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Probabilistic Properties of Near-Optimal Trajectories of an Agent Moving Over a Lattice

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Abstract
The paper considers probabilistic properties of the trajectory of a moving agent. The agent finds a route close to the optimal one on a lattice consisting of cells with different impassabilities. We study the distribution of the agent's exit time to the end point for random landscapes of different types using a special sort of simulation. After that, we compare the obtained empirical probability density function with the probability density function derived from theoretical considerations. We also obtain the probability density function for the ratio of Rician and uniform random variables. Finally, the probability distribution of the agent's residence in a given cell at a given moment of time for random landscapes of different types is analyzed.

Keywords Agents · Lattice · Pathfinding · Brownian bridge · Rician distribution · Extreme values distribution

Mathematics Subject Classification 49J55 · 49M30 · 37B15

1 Introduction

Previously, the first author has developed the cellular automaton simulating a motion of hierarchically organized agent's swarms and formations over a rough terrain [1–3]. Also, he studied a connection between the automaton discrete model and the corresponding continuous optimization problem [4].

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The "terrain" consists of randomly placed cells with different passabilities. The method for the random "terrain" or random landscape generation, as well as landscape properties, was studied in [5]. Agents can be people or unmanned aerial vehicle (UAV) moving through the battlefield or, as in the article [6], macromolecules or cells in overcrowded environments.

The movement of agent over a random landscape can be seen as Brownian bridge realization. In the work [7], a Brownian bridge motion model (BBMM) was studied as a biologically grounded approximation for an animal movement path. The approximation was obtained from the discrete animal's location data. BBMM is a powerful mean to estimate a utilization distribution.

The use of cellular automata for the optimal pathfinding, for example, in chemical or biological processors, is well known [8,9]. However, in such studies, e.g., [10–12], the formal logical approach prevails, in the spirit of computability theory. Statistical and, speaking in general, analytical properties of such systems have been studied little ever for simplest cases.

Consider the case of the randomized movement of an agent over a fixed landscape from a fixed start point to a fixed destination. The set of several realizations of such movements can be compared with a motion of agents' swarm. If the agent can, at each subsequent realization of its movement, change the properties of the landscape in such way that this will affect the next realization, then it is quite consistent with the movement of the swarm of interacting agents.

Swarms, even consisting of unicellular organisms, can solve logistic and transport problems, including NP-hard ones, very effectively. For instance, there is a collective navigation of bacterial swarms, an effective pathfinding by amoebae and *Physarum polycephalum*, and a possibility of traffic optimization by them. Ants can solve mazes, ants and amoebae can solve the traveling salesman problem, amoebae can solve the Steiner tree problem, bees can solve the generalized assignment problem, and so on. By these reasons, the swarm intelligence is actively studied now.

In the present work, we study statistical properties of near-optimal trajectories of an agent, obtained from the cellular automaton, previously proposed by the first author. These trajectories are interpreted as Brownian bridge realizations. We use the hypothesis, that an agent's speed has Rician distribution as in [7]. The approach described in the present article can be extended to the study of the probabilistic properties of swarm optimization.

2 Continuous Optimization Problem and Its Discretization

Let $\Omega \subseteq \mathbb{R}^2$, $\mathbb{R}_+$ be the set of real nonnegative numbers, and consider the motion problem for an agent seeking the quickest route $\mathbf{r} = \frown{\mathbf{AB}}$ from the point $\mathbf{A} \in \Omega$ to the point $\mathbf{B} \in \Omega$. This problem can be formulated as

$$\min T, \text{ subject to} \tag{1}$$

$$\|\dot{\mathbf{r}}(t)\| = u^c(t, \mathbf{r}(t)), \quad t \in [0, T_{\max}], \tag{2}$$

$$\mathbf{r}(0) = \mathbf{A}, \quad \mathbf{r}(T) = \mathbf{B}, \quad 0 \le T \le T_{\max}, \tag{3}$$

where $\dot{\mathbf{r}}$ is the first derivative of $\mathbf{r}$ with respect to t, $u^c : \mathbb{R}_+ \times \Omega \to \mathbb{R}_+$ is a function of the domain Ω passability, which defines the maximum possible speed $u^c(t, \boldsymbol{\rho})$ in the point $\boldsymbol{\rho} \in \Omega$ at the moment of time t, $\|\cdot\|$ is the Euclidian norm,

$$T(\mathbf{r}) := \min\{t \in \mathbb{R}_+ : \mathbf{r}(t) = \mathbf{B}\}.$$

From (1) to (3), it is possible to obtain the expression for

$$T : C([0, T_{\max}]; \mathbb{R}^2) \to \mathbb{R}_+$$

as the following one

$$T(\mathbf{r}) = \int_0^T \frac{\|\dot{\mathbf{r}}(t)\|\mathrm{d}t}{u^c(t, \mathbf{r}(t))} = \int_{\widehat{\mathbf{AB}}} \frac{\mathrm{d}l}{u^c(t(l), \mathbf{r}(t(l)))}, \tag{4}$$

where $t \in [0, T]$ is the time variable and l is the natural parameter.

Clearly, it is impossible to obtain explicit form of T from (4) in common case. The exception is, for example, the case when $\mathbf{r}$ is the graph of some function $y \in C^2[a, b]$, i.e.,

$$\mathbf{r} := \{(x, y(x)) : x \in [a, b]\},$$

and u^c does not depend on time. However, one can obtain various properties of the functional T, prove the solvability of the minimization problem (1)–(3) in a class $\mathcal{Y} \subset W_p^1([0, T_{\max}]; \mathbb{R}^2)$, and construct an algorithm for the problem's approximate solution with estimates for it [4].

Let us denote $u^c_{\max} := \max\limits_{(t,\boldsymbol{\rho})\in[0,T_{\max}]\times\Omega} u^c(t, \boldsymbol{\rho})$. Introduce the impassability function

$$u(t, \boldsymbol{\rho}) := \frac{u^c_{\max}}{u^c(t, \boldsymbol{\rho})}.$$

Let $\Omega_h \supseteq \Omega$ be a tessellation of Ω, $\Omega_h = \{\omega_{ij} : (i, j) \in \mathbb{Z}^2\}$, and $\mathbf{r}_h \supseteq \mathbf{r}$ is a cellular route, i.e., minimal covering of the route $\mathbf{r}$ by cells from Ω_h. We denote the set of all cellular routes which approximate routes from $\mathcal{Y}$ by $\mathcal{Y}_h$.

Denote as $\mathbf{r}_h[k]$ the kth by the order of passing cell of the route $\mathbf{r}_h$. The functional T has discrete analog

$$T_h(\mathbf{r}_h) := \sum_{(i,j)\in\mathbf{r}_h} u_{ij}(t)\|\mathbf{d}_{ij}\|, \tag{5}$$

where $u_{ij}(t) = \max\limits_{\boldsymbol{\rho}\in\omega_{ij}} u(t, \boldsymbol{\rho})$ is the impassability of the cell (i, j) in the time moment t,

$$\mathbf{d}_{ij} := \mathbf{d}_k := \mathbf{r}_h[k+1] - \mathbf{r}_h[k], \quad \mathbf{r}_h[k] := (i, j).$$

When the grid step h is decreased, the sequence of cell routes $\{\mathbf{r}_h\}$ minimizing the functional (5) will converge in some sense to the route $\mathbf{r}$ minimizing the functional (4). Next, we will use the notation

$$u_{ij}(t) := u(t, (i, j)) := \max_{\boldsymbol{\rho} \in \omega_{ij}} u(t, \boldsymbol{\rho}).$$

For example, assume that $u^c_{\max} = 8$ m/s and $u^c(t, (1, 1)) = 2$ m/s, $t \in [0, T]$. It means that the maximum agent's speed at the point (1, 1) is always 2 m/s and the maximum possible speed in the whole Ω is 8 m/s. We obtain that $u(t, (1, 1)) = 4$. If $(1, 1) \in \omega_{11}$, $\omega_{11} \in \Omega_h$, and

$$\max_{t \in [0,T]} \max_{\boldsymbol{\rho} \in \omega_{11}} u(t, \boldsymbol{\rho}) = u(t, (1, 1)) = 4,$$

then an agent can pass the cell ω_{11} in 4 quanta of discrete time. The simplest example of Ω_h is a square grid with equal-sized cells.

It is quite common that we can measure impassability of a landscape only in a sufficiently small number of points, and these points do not form any uniform grid. Therefore, the more advanced case of Ω_h is the Voronoi diagram of a set of points with a given impassability. The Voronoi diagram is a natural partition of a landscape into cells with different properties. The example of a cellular automaton using the Voronoi cells is studied in [13].

Let us denote by $\mathbf{r}_h(t)$ the cell of a cell route, where agent is situated in the time moment t. Note that for $\chi \le \kappa d(t) u(t, \mathbf{r}_h(t))$, we obtain the discrete analog of (1)

$$\|\mathbf{r}_h(t + \chi) - \mathbf{r}_h(t)\| = \|\mathbf{d}(t)\| \left\lfloor \frac{\chi}{\kappa \|\mathbf{d}(t)\| u(t, \mathbf{r}_h(t))} \right\rfloor, \tag{6}$$

where $\mathbf{d}(t) \in \mathcal{D} := \{(i, j) : i, j = \overline{-1, 1}\}$ is the direction vector of the agent, κ is the number of seconds in one discrete time quantum, and $\lfloor x \rfloor$ is the floor function of x.

We previously designed a cellular automaton, where agents moving in accordance with (6) sought the shortest route to the destination, minimizing the functional (5) in some agent's neighborhood, and then concatenated a near-optimal route from the resulting fragments [1]. As an alternative, the probability of choosing the locally optimal route by the agent is maximal, and less the route is optimal, the less likely that the agent wants to choose it.

At the same time, in each neighborhood of the agent in the cell ω_{ij}, $U_o(i, j) := \{\omega_{ij} : \|(i, j)\| \le o, \omega_{ij} \in \Omega_h\}$, $\mathbf{B} \notin U_o(i, j)$, the agent minimizes the time of the path from the point $\mathbf{B}_i$ of the intersection of the boundary of $U_o(i, j)$ with the closed segment connecting the agent's location and $\mathbf{B}$.

3 Dependencies of Statistical Properties of the Near-Optimal Route on Characteristics of a Terrain

Definition 1 The *landscape* in the time moment t is the set of tessellation's cells impassabilities

$$\mathcal{L}(\Omega_h, l) := \{u_{ij} : u_{ij} = u(t, \omega_{ij}), \omega_{ij} \in \Omega_h\},$$

such as u takes on $\{\omega_{ij} : i = \overline{1,n},\ j = \overline{1,m}\}$ no more than l possible values, and to the impassability class i belongs N_i cells, i.e., $\sum_{i=1}^{l} N_i = M$, $M = |\mathcal{L}(\Omega_h, l)|$.

Denote as "$\overset{\approx}{\min}$" a *near-minimum* of functionals T and T_h obtained in the aforementioned way. We call this solution *near-minimum*, because it can be used as an easily computable substitution of the true minimum. If a *landscape* consists of cells whose impassabilities are randomly distributed, then each near-optimal route found by the agent in a certain random *landscape* can be viewed as the realization of a random walk on a lattice, and more precisely, as a Brownian bridge. In this case,

$$\tau = \overset{\approx}{\min_{\mathbf{r}\in\mathcal{Y}}} T(\mathbf{r}), \quad \tau_h = \overset{\approx}{\min_{\mathbf{r}_h\in\mathcal{Y}_h}} T_h(\mathbf{r}), \quad \tau_{\min} \le \tau_h \le \tau_{\max}$$

will be times to exit to the end point of the agent in the continuous and the discrete cases correspondingly. Next, we will try to determine the distribution law for random variables τ, τ_h and

$$\boldsymbol{\varrho} := \arg \overset{\approx}{\min_{\mathbf{r}\in\mathcal{Y}}} T(\mathbf{r}), \quad \boldsymbol{\varrho}_h := \arg \overset{\approx}{\min_{\mathbf{r}_h\in\mathcal{Y}_h}} T(\mathbf{r}_h)$$

both experimentally and from theoretical considerations and compare results.

We will take previously introduced by the first author "natural" and "uniform" *landscapes* [5] as random *landscapes*. In the case of a "uniform" *landscape*, the random variable $u(t, \boldsymbol{\rho}) \in [1, u_{\max}]$,

$$u_{\max} := \max_{(t,\boldsymbol{\rho})\in[0,T]\times\Omega} u(t, \boldsymbol{\rho}),$$

is close to a uniformly distributed random variable for all $(t, \boldsymbol{\rho}) \in \mathbb{R}_+ \times \Omega$. So,

$$u^c(t, \boldsymbol{\rho}) := \frac{u^c_{\max}}{u(t, \boldsymbol{\rho})}$$

has the inverse uniform distribution with the density function

$$\operatorname{invu}(x) = \begin{cases} \frac{u^c_{\max}}{u_{\max}-1} x^{-2}, & x \in \left[\frac{u^c_{\max}}{u_{\max}}, u^c_{\max}\right], \\ 0, & x \notin \left[\frac{u^c_{\max}}{u_{\max}}, u^c_{\max}\right]. \end{cases}$$

Assume that the random value $\|\dot{\mathbf{r}}(t)\|$ has Rician distribution with the probability density function (PDF)

$$\operatorname{rice}(\nu(t), \sigma(t); x) := \frac{x}{\sigma^2(t)} \exp\left(\frac{-(x^2 + \nu^2(t))}{2\sigma^2(t)}\right) I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right), \quad x \ge 0,$$

where I_α is the modified Bessel function

$$I_\alpha(x) := i^{-\alpha} J_\alpha(ix) := \sum_{m=0}^{\infty} \frac{1}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha} \tag{7}$$

for all $t \in [0, T_{\max}]$. This assumption holds if the agent's velocity is a circular bivariate normal random variable, as in [7]. Denote the function under the integral (4) as

$$\zeta(t) := \frac{\|\dot{\mathbf{r}}(t)\|}{u^c(t, \mathbf{r}(t))} = \frac{1}{u^c_{\max}} \|\dot{\mathbf{r}}(t)\| u(t, \mathbf{r}(t)).$$

From the above, the cumulative distribution function (CDF) of the integrand $\zeta(t)$ has the form

$$\mathbb{P}\{\zeta(t) < z\} = \iint_{D_z} \mathrm{rice}(\nu(t), \sigma(t); x) \Pi(y) \mathrm{d}x \mathrm{d}y, \tag{8}$$

where $D_z := \{(x, y) \in \mathbb{R}^2 : xy < z\}$,

$$\Pi(y) := \begin{cases} 1, & y \in \left[\frac{1}{u^c_{\max}}, \frac{u_{\max}}{u^c_{\max}}\right], \\ 0, & y \notin \left[\frac{1}{u^c_{\max}}, \frac{u_{\max}}{u^c_{\max}}\right]. \end{cases}$$

4 Analytical Approach to Finding the Distribution of the End Point Reaching Time

In this section, we will find the cumulative distribution function (8) of the integrand $\zeta(t)$ and its distribution density function analytically. Let $\frac{1}{u^c_{\max}} = 1$ and $\frac{u_{\max}}{u^c_{\max}} = b$. We have

$$\begin{aligned} \mathrm{CDF}_\zeta(z) := \mathbb{P}\{\zeta(t) \le z\} &= \iint_{D_z} \mathrm{rice}(\nu(t), \sigma(t); x) \Pi(y) \mathrm{d}x \mathrm{d}y \\ &= \frac{1}{\sigma^2(t)} \iint_{D_z} x \exp\left(\frac{-(x^2 + \nu^2(t))}{2\sigma^2(t)}\right) I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right) \Pi(y) \mathrm{d}x \mathrm{d}y, \end{aligned}$$

where

$$\Pi(y) = \begin{cases} 1, & y \in [1, b], \\ 0, & y \notin [1, b]. \end{cases}$$

Therefore,

$$
\begin{aligned}
\mathrm{CDF}_\zeta(z) &= \frac{1}{\sigma^2(t)}\,\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}} \int_1^b \mathrm{d}y \int_0^{\frac{z}{y}} x\mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x \\
&= \frac{1}{\sigma^2(t)}\,\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}} \left(\int_0^{\frac{z}{b}} x\mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x \int_1^b \mathrm{d}y \right. \\
&\quad \left. + \int_{\frac{z}{b}}^{z} x\mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x \int_1^{\frac{z}{x}} \mathrm{d}y \right) \\
&= \frac{1}{\sigma^2(t)}\,\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}} \left((b-1) \int_0^{\frac{z}{b}} x\mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x \right. \\
&\quad \left. + \int_{\frac{z}{b}}^{z} (z-x)\mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x \right). \qquad (9)
\end{aligned}
$$

Using series expansion $I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\left(\frac{\nu(t)}{\sigma^2(t)}\right)^{2n} x^{2n}$, we obtain

$$
\begin{aligned}
\mathrm{CDF}_\zeta(z) = \mathrm{e}^{-\frac{\nu^2}{2\sigma^2}} &\sum_{n=0}^{\infty} \frac{\nu^{2n}(t)}{2^n (n!)^2 \sigma^{2n}(t)} \\
&\times \left[\frac{z}{\sqrt{2}\sigma(t)} \left(\Gamma\left(n+\frac{1}{2}, \frac{z^2}{2b^2\sigma^2}\right) - \Gamma\left(n+\frac{1}{2}, \frac{z^2}{2\sigma^2}\right) \right) \right. \\
&+ \Gamma\left(n+1, \frac{z^2}{2\sigma^2}\right) - \Gamma\left(n+1, \frac{z^2}{2b^2\sigma^2}\right) \\
&\left. + (b-1)\left(\Gamma(n+1) - \Gamma\left(n+1, \frac{z^2}{2b^2\sigma^2}\right) \right) \right]. \qquad (10)
\end{aligned}
$$

Now, let us find distribution density function. Using (9), we obtain

$$
\begin{aligned}
\mathrm{PDF}_\zeta(z) &:= \frac{\mathrm{d}}{\mathrm{d}z}\mathbb{P}\{\zeta(t) \le z\} \\
&= \frac{1}{\sigma^2(t)}\,\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}} \left((b-1)\frac{\mathrm{d}}{\mathrm{d}z} \int_0^{\frac{z}{b}} x\mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x \right.
\end{aligned}
$$

$$+\frac{\mathrm{d}}{\mathrm{d}z} z \int_{\frac{z}{b}}^{z} \mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}z}\int_{\frac{z}{b}}^{z} x\mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x\Bigg)$$

$$= \frac{1}{\sigma^2(t)}\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}}\Bigg[\left(1-\frac{1}{b}\right)\frac{z}{b}\mathrm{e}^{-\frac{z^2}{2b^2\sigma^2(t)}} I_0\left(\frac{z\nu(t)}{b\sigma^2(t)}\right)$$

$$+ z\mathrm{e}^{-\frac{z^2}{2\sigma^2(t)}} I_0\left(\frac{z\nu(t)}{\sigma^2(t)}\right) - \frac{z}{b}\mathrm{e}^{-\frac{z^2}{2b^2\sigma^2(t)}} I_0\left(\frac{z\nu(t)}{b\sigma^2(t)}\right)$$

$$- z\mathrm{e}^{-\frac{z^2}{2\sigma^2(t)}} I_0\left(\frac{z\nu(t)}{\sigma^2(t)}\right) + \frac{z}{b^2}\mathrm{e}^{-\frac{z^2}{2b^2\sigma^2(t)}} I_0\left(\frac{z\nu(t)}{b\sigma^2(t)}\right)$$

$$+ \int_{\frac{z}{b}}^{z} \mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x\Bigg] = \frac{1}{\sigma^2(t)}\int_{\frac{z}{b}}^{z} \mathrm{e}^{-\frac{x^2+\nu^2(t)}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x.$$

Therefore,

$$\mathrm{PDF}_\zeta(z) = \frac{1}{\sigma^2(t)}\int_{\frac{z}{b}}^{z} \mathrm{e}^{-\frac{x^2+\nu^2(t)}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x$$

$$= \int_{\frac{z}{b}}^{z} \frac{1}{x}\mathrm{rice}(\nu(t), \sigma(t); x)\mathrm{d}x. \tag{11}$$

Let us represent the $\mathrm{PDF}_\zeta(z)$ as a series. Using series expansion

$$\mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2\sigma^2(t))^n n!}x^{2n},$$

we obtain

$$\mathrm{PDF}_\zeta(z) = \frac{\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}}}{\sigma^2(t)}\int_{\frac{z}{b}}^{z} \mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x$$

$$= \frac{\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}}}{\sigma^2(t)}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2\sigma^2(t))^n n!}\int_{\frac{z}{b}}^{z} x^{2n} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x$$

$$= \frac{\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}}}{\sigma^2(t)}\sum_{n=0}^{\infty}\frac{(-1)^n z^{2n+1}}{(2\sigma^2(t))^n n!(2n+1)}\left({}_1F_2\left(n+\frac{1}{2}; 1, n+\frac{3}{2}; \frac{z^2\nu^2}{4\sigma^4}\right)\right.$$

$$\left.-\frac{1}{b^{2n+1}}\,{}_1F_2\left(n+\frac{1}{2}; 1, n+\frac{3}{2}; \frac{z^2\nu^2}{4b^2\sigma^4}\right)\right). \tag{12}$$

Now, using series representation of I_0

$$I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{\nu(t)}{2\sigma^2(t)}\right)^{2k} x^{2k},$$

we get

$$\begin{aligned}
\mathrm{PDF}_\zeta(z) &= \frac{\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}}}{\sigma^2(t)} \int\limits_{\frac{z}{b}}^{z} \mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}} I_0\left(\frac{x\nu(t)}{\sigma^2(t)}\right)\mathrm{d}x \\
&= \frac{\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}}}{\sigma^2(t)} \sum_{k=0}^{\infty} \frac{1}{(k!)^2}\left(\frac{\nu(t)}{2\sigma^2(t)}\right)^{2k} \int\limits_{\frac{z}{b}}^{z} x^{2k}\mathrm{e}^{-\frac{x^2}{2\sigma^2(t)}}\mathrm{d}x \\
&= \frac{\mathrm{e}^{-\frac{\nu^2(t)}{2\sigma^2(t)}}}{\sqrt{2}} \sum_{k=0}^{\infty} \frac{\nu^{2k}(t)}{2^k(k!)^2\sigma^{2k+1}(t)}\left(\Gamma\left(k+\frac{1}{2}, \frac{z^2}{2b^2\sigma^2(t)}\right)\right. \\
&\quad \left. - \Gamma\left(k+\frac{1}{2}, \frac{z^2}{2\sigma^2(t)}\right)\right).
\end{aligned} \tag{13}$$

We will further denote PDF of ζ with arbitrary parameters ν and σ by $\mathrm{PDF}_\zeta(\nu, \sigma; z)$.

Let $\nu(t) > 0, \sigma(t) > 0$ being constants. From (4) and the first mean value theorem for definite integrals, we have that

$$T(\mathbf{r}) = T\frac{\|\dot{\mathbf{r}}(t_0)\|}{u^c(t_0, \mathbf{r}(t_0))}, \quad t_0 \in (0, T).$$

The idea is to replace non-random $T(\mathbf{r})$ with the random variable $\zeta(t_0)T$.

Taking into account (11) and assuming that

$$\mathrm{CDF}_\tau(t) := \mathbb{P}\{\tau < t\} = \mathbb{P}\{\zeta(t_0)T < t\} = \mathbb{P}\left\{\zeta(t_0) < \frac{t}{T}\right\} = \mathrm{CDF}_\zeta\left(\frac{t}{T}\right),$$

we can conclude that the PDF of the random variable τ approximately has the form

$$\begin{aligned}
\mathrm{PDF}_\tau(t) &= \frac{1}{T}\mathrm{PDF}_\zeta\left(\nu, \sigma; \frac{t}{T}\right) = \frac{1}{T}\int\limits_{\frac{t}{bT}}^{\frac{t}{T}} \frac{1}{x}\mathrm{rice}(\nu, \sigma; x)\mathrm{d}x = \{y = Tx\} \\
&= \frac{1}{T}\int\limits_{\frac{t}{b}}^{t} \frac{1}{y}\mathrm{rice}\left(\nu, \sigma; \frac{y}{T}\right)\mathrm{d}y
\end{aligned}$$

$$= \int_{\frac{t}{b}}^{t} \frac{1}{y} \frac{y}{T^2\sigma^2} \exp\left(\frac{-(y^2+T^2\nu^2)}{2T^2\sigma^2}\right) I_0\left(\frac{y\nu}{T\sigma^2}\right) \mathrm{d}y$$

$$= \int_{\frac{t}{b}}^{t} \frac{1}{y} \mathrm{rice}(\nu T, \sigma T; y) \mathrm{d}y = \mathrm{PDF}_{\zeta}(\nu T, \sigma T; t), \tag{14}$$

where $t \in [0, T]$, $\tilde{\nu} > 0$, $\tilde{\sigma} > 0$. Finally, from (14) and empirical considerations, we look for the PDF of τ in the form

$$\mathrm{PDF}_{\tau}(t) = T_0 \mathrm{PDF}_{\zeta}(\tilde{\nu}, \tilde{\sigma}; t), \tag{15}$$

where $T_0 > 0$, $\tilde{\nu} = \nu T$, $\tilde{\sigma} = \sigma T$.

5 Computational Experiment for the Time of the End Point Reaching

Definition 2 The configuration entropy of the *landscape* $\mathcal{L}(\Omega_h, l)$ is defined as the following:

$$S(\mathcal{L}(\Omega_h, l)) := -\sum_{i=1}^{l} \frac{N_i}{M} \ln \frac{N_i}{M},$$

and characterizes the heterogeneity of the *landscape* in general.

We perform 10,000 experiments with the generation of a random *landscape* $\mathcal{L}(100 \times 100, 9)$ with configuration entropy $S = 2.125$ to estimate distribution parameters and additional 1000 experiments to test the hypothesis concerning the type of distribution. In each experiment, the agent moved from the cell (1, 1) to the cell (98, 98). The obtained results are shown in the form of histograms in Fig. 1. Figure 1a shows the histograms for the exit time to the end point τ_h for the optimal (left) and near-optimal (right) routes for uniform random *landscapes*. Figure 1b shows the same for "natural" random *landscapes*. Further, we will assume that agent's coordinates (x, y) coincide with cell's coordinates (i, j) if $x, y \in \mathbb{Z}$.

We obtain for uniform random *landscapes* with the sample from 10,000 experiments

$$\tau_{\min} = 204, \quad \tau_{\max} = 1999, \quad \overline{\tau} = 527.923,$$

where $\tau_{\min}$ is the sample's minimum, $\tau_{\max}$ is the sample's maximum, and $\overline{\tau}$ is the sample's mean.

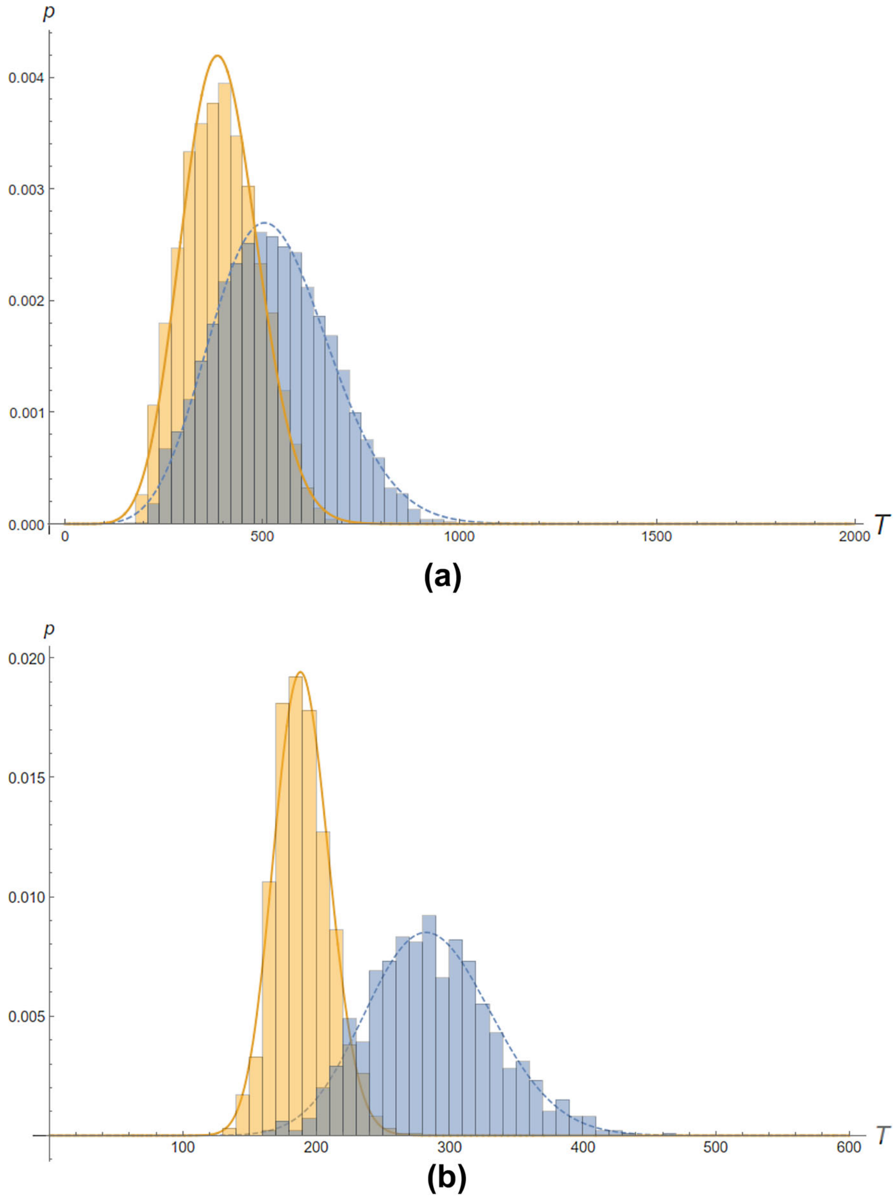

Fig. 1 Histograms of the τ_h distribution for uniform (**a**) and "natural" (**b**) *landscapes*

It was found out that histograms data are best corresponded (with p values approximately equal to 0.7) to the Norton–Rice distribution with PDF

$$\operatorname{nrice}(m, \nu, \sigma; x) := \begin{cases} \dfrac{\nu m \left(\frac{x}{\nu}\right)^m \mathrm{e}^{-\frac{m\left(\nu^2+x^2\right)}{2\sigma^2}} I_{m-1}\left(\frac{m x \nu}{\sigma^2}\right)}{\sigma^2}, & x > 0, \\ 0, & x \leq 0, \end{cases} \tag{16}$$

where I_k is the modified Bessel functions (7). The search was carried out among distributions of various types. The hypothesis that the distribution we look for is exactly the Norton–Rice distribution arose from the form of the theoretical density function (11). From (9), we can conclude that the distribution function is expressed by a series (10).

For "uniform" *landscapes*, this coincidence is manifested especially good. We used the Cramér–von Mises criterion ω^2 with statistic

$$n\omega_n^2 := \frac{1}{12n} + \sum_{i=1}^{n} \left(F(x_i, \boldsymbol{\theta}) - \frac{2i-1}{2n} \right)^2,$$

to test the distribution hypothesis, where $F(x, \boldsymbol{\theta})$ is a theoretical CDF with the parameter vector $\boldsymbol{\theta}$.

In accordance with the Cramér–von Mises criterion, the value of statistic 0.0802693 was obtained for "uniform" *landscapes*. Such a value for the statistics from 1000 experiments allows us to suppose that the hypothesis about the distribution is not rejected at the 15% significance level. However, in reality the nature of the studied distribution is more complicated.

We obtain parameter values

$$m = 3.33579, \quad \nu = 11.4566, \quad \sigma = 387.896 \tag{17}$$

for "uniform" *landscapes*, and parameter values

$$m = 9.47719, \quad \nu = 13.3365, \quad \sigma = 205.047,$$

for "natural" *landscapes*.

6 Comparison of the Analytically Obtained Distribution and the Numerical Experiment's Data

Let us compare the PDF (15) obtained analytically with the PDF (16) obtained via the numerical experiment. Because parameters of (15) are sufficiently hard to compute, we will use the segment of the series (13) instead. Note that the segment of the series (12) poorly approximates the function under study for large values of the argument. We can see from Fig. 2 that the PDF (15) with parameters

$$\tilde{\nu} = 419.25, \quad \tilde{\sigma} = 121.544, \quad b = 1.41866, \quad T_0 = 2.4386 \tag{18}$$

which is designated by the continuous thick line and the PDF (16) with parameters (17) which is designated by the dashed line are sufficiently close to each other and to the empirical PDF. To be precise, the value of the Cramér–von Mises statistic for the PDF (15) with parameters (18) is $n\omega_n^2 = 0.0540313$, $n = 1000$, and this is much less than the value of the statistics for the PDF (16).

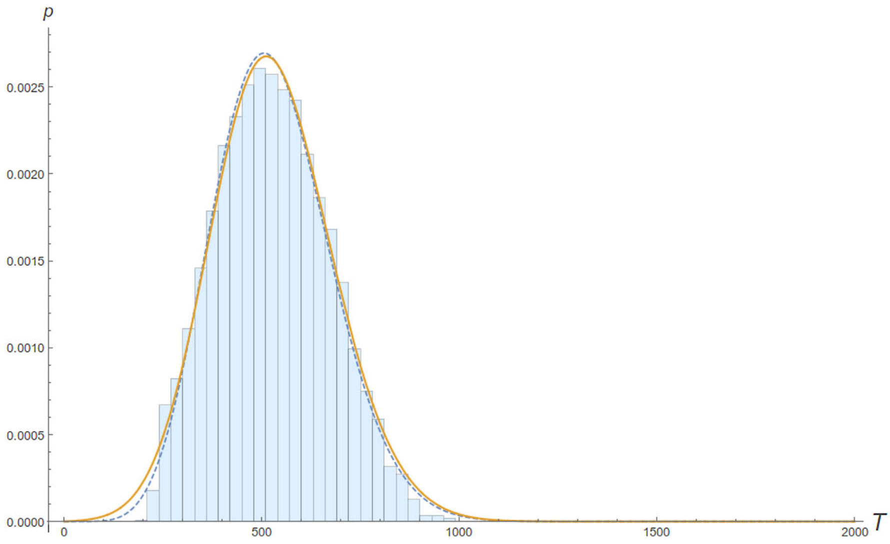

Fig. 2 Analytically and numerically obtained PDFs

7 Computational Experiment for Finding the Distribution of Agent's Coordinates

The histograms for the distribution of agents' coordinates $\varrho_h(t) = (x_h(t), y_h(t))$ are shown in Fig. 3. The selection of distributions showed that these histograms (which are equal both for x and y) are well approximated by the extreme value distribution (exactly, type I distribution of maxima) [14]. This distribution is the asymptotic distribution of maxima from the set of identically distributed random variables with the following PDF

$$evd(\alpha(t), \beta(t); x) := \frac{e^{\frac{\alpha(t)-x}{\beta(t)} - e^{\frac{\alpha(t)-x}{\beta(t)}}}}{\beta(t)}.$$

It is well known that the mean value of this extreme value distribution can be expressed as

$$\mathbb{E}x(t) = \alpha(t) + \gamma\beta(t),$$

where γ is the Euler constant. The functions $\alpha(t)$ and $\beta(t)$ themselves are approximated for $t > 20$ by linear functions

$$\alpha(t) + \gamma\beta(t) = \mathbb{E}x(t) = 0.979616 + 0.199427t, \quad r^2 = 0.999997,$$
$$\alpha(t) = 0.352148 + 0.170925t, \quad r^2 = 0.999965,$$

where r^2 is the coefficient of determination.

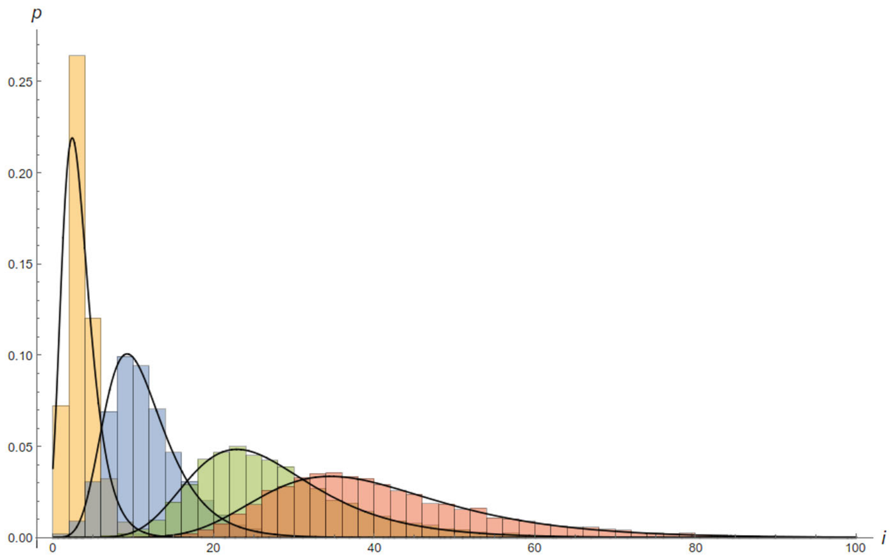

Fig. 3 Distribution of the i coordinate for the agent at time moments $t = 12, 52, 132, 200$

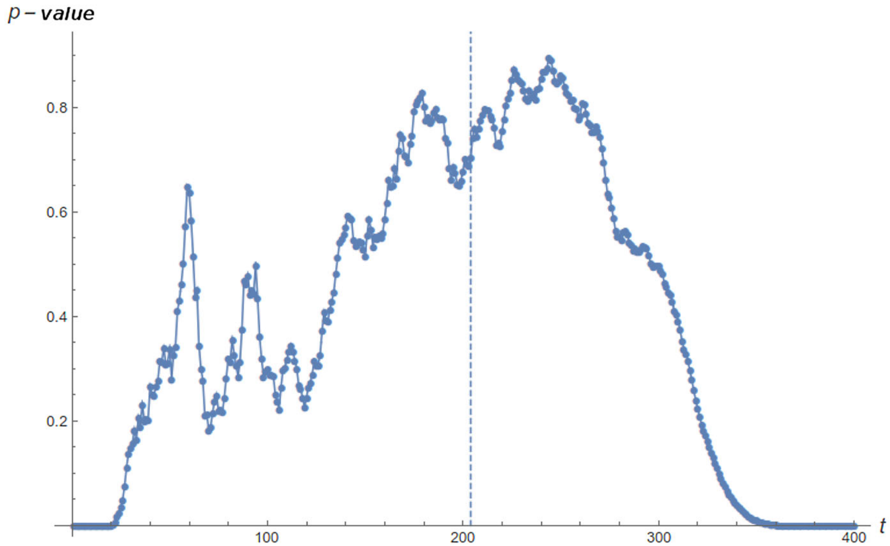

Fig. 4 Dependence of the p value on time

The quality of approximation falls only around the area where agent stops or starts motion. The plot of the dependence on the hypothesis' p value on time is presented in Fig. 4. The dashed line indicates the value of $\tau_{\min}$.

Histograms near the end of the route are presented in Fig. 5.

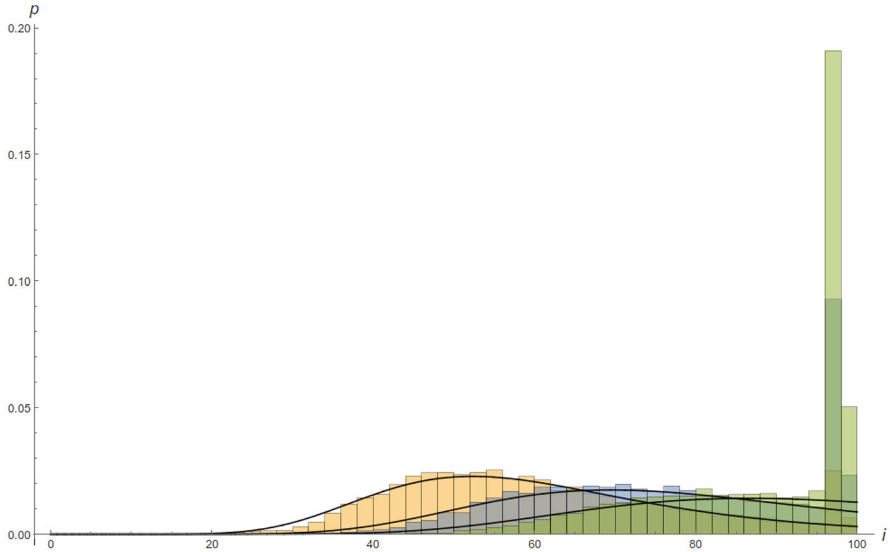

Fig. 5 Form of histograms near the end of a route, $t = 304, 404, 504$

We can assume that unknown distribution function is

$$rd(t, x) := \Theta(98 - x)evd(\alpha(t), \beta(t); x) + \varphi(t, x),$$
$$\Theta(x) := \begin{cases} 0, & x < 0, \\ 1, & x \geq 0. \end{cases}$$

$\varphi(t, x) = 0$, $x < 94$, $t < \tau_{\min}$. Behavior of φ in accordance with the data of the computational experiment is shown in Fig. 6. The left and right dashed lines indicate the values of $\tau_{\min}$ and $\overline{\tau}$ correspondingly.

It is obvious that function φ should satisfy the following conditions

$$1 = \int_0^\infty rd(t, x)\mathrm{d}x = \int_0^{98} evd(\alpha(t), \beta(t); x)\mathrm{d}x + \int_0^{98} \varphi(t, x)\mathrm{d}x.$$
$$\varphi(x, 0) = 0,$$
$$\lim_{t \to \infty} \varphi(x, t) = \begin{cases} 1/98, & x = 98, \\ 0, & x \neq 98. \end{cases}$$

It is followed from Sect. 1 that in the cell (i_0, j_0) an agent chooses ordered set $\{u_{ij}^k | (i, j) = \mathbf{r}_h[k]\}$, such that

$$T_h(\mathbf{r}_h) = \min_{\mathbf{r}_h \in \mathcal{Y}_h \cap U_o(i_0, j_0)} \sum_{(i,j) \in \mathbf{r}_h} u_{ij}^k \|\mathbf{d}_{ij}\|.$$

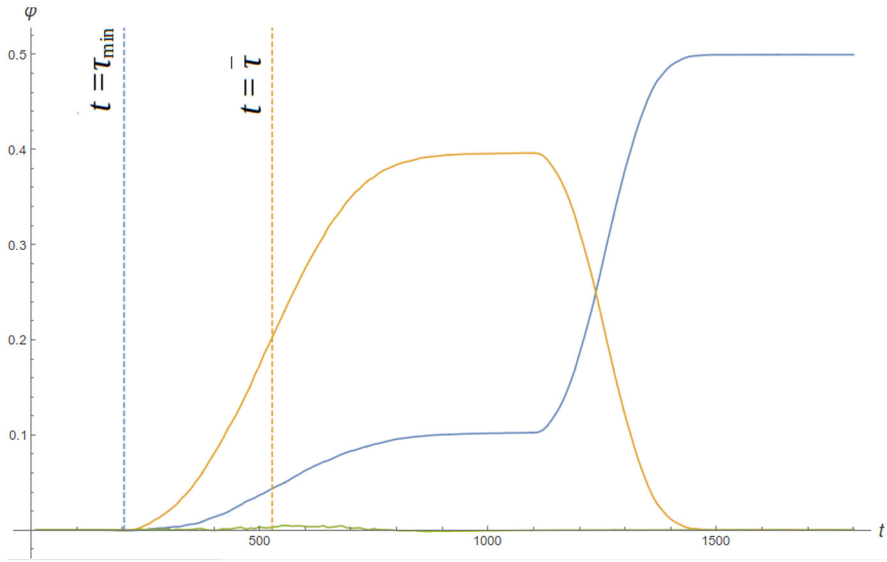

Fig. 6 Behavior of the function $\varphi(t, x)$, $x = 94, 96, 98$

It can be assumed that distribution $\boldsymbol{\varrho}(t)$ is close to the extreme value distribution because agent always chooses the set of cells $\mathbf{r}_h = \{\mathbf{r}_h[1], \ldots, \mathbf{r}_h[m+1]\}$ (or curve $\mathbf{r}$) for moving in order to minimize $T(\mathbf{r})$.

Put $\mathbf{r}_h = \{\mathbf{r}_h[1], \ldots, \mathbf{r}_h[m+1]\}$, $\chi_m = m\kappa$. When applying (6), we get

$$\begin{aligned} \|\mathbf{r}_h(t+\chi_m) - \mathbf{r}_h(t)\| &\leq \sum_{k=1}^{m} \|\mathbf{r}_h(t+\chi_k) - \mathbf{r}_h(t+\chi_{k-1})\| \\ &= \sum_{k=1}^{m} \|\mathbf{d}_k\| \left\lfloor \frac{\kappa}{\kappa \|\mathbf{d}(t)\| u(t, \mathbf{r}_h(t))} \right\rfloor = \xi(t+\chi_m) \\ &= \sum_{(i,j)\in \mathbf{r}_h} \|\mathbf{d}_k\| \left\lfloor \frac{1}{\|\mathbf{d}_k\| u_{ij}^k} \right\rfloor \to \max . \end{aligned} \tag{19}$$

Thus, at each step the agent chooses such a direction of motion that

$$\xi(t+\chi_m) = \max_{r_h \in \mathcal{Y}_h \cap U_o(i_0, j_0)} \sum_{(i,j)\in r_h} \|\mathbf{d}_k\| \left\lfloor \frac{1}{\|\mathbf{d}_k\| u_{ij}^k} \right\rfloor .$$

Consequently, we obtain the distribution which is close enough to the type I maximum extreme value distribution. From experimental data, we can see that estimate (19) is quite exact.

8 Conclusions

In the article, we analytically find the PDF and the CDF of the ratio of a Rician and uniform random variables. We established that exit time of the agent seeking the quickest route through random *landscape* has the distribution close to the distribution of such ratio. Also, we found simpler approximation of the exit times PDF with PDF of the Norton–Rice distribution via numerical experiment. We obtain that the distribution of the agent's coordinates is close to the extreme value distribution of maxima. This follows from both the computational experiment and the design of the cellular automaton that simulates the movement of the agent.

In future works, the authors plan to clarify relations between the parameters of the PDF constructed with the analytical approach.

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