{\alpha}$, $n + |\nu| - 1 < \alpha$ and $k \in \mathbb{N}$, the formula*

$$I_{s,\nu}^{\alpha+2k}\Big(\frac{\partial^2}{\partial t^2} - \Delta_\nu\Big)^k f = I_{s,\nu}^{\alpha} f \tag{2.10}$$

is valid if the function $f = f(t,x)$ is such that

$$\frac{\partial^m f}{\partial t^m}\Big|_{t=0,x=0} = 0, \quad \frac{\partial^m f}{\partial x_i^m}\Big|_{t=0,x=0} = 0 \quad \textit{for all } m = 0,\dots,2k. \tag{2.11}$$

Proof. Using [3, formula (1.8.3)] in the form ${}^{\nu_i}T_{x_i}^{y_i}(B_{\nu_i})_{x_i} = (B_{\nu_i})_{x_i}^{\nu_i}T_{x_i}^{y_i}$, integrating by parts and applying (2.11), we obtain

$$\begin{aligned}(I_{s,\nu}^{\alpha+2k})\Big(\frac{\partial^2}{\partial t^2} - \Delta_\nu\Big)^k f(t,x) &= \int_{-\infty}^{\infty}\bigg(\int_{\mathbb{R}_+^n} ({}^\nu\mathbf{T}_x^y)\Big(\frac{\partial^2}{\partial t^2} - (\Delta_\nu)_x\Big)^k s_+^{\frac{\alpha-n-|\nu|-1}{2}+k}(t-\tau,x) f(\tau,y) y^\nu\,dy\bigg)\,d\tau \\ &= (I_{s,\nu}^{\alpha})f(t,x).\end{aligned}$$

The proof is complete. □

If an analytic continuation of $I_{s,\nu}^\alpha$ for the values $0 < \alpha$ exists and for this analytic continuation it is true that $\lim_{\alpha\to 0}(I_{s,\nu}^\alpha u)(t,x) = u(t,x)$, then from formulas (2.9) and (2.10), it follows that mixed hyperbolic Riesz B-potentials can be used for finding solutions to iterated nonhomogeneous Euler–Poisson–Darboux equations

$$\Big(\frac{\partial^2}{\partial t^2} - \Delta_\nu\Big)^k u(t,x) = f(t,x), \tag{2.12}$$

under conditions (2.11). Namely, applying formally $I_{s,\nu}^\alpha$ to (2.12), we get

$$(I_{s,\nu}^\alpha u)(t,x) = (I_{s,\nu}^{\alpha+2k} f)(t,x).$$

Furthermore, passing to the limit as $\alpha \to 0$, formally, we obtain the solution to the problem in the form $u(t,x) = (I_{s,\nu}^{2k} f)(t,x)$. However, the integral $I_{s,\nu}^\alpha f$ converges absolutely only for $\alpha > n + |\nu| - 1$ (see Theorem 2.1), therefore it is necessary to construct an analytic continuation of these operators by extending the range of order α.

3 Analytic continuation of mixed hyperbolic Riesz B-potentials

Here first we write a convenient representation for $I_{s,\nu}^\alpha$, and then, using this representation, we obtain an analytic continuation of $I_{s,\nu}^\alpha$.

Lemma 3.1. *For $n + |\nu| - 1 < \alpha$, the following representation of $I_{s,\nu}^\alpha$ is valid:*

$$(I_{s,\nu}^\alpha f)(t,x) = \int_{-\infty}^{\infty}\bigg(\int_{\mathbb{R}_+^n} \frac{\tau^{\alpha-n-|\nu|}}{\sqrt{|y|^2+\tau^2}}\,{}^\nu\mathbf{T}_x^y f\big(t - \sqrt{|y|^2+\tau^2}, x\big) y^\nu\,dy\bigg)\,d\tau. \tag{3.1}$$

Proof. We have

$$\begin{aligned}(I_{s,\nu}^\alpha f)(t,x) &= \int_{-\infty}^{\infty}\bigg(\int_{\mathbb{R}_+^n} s_+^{\frac{\alpha-n-|\nu|-1}{2}}(\tau,y)({}^\nu\mathbf{T}_x^y)f(t-\tau,x)y^\nu\,dy\bigg)\,d\tau \\ &= \int_0^\infty d\tau\bigg(\int_{\{|y|^2<\tau^2\}^+} (\tau^2-|y|^2)^{\frac{\alpha-n-|\nu|-1}{2}}({}^\nu\mathbf{T}_x^y)f(t-\tau,x)y^\nu\,dy\bigg)\end{aligned}$$

$$= C(\nu)\int_0^\infty d\tau \int_{\{|y|^2<\tau^2\}^+} (\tau^2-|y|^2)^{\frac{\alpha-n-|\nu|-1}{2}} y^\nu\, dy$$
$$\times \int_0^\pi \cdots \int_0^\pi f\Big(t-\tau, \sqrt{(y_1\cos\varphi_1+x_1)^2+y_1^2\sin^2\varphi_1}, \dots, \sqrt{(y_n\cos\varphi_n+x_n)^2+y_n^2\sin^2\varphi_n}\Big)$$
$$\times \prod_{i=1}^n \sin^{\nu_i-1}\varphi_i\, d\varphi_i.$$

Introducing the new variables

$$z_1 = y_1\cos\varphi_1,\ z_2 = y_1\sin\varphi_1, \dots, z_{2n-1} = y_n\cos\varphi_n,\ z_{2n} = y_n\sin\varphi_n, \quad 0 \le \varphi_i \le \pi,\ i = 1,\dots,n, \tag{3.2}$$

we obtain

$$(I^\alpha_{s,\nu}f)(t,x) = C(\nu)\int_0^\infty d\tau \int_{\{|z|^2<\tau^2\}^+} (\tau^2-|z|^2)^{\frac{\alpha-n-|\nu|-1}{2}}$$
$$\times f\Big(t-\tau, \sqrt{(z_1+x_1)^2+z_2^2}, \dots, \sqrt{(z_{2n-1}+x_n)^2+z_{2n}^2}\Big)\prod_{i=1}^n z_{2i}^{\nu_i-1}\, dz,$$

where

$$\{|z|^2<\tau^2\}^+ = \{z_{2i-1}\in\mathbb{R}, z_{2i}\in\mathbb{R}_+, i = 1,\dots,n-1 : |z|^2<\tau^2\}.$$

Changing variables by using the formula $(z_{2i-1}+x_{i+1}) \to z_{2i-1}$, $i = 1,\dots,n$, and using the designation

$$|\widetilde{z}| = \sqrt{(z_1-x_1)^2+z_2^2+\dots+(z_{2n-1}-x_n)^2+z_{2n}^2},$$

we can write

$$(I^\alpha_{s,\nu}f)(t,x) = C(\nu)\int_0^\infty d\tau \int_{\{|\widetilde{z}|^2<\tau^2\}^+} (\tau^2-|\widetilde{z}|^2)^{\frac{\alpha-n-|\nu|-1}{2}} f\Big(t-\tau, \sqrt{z_1^2+z_2^2}, \dots, \sqrt{z_{2n-1}^2+z_{2n}^2}\Big)\prod_{i=1}^n z_{2i}^{\nu_i-1}\, dz.$$

Let us consider the part of the sphere $|\widetilde{z}|^2+\xi^2 = \tau^2$ in a space of dimension $2n+1$ for points $(z_1,\dots,z_{2n},\xi)$, with center at the origin and of radius τ. The projection of this part of the sphere onto the plane $\xi = 0$ is $|\widetilde{z}|^2 \le \tau^2$ and $dS = \frac{\tau}{\xi}\, dz$, hence the integral can be rewritten in the form

$$(I^\alpha_{s,\nu}f)(t,x) = C(\nu)\int_0^\infty \frac{d\tau}{\tau} \int_{\{|\widetilde{z}|^2+\xi^2=\tau^2\}^+} \xi^{\alpha-n-|\nu|} f\Big(t-\tau, \sqrt{z_1^2+z_2^2}, \dots, \sqrt{z_{2n-1}^2+z_{2n}^2}\Big)\prod_{i=1}^n z_{2i}^{\nu_i-1}\, dS$$
$$= C(\nu)\int_0^\infty \frac{d\tau}{\tau} \int_{\{|z|^2+\xi^2=\tau^2\}^+} \xi^{\alpha-n-|\nu|} f\Big(t-\tau, \sqrt{(z_1-x_1)^2+z_2^2}, \dots, \sqrt{(z_{2n-1}-x_n)^2+z_{2n}^2}\Big)\prod_{i=1}^n z_{2i}^{\nu_i-1}\, dS$$
$$= C(\nu)\int_{\mathbb{R}_+^{2n+1}} \frac{\xi^{\alpha-n-|\nu|}}{\sqrt{|z|^2+\xi^2}} f\Big(t-\sqrt{|z|^2+\xi^2}, \sqrt{(z_1-x_1)^2+z_2^2}, \dots, \sqrt{(z_{2n-1}-x_n)^2+z_{2n}^2}\Big)\prod_{i=1}^n z_{2i}^{\nu_i-1}\, dz\, d\xi.$$

Returning to the variables $y_1,\dots,y_n,\varphi_1,\dots,\varphi_n$, by formulas (3.2), we get

$$(I^\alpha_{s,\nu}f)(t,x) = \int_{\mathbb{R}_+^{n+1}} \frac{\xi^{\alpha-n-|\nu|}}{\sqrt{|y|^2+\xi^2}}\, {}^\nu\mathbf{T}_x^y f\Big(t-\sqrt{|y|^2+\xi^2}, x\Big) y^\nu\, dy\, d\xi.$$

Renaming the variable ξ through τ, we get (3.1). □

Using representation (3.1), we extend $I^\alpha_{s,\nu}f$, $f \in S_{\mathrm{ev}}$ for values $\alpha > n+|\nu|-3$.

Theorem 3.2. *For $f \in S_{\mathrm{ev}}$, the analytical continuation of $I^\alpha_{s,\nu} f$ for $n > 2$ and $\alpha \in (n + |\nu| - 3, n + |\nu| - 1]$ is given by the formula*

$$
\begin{aligned}
(I^\alpha_{s,\nu} f)(t,x) &= C(\nu) \int\limits_{\mathbb{R}^n_+} \prod_{i=1}^{n} z_{2i}^{\nu_i - 1}\, dz' \int\limits_0^\infty \frac{\rho^{\alpha - |\nu|}}{\sqrt{\rho^2 + |z'|^2}}\, d\rho \\
&\quad \times \int\limits_0^\pi \cdots \int\limits_0^\pi \sin^{n-2}\varphi_2 \cdots \sin\varphi_{n-1}\, d\varphi_2 \cdots d\varphi_{n-1} \int\limits_0^{2\pi} d\varphi_n \int\limits_0^{\pi/2} \cos^{\alpha - n - |\nu|}\varphi_1 \sin^{n-1}\varphi_1 \mathcal{F}\, d\varphi_1 \\
&= \frac{1}{n + |\nu| - \alpha - 1} \int\limits_0^{\pi/2} \cos^{\alpha - n - |\nu| + 2}\varphi_1 \big[(n-2)\sin^{n-3}\varphi_1 \mathcal{F} + \sin^{n-2}\varphi_1 G(\rho,\varphi)\big]\, d\varphi_1,
\end{aligned}
\tag{3.3}
$$

where $z' = (z_2, \dots, z_{2n})$,

$$
\mathcal{F} = f\Big(t - \sqrt{\rho^2 + |z'|^2}, \sqrt{(\rho\sigma_1 - x_1)^2 + z_2^2}, \dots, \sqrt{(\rho\sigma_n - x_n)^2 + z_{2n}^2}\Big),
$$

and

$$
G(\rho,\varphi) = \rho \cdot \Bigg(\mathcal{F}_2' \frac{\rho\sigma_2 - x_2}{\sqrt{(\rho\sigma_2 - x_2)^2 + z_2^2}} \cos\varphi_2 + \cdots + \mathcal{F}_n' \frac{\rho\sigma_n - x_n}{\sqrt{(\rho\sigma_n - x_n)^2 - z_{2n}^2}} \sin\varphi_2 \cdots \sin\varphi_{n-1} \sin\varphi_n\Bigg).
$$

Proof. Let us consider representation (3.1) of $I^\alpha_{s,\nu} f$, i.e.,

$$
\begin{aligned}
(I^\alpha_{s,\nu} f)(t,x) &= \int\limits_{\mathbb{R}^{n+1}_+} \frac{\tau^{\alpha - n - |\nu|}}{\sqrt{|y|^2 + \tau^2}} \overset{\nu}{\mathbf{T}}{}^y_x f\Big(t - \sqrt{|y|^2 + \tau^2}, x\Big) y^\nu \, dy\, d\tau \\
&= C(\nu) \int\limits_{\mathbb{R}^{2n+1}_+} \frac{\xi^{\alpha - n - |\nu|}}{\sqrt{|z|^2 + \xi^2}} f\Big(t - \sqrt{|z|^2 + \xi^2}, \sqrt{(z_1 - x_1)^2 + z_2^2}, \dots, \sqrt{(z_{2n-1} - x_n)^2 + z_{2n}^2}\Big) \\
&\quad \times \prod_{i=1}^{n} z_{2i}^{\nu_i - 1}\, dz\, d\xi.
\end{aligned}
$$

Passing to spherical coordinates only with respect to the variables $\xi, z_1, \dots, z_{2n-1}$, by the formulas

$$
\begin{aligned}
\xi &= \rho\cos\varphi_1, \\
z_1 &= \rho\sin\varphi_1\cos\varphi_2 = \rho\sigma_1, \\
z_3 &= \rho\sin\varphi_1\sin\varphi_2\cos\varphi_3 = \rho\sigma_2, \\
&\ \ \vdots \\
z_{2n-3} &= \rho\sin\varphi_1\cdots\sin\varphi_{n-1}\cos\varphi_n = \rho\sigma_{n-1}, \\
z_{2n-1} &= \rho\sin\varphi_1\cdots\sin\varphi_{n-1}\sin\varphi_n = \rho\sigma_n,
\end{aligned}
$$

where $\rho = \sqrt{\xi^2 + z_1^2 + \cdots + z_{2n-1}^2} > 0$, $0 < \varphi_1 < \pi/2$, $0 < \varphi_2 < \pi, \dots, 0 < \varphi_{n-1} < \pi$ and $0 < \varphi_n < 2\pi$, with Jacobian

$$
\mathcal{J} = \rho^n \sin^{n-1}\varphi_1 \sin^{n-2}\varphi_2 \cdots \sin\varphi_{n-1},
$$

we obtain

$$
\begin{aligned}
(I^\alpha_{s,\nu} f)(t,x) &= C(\nu) \int\limits_{\mathbb{R}^n_+} \prod_{i=1}^{n} z_{2i}^{\nu_i - 1}\, dz' \int\limits_0^\infty \frac{\rho^{\alpha - |\nu|}}{\sqrt{\rho^2 + |z'|^2}}\, d\rho \int\limits_0^\pi \cdots \int\limits_0^\pi \sin^{n-2}\varphi_2 \cdots \sin\varphi_{n-1}\, d\varphi_2 \cdots d\varphi_{n-1} \int\limits_0^{2\pi} d\varphi_n \\
&\quad \times \int\limits_0^{\pi/2} \cos^{\alpha - n - |\nu|}\varphi_1 \sin^{n-1}\varphi_1 \mathcal{F}\, d\varphi_1,
\end{aligned}
\tag{3.4}
$$

where

$$\mathcal{F} = f\Big(t - \sqrt{\rho^2 + |z'|^2}, \sqrt{(\rho\sigma_1 - x_1)^2 + z_2^2}, \dots, \sqrt{(\rho\sigma_n - x_n)^2 + z_{2n}^2}\Big), \quad z' = (z_2, \dots, z_{2n}).$$

By integrating the integral

$$\int_0^{\pi/2} \cos^{\alpha-n-|\nu|}\varphi_1 \sin^{n-1}\varphi_1 \mathcal{F} d\varphi_1$$

by parts, putting

$$v = -\frac{\cos^{\alpha-n-|\nu|+1}\varphi_1}{\alpha - n - |\nu| + 1}, \quad dv = \cos^{\alpha-n-|\nu|}\varphi_1 \sin\varphi_1 d\varphi_1,$$
$$u = \sin^{n-2}\varphi_1 \mathcal{F}, \qquad du = \frac{\partial}{\partial\varphi_1}(\sin^{n-2}\varphi_1\mathcal{F})\, d\varphi_1 \quad \text{for } n > 2,\ n + |\nu| - 1 < \alpha,$$

we get

$$\int_0^{\pi/2} \cos^{\alpha-n-|\nu|}\varphi_1 \sin^{n-1}\varphi_1 \mathcal{F}\, d\varphi_1 = -\frac{1}{\alpha - n - |\nu| + 1}\int_0^{\pi/2} \cos^{\alpha-n-|\nu|+1}\varphi_1 \frac{\partial}{\partial\varphi_1}(\sin^{n-2}\varphi_1\mathcal{F})\, d\varphi_1.$$

The resulting integral converges already for $n + |\nu| - 3 < \alpha$, since

$$\begin{aligned}
\frac{\partial}{\partial\varphi_1}(\sin^{n-2}\varphi_1\mathcal{F}) &= (n-2)\sin^{n-3}\varphi_1\cos\varphi_1\mathcal{F} + \sin^{n-2}\varphi_1\mathcal{F}'_{\varphi_1} \\
&= \cos\varphi_1[(n-2)\sin^{n-3}\varphi_1\mathcal{F} + \sin^{n-2}\varphi_1 G(\rho,\varphi)], \\
\mathcal{F}'_{\varphi_1} &= \cos\varphi_1 G(\rho,\varphi), \\
G(\rho,\varphi) &= \rho\cdot\Bigg(\mathcal{F}'_2\frac{\rho\sigma_2 - x_2}{\sqrt{(\rho\sigma_2 - x_2)^2 + z_2^2}}\cos\varphi_2 + \dots + \mathcal{F}'_n\frac{\rho\sigma_n - x_n}{\sqrt{(\rho\sigma_n - x_n)^2 - z_{2n}^2}}\sin\varphi_2\cdots\sin\varphi_{n-1}\sin\varphi_n\Bigg)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^{\pi/2} \cos^{\alpha-n-|\nu|}\varphi_1 \sin^{n-1}\varphi_1\mathcal{F}d\varphi_1 \\
&\qquad = \frac{1}{n + |\nu| - \alpha - 1}\int_0^{\pi/2} \cos^{\alpha-n-|\nu|+2}\varphi_1[(n-2)\sin^{n-3}\varphi_1\mathcal{F} + \sin^{n-2}\varphi_1 G(\rho,\varphi)]\, d\varphi_1.
\end{aligned}$$

Substituting the resulting integral into (3.4), we obtain the formula for $I^\alpha_{s,\nu}f$, for $\alpha > n + |\nu| - 3$. □

Remark. Further integration by parts of the integral (3.3) with respect to φ_1 will allow us to construct an analytic continuation for $\alpha \in (n + |\nu| - 5, n + |\nu| - 3]$ etc. The cases $n = 1$ and $n = 2$ are treated similarly.

4 Image spaces for a mixed hyperbolic Riesz B-potentials

In this section we first give theorems related to the inversion of the operator $I^\alpha_{s,\nu}$, and then, by using these theorems, we describe the space of images for mixed hyperbolic Riesz B-potentials.

As for the inversion of potential (2.2), an approach based on the idea of approximative inverse operators was used in [6]. This method gives an inverse operator as a limit of regularized operators. Namely, taking into account formula (2.6), we will construct the inverse operator for potential (2.2) in the form

$$(I^\alpha_{s,\nu})^{-1}f = \lim_{\varepsilon\to 0}\Big(\mathcal{F}_\nu^{-1}(q|\tau^2 - |\xi|^2|^{\frac{\alpha}{2}} e^{-\varepsilon|\tau| - \varepsilon|\xi|}) * f\Big)_\nu,$$

where the limit is understood in the norm L_p^ν or almost everywhere.

Let

$$g_{\alpha,\nu,\varepsilon}(t,x) = \mathcal{F}_\nu^{-1}(q^{-1}|\tau^2 - |\xi|^2|^{\frac{\alpha}{2}} e^{-\varepsilon|\tau|-\varepsilon|\xi|})(t,x).$$

Then

$$((I^\alpha_{s,\nu})^{-1}_\varepsilon f)(t,x) = \int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mathbb{R}^n_+}({}^\nu\mathbf{T}^y_x)g_{\alpha,\nu,\varepsilon}(t-\tau,x)\cdot f(\tau,y)y^\nu\,dy\Big)\,d\tau.$$

Now we introduce the homogenizing kernel $N_\nu(t,x,\varepsilon)$, which is defined as follows:

$$N_\nu(t,x,\varepsilon) = \frac{C(n,\nu,\varepsilon)}{(t^2+\varepsilon^2)(|x|^2+\varepsilon^2)^{\frac{n+|\nu|}{2}}},$$

where $x = (x_1,\dots,x_n)\mathbb{R}^n_+$, $t \in \mathbb{R}$, $\varepsilon > 0$, and

$$C(n,\nu,\varepsilon) = \frac{2^n\varepsilon^2\Gamma(\frac{n+1+|\nu|}{2})}{\pi^{\frac{3}{2}}\prod_{i=1}^n\Gamma(\frac{\nu_i+1}{2})}.$$

We give some properties for the function $N_\nu(t,x,\varepsilon)$, proved in [12] for a more general case.

Theorem 4.1. *The homogenizing kernel $N_\nu(t,x,\varepsilon)$ has the properties*
(i) $\mathcal{F}_\nu[N_\nu(t,x,\varepsilon)](\xi) = e^{-\varepsilon\tau-\varepsilon|\xi|}$,
(ii) $\int_{-\infty}^{\infty}(\int_{\mathbb{R}^n_+}N_\nu(t,x,\varepsilon)x^\nu\,dx)\,dt = \int_{-\infty}^{\infty}(\int_{\mathbb{R}^n_+}N_\nu(t,x,1)x^\nu\,dx)\,dt = 1$,
(iii) $N_\nu(t,x,\varepsilon) \in L^\nu_p$, $1 \le p \le \infty$.

Theorem 4.2. *Let $f \in L^\nu_p$ and*

$$(N_{\nu,\varepsilon}f)(\tau,y) = \int\limits_{-\infty}^{\infty}\Big(\int\limits_{\mathbb{R}^n_+}N_\nu(t,x,\varepsilon)T^y_x f(\tau-t,x)x^\nu\,dx\Big)\,dt.$$

Then

$$\lim_{\varepsilon\to 0}(N_{\nu,\varepsilon}f)(\tau,y) = f(\tau,y)\quad a.e.$$

Theorem 4.3. *Let $n+|\nu|-1 < \alpha < n+1+|\nu|$, $1 < p < \frac{n+1+|\nu|}{\alpha}$, with the additional restriction*

$$p < \frac{2(n+1+|\nu|)(n+|\nu|)}{n+|\nu|+3\alpha(n+|\nu|)}\quad \textit{when } n+|\nu|-1<\alpha<n+|\nu| \textit{ and } n \textit{ is odd.}$$

Then

$$((I^\alpha_{s,\nu})^{-1}_\varepsilon I^\alpha_{s,\nu}f)(t,x) = (N_{\nu,\varepsilon}f)(t,x),\qquad f(x) \in L^\nu_p,$$

where

$$(I^\alpha_{s,\nu})^{-1}_\varepsilon f = (\mathcal{F}_\nu^{-1}(q|\tau^2-|\xi|^2|^{\frac{\alpha}{2}}e^{-\varepsilon|\tau|-\varepsilon|\xi|}) * f)_\nu.$$

Theorem 4.4. *Let $n+|\nu|-1 < \alpha < n+1+|\nu|$, $1 < p < \frac{n+1+|\nu|}{\alpha}$, with the additional restriction*

$$p < \frac{2(n+1+|\nu|)(n+|\nu|)}{n+1+|\nu|+2\alpha(n+|\nu|)}\quad \textit{when } n+|\nu|-1<\alpha<n+|\nu| \textit{ and } n \textit{ is odd.}$$

Then

$$((I^\alpha_{s,\nu})^{-1}I^\alpha_{s,\nu}f)(t,x) = f(t,x),\qquad f(t,x) \in L^\nu_p,$$

where

$$(I^\alpha_{s,\nu})^{-1}f = \lim_{\varepsilon\to 0}(I^\alpha_{s,\nu})^{-1}_\varepsilon f.$$

Theorems 4.3–4.4 were proved in [13].

Let $I^\alpha_{s,\nu}(L^\nu_p)$ be the images of potentials (2.3), i.e.,

$$I^\alpha_{s,\nu}(L^\nu_p) = \{f(t,x) : f(t,x) = (I^\alpha_{s,\nu}\varphi)(t,x),\ \varphi(t,x) \in L^\nu_p\},$$

$1 < p < \frac{n+1+|\nu|}{\alpha}$, $n+|\nu|-1 < \alpha < n+|\nu|+1$. Assume

$$\|f\|_{I^\alpha_{s,\nu}(L_p\nu)} = \|\varphi\|_{p,\nu},$$

where $(I^\alpha_{s,\nu}\varphi)(t,x) = f(t,x)$.

Theorem 4.5. *Let $n + |\nu| - 1 < \alpha < n + 1 + |\nu|$, $1 < p < \frac{n+1+|\nu|}{\alpha}$, with the additional restriction*

$$p < \frac{2(n+1+|\nu|)(n+|\nu|)}{n+1+|\nu|+2\alpha(n+|\nu|)} \quad \text{when } n+|\nu|-1 < \alpha < n+|\nu| \text{ and } n+|\nu| \text{ is odd.}$$

Then, in order that $f \in I^\alpha_{s,\nu}(L^\nu_p)$, it is necessary and sufficient that $f \in L^\nu_r$, $r = \frac{(n+1+|\nu|)p}{n+1+|\nu|-\alpha p}$ and the limit

$$\lim_{\varepsilon\to 0}(I^\alpha_{s,\nu})^{-1}_\varepsilon f = (I^\alpha_{s,\nu})^{-1} f$$

exists in the L^ν_p-sense. Whence

$$\|f\|_{I^\alpha_{s,\nu}(L_p\nu)} = \|(I^\alpha_{s,\nu})^{-1} f\|_{p,\nu}. \tag{4.1}$$

Proof. The necessity follows from Theorems 2.3 and 4.4.

Let us show sufficiency. We should demonstrate that for almost all $t \in \mathbb{R}$, $x \in \mathbb{R}^n_+$, we have

$$f(t,x) = (I^\alpha_{s,\nu}(I^\alpha_{s,\nu})^{-1} f)(t,x). \tag{4.2}$$

From (4.2) will follow that $f \in I^\alpha_{s,\nu}(L^\nu_p)$ and (4.1). Let us establish equality (4.2) first in the sense of Φ'_V, where $V = \{\tau \in \mathbb{R}, \xi \in \mathbb{R}^n_+ : \tau^2 - |\xi|^2 = 0\}$, and then we will proceed to the equality almost everywhere. For any $\omega(t,x) \in \overset{\circ}{C}{}^\infty_{\mathrm{ev}}$, we choose a sequence $\omega_N(t,x) \in \Phi_V$ approximating $\omega(t,x)$ by norm in $L^\nu_{r'}$, $r' = \frac{r}{r-1}$. Then

$$(I^\alpha_{s,\nu}(I^\alpha_{s,\nu})^{-1} f, \omega_N)_\nu = ((I^\alpha_{s,\nu})^{-1} f, \overline{I}^\alpha_{s,\nu}\omega_N)_\nu,$$

where

$$(\overline{I}^\alpha_{s,\nu} f)(t,x) = \int_{-\infty}^{\infty}\Big(\int_{\mathbb{R}^n_+} \overline{s}_+^{\frac{\alpha-n-|\nu|-1}{2}}(\tau,y)({}^\nu\mathbf{T}^y_x) f(t-\tau,x) y^\nu\,dy\Big)\,d\tau$$

and

$$\overline{s}^\lambda(t,x) = \begin{cases} \frac{(t^2-|x|^2)^\lambda}{N(\alpha,\nu,n)} & \text{when } t^2 \ge |x|^2 \text{ and } t \le 0,\\ 0 & \text{when } t^2 < |x|^2 \text{ or } t > 0.\end{cases}$$

We obtain

$$\begin{aligned}((I^\alpha_{s,\nu})^{-1} f, \overline{I}^\alpha_{s,\nu}\omega_N)_\nu &= \Big(\lim_{\varepsilon\to 0}(I^\alpha_{s,\nu,\varepsilon})^{-1} f, \overline{I}^\alpha_{s,\nu}\omega_N\Big)_\nu = \lim_{\varepsilon\to 0}((I^\alpha_{s,\nu,\varepsilon})^{-1} f, \overline{I}^\alpha_{s,\nu}\omega_N)_\nu\\ &= \lim_{\varepsilon\to 0}((f, (\overline{I}^\alpha_{s,\nu,\varepsilon})^{-1}\overline{I}^\alpha_{s,\nu}\omega_N)_\nu = \lim_{\varepsilon\to 0}(f, N_\varepsilon\omega_N)_\nu = \lim_{\varepsilon\to 0}(N_\varepsilon f, \omega_N)_\nu,\end{aligned}$$

where

$$((\overline{I}^\alpha_{s,\nu,\varepsilon})^{-1} f)(t,x) = \int_{-\infty}^{\infty}\Big(\int_{\mathbb{R}^n_+}({}^\nu\mathbf{T}^y_x) g_{\alpha,\nu,\varepsilon}(\tau - t, x)\cdot f(\tau,y) y^\nu\,dy\Big)\,d\tau.$$

Taking into account that $\lim_{\varepsilon\to 0}(N_{\nu,\varepsilon} f)(\tau,y) = f(\tau,y)$ in L^ν_p (see Theorem 4.2), we obtain

$$(I^\alpha_{s,\nu}(I^\alpha_{s,\nu})^{-1} f, \omega_N)_\nu = (f,\omega_N)_\nu, \quad \omega_N(t,x) \in \Phi_V.$$

Passing to the limit as N tends to ∞ in the last equality, we obtain

$$(I^\alpha_{s,\nu}(I^\alpha_{s,\nu})^{-1} f, \omega)_\nu = (f,\omega)_\nu \quad \text{for all } \omega \in \overset{\circ}{C}{}^\infty_{\mathrm{ev}},$$

whence (4.2) follows. □

References

[1] S. Helgason, *Groups and Geometric Analysis*, Math. Surveys Monogr. 83, American Mathematical Society, Providence, 2000.

[2] S. Helgason, *Integral Geometry and Radon Transforms*, Springer, New York, 2011.

[3] I. A. Kipriyanov, *Singular Elliptic Boundary Value Problems* (in Russian), Nauka, Moscow, 1997.

[4] B. M. Levitan, Expansion in Fourier series and integrals with Bessel functions (in Russian), *Uspehi Matem. Nauk (N. S.)* **6** (1951), no. 2(42), 102–143.
[5] L. N. Lyakhov, I. P. Polovinkin and E. L. Shishkina, Formulas for the solution of the Cauchy problem for a singular wave equation with a Bessel time operator, *Dokl. Math.* **90(3)** (2014), 737–742.
[6] V. A. Nogin and E. V. Sukhinin, Inversion and description of hyperbolic potentials with L_p-densities (in Russian), *Dokl. Akad. Nauk* **329** (1993), no. 5, 550–552.
[7] A. P. Prudnikov, Y. A. Brychkov and O. I. Marichev, *Integrals and Series. Vol. 2: Special Functions*, Gordon & Breach Science, New York, 1990.
[8] M. Riesz, Intégrale de Riemann–Liouville et solution invariantive du probléme de Cauchy pour l'équation de sondes, *Compt. Rend. Congres Int. Math.* **2** (1936), 44–45.
[9] M. Riesz, L'intégrale de Riemann–Liouville et le problème de Cauchy, *Acta Math.* **81** (1949), 1–223.
[10] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives - Theory and Applications*, Gordon & Breach Science, Amsterdam, 1987.
[11] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.
[12] E. L. Shishkina, About properties of the averaging kernel in the Lebesgue weighted class (in Russian), *Sci. Bull. Belgorod State Univ. Ser. Math. Phys.* **42** (2016), no. 6(227), 12–19.
[13] E. L. Shishkina, Inversion of the mixed Riesz hyperbolic B-potentials, *Int. J. Appl. Math.* **30** (2017), no. 6, 487–500.
[14] E. L. Shishkina, Properties of mixed hyperbolic B-potential, *Progr. Fract. Differ. Appl.* **4** (2018), no. 2, 83–98.
[15] E. L. Shishkina and S. M. Sitnik, General form of the Euler–Poisson–Darboux equation and application of the transmutation method, *Electron. J. Differential Equations* **2017** (2017), Paper No. 177.
[16] E. M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton Math. Ser. 30, Princeton University Press, Princeton, 1970.
[17] Y. I. Žitomirskiĭ, Cauchy's problem for systems of linear partial differential equations with differential operators of Bessel type, *Mat. Sb. (N. S.)* **36(78)** (1955), 299–310.