Holographic Correlators and Effective Actions for Quantum Gravity

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Abstract:

In this thesis we study holographic correlators in three different examples of the AdS/CFT correspondence. In particular, we consider the low-energy effective actions of quantum gravity, where the leading term describes supergravity and the corrections correspond to higher-derivative interactions. Firstly, we consider the duality between string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ and 4d maximally supersymmetric Yang-Mills theory ($\mathcal{N}=4$ SYM). We propose a systematic procedure for obtaining all single-trace half-BPS correlators in $\mathcal{N}=4$ SYM corresponding to the four-point tree-level amplitude for type IIB string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$. The underlying idea is to compute generalised ten-dimensional contact Witten diagrams, treating AdS and S on equal footing, which are obtained from a 10d scalar effective field theory in $\mathrm{AdS}_5\times\mathrm{S}^5$. Next, we study holographic correlators in $\mathrm{AdS}_7\times\mathrm{S}^4$. M-theory in this background is dual to the 6d $(2,0)$ theory. In particular, we derive recursion relations for the anomalous dimensions of double-trace operators occurring in the conformal block expansion of four-point stress tensor correlators in the 6d $(2,0)$ theory. These anomalous dimensions encode higher-derivative corrections to supergravity in $\mathrm{AdS}_7\times\mathrm{S}^4$ arising from M-theory. Finally, we consider quantum gravity in $\mathrm{AdS}_2\times\mathrm{S}^2$ whose dual CFT

has superconformal group $SU(1,1|2)$. Firstly, we propose an AdS$_2$×S^2 effective action which describes both supergravity and higher-derivative corrections and compute the four-point half-BPS correlators using generalised 4d Witten diagrams, analogous to AdS$_5$×S^5 above. Moreover, it was recently shown that IIB supergravity in AdS$_5$×S^5 enjoys 10d conformal symmetry. We adapt this approach, which is complementary to the effective action approach, to quantum gravity in AdS$_2$×S^2. We show that the 1d supergravity and free theory correlators exhibit 4d conformal symmetry and discuss implications for higher-derivative corrections where the symmetry is generically broken, except for specific cases.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. The results in chapters 3, 4 and 5 are based on the following collaborative works

- Theresa Abl, Paul Heslop and Arthur E. Lipstein. 'Towards the Virasoro-Shapiro amplitude in $AdS_5 \times S^5$'. In: *JHEP* 04 (2021), p. 237. doi:10.1007/JHEP04(2021)237. arXiv:2012.12091 [hep-th].
- Theresa Abl, Paul Heslop and Arthur E. Lipstein. 'Recursion relations for anomalous dimensions in the 6d $(2,0)$ theory'. In: *JHEP* 04 (2019), p. 038. doi:10.1007/JHEP04(2019)038. arXiv:1902.00463 [hep-th].
- Theresa Abl, Paul Heslop and Arthur E. Lipstein. In preparation.

No part of this thesis has been submitted elsewhere for any degree or qualification.

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Dedicated to

Severin

Contents

Chapter 1

Introduction

Correlation functions are amongst the most fundamental observables in quantum field theories (QFTs). They characterise the theory and once all correlation functions of gauge invariant operators are known, the theory is considered solved. Conformal correlators, correlation functions in conformal field theories (CFTs), are especially interesting. In addition to Poincaré invariance, conformal correlators are also invariant under scaling and special conformal transformations. Therefore, these observables are strongly constrained by symmetries and show interesting structures.

In particular, conformal correlators can be studied through the AdS/CFT correspondence [1]. This is a remarkable duality between certain conformal field theories and quantum gravity in a specific curved background. The starting point is a stack of N D- or M-branes and in the low-energy limit the worldvolume theory describing this stack of branes is a conformal field theory. On the other hand, the stack of branes curves the spacetime and in the near-horizon limit the resulting geometry is Anti-de Sitter-space times a sphere (AdS×S) where the stack of branes is in the boundary of AdS. Therefore, string theory or M-theory in AdS×S is dual to the CFT in the boundary. This correspondence is very powerful as it relates quantum gravity in AdS to a non-gravitational theory in the boundary, and is therefore holographic. Gravity amplitudes in the so-called bulk are dual to stress tensor correlators in the boundary CFT. Studying these so-called holographic correlators can be of great interest from the point of view of both sides of the duality. In theories with enough supersymmetry, which will be the focus of this thesis, stress tensor correlators can be related to certain scalar operators using supersymmetry. This is technically advantageous since correlators of scalars are much simpler than those of tensors. In this thesis, we will investigate different aspects of holographic correlators in three different examples of the AdS/CFT correspondence. In all of the considered examples the boundary theory is a supersymmetric CFT (SCFT) and in particular, we study four-point functions of half-BPS operators. These are operators in special representations of the superconformal group which are annihilated by half of the supersymmetry generators.

In the low-energy limit, quantum gravity can be described by an effective action, where the leading contribution is Einstein gravity coupled to matter. The matter and couplings can be chosen to give supergravity, which generally arises from the low-energy limit of string theory or M-theory. The subleading corrections take the form of interaction terms with derivatives and we will refer to them as higher-derivative corrections. In this thesis, we will study these effective actions and in particular the higher-derivative corrections in three different examples of the AdS/CFT correspondence. In the original AdS/CFT paper [1], three canonical examples of the correspondence are considered:

- CFT on D3-branes $\leftrightarrow$ type IIB string theory in $\mathrm{AdS}_5{\times}\mathrm{S}^5$,
- CFT on M2-branes $\leftrightarrow$ M-theory in $\mathrm{AdS}_4{\times}\mathrm{S}^7$,
- CFT on M5-branes $\leftrightarrow$ M-theory in $\mathrm{AdS}_7{\times}\mathrm{S}^4$.

In each of these dualities, the number of branes on the CFT side is N and moreover, there are N units of flux through the sphere on the gravity side. The first canonical example is the most studied and best understood of the three. It is a duality between type IIB superstring theory in $\mathrm{AdS}_5{\times}\mathrm{S}^5$ and maximally supersymmetric Yang-Mills theory in four dimensions ($\mathcal{N} = 4$ SYM). We will consider holographic correlators in this duality in chapter 3. In particular we consider a 10d effective action describing higher-derivative corrections to the supergravity approximation arising from tree-level string theory.

The second canonical example is a duality between the worldvolume theory of N M2-branes and M-theory in $AdS_4 \times S^7$. M-theory is not well understood yet, but in the low-energy limit it can be described by 11d supergravity [2]. Its fundamental degrees of freedom are 2d and 5d objects, the M2- and M5-branes. The M2-brane worldvolume theory is understood by the ABJM theory, a 3d CFT that is dual to M-theory in $AdS_4 \times S^7$ [3], while the worldvolume theory of M5-branes is still mysterious. It is expected to be a 6d conformal field theory with $(2,0)$ supersymmetry, the 6d (2,0) theory. We study aspects of holographic correlators in $AdS_7 \times S^4$ in chapter 4, which is the third of the canonical examples of the AdS/CFT correspondence. In particular we focus on higher-derivative corrections to the low-energy effective action of M-theory. This duality is less well understood than the previous examples since the boundary theory has no Lagrangian description and the bulk theory is not known beyond the supergravity approximation.

Finally, in chapter 5 we will apply the knowledge and techniques from previous considerations to the study of holographic correlators in $AdS_2 \times S^2$ which is expected to be dual to a one-dimensional SCFT in the boundary. We use superconformal symmetry, crossing and higher-dimensional symmetries to reconstruct tree-level supergravity and higher-derivative corrections which describe the low-energy limit of any theory of quantum gravity in this background. This duality is less well understood but of great interest because $AdS_2 \times S^2$ is the near-horizon geometry of extremal black holes in four dimensions [4, 5] and thus by studying holographic correlators in this background one can hope to gain insight that could be adapted to real world physics.

Note that in each of these examples we restrict to tree-level correlators, therefore we suppress all loop corrections by taking the Newton constant $G_N \to 0$. Trough the AdS/CFT correspondence $G_N \sim 1/c$ where c is the central charge of the boundary

CFT. The central charge is proportional to positive powers of N depending on the CFT, where N is the number of branes, therefore taking $N \to \infty$ suppresses all loop corrections. We will now briefly motivate the main ideas of each of the three main chapters, followed by a description of the structure of this thesis. More detailed introductions will then be given at the beginning of each chapter.

Holographic correlators in $\mathrm{AdS}_5\times\mathrm{S}^5$

First, we study holographic correlators in $\mathrm{AdS}_5\times\mathrm{S}^5$ which is the most studied canonical example of the AdS/CFT correspondence. Half-BPS correlators in $\mathcal{N}=4$ SYM are dual to type IIB string theory scattering amplitudes in $\mathrm{AdS}_5\times\mathrm{S}^5$. In flat space, the four-point tree-level amplitude of closed string theory takes a very compact form known as the Virasoro-Shapiro (VS) amplitude[1] [6, 8]. This formula encodes many essential properties of string theory such as a Regge trajectory describing massive states with arbitrarily high spin, and exponential suppression at high-energy which was one of the earliest indications that string theory could be a promising candidate for quantum gravity. Given that the effects of quantum gravity are expected to become most important in curved backgrounds like the interior of black holes and the early Universe, it is therefore very important to understand how to generalise the VS amplitude beyond the flat space limit. At present it is technically challenging to calculate string amplitudes in curved backgrounds from first principles, but progress can be made in AdS backgrounds using holographic methods. In the limit $\alpha' \to 0$, where α' is related to the square of the string length, string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ can be approximated by supergravity. The subleading terms describe string corrections and take the form of higher-derivative interactions. We can also write the flat space VS amplitude as an infinite series in α', where the leading term will describe supergravity while higher-order terms describe string corrections. These corrections

[1] The Virasoro amplitude is the amplitude for four tachyonic scalars in bosonic string theory found by Virasoro [6] and generalised to n points by Shapiro [7]. The tree-level four-point amplitude in IIB string theory [8] which we will be considering here is given by the Virasoro amplitude multiplied by a kinematic factor and it has become the convention to refer to it as the Virasoro-Shapiro amplitude.

can be derived at all orders in α' from a simple effective field theory consisting of a scalar field with quartic interactions where the coefficients are fixed by comparing to the VS amplitude.

The goal is to generalise this to curved spacetime and remarkably, we will find that the interacting part of all single-trace half-BPS correlators in $\mathcal{N}=4$ SYM can be obtained from a similar scalar effective action describing tree-level IIB string theory on AdS$_5$×S^5 (rather than a flat) background. The key idea that allows us to derive correlation functions from the 10d effective field theory is the use of a natural generalisation of contact Witten diagrams [9] (which are integrals over AdS space) to integrals over the full AdS×S space, treating AdS and S on equal footing. These manifestly 10d contact diagrams generate all four-point half-BPS correlators described by tree-level string theory, corresponding to string corrections at any order in α'. Note that we do not prove the existence of this 10d effective field theory in AdS$_5$×S^5. We propose its existence and derive the four-point correlators of single-trace half-BPS operators at different orders in α'. We show that it reproduces the known results for α'^3 and α'^5 corrections, which were previously obtained in [10–15] using constraints imposed by superconformal and crossing symmetry as well as simplifications of the spectrum predicted by AdS/CFT. We also present a general algorithm for extending these predictions to arbitrarily high order in α' and use it to obtain new predictions at α'^6 and α'^7. At the same time as we completed our work, the authors of [16] also obtained higher-order α' corrections in AdS$_5$×S^5, which nicely complements our results.

Holographic correlators in AdS$_7$×S^4

Next, we consider the correspondence between the M5-brane worldvolume theory, the 6d $(2,0)$ theory, and M-theory in AdS$_7$×S^4. Correlators in the 6d $(2,0)$ theory are dual to M-theory amplitudes in AdS$_7$×S^4. The formulation of M-theory is one of the most important open questions in string theory and it is not well understood yet. It is an 11-dimensional theory of quantum gravity which, as mentioned before, is approximated by 11d supergravity in the low-energy limit. It arises in the strong coupling limit of type IIA string theory, where the size of the 11th dimension is proportional to the string coupling [17, 18]. Therefore, there is no tunable coupling constant and the only tunable parameter is the number of branes N. Further, there are no strings but its fundamental degrees of freedom are believed to be 2d and 5d objects, the M2- and M5-branes. Since there is no string length in this theory, the only length scale is the Planck length l_P which is related to the Newton constant as $G_N \sim l_P^9$. And because of the AdS/CFT identification $G_N \sim 1/c$, where $c \sim N^3$ is the central charge, the supergravity approximation $N \to \infty$ is also like taking $l_P \to 0$ which is like a low-energy limit. Moreover, this limit also implies $G_N \to 0$ which suppresses all loop corrections. Therefore, both, higher-derivative corrections and loop corrections are suppressed by $N \to \infty$ and the leading contribution is tree-level supergravity. Understanding the worldvolume theory of a stack of M5-branes is an important open question in string theory and very little is known about it since it is intrinsically strongly coupled and it is believed to have no Lagrangian description in six dimensions. In [19] a 5d Lagrangian was proposed which is believed to describe the full 6d physics of the theory. Furthermore, the 6d $(2,0)$ theory is also important because dimensional reduction along various manifolds gives various lower-dimensional theories, such as $\mathcal{N} = 4$ SYM. It also provides a geometric interpretation for their dualities, like S-duality in $\mathcal{N} = 4$ SYM [20]. A lot of progress in understanding the 6d $(2,0)$ theory has been made by dimensionally reducing the theory or computing quantities protected by supersymmetry, but ultimately one wants to compute unprotected quantities in six dimensions.

A promising strategy to study the 6d $(2,0)$ theory is the conformal bootstrap, where we try to use principle properties of the theory such as superconformal and crossing symmetry to constrain the correlators. This program was originally proposed in [21–23] and brought back more than 30 years later in [24]. This approach was first applied to the 6d $(2,0)$ theory in [25]. Our goal is to study the 6d $(2,0)$ theory away from the strict large-N (or large central charge c) limit and thus study M-theory beyond the supergravity approximation. Important recent progress in this research area has been made in [26–28]. Moreover, the M-theory effective action can also be deduced from correlators of the ABJM theory [3], which is dual to M-theory in $AdS_4 \times S^7$ [27, 29].

We analyse four-point correlation functions of stress tensor multiplets in the 6d $(2,0)$ theory in the large-c limit. The main aim is to derive the higher-derivative corrections to the effective action using conformal bootstrap methods without requiring knowledge of the explicit form of the correlators, which gives a very direct way of deriving higher-derivative corrections. We follow the strategy described in [30]. Starting from a $1/c$ expansion of the crossing equations, we derive recursion relations for the anomalous dimensions ($1/c$ corrections to the scaling dimensions) of the operators in the conformal block expansion of the correlators. These anomalous dimensions encode the higher-derivative corrections to the supergravity effective action arising from M-theory.

Holographic correlators in $\mathbf{AdS_2 \times S^2}$

Finally, we investigate holographic correlators in $AdS_2 \times S^2$ where the superconformal group of the boundary theory is $SU(1,1|2)$. Hence, the 1d boundary SCFT is quarter-maximal. We will apply techniques from other examples of the AdS/CFT correspondence to this less well studied case. $AdS_2 \times S^2$ is of great interest because it is the near-horizon geometry of extremal black holes in four dimensions as mentioned before. Hence, understanding holographic correlators in this background can help to develop a description of quantum gravity in the real world. The AdS_2/CFT_1 correspondence has been studied e.g. in [31, 32]. More recently there has been a lot of interest in AdS_2/CFT_1 due to the relation between the 2d Jackiw-Teitelboim (JT) gravity [33, 34] and the 1d Sachdev-Ye-Kitaev (SYK) model [35, 36], e.g. in [37, 38], see [39] for a review. The relation to black holes in nature has been discussed in [40]. Another motivation to study holographic correlators in $AdS_2 \times S^2$ is that they are in many ways simpler than higher-dimensional analogues and hence, $AdS_2 \times S^2$ can act as a toy model for various aspects of holography.

It was recently observed in [41] that four-point tree-level correlators in $AdS_5 \times S^5$ exhibit a ten-dimensional conformal symmetry. The conjecture is that free theory and supergravity correlators can be obtained from a single object invariant under 10d conformal symmetry. This is a consequence of the fact that $AdS_5 \times S^5$ is conformally flat and further that the four-point tree-level supergravity amplitude is scale invariant and can therefore be transformed to flat space by conformal transformation. We aim to understand this higher-dimensional conformal symmetry more systematically and will start by showing that free theory and supergravity holographic correlators in $AdS_2 \times S^2$ exhibit 4d conformal symmetry. Furthermore we investigate higher-derivative-corrections. For both, supergravity and higher-derivative corrections, we perform conformal block analyses of four-point half-BPS correlators. Generally, there are many different operators contributing to the same conformal block, i.e. there are many operators with the same scaling dimension and R-symmetry charge contributing to the spectrum. Thus the conformal block coefficients, and in particular the anomalous dimensions of exchanged operators, are degenerate and when unmixing these contributions one lifts the degeneracy which is called solving the mixing problem. After unmixing, the anomalous dimensions of double-trace operators in the spectrum exhibit a simple structure which can be interpreted in terms of the 4d conformal symmetry. The higher-dimensional conformal symmetry is generally broken for higher-derivative corrections, as was observed in [14] in the context of unmixing anomalous dimensions. We show that nevertheless an infinite set of specific correlators can be constructed from it.

On the other hand, we obtain all higher-derivative corrections from a 4d scalar effective action analogous to the one proposed above for $AdS_5 \times S^5$. We assume the existence of this effective action which, in the $AdS_2 \times S^2$ case, describes supergravity as well as higher-derivative corrections and justify it by comparing the results to ones obtained from other methods like the 4d conformal symmetry. These two approaches are complementary, as the 4d conformal symmetry describes all free theory and supergravity tree-level half-BPS four-point correlators but not general higher-derivative corrections, while the 4d effective action generates all supergravity correlators and all higher-derivative corrections but does not describe free theory. Note that the effective action we found for $AdS_5{\times}S^5$ describes higher-derivative corrections but not supergravity. In contrast, the effective action proposed for $AdS_2{\times}S^2$ treats supergravity and higher-derivative corrections on equal footing.

Structure of this thesis

We begin with a review of important concepts in chapter 2, starting with a review of conformal correlators and the conformal bootstrap in section 2.1. Next, we will briefly review the AdS/CFT correspondence in section 2.2 followed by a discussion of holographic correlators in section 2.3. Specifically we review the evaluation of correlators from Witten diagrams in AdS and discuss higher-derivative corrections to the supergravity approximation. There is one appendix related to this chapter, appendix A, where we review some important concepts in Mellin space, in particular contact Witten diagrams.

The rest of this thesis consists of three chapters, each of which focusing on one of the examples of the AdS/CFT correspondence introduced above. In general, each of the chapters can be read independently, however some sections of chapter 5 make use of material presented in chapter 3. Where this is the case we will clearly refer to the relevant sections.

Chapter 3 is based on [42] and is organised as follows. In section 3.1 we provide an overview of the general strategy including a general discussion of the effective action and define generalised contact diagrams in AdS×S as well as their Mellin transforms. In section 3.2 we use these techniques to compute the leading correction to half-BPS correlators which occurs at α'^3. In section 3.3 we develop an algorithm for extending these calculations to arbitrary order in α'. Using this algorithm, we reproduce previous results at α'^5 in section 3.4, and obtain new predictions at α'^6

and α'^7 in sections 3.5 and 3.6, respectively. We present conclusions and future directions in section 3.7. There are also several appendices related to this chapter. In appendix B, we present more details about the parametrisation of half-BPS correlators. In appendix C we discuss a relation between contact diagrams in AdS×S and AdS and present its implications for the tree-level supergravity prediction and in appendix D we list further results at order α'^7.

Chapter 4 is based on [43] and its structure is described in the following. To start with, we review some important concepts, including M-theory and 11d supergravity, in section 4.1. In section 4.2 we derive recursion relations for anomalous dimensions in a toy 6d model and match the solutions against the conformal block expansion of Witten diagrams in AdS_7. In section 4.3, we then adapt this analysis to the 6d $(2,0)$ theory, and match the solutions of the supersymmetric recursion relations with the results obtained in [26]. In section 4.4 we present our conclusions and future directions. There are also several appendices. In appendix E, we provide formulas for the conformal blocks in terms of hypergeometric functions and in appendix F we derive inner products for these functions. Furthermore, in appendix G we perform the conformal block analysis of the supergravity solution.

Chapter 5 is based on [44], which at the time of submission of this thesis is in preparation for publication. We start by reviewing the ten-dimensional hidden conformal symmetry of $\mathrm{AdS}_5\times\mathrm{S}^5$ in section 5.1 before going on to introduce the formalism for 1d superconformal correlators studied throughout this chapter in section 5.2. We will review the half-BPS correlators considered, the conformal blocks and conformal Casimirs relevant for the subsequent sections as well as propose a 4d scalar effective action from which we deduce the higher-derivative corrections to supergravity in $\mathrm{AdS}_2\times\mathrm{S}^2$, similar to the one in chapter 3. Next, we discuss the free disconnected theory in the context of the higher-dimensional conformal symmetry in section 5.3. In section 5.4 we derive the supergravity correlator of lowest charge and then obtain all higher-charge correlators from the 4d conformal symmetry as well as from the 4d effective action point of view. In section 5.5 we discuss the double-trace spectrum of the conformal block decomposition of half-BPS correlators and then explain how to solve the mixing problem at different orders in the central charge and for higher-derivative corrections. In section 5.6 we solve the mixing problem for supergravity. Finally, we study higher-derivative corrections in section 5.7, where we derive general higher-derivative corrections from crossing symmetry before obtaining predictions for the correlators from the effective action approach. We also discuss the breaking of the four-dimensional conformal symmetry. In section 5.8 we solve the mixing problem for the four-derivative corrections, we can then analyse the anomalous dimensions in terms of a 4d effective spin. There are two appendices for this chapter. In appendix H we describe the derivation of the quadratic super Casimir of $SU(1,1|2)$, which plays an important role in the 4d conformal symmetry, and will compute the correlator of descendants of the superconformal primaries. Finally, in appendix I we present further result from unmixing of four-derivative corrections.

Chapter 2

Review

In this chapter we review several concepts which play an important role throughout this thesis where the focus lies on four-point correlation functions. We start by reviewing conformal correlators, the operator-product expansion and the conformal bootstrap. Subsequently, we briefly review the AdS/CFT correspondence and then discuss holographic correlators including higher-derivative corrections, in particular their evaluation from Witten diagrams in AdS.

2.1 Conformal Correlation Functions

Firstly, let us review correlation functions in conformal field theories in more than two dimensions, for a more detailed review see e.g. [45]. For simplicity, we will consider scalar operators ϕ_i with scaling dimension Δ_i in what follows. The two-point functions are completely fixed by the symmetries of the theory, i.e. Poincaré invariance, scaling invariance and invariance under special conformal transformations. They are given by

$$\langle \phi_i(x)\phi_j(y)\rangle = \frac{\delta_{ij}}{(x-y)^{2\Delta}} \ , \tag{2.1.1}$$

if ϕ_i and ϕ_j have the same scaling dimension Δ and zero otherwise. Similarly, three-point functions can be fixed by symmetries up to a constant as follows

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\rangle = \frac{\lambda_{123}}{|x_{12}|^{2\alpha_{123}}|x_{13}|^{2\alpha_{132}}|x_{23}|^{2\alpha_{231}}} , \tag{2.1.2}$$

where $x_{ij} = x_i - x_j$ and $\alpha_{ijk} = \frac{\Delta_i+\Delta_j-\Delta_k}{2}$.

The four-point functions can no longer be fixed by symmetries but they can be significantly reduced to a function of only two conformally invariant variables u and v. Let us consider correlators of four identical operators for simplicity, the four-point functions are then given by

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \frac{f(u,v)}{x_{12}^{2\Delta}x_{34}^{2\Delta}} , \tag{2.1.3}$$

where u and v are conformal cross-ratios defined as

$$u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} , \qquad v = (1-z)(1-\bar{z}) = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} . \tag{2.1.4}$$

It is useful to introduce the parameters $(z,\bar{z})$ and we will use (u,v) and $(z,\bar{z})$ interchangeably in later chapters. Note that in one-dimensional conformal field theories, which we discuss in chapter 5, the cross-ratios are not independent and reduce to

$$u_{\bar{z}\to z} = z^2 , \qquad v_{\bar{z}\to z} = (1-z)^2 , \tag{2.1.5}$$

see section 5.2.1 for more details. The rest of the discussion in this section is still valid for 1d CFTs if one keeps in mind that the cross-ratios are not independent and can be written in terms of a single variable z.

Importantly, the four-point function (2.1.3) is crossing symmetric, i.e. it is invariant under transformations of the four-point crossing symmetry group S_3 which is generated by

$$(u,v) \to (v,u) , \qquad (u,v) \to \left(\frac{u}{v},\frac{1}{v}\right) . \tag{2.1.6}$$

Note that this follows from the fact that we consider four identical operators. As a consequence also $f(u,v)$ has to satisfy a specific crossing constraint. This can easily be seen by considering (2.1.3) which groups the external points (12) and (34) together. However, this choice is not unique and one can just as well group together (14) and (23). From (2.1.4) it is obvious that the exchange of $2 \leftrightarrow 4$ corresponds to $(u \leftrightarrow v)$ and thus

$$\frac{1}{x_{12}^{2\Delta} x_{34}^{2\Delta}} f(u,v) = \frac{1}{x_{14}^{2\Delta} x_{23}^{2\Delta}} f(v,u) \ . \tag{2.1.7}$$

Multiplying this by $x_{14}^{2\Delta} x_{23}^{2\Delta}$ yields the following crossing equation for $f(u,v)$:

$$\left(\frac{v}{u}\right)^{\Delta} f(u,v) = f(v,u) \ , \tag{2.1.8}$$

which will play an important role in the following.

2.1.1 Operator-Product Expansion, Conformal Blocks and Bootstrap

An important concept in conformal field theories is the *operator-product expansion* (OPE). First, recall that primary operators are the lowest-weight operators in a representation of the conformal algebra, they are local operators annihilated by all generators of special conformal transformations and acting with translation generators on them gives the so-called descendants (derivatives of primaries)[1]. A product of two primary operators can be expanded as a sum of all the primaries and descendants in the theory as follows:

$$\phi_1(x)\phi_2(0) \left|0\right\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\, C_{\mathcal{O}}(x, \partial_y)\, \mathcal{O}(y)|_{y=0} \left|0\right\rangle \ , \tag{2.1.9}$$

where we sum over all primary operators of the CFT. The coefficients $\lambda_{12\mathcal{O}}$ are called OPE coefficients and are the same as the constants λ_{123} in the three-point function (2.1.2). Furthermore, $C_{\mathcal{O}}(x, \partial_y)$ are polynomials of partial derivatives with

[1] In fact we will study superconformal theories in this thesis, hence the operators we consider are superconformal primaries which in addition to the conformal symmetries are also annihilated by half of the supersymmetry generators.

respect to y acting on the primaries, which will generate all the descendants in the theory. Note that one can perform an OPE in any quantum field theory, but in general (2.1.9) only holds in the limit where the operators are close together, i.e. when $x \to 0$ whereas for a CFT the statement is much stronger. Since CFTs are scale-invariant we do not have a notion of distance and the OPE is valid for finite x.

Conformal field theories can be defined by its so-called *CFT data* which consists of a list of scaling dimensions Δ_i of all local primaries of the theory together with all OPE coefficients λ_{ijk} for any three primaries. Once one knows the whole CFT data one has solved the CFT in question. In principle we can construct all n-point correlation functions from the three-point functions, which are completely fixed up to coefficients, using the OPE expansion. In practice this is very difficult to realise and this is why we have to use other methods to constrain the CFT data. One such method is the *conformal bootstrap* which uses fundamental properties of the CFT, such as crossing symmetry and conformal invariance, to constrain the CFT data and fix the correlation functions.

The starting point is the four-point function, where we again consider four identical fields, written in terms of OPEs of two pairs of primaries as

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\,\lambda_{34\mathcal{O}}[C_{\mathcal{O}}(x_{12},\partial_y)\,C_{\mathcal{O}}(x_{34},\partial_z)\langle\mathcal{O}(y)\,\mathcal{O}(z)\rangle]\ , \tag{2.1.10}$$

where $y = \frac{x_1+x_2}{2}$, $z = \frac{x_3+x_4}{2}$ and the functions in the square brackets are completely fixed by conformal symmetry. Since the form of the four-point function is known to be $\frac{f(u,v)}{x_{12}^{2\Delta}x_{34}^{2\Delta}}$, it follows that the right hand side of (2.1.10) has to have the same transformation properties which yields

$$\langle\phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\,\lambda_{34\mathcal{O}}\frac{G_{\mathcal{O}}(u,v)}{x_{12}^{2\Delta}x_{34}^{2\Delta}}\ . \tag{2.1.11}$$

The conformally invariant functions $G_{\mathcal{O}}(u,v)$ are called *conformal blocks*. Decomposing a four-point function into conformal blocks, the blocks represent the contributions of the separate conformal primary operators in the theory together with their descendants, where the operators and their corresponding blocks are labelled by their scaling dimension and spin. To make this more precise, the conformal block expansion of a correlator of four operators with scaling dimensions Δ_i, where $i = 1, \ldots, 4$, can be written as

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \mathcal{P}^{\Delta_i} \sum_{\Delta,l} A^{\Delta_i}_{\Delta,l} G^{\Delta_i}_{\Delta,l}(u,v) \ , \tag{2.1.12}$$

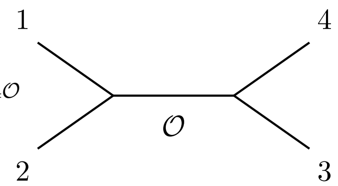
where $G^{\Delta_i}_{\Delta,l}(u,v)$ are the conformal blocks describing the contributions of operators in the theory belonging to a multiplet whose conformal primary has scaling dimension Δ and spin l. The coefficients $A^{\Delta_i}_{\Delta,l}$ are squares of OPE coefficients and $\mathcal{P}^{\Delta_i}$ is a prefactor depending on the spacetime coordinates. The labels Δ_i indicate a dependence on the scaling dimensions of the four external operators. As an example of operators contributing to the conformal block expansion, consider double-trace operators which will be studied in detail in later chapters. These operators are constructed from two half-BPS operators with dimensions p, q and have the schematic form $\mathcal{O}_p \partial^l \Box^n \mathcal{O}_q$. Their classical scaling dimension is $p + q + 2n + l$, where the spin l counts the number of partial derivatives and n counts the number of boxes acting on one of the half-BPS operators. It is also common to refer to the twist of the exchanged operator which is defined as its scaling dimension minus its spin. Double-trace operators of the above form have twist $p + q + 2n$.

Conformal blocks have been computed explicitly for many theories. For four-point correlators of scalar operators of arbitrary scaling dimensions in any even dimension they were derived in [46]. The authors obtained the conformal blocks by solving for the eigenfunctions of the Casimir operator of the conformal group $SO(d,2)$ in d dimensions. The blocks for $d = 4$ and $d = 6$ can be written in terms of hypergeometric functions in a simple way (see e.g. section 5.4.2 for 4d blocks and appendix E for blocks in 6d), whereas the blocks for general even dimensions are given as an infinite sum of so-called Jack polynomials. Superconformal blocks in six dimensions were given in [25, 47]. Conformal blocks for odd dimensions were obtained in [48]. Finally, the superconformal blocks relevant for 1d supersymmetric CFTs which we study in chapter 5 were derived in [49] (see section 5.2.2).

Once the blocks are known, this allows us to perform a so-called conformal block analysis of the theory. Decomposing the four-point correlators into conformal blocks gives information about the spectrum of operators in the theory, their scaling dimensions and spins. We will perform conformal block analyses of correlators in 1d CFTs in chapter 5 where we compute three-point coefficients and anomalous dimensions of operators in the double-trace spectrum of the theory in the context of a higher-dimensional conformal symmetry as mentioned in the introduction. We also perform a conformal block analysis of correlators in a 6d SCFT in chapter 4 where we compute anomalous dimensions which encode information about higher-derivative corrections to the bulk low-energy effective action. However, the main focus of chapter 4 is to compute anomalous dimensions of double-trace operators in the spectrum of half-BPS correlators without requiring knowledge of the explicit form of the four-point functions but rather deriving a recursion relation for anomalous dimensions using the conformal bootstrap, which we describe in the following.

Let us start by writing the above equation (2.1.11) diagrammatically:

$$\langle \phi_1(x_1)\phi_2(x_2)\phi_3(x_3)\phi_4(x_4)\rangle = \sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\lambda_{34\mathcal{O}}$$

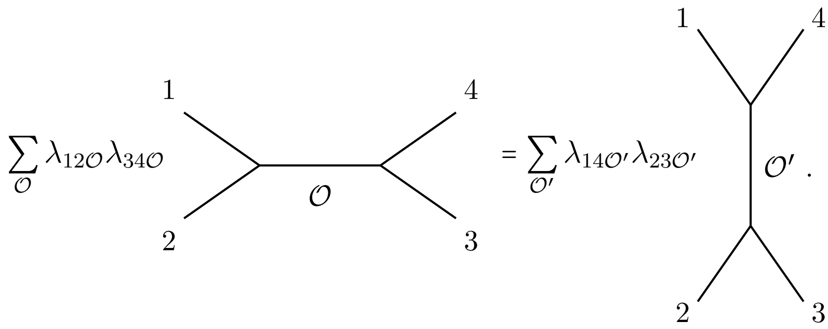


.

Notice that in the double OPE of the four-point function we chose the OPE channel (12)(34), we could as well have chosen the channel (14)(23) and the result must be the same. This leads to the conformal bootstrap or crossing symmetry condition

$$\sum_{\mathcal{O}} \lambda_{12\mathcal{O}}\, \lambda_{34\mathcal{O}} \frac{G_{\mathcal{O}}(u,v)}{x_{12}^{2\Delta} x_{34}^{2\Delta}} = \sum_{\mathcal{O}'} \lambda_{14\mathcal{O}'}\, \lambda_{23\mathcal{O}'} \frac{G_{\mathcal{O}'}(v,u)}{x_{14}^{2\Delta} x_{23}^{2\Delta}} \;, \tag{2.1.13}$$

or diagrammatically:



After imposing these conditions on all four-point functions of a theory no more conditions will arise from higher-point functions. The bootstrap conditions (also called OPE associativity or crossing equations) are used to classify CFTs using the following statement.

A CFT is a set of CFT data which satisfies the crossing equations for all four-point functions [21, 23].

To summarise, crossing and (super)conformal symmetry impose powerful constraints on four-point correlators which can be used to determine CFT data analytically or numerically. When a holographic dual exists, the CFT data can be constrained even more by combining the conformal bootstrap with knowledge about the holographic dual.

2.2 The AdS/CFT Correspondence

In this section, we briefly review the AdS/CFT correspondence [1, 9, 50]. For a detailed review see e.g. [51, 52].

Through the AdS/CFT correspondence quantum gravity in AdS is described by a CFT in the boundary. Considering a stack of D- or M-branes, in the low-energy limit, the worldvolume quantum field theory describing the stack of branes is a CFT. Besides, the stack of branes warps the surrounding spacetime and in the near-horizon limit the geometry of this curved space is $\mathrm{AdS}_p \times \mathrm{S}^q$ and the stack of branes is located in the boundary of AdS. Hence, the boundary CFT is dual to string or M-theory in $\mathrm{AdS}_p \times \mathrm{S}^q$, which is a very remarkable duality between a theory of gravity and a quantum field theory without gravity in different dimensions. It is conjectured that the two theories in the duality are equivalent to each other, i.e. its operator observables, states, correlation functions as well as the full dynamics of the theories are equivalent. This means that each of these objects can be computed from two completely different calculations and both will lead to the same result. Because the correspondence is between a theory in AdS×S and a CFT on the boundary of AdS, it is also called a holographic duality. The stack of N branes creates a flux through the sphere, thus integrating the flux and using Gauss' law gives the number of branes. When the stack is coincident and in flat space, the gauge group of the boundary theory is expected to be $SU(N)$ in the case of $\mathcal{N} = 4$ SYM.

In Maldacena's original paper [1] there are three canonical examples, as we have seen in the introduction. The most well studied duality is between type IIB string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$ and $\mathcal{N} = 4$ SYM in the boundary which we study in chapter 3. Another canonical example is the focus of chapter 4, the correspondence between the worldvolume theory of N M5-branes and M-theory in $\mathrm{AdS}_7 \times \mathrm{S}^4$. The third canonical example is a duality between M-theory in $\mathrm{AdS}_4 \times \mathrm{S}^7$ and the worldvolume theory of N M2-branes which can be described by a 3d CFT, the ABJM theory [3]. In chapter 5 we focus on quantum gravity in $\mathrm{AdS}_2 \times \mathrm{S}^2$ which is expected to be dual to a 1d SCFT with superconformal symmetry group $SU(1,1|2)$.

Let us have a closer look at the two canonical examples we study in later chapters and their parameters, starting with $\mathcal{N} = 4$ SYM and string theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$. The parameters of the two theories are identified with one another as

$$g_{\mathrm{YM}}^2 = 4\pi g_s \ , \qquad R^4 = 4\pi g_s N (\alpha')^2 \ , \tag{2.2.1}$$

where g_{YM} is the Yang-Mills coupling, g_s is the closed string coupling, R is the curvature radius of the AdS_5 space and the S^5 sphere and α' is related to the string length l_s as $\alpha' = l_s^2$. With the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ this yields

$$R = \lambda^{1/4} l_s \ , \tag{2.2.2}$$

which relates R/l_s, the AdS and S radius in units of string length, on the bulk side of the duality to the 't Hooft coupling of the boundary CFT on the other side. This makes the duality very powerful since it relates a weakly coupled field theory which can be studied perturbatively to a string theory in a strongly curved background which makes computations very difficult. On the other hand, when the string theory is in a weakly curved background and thus calculations are simpler because it can be approximated by supergravity, the boundary CFT is strongly coupled. Thus, problems that are intractable on the one side of the correspondence can be much simpler to compute on the other side. Note that even when both sides of the duality are non-perturbative, as is the case in the M5-brane case (see chapter 4), this duality is very useful as for example in the CFT we can use conformal bootstrap methods to study correlation functions which in turn gives insight into the dual quantum gravity theory where computations are very difficult.

The second relation in (2.2.1) is derived by constructing extremal black D3-brane solutions, where the near-horizon geometry of N coincident D3-branes is $\mathrm{AdS}_5 \times \mathrm{S}^5$. A similar relation for M5-branes is obtained by constructing extremal black M5-brane solutions:

$$r_5^3 = \pi N_5 l_P^3 \ . \tag{2.2.3}$$

The near-horizon geometry of N_5 coincident M5-branes is $\mathrm{AdS}_7 \times \mathrm{S}^4$, where the radius of the four-sphere is r_5 and the radius of AdS_7 is $2\, r_5$. Note that there are no strings in M-theory and thus no concept of string length, therefore M-theory is manifestly non-perturbative and has only one length scale, the Planck length l_P.

The AdS/CFT correspondence is complicated when considering the full quantum gravity theory in the bulk, however there are significant simplifications when considering certain limits. Taking the limit $N \to \infty$ is like taking the Newton constant $G_N \to 0$ and therefore this limit suppresses all loop corrections and restricts to tree-level gravity. This can be seen from the AdS/CFT identification $G_N \sim 1/c$, where c is the central charge of the CFT which is proportional to positive powers of N depending on the theory. In all of the following chapters we will restrict to this limit and only consider tree-level correlators.

Note that in the case of $\mathcal{N} = 4$ SYM one can take $N \to \infty$ while keeping the 't Hooft coupling λ constant, this corresponds to a topological expansion in $1/N$ where the leading contribution is the planar limit. Furthermore, it restricts to tree-level string theory where quantum corrections described by string loops are suppressed since $g_s \to 0$. Additionally, one can consider the low-energy limit $\alpha' \to 0$ which corresponds to the strong-coupling limit of $\mathcal{N} = 4$ SYM, where $\lambda \to \infty$, and describes supergravity with all stringy corrections suppressed. When studying $\mathcal{N} = 4$ SYM away from the strict $\alpha' \to 0$ limit one adds stringy corrections to the supergravity approximation. Hence, a particularly simple limit of the duality is accessible in $\mathcal{N} = 4$ SYM, a duality between a strongly coupled planar CFT and weakly coupled classical supergravity.

In the case of M-theory, these limits are not available separately since there is only one length-scale, the Planck length l_P. As described before, taking $N \to \infty$ suppresses all loop corrections. Since the Newton constant is proportional to l_P, $G_N^{11d} \sim l_P^9$ in eleven dimensions, the $N \to \infty$ limit also corresponds to $l_P \to 0$ and is thus a low-energy limit. Further, M-theoretic corrections are also suppressed in the limit $l_P \to 0$ (or $N \to \infty$), since this is the only length scale available. Therefore, both, loop corrections and higher-derivative corrections are described by a large-N expansion and $N \to \infty$ corresponds to the tree-level supergravity approximation. In this thesis we will not discuss loop corrections but focus on tree-level supergravity and higher-derivative corrections arising from quantum gravity.

2.3 Holographic Correlators and Witten Diagrams

In each of the following chapters we consider four-point correlation functions of half-BPS operators, which are annihilated by half of the supercharges and thus form a short multiplet. On the quantum gravity side, in the large-N limit, these short multiplets correspond to supergravity multiplets plus an infinite tower of Kaluza-Klein (KK) excitations on the sphere. These excitations arise from dimensionally reducing the corresponding theory of quantum gravity on the sphere. Thus, in the 4d, 6d and 1d examples considered in the chapters below the half-BPS operators are dual to harmonics on the S^5, S^4 and S^2 sphere respectively. These spherical harmonics transform as totally symmetric and traceless rank-n tensors under $SO(6)$, $SO(5)$ and $SO(3)$ transformations in the case of $\mathrm{AdS}_5\times\mathrm{S}^5$, $\mathrm{AdS}_7\times\mathrm{S}^4$ and $\mathrm{AdS}_2\times\mathrm{S}^2$ respectively.

A supermultiplet can be characterised by its primary operator, which is annihilated by all generators of special conformal transformations and conformal supercharges. The descendants in the supermultiplet are obtained by acting on the primary with Poincaré supercharges. In the case of short multiplets, the primary operators, in addition to being annihilated by the conformal symmetries, are also annihilated by half of the supercharges. These primaries are called *chiral primary operators* (half-BPS operators). It will be useful to describe schematically how chiral primary operators are constructed in general superconformal theories. To construct a chiral primary operator, consider the trace of a product of n scalar operators

$$\mathcal{O}^{I_1 I_2 \dots I_n} = \mathrm{Tr}(\phi^{I_1}\phi^{I_2}\cdots\phi^{I_n}) \ , \tag{2.3.1}$$

with R-symmetry indices I_i, where in supersymmetric theories R-symmetry transforms different supercharges into each other. Next, symmetrise all n indices in (2.3.1) and remove all traces to make a traceless symmetric tensor. Thus, consider operators of the form $\phi^{(I_1}\phi^{I_2}\cdots\phi^{I_n)}$, which are totally symmetric and it is understood that all traces are removed. More precisely, in the case of $\mathcal{N} = 4$ SYM the scalar operators ϕ^{I_i} are the six real scalars of $\mathcal{N} = 4$ SYM (see subsection 3.1.1) where the interacting fields can be written as perturbations around the free fields. Moreover, the trace in $\mathcal{N} = 4$ SYM is defined in terms of the gauge group $SU(N)$. In the 6d $(2,0)$ theory on the other hand, there is no gauge group and there is no expression for the interacting fields in terms of the free fields due to the lack of a small coupling constant, nevertheless these operators are expected to exist. Importantly, nowhere in chapter 4 is the definition of these operators required and it will be enough to know the superconformal blocks for this theory. Let us now go back to the general case.

The operator $\phi^{(I}\phi^{J)}$ is the superconformal primary of the stress tensor multiplet and, through the AdS/CFT correspondence, encodes the graviton in AdS. Traceless symmetric operators of the form $\phi^{(I_1}\phi^{I_2}\cdots\phi^{I_n)}$ with more than two indices are higher-charge operators which encode higher KK modes on the sphere. These operators are so-called single-trace operators and (2.3.1) forms a complete list of single-trace chiral primary operators for $n \leq N$. For $n > N$ they can be related to so-called multi-trace operators which are products of single-trace operators. We will study correlation functions of single-trace chiral primary operators in the large-N limit in the following chapters.

Through the AdS/CFT correspondence, the dimensions Δ of any scalar operator on the conformal field theory side correspond to the mass of the dual bulk state on the gravity side. The relation between these quantities is given by

$$\Delta(\Delta - d) = m^2 R^2 \ , \tag{2.3.2}$$

where m is the mass of the scalar field in the bulk, R is the AdS radius and d is the spacetime dimension of the CFT. Considering the masses of Kaluza-Klein excitations on the sphere and comparing to the dimensions of the chiral primary operators in the dual conformal field theory will satisfy this relation.

Finally, correlation functions of half-BPS operators correspond to scattering amplitudes of scalar fields in the bulk with mass m (2.3.2). Therefore, by studying AdS scattering amplitudes we can learn more about CFT correlation functions and vice versa [53–56]. As mentioned before, we are interested in studying the tree-level supergravity approximation of quantum gravity. This is described by an effective action whose leading contribution describes supergravity and the subleading corrections are contained in an infinite tower of higher-derivative terms. These corrections correspond to local interaction vertices in the tree-level low-energy effective action in AdS, where we focus on quartic interactions in this thesis. These quartic interactions can be computed from contact Witten diagrams in AdS and we will review them in the following subsections starting with interaction vertices without derivatives followed by a discussion of Witten diagrams for higher-derivative corrections.

2.3.1 Contact Witten Diagrams

Supergravity scattering in AdS can be computed from Witten diagrams. They are most conveniently expressed using embedding coordinates for AdS:

$$\hat{X}^2 = -\left(\hat{X}^{-1}\right)^2 - \left(\hat{X}^0\right)^2 + \sum_{i=1}^{d}\left(\hat{X}^i\right)^2 = -1 \ , \tag{2.3.3}$$

where d is the spacetime dimension of the boundary CFT. In terms of these coordinates, covariant derivatives can be defined using projection tensors

$$\mathcal{P}_A^B = \delta_A^B + \hat{X}_A\hat{X}^B \ , \tag{2.3.4}$$

which satisfy the useful identities

$$\mathcal{P}_A^B\hat{X}^A = 0 \ , \quad \mathcal{P}_A^B\mathcal{P}_B^C = \mathcal{P}_A^C \ . \tag{2.3.5}$$

In particular, the covariant derivative of a tensor is given by [57, 58]

$$\nabla_A \mathrm{T}_{\mathrm{A}_1\ldots\mathrm{A}_\mathrm{N}} = \mathcal{P}_\mathrm{A}^\mathrm{C}\mathcal{P}_{\mathrm{A}_1}^{\mathrm{C}_1}\ldots\mathcal{P}_{\mathrm{A}_\mathrm{N}}^{\mathrm{C}_\mathrm{N}}\frac{\partial}{\partial\hat{\mathrm{X}}^\mathrm{C}}\left(\mathcal{P}_{\mathrm{C}_1}^{\mathrm{E}_1}\ldots\mathcal{P}_{\mathrm{C}_\mathrm{N}}^{\mathrm{E}_\mathrm{N}}\mathrm{T}_{\mathrm{E}_1\ldots\mathrm{E}_\mathrm{N}}\right) \ . \tag{2.3.6}$$

As an application, let us consider two transverse tensors T and U of rank $N+1$ and N, respectively. Using the chain rule, we see that

$$\mathrm{T}^{\mathrm{BA}_1\ldots\mathrm{A}_\mathrm{N}}\nabla_\mathrm{B}\mathrm{U}_{\mathrm{A}_1\ldots\mathrm{A}_\mathrm{N}} = -\nabla_\mathrm{B}\mathrm{T}^{\mathrm{BA}_1\ldots\mathrm{A}_\mathrm{N}}\mathrm{U}_{\mathrm{A}_1\ldots\mathrm{A}_\mathrm{N}} + \ldots \tag{2.3.7}$$

where the ellipsis denotes

$$\partial_C\left(\mathrm{T}^{\mathrm{BA}_1\ldots\mathrm{A}_\mathrm{N}}\mathcal{P}^\mathrm{C}_\mathrm{B}\mathrm{U}_{\mathrm{A}_1\ldots\mathrm{A}_\mathrm{N}}\right) - \mathrm{T}^{\mathrm{BA}_1\ldots\mathrm{A}_\mathrm{N}}\partial_\mathrm{C}\left(\mathcal{P}^\mathrm{C}_\mathrm{B}\mathcal{P}^{\mathrm{C}_1}_{\mathrm{A}_1}\ldots\mathcal{P}^{\mathrm{C}_\mathrm{N}}_{\mathrm{A}_\mathrm{N}}\right)\mathrm{U}_{\mathrm{C}_1\ldots\mathrm{C}_\mathrm{N}} \ . \tag{2.3.8}$$

When we act on projection tensors with derivatives, this gives terms which vanish when contracted with the transverse tensors, so the second term vanishes. Since the first term is a total derivative, (2.3.7) implies that Lagrangians written in embedding coordinates enjoy the same equivalence relations as flat space Lagrangians under integration by parts.

The AdS contact Witten diagrams in embedding space are defined as integrals over AdS_{d+1} of products of bulk-to-boundary propagators. These propagators are given by

$$G(\hat{X}, X_i) = \frac{\mathcal{C}_{\Delta_i}}{(-2\hat{X}.X_i)^{\Delta_i}} \ , \tag{2.3.9}$$

properly normalised to yield a delta-function at the boundary, the bulk-to-boundary propagator will include a normalisation[2] [57, 59]

$$\mathcal{C}_\Delta = \frac{\Gamma(\Delta)}{2\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2} + 1)} \ . \tag{2.3.10}$$

Note that $x_{ij}^2 = -2X_i.X_j$ and the X_i are boundary points which satisfy

$$X_i^2 = 0 \ . \tag{2.3.11}$$

Using the definition (2.3.6) and acting on the bulk-to-boundary propagator we get the equations of motion

$$\nabla^2 G = \Delta\,(\Delta - d)\,G \ , \tag{2.3.12}$$

[2] These are normally omitted from the definition of the contact diagrams or D-functions and we do so here. We will also later absorb factors of $\mathcal{C}_{\Delta_i}$ into the definition of the Mellin amplitude in chapter 3.

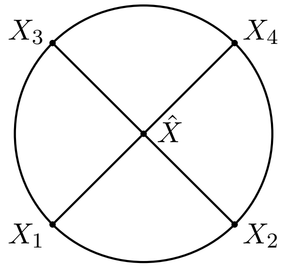


Figure 2.1: Contact Witten diagram for a conformal four-point function dual to a quartic contact interaction of the bulk scalar fields. Bulk-to-boundary propagators $G(\hat{X}, X_i)$ connect the boundary points X_i to the bulk interaction point $\hat{X}$. A general quartic interaction vertex includes derivatives acting on the bulk-to-boundary propagators (see section 2.3.2).

which agrees with the mass from the AdS/CFT prediction (2.3.2) (where we set $R = 1$). For $d = \Delta$ the propagator obeys massless equations of motion $\nabla^2 G = 0$.

To compute a four-point function associated with a ϕ^4 quartic interaction in AdS, we take the AdS integral over four bulk-to-boundary propagators as follows:

$$D^{(d)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i) = \frac{1}{(-2)^{2\Sigma_\Delta}} \int_{\mathrm{AdS}} \frac{d^{d+1}\hat{X}}{(\hat{X}.X_1)^{\Delta_1}(\hat{X}.X_2)^{\Delta_2}(\hat{X}.X_3)^{\Delta_3}(\hat{X}.X_4)^{\Delta_4}} \,, \qquad (2.3.13)$$

where $\Sigma_\Delta = (\Delta_1+\Delta_2+\Delta_3+\Delta_4)/2$. This is the definition of the D-functions. They describe quartic contact diagrams and will play an important role in each of the following chapters. The powers of minus 2 can be absorbed into the propagators as $(-2\hat{X}.X_i)$, but for notational simplicity we pull them out. In figure 2.1 a general four-point Witten diagram is illustrated.

There is a particularly useful representation for contact diagrams, the Mellin representation. The above D-functions have the following form in Mellin space [57] (see appendix A for more details on Mellin space and contact diagrams)

$$D^{(d)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i) = \mathcal{N}^{\mathrm{AdS}_{d+1}}_{\Delta_i} \times \int \frac{d\delta_{ij}}{(2\pi i)^2} \prod_{i<j} \frac{\Gamma(\delta_{ij})}{(X_i.X_j)^{\delta_{ij}}} \,, \quad \text{with } \sum_{i<j} \delta_{ij} = \Delta_j \,, \qquad (2.3.14)$$

where the normalisation is given by

$$\mathcal{N}^{\mathrm{AdS}_{d+1}}_{\Delta_i} = \frac{\frac{1}{2}\pi^{d/2}\Gamma(\Sigma_\Delta - d/2)}{(-2)^{\Sigma_\Delta}\prod_i \Gamma(\Delta_i)} \,. \qquad (2.3.15)$$

For later use we define normalised D-functions without the factor $\mathcal{N}^{\mathrm{AdS}_{d+1}}_{\Delta_i}$ as

$$D_{\Delta_i}(X_i) = \frac{1}{\mathcal{N}^{\mathrm{AdS}_{d+1}}_{\Delta_i}} D^{(d)}_{\Delta_i}(X_i) \ . \tag{2.3.16}$$

Note that the normalised D-functions are independent of the spacetime dimension d as can be seen from (2.3.14) and they are distinguished by the presence or not of the superscript (d).

To perform explicit calculations in position space it is useful to rewrite the $D^{(d)}$-functions in terms of $\bar{D}$-functions, which only depend on the conformal cross-ratios u, v [60]:

$$\frac{\prod_{i=1}^4 \Gamma(\Delta_i)}{\Gamma(\Sigma_\Delta - \frac{d}{2})} \frac{2}{\pi^{\frac{d}{2}}} D^{(d)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(x_i) = \frac{(x_{14}^2)^{\Sigma_\Delta - \Delta_1 - \Delta_4} (x_{34}^2)^{\Sigma_\Delta - \Delta_3 - \Delta_4}}{(x_{13}^2)^{\Sigma_\Delta - \Delta_4} (x_{24}^2)^{\Delta_2}} \bar{D}_{\Delta_1\Delta_2\Delta_3\Delta_4}(u,v) \ . \tag{2.3.17}$$

The simplest of these functions $\bar{D}_{1111}$ is given by the standard four-dimensional box integral which can be written explicitly in terms of dilogarithms depending on u, v. Higher $\bar{D}$-functions can be computed by acting with differential operators on the box integral. For more details on these functions we refer the reader to e.g. [60].

2.3.2 Higher-Derivative Corrections

In this subsection we discuss higher-derivative corrections to the low-energy effective action of the quantum gravity theory under consideration which correspond to subleading terms in the low-energy expansion of correlators in the boundary CFT. Let us illustrate this with a schematic low-energy effective action. Supergravity is described by a supersymmetric version of the Einstein-Hilbert action which contains the Riemann curvature tensor $\mathcal{R}$. Including quantum gravity corrections corresponds to including higher-derivative interaction terms where we restrict to quartic interactions of the schematic form $\mathcal{D}^{2k}\mathcal{R}^4$:

$$\mathcal{L} \sim \frac{1}{G_N}\left(\mathcal{R} + \sum_k c_k \, \mathcal{D}^{2k}\mathcal{R}^4 + \dots\right) \ , \tag{2.3.18}$$

with the Newton constant G_N. Recall that the Riemann tensor contains two derivatives, therefore the first correction $\mathcal{R}^4$ has six more derivatives than supergravity. Note that in later chapters $\mathcal{R}^4$ will be referred to as the zero-derivative correction, even though it has more derivatives than supergravity, as we label the corrections in terms of the number of derivatives acting on $\mathcal{R}^4$.

In [30] the authors considered higher-derivative corrections for generic 2d and 4d CFTs in the large-N expansion at first subleading order which corresponds to tree-level supergravity plus higher-derivative corrections. Their strategy was to solve the crossing equations for four-point correlators by performing a conformal block analysis (as in (2.1.12)) and truncating the expansion to spin L. They found that there are $(L{+}2)(L{+}4)/8$ solutions to the crossing equations for each spin-L truncation. Furthermore, they provided holographic arguments and came to the same conclusion from a bulk point of view. In particular, the authors considered a massive scalar field in AdS with local quartic interactions (which can be thought of as a toy model for the low-energy effective action of quantum gravity in AdS, schematically described in (2.3.18)) and showed that, up to integration by parts and equations of motion, the quartic bulk interactions are in one-to-one correspondence with the solutions to the crossing equations described above. There are $L/2+1$ independent interactions which can create or annihilate a state of at most spin L, with the total number of derivatives ranging from $2L$ to $3L$ in intervals of two. These can be written as

$$\left(\nabla^{L/2}_{\underline{\mu}}\phi\right)\left(\nabla^{L/2}_{\underline{\nu}}\phi\right)\left(\nabla^{k}_{\underline{\rho}}\phi\right)\left(\nabla^{L+k}_{\underline{\mu\nu\rho}}\phi\right)\ , \quad k=0,1,\ldots,L/2\ , \tag{2.3.19}$$

where the underscores denote sets of Lorentz indices. Note that the first two scalars in isolation have L free Lorentz indices as do the last two and so they can create a spin-L state. Hence, there is for example one spin-0 interaction vertex ϕ^4, and two spin-2 interaction vertices equivalently written $\phi^2\left(\nabla_\mu\nabla_\nu\phi\right)^2$ and $\phi^2\left(\nabla_\mu\nabla_\nu\nabla_\rho\phi\right)^2$ which contain four and six derivatives, respectively. The total number of interactions up to spin L is then given by $\sum_{l=0}^{L/2}(l+1)=(L+2)(L+4)/8$. The diagram in (2.3.13) describes a zero-derivative interaction and thus corresponds to a spin-0 interaction vertex ϕ^4 (or to $\mathcal{R}^4$ in (2.3.18)).

The counting of independent interaction vertices can be nicely seen in Mellin space. In [61] a basis of solutions for higher-derivative corrections is given in Mellin space by

$$P_{p,q} = \sigma_2^p \sigma_3^q \ , \quad \text{with} \quad \sigma_2 = s^2 + t^2 + u^2 \ , \quad \sigma_3 = s^3 + t^3 + u^3 \ , \tag{2.3.20}$$

with non-negative integers p, q and s, t, u are Mellin variables, analogous to Mandelstam variables for four-particle scattering. The solutions $P_{p,q}$ correspond to solutions of the crossing equations with conformal block expansions truncated to spin $L = 2(p+q)$, hence for each L there are $L/2+1$ solutions.

Remarkably, in [30] the authors argued that the number of derivatives in the bulk interaction vertex can be deduced from the large-twist behaviour of the anomalous dimensions of double-trace operators in the conformal block expansions of half-BPS correlators. From this analysis it is possible to obtain the large-N scaling of the solutions to the crossing equations by dimensional analysis from comparing the large-twist limit of the anomalous dimensions corresponding to spin-L corrections to those of supergravity. This was implemented for $\mathcal{N} = 4$ SYM in [61] and for the 6d $(2,0)$ theory in [26, 43] (see chapter 4 for more details).

In the previous subsection we have seen how to obtain contact Witten diagrams corresponding to a quartic bulk interaction with no derivatives (2.3.14). Let us illustrate the evaluation of contact diagrams with covariant derivatives acting on the bulk-to-boundary propagators in a couple of simple examples. The first non-trivial quartic interaction with derivatives has four derivatives

$$(\nabla\phi)^2 (\nabla\phi)^2 \ , \tag{2.3.21}$$

which corresponds to a spin-2 solution, see (2.3.19). This is a correction with four more derivatives than the ϕ^4 interaction and thus corresponds to the $D^4\mathcal{R}^4$ term in the low-energy effective action. The interaction with two derivatives can be reduced to ϕ^4 by using integration by parts and equations of motion (2.3.12). The contact diagram for a quartic interaction with four derivatives acting on the bulk-to-boundary propagator is then

$$\sum_{\text{perms}} \int_{\text{AdS}} d\hat{X} \nabla_A \left(-2\hat{X}.X_1\right)^{-\Delta} \nabla^A \left(-2\hat{X}.X_2\right)^{-\Delta} \nabla_B \left(-2\hat{X}.X_3\right)^{-\Delta} \nabla^B \left(-2\hat{X}.X_4\right)^{-\Delta}$$
$$\propto \int_{\text{AdS}} d\hat{X} \prod_{i=1}^{4} (-2\hat{X}.X_i)^{-\Delta} \sum_{\text{perms}} \left(\frac{X_1.X_2\, X_3.X_4}{\hat{X}.X_1 \hat{X}.X_2 \hat{X}.X_3 \hat{X}.X_4} + \dots \right), \tag{2.3.22}$$

where we use (2.3.6) and the ellipsis denotes terms that can be reduced to ϕ^4 by use of integration by parts and equations of motion.[3] The new contribution can be written in terms of $\bar{D}$-functions as

$$(1+u+v)\bar{D}_{\Delta+1\,\Delta+1\,\Delta+1\,\Delta+1}\,. \tag{2.3.23}$$

Performing a conformal block analysis of this correlator one finds that indeed the sum truncates to spin-2. Furthermore, analysing the large-twist behaviour of the anomalous dimensions gives insight into the large-N scaling of this solution to the crossing equations, see chapter 4.

The next higher-derivative correction is given by a six-derivative interaction vertex $(\nabla\phi)^2 \left(\nabla_\mu\nabla_\nu\phi\right)^2$ which corresponds to another spin-2 solution as explained above. The contact diagram is

$$\sum_{\text{perms}} \int_{\text{AdS}} d\hat{X} \nabla_A \left(-2\hat{X}.X_1\right)^{-\Delta} \nabla^A \left(-2\hat{X}.X_2\right)^{-\Delta} \nabla_B\nabla_C \left(-2\hat{X}.X_3\right)^{-\Delta} \nabla^B\nabla^C \left(-2\hat{X}.X_4\right)^{-\Delta}$$
$$\propto \int_{\text{AdS}} d\hat{X} \prod_{i=1}^{4} \left(-2\hat{X}.X_i\right)^{-\Delta} \sum_{\text{perms}} \left(\frac{X_1.X_2 \left(X_3.X_4\right)^2}{\hat{X}.X_1\, \hat{X}.X_2 \left(\hat{X}.X_3\right)^2 \left(\hat{X}.X_4\right)^2} + \dots \right), \tag{2.3.24}$$

where the ellipsis denotes terms that can be reduced to the zero- and four-derivative contributions. The new six-derivative contribution is given in terms of $\bar{D}$-functions as follows

$$\bar{D}_{\Delta+2\,\Delta+1\,\Delta+2\,\Delta+1} + \bar{D}_{\Delta+1\,\Delta+2\,\Delta+1\,\Delta+2} + u^2\,\bar{D}_{\Delta+2\,\Delta+2\,\Delta+1\,\Delta+1}$$
$$+ u\,\bar{D}_{\Delta+1\,\Delta+1\,\Delta+2\,\Delta+2} + v^2\,\bar{D}_{\Delta+1\,\Delta+2\,\Delta+2\,\Delta+1} + v\,\bar{D}_{\Delta+2\,\Delta+1\,\Delta+1\,\Delta+2}\,. \tag{2.3.25}$$

[3] These additional terms will become important in chapter 3 where we introduce higher-dimensional generalised bulk-to-boundary propagators which treat AdS and S on equal footing. These propagators lead to new AdS×S contact Witten diagrams and the corresponding covariant derivatives no longer commute, therefore ambiguities appear in the place of the ellipsis above and they can no longer be disregarded.

Again, performing the conformal block expansion one finds that indeed the sum truncates to spin-2. This analysis can be generalised to higher derivatives by acting with the appropriate number of covariant derivatives on the bulk-to-boundary propagators as in the above example. The results can be represented in position space in terms of D-functions or in Mellin space according to appendix A.

We will encounter such local quartic bulk interactions dual to conformal correlators with spin-truncated conformal block expansions again in all three main chapters. Such correlators can be obtained from contact Witten diagrams only and no exchange diagrams need to be considered to compute higher-derivative corrections [30]. This makes the calculation much simpler and all Witten diagrams (with any number of derivatives) are given in terms of D-functions (or their Mellin transforms). Note that for supergravity the conformal block expansion does not generally truncate in spin and therefore it is not included in the above discussion. However, in the 1d case considered in chapter 5 supergravity is actually described by a ϕ^4 bulk interaction and can thus be obtained from contact diagrams.

We have now reviewed important concepts which will play a role in the following chapters. In the next chapter we discuss higher-derivative corrections to the low-energy effective action of type IIB string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ dual to half-BPS correlators in the small α' expansion in $\mathcal{N}=4$ SYM.

Chapter 3

Towards the Virasoro-Shapiro Amplitude in $\mathrm{AdS}_5\times\mathrm{S}^5$

This chapter is based on [42] and we follow the paper closely. As explained in the introduction 1, in flat space, four-point amplitudes of closed strings are given by the Virasoro-Shapiro amplitude. It is of great interest to generalise this to curved spacetime and our aim is to obtain an analogue of the flat space VS amplitude in $\mathrm{AdS}_5\times\mathrm{S}^5$.

The AdS/CFT correspondence relates IIB gravity amplitudes to $\mathcal{N}=4$ SYM single-trace[1] half-BPS correlators. From the early days of the AdS/CFT correspondence, many direct calculations of four-point AdS amplitudes at tree-level and in the supergravity limit have been performed, resulting in predictions for the corresponding correlators on the CFT side [60, 63, 65–74]. Although the action for superstrings in $\mathrm{AdS}_5\times\mathrm{S}^5$ is known using the Green-Schwarz [75, 76] and pure spinor [77] formalisms, explicit construction of vertex operators is not fully understood. Hence, computing amplitudes beyond the supergravity approximation in this background directly from string theory remains challenging (see [78–80] for recent progress). On the other

[1]In fact the operators dual to supergravity are only single-trace in the large-N limit but have multi-trace corrections at subleading order [62, 63]. These have recently been given explicitly to all orders in N [64]. Here however, we work at leading order and so these multi-trace corrections will not play a role.

hand, a great deal of progress has recently been achieved on the CFT side despite the CFT being strongly coupled, using the constraints imposed by superconformal and crossing symmetry as well as the simplification of the spectrum predicted by AdS/CFT (hereby summarised as 'bootstrap methods'). All tree-level single-trace half-BPS correlators in the supergravity limit have been obtained in this way [81–83] and more recently string corrections have also been bootstrapped [10–16] with groundwork laid in [30, 61]. Loop corrections to four-point AdS amplitudes have also been obtained via bootstrap methods both in the supergravity limit [84–89] as well as string corrections [11, 90–92]. The more recent of these works have also made use of a hidden 10d conformal symmetry [41] which we will discuss further in the context of $AdS_2 \times S^2$ in chapter 5.

The program in this chapter can be viewed as partly going back to the direct calculation approach but in a hugely simplified form. We notice that the tree-level string corrections obtained via bootstrap methods can be obtained via AdS×S contact diagrams arising from a simple 10d scalar effective action. The starting point is the observation that if we write the flat space VS amplitude as an infinite series in α', the leading term will describe supergravity while higher-order terms describe string corrections. These corrections can be derived from a simple effective field theory consisting of a scalar field with quartic interactions. For example, the first string correction is simply a constant proportional to α'^3 which arises from a ϕ^4 interaction, and the next correction is $\mathcal{O}(\alpha'^5)$ and quadratic in the Mandelstam variables so can be derived from a four-derivative interaction $(\partial\phi.\partial\phi)^2$. In this way, we can construct the four-field piece of the linearised (about flat space) effective action at all orders in α', fixing coefficients by comparing to the VS amplitude. This can be made more precise. All the fields of type IIB supergravity can be described with a chiral scalar superfield, ϕ, in 10d $\mathcal{N} = 2$ superspace [93], and it is this scalar superfield that appears in the superaction. The Virasoro-Shapiro amplitude for IIB string theory is a superamplitude containing a factor $\delta^{16}(Q)$ [94]. Similarly the corresponding linearised effective action is a superaction and one integrates a scalar superfield (prepotential) over 16 Grassmann odd variables $\int d^{16}\theta$ [95]. The action of four Grassmann derivatives on the scalar produces the Riemann curvature and so ϕ^4 in the effective superpotential produces the familiar $\mathcal{R}^4$ correction to supergravity.

We propose a generalisation of the flat space VS amplitude by uplifting its small α' expansion to $\mathrm{AdS}_5\times\mathrm{S}^5$ by replacing the derivatives in flat space by covariant derivatives in AdS_5 and S^5. We will find that this 10d scalar effective action describing tree-level string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ generates the interacting part of all single-trace half-BPS correlators. The resulting correlators are naturally packaged together into a 10d structure. This 10d structure is very reminiscent of and indeed was partly inspired by the 10d conformal structure of these correlators observed in [41]. However, here the 10d *conformal* structure is not apparent and does not play a role. We can read off some coefficients of the AdS×S effective action directly from the flat space one, but not all terms can be read off in this way. Firstly, since covariant derivatives will no longer commute in general, there is the possibility of commutator terms which vanish in flat space. Furthermore, it is also possible to add terms proportional to the curvature which vanish in the flat space limit. The effective action will therefore have additional terms with unfixed coefficients.

We do not here prove the existence of the effective field theory in $\mathrm{AdS}_5\times\mathrm{S}^5$, but justify it a posteriori by showing that it reproduces all known results for four-point correlators of single-trace half-BPS operators at orders α'^3 and α'^5, which were previously obtained via bootstrap methods in [10–15]. Furthermore, we present a general algorithm to obtain four-point correlators to any order in α' and use it to derive new predictions at α'^6 and α'^7. At the same time as we completed our work, the authors of [16] also obtained higher-order α' corrections in $\mathrm{AdS}_5\times\mathrm{S}^5$, using bootstrap methods in Mellin space to arrive at the higher-derivative corrections. Their results nicely complement ours.

As mentioned in the introduction, the key technical tool to derive correlation functions from the 10d effective field theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ are generalised 10d Witten diagrams which treat AdS and S on equal footing. Usual contact Witten diagrams describe supergravity scattering and are defined as integrals over AdS space, we will generalise them to integrals over the full AdS×S space to manifestly include the spherical harmonics on the five-sphere. We are not aware of such generalised Witten diagrams directly appearing in the literature before, although similar structures on the sphere are given in [96] where analogues of geodesic Witten diagrams (which give conformal blocks) on the sphere were considered. The generalised Witten diagrams involve introducing propagators connecting the $(5+5)$-dimensional bulk of $\mathrm{AdS}_5\times\mathrm{S}^5$ to a generalised notion of a boundary. Although the five-sphere is compact, we can formally define its boundary using embedding coordinates analogous to those of AdS_5. This definition is physically sensible when describing half-BPS operators since it essentially encodes the condition that they are traceless and symmetric in R-symmetry indices. Expanding the 10d Witten diagrams in modes on the S^5 then gives a prediction for all four-point correlators of single-trace half-BPS operators corresponding to a fixed order in the α' expansion of tree-level string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$. Comparing these results to those obtained using localisation techniques[2] [12, 97, 98] allows us to fix some ambiguities in the effective action.

Before we go on and compute these higher-derivative corrections from the effective action we describe the general setup for our analysis in the following section.

3.1 General Setup

In this section we describe the basic ingredients that we will use in this chapter. We start with a review of $\mathcal{N}=4$ super Yang-Mills theory, type IIB string theory and the corresponding holographic correlators. This is followed by a review of half-BPS correlators in $\mathcal{N}=4$ SYM, which will be the analogue of the Virasoro-Shapiro amplitude in $\mathrm{AdS}_5\times\mathrm{S}^5$. Next, we describe our strategy for deducing an effective action from the VS amplitude in flat space and translating it to $\mathrm{AdS}_5\times\mathrm{S}^5$. Subsequently, we

[2] Supersymmetric localisation is a method to exactly compute correlation functions of supersymmetric operators in certain supersymmetric quantum field theories by reducing the path integral to finite-dimensional integrals.

introduce embedding space formalism for the sphere together with sphere-analogues of AdS contact diagrams. In the next subsection we then show how to compute contact diagrams directly in AdS×S using novel bulk-to-boundary propagators which are manifestly ten-dimensional. For a given order in the α' expansion of the VS amplitude, this will allow us to compute the infinite tower of half-BPS correlators by computing Witten diagrams from a 10d effective action and expanding them in modes on the sphere. The correlators are most elegantly expressed in Mellin space, which we review in the last subsection. In particular, we find that expanding our 10d Witten diagrams in terms of spherical coordinates gives rise to a spherical analogue of the Mellin transform and implies a generalised Mellin amplitude where AdS_5 and S^5 are on equal footing. The question of stringy corrections will be addressed in subsequent sections.

3.1.1 Review: $\mathcal{N}=4$ SYM/IIB Superstring Theory in $\mathbf{AdS}_5\times\mathbf{S}^5$

Let us start by reviewing the ingredients of the correspondence between $\mathcal{N}=4$ SYM and string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$. We start by introducing the field content and action of $\mathcal{N}=4$ SYM before considering type IIB supergravity. For more detailed reviews of these concepts see e.g [52, 99, 100]. Finally, we discuss holographic correlators in this example of the AdS/CFT correspondence.

$\mathcal{N}=4$ super Yang-Mills theory

The conformal field theory we study in this chapter, $\mathcal{N}=4$ supersymmetric Yang-Mills theory, is a theory of great interest to the scientific community. On the one hand, it is the interacting four-dimensional gauge theory with the highest amount of supersymmetry and it is conformal even at the quantum level [101, 102]. It is also dual to IIB string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ and is the classical example of the AdS/CFT correspondence. Moreover, it is believed to be integrable, i.e. exactly solvable at all values of the coupling constant, at least in the planar limit [103]. On the other hand, $\mathcal{N}=4$ SYM is a four-dimensional interacting theory and its tree-level scattering amplitudes are equivalent to those of massless QCD [104], therefore it is a useful toy model for developing techniques that can be applied to less symmetric QFTs, such as QCD. We will briefly introduce the field theory in the following.

Denoted by $\mathcal{N}=4$ SYM is the maximally supersymmetric four-dimensional Yang-Mills theory with gauge group $SU(N)$. The field content of $\mathcal{N}=4$ SYM consists of one gluon field A^a_μ, with $a=1,\dots N^2-1$ with four fermions as superpartners, $\psi^a_{\alpha A}$, $\bar{\psi}^{\dot{\alpha}\,a}_A$ with $\alpha,\dot{\alpha}=1,2$ and $A=1,2,3,4$. Additionally, closure of the supersymmetry algebra requires that there are six real scalars $\phi^{a\,I}_{\rm YM}$, with $I=1,\dots,6$, transforming in the fundamental representation of $SO(6)\sim SU(4)$. Hence, there are 4×2 fermionic and $6+2$ bosonic degrees of freedom. Due to supersymmetry, all fields have to transform in the adjoint representation of $SU(N)$. The action of $\mathcal{N}=4$ SYM was obtained in [105]

$$S=\int d^4x\,\mathrm{Tr}\Big(-\frac{1}{4g^2_{\rm YM}}F_{\mu\nu}F^{\mu\nu}+\frac{\theta}{16\pi^2}F_{\mu\nu}\tilde{F}^{\mu\nu}-(D_\mu\varphi_{AB})(D^\mu\varphi^{AB})+i\bar{\psi}^A_{\dot{\alpha}}\sigma^{\dot{\alpha}\alpha}_\mu D^\mu\psi_{\alpha A}$$
$$-\frac{i\,g_{\rm YM}}{2}\psi^\alpha_A[\varphi^{AB},\psi_{\alpha B}]-\frac{i\,g_{\rm YM}}{2}\bar{\psi}^A_{\dot{\alpha}}[\varphi_{AB},\bar{\psi}^{\dot{\alpha}B}]-\frac{g^2_{\rm YM}}{2}[\varphi_{AB},\varphi_{CD}][\varphi^{AB},\varphi^{CD}]\Big)\,, \tag{3.1.1}$$

where we grouped the six real scalar fields $\phi^{a\,I}_{\rm YM}$ into six complex scalar fields $\varphi^{a\,AB}=-\varphi^{a\,BA}$ which transform in the fully antisymmetric two-index representation of $SU(4)$. The field strength is defined as $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+i\,g_{\rm YM}\,[A_\mu,A_\nu]$ and the covariant derivative is $D_\mu=\partial_\mu-i\,g_{\rm YM}\,[A_\mu,A_\nu]$. In the second term, θ is a real coupling, the so-called Yang-Mills theta-angle, and $\tilde{F}^{\mu\nu}$ is the Hodge dual of the field strength $\tilde{F}^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}$. It is standard to combine the real couplings $g_{\rm YM}$ and θ into the single complex Yang-Mills coupling

$$\tau=\frac{\theta}{2\pi}+\frac{4\pi\,i}{g^2_{\rm YM}}\,. \tag{3.1.2}$$

The matrices $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha}=(\mathrm{Id},-\sigma)^{\dot{\alpha}\alpha}$ and $(\sigma^\mu)_{\alpha\dot{\alpha}}=\epsilon_{\alpha\beta}\epsilon_{\dot{\alpha}\dot{\beta}}(\bar{\sigma})^{\dot{\beta}\beta}=(\mathrm{Id},\sigma)_{\alpha\dot{\alpha}}$, where σ^μ are the usual Pauli matrices and ϵ_{ab} is the Levi-Civita tensor. The only tunable parameters in the action are the gauge coupling g_{YM} and the rank of the gauge group $SU(N)$. Taking the limit $N \to \infty$, $g_{\mathrm{YM}} \to 0$, while holding the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ fixed, gives a topological expansion in $1/N$ where the leading term corresponds to the planar limit.

A remarkable property of this theory is that it is ultraviolet finite, i.e. that the coupling g_{YM} is not renormalised at the quantum level. Hence, $\mathcal{N} = 4$ SYM is a conformal field theory even at the quantum level [101, 102]. Thus, the theory is invariant under the conformal group $SO(2,4) \sim SU(2,2)$ and supersymmetry enhances this to the superconformal group $PSU(2,2|4)$. The bosonic part of $PSU(2,2|4)$ then consists of the conformal group $SU(2,2)$ and the R-symmetry group $SU(4)$. The fermionic part of the supergroup are the four supersymmetry generators and their superconformal partners.

Type IIB supergravity

The holographic dual of $\mathcal{N} = 4$ SYM is type IIB superstring theory in $\mathrm{AdS}_5 \times \mathrm{S}^5$. Type IIB string theory was found in [106] and in the low-energy limit it can be approximated by type IIB supergravity [107]. Let us review the field content and action of 10d IIB supergravity here.

First, since it is a theory of gravity, there is a graviton, hence the metric $g_{\mu\nu}$ which is a supersymmetric traceless tensor of the symmetry group $SO(8)$ and thus has 35 bosonic degrees of freedom (counting the number of independent components of a symmetric 8×8 matrix and subtracting one due to tracelessness). The rest of the bosonic part of the spectrum consists of the axion-dilaton $C + i\Phi$ with two bosonic degrees of freedom, a rank-2 antisymmetric tensor $B_{\mu\nu} + iA_{2\,\mu\nu}$ with 56, and a rank-4 antisymmetric tensor $A^{+}_{4\,\mu\nu\rho\sigma}$ with 35 bosonic degrees of freedom. Furthermore, there are two Majorana-Weyl gravitinos $\psi^{I}_{\mu\alpha}$, where $I = 1, 2$, with 112 fermionic degrees of freedom and two Majorana-Weyl dilatinos λ^{I}_{α}, $I = 1, 2$, with 16 fermionic degrees of freedom. Thus, there are 128 bosonic and 128 fermionic degrees of freedom, as required by supersymmetry. The + superscript of the rank-4 antisymmetric tensor $A^+_{4\,\mu\nu\rho\sigma}$ indicates that it has a self-dual field strength, the five-form field strength $\tilde{F}_5$ defined below. The two gravitinos have the same chirality and so do the two dilatinos, however theirs is opposite to that of the gravitinos. Therefore, the theory is chiral.

The type IIB supergravity action was found in [93, 107]. Because of the self-dual field strength $\tilde{F}_5$ it is not straightforward to write down a classical action. This is because the term $|\tilde{F}_5|^2$ in the action counts twice the desired amount of physical degrees of freedom since it does not contain the self-duality constraint. This can be resolved by imposing the self-duality constraint as an additional field equation. The action is then given by

$$\begin{aligned} 4\kappa_B^2\, S = &\int d^{10}x\sqrt{-g}\, e^{-\Phi}\left(2\mathcal{R} + 8\partial_\mu\Phi\partial^\mu\Phi - |H_3|^2\right) \\ &- \int\left[d^{10}x\sqrt{-g}\left(|F_1|^2 + |\tilde{F}_3|^2 + \frac{1}{2}|\tilde{F}_5|^2\right) + A_4^+ \wedge H_3 \wedge F_3\right] + \text{fermions}\ , \end{aligned} \tag{3.1.3}$$

with the scalar curvature $\mathcal{R}$, $g = \det g_{\mu\nu}$ and the coupling constant κ_B is related to the string length l_s and the 10d Newton constant as follows:

$$2\kappa_B^2 = \frac{1}{2\pi}\left(2\pi l_s\right)^8\ , \qquad 16\pi G_N^{10d} = \frac{1}{2\pi}\left(2\pi l_s\right)^8 g_s^2 = 2\kappa_B^2\, g_s^2\ . \tag{3.1.4}$$

Moreover, the field strengths are defined as:

$$\begin{aligned} &F_1 = dC\ ,\quad H_3 = dB\ ,\quad F_3 = dA_2\ ,\quad F_5 = dA_4^+\ , \\ &\tilde{F}_3 = F_3 - C\,H_3\ ,\quad \tilde{F}_5 = F_5 - \frac{1}{2}A_2 \wedge H_3 + \frac{1}{2}B \wedge F_3\ , \end{aligned} \tag{3.1.5}$$

and the supplementary self-duality constraint is

$$\ast\,\tilde{F}_5 = \tilde{F}_5\ . \tag{3.1.6}$$

Finally, the quantities $|F_p|^2$ are defined as

$$|F_p|^2 = \frac{1}{p!}g^{\mu_1\nu_1}\dots g^{\mu_p\nu_p}\bar{F}_{\mu_1\cdots\mu_p}F_{\nu_1\dots\nu_p}\ , \tag{3.1.7}$$

where $\bar{F}$ is the complex conjugate of F. It is worth noting that by dimensional analysis this theory is non-renormalisable which generally speaking is true for all effective supergravity actions. This suggests that they do not approximate well-defined quantum theories. However, this is no problem when considering these theories as low-energy effective actions of more fundamental quantum theories (like string theory or M-theory). The non-renormalisability can be seen from conventional power counting. In the example of the 10d theory above this goes as follows. The curvature $\mathcal{R}$ has dimension (length)$^{-2}$. By dimensional analysis the square of the gravitational coupling constant κ_B^2 then has to have dimensions (length)^8, or for general dimensions d, (length)$^{d-2}$. It is now easy to see that the coupling constant has negative mass dimension for $d > 2$ and is therefore non-renormalisable. As mentioned before, this is no problem because we only consider it as an effective theory describing the low-energy physics of string theory.

Type IIB supergravity has $SL(2,\mathbb{R})$ symmetry. This is most manifest in the Einstein frame, where we redefine fields from the string metric $g_{\mu\nu}$ to the Einstein metric $g^E_{\mu\nu}$

$$g_{\mu\nu} = e^{\Phi/2} g^E_{\mu\nu} \ , \tag{3.1.8}$$

which leads to a transformation of the scalar curvature term such that it contains the usual Einstein-Hilbert term. Furthermore, combining the axion C and dilaton Φ into a new complex scalar field gives the axion-dilaton field

$$\tau \equiv C + i\, e^{-\Phi} \ . \tag{3.1.9}$$

The metric and A_4^+ fields are invariant under $SL(2,\mathbb{R})$ transformations and the axion-dilaton field transforms under a Möbius transformation

$$\tau \to \frac{a\tau + b}{c\tau + d} \ , \quad \text{with } ad - bc = 1 \ , \quad a,b,c,d \in \mathbb{R} \ . \tag{3.1.10}$$

Moreover, the fields $B_{\mu\nu}$ and $A_{2\,\mu\nu}$ rotate into each other under the above linear transformation. When considering the full quantum theory, there is a quantisation condition $\tau \sim \tau + 1$. Thus, the symmetry group of type IIB superstring theory is the $SL(2,\mathbb{Z})$ subgroup of $SL(2,\mathbb{R})$, where $a,b,c,d \in \mathbb{Z}$ in the Möbius transformation.

Holographic correlators

We have discussed general holographic correlators in section 2.3, were we have seen that correlators of chiral primary (half-BPS) operators are dual to scattering amplitudes of bulk scalars in AdS. In particular, we have described how these correlators can be computed from AdS Witten diagrams. In this chapter we will study four-point correlators of chiral primary operators in $\mathcal{N} = 4$ SYM. We have described the form of general chiral primary operators in section 2.3. Now consider the superconformal primary of the stress tensor multiplet in $\mathcal{N} = 4$ SYM. It is a scalar operator of protected dimension two and transforms in the two-index traceless symmetric representation **20** of the R-symmetry group $SO(6)$. It is constructed from scalar fields as follows

$$\mathcal{O}_2 = \mathrm{Tr}(\phi^I_{\mathrm{YM}}\phi^J_{\mathrm{YM}}) - \frac{\delta_{IJ}}{6}\mathrm{Tr}(\phi^K_{\mathrm{YM}}\phi_{\mathrm{YM}\,K}) \, . \tag{3.1.11}$$

Its supergravity dual is a scalar in the graviton multiplet in AdS$_5$ with mass $m_2^2 = -4$ (see (2.3.2), where we set the AdS radius $R = 1$). Traceless symmetric operators of the form $\phi^{(I_1}_{\mathrm{YM}}\phi^{I_2}_{\mathrm{YM}}\dots\phi^{I_p)}_{\mathrm{YM}}$ with more than two indices, where all p R-symmetry indices are symmetrised and all traces removed, are dual to higher Kaluza-Klein modes on the five-sphere. These modes arise from dimensionally reducing type IIB gravity on S^5 which leads to an infinite tower of protected scalar operators with scaling dimension $\Delta = p$, where p is the $SO(6)$ R-symmetry charge describing the KK modes on the sphere. A chiral primary operator with $SO(6)$ charge p is thus dual to a KK mode with mass $m_p^2 = p(p-4)$.

In the following, we consider the low-energy approximation of string theory, which corresponds to a small α' expansion, where the leading contribution is supergravity and the subleading terms describe stringy corrections. These higher-derivative corrections to the supergravity approximation are the focus of this chapter. Furthermore, taking $N \to \infty$ (which is like taking $G_N \to 0$) suppresses loop corrections and restricts to classical gravity. Through the AdS/CFT correspondence $G_N \sim 1/c$, where c is the central charge, and in the present theory $c = (N^2 - 1)/4$. Hence, we consider a large-c expansion. See the review in 2.2 for further discussions of these limits and the relations between the parameters of the bulk and boundary theory.

We now understand the correlators from a holographic point of view, in the next subsection we consider the half-BPS correlators further in the specific context of the considerations in this chapter.

3.1.2 Half-BPS Correlators

As we have seen, there are six real scalars in $\mathcal{N} = 4$ SYM transforming in the adjoint representation of $SU(N)$ and the fundamental of $SO(6)$, $\phi^{a\,I}_{\mathrm{YM}}(X)$[3]. Here we view the 4d Minkowski space via null 6d embedding coordinates X^A with $X.X = 0$ manifesting the conformal $SO(2,4)$ symmetry (see section 2.3.1 for a review on AdS embedding space). We also project with a null 6d coordinate Y_I, $Y.Y = 0$ to obtain $\phi_{\mathrm{YM}}(X,Y) = \phi^I_{\mathrm{YM}}(X)Y_I$ manifesting the internal $SO(6)$ symmetry. Then the single-trace half-BPS operators are defined as

$$\mathcal{O}_p(X,Y) = \frac{1}{pN^{p/2}}\mathrm{Tr}(\phi^p_{\mathrm{YM}}) \;, \tag{3.1.12}$$

which looks like the chiral primary operators discussed in the previous subsection but with the R-symmetry indices contracted and with an additional normalisation. Note that we normalise the operators with an additional factor of $1/\sqrt{p}$ compared to the normalisation giving a normalised two-point function, first derived in [108]. This normalisation is inspired by the ten-dimensional conformal symmetry of [41] and will be discussed further in chapter 5.

It is then useful to collect together the four-point functions of all single-trace half-BPS operators $\mathcal{O}_p(X,Y)$ into a single object $\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle$ as follows

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle = \sum_{p,q,r,s=2}^{\infty} \langle \mathcal{O}_p\mathcal{O}_q\mathcal{O}_r\mathcal{O}_s\rangle_{\mathrm{int}} \;, \tag{3.1.13}$$

[3] We drop the index a from now on and it will be understood that the $\phi^I_{\mathrm{YM}}(X)$ transform in the adjoint of $SU(N)$.

where $\langle\mathcal{O}_p\mathcal{O}_q\mathcal{O}_r\mathcal{O}_s\rangle_{\text{int}}$ represents the interacting part of the correlator, which always contains a particular factor $I(X_i,Y_i)$ due to superconformal symmetry [109] which we thus divide out

$$\langle\mathcal{O}_p\mathcal{O}_q\mathcal{O}_r\mathcal{O}_s\rangle_{\text{int}} = \frac{\langle\mathcal{O}_p\mathcal{O}_q\mathcal{O}_r\mathcal{O}_s\rangle - \langle\mathcal{O}_p\mathcal{O}_q\mathcal{O}_r\mathcal{O}_s\rangle_{\text{free}}}{I(X_i,Y_i)}\ . \tag{3.1.14}$$

From now on we will usually drop the explicit 'int' subscript at the end of the correlators. Here I is a polynomial in X_i and Y_i, the so-called Intriligator polynomial, which is a common factor of all interacting half-BPS four-point functions [109]. It is the counterpart of the $\delta^{16}(Q)$ factor of flat space superamplitudes [110] and we give its explicit form in appendix B.

Now that we have specified what correlators we are studying in this chapter we can go on and discuss the Virasoro-Shapiro amplitude that describes string scattering in flat space and how to lift it to curved spacetime.

3.1.3 Effective Action

As explained in the introduction, the four-point amplitude of closed string theory takes a very compact form in flat space, the Virasoro-Shapiro amplitude:

$$A_{\text{VS}}(S,T) = \frac{1}{STU}\frac{\Gamma(1-\frac{\alpha' S}{4})\Gamma(1-\frac{\alpha' T}{4})\Gamma(1-\frac{\alpha' U}{4})}{\Gamma(1+\frac{\alpha' S}{4})\Gamma(1+\frac{\alpha' T}{4})\Gamma(1+\frac{\alpha' U}{4})}\ , \qquad S+T+U=0\ , \tag{3.1.15}$$

where S,T,U are the standard four-point kinematic invariants. Note that we have factored out a supermomentum delta-function which encodes all the external supergravity states. In $\mathrm{AdS}_5{\times}\mathrm{S}^5$ the analogue is to factor out an Intriligator polynomial from the interacting part of half-BPS correlators in the boundary, as we explained in the previous subsection. Our goal will then be to construct a bosonic 10d effective action which describes the remaining quantity. A priori it is not obvious that such an effective action should exist in curved background, but we justify it by showing that it reproduces previous results. It is important to note that the fact that we can factor out the supermomentum delta-function, or the Intriligator polynomial which contains all the supersymmetry properties of the correlators, is the crucial aspect of the theory that allows us to obtain correlators using a higher-dimensional effective action.

The flat space VS amplitude in (3.1.15) has expansion

$$\begin{aligned} A_{VS}(S,T) &= \frac{1}{STU}\exp\left(\sum_{n=1}^{\infty} 2\left(\frac{\alpha'}{4}\right)^{2n+1}\frac{\zeta_{2n+1}}{2n+1}(S^{2n+1}+T^{2n+1}+U^{2n+1})\right) \\ &= \frac{1}{STU} + 2\zeta_3(\tfrac{\alpha'}{4})^3 + (S^2+T^2+U^2)\zeta_5(\tfrac{\alpha'}{4})^5 + 2STU(\zeta_3)^2(\tfrac{\alpha'}{4})^6 \\ &\quad + \tfrac{1}{2}(S^2+T^2+U^2)^2\zeta_7(\tfrac{\alpha'}{4})^7 + \dots\ , \end{aligned} \tag{3.1.16}$$

where ζ_n are Riemann-zeta functions. Excluding the first term, which corresponds to supergravity, we can view the remaining terms as arising from a scalar effective action. From this point of view, the α'^3 correction which gives a constant contribution to the four-point amplitude, comes from a ϕ^4 interaction. Higher-order terms can then be obtained by applying derivatives to the ϕ^4 interaction corresponding to the invariants S,T,U. So $S = -2k_1.k_2 \to 2\partial_\mu\phi\partial^\mu\phi\phi^2$, $T = -2k_1.k_3 \to 2\partial_\mu\phi\phi\partial^\mu\phi\phi$ etc.

Specifically then the VS amplitude is equivalent to the following four-field terms in an effective superpotential for supergravity linearised about flat space:

$$\begin{aligned} V_{\rm VS}^{\rm flat}(\phi) = \tfrac{1}{2^3\times 4!}\Big(&2\zeta_3(\tfrac{\alpha'}{2})^3\phi^4 + 3\zeta_5(\tfrac{\alpha'}{2})^5(\partial\phi.\partial\phi)^2 + 2(\zeta_3)^2(\tfrac{\alpha'}{2})^6(\partial\phi.\partial\phi)(\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi) \\ &+ 3\zeta_7(\tfrac{\alpha'}{2})^7(\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi)^2 + \dots\Big)\ . \end{aligned} \tag{3.1.17}$$

We now uplift the effective superpotential to an AdS$_5$×S^5 background by replacing the flat derivatives with covariant AdS×S derivatives. This uplift is not unique however. Firstly the covariant derivatives no longer commute with each other leading to ambiguities. Secondly there could be terms involving lower number of derivatives, compensated by the AdS radius R which would vanish in the flat space limit.

So to $\mathcal{O}(\alpha'^7)$ the superpotential translates to

$$\begin{aligned}
V_{\rm VS}^{\rm AdS\times S}(\phi) = \tfrac{1}{8\times 4!}\Big[&(\tfrac{\alpha'}{2})^3 A\phi^4 + (\tfrac{\alpha'}{2})^5\Big(3B(\nabla\phi.\nabla\phi)^2 + 6C\nabla^2\nabla_\mu\phi\nabla^\mu\phi\phi^2\Big) \\
&+ (\tfrac{\alpha'}{2})^6\Big(D(\nabla\phi.\nabla\phi)(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi) + 6E\nabla_\mu\nabla^2\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\phi^2\Big) \\
&+ (\tfrac{\alpha'}{2})^7\Big(6F(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi)^2 \\
&+ 6G_1\,(\nabla^\mu\nabla^\nu\nabla_\mu\nabla^\rho\nabla^\sigma\nabla_\rho\phi)\,(\nabla_\nu\nabla_\sigma\phi)\,\phi^2 + \dots\Big) + \dots\Big]\,. \qquad (3.1.18)
\end{aligned}$$

There are four more eight-derivative terms with coefficients G_2, G_3, G_4, G_5 whose explicit expressions are given in (3.6.2) and appendix D. The ambiguities with coefficients C, E, G_i are multiplied by symmetry factors for later convenience. Here, unlike in flat space, the coefficients $A, B, C, \dots$ themselves can have an expansion in the dimensionless parameter α'/R^2 where R is the radius of AdS (or S). So whereas in flat space $2k$-derivative terms only occur at order α'^{k+3}, in AdS×S, $2k$-derivative terms occur at α'^{k+3} and all higher orders in principle.

One could also imagine replacing the coefficient of $1/STU$ in (3.1.16) with an expansion in α'/R^2, which is not included in (3.1.18), however this is forbidden by superconformal symmetry of $\mathcal{N} = 4$ SYM correlators. In more detail, the non-renormalisation results of [109] imply that supergravity correlators must contain a contribution from free theory and there is a non-trivial cancellation between the two terms which links them together. Since free theory does not receive α' corrections, there cannot be α'/R^2 corrections to the coefficient of $1/STU$.

The zeroth-order terms in the expansion of A, B, D, F are then determined by the Virasoro-Shapiro amplitude. Specifically,

$$\begin{aligned}
A(\alpha') &= 2\zeta_3 + A_1\tfrac{\alpha'}{2R^2} + A_2\left(\tfrac{\alpha'}{2R^2}\right)^2 + \dots & E(\alpha') &= E_0 + E_1\tfrac{\alpha'}{2R^2} + \dots \\
B(\alpha') &= \zeta_5 + B_1\tfrac{\alpha'}{2R^2} + \dots & F(\alpha') &= \tfrac{1}{2}\zeta_7 + F_1\tfrac{\alpha'}{2R^2} + \dots \\
C(\alpha') &= C_0 + C_1\tfrac{\alpha'}{2R^2} + \dots & G_i(\alpha') &= G_{i;0} + G_{i;1}\tfrac{\alpha'}{2R^2} + \dots \text{ for } i = 1,\dots,5\,. \\
D(\alpha') &= 2(\zeta_3)^2 + D_1\tfrac{\alpha'}{2R^2} + \dots & & \qquad (3.1.19)
\end{aligned}$$

For simplicity, we will set $R = 1$ from now on throughout this chapter, but it will be understood that these higher-order terms vanish in the flat space limit.

Computing 10d Witten diagrams using novel generalised bulk-to-boundary propagators and expanding them in terms of S^5 coordinates will give all single-trace half-BPS four-point correlators in $\mathcal{N}=4$ SYM described by tree-level string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$. We introduce the generalised bulk-to-boundary propagators and Witten diagrams in the following subsection.

3.1.4 Generalised Contact Witten Diagrams

Standard AdS Witten diagrams were reviewed in section 2.3 and in this subsection we define analogous objects on the sphere (following similar ideas in [96]) and finally we introduce a generalisation of Witten diagrams using bulk-to-boundary propagators which are intrinsically ten-dimensional and treat AdS and S on equal footing. This will have a big pay-off since we will obtain the whole tower of half-BPS correlators by expanding the Witten diagrams in spherical coordinates.

As we have seen in subsection 2.3.1, Witten diagrams are most conveniently expressed in embedding coordinates. In addition to embedding coordinates in AdS_{d+1} we also introduce embedding coordinates for S^{d+1}:

$$\hat{Y}^2=\sum_{i=-1}^{d}\left(\hat{Y}^i\right)^2=1\ . \tag{3.1.20}$$

In the present context, $d=4$. As for AdS (2.3.4), covariant derivatives in terms of embedding coordinates are defined using projection tensors

$$\mathcal{P}_I^J=\delta_I^J-\hat{Y}_I\hat{Y}^J\ , \tag{3.1.21}$$

which satisfy the useful identities

$$\mathcal{P}_I^J\hat{Y}^J=0\ ,\qquad \mathcal{P}_I^J\mathcal{P}_J^K=\mathcal{P}_I^K\ . \tag{3.1.22}$$

To distinguish AdS and S projection tensors we use labels $A,B,C\dots$ for AdS in section 2.3.1 and $I,J,K\dots$ for the sphere analogues here. Recall that the covariant derivative of a tensor is then given by [57, 58]

$$\nabla_A \mathrm{T}_{\mathrm{A}_1\ldots\mathrm{A}_\mathrm{N}} = \mathcal{P}_{\mathrm{A}}^{\mathrm{C}}\mathcal{P}_{\mathrm{A}_1}^{\mathrm{C}_1}\ldots\mathcal{P}_{\mathrm{A}_\mathrm{N}}^{\mathrm{C}_\mathrm{N}}\partial_{\mathrm{C}}\left(\mathcal{P}_{\mathrm{C}_1}^{\mathrm{E}_1}\ldots\mathcal{P}_{\mathrm{C}_\mathrm{N}}^{\mathrm{E}_\mathrm{N}}\mathrm{T}_{\mathrm{E}_1\ldots\mathrm{E}_\mathrm{N}}\right) \ . \tag{3.1.23}$$

This can equally be applied to the sphere case by simply sending $A, B, C, \ldots$ indices to $I, J, K, \ldots$ etc.

Now first recall that the standard AdS contact Witten diagrams in embedding space are defined as integrals over AdS$_{d+1}$ of products of bulk-to-boundary propagators $G(\hat{X}, X_i) = \frac{\mathcal{C}_{\Delta_i}}{(-2\hat{X}.X_i)^{\Delta_i}}$, see (2.3.13). At four points this yields:

$$D^{(d)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i) = \frac{1}{(-2)^{2\Sigma_\Delta}}\int_{\mathrm{AdS}}\frac{d^{d+1}\hat{X}}{(\hat{X}.X_1)^{\Delta_1}(\hat{X}.X_2)^{\Delta_2}(\hat{X}.X_3)^{\Delta_3}(\hat{X}.X_4)^{\Delta_4}} \ . \tag{3.1.24}$$

Recall that $\Sigma_\Delta = (\Delta_1{+}\Delta_2{+}\Delta_3{+}\Delta_4)/2$. These D-functions have the following form in Mellin space [57] (see also appendix A)

$$D^{(d)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i) = \mathcal{N}^{\mathrm{AdS}_{d+1}}_{\Delta_i} \times \int \frac{d\delta_{ij}}{(2\pi i)^2}\prod_{i<j}\frac{\Gamma(\delta_{ij})}{(X_i.X_j)^{\delta_{ij}}} \ , \qquad \text{with } \sum_{i<j}\delta_{ij} = \Delta_j \ , \tag{3.1.25}$$

where the normalisation is given in (2.3.15) and for later use we define normalised D-functions without the factor $\mathcal{N}^{\mathrm{AdS}_{d+1}}_{\Delta_i}$ which are independent of the spacetime dimension d :

$$D_{\Delta_i}(X_i) = \frac{1}{\mathcal{N}^{\mathrm{AdS}_{d+1}}_{\Delta_i}}D^{(d)}_{\Delta_i}(X_i) \ . \tag{3.1.26}$$

We can also consider direct analogues of these contact diagrams on the sphere. Bulk-to-boundary propagators on the sphere were introduced in [96]

$$G(\hat{Y}, Y_i) \propto (-2\hat{Y}.Y_i)^{p_i} \tag{3.1.27}$$

and in this context it is then very natural to introduce functions $B^{(d)}_{p_1p_2p_3p_4}(Y_i)$, spherical analogues of the contact Witten diagrams $D^{(d)}_{\Delta_1\Delta_2\Delta_3\Delta_4}$, as:

$$B^{(d)}_{p_1p_2p_3p_4}(Y_i) = (-2)^{2\Sigma_p}\int_{\mathrm{S}} d^{d+1}\hat{Y}(\hat{Y}.Y_1)^{p_1}(\hat{Y}.Y_2)^{p_2}(\hat{Y}.Y_3)^{p_3}(\hat{Y}.Y_4)^{p_4} \ , \tag{3.1.28}$$

where $\Sigma_p = (p_1{+}p_2{+}p_3{+}p_4)/2$. Even though the sphere is compact, we can formally define a boundary when describing half-BPS operators in $\mathcal{N}=4$ SYM since the condition $Y.Y=0$ simply encodes tracelessness of the R-symmetry indices. The B-functions are polynomials in the Y_i and can be explicitly evaluated purely combinatorially, following similar techniques to those found in the appendix of [96] (where the two- and three-point analogues were obtained):

$$B^{(d)}_{p_1p_2p_3p_4}(Y_i)=\mathcal{N}^{\mathrm{S}^{d+1}}_{p_i}\sum_{\{d_{ij}\}}\prod_{i<j}\frac{(Y_i.Y_j)^{d_{ij}}}{\Gamma(d_{ij}+1)}\ , \qquad \text{with } \sum_{i<j}d_{ij}=p_j\ , \tag{3.1.29}$$

where

$$\mathcal{N}^{\mathrm{S}^{d+1}}_{p_i}=2\times 2^{\Sigma_p}\frac{\pi^{d/2+1}\prod_i\Gamma(p_i+1)}{\Gamma(\Sigma_p+d/2+1)}\ . \tag{3.1.30}$$

For later use let us also define a normalised B-function which does not depend on the dimension d:

$$B_{p_i}(Y_i)=\frac{1}{\mathcal{N}^{\mathrm{S}^{d+1}}_{p_i}}B^{(d)}_{p_i}(Y_i)\ . \tag{3.1.31}$$

In (3.1.29) the sum is over all sets of numbers $d_{ij}=d_{ji}$ such that

$$\Big\{(d_{12},d_{13},d_{14},d_{23},d_{24},d_{34}):0\le d_{ij}=d_{ji},\quad d_{ii}=0,\quad \sum_{i=1}^{4}d_{ij}=p_j\Big\}\ . \tag{3.1.32}$$

These constraints on d_{ij} leave just two free parameters. Note the close similarity this explicit expansion of the B-functions (3.1.29) has with the Mellin transform of the AdS contact terms (3.1.25). The B-function can be seen as a discrete D-function, which is expected since it lives in a compact space. We can thus view the expansion parameters d_{ij} as analogues of the Mellin variables δ_{ij}.

It is now natural to combine the above AdS and S bulk-to-boundary propagators into one 10d object, which we refer to as a generalised bulk-to-boundary propagator in AdS×S:

$$G(\hat{X},\hat{Y};X,Y)=\left(-2\hat{X}.X-2\hat{Y}.Y\right)^{-\Delta}\ , \tag{3.1.33}$$

where X and Y satisfy

$$X^2=Y^2=0\ . \tag{3.1.34}$$

Using the definition in (3.1.23), we see that

$$\nabla^2 G = \left(\nabla^2_{\hat{X}} + \nabla^2_{\hat{Y}}\right) G = \Delta(\Delta - d)\left((-2\hat{X}.X)^2 - (-2\hat{Y}.Y)^2\right)\left(-2\hat{X}.X - 2\hat{Y}.Y\right)^{-\Delta-2} . \tag{3.1.35}$$

Hence, the propagator obeys massless equations of motion when $d = \Delta$:

$$\nabla^2 G = 0 , \tag{3.1.36}$$

which will become important in later sections when computing ambiguities. Whereas X describes the boundary of AdS, Y is not a boundary point since the sphere is compact.

We will derive predictions for four-point correlators of half-BPS operators from an effective action by computing analogues of Witten diagrams directly in the product geometry AdS×S. For now we will just develop some general properties of AdS$_{d+1}$×S^{d+1} contact Witten diagrams which are defined simply as [4]

$$D^{\mathrm{AdS}_{d+1}\times \mathrm{S}^{d+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i, Y_i) = \frac{1}{(-2)^{2\Sigma_\Delta}} \int_{\mathrm{AdS}\times\mathrm{S}} \frac{d^{d+1}\hat{X} d^{d+1}\hat{Y}}{(P_1+Q_1)^{\Delta_1}(P_2+Q_2)^{\Delta_2}(P_3+Q_3)^{\Delta_3}(P_4+Q_4)^{\Delta_4}} , \tag{3.1.37}$$

where we introduce the shorthand

$$P_i = \hat{X}.X_i , \qquad Q_i = \hat{Y}.Y_i . \tag{3.1.38}$$

The contact diagrams in AdS×S are related to the standard AdS contact diagrams in an intriguing way. This relation is explained in appendix C together with a discussion of tree-level supergravity which is not expected to arise from a superpotential[5] but can be discussed using the aforementioned relation.

It is now straightforward to expand this AdS×S contact diagram into an infinite number of standard AdS contact diagrams multiplied by sphere analogues. In

[4] We keep d and Δ_i general here but we will be focussing on the case $\Delta_i = d = 4$ later in this chapter. The case $\Delta_i = d = 1$ will be studied in chapter 5.

[5] Remarkably, in the case of 1d CFTs studied in chapter 5 we find that supergravity correlators correspond to a ϕ^4 interaction and can therefore be computed from a 4d effective superpotential.

particular, using

$$\frac{1}{(P+Q)^{\Delta}}=\sum_{p=0}^{\infty}(-1)^{p}\frac{(p+1)_{\Delta-1}}{\Gamma(\Delta)}\frac{Q^{p}}{P^{p+\Delta}} \tag{3.1.39}$$

four times and then inserting (3.1.24) and (3.1.28) gives the expansion:

$$\begin{aligned} D^{\mathrm{AdS}\times\mathrm{S}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,Y_i) &= \sum_{p_i=0}^{\infty}\prod_{i=1}^{4}(-1)^{p_i}\frac{(p_i+1)_{\Delta_i-1}}{\Gamma(\Delta_i)} \\ &\quad\times D^{(d)}_{p_1+\Delta_1\,p_2+\Delta_2\,p_3+\Delta_3\,p_4+\Delta_4}(X_i)B^{(d)}_{p_1p_2p_3p_4}(Y_i)\ . \end{aligned} \tag{3.1.40}$$

Expanding the 10d Witten diagrams in spherical modes gives all half-BPS correlators and in (3.1.40) it is given in terms of AdS and S contact diagrams for a quartic interaction with no derivatives. We will discuss derivative interactions in section 3.3 but first let us discuss how to express the higher-dimensional Witten diagrams in Mellin space.

3.1.5 AdS×S Contact Diagrams in Mellin Space

Inserting the expression for the AdS contact term, $D^{(d)}$, as a Mellin integral (3.1.25) and the sphere analogue $B^{(d)}$ as an expansion (3.1.29) into the expression for the AdS×S contact term (3.1.40), we get, after some simplifications, a Mellin representation for the AdS×S contact term:

$$\begin{aligned} &D^{\mathrm{AdS}_{d+1}\times\mathrm{S}^{d+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,Y_i) = \frac{\pi^{d+1}}{(-2)^{\Sigma_\Delta}\prod_i\Gamma(\Delta_i)} \\ &\times\sum_{p_i=0}^{\infty}(-1)^{\Sigma_p}\int\frac{d\delta_{ij}}{(2\pi i)^2}\sum_{\{d_{ij}\}}\left(\prod_{i<j}\frac{(Y_i.Y_j)^{d_{ij}}}{(X_i.X_j)^{\delta_{ij}}}\frac{\Gamma(\delta_{ij})}{\Gamma(d_{ij}+1)}\right)\times(\Sigma_p+d/2+1)_{\Sigma_\Delta-d-1}\ , \\ &\text{where}\qquad \sum_{i<j}\delta_{ij}=p_j+\Delta_j\ ,\qquad \sum_{i<j}d_{ij}=p_j\ , \end{aligned} \tag{3.1.41}$$

and $x_n=\Gamma(x+n)/\Gamma(x)$ is the Pochhammer-symbol.

We thus define the AdS$_{d+1}$×S^{d+1} Mellin amplitude, $\mathcal{M}_{\Delta_i}[f](\delta_{ij},d_{ij})$, for any such four-point expression, $f(X_i,Y_i)$, via a similar expression

$$\begin{aligned} f(X_i,Y_i) =& \frac{1}{4!}\frac{\pi^{d+1}}{(-2)^{\Sigma_\Delta}}\left(\prod_i \frac{\mathcal{C}_{\Delta_i}}{\Gamma(\Delta_i)}\right) \\ &\times \sum_{p_i=0}^{\infty}(-1)^{\Sigma_p}\int \frac{d\delta_{ij}}{(2\pi i)^2}\sum_{\{d_{ij}\}}\left(\prod_{i<j}\frac{(Y_i.Y_j)^{d_{ij}}}{(X_i.X_j)^{\delta_{ij}}}\frac{\Gamma(\delta_{ij})}{\Gamma(d_{ij}+1)}\right)\times \mathcal{M}_{\Delta_i}[f]\ , \\ &\text{where} \qquad \sum_{i<j}\delta_{ij} = p_j+\Delta_j\ , \qquad \sum_{i<j} d_{ij} = p_j\ . \end{aligned} \tag{3.1.42}$$

Hence, the Mellin amplitude of an AdS×S contact diagram with no derivatives is not in general a constant as for the AdS case, but rather a Pochhammer:

$$\frac{1}{4!}\left(\prod_i \mathcal{C}_{\Delta_i}\right)\times D^{\mathrm{AdS}_{d+1}\times \mathrm{S}^{d+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i,Y_i) \qquad \leftrightarrow \qquad \mathcal{M}_{\Delta_i}(\delta_{ij},d_{ij}) = (\Sigma_p{+}d/2{+}1)_{\Sigma_\Delta-d-1}\ . \tag{3.1.43}$$

We now have all the tools at our disposal to discuss the first correction in the α' expansion of the correlators.

3.2 α'^3 Corrections

Having outlined the general procedure for computing stringy corrections to tree-level four-point half-BPS correlators in $\mathcal{N}=4$ SYM using an effective action in AdS$_5$×S^5, we will now illustrate how this works for the first correction to the supergravity prediction which occurs at order α'^3.

In particular, the first term of the effective action (3.1.18) is just a ϕ^4 interaction:

$$S_{\alpha'^3} = \frac{1}{8\times 4!}\left(\frac{\alpha'}{2}\right)^3 \times 2\zeta_3 \times \int_{\mathrm{AdS}\times\mathrm{S}} d^5\hat{X}d^5\hat{Y}\phi(\hat{X},\hat{Y})^4\ . \tag{3.2.1}$$

To obtain the corresponding CFT correlators we mimic the standard AdS/CFT procedure for obtaining correlators from AdS, but in a fully 10d covariant way, including the sphere manifestly. Using the generalised bulk-to-boundary propagators in (3.1.33) we obtain the AdS×S Witten diagram for this contact interaction, yielding the following proposal for the α'^3 corrections to the correlators:

$$
\begin{aligned}
&\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^3} \\
&= \frac{1}{8\times 4!}\left(\frac{\alpha'}{2}\right)^3 \times 2\zeta_3 \times \frac{(\mathcal{C}_4)^4}{(-2)^{16}} \int_{\mathrm{AdS}\times\mathrm{S}} \frac{d^5\hat{X} d^5\hat{Y}}{(P_1+Q_1)^4(P_2+Q_2)^4(P_3+Q_3)^4(P_4+Q_4)^4} \\
&= \frac{1}{8\times 4!}\left(\frac{\alpha'}{2}\right)^3 (\mathcal{C}_4)^4 \times 2\zeta_3 \times D_{4444}^{\mathrm{AdS}_5\times\mathrm{S}^5} \ .
\end{aligned}
\tag{3.2.2}
$$

We can now extract any specific half-BPS correlator from (3.2.2) by expanding to the appropriate power in Y_i (see (3.1.13)). First note that the 10d bulk-to-boundary propagator Taylor expands as

$$
(P_i+Q_i)^{-4} = \sum_{p=2}^{\infty}(-1)^p \frac{(p-1)_3}{3!}(P_i)^{-p-2}(-Q_i)^{p-2} \ . \tag{3.2.3}
$$

So the individual correlators are given by[6]:

$$
\begin{aligned}
&\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle|_{\alpha'^3} \\
&= \frac{1}{8\times 4!}\left(\frac{\alpha'}{2}\right)^3 \times 2\zeta_3 \times \frac{(\mathcal{C}_4)^4}{(-2)^{16}} \prod_i \frac{(p_i-1)_3}{3!} \int_{\mathrm{AdS}} d^5\hat{X} \prod_i \frac{1}{(P_i)^{p_i+2}} \times \int_{\mathrm{S}} d^5\hat{Y} \prod_i (Q_i)^{p_i-2} \\
&= \frac{1}{8\times 4!}\left(\frac{\alpha'}{2}\right)^3 (\mathcal{C}_4)^4 \times 2\zeta_3 \times \left(\prod_i \frac{(p_i-1)_3}{3!}\right) \\
&\quad \times D^{(4)}_{p_1+2\,p_2+2\,p_3+2\,p_4+2}(X_i) \times B^{(4)}_{p_1-2\,p_2-2\,p_3-2\,p_4-2}(Y_i) \ .
\end{aligned}
\tag{3.2.4}
$$

To see what it looks like in Mellin space we plug the Mellin transform of D (3.1.25) and the expansion of B (3.1.29) into this expression (or just use (3.1.41)) giving the Mellin amplitude (defined in (3.1.42))

$$
\mathcal{M}_{\alpha'^3} = \tfrac{1}{8}\left(\tfrac{\alpha'}{2}\right)^3 \times 2\zeta_3 \times (\Sigma_p - 1)_3 \ . \tag{3.2.5}
$$

This correctly reproduces the results of [10, 11, 13] for the Mellin amplitude of half-BPS correlators at this order.

[6] This is (3.1.40) with $\Delta_i = d = 4$ and with $p_i \to p_i - 2$ to account for the fact that the lowest correlator is labelled with $p_i = 2$ rather than $p_i = 0$. We do not need to worry about the minus signs in the factors $(-1)^p$ in (3.2.3) since $B_{p_1p_2p_3p_4} = 0$ if $p_1+p_2+p_3+p_4$ is odd .

3.3 Algorithm for Computing General α' Corrections

At higher orders in α' the effective action (3.1.18) has terms with covariant derivatives acting on the scalar field. Thus before proceeding, we describe an efficient way to evaluate generalised contact diagrams with derivatives in AdS×S in position space. Then we present a general formula for converting them to Mellin space.

3.3.1 Generalised Witten Diagrams

Computing the action of the covariant derivatives quickly becomes complicated and so it is useful to develop an algorithm to do this automatically. We will motivate the algorithm by building up from simple cases. First, we consider the application of multiple covariant derivatives at a single point in AdS. From (3.1.23) this is given recursively as

$$\nabla_A \nabla_B \ldots \nabla_C \phi = \mathcal{P}_A^{A'} \mathcal{P}_B^{B'} \ldots \mathcal{P}_C^{C'} \partial_{A'} \left(\nabla_{B'} \ldots \nabla_{C'} \phi \right) \ , \qquad \nabla_A \phi = \mathcal{P}_A^{A'} \partial_{A'} \phi \ . \tag{3.3.1}$$

So the application of two covariant derivatives gives

$$\nabla_B \nabla_A \phi = \mathcal{P}_B^{B'} \mathcal{P}_A^{A'} \partial_{B'} (\mathcal{P}_{A'}^{A''} \partial_{A''} \phi) = \mathcal{P}_B^{B'} \mathcal{P}_A^{A'} \partial_{B'} \partial_{A'} \phi + \mathcal{P}_{BA} \hat{X} . \partial \phi \ . \tag{3.3.2}$$

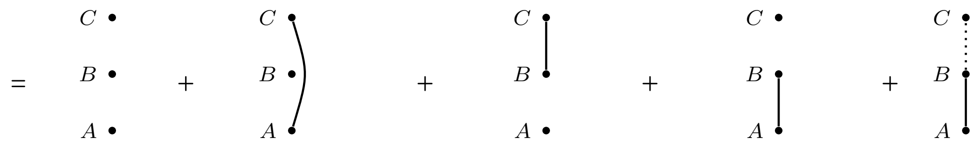
The first term arises from the partial derivative $\partial_{B'}$ being commuted through $\mathcal{P}_{A'}^{A''}$ whereas the second term arises from the partial derivative hitting $\mathcal{P}_{A'}^{A''}$. To arrive at this form, one then uses the definition of $\mathcal{P}$ given in (2.3.4) as well as the useful formulae (2.3.5). We denote this result graphically as

$$\nabla_B \nabla_A =$$  , (3.3.3)

where each vertex corresponds to an index ordered vertically such that the bottom one is the index of the first derivative to act. An isolated vertex at position A denotes $(\mathcal{P}.\partial)_A$ (with the understanding that the derivative has been commuted all the way to the right) whereas an edge between vertices A and B denotes $\mathcal{P}_{AB}\hat{X}.\partial$.

Next, consider three covariant derivatives. Here we obtain

$$
\begin{aligned}
&\nabla_C \nabla_B \nabla_A \phi \\
&= \mathcal{P}_C^{C'} \mathcal{P}_B^{B'} \mathcal{P}_A^{A'} \partial_{C'} \left(\mathcal{P}_{B'}^{B''} \mathcal{P}_{A'}^{A''} \partial_{B''} \partial_{A''} + \mathcal{P}_{B'A'} \hat{X}.\partial \right) \phi \\
&= \Big(\mathcal{P}_C^{C'} \mathcal{P}_B^{B'} \mathcal{P}_A^{A'} \partial_{C'} \partial_{B'} \partial_{A'} + \mathcal{P}_{AC} \mathcal{P}_B^{B'} \hat{X}.\partial\partial_{B'} + \mathcal{P}_{CB} \mathcal{P}_A^{A'} \hat{X}.\partial\partial_{A'} + \mathcal{P}_{AB} \mathcal{P}_C^{C'} \hat{X}.\partial\partial_{C'} \\
&\quad + \mathcal{P}_{AB} \mathcal{P}_C^{C'} \partial_{C'} \Big) \phi
\end{aligned}
$$



, (3.3.4)

and we give the corresponding diagrammatic form in the same order as the terms above. All terms apart from the last arise either from the derivative, ∂_C, hitting a $\mathcal{P}$ (which we denote with a solid line) or commuting through (leaving an isolated vertex at C). The last term arises from the derivative, ∂_C, hitting the $\hat{X}.\partial$ term associated with the solid line between A and B. We denote this by a dotted line from C to B. Thus, a solid line with a dotted line attached to the top of it loses its decoration, $\hat{X}.\partial$. For the general case of several derivatives acting at a point we can work recursively: each additional derivative either commutes through everything, corresponding to an isolated vertex, or it hits a $\mathcal{P}$ corresponding to a solid line, or it hits a $\hat{X}.\partial$, denoted by a dotted line. We add all such lines in all possible ways. Hence, the n-derivative term is given diagrammatically by summing all graphs containing n vertices in a vertical line, with any number of solid edges between any two points, such that no vertex is attached to more than one solid edge, and with any number of dotted edges from the vertex at the top of a solid edge to a higher vertex either isolated or at the bottom of a solid edge.

The above examples are already enough to illustrate the key ingredients of the general algorithm for obtaining an explicit expression for several covariant derivatives at a point, $\nabla_{A_1} \nabla_{A_2} \ldots \nabla_{A_n}$ by summing over all possible graphs.

Algorithm for $\nabla_{A_1}\nabla_{A_2}\dots\nabla_{A_n}\phi$

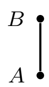
1. Draw n vertices vertically. Each corresponds to an embedding space index ordered so the bottom one corresponds to A_n and the top one to A_1.

2. Draw any number of solid edges between any two vertices such that each vertex is connected to at most one solid edge.

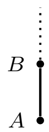
3. Draw any number of dotted edges from the upper vertex of a solid edge up to either a higher disconnected vertex or a higher vertex that is the lower vertex of a solid edge. No vertex can be attached to more than one dotted edge.

4. Sum over all the resulting graphs with the following interpretation:

$$A \bullet = \mathcal{P}_A^{A'}\partial_{A'}$$



$$= \mathcal{P}_{AB}\hat{X}.\partial$$

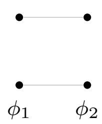


$$= \mathcal{P}_{AB} \tag{3.3.5}$$

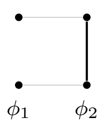
So solid edges come with a decoration $\hat{X}.\partial$ unless they have a dotted line attached to the top in which case the decoration is removed (otherwise the dotted lines can be ignored).

A general derivative interaction term consists of covariant derivatives acting on different scalars with indices contracted together pairwise. We denote this graphically by putting together two or more of the above vertical graphs and adding grey edges corresponding to the contractions. So for example we obtain $\nabla^B\nabla^A\phi_1\,\nabla_B\nabla_A\phi_2$ by taking two copies of all the two-derivative diagrams (3.3.3) and gluing the corresponding vertices together

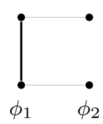
$$\nabla^B\nabla^A\phi_1\,\nabla_B\nabla_A\phi_2 =$$



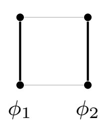
$$+$$



$$+$$



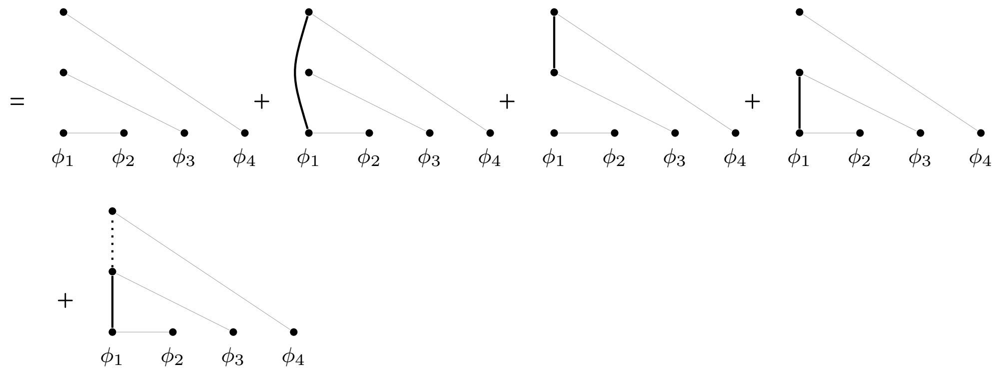
$$+$$



$$\begin{aligned} &= \mathcal{P}^{AA'}\mathcal{P}^{BB'}(\partial_A\partial_B\phi_1)(\partial_{A'}\partial_{B'}\phi_2) + \mathcal{P}^{AB}(\partial_A\partial_B\phi_1)\hat{X}.\partial\phi_2 \\ &\quad + \mathcal{P}^{AB}(\partial_A\partial_B\phi_2)\hat{X}.\partial\phi_1 + \mathcal{P}_A^A(\hat{X}.\partial\phi_1)(\hat{X}.\partial\phi_2)\ . \end{aligned} \tag{3.3.6}$$

Similarly we obtain $\nabla^C\nabla^B\nabla^A\phi_1\,\nabla_A\phi_2\,\nabla_B\phi_3\,\nabla_C\phi_4$ by taking the three-derivative diagram (3.3.4) together with three more vertices to the right and gluing the vertices correspondingly

$$\nabla^C\nabla^B\nabla^A\phi_1\,\nabla_A\phi_2\,\nabla_B\phi_3\,\nabla_C\phi_4$$



$$\begin{aligned}
=&\mathcal{P}^{AA'}\mathcal{P}^{BB'}\mathcal{P}^{CC'}(\partial_A\partial_B\partial_C\phi_1)(\partial_{A'}\phi_2)(\partial_{B'}\phi_3)(\partial_{C'}\phi_4)\\
&+\mathcal{P}^{BB'}\mathcal{P}^{AC}(\hat{X}.\partial\partial_B\phi_1)(\partial_A\phi_2)(\partial_{B'}\phi_3)(\partial_C\phi_4)\\
&+\mathcal{P}^{AA'}\mathcal{P}^{BC}(\hat{X}.\partial\partial_A\phi_1)(\partial_{A'}\phi_2)(\partial_B\phi_3)(\partial_C\phi_4)\\
&+\mathcal{P}^{CC'}\mathcal{P}^{AB}(\hat{X}.\partial\partial_C\phi_1)(\partial_A\phi_2)(\partial_B\phi_3)(\partial_{C'}\phi_4)\\
&+\mathcal{P}^{CC'}\mathcal{P}^{AB}(\partial_C\phi_1)(\partial_A\phi_2)(\partial_B\phi_3)(\partial_{C'}\phi_4)\ .
\end{aligned} \tag{3.3.7}$$

The general algorithm for interaction terms is then a straightforward extension of the one above for covariant derivatives acting on a single scalar.

Algorithm for contact interactions in AdS

1. For each scalar ϕ_i with n_i covariant derivatives acting on it, draw all the corresponding contributing vertical graphs using the above algorithm. Place the graphs for each scalar next to each other horizontally (taking the outer product over the list of graphs at each point).

2. Draw grey lines between corresponding contracted vertices in the interaction term.

3. Finally sum over all the resulting graphs with the following interpretation:

4. Each connected path of solid and grey lines with end points in the vertical line above ϕ_i and ϕ_j corresponds to $\mathcal{P}^{AB}\partial_A\phi_i\partial_B\phi_j$.

5. Each solid line above ϕ_i corresponds to $\hat{X}.\partial\phi_i$, *as long as it does not have a dotted line attached to its upper vertex* (if it does have such a dotted line it has no additional contribution).

See the above two examples (3.3.6) and (3.3.7).

So far we have only discussed AdS covariant derivatives. The above rules can be used with the obvious modifications if instead we are viewing the action on a sphere (i.e. A, B indices become I, J indices, $\hat{X} \to \hat{Y}$ and $\mathcal{P}^{AB} \to \mathcal{P}^{IJ}$ in (2.3.4)). However, our main purpose here is to consider AdS×S covariant derivatives. Thus, each vertex now represents a 10d index $\mu = (A, I)$ but there needs to be some non-trivial re-interpretation in the case of the product geometry.

Algorithm for contact interactions in AdS×S

The first three steps of the algorithm are as for the AdS case above. Then

4. Each connected path of solid and grey lines with end points in the vertical line above ϕ_i and ϕ_j respectively corresponds to $\mathcal{P}^{\mu\nu}\partial_\mu\phi_i\partial_\nu\phi_j$, but:

5. Each solid line above ϕ_i, *as long as it does not have a dotted line attached to its upper vertex*, breaks this manifest 10d structure by contributing a multiplicative factor $\hat{X}^A\partial_A\phi_i$, if the index running through it is in AdS or $-\hat{Y}^I\partial_I\phi_i$ if the index running through is in the sphere. (The minus sign appears in the latter case since this term arises from a derivative hitting $\mathcal{P}$ in (2.3.4) or (3.1.21) which has a minus sign for the internal case.)

6. Finally, there is an additional subtlety related to the dotted lines. The dotted line ties together the index type corresponding to the otherwise potentially disconnected parts of the graph, and then contributes a factor of +1 if the

index running through is in AdS or −1 if the index running through is in the sphere. (Recall that the dotted lines arise from derivatives $\partial_{\hat{X}}$ or $\partial_{\hat{Y}}$ hitting the decoration $\hat{X}^A\partial_A\phi_i$ or $-\hat{Y}^I\partial_I\phi_i$. Thus firstly, this vanishes unless the derivative type (AdS or S) is the same as that of the solid line (hence tying together the index type), and secondly it gives ±1 depending on whether it is AdS or S.)

Thus, for example the AdS×S covariant version of (3.3.7) is, with each of the five lines corresponding to the five graphs in (3.3.7),

$$
\begin{aligned}
&\nabla^\rho\nabla^\nu\nabla^\mu\phi_1\,\nabla_\mu\phi_2\,\nabla_\nu\phi_3\,\nabla_\rho\phi_4\\
&=\mathcal{P}^{\mu\mu'}\mathcal{P}^{\nu\nu'}\mathcal{P}^{\rho\rho'}(\partial_\mu\partial_\nu\partial_\rho\phi_1)(\partial_{\mu'}\phi_2)(\partial_{\nu'}\phi_3)(\partial_{\rho'}\phi_4)\\
&+\mathcal{P}^{\nu\nu'}\mathcal{P}^{AC}(\hat{X}.\partial_{\hat{X}}\partial_\nu\phi_1)(\partial_A\phi_2)(\partial_{\nu'}\phi_3)(\partial_C\phi_4)-\mathcal{P}^{\nu\nu'}\mathcal{P}^{IK}(\hat{Y}.\partial_{\hat{Y}}\partial_\nu\phi_1)(\partial_I\phi_2)(\partial_{\nu'}\phi_3)(\partial_K\phi_4)\\
&+\mathcal{P}^{\mu\mu'}\mathcal{P}^{BC}(\hat{X}.\partial_{\hat{X}}\partial_\mu\phi_1)(\partial_{\mu'}\phi_2)(\partial_B\phi_3)(\partial_C\phi_4)-\mathcal{P}^{\mu\mu'}\mathcal{P}^{JK}(\hat{Y}.\partial_{\hat{Y}}\partial_\mu\phi_1)(\partial_{\mu'}\phi_2)(\partial_J\phi_3)(\partial_K\phi_4)\\
&+\mathcal{P}^{\rho\rho'}\mathcal{P}^{AB}(\hat{X}.\partial_{\hat{X}}\partial_\rho\phi_1)(\partial_A\phi_2)(\partial_B\phi_3)(\partial_{\rho'}\phi_4)-\mathcal{P}^{\rho\rho'}\mathcal{P}^{IJ}(\hat{Y}.\partial_{\hat{Y}}\partial_\rho\phi_1)(\partial_I\phi_2)(\partial_J\phi_3)(\partial_{\rho'}\phi_4)\\
&+\mathcal{P}^{CC'}\mathcal{P}^{AB}(\partial_C\phi_1)(\partial_A\phi_2)(\partial_B\phi_3)(\partial_{C'}\phi_4)-\mathcal{P}^{KK'}\mathcal{P}^{IJ}(\partial_K\phi_1)(\partial_I\phi_2)(\partial_J\phi_3)(\partial_{K'}\phi_4)\ .
\end{aligned}
\tag{3.3.8}
$$

In particular, note that only the first line is manifestly 10d covariant (has only 10d μ,ν indices). Also compare carefully the penultimate with the final line. These arise from similar graphs (the last two in (3.3.7)) but one with a dotted line and one without. In the final line, as well as the decoration $\hat{X}.\partial_{\hat{X}}$ or $\hat{Y}.\partial_{\hat{Y}}$ being absent, the dotted line has tied together the two otherwise disconnected parts of the graph, meaning for example that all indices are either AdS or S, with no mixed ones, unlike the penultimate line.

Finally, note that in practice for our purposes here the derivatives will always be acting on bulk-to-boundary propagators (3.1.33) and thus partial derivatives acting on a single scalar, $\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_{n_i}}\phi_i$, give $(-1)^{n_i}(\Delta_i)_{n_i}X^{\mu_1}\ldots X^{\mu_{n_i}}$ etc.

3.3.2 Mellin Space

The previous subsection gave an algorithm for obtaining explicit expressions for the integrands of generalised Witten diagrams in AdS×S coming from contact interactions with derivatives. This will result in integrands corresponding to decorations of the (no-derivative) contact diagram D (3.1.37). The decorations are in the form of polynomials in $X_i.X_j$, $Y_i.Y_j$, Q_i and P_i which are homogeneous at each point (i.e. scale the same under the local scaling $X_i.X_j \to e_ie_jX_i.X_j$, $Y_i.Y_j \to e_ie_jY_i.Y_j$, $Q_i \to e_iQ_i$ and $P_i \to e_iP_i$). Each term of such a decoration thus has the form

$$\frac{1}{4!}\frac{\prod_i \mathcal{C}_{\Delta_i}}{(-2)^{2\Sigma_\Delta}}\int_{\mathrm{AdS}\times\mathrm{S}} d^{d+1}\hat{X}d^{d+1}\hat{Y}\left(\prod_i \frac{Q_i^{n_i^Q}P_i^{n_i^P}\times(\Delta_i)_{n_i}}{(P_i+Q_i)^{\Delta_i+n_i}}\right)\times\left(\prod_{i<j}(X_i.X_j)^{n_{ij}^X}(Y_i.Y_j)^{n_{ij}^Y}\right) \ , \tag{3.3.9}$$

with $n_i = n_i^P + n_i^Q + \sum_j n_{ij}^X + \sum_j n_{ij}^Y$. We define Σ_X, Σ_Y to represent the sum of all the n_{ij}^X, n_{ij}^Y respectively, Σ_Q, Σ_P represents *half* the sum of all the n_i^Q, n_i^P and Σ_n half the sum of the n_i, so $\Sigma_n = \Sigma_P + \Sigma_Q + \Sigma_X + \Sigma_Y$. Such a decorated integral will modify (3.1.40) to

$$\begin{aligned}(-2)^{2\Sigma_X+2\Sigma_Y}\sum_{p_i=0}^{\infty}&\left(\prod_{i=1}^{4}(-1)^{p_i}\frac{(p_i+1)_{\Delta_i+n_i-1}}{\Gamma(\Delta_i)}D^{(d)}_{p_i+\Delta_i+n_i-n_i^P}(X_i)B^{(d)}_{p_i+n_i^Q}(Y_i)\right)\\&\times\left(\prod_{i<j}(X_i.X_j)^{n_{ij}^X}(Y_i.Y_j)^{n_{ij}^Y}\right) \ .\end{aligned} \tag{3.3.10}$$

Inserting the Mellin transform of D (3.1.25) and expansion of B (3.1.29) and performing some re-definitions and simplifications then gives the Mellin amplitude (defined in (3.1.42)):

$$\begin{aligned}\mathcal{M}_{\Delta_i}[(3.3.9)] =&(-2)^{\Sigma_X}2^{\Sigma_Y}(-1)^{2\Sigma_Q}\left(\prod_{i<j}(\delta_{ij})_{n_{ij}^X}(d_{ij}-n_{ij}^Y+1)_{n_{ij}^Y}\right)\\&\times\left(\prod_i(p_i+n_i^X+\Delta_i)_{n_i^P}(p_i-n_i^Q-n_i^Y+1)_{n_i^Q}\right)(\Sigma_p-\Sigma_Y+\tfrac{d}{2}+1)_{\Sigma_\Delta-d-1+\Sigma_X+\Sigma_Y} \ ,\\&\text{where} \quad \sum_{i<j}\delta_{ij} = p_j+\Delta_j \ , \qquad \sum_{i<j} d_{ij} = p_j \ .\end{aligned} \tag{3.3.11}$$

We will use this general formula, in conjunction with the algorithm of the previous subsection, to compute higher-order terms in the α' expansion of half-BPS correlators in the next sections.

3.4 α'^5 Corrections

After α'^3, the next terms in the effective action for string corrections occur at α'^5. In the flat space limit, such terms contain four derivatives, so first we consider all the possible terms in the effective action in AdS×S involving four derivatives. At first there are many terms one can write down, but using integration by parts as well as the equations of motion (3.1.36) reduces the number down quickly. We find that in fact there are only two linearly independent terms one can write down involving four derivatives:

$$(\nabla\phi.\nabla\phi)^2 \qquad \text{and} \qquad \nabla^2\nabla_\mu\phi\nabla^\mu\phi\phi^2 \; . \tag{3.4.1}$$

These are the two terms appearing in the effective action (3.1.18). Any other four-derivative term can be written in terms of these, using integration by parts and the equations of motion. For example

$$\begin{aligned} \nabla_\mu\nabla_\nu\phi\nabla^\mu\phi\nabla^\nu\phi\phi &\sim -\frac{1}{2}(\nabla\phi.\nabla\phi)^2 \; , \\ (\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi)\,\phi^2 &\sim (\nabla\phi.\nabla\phi)^2 - \nabla^2\nabla_\mu\phi\nabla^\mu\phi\phi^2 \; . \end{aligned} \tag{3.4.2}$$

Although at this level the independent integrands can be obtained by hand, they can also be nicely checked on a computer by using the algorithm of the previous section and converting to Mellin space where the integration by parts identities are made manifest. Simply list all possible four-derivative integrands on the computer, use the algorithm to obtain the corresponding integrand, convert them to Mellin amplitudes, and then solve for the independent ones.

We see here for the first time that the effective action has an ambiguity - a term not determined by the Virasoro-Shapiro amplitude: in the flat space limit the second integrand in (3.4.1) will vanish (as we can commute the Laplacian through so it acts directly on ϕ giving zero by the equations of motion) and so remains undetermined.

The complete effective action at this order is thus (see (3.1.18))

$$S_{\alpha'^5} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^5 \left(\zeta_5 S^{\text{main}}_{\alpha'^5} + C_0 S^{\text{amb}}_{\alpha'^5} + A_2 S^{\text{main}}_{\alpha'^3}\right) , \tag{3.4.3}$$

where

$$\begin{aligned}
S^{\text{main}}_{\alpha'^5} &= \frac{3}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X} d^5\hat{Y} (\nabla\phi.\nabla\phi)(\nabla\phi.\nabla\phi) ,\\
S^{\text{amb}}_{\alpha'^5} &= \frac{6}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X} d^5\hat{Y} \nabla^2\nabla_\mu\phi\nabla^\mu\phi\phi^2 ,\\
S^{\text{main}}_{\alpha'^3} &= \frac{1}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X} d^5\hat{Y} \phi^4 .
\end{aligned} \tag{3.4.4}$$

The ambiguity in the third line of (3.4.4) vanishes in the flat space limit since it comes from a $1/R^2$ expansion of the coefficient of the α'^3 correction in (3.1.18).

Replacing the scalar fields by bulk-to-boundary propagators and applying the covariant derivatives directly on them then gives a prediction for the half-BPS correlators at this order in α'. First consider the main contribution (3.4.4)

$$\begin{aligned}
&\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^5;\text{main}}\\
&= \frac{1}{4!}\frac{(\mathcal{C}_4)^4}{(-2)^{16}}\int_{\text{AdS}\times\text{S}} d^5\hat{X} d^5\hat{Y} \frac{N_{12}N_{34} + N_{13}N_{24} + N_{14}N_{23}}{(P_1+Q_1)^5(P_2+Q_2)^5(P_3+Q_3)^5(P_4+Q_4)^5} \times 4^4 ,
\end{aligned} \tag{3.4.5}$$

where

$$N_{ij} = X_i.X_j + Y_i.Y_j + P_iP_j - Q_iQ_j . \tag{3.4.6}$$

This can then be straightforwardly expanded to give any correlator directly and explicitly in position space in terms of AdS and S contact diagram functions, as is done for a general integral in (3.3.10). The corresponding Mellin amplitude can also be read off directly from (3.3.11)

$$\begin{aligned}
\mathcal{M}^{\text{main}}_{\alpha'^5} = 4\Big[&(\Sigma_p-1)_5 \left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right)\\
&+(\Sigma_p-1)_4\left(-10\left(\tilde{s}\,\mathbf{s}+\tilde{t}\,\mathbf{t}+\tilde{u}\,\mathbf{u}\right)-5\left(c_s\,\mathbf{s}+c_t\,\mathbf{t}+c_u\,\mathbf{u}\right)\right)\\
&+(\Sigma_p-1)_3\left(20\left(\tilde{s}^2+\tilde{t}^2+\tilde{u}^2\right)+4\left(c_s^2+c_t^2+c_u^2\right)+20\left(\tilde{s}\,c_s+\tilde{t}\,c_t+\tilde{u}\,c_u\right)\right)\\
&+(\Sigma_p-1)_3\left(-12\,\Sigma_p^2\right)\Big] .
\end{aligned} \tag{3.4.7}$$

Here we have used (3.3.11) to obtain the Mellin amplitude (with $\Delta_i = 4, d = 4$ and

$p_i \to p_i - 2$) and then solved the constraints

$$\sum_{i<j} \delta_{ij} = p_j + 2 \ , \qquad \sum_{i<j} d_{ij} = p_j - 2 \ , \tag{3.4.8}$$

in terms of new variables (s,t,u) and $(\tilde{s},\tilde{t},\tilde{u})$, which are defined as follows [15]:

$$\begin{aligned}
&\delta_{12} = -s + c_s \ , && \delta_{14} = -t + c_t \ , && \delta_{13} = -u \ , \\
&\delta_{23} = -t \ , && \delta_{24} = -u + c_u \ , && \delta_{34} = -s \ , \\
&d_{12} = \tilde{s} + c_s \ , && d_{14} = \tilde{t} + c_t \ , && d_{13} = \tilde{u} \ , \\
&d_{23} = \tilde{t} \ , && d_{24} = \tilde{u} + c_u \ , && d_{34} = \tilde{s} \ , \\
&\mathbf{s} = s + \tilde{s} \ , && \mathbf{t} = t + \tilde{t} \ , && \mathbf{u} = u + \tilde{u} \ ,
\end{aligned} \tag{3.4.9}$$

where $s + t + u = -p_3 - 2\,,\ \tilde{s} + \tilde{t} + \tilde{u} = p_3 - 2$ and $\mathbf{s} + \mathbf{t} + \mathbf{u} = -4$. We also define

$$c_s = \frac{p_1 + p_2 - p_3 - p_4}{2} \ , \quad c_t = \frac{p_1 + p_4 - p_2 - p_3}{2} \ , \quad c_u = \frac{p_2 + p_4 - p_3 - p_1}{2} \ . \tag{3.4.10}$$

Note that for any CFT with a string theory or M-theory dual, the leading terms of the Mellin amplitude in the limit $s,t \to \infty$ can be compared to the appropriate higher-dimensional string-/M-theory scattering amplitude. In our case, the first line in (3.4.7) is leading in the flat space limit and is fixed from the flat space VS amplitude.

Now let us take a closer look at the ambiguity in the second line of (3.4.4). Using the equations of motion (3.1.36), the integrand can be written as

$$\nabla^2 \nabla_\mu \phi \nabla^\mu \phi \phi^2 = \left[\nabla^2_{\hat{X}}, \nabla_A\right] \phi \nabla^A \phi \phi^2 + \left[\nabla^2_{\hat{Y}}, \nabla_I\right] \phi \nabla^I \phi \phi^2 \ . \tag{3.4.11}$$

Moreover, after some algebra we find that

$$\left[\nabla^2_{\hat{X}}, \nabla_A\right] \phi = -d \nabla_A \phi \ , \quad \left[\nabla^2_{\hat{Y}}, \nabla_I\right] \phi = d \nabla_I \phi \ , \tag{3.4.12}$$

so the ambiguity can be written as

$$\nabla^2 \nabla_\mu \phi \nabla^\mu \phi \phi^2 = -d \left((\nabla_{\hat{X}} \phi)^2 - (\nabla_{\hat{Y}} \phi)^2 \right) \phi^2 \ . \tag{3.4.13}$$

The corresponding Witten diagram expression is given by

$$\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^5;\text{amb}} = -\frac{1}{4!}\frac{(\mathcal{C}_4)^4}{(-2)^{16}}\int_{\text{AdS}\times\text{S}}\frac{d^5\hat{X}d^5\hat{Y}}{\prod_i\left(P_i+Q_i\right)^4}\sum_{i<j}\frac{L_{ij}}{\left(P_i+Q_i\right)\left(P_j+Q_j\right)}\times 4^3\ , \tag{3.4.14}$$

where

$$L_{ij} = X_i.X_j + P_iP_j - Y_i.Y_j + Q_iQ_j\ . \tag{3.4.15}$$

This takes a very simple form in Mellin space

$$\mathcal{M}^{\text{amb}}_{\alpha'^5} = 4\left(\Sigma_p-1\right)_3\left(c_s^2+c_t^2+c_u^2+\Sigma_p^2-16\right)\ . \tag{3.4.16}$$

Moreover, after multiplying the α'^3 term in (3.2.5) by $(\alpha'/(2R^2))^2$ (where we set $R=1$), it can be thought of as an additional ambiguity at α'^5, which is the origin of the third line in (3.4.4) as we have seen before. Restoring the prefactors in (3.4.3), the α'^5 correction to the Mellin amplitude for half-BPS correlators can be written as a sum over three terms:

$$\mathcal{M}_{\alpha'^5} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^5\left(\zeta_5\mathcal{M}^{\text{main}}_{\alpha'^5}+C_0\mathcal{M}^{\text{amb}}_{\alpha'^5}+A_2\mathcal{M}^{\text{main}}_{\alpha'^3}\right)\ , \tag{3.4.17}$$

where $\mathcal{M}^{\text{main}}_{\alpha'^3} = (\Sigma_p-1)_3$ (it is given in (3.2.5) but here we take it without the explicit normalisation there). The coefficients of the subleading terms can be fixed by comparing to the localisation result in [12] and are given by

$$C_0 = -\frac{3}{2}\zeta_5\ ,\quad A_2 = -30\,\zeta_5\ . \tag{3.4.18}$$

We find perfect agreement with the results from bootstrap methods of [14] (rewritten in this notation in [15])[7].

3.5 α'^6 Corrections

At order α'^6 we have to consider all possible terms in the effective action involving six derivatives. Using a computer, it is straightforward to enumerate all possibilities

[7] We thank Francesco Aprile for explicitly checking this agreement.

and compute their Mellin amplitudes using the algorithm explained in section 3.3 to find all linearly independent terms. After doing so, we find that there are only two linearly independent terms involving six derivatives:

$$(\nabla\phi.\nabla\phi)(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi) \quad \text{and} \quad \nabla_\mu\nabla^2\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\phi^2 \ , \tag{3.5.1}$$

which appear in the effective action in (3.1.18). The first term is the main correction at α'^6 while the second term is an ambiguity which vanishes in the flat space limit and is thus not determined by the flat space Virasoro-Shapiro amplitude.

The complete action at order α'^6 is given by (see (3.1.18))

$$S_{\alpha'^6} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^6 \left(2(\zeta_3)^2 S^{\text{main}}_{\alpha'^6} + E_0 S^{\text{amb}}_{\alpha'^6} + B_1 S^{\text{main}}_{\alpha'^5} + C_1 S^{\text{amb}}_{\alpha'^5} + A_3 S^{\text{main}}_{\alpha'^3}\right) \ , \tag{3.5.2}$$

where

$$\begin{aligned} S^{\text{main}}_{\alpha'^6} &= \frac{1}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X} d^5\hat{Y} (\nabla\phi.\nabla\phi)(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi) \ , \\ S^{\text{amb}}_{\alpha'^6} &= \frac{6}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X} d^5\hat{Y} \nabla_\mu\nabla^2\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\phi^2 \ , \end{aligned} \tag{3.5.3}$$

and the rest was defined in (3.4.4) (in particular, they arise from taking all the terms contributing at α'^5 and multiplying them with $\alpha'/(2R^2)$ with unfixed numerical coefficients and setting $R=1$). We then find that the main contribution to half-BPS correlators at this order is

$$\begin{aligned} &\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^6;\text{main}} \\ &= \frac{1}{4!}\frac{1}{6}\frac{(\mathcal{C}_4)^4}{(-2)^{16}}\int_{\text{AdS}\times\text{S}} \frac{d^5\hat{X}d^5\hat{Y}}{\prod_i (P_i+Q_i)^5}\left[\frac{N_{12}M_{34}}{(P_3+Q_3)(P_4+Q_4)} + \text{perms}\right]\times 4^4\times 5^2 \ , \end{aligned} \tag{3.5.4}$$

where the correlator is understood to come from the first line of (3.5.3), N_{ij} was defined in (3.4.6) and

$$M_{ij} = (X_i.X_j + P_iP_j + Y_i.Y_j - Q_iQ_j)^2 - \frac{1}{5}(P_iP_j - Q_iQ_j)^2 \ . \tag{3.5.5}$$

Before discussing the Mellin amplitude of the main contribution, let us briefly describe the ambiguity whose integrand can be written as

$$\nabla_\mu \nabla^2 \nabla_\nu \phi \nabla^\mu \nabla^\nu \phi \phi^2 = -d\left((\nabla_A \nabla_B \phi)^2 - (\nabla_I \nabla_J \phi)^2 \right) \phi^2 \ , \tag{3.5.6}$$

where A and I indices label $\hat{X}$ and $\hat{Y}$ coordinates, respectively. We obtained the right hand side by commuting the ∇^2 with ∇_ν and using the equations of motion as we did in the previous subsection. The Witten diagram expression associated with (3.5.6) is

$$\begin{aligned} &\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle|_{\alpha'^6;\mathrm{amb}} \\ &= \frac{1}{4!} \frac{(\mathcal{C}_4)^4}{(-2)^{16}} \int_{\mathrm{AdS}\times\mathrm{S}} \frac{d^5\hat{X} d^5\hat{Y}}{\prod_i (P_i+Q_i)^4} \sum_{i<j} \frac{K_{ij}}{(P_i+Q_i)^2 (P_j+Q_j)^2} \times 4^3 \times 5^2 \ , \end{aligned} \tag{3.5.7}$$

where

$$K_{ij} = (X_i.X_j + P_i P_j)^2 - (Y_i.Y_j - Q_i Q_j)^2 - \frac{1}{5}\left((P_i P_j)^2 - (Q_i Q_j)^2 \right) \ . \tag{3.5.8}$$

Converting this to Mellin space gives the ambiguity

$$\begin{aligned} \mathcal{M}^{\mathrm{amb}}_{\alpha'^6} = -32\Big[& (\Sigma_p - 1)_5 \left(\mathbf{s}^2 + \mathbf{t}^2 + \mathbf{u}^2 \right) \\ & + (\Sigma_p - 1)_4 \, \frac{1}{2} \left(\mathbf{s}\, c_s^2 + \mathbf{t}\, c_t^2 + \mathbf{u}\, c_u^2 \right) \\ & - (\Sigma_p - 1)_4 \, (\Sigma_p + 3) \left[2\left(\mathbf{s}\,\tilde{s} + \mathbf{t}\,\tilde{t} + \mathbf{u}\,\tilde{u} \right) + \left(\mathbf{s}\, c_s + \mathbf{t}\, c_t + \mathbf{u}\, c_u \right) \right] \\ & - (\Sigma_p - 1)_3 \left(\left(c_s^3 + c_t^3 + c_u^3 \right) + 2\left(c_s^2\,\tilde{s} + c_t^2\,\tilde{t} + c_u^2\,\tilde{u} \right) - 10\,\Sigma_p \left(\tilde{s}^2 + \tilde{t}^2 + \tilde{u}^2 \right) \right) \\ & + (\Sigma_p - 1)_3 \left(10\,\Sigma_p \left(\tilde{s}\, c_s + \tilde{t}\, c_t + \tilde{u}\, c_u \right) + 3\,\Sigma_p \left(c_s^2 + c_t^2 + c_u^2 \right) \right) \\ & + (\Sigma_p - 1)_3 \left(-2\,\Sigma_p^3 - 16\,\Sigma_p \right) \Big] \ . \end{aligned} \tag{3.5.9}$$

Converting (3.5.4) to Mellin space, the Mellin amplitude of the main contribution is

$$\mathcal{M}^{\mathrm{main}}_{\alpha'^6} = \hat{\mathcal{M}}^{\mathrm{main}}_{\alpha'^6} - \frac{1}{12} \mathcal{M}^{\mathrm{amb}}_{\alpha'^6}, \tag{3.5.10}$$

where

$$\begin{aligned} \hat{\mathcal{M}}^{\mathrm{main}}_{\alpha'^6} = \frac{8}{3}\Big[& (\Sigma_p - 1)_6 \left(\mathbf{s}^3 + \mathbf{t}^3 + \mathbf{u}^3 \right) \\ & + (\Sigma_p - 1)_5 \left(6\,\Sigma_p \left(\mathbf{s}^2 + \mathbf{t}^2 + \mathbf{u}^2 \right) - 18\left(\mathbf{s}^2\,\tilde{s} + \mathbf{t}^2\,\tilde{t} + \mathbf{u}^2\,\tilde{u} \right) \right) \end{aligned}$$

$$
\begin{aligned}
&+(\Sigma_p{-}1)_5\left(-9\left(\mathbf{s}^2\,c_s+\mathbf{t}^2\,c_t+\mathbf{u}^2\,c_u\right)\right)\\
&+(\Sigma_p{-}1)_4\left(90\left(\mathbf{s}\,\tilde{s}^2+\mathbf{t}\,\tilde{t}^2+\mathbf{u}\,\tilde{u}^2\right)+\frac{39}{2}\left(\mathbf{s}\,c_s^2+\mathbf{t}\,c_t^2+\mathbf{u}\,c_u^2\right)\right)\\
&+(\Sigma_p{-}1)_4\;90\left(\mathbf{s}\,\tilde{s}\,c_s+\mathbf{t}\,\tilde{t}\,c_t+\mathbf{u}\,\tilde{u}\,c_u\right)\\
&+(\Sigma_p{-}1)_4\left(-60\,\Sigma_p\left(\mathbf{s}\,\tilde{s}+\mathbf{t}\,\tilde{t}+\mathbf{u}\,\tilde{u}\right)-30\,\Sigma_p\left(\mathbf{s}\,c_s+\mathbf{t}\,c_t+\mathbf{u}\,c_u\right)\right)\\
&+(\Sigma_p{-}1)_3\left(-120\left(\tilde{s}^3+\tilde{t}^3+\tilde{u}^3\right)-9\left(c_s^3+c_t^3+c_u^3\right)\right)\\
&+(\Sigma_p{-}1)_3\left(-180\left(\tilde{s}^2\,c_s+\tilde{t}^2\,c_t+\tilde{u}^2\,c_u\right)-78\left(c_s^2\,\tilde{s}+c_t^2\,\tilde{t}+c_u^2\,\tilde{u}\right)\right)\\
&+(\Sigma_p{-}1)_3\left(120\,\Sigma_p\left(\tilde{s}^2+\tilde{t}^2+\tilde{u}^2\right)+27\,\Sigma_p\left(c_s^2+c_t^2+c_u^2\right)\right)\\
&+(\Sigma_p{-}1)_3\left(120\,\Sigma_p\left(\tilde{s}\,c_s+\tilde{t}\,c_t+\tilde{u}\,c_u\right)-50\,\Sigma_p^3-16\,\Sigma_p\right)\Big]\,. \qquad (3.5.11)
\end{aligned}
$$

This Mellin amplitude shows a similar structure as (3.4.7). Every line is multiplied by a Pochhammer depending on the power of $\{\mathbf{s},\mathbf{t},\mathbf{u}\}$ and the rest is at most cubic in the variables $\{\mathbf{s},\mathbf{t},\mathbf{u},\tilde{s},\tilde{t},\tilde{u},c_s,c_t,c_u,\Sigma_p\}$. Additionally, after multiplying the three terms which span the α'^5 correction in (3.4.17) by α', they become additional ambiguities at α'^6, see the expansion (3.1.19). The complete Mellin amplitude for half-BPS correlators at order α'^6 can then be written as a sum over five terms:

$$
\mathcal{M}_{\alpha'^6}=\frac{1}{8}\left(\frac{\alpha'}{2}\right)^6\left(2\,(\zeta_3)^2\mathcal{M}^{\text{main}}_{\alpha'^6}+E_0\mathcal{M}^{\text{amb}}_{\alpha'^6}+B_1\mathcal{M}^{\text{main}}_{\alpha'^5}+C_1\mathcal{M}^{\text{amb}}_{\alpha'^5}+A_3\mathcal{M}^{\text{main}}_{\alpha'^3}\right)\,, \qquad (3.5.12)
$$

where we restore the coefficients from (3.5.2).[8]

We can fix two of the coefficients by comparing the Mellin amplitude to the result from localisation in [97, 98]. To compare (3.5.12) to [97] we take $s\to\frac{s}{2}-2$, $t\to\frac{t}{2}-2$ and specialise to $p_i=2$ (where $\tilde{s}=\tilde{t}=\tilde{u}=0$):

$$
\begin{aligned}
\mathcal{M}^{2222}_{\alpha'^6}=\frac{1}{8}\left(\frac{\alpha'}{2}\right)^6\times 60\Big(&672(\zeta_3)^2\,s\,t\,u+14\left(3\,B_1+4\left((\zeta_3)^2-6\,E_0\right)\right)\left(s^2+t^2+u^2\right)\\
&+A_3-96\,B_1+768\,E_0-3200(\zeta_3)^2\Big)\,, \qquad (3.5.13)
\end{aligned}
$$

where $u=4-s-t$. We can now compare this expression to the result in [97] and

[8] Note that the number of ambiguities is consistent with the number obtained via the bootstrap method. We thank Francesco Aprile, James Drummond, Hynek Paul and Michele Santagata for discussions on this.

partially fix the coefficients to

$$E_0 = \frac{B_1}{8} + \frac{2(\zeta_3)^2}{3}\ , \quad A_3 = 0\ , \tag{3.5.14}$$

which leads to the α'^6 correction to the correlator for $p_i = 2$:

$$\mathcal{M}^{2222}_{\alpha'^6} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^6 \times 2\,(\zeta_3)^2 \times (3)_6 \left[s\,t\,u - \frac{1}{4}\left(s^2+t^2+u^2\right) - 4\right]\ . \tag{3.5.15}$$

It is noteworthy that localisation predicts the coefficient $A_3 = 0$. Localisation also predicts the absence of any α'^4 corrections i.e. $A_1 = 0$ [12]. Indeed, as we discuss in the conclusions, it is natural to expect all odd terms in the expansion of the coefficients in α'/R^2 (see (3.1.19)) to vanish in which case we would have $B_1 = C_1 = 0$ and then the α'^6 correction to the Mellin amplitude in (3.5.12) is completely fixed.

3.6 α'^7 Corrections

Using the algorithm explained in section 3.3 to find all linearly independent terms in the effective action involving eight derivatives, we find that there are six independent terms, notably the main contribution

$$(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi)^2\ , \tag{3.6.1}$$

and five ambiguities:

$$\begin{aligned}
&(\nabla^\mu\nabla^\nu\nabla_\mu\nabla^\rho\nabla^\sigma\nabla_\rho\phi)\,(\nabla_\nu\nabla_\sigma\phi)\,\phi^2\ , \quad \left(\nabla^2\nabla^\mu\nabla^\nu\nabla^\rho\nabla_\nu\phi\right)(\nabla_\mu\nabla_\rho\phi)\,\phi^2\ ,\\
&\left(\nabla^2\nabla^\mu\nabla^\nu\nabla^\rho\nabla_\nu\phi\right)(\nabla_\rho\phi)\,(\nabla_\mu\phi)\,\phi\ , \quad (\nabla^\mu\nabla^\nu\nabla^\rho\nabla_\nu\nabla_\rho\phi)\,(\nabla^\sigma\nabla_\mu\phi)\,(\nabla_\sigma\phi)\,\phi\ ,\\
&(\nabla^\mu\nabla^\nu\nabla^\rho\nabla^\sigma\nabla_\rho\phi)\,(\nabla_\mu\nabla_\sigma\phi)\,(\nabla_\nu\phi)\,\phi\ .
\end{aligned} \tag{3.6.2}$$

See appendix D for details on the ambiguities. The complete effective action at this order is then given by (see (3.1.18))

$$\begin{aligned}
S_{\alpha'^7} = \frac{1}{8}\left(\frac{\alpha'}{2}\right)^7 \Big(&\tfrac{1}{2}\zeta_7 S^{\text{main}}_{\alpha'^7} + G_{1;0}S^{\text{amb}_1}_{\alpha'^7} + G_{2;0}S^{\text{amb}_2}_{\alpha'^7} + G_{3;0}S^{\text{amb}_3}_{\alpha'^7} + G_{4;0}S^{\text{amb}_4}_{\alpha'^7} + G_{5;0}S^{\text{amb}_5}_{\alpha'^7}\\
&+ D_1 S^{\text{main}}_{\alpha'^6} + E_1 S^{\text{amb}}_{\alpha'^6} + B_2 S^{\text{main}}_{\alpha'^5} + C_2 S^{\text{amb}}_{\alpha'^5} + A_4 S^{\text{main}}_{\alpha'^3}\Big)\ ,
\end{aligned} \tag{3.6.3}$$

where the main contribution is

$$S^{\text{main}}_{\alpha'^7} = \frac{6}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X}d^5\hat{Y}\,(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi)^2\ , \tag{3.6.4}$$

and the contributions from the five α'^7 ambiguities in (3.6.2) to the effective action are given in appendix D together with their Witten diagram expressions and Mellin amplitudes. The contributions from lower α' orders were defined in (3.4.4) and (3.5.3). The prediction for the main contribution to the half-BPS correlator in position space at this order is:

$$\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^7;\text{main}} = \frac{2}{4!}\frac{(\mathcal{C}_4)^4}{(-2)^{16}}\int_{\text{AdS}\times\text{S}}\frac{d^5\hat{X}d^5\hat{Y}}{\prod_i(P_i+Q_i)^6}\left[M_{12}M_{34}+\text{perms}\right]\times 4^4\times 5^4\,, \tag{3.6.5}$$

where M_{ij} was defined in (3.5.5). The Mellin amplitude of the main contribution is

$$\begin{aligned}\mathcal{M}^{\text{main}}_{\alpha'^7} = \hat{\mathcal{M}}^{\text{main}}_{\alpha'^7} &+ \frac{63}{8}\mathcal{M}^{\text{amb}_1}_{\alpha'^7} - \frac{31}{4}\mathcal{M}^{\text{amb}_2}_{\alpha'^7} - \frac{25}{32}\mathcal{M}^{\text{amb}_3}_{\alpha'^7} - \mathcal{M}^{\text{amb}_4}_{\alpha'^7} - 32\mathcal{M}^{\text{main}}_{\alpha'^5}\\ &- \frac{85}{2}\mathcal{M}^{\text{amb}}_{\alpha'^5} - 1024\mathcal{M}^{\text{main}}_{\alpha'^3}\ ,\end{aligned} \tag{3.6.6}$$

where $\hat{\mathcal{M}}^{\text{main}}_{\alpha'^7}$ is:

$$\begin{aligned}\hat{\mathcal{M}}^{\text{main}}_{\alpha'^7} = \ 32\Big[&(\Sigma_p-1)_7\left(\mathbf{s}^4+\mathbf{t}^4+\mathbf{u}^4\right)\\
&+(\Sigma_p-1)_6\left(8\,\Sigma_p\left(\mathbf{s}^3+\mathbf{t}^3+\mathbf{u}^3\right)-28\left(\mathbf{s}^3\,\tilde{s}+\mathbf{t}^3\,\tilde{t}+\mathbf{u}^3\,\tilde{u}\right)-14\left(\mathbf{s}^3\,c_s+\mathbf{t}^3\,c_t+\mathbf{u}^3\,c_u\right)\right)\\
&+(\Sigma_p-1)_5\Big(\Sigma_p\,(26\,\Sigma_p+9)\left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right)+252\left(\mathbf{s}^2\,\tilde{s}^2+\mathbf{t}^2\,\tilde{t}^2+\mathbf{u}^2\,\tilde{u}^2\right)\\
&\qquad+252\left(\mathbf{s}^2\,\tilde{s}\,c_s+\mathbf{t}^2\,\tilde{t}\,c_t+\mathbf{u}^2\,\tilde{u}\,c_u\right)+57\left(\mathbf{s}^2\,c_s^2+\mathbf{t}^2\,c_t^2+\mathbf{u}^2\,c_u^2\right)\\
&\qquad-144\,\Sigma_p\left(\mathbf{s}^2\,\tilde{s}+\mathbf{t}^2\,\tilde{t}+\mathbf{u}^2\,\tilde{u}\right)-72\,\Sigma_p\left(\mathbf{s}^2\,c_s+\mathbf{t}^2\,c_t+\mathbf{u}^2\,c_u\right)\Big)\\
&+(\Sigma_p-1)_4\Big(-840\left(\mathbf{s}\,\tilde{s}^3+\mathbf{t}\,\tilde{t}^3+\mathbf{u}\,\tilde{u}^3\right)-75\left(\mathbf{s}\,c_s^3+\mathbf{t}\,c_t^3+\mathbf{u}\,c_u^3\right)\\
&\qquad-1260\left(\mathbf{s}\,\tilde{s}^2\,c_s+\mathbf{t}\,\tilde{t}^2\,c_t+\mathbf{u}\,\tilde{u}^2\,c_u\right)-570\left(\mathbf{s}\,c_s^2\,\tilde{s}+\mathbf{t}\,c_t^2\,\tilde{t}+\mathbf{u}\,c_u^2\,\tilde{u}\right)\\
&\qquad+720\,\Sigma_p\left[\left(\mathbf{s}\,\tilde{s}^2+\mathbf{t}\,\tilde{t}^2+\mathbf{u}\,\tilde{u}^2\right)+\left(\mathbf{s}\,\tilde{s}\,c_s+\mathbf{t}\,\tilde{t}\,c_t+\mathbf{u}\,\tilde{u}\,c_u\right)\right]\\
&\qquad+\frac{1}{2}(336\,\Sigma_p-1)\left(\mathbf{s}\,c_s^2+\mathbf{t}\,c_t^2+\mathbf{u}\,c_u^2\right)\\
&\qquad-\Sigma_p\,(139\,\Sigma_p+27)\left[2\left(\mathbf{s}\,\tilde{s}+\mathbf{t}\,\tilde{t}+\mathbf{u}\,\tilde{u}\right)+\left(\mathbf{s}\,c_s+\mathbf{t}\,c_t+\mathbf{u}\,c_u\right)\right]\Big)\\
&+(\Sigma_p-1)_3\Big(840\left(\tilde{s}^4+\tilde{t}^4+\tilde{u}^4\right)+\frac{191}{8}\left(c_s^4+c_t^4+c_u^4\right)+1680\left(\tilde{s}^3\,c_s+\tilde{t}^3\,c_t+\tilde{u}^3\,c_u\right)\end{aligned}$$

$$
\begin{aligned}
&+1140\left(\tilde{s}^2\, c_s^2+\tilde{t}^2\, c_t^2+\tilde{u}^2\, c_u^2\right)+300\left(c_s^3\, \tilde{s}+c_t^3\, \tilde{t}+c_u^3\, \tilde{u}\right) \\
&-960\, \Sigma_p\left(\tilde{s}^3+\tilde{t}^3+\tilde{u}^3\right)-\frac{191}{2}\, \Sigma_p\left(c_s^3+c_t^3+c_u^3\right) \\
&-1440\, \Sigma_p\left(\tilde{s}^2\, c_s+\tilde{t}^2\, c_t+\tilde{u}^2\, c_u\right)-671\, \Sigma_p\left(c_s^2\, \tilde{s}+c_t^2\, \tilde{t}+c_u^2\, \tilde{u}\right) \\
&+610\, \Sigma_p^2\left[\left(\tilde{s}^2+\tilde{t}^2+\tilde{u}^2\right)+\left(\tilde{s}\, c_s+\tilde{t}\, c_t+\tilde{u}\, c_u\right)\right] \\
&+\frac{573}{4}\, \Sigma_p^2\left(c_s^2+c_t^2+c_u^2\right)-\frac{1471}{8}\, \Sigma_p^4-116\, \Sigma_p^2\Big)\Big]\,, \qquad (3.6.7)
\end{aligned}
$$

and the Mellin amplitudes of the ambiguities are given in appendix D. Note that this exhibits a similar structure to (3.4.7) and (3.5.11), since every line is multiplied by a Pochhammer depending on the power of $\{\mathbf{s},\mathbf{t},\mathbf{u}\}$ and the rest is at most quartic in the variables $\{\mathbf{s},\mathbf{t},\mathbf{u},\tilde{s},\tilde{t},\tilde{u},c_s,c_t,c_u,\Sigma_p\}$. Collecting all possible contributions at this order, the complete Mellin amplitude for the half-BPS correlator at α'^7 is given by eleven terms:

$$
\begin{aligned}
\mathcal{M}_{\alpha'^7}=\frac{1}{8}\left(\frac{\alpha'}{2}\right)^7\Big(&\tfrac{1}{2}\,\zeta_7\mathcal{M}^{\text{main}}_{\alpha'^7} \\
&+G_{1;0}\mathcal{M}^{\text{amb}_1}_{\alpha'^7}+G_{2;0}\mathcal{M}^{\text{amb}_2}_{\alpha'^7}+G_{3;0}\mathcal{M}^{\text{amb}_3}_{\alpha'^7}+G_{4;0}\mathcal{M}^{\text{amb}_4}_{\alpha'^7}+G_{5;0}\mathcal{M}^{\text{amb}_5}_{\alpha'^7} \\
&+D_1\mathcal{M}^{\text{main}}_{\alpha'^6}+E_1\mathcal{M}^{\text{amb}}_{\alpha'^6}+B_2\mathcal{M}^{\text{main}}_{\alpha'^5}+C_2\mathcal{M}^{\text{amb}}_{\alpha'^5}+A_4\mathcal{M}^{\text{main}}_{\alpha'^3}\Big)\ . \qquad (3.6.8)
\end{aligned}
$$

The coefficients of the subleading terms remain unfixed at this order, to fix them we would need additional information. As an example, let us look at the lowest-charge correlator with $p_i=2$ (as in the previous section we shift $s\to\frac{s}{2}-2$, $t\to\frac{t}{2}-2$):

$$
\mathcal{M}^{2222}_{\alpha'^7}=\frac{1}{8}\left(\frac{\alpha'}{2}\right)^7\times 60\Big(a_1\,\left(s^2+t^2+u^2\right)^2+a_2\, s\, t\, u\,+a_3\,\left(s^2+t^2+u^2\right)+a_4\Big)\,, \qquad (3.6.9)
$$

with $u=4-s-t$ and

$$
\begin{aligned}
&a_1=1512\,\zeta_7,\quad a_2=336\,\left(D_1+48\left(G_{5;0}-2\,\zeta_7\right)\right)\,, \\
&a_3=42\,B_2+28\left(D_1-6\left(2\,E_1-18\,G_{1;0}-20\,G_{2;0}+40\,G_{3;0}-12\,G_{5;0}+23\,\zeta_7\right)\right)\,, \\
&a_4=A_4-32\left(3\,B_2+50\,D_1-12\left(2\,E_1-18\,G_{1;0}-20\,G_{2;0}+40\,G_{3;0}-204\,G_{5;0}+335\,\zeta_7\right)\right)\,. \\
&\qquad (3.6.10)
\end{aligned}
$$

3.7 Conclusions and Future Directions

To conclude, in this chapter, we have postulated a simple effective field theory in ten dimensions describing all four-point tree-level string interactions in $\mathrm{AdS}_5\times\mathrm{S}^5$. To obtain the interaction terms in the $\mathrm{AdS}_5\times\mathrm{S}^5$ effective action we start from the flat space VS amplitude written as an infinite series in small α', where the leading term is supergravity and all subleading terms describe string corrections and can be obtained from a scalar effective action in flat space. Lifting the interaction terms in this flat space action to $\mathrm{AdS}_5\times\mathrm{S}^5$ by replacing the flat space derivatives with 10d covariant derivatives containing both AdS and S derivatives we obtain the curved space analogue of the scalar effective field theory. The main new tool to obtain correlators from this effective action was to use a new formulation of Witten diagrams and the Mellin transform which is manifestly 10d and treats AdS and S on equal footing. We have shown that this simple description reproduces previous results for all four-point correlators of half-BPS operators in $\mathcal{N}=4$ SYM up to order α'^5, and have proposed a general algorithm for extending this to arbitrarily high order. From this algorithm we obtained new predictions at α'^6 and α'^7. The coefficients of the effective action can be determined by comparing to the flat space VS amplitude, although there are curvature-dependent ambiguities which cannot be fixed in this way and need additional input from other methods such as localisation. After fixing all the coefficients in the effective action, the 10d Mellin amplitudes derived from it can be thought of as the analogue of the VS amplitude in $\mathrm{AdS}_5\times\mathrm{S}^5$.

Note that in the considerations above we have focused on the limit of tree-level string theory for which all orders in the α' effective action are known in flat space. However, the coefficients of the first three terms in the flat space effective action (3.1.17) (i.e. up to $\partial^6\phi^4$) are actually known at the full non-perturbative level as functions of the string coupling [111–115][9]. These results imply that the coefficients in (3.1.19) can be promoted to full functions of the complex Yang-Mills coupling $\tau=\theta/(2\pi)+4\pi i N\alpha'^2$

[9] We thank Congkao Wen for drawing our attention to this.

(see (3.1.2), where θ is the Yang-Mills theta-angle and we recall that $N\alpha'^2 = g_{YM}^{-2}$ if we set the AdS radius $R = 1$). Specifically they are promoted as

$$
\begin{aligned}
\left(\tfrac{\alpha'}{2}\right)^3 A_0 = \left(\tfrac{\alpha'}{2}\right)^3 2\zeta_3 \quad &\to \quad \frac{1}{(2^4\pi N)^{3/2}} \times E(\tfrac{3}{2},\tau,\bar\tau) \;, \\
\left(\tfrac{\alpha'}{2}\right)^5 B_0 = \left(\tfrac{\alpha'}{2}\right)^5 \zeta_5 \quad &\to \quad \frac{1}{(2^4\pi N)^{5/2}} \times \tfrac{1}{2} E(\tfrac{5}{2},\tau,\bar\tau) \;, \\
\left(\tfrac{\alpha'}{2}\right)^6 D_0 = \left(\tfrac{\alpha'}{2}\right)^6 2(\zeta_3)^2 \quad &\to \quad \frac{1}{(2^4\pi N)^{3}} \times 3\,\mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2},\tau,\bar\tau) \;,
\end{aligned}
\tag{3.7.1}
$$

where A_0, B_0, D_0 are the leading coefficients in the first, second, and fourth lines of the left column in (3.1.19),

$$
E(s,\tau,\bar\tau) = 2\zeta_{2s}(\Im(\tau))^s(1+\dots) \tag{3.7.2}
$$

are non-holomorphic Eisenstein series and

$$
\mathcal{E}(3,\tfrac{3}{2},\tfrac{3}{2},\tau,\bar\tau) = \tfrac{2}{3}(\zeta_3)^2(\Im(\tau))^3(1+\dots) \tag{3.7.3}
$$

is a generalised Eisenstein series. In the above two equations the ellipses denote perturbative and non-perturbative terms which vanish when $\Im(\tau)\to\infty$. The precise definitions of the functions can be found for example in [98]. The modular functions in (3.7.1) are a consequence of the $SL(2,\mathbb{Z})$ symmetry of IIB string theory (see subsection 3.1.1), which can be understood from compactifying M-theory on a torus and identifying the IIB coupling τ with the complex structure of the torus [116, 117].

Furthermore, recently in [98, 110, 118] the corresponding dual (but lowest-charge only) correlators were considered and completely fixed via localisation to all orders. This then fixes the remaining ambiguities at this order (assuming $B_1 = C_1 = 0$ as we discuss around (3.5.13) and the second bullet point below) in terms of the above functions as

$$
C_0 = -\frac{3}{2}B_0 \;, \qquad A_2 = -30B_0 \;, \qquad E_0 = \frac{D_0}{3} \;. \tag{3.7.4}
$$

These relations follow from the earlier results in (3.4.18) and (3.5.14), respectively.

In summary, the 10d effective action in (3.1.18) appears to be a very useful way to describe IIB string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ and a powerful tool for computing four-point correlators in $\mathcal{N}=4$ SYM. We will propose a similar effective action for $\mathrm{AdS}_2\times\mathrm{S}^2$ in chapter 5.

Future directions

Let us now discuss some interesting questions for future research.

- Firstly, we have not proven the existence of the 10d scalar effective field theory but rather justified it by showing that it reproduces known results for string corrections to IIB supergravity in $\mathrm{AdS}_5\times\mathrm{S}^5$ which were previously obtained using bootstrap methods in [11–15]. Hence, it would be very interesting to prove the existence of this local effective field theory and to do so one would consider the superspace formalism. As on flat background [93], IIB supergravity linearised on the AdS×S superspace background is again described by a chiral scalar superfield with a certain fourth-order constraint [119, 120]. It presumably then makes sense to integrate a superpotential consisting of a holomorphic function of this scalar in chiral AdS×S superspace. This then leads to the question of the existence of an effective chiral superpotential describing the full nonlinear theory. Such an object has been discussed before [121–123] and it would be interesting to explore this point further.

- As we have seen above, the effective action has ambiguities corresponding to curvature corrections which vanish in the flat space limit. For low orders in α', we find that these ambiguities can be fixed by comparing to results from localisation. It would be interesting to understand whether one could find a systematic way to fix all the ambiguities. If this were possible, the next question would be whether we can resum the α' expansion to obtain a compact form analogous to the flat space VS amplitude. If so, how does the analytic structure become modified in curved background? Note here that, as observed below (3.5.13), the explicit results for these ambiguities obtained via localisation are completely consistent with all odd powers in the expansion of α'/R^2 vanishing. Since the curvature has opposite sign for AdS ($\sim -1/R^2$) and S ($\sim 1/R^2$), it is perhaps quite natural to expect that only even powers of the curvature should contribute.

- Another important direction would be to extend this approach to other backgrounds. As explained above, in $\mathcal{N}=4$ SYM the supersymmetry factors out of certain correlators in a very simple way making it possible to derive them from a 10d scalar theory in the bulk. We expect this factorisation to hold when the bulk geometry is AdS$_p$×S^q with $p=q$, but not when $p\neq q$. And we will propose the existence of such an action for AdS$_2$×S^2 in chapter 5 and deduce the consequences. For AdS$_q$×S^q with $q=3,5$, it was recently shown that supergravity correlators enjoy conformal symmetry which can be used to lift the lowest-charge half-BPS four-point correlator to all higher-charge correlators [41, 124, 125]. It would be interesting to investigate the relation of this higher-dimensional conformal symmetry with the explicit higher-dimensional integrals (AdS×S Witten diagrams) we write down here. We will investigate this further in chapter 5 where we study holographic correlators in AdS$_2$×S^2 in the context of a 4d hidden conformal symmetry as well as derive higher-derivative corrections from a 4d scalar effective action analogous to the one in this chapter.

- It would also be interesting to extend this approach to higher-point correlators. The four-point AdS×S contact diagrams have a direct generalisation to n points. Note that an important feature of half-BPS four-point correlators in $\mathcal{N}=4$ SYM that allowed us to write down a simple effective action was the ability to factor out a polynomial which encodes all the supersymmetry. This is analogous to factoring out a supersymmetric delta-function $\delta^{16}(Q)$ from a maximally supersymmetric four-point superamplitude in flat space. Therefore, it is not obvious how to generalise this approach to generic n-point functions.

However, there are specific cases where similar properties hold as for example for the n-point maximally nilpotent correlators – those with fermionic degree $n-4$ [126–128] which have recently been studied at strong coupling in [110]. Thus, one might expect them to be computable from a 10d scalar effective action just as for the four-point ones.

- It would also be conceptually very satisfying to derive the effective action directly from CFT without assuming local spacetime description in the bulk. A systematic approach to such a derivation was achieved in the context of a toy model consisting of a scalar field in AdS in [30] using crossing and conformal symmetry of boundary CFT correlators. This calculation was adapted to stress tensor correlators in $\mathcal{N}=4$ SYM in [61]. The fact that IIB string theory in $AdS_5 \times S^5$ can be reduced to a simple 10d effective field theory therefore suggests that this program might be realised for a full-blown theory of quantum gravity.

- In recent years, loop corrections to the amplitudes have been obtained via bootstrap methods on the CFT side of the duality [11, 85, 87–92]. It would be very interesting to perform loop computations directly on the gravitational side. This is usually technically very difficult and one could hope that the simplicity at tree-level uncovered from the approach in this chapter could help to give new insight also at loop-level.

Chapter 4

AdS$_7$×S^4: Recursion Relations for Anomalous Dimensions in the 6d $(2,0)$ Theory

This chapter is based on [43] and we mainly follow the structure of this paper. We study anomalous dimensions of double-trace operators in the conformal block expansion of stress tensor correlators in the 6d $(2,0)$ theory. As mentioned in the introduction, the 6d $(2,0)$ theory is dual to M-theory in AdS$_7$×S^4 and studying the holographic correlators is a promising way of understanding M-theory better. In this chapter we are interested in the stable 5d objects whose worldvolume theory is the 6d $(2,0)$ theory, the M5-branes. In the low-energy limit, $l_P \to 0$, M-theory can be approximated by 11d supergravity which on the CFT side of the duality corresponds to studying correlation functions in the limit $N \to \infty$, since in AdS$_7$×S^4 we have the relation $l_P \sim N^{-1/3}$ and N is the number of M5-branes. Recent progress has been made in computing all supergravity tree-level correlators in AdS$_7$×S^4 in Mellin space using constraints from the symmetry of the problem and the analytic properties of the amplitudes [129, 130]. When going away from the strict large-N limit and studying subleading terms we can learn something about higher-derivative corrections to the 11d supergravity effective action. Since the CFT in question is non-Lagrangian and can therefore not be studied perturbatively, using conformal bootstrap methods is a very promising approach (see chapter 2 for a review on conformal bootstrap). We will use these methods to compute anomalous dimensions of the operators in the conformal block expansion which will in turn tell us something about the higher-derivative corrections.

Our main strategy, adapted from the seminal work of [30], is to expand the crossing equations in the inverse central charge c, where it has been shown using holographic methods, that $c \sim N^3$ [131]. Next, one takes a certain limit of the conformal cross-ratios, the so-called light-cone limit, to isolate the terms in the conformal block expansion corresponding to anomalous dimensions of double-trace operators. We then truncate the conformal block expansion in spin and use an orthogonality relation of the hypergeometric functions in the superconformal blocks to derive a recursion relation for the anomalous dimensions. For truncated spin L, we find that the solution to the recursion relation depends on $(L+2)(L+4)/8$ free parameters, in agreement with holographic arguments of [30] (see section 2.3.2) and with the explicit four-point functions found in [26]. In particular, they can be thought of as the coefficients of higher-derivative corrections to supergravity in AdS$_7$×S^4 arising from M-theory [26] (see [10, 61] for similar results in $\mathcal{N}=4$ SYM).

A strategy for fixing these coefficients using a chiral algebra conjecture [132] was proposed in [28]. Moreover, the M-theory effective action can also be deduced from correlators of the ABJM theory [3], which is dual to M-theory in AdS$_4$×S^7 [27, 29]. As a warm-up for our analysis in the $(2,0)$ theory, we first derive recursion relations for anomalous dimensions in an abstract non-supersymmetric 6d CFT, which we match against the conformal block expansion of Witten diagrams for a massive scalar field in AdS$_7$. The recursion relations we obtain for both the toy model and the $(2,0)$ theory can be efficiently solved using a computer.

We start with a brief review of the 6d $(2,0)$ Theory/M-Theory in AdS$_7$×S^4 correspondence, followed by the derivation of recursion relations for anomalous dimensions in a 6d toy model. Finally, we consider the 6d $(2,0)$ theory, where we compute anomalous dimensions which encode the higher-derivative corrections to the supergravity effective action.

4.1 Review: 6d $(2,0)$ Theory/M-Theory in AdS$_7\times$S^4

In this section we briefly review some aspects of the correspondence between the 6d $(2,0)$ theory and M-theory in AdS$_7\times$S^4. Other important concepts used in this chapter are conformal bootstrap methods and higher-derivative corrections which were both reviewed in chapter 2. For a more detailed review on M-theory see e.g. [51].

The 6d $(2,0)$ theory is the six-dimensional worldvolume theory of M5-branes with $OSp(8^*|4)$ symmetry, which is the maximal supersymmetry in 6d. The worldvolume theory of a single M5-brane can be formulated in terms of an abelian $(2,0)$ tensor multiplet [133–135]. The field content in this case consists of five scalars ϕ^I, eight fermions and a self-dual two-form gauge field. Generalising this to a higher number of branes and thus considering the interacting 6d $(2,0)$ theory is very difficult. It is believed to be non-Lagrangian, since a 6d local Lagrangian can be ruled out by powercounting as it would contain non-renormalisable and unbounded interactions. In [19] a 5d Lagrangian, which is believed to capture the full 6d physics, was proposed. The 6d $(2,0)$ theory is manifestly non-perturbative and the only tunable parameter is N, which is the number of M5-branes. Therefore, a very promising approach to study correlators in this theory is to constrain the CFT data using conformal bootstrap methods. This in turn will tell us something about the bulk dual, M-theory in AdS$_7\times$S^4, or more specifically about the higher-derivative corrections to the low-energy approximation, which is 11d supergravity. Note that higher-derivative corrections can also be deduced from uplifting string calculations [111, 136, 137]. Let us briefly review the field content and action of 11d supergravity before we discuss holographic correlators in this example of the AdS/CFT duality.

4.1.1 11d Supergravity

We start by considering the field content of 11d supergravity, where we focus on the bosonic fields for our considerations. Firstly, the graviton which is a supersymmetric traceless tensor of the symmetry group $SO(9)$ has 44 bosonic degrees of freedom, which counts the number of independent components of a symmetric 9×9 matrix subtracting one because of tracelessness. For the theory to be supersymmetric, the number of physical bosonic and fermionic degrees of freedom needs to be the same. The only fermion field in the theory is the gravitino Ψ_M which has, in addition to the vector index, an implicit spinor index. It is a 32-component Majorana spinor for each value of the index M and has 128 fermionic degrees of freedom (see e.g. [51] for more details on the fermionic degrees of freedom). To get the right number of bosonic degrees of freedom (compared to 128 fermionic ones) one needs to include a rank-3 antisymmetric tensor, represented by a three-form A_3. The theory is then invariant under gauge transformations $A_3 \to A_3 + d\Lambda_2$, where Λ_2 is a two-form. As a consequence of the gauge invariance the indices for the independent physical polarisations are transverse (as for any antisymmetric tensor gauge field). Hence, the three-form in 11 dimensions has $9 \times 8 \times 7/3! = 84$ degrees of freedom and combining this with the graviton gives $44 + 84 = 128$ bosonic degrees of freedom which agrees with the fermionic ones.

The bosonic part of the action of 11d supergravity is then given by

$$2\kappa_{11}^2 S = \int d^{11}x\sqrt{-G}\left(\mathcal{R} - \frac{1}{2}|F_4|^2\right) - \frac{1}{6}\int A_3 \wedge F_4 \wedge F_4 \ , \tag{4.1.1}$$

with the scalar curvature $\mathcal{R}$, the field strength $F_4 = dA_3$ and the 11d gravitational coupling constant κ_{11}. The coupling constant is related to the 11d Newton constant G_N^{11d} and the 11d Planck length l_P as follows:

$$16\pi G_N^{11d} = 2\kappa_{11}^2 = \frac{1}{2\pi}(2\pi l_P)^9 \ . \tag{4.1.2}$$

G is defined as $\det G_{MN}$ where G_{MN} is the metric combination

$$G_{MN} = \eta_{AB} E^A_M E^B_N \ , \tag{4.1.3}$$

with η_{AB} being the flat metric and E^A_M a vielbein field where the indices $M, N, \dots$ describe curved base-space vectors in 11d and transform non-trivially under general coordinate transformations. The indices $A, B, \dots$ on the other hand describe flat tangent-space vectors which transform non-trivially under local Lorentz transformations. The quantity $|F_4|^2$ is defined as

$$|F_4|^2 = \frac{1}{4!} G^{M_1 N_1} \dots G^{M_4 N_4} F_{M_1 \cdots M_4} F_{N_1 \dots N_4} \ . \tag{4.1.4}$$

Note that the goal of this chapter is to go beyond the supergravity approximation and obtain the form of higher-derivative corrections to the low-energy effective action (4.1.1). We approach this by computing anomalous dimensions of operators in the conformal block decomposition of the four-point stress tensor correlator in the dual conformal field theory.

4.1.2 Holographic Correlators

We are interested in the study of four-point stress tensor correlators. The stress tensor belongs to a half-BPS multiplet whose superconformal primary, T_{IJ}, is a dimension-4 scalar in the two-index symmetric traceless representation **14** of the R-symmetry group $SO(5)$. As described in the review 2.3, general chiral primary operators are constructed from scalar fields as $\phi^{(I_1} \phi^{I_2} \dots \phi^{I_k)}$, which is totally symmetric and all traces are understood to be removed. The superconformal primary of the stress tensor multiplet in the 6d $(2,0)$ theory is then constructed from the scalar fields ϕ^I in the abelian $(2,0)$ tensor multiplet as follows

$$T_{IJ} = \mathrm{Tr}(\phi^I \phi^J) - \frac{\delta_{IJ}}{5} \mathrm{Tr}(\phi^K \phi_K) \ , \tag{4.1.5}$$

where $k = 2$ and its scaling dimension is $2k = 4$. Let us now consider the holographic duals of these operators. Dimensionally reducing the bulk dual on the four-sphere yields a Kaluza-Klein tower of scalars in AdS$_7$ with masses $m_k^2 = 4k(k-3)$ in units of the inverse AdS radius [138], which agrees with (2.3.2). The KK modes are the holographic duals of chiral primary operators with k indices which have scaling dimension $2k$. The superconformal primaries of the stress tensor multiplet T_{IJ} (4.1.5), which are the focus of this chapter, correspond to the bottom of the tower, $k = 2$, and are dual to bulk scalars with mass $m^2 = -8$.

Four-point correlators of stress tensor multiplets were computed in the supergravity approximation in [139], and a conformal block decomposition of these results was subsequently carried out in [47]. More recently, corrections to the supergravity approximation were deduced in [26] by constructing solutions to the crossing equations whose conformal block expansion is truncated in spin. In section 4.3, we will derive recursion relations for the anomalous dimensions appearing in the conformal block expansions of these solutions. These recursion relations allow one to directly compute the OPE data of these solutions without having to know them explicitly, and can be straightforwardly implemented on a computer.

Before we go on and study the four-point stress tensor correlator in the $(2,0)$ theory we consider a general bosonic six-dimensional CFT to illustrate our strategy of deriving recursion relations for anomalous dimensions and interpreting them to give insight into the higher-derivative corrections to the low-energy effective action.

4.2 Toy Model

In [30] the authors considered four-point correlators of scalar operators in an abstract non-supersymmetric CFT in two and four dimensions, and showed that the solutions to the crossing equations whose conformal block expansion is truncated in spin are in one-to-one correspondence with local quartic interactions of a massive scalar field in AdS (modulo integration by parts and equations of motion). In this section, we will carry out a similar analysis for a toy model in six dimensions as a warm up for our analysis of the 6d $(2,0)$ theory in the next section. In particular, we will analyse four-point correlators of a scalar operator $\mathcal{O}$ with classical dimension Δ_0. Recall the form of four-point correlators [46]

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \frac{F(u,v)}{(x_{12}^2)^{\Delta_0}(x_{34}^2)^{\Delta_0}} \ , \tag{4.2.1}$$

where x_i is the position of the ith operator, $x_{ij}^2 = (x_i - x_j)^2$, and F is a function of the conformal cross-ratios

$$u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} = z\bar{z} \ , \qquad v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2} = (1-z)(1-\bar{z}) \ , \tag{4.2.2}$$

where we will use the variables (u,v) interchangeably with $(z,\bar{z})$, see also section 2.1. Note that exchanging x_2 with x_4 corresponds to exchanging u and v, or $(z,\bar{z})$ with $(1-z, 1-\bar{z})$. Invariance of the correlator under this exchange (known as crossing symmetry, see (2.1.6)) then implies the following constraint on F:

$$v^{\Delta_0}F(u,v) = u^{\Delta_0}F(v,u) \ . \tag{4.2.3}$$

In this model, the primary double-trace operators are schematically

$$\mathcal{O}_{n,l} = \mathcal{O}\partial_{\mu_1}...\partial_{\mu_l}\partial^{2n}\mathcal{O} \ , \tag{4.2.4}$$

which have scaling dimension $\Delta = 2n + l + 2\Delta_0 + \mathcal{O}(1/c)$, spin l and naive twist $2n + 2\Delta_0$.

The conformal block expansion of $F(u,v)$ is then given by the following sum over primary operators:

$$F(u,v) = \sum_{n,l\geq 0} A_{n,l}\, G^{\mathrm{B}}_{\Delta,l}(z,\bar{z}) \ , \tag{4.2.5}$$

where $A_{n,l}$ are OPE coefficients and $G^{\mathrm{B}}_{\Delta,l}$ are six-dimensional bosonic conformal blocks, which implicitly depend on n through the scaling dimensions of the conformal primary operator Δ. They are given in terms of hypergeometric functions and we spell them out in appendix E. Note that $A_{n,l} = 0$ when l is odd since operators with an odd number of derivatives in the OPE of two identical operators correspond to descendants.

The function F can be obtained from tree-level Witten diagrams for supergravity (see [30], and section 2.3). However, the main point of this chapter is to derive the anomalous dimensions that appear in the conformal block expansion of the correlator directly from a recursion relation derived from (4.2.5) without requiring knowledge of the explicit form of the correlators.

The first step is to expand the OPE data in $1/c$:

$$A_{n,l} = A^{(0)}_{n,l} + \frac{1}{c}A^{(1)}_{n,l} + \dots \,, \qquad \Delta = 2n + l + 2\Delta_0 + \frac{1}{c}\gamma_{n,l} + \dots \tag{4.2.6}$$

Firstly, let us compute the free OPE coefficients $A^{(0)}_{n,l}$ from the conformal block expansion of the free disconnected part of the four-point correlator which is given by

$$F^{\text{free}}(u,v) = 1 + u^{\Delta_0} + \frac{u^{\Delta_0}}{v^{\Delta_0}} \,. \tag{4.2.7}$$

The leading contribution to the OPE coefficients is then given by

$$\begin{aligned} A^{(0)}_{n,l} = & \frac{2\,(l+2)\,(2\Delta_0+l+2n-2)\,(2\Delta_0+l+2n-3)\,((\Delta_0+n-3)!)^2}{((\Delta_0-3)!)^2\,((\Delta_0-1)!)^2\,n!\,(l+n+2)!\,(2\Delta_0+2n-6)!\,(2\Delta_0+2l+2n-2)!} \\ & \times ((\Delta_0+l+n-1)!)^2\,(2\Delta_0+n-6)!\,(2\Delta_0+l+n-4)! \,, \end{aligned} \tag{4.2.8}$$

for even l and zero otherwise. In the next subsection, we will derive recursion relations for the anomalous dimensions $\gamma_{n,l}$ in (4.2.6). After solving the recursion relations, we can then deduce the $1/c$ correction to the OPE coefficients $A^{(1)}_{n,l}$ using the following formula:

$$A^{(1)}_{n,l} = \frac{1}{2}\partial_n\left(A^{(0)}_{n,l}\gamma_{n,l}\right) \,. \tag{4.2.9}$$

This formula was first found in two and four dimensions [30, 140] and was subsequently observed to hold in six dimensions [26].

4.2.1 Recursion Relations

In this subsection, we will derive a formula for the anomalous dimensions of double-trace operators in the toy 6d CFT described above following the method developed for 2d and 4d CFTs in [30]. This formula will be written as a sum over the spin of the operators and will depend on two non-negative integers p and q. Truncating the sum over spin to maximum spin L and choosing p and q appropriately will then give rise to recursion relations for the anomalous dimensions, which can be solved for arbitrary twist and spin $l \leq L$ in terms of $(L+2)(L+4)/8$ free parameters, in agreement with counting of solutions in lower dimensions and holographic arguments, as we will describe in subsection 4.2.2.

After expanding the OPE data in $1/c$ in (4.2.6), expanding the conformal block decomposition (the right hand side of (4.2.5)) in $1/c$ and inserting this into the crossing equation (4.2.3) then gives

$$v^{\Delta_0} \sum_{n,l\geq 0} \left[A^{(1)}_{n,l}\, G^{\mathrm{B}}_{\Delta,l}(z,\bar{z}) + \frac{1}{2} A^{(0)}_{n,l}\, \gamma_{n,l}\, \partial_n G^{\mathrm{B}}_{\Delta,l}(z,\bar{z}) \right] - (u \leftrightarrow v) = 0 \ . \tag{4.2.10}$$

Note that in general there will be degeneracy in the free theory, i.e. more than one operator with each given naive dimension and spin. Consequently, in (4.2.10) the free conformal block coefficient gives a sum over these operators of three-point coefficients squared, $A^{(0)}_{n,l} = \sum_i \langle \mathcal{O}_{\Delta_0} \mathcal{O}_{\Delta_0} \mathcal{O}_i \rangle^2$. Then $\gamma_{n,l}$ is in reality the so-called 'averaged anomalous dimension' $\gamma_{n,l} = (\sum_i \langle \mathcal{O}_{\Delta_0} \mathcal{O}_{\Delta_0} \mathcal{O}_i \rangle^2 \gamma_{n,l,i}) / (\sum_i \langle \mathcal{O}_{\Delta_0} \mathcal{O}_{\Delta_0} \mathcal{O}_i \rangle^2)$ where $\gamma_{n,l,i}$ are the anomalous dimensions of the individual operators. To obtain the individual anomalous dimensions requires more data, for example four-point functions of operators with different dimensions. We will solve mixing problems like these for 1d correlators in chapter 5 but in the present chapter our focus lies on the averaged anomalous dimensions since they contain information about the higher-derivative terms in the low-energy effective action.

The exact form of the conformal blocks is given in appendix E, but they are given as a sum of products of hypergeometrics with the following schematic form

$$G^{\mathrm{B}}_{\Delta,l}(z,\bar{z}) \sim \sum \frac{u^n}{(z-\bar{z})^3} k_\alpha(z) k_\beta(\bar{z}) \ , \tag{4.2.11}$$

where

$$k_\beta(z) = {}_2F_1\left(\frac{\beta}{2}, \frac{\beta}{2}, \beta, z\right) \ . \tag{4.2.12}$$

From this we see that $\partial_n G^{\mathrm{B}}_{\Delta,l}(z,\bar{z})$ gives a contribution of the form $\log(u) = \log(z\bar{z})$, and the analogous term in the cross channel will contribute $\log((1-z)(1-\bar{z}))$. As a result, we can isolate the terms containing the anomalous dimensions in both channels simultaneously by taking the $\log(z)\log(1-\bar{z})$ coefficient of the crossing equation as $z\to 0$ and $\bar{z}\to 1$. This limit is referred to as the light-cone limit. In order for the crossing equation to be consistent, the $\log(z)$ coming from $\partial_n G^{\mathrm{B}}_{\Delta,l}(z,\bar{z})$ must thus be accompanied by a $\log(1-\bar{z})$. Such terms indeed arise from the hypergeometrics depending on $\bar{z}$ after making use of the relation

$$k_\beta(\bar{z}) = \log(1-\bar{z})\,\tilde{k}_\beta(1-\bar{z}) + \text{holomorphic at } \bar{z}=1\ , \tag{4.2.13}$$

where

$$\tilde{k}_\beta(z) = -\frac{\Gamma(\beta)}{\Gamma\left(\frac{\beta}{2}\right)^2}\,{}_2F_1\left(\frac{\beta}{2},\frac{\beta}{2},1,z\right)\ . \tag{4.2.14}$$

Similarly, the hypergeometrics depending on $1-z$ in the cross channel will give rise to $\log(z)$.

In summary, we take the $\log(z)\log(1-\bar{z})$ coefficient of (4.2.10) as $z\to 0$ and $\bar{z}\to 1$ yielding the refined crossing equation:

$$\begin{aligned} &v^{\Delta_0}\sum_{n,l\geq 0} A^{(0)}_{n,l}\,\gamma_{n,l}\left(\partial_n G^{\mathrm{B}}_{2n+l+2\Delta_0,l}(z,\bar{z})\right)\Big|_{\log z\log(1-\bar{z})} = \\ &u^{\Delta_0}\sum_{n,l\geq 0} A^{(0)}_{n,l}\,\gamma_{n,l}\left(\partial_n G^{\mathrm{B}}_{2n+l+2\Delta_0,l}(1-z,1-\bar{z})\right)\Big|_{\log z\log(1-\bar{z})}\ , \end{aligned} \tag{4.2.15}$$

into which we insert (the precise forms of) (4.2.11) and (4.2.13) to obtain sums of terms of the form $k_\alpha(z)\tilde{k}_\beta(1-\bar{z})$ and $k_\alpha(1-\bar{z})\tilde{k}_\beta(z)$. To extract a purely numerical recursion relation we use an orthogonality relation between hypergeometric functions obtained in [30]

$$\delta_{m,m'} = \oint \frac{dz}{2\pi i}\, z^{m-m'-1}\, k_{2m+4}(z)\, k_{-2m'-2}(z)\ , \tag{4.2.16}$$

for more details on this relation please see appendix F. To use it, one has to multiply (4.2.15) in terms of sums of the schematic form $k\,\tilde{k}$ by

$$\frac{k_{-2q}(z)}{z^{5-\Delta_0+q}}\times\frac{k_{-2p}(1-\bar{z})}{(1-\bar{z})^{5-\Delta_0+p}}\ , \tag{4.2.17}$$

which leads to a sum of terms of the schematic form $k\,k\,k\,\tilde{k}$, where p and q are arbitrary non-negative integers. Then, performing the contour integrals $\oint \frac{dz}{2\pi i} \oint \frac{d\bar{z}}{2\pi i}$, where the contours encircle $(z,\bar{z}) = (0,1)$, one can use the orthogonality relation (4.2.16), which takes two of the k's per term and turns them into numerical expressions. Additionally, to make sense of the terms of the form $k\,\tilde{k}$ we define the following integral

$$\mathcal{I}_{m,m'} = \oint \frac{dz}{2\pi i} \frac{(1-z)^{m-\Delta_0+3}}{z^{m'-\Delta_0+5}} \tilde{k}_{2m}(z)\, k_{-2m'}(z)\ . \tag{4.2.18}$$

Finally, we arrive at the following equation:

$$\begin{aligned}
0 = \sum_{l=0}^{L} \sum_{n=0}^{\infty} A^{(0)}_{n,l} \gamma_{n,l} \Big[&(l+1) \left(\delta_{q,l+n+3} \mathcal{I}_{\Delta_0+n-3,p+\Delta_0-4} - \delta_{q,n} \mathcal{I}_{\Delta_0+l+n,p+\Delta_0-4} \right) \\
&+ (l+3) \left(\delta_{q,n+1} \mathcal{I}_{\Delta_0+l+n-1,p+\Delta_0-4} - \delta_{q,l+n+2} \mathcal{I}_{\Delta_0+n-2,p+\Delta_0-4} \right) \\
&+ P_{n,l} \left(\delta_{q,l+n+3} \mathcal{I}_{\Delta_0+n-1,p+\Delta_0-4} - \delta_{q,n+2} \mathcal{I}_{\Delta_0+l+n,p+\Delta_0-4} \right) \\
&+ Q_{n,l} \left(\delta_{q,n+1} \mathcal{I}_{\Delta_0+l+n+1,p+\Delta_0-4} - \delta_{q,l+n+4} \mathcal{I}_{\Delta_0+n-2,p+\Delta_0-4} \right) - (q \leftrightarrow p) \Big] \ ,
\end{aligned} \tag{4.2.19}$$

where

$$\begin{aligned}
P_{n,l} &= \frac{(l+3)\,(\Delta_0+n-2)^2\,(2\Delta_0+l+2n-4)}{4\,(2\Delta_0+2n-5)\,(2\Delta_0+2n-3)\,(2\Delta_0+l+2n-2)}\ , \\
Q_{n,l} &= \frac{(l+1)\,(\Delta_0+l+n)^2\,(2\Delta_0+l+2n-4)}{4\,(2\Delta_0+l+2n-2)\,(2\Delta_0+2l+2n-1)\,(2\Delta_0+2l+2n+1)}\ .
\end{aligned} \tag{4.2.20}$$

Note that we have truncated the sum over spins in (4.2.19) to a maximum spin L. Recursion relations for the anomalous dimensions are then obtained by making particular choices of p and q, and the solutions are labelled by L. In the next subsection we will explain how to solve the recursion relations for $L = 0, 2$ before describing a general algorithm to solve the recursion relations for any L.

4.2.2 Solutions

Let us first consider the $L=0$ spin truncation in (4.2.19). In this case, setting $q=0$ leads to the following recursion relation in terms of p:

$$\mathcal{I}_{\Delta_0,p+\Delta_0-4}A^{(0)}_{0,0}\gamma_{0,0}=\sum_{a=0}^{4}C_aA^{(0)}_{p-a,0}\gamma_{p-a,0}\ , \tag{4.2.21}$$

where

$$\begin{aligned}
C_0&=\mathcal{I}_{p+\Delta_0,\Delta_0-4}\ ,\\
C_1&=-3\mathcal{I}_{p+\Delta_0-2,\Delta_0-4}-\frac{(p+\Delta_0-1)^2(p+\Delta_0-3)\,\mathcal{I}_{p+\Delta_0,\Delta_0-4}}{4(p+\Delta_0-2)(2p+2\Delta_0-1)(2p+2\Delta_0-3)}\ ,\\
C_2&=3\mathcal{I}_{p+\Delta_0-4,\Delta_0-4}+\frac{3(p+\Delta_0-4)^3\,\mathcal{I}_{p+\Delta_0-2,\Delta_0-4}}{4(p+\Delta_0-3)(2p+2\Delta_0-7)(2p+2\Delta_0-9)}\ ,\\
C_3&=-\mathcal{I}_{p+\Delta_0-6,\Delta_0-4}-\frac{3(p+\Delta_0-5)^3\,\mathcal{I}_{p+\Delta_0-4,\Delta_0-4}}{4(p+\Delta_0-4)(2p+2\Delta_0-9)(2p+2\Delta_0-11)}\ ,\\
C_4&=\frac{(p+\Delta_0-4)^2(p+\Delta_0-6)\,\mathcal{I}_{p+\Delta_0-6,\Delta_0-4}}{4(p+\Delta_0-5)(2p+2\Delta_0-7)(2p+2\Delta_0-9)}\ .
\end{aligned}\tag{4.2.22}$$

This recursion relation can be solved for all $\gamma_{n,0}$ with $n>0$ in terms of $\gamma_{0,0}$ as follows:

$$\begin{aligned}
\gamma^{\text{spin-0}}_{n,0}(\Delta_0)\ &=\gamma_{0,0}\frac{(2\Delta_0-3)(2\Delta_0-1)(n+1)(n+2)(\Delta_0+n-2)(\Delta_0+n-1)}{8(\Delta_0-2)^2(\Delta_0-1)(2\Delta_0+2n-5)(2\Delta_0+2n-3)}\\
&\quad\times\frac{(2\Delta_0+n-5)(2\Delta_0+n-4)}{(2\Delta_0+2n-1)}\ ,
\end{aligned}\tag{4.2.23}$$

where we divided by $A^{(0)}_{n,0}$, see (4.2.8).

For $L=2$, first choose $(p,q)=(1,0)$ to obtain $\gamma_{1,0}$ in terms of three unfixed parameters $\{\gamma_{0,0},\gamma_{0,2},\gamma_{1,2}\}$. For $p>1$, one can then solve the equations with $q\in\{0,1\}$ for $\gamma_{p,l}$ with $l\in\{0,2\}$ in terms of $\gamma_{p',l'}$ with $p'<p$ and $l'\in\{0,2\}$. In the end, we obtain a solution for all $\gamma_{n,l}$ with $l\in\{0,2\}$ in terms of $\{\gamma_{0,0},\gamma_{0,2},\gamma_{1,2}\}$. We find the following solutions for general scaling dimension Δ_0:

$$\gamma^{\text{spin-2}}_{n,0}(\Delta_0)=\frac{\gamma^{\text{spin-0}}_{n,0}(\Delta_0)}{\gamma_{0,0}}\left(\gamma_{0,0}+\gamma_{0,2}\,f_1(n,\Delta_0)+\gamma_{1,2}\,f_2(n,\Delta_0)\right)\ , \tag{4.2.24}$$

$$\gamma_{n,2}^{\text{spin-2}}(\Delta_0) = -\frac{\gamma_{n,0}^{\text{spin-0}}(\Delta_0)}{\gamma_{0,0}} \frac{(2\Delta_0+1)^2(2\Delta_0+3)(n-1)(n+3)(n+4)}{4(\Delta_0-1)\Delta_0^4(\Delta_0+1)^2}$$
$$\times \frac{(\Delta_0+n)(\Delta_0+n+1)(2\Delta_0+n)(2\Delta_0+n-3)(2\Delta_0+n-2)}{(2\Delta_0-3)(2\Delta_0+2n+1)(2\Delta_0+2n+3)}$$
$$\times \left(\gamma_{0,2} - \gamma_{1,2}\frac{4\Delta_0(2\Delta_0+3)(2\Delta_0+5)n(2\Delta_0+n-1)}{(\Delta_0+1)(\Delta_0+2)^2(2\Delta_0-1)(2\Delta_0+1)(n-1)(2\Delta_0+n)}\right), \tag{4.2.25}$$

where

$$f_1(n,\Delta_0) = \frac{(2\Delta_0+1)^2(2\Delta_0+3)n(2\Delta_0+n-3)}{(\Delta_0-1)\Delta_0^4(\Delta_0+1)^2(2\Delta_0-3)(2\Delta_0+2n-7)(2\Delta_0+2n+1)}$$
$$\times\Bigg(5n^6+15(2\Delta_0-3)n^5+\left(89\Delta_0^2-161\Delta_0+127\right)n^4+(2\Delta_0-3)$$
$$\times\left(78\Delta_0^2-22\Delta_0+29\right)n^3+2\left(82\Delta_0^4-143\Delta_0^3-107\Delta_0^2+117\Delta_0-39\right)n^2$$
$$+(2\Delta_0-3)\left(48\Delta_0^4-14\Delta_0^3-215\Delta_0^2-33\Delta_0-6\right)n$$
$$+\frac{6(\Delta_0-1)\Delta_0^2(2\Delta_0-7)\left(4\Delta_0^3+12\Delta_0^2+5\Delta_0-1\right)}{2\Delta_0+1}\Bigg),$$

$$f_2(n,\Delta_0) = \frac{(2\Delta_0+1)(2\Delta_0+3)^2(2\Delta_0+5)n(2\Delta_0+n-3)}{(3-2\Delta_0)(\Delta_0-1)\Delta_0^3(\Delta_0+1)^3(\Delta_0+2)^2(2\Delta_0-1)(2\Delta_0+2n-7)}$$
$$\times\frac{1}{(2\Delta_0+2n+1)}\Big(20n^6+60(2\Delta_0-3)n^5+(4\Delta_0(89\Delta_0-199)+508)n^4$$
$$+4(2\Delta_0-3)\left(78\Delta_0^2-98\Delta_0+29\right)n^3+8(\Delta_0(2\Delta_0(\Delta_0(41\Delta_0-131)+104)$$
$$-27)-39)n^2+4(2\Delta_0-3)(\Delta_0(\Delta_0(4\Delta_0(12\Delta_0-25)-41)+21)-6)n$$
$$+24(\Delta_0-1)\Delta_0^2(\Delta_0+1)(2\Delta_0-7)(2\Delta_0-1)\Big). \tag{4.2.26}$$

We will discuss these solutions and solutions with general L below, but first let us describe an algorithm to solve recursion relations for general L.

Algorithm to solve recursion relations for any spin-L truncation

Recursion relations for the anomalous dimensions of double-trace operators for a general spin-L truncation are encoded in (4.2.19) for the bosonic toy model and below in (4.3.19) for the $(2,0)$ theory, respectively. They are obtained by specifying a spin truncation L and making appropriate choices of non-negative integers p and

q. The general algorithm for solving the recursion relation for any L is as follows:

- For each $1 \leq p \leq L/2$, write down the equations for $0 \leq q \leq p-1$.
- Solve these equations for $\gamma_{p,l}$ with $0 \leq l \leq 2p-2$ in terms of $\gamma_{p',l'}$ with $(p' \leq p-1,\, l' \leq L)$ and $(p' = p,\, 2p \leq l' \leq L)$.
- For each $p \geq L/2+1$, write down the equations for $0 \leq q \leq L/2$.
- Solve these equations for $\gamma_{p,l}$ with $0 \leq l \leq L$ in terms of $\gamma_{p',l'}$ with $(p' \leq p-1,\, l' \leq L)$.

In the end, this algorithm will give all $\gamma_{n,l}$ with $l \leq \min(2n-2, L)$ in terms of all $\gamma_{n',l'}$ with $2n' \leq l' \leq L$, which correspond to $(L+2)(L+4)/8$ free parameters as depicted in figure 3 of [30] (for a review of the holographic arguments of [30] see subsection 2.3.2). This algorithm can easily be implemented on a computer[1] by generating all the free parameters for a given L, writing down the equations for every $p \geq 1$ and $0 \leq q \leq \min(p-1, L/2)$, replacing $(L+2)(L+4)/8$ of the anomalous dimensions by the free parameters, and solving these equations for the remaining anomalous dimensions.

Discussion of solutions

For a spin-L truncation, we find that the solution depends on $(L+2)(L+4)/8$ unfixed parameters, as described above for general L and illustrated in the $L = 0, 2$ cases. This agrees with the holographic arguments of [30] (see subsection 2.3.2). Recall that the four-point correlators are dual to local quartic interactions in the bulk. Specifically, there are $L/2+1$ independent interactions which can create or annihilate a state of at most spin L, with the total number of derivatives ranging from $2L$ to $3L$ in intervals of two. For the cases we considered in the discussion of the recursion relation above, there is one spin-0 interaction vertex with no derivatives ϕ^4, and

[1] This can be seen in the auxiliary Mathematica file `6drecursion.nb` of [43].

two spin-2 interaction vertices $\phi^2 \left(\nabla_\mu \nabla_\nu \phi\right)^2$ and $\phi^2 \left(\nabla_\mu \nabla_\nu \nabla_\rho \phi\right)^2$ with four and six derivatives, respectively, see (2.3.19). The total number of interactions up to spin L is then given by $\sum_{l=0}^{L/2}(l+1) = (L+2)(L+4)/8$.

Note that this is the same number as there are unfixed parameters in the solutions to the recursion relations above. Thus, these unfixed parameters can be identified with coefficients of the bulk interaction vertices. Indeed, we have verified that the solution in (4.2.23) reproduces the anomalous dimensions in the conformal block expansion of a Witten diagram for a ϕ^4 interaction (see [30] and subsection 2.3.1)

$$F^{\text{spin-0}}(u,v) = C^{(0)} \bar{D}_{\Delta_0 \Delta_0 \Delta_0 \Delta_0} \ , \tag{4.2.27}$$

for the following choice of the free parameter:

$$\gamma_{0,0} = -\frac{C^{(0)} \left((\Delta_0 - 1)!\right)^4}{(2\Delta_0 - 1)!} \ , \tag{4.2.28}$$

where the coefficient $C^{(0)}$ is unfixed and the definition of $\bar{D}$-functions was reviewed in chapter 2 around (2.3.17). Note that the anomalous dimensions of $F^{\text{spin-0}}$ are obtained by expanding this function according to (4.2.10) but without considering the crossing equation.

Moreover, the $L = 2$ solution encodes the anomalous dimensions in the conformal block expansion of Witten diagrams with four- and six-derivative interactions (again see [30] and section 2.3.2)

$$F_4^{\text{spin-2}}(u,v) = C_4^{(2)} (1+u+v)\, \bar{D}_{\Delta_0+1\, \Delta_0+1\, \Delta_0+1\, \Delta_0+1} \ , \tag{4.2.29}$$

$$\begin{aligned} F_6^{\text{spin-2}}(u,v) = C_6^{(2)} \Big(&\bar{D}_{\Delta_0+2\, \Delta_0+1\, \Delta_0+2\, \Delta_0+1} + \bar{D}_{\Delta_0+1\, \Delta_0+2\, \Delta_0+1\, \Delta_0+2} \\ &+ u^2\, \bar{D}_{\Delta_0+2\, \Delta_0+2\, \Delta_0+1\, \Delta_0+1} + u\, \bar{D}_{\Delta_0+1\, \Delta_0+1\, \Delta_0+2\, \Delta_0+2} \\ &+ v^2\, \bar{D}_{\Delta_0+1\, \Delta_0+2\, \Delta_0+2\, \Delta_0+1} + v\, \bar{D}_{\Delta_0+2\, \Delta_0+1\, \Delta_0+1\, \Delta_0+2} \Big) \ , \end{aligned} \tag{4.2.30}$$

for the following choice of free parameters:

$$\{\gamma_{0,0},\gamma_{0,2},\gamma_{1,2}\}_4 = C_4^{(2)}\left\{-\frac{4\,(\Delta_0!)^3\,(\Delta_0+1)!}{(2\Delta_0+2)!}\,,\,-\frac{2\,\Delta_0!\,((\Delta_0+1)!)^2\,(\Delta_0+2)!}{3\,(2\Delta_0+1)\,(2\Delta_0+4)!}\,,\right.$$
$$\left.-\frac{(\Delta_0+1)\,(2\Delta_0-1)\,(\Delta_0-1)!\,((\Delta_0+2)!)^2\,(\Delta_0+3)!}{3\,(2\Delta_0+3)\,(2\Delta_0+6)!}\right\}\,, \tag{4.2.31}$$

$$\{\gamma_{0,0},\gamma_{0,2},\gamma_{1,2}\}_6 = C_6^{(2)}\left\{-\frac{4\,(\Delta_0!)^2\,((\Delta_0+1)!)^2}{(2\Delta_0+2)!}\,,\right.$$
$$-\frac{2\,(3\Delta_0+2)\,\Delta_0!\,((\Delta_0+1)!)^2\,(\Delta_0+2)!}{3\,(2\Delta_0+1)\,(2\Delta_0+4)!}\,,$$
$$\left.-\frac{(\Delta_0+1)\,(6\Delta_0^2+7\Delta_0-2)\,(\Delta_0-1)!\,((\Delta_0+2)!)^2\,(\Delta_0+3)!}{3\,(2\Delta_0+3)\,(2\Delta_0+6)!}\right\}\,, \tag{4.2.32}$$

where the coefficients $C_{4,6}^{(2)}$ are unfixed. Even though the contact interactions can be obtained from Witten diagrams in AdS case by case for spin-L truncations and this could be used to obtain the anomalous dimensions of double-trace operators in the OPE, computing the anomalous dimensions from a recursion relation is much more efficient. The recursion relation does not require the knowledge of the exact form of the correlators, one only needs the conformal blocks (and the free theory coefficients) and can derive the averaged anomalous dimensions for any spin-L truncation easily. Furthermore, the algorithm for general L solutions described in subsection 4.2.2 can be implemented on a computer which makes it very efficient to obtain anomalous dimensions for any L.

Importantly, the number of derivatives in the bulk interactions can be read off from the large-twist behaviour of the corresponding anomalous dimensions. Indeed, the anomalous dimensions of $F^{\text{spin-0}}$ scale like n^3 in the large-n limit, while those of $F_4^{\text{spin-2}}$ and $F_6^{\text{spin-2}}$ scale like n^7 and n^9, respectively. In other words, the anomalous dimensions associated with four- and six-derivative interactions scale like n^4 and n^6 compared to those of the ϕ^4 interaction. Studying the anomalous dimensions obtained from the recursion relations, there is a subtlety for $L>0$. The solutions (4.2.24) and (4.2.25) both scale like n^9, so they both correspond to six-derivative interactions. We know however, that for $L=2$ there should also be a four-derivative interaction corresponding to anomalous dimensions scaling like n^7 and it turns out that this is included in the solutions as follows. For a specific choice of the ratio between the coefficients $\gamma_{0,2}$ and $\gamma_{1,2}$, the scaling of (4.2.25) reduces to n^7, which can be easily deduced by imposing the vanishing of the large-n limit of the last line in (4.2.25). The choice of free parameters is

$$\gamma_{1,2} = \frac{(\Delta_0+1)(\Delta_0+2)^2(2\Delta_0-1)(2\Delta_0+1)}{4\Delta_0(2\Delta_0+3)(2\Delta_0+5)}\gamma_{0,2}\ . \tag{4.2.33}$$

Note that the solution in (4.2.31) is consistent with this constraint. More generally, for a spin-L solution one can deduce $L/2$ constraints on the coefficients (corresponding to bulk interactions with the number of derivatives ranging from $2L$ to $3L-2$ in intervals of 2) by analysing the large-twist limit. Unconstrained coefficients then encode the freedom to add solutions with lower spin or subleading large-twist behaviour.

We will discuss this further in the 6d $(2,0)$ case in subsection 4.3.2 where the anomalous dimensions encode information about the higher-derivative corrections to the low-energy 11d supergravity action. Let us now go on and derive recursion relations for the supersymmetric theory.

4.3 $(2,0)$ Theory

In this section we will adapt the analysis of the previous section to four-point stress tensor correlators of the 6d $(2,0)$ theory. Recall from the review in section 4.1.2 that the superconformal primary of the half-BPS multiplet is the dimension-four scalar T_{IJ} in the two-index symmetric traceless representation of the R-symmetry group $SO(5)$. Its holographic dual corresponds to the bottom of the KK tower, $k=2$, and has mass $m^2=-8$.

As shown in [47, 141], superconformal symmetry constrains the four-point function of stress tensor multiplets in the 6d $(2,0)$ theory in terms of a prepotential $F(z,\bar{z})$

as follows:

$$(z-\bar{z})^4\left(g_{13}g_{24}\right)^{-2}\langle T_1T_2T_3T_4\rangle = \mathcal{D}\left(\mathcal{S}F\left(z,\bar{z}\right)\right)+\mathcal{S}_1^2F\left(z,z\right)+\mathcal{S}_2^2F\left(\bar{z},\bar{z}\right)\ , \tag{4.3.1}$$

where $\mathcal{D}=-\left(\partial_z-\partial_{\bar{z}}+(z-\bar{z})\partial_z\partial_{\bar{z}}\right)(z-\bar{z})$ and the variables $z,\bar{z}$ are defined in terms of the spacetime cross-ratios (4.2.2). We have introduced auxiliary variables Y^I to contract the $SO(5)$ indices of T_{IJ} via $T_i=T_{IJ}Y_i^IY_i^J$. Using these internal coordinates, we then define superpropagators $g_{ij}=Y_i\cdot Y_j/x_{ij}^4$ and internal conformal cross-ratios

$$y\bar{y}=\frac{Y_1\cdot Y_2Y_3\cdot Y_4}{Y_1\cdot Y_3Y_2\cdot Y_4}\ , \qquad (1-y)(1-\bar{y})=\frac{Y_1\cdot Y_4Y_2\cdot Y_3}{Y_1\cdot Y_3Y_2\cdot Y_4}\ , \tag{4.3.2}$$

in terms of which we define $\mathcal{S}_1=(z-y)(z-\bar{y})$, $\mathcal{S}_2=(\bar{z}-y)(\bar{z}-\bar{y})$, and $\mathcal{S}=\mathcal{S}_1\mathcal{S}_2$. Crossing symmetry implies that

$$F(u,v)=F(v,u)\ . \tag{4.3.3}$$

This is important because as there is a crossing symmetric prepotential in terms of which the correlator is constrained, one does not need to consider the full correlator but studying the much simpler prepotential is enough. Moreover, we can write $F(u,v)$ as

$$F(z,\bar{z})=\frac{A}{u^2}+\frac{g(z)-g(\bar{z})}{u\,(z-\bar{z})}+(z-\bar{z})\,G(z,\bar{z})\ , \tag{4.3.4}$$

where each function in this decomposition encodes certain contributions to the OPE. Roughly speaking, A encodes the unit operator, g encodes protected operators, and G encodes non-protected double-trace operators, which will be our main interest. In more detail, these operators have the schematic form $T\partial^l\,\Box^n\,T$ with $n\geq 0$ and scaling dimension $\Delta=2n+l+8+\mathcal{O}(1/c)$. They contribute to the conformal block expansion of G as follows [2]

$$(z-\bar{z})^2G(z,\bar{z})=\sum_{n,l\geq 0}A_{n,l}\,G^{\mathrm{S}}_{\Delta,l}(z,\bar{z})\ , \tag{4.3.5}$$

where the supersymmetric conformal blocks $G^{\mathrm{S}}_{\Delta,l}(z,\bar{z})$ are given in appendix E, and

[2] Note that the conformal block expansion of G also contains protected double-trace operators, which correspond to $n\in\{-1,-2\}$ in our conventions, but we will not need to consider these operators. For more details, see [47].

implicitly depend on n through Δ. Note that equations (4.3.3) and (4.3.4) imply that G obeys the following crossing relation:

$$G(z,\bar{z}) = -G(1-z,1-\bar{z}) \ . \tag{4.3.6}$$

Again we start by computing the leading free theory coefficients which are obtained from the free disconnected part of the four-point correlator. This can be computed in the abelian theory and corresponds to the following prepotential:

$$F^{\text{free-disc}}(u,v) = 1 + \frac{1}{u^2} + \frac{1}{v^2} \ . \tag{4.3.7}$$

Decomposing this function according to (4.3.4) and computing the conformal block expansion of the piece encoding the non-protected operators according to (4.3.5) then gives the following formula for the leading contribution to the OPE coefficients:

$$A^{(0)}_{n,l} = \frac{(l+2)\,(n+3)!\,(n+4)!\,(l+2n+9)\,(l+2n+10)\,(l+n+5)!\,(l+n+6)!}{72\,(2n+5)!\,(2l+2n+9)!} \ . \tag{4.3.8}$$

As we have established, our focus in this chapter are the higher-derivative corrections to the supergravity approximation which we will discuss in the following subsections. Nevertheless, it is interesting to analyse the tree-level supergravity result as well, and this is done in appendix G.

4.3.1 Recursion Relations

To derive recursion relations for the anomalous dimensions of the double-trace operators described above, we follow the same procedure as section 4.2. First, expand the OPE data in $1/c$:

$$A_{n,l} = A^{(0)}_{n,l} + \frac{1}{c}A^{(1)}_{n,l} + \dots \ , \qquad \Delta = 2n + l + 8 + \frac{1}{c}\gamma_{n,l} + \dots \tag{4.3.9}$$

Focusing on the part of the prepotential which describes non-protected operators and expanding the crossing equation (4.3.6) to first order in $1/c$ then gives[3]

$$\sum_{n,l\geq 0}\left[A^{(1)}_{n,l}\,G^{\rm S}_{\Delta,l}(z,\bar z)+\frac{1}{2}A^{(0)}_{n,l}\,\gamma_{n,l}\,\partial_n G^{\rm S}_{\Delta,l}(z,\bar z)\right]+(u\leftrightarrow v)=0\;. \tag{4.3.10}$$

In the supersymmetric case, the conformal blocks have the schematic form

$$G^{\rm S}_{\Delta,l}(z,\bar z)\sim\sum u^n h_\alpha(z)h_\beta(\bar z)\;, \tag{4.3.11}$$

where

$$h_\beta(z)={}_2F_1\left(\frac{\beta}{2},\frac{\beta}{2}-1,\beta,z\right)\;, \tag{4.3.12}$$

see appendix E for the explicit form of the superconformal blocks. Following the same reasoning described in the previous section, the term $\partial_n G^{\rm S}_{\Delta,l}(z,\bar z)$ in (4.3.10) gives a contribution proportional to $\log(z)$ and the analogous term in the cross channel will give $\log(1-\bar z)$, so we can isolate the terms containing anomalous dimensions by taking the light-cone limit $z\to 0$ and $\bar z\to 1$. In this case, the hypergeometrics depending on $\bar z$ and $1-z$ will give rise to $\log(1-\bar z)$ and $\log(z)$ using the relation

$$h_\beta(\bar z)=\log(1-\bar z)\,(1-\bar z)\,\tilde h_\beta(1-\bar z)+\text{holomorphic at }\bar z=1\;, \tag{4.3.13}$$

where

$$\tilde h_\beta(z)=\frac{\Gamma(\beta)}{\Gamma\left(\frac{\beta}{2}\right)\Gamma\left(\frac{\beta}{2}-1\right)}\,{}_2F_1\left(\frac{\beta}{2}+1,\frac{\beta}{2},2,z\right)\;. \tag{4.3.14}$$

We thus consider the $\log(z)\log(1-\bar z)$ coefficient of (4.3.10) in the limit $z\to 0$ and $\bar z\to 1$:

$$\begin{aligned}&\sum_{n,l\geq 0}A^{(0)}_{n,l}\,\gamma_{n,l}\left(\partial_n G^{\rm S}_{\Delta,l}(z,\bar z)\right)\Big|_{\log z\log(1-\bar z)}=\\ &-\sum_{n,l\geq 0}A^{(0)}_{n,l}\,\gamma_{n,l}\left(\partial_n G^{\rm S}_{\Delta,l}(1-z,1-\bar z)\right)\Big|_{\log z\log(1-\bar z)}\;,\end{aligned} \tag{4.3.15}$$

into which we insert the (precise forms of) (4.3.11) and (4.3.13) to obtain sums of terms involving $h_\alpha(z)\tilde h_\beta(1-\bar z)$ and $h_\alpha(1-\bar z)\tilde h_\beta(z)$. To obtain a purely numeric

[3]Note that as in the toy model case, here $\gamma_{n,l}$ are averaged anomalous dimensions.

crossing equation, first multiply this equation by

$$\frac{h_{-2q}(z)}{z^q\,(1-z)} \times \frac{h_{-2p}(1-\bar{z})}{(1-\bar{z})^p\,\bar{z}}\ , \tag{4.3.16}$$

where p and q are arbitrary non-negative integers, and perform the contour integrals $\oint \frac{dz}{2\pi i} \oint \frac{d\bar{z}}{2\pi i}$, which encircle $(z,\bar{z}) = (0,1)$. Again we use an orthogonality relation of the hypergeometric functions and we prove this new relation in appendix F. It is given by

$$\delta_{m,m'} = \oint \frac{dz}{2\pi i}\,\frac{z^{m-m'-1}}{1-z}\,h_{2m+4}(z)\,h_{-2m'-2}(z)\ , \tag{4.3.17}$$

and defining the integral

$$\mathcal{I}_{m,m'} = \oint \frac{dz}{2\pi i}\,\frac{(1-z)^{m-3}}{z^{m'-1}}\,\tilde{h}_{2m}(z)\,h_{-2m'}(z)\ , \tag{4.3.18}$$

finally leads to the following equation:

$$\begin{aligned}
0 = \sum_{l=0}^{L}\sum_{n=0}^{\infty} A^{(0)}_{n,l}\gamma_{n,l}\Big[& P_{n,l}\left(\delta_{q,n}\mathcal{I}_{n+l+6,p+2} - \delta_{q,n+l+3}\mathcal{I}_{n+3,p+2}\right)\\
&+ Q_{n,l}\left(\delta_{q,n+2}\mathcal{I}_{n+l+6,p+2} - \delta_{q,n+l+3}\mathcal{I}_{n+5,p+2}\right)\\
&+ R_{n,l}\left(\delta_{q,n+l+2}\mathcal{I}_{n+4,p+2} - \delta_{q,n+1}\mathcal{I}_{n+l+5,p+2}\right)\\
&+ S_{n,l}\left(\delta_{q,n+l+4}\mathcal{I}_{n+4,p+2} - \delta_{q,n+1}\mathcal{I}_{n+l+7,p+2}\right) - (q \leftrightarrow p)\Big]\ ,
\end{aligned} \tag{4.3.19}$$

where we have truncated the sum over spins and defined

$$\begin{aligned}
P_{n,l} &= \frac{l+1}{(n+3)\,(n+l+5)}\ , \quad Q_{n,l} = \frac{(l+3)\,(n+5)\,(2n+l+8)}{4\,(2n+7)\,(2n+9)\,(n+l+5)\,(2n+l+10)}\ ,\\
R_{n,l} &= \frac{l+3}{(n+3)\,(n+l+5)}\ , \quad S_{n,l} = \frac{(l+1)\,(n+l+7)\,(2n+l+8)}{4\,(n+3)\,(2n+l+10)\,(2n+2l+11)\,(2n+2l+13)}\ .
\end{aligned} \tag{4.3.20}$$

As we explain in the next subsection and in 4.2.2, recursion relations for the anomalous dimensions are obtained from (4.3.19) by making appropriate choices for p and q, and solutions are labelled by the spin truncation L.

4.3.2 Solutions

In this subsection, we will describe solutions to the recursion relations for low spin truncations and match them with results previously obtained in [26], an algorithm to obtain solutions for any L was described in subsection 4.2.2. For spin truncation $L=0$, setting $q=0$ in (4.3.19) gives the following recursion relation in terms of p:

$$\frac{1}{15}\mathcal{I}_{6,p+2}A^{(0)}_{0,0}\gamma_{0,0}=\sum_{a=0}^{4}C_a A^{(0)}_{p-a,0}\gamma_{p-a,0}\ , \tag{4.3.21}$$

where

$$\begin{aligned}
C_0 &= \frac{\mathcal{I}_{p+6,2}}{(p+3)(p+5)}\ ,\\
C_1 &= -\frac{3\mathcal{I}_{p+4,2}}{(p+2)(p+4)}-\frac{(p+3)(p+6)\mathcal{I}_{p+6,2}}{4(p+2)(p+4)(2p+9)(2p+11)}\ ,\\
C_2 &= \frac{3\mathcal{I}_{p+2,2}}{(p+1)(p+3)}+\frac{3(p+2)\mathcal{I}_{p+4,2}}{4(p+3)(2p+3)(2p+5)}\ ,\\
C_3 &= -\frac{\mathcal{I}_{p,2}}{p(p+2)}-\frac{3(p+1)\mathcal{I}_{p+2,2}}{4(p+2)(2p+1)(2p+3)}\ ,\\
C_4 &= \frac{p(p+3)\mathcal{I}_{p,2}}{4(p-1)(p+1)(2p+3)(2p+5)}\ .
\end{aligned} \tag{4.3.22}$$

This can be solved for all $\gamma_{n,0}$ in terms of $\gamma_{0,0}$ to give

$$\gamma^{\text{spin}-0}_{n,0}=\gamma_{0,0}\frac{11(n+1)_8(n+2)_6}{2304000(2n+7)(2n+9)(2n+11)}\ , \tag{4.3.23}$$

where we divided by $A^{(0)}_{n,0}$, see (4.3.8), and recall that $x_n=\Gamma(x+n)/\Gamma(x)$ is the Pochhammer symbol. It is interesting to compare this with the bosonic solution (4.2.23) for $\Delta_0=4$:

$$(\gamma_{\text{bos}})^{\text{spin-0}}_{n,0}=(\gamma_{\text{bos}})_{0,0}\frac{35(n+1)_4(n+2)_2}{96(2n+3)(2n+5)(2n+7)}\ . \tag{4.3.24}$$

Similarly, following the procedure described in subsection 4.2.2, we obtain solutions for $L=2$ in terms of three unfixed parameters $\{\gamma_{0,0},\gamma_{0,2},\gamma_{1,2}\}$, which are :

$$\gamma^{\text{spin-2}}_{n,0}=\frac{\gamma^{\text{spin-0}}_{n,0}}{\gamma_{0,0}}\left(\gamma_{0,0}+\gamma_{0,2}f_1(n)+\gamma_{1,2}f_2(n)\right)\ , \tag{4.3.25}$$

$$\gamma^{\text{spin-2}}_{n,2} = -\frac{\gamma^{\text{spin-0}}_{n,0}}{\gamma_{0,0}} \frac{845\,(n-1)\,(n+5)\,(n+6)\,(n+8)\,(n+9)^2\,(n+10)\,(n+12)}{4064256\,(2n+13)\,(2n+15)}$$
$$\times\left(\gamma_{0,2} - \gamma_{1,2}\,\frac{51n\,(n+11)}{364\,(n-1)\,(n+12)}\right)\,, \tag{4.3.26}$$

where

$$f_1\,(n) = \frac{325n\,(n+9)}{1016064\,(2n+5)\,(2n+13)}$$
$$\times\left(13n^6 + 351n^5 + 6201n^4 + 64233n^3 + 385476n^2 + 1251666n + 1512620\right)\,,$$
$$f_2\,(n) = -\frac{1105n\,(n+9)}{9483264\,(2n+5)\,(2n+13)}$$
$$\times\left(5n^6 + 135n^5 + 2157n^4 + 20601n^3 + 117468n^2 + 370494n + 441700\right)\,. \tag{4.3.27}$$

We discuss these solutions and what they tell us about the bulk interactions in the following.

Discussion of solutions

Again, for spin truncation L, the solution will depend on $(L+2)(L+4)/8$ free parameters, in agreement with the counting of solutions in section 4.2.2. Moreover, our results for the anomalous dimensions agree with those obtained in [26], which deduced solutions to the crossing equations whose conformal block expansions are truncated in spin. In particular, the anomalous dimensions in (4.3.23) can be obtained from the conformal block expansion of [26]

$$F^{\text{spin-0}}(u,v) = C^{(0)}(z-\bar{z})^2 uv\bar{D}_{5755}\,, \tag{4.3.28}$$

where the coefficient $C^{(0)}$ is unfixed. Decomposing $F^{\text{spin-0}}$ according to (4.3.4) and performing the conformal block expansion according to (4.3.10) gives the anomalous dimensions in (4.3.23) if we choose the free parameter to be

$$\gamma_{0,0} = -\frac{7200\,C^{(0)}}{77}\,. \tag{4.3.29}$$

Following the holographic arguments of [26, 30, 61], which were reviewed in section 2.3.2, $F^{\text{spin-0}}$ should arise from an $\mathcal{R}^4$ correction to supergravity in AdS$_7$×S^4, where $\mathcal{R}$ is the Riemann tensor. This can be seen by comparing the large-n limit of the anomalous dimensions of the spin-L solutions to the anomalous dimensions of the tree-level supergravity solutions at large n, see appendix G where one can see that for supergravity the anomalous dimensions scale like n^5. The difference in the powers of n then predicts how many derivatives the spin-L interaction has compared to supergravity. Note that in the large-n limit the anomalous dimensions scale like n^{11} which is n^6 times the anomalous dimensions obtained in the supergravity approximation, indicating that the corresponding interaction vertex has six more derivatives than supergravity. Since $\mathcal{R}$ has two derivatives and we are only considering quartic interactions, the spin-0 correction is connected to a $\mathcal{R}^4$ term.

Note that F is a prepotential from which many four-point component correlators (corresponding to different choices of Y_i) are obtained by applying a differential operator according to (4.3.1). This differential operator can be rewritten in terms of u, v derivatives and if the prepotential is expressed in terms of $\bar{D}$-functions then so will all the component correlators. Whilst this does not prove that the prepotential can always be expressed in terms of $\bar{D}$-functions, this property holds in all the examples we have considered, and it is natural to conjecture that it should hold in general. A similar conjecture was made in [129] for four-point correlators of more general half-BPS operators in the supergravity approximation.

For the $L = 2$ spin truncation, [26] found the following solutions to the crossing equation:

$$F_4^{\text{spin-2}}(u,v) = 2\,C_4^{(2)}\,(z-\bar{z})^2 uv\left(\bar{D}_{6776} + \bar{D}_{7676} + \bar{D}_{7766}\right)\ , \tag{4.3.30}$$

$$F_6^{\text{spin-2}}(u,v) = 6\,C_6^{(2)}\,(z-\bar{z})^2 uv\bar{D}_{7777}\ , \tag{4.3.31}$$

where the coefficients $C_{4,6}^{(2)}$ are unfixed and the subscripts indicate the number of additional derivatives compared to the bulk interaction vertex associated with the $L = 0$ solution. The first solution corresponds to a $D^4\mathcal{R}^4$ correction and the second one corresponds to a $D^6\mathcal{R}^4$ correction to supergravity in AdS$_7\times$S^4, which can be read off from the large-twist behaviour of the corresponding anomalous dimensions. In the large-n limit, the first solution scales like n^{15} while the second one scales like n^{17}, which corresponds to four and six more derivatives than the spin-0 solution which goes like n^{11}. The anomalous dimensions of these two solutions are reproduced from the general solution in (4.3.25) and (4.3.26) for the choice of parameters

$$\{\gamma_{0,0},\gamma_{0,2},\gamma_{1,2}\}_4 = C_4^{(2)}\left\{-\frac{5\times 72000}{1001}\,,\,\frac{80640}{1859}\,,\,\frac{5\times 150528}{2431}\right\}\,, \tag{4.3.32}$$

$$\{\gamma_{0,0},\gamma_{0,2},\gamma_{1,2}\}_6 = C_6^{(2)}\left\{\frac{54\times 72000}{1001}\,,\,-\frac{3\times 80640}{1859}\,,\,-\frac{33\times 150528}{2431}\right\}\,. \tag{4.3.33}$$

Note that in both cases, $\gamma_{0,0}$ has the opposite sign of $\gamma_{0,2}$ and $\gamma_{1,2}$, in contrast to what we found for the toy model in (4.2.31) and (4.2.32), where all three parameters had the same sign.

Similar to the toy model, considering the anomalous dimensions from the recursion relations for $L=2$, both solutions scale like n^{17} in the large-n limit. It follows that the corresponding bulk interactions have six more derivatives than the spin-0 interaction. However, we also expect to find solutions corresponding to interactions with four more derivatives, which scale like n^{15} for large n. We obtain these solutions for the choice of free parameters

$$\gamma_{1,2} = \frac{364}{51}\gamma_{0,2}\,, \tag{4.3.34}$$

which comes from imposing that the last line of (4.3.26) vanishes in the large-n limit. The solution in (4.3.32) is consistent with this constraint. Having discussed the solutions for $L=0,2$ using the algorithm in section 4.2.2 one can solve the recursion relations up to any desired spin truncation for any twist.

Although solutions to the recursion relations have unfixed coefficients, it is possible to deduce their leading $1/c$-dependence using holographic reasoning, as described in [26]. First, note that since we solve the recursion relations by truncating in spin, this restricts to contact interactions in the bulk (interactions involving bulk-to-bulk propagators will not truncate in spin). The effective action then has the schematic form

$$\mathcal{L} \sim \frac{1}{G_N^{11d}}\left[(\partial\phi)^2 + \sum_D l_P^{D-2}\partial^D\phi^4\right], \tag{4.3.35}$$

where ϕ represents a graviton field, G_N^{11d} is the 11d Newton constant, and the Planck length l_P is inserted by dimensional analysis. After rescaling the graviton by $\sqrt{G_N^{11d}}$ in order to have canonical kinetic terms, the four-point interactions will acquire a factor of $G_N^{11d} \sim 1/c$ (this is the origin of the $1/c$ in (4.3.9)). Recalling that $G_N^{11d} \sim l_P^9$ in eleven dimensions (see (4.1.2)), we see that a four-point contact interaction with D derivatives must therefore have coefficient $G_N^{11d} l_P^{D-2} \sim c^{-(D+7)/9}$. Moreover, the number of derivatives in a contact interaction can be read off from the large-twist behaviour of the corresponding solution to the crossing equations [30]. In particular, if the anomalous dimensions of the solution scale like n^α, then the corresponding bulk interaction must have $D = (\alpha-5)+2 = \alpha-3$ derivatives (recalling that anomalous dimensions scale like n^5 in the supergravity approximation).

In summary, a solution whose anomalous dimensions scale like n^α must have a coefficient $c^{-(\alpha+4)/9}$. For example, the spin-0 solution in (4.3.23) will have a coefficient of $c^{-5/3}$ and the spin-2 solutions (4.3.25) and (4.3.26) which scale like n^{15} and n^{17} (with a specific choice of parameters) will have coefficients of $c^{-19/9}$ and $c^{-7/3}$, respectively. Note that the spin-0 correction is the leading correction to the low-energy tree-level effective action, whereas the spin-2 corrections are subleading to the one-loop supergravity correction which goes like $(G_N^{11d})^2 \sim c^{-2}$.

Similar reasoning applies to conformal field theories with string theory duals, like $\mathcal{N}=4$ SYM with any fixed finite value of the string coupling. In that case, a contact interaction with D derivatives will have a coefficient of $G_N \alpha'^{(D-2)/2}$, where α' is related to the square of the string length. Writing this prefactor in terms of the central charge and string coupling, and fixing the latter at some finite value will then give an expansion in $1/c$ analogous to M-theory.

4.4 Conclusions and Future Directions

In this chapter, we derive recursion relations for anomalous dimensions of double-trace operators in the 6d $(2,0)$ theory. Given that no Lagrangian description is presently known for this model, our strategy is to use superconformal and crossing symmetry of four-point correlators of stress tensor multiplets. In particular, we expand the crossing equation to first order in the inverse central charge and then take the light-cone limit of the conformal cross-ratios to isolate the terms containing anomalous dimensions. Recursion relations then follow from truncating the conformal block expansion in spin and taking inner products of the resulting equation with certain hypergeometric functions, where we make use of an orthogonality relation of the hypergeometrics. These recursion relations can then be solved to obtain anomalous dimensions for arbitrary twist and spin, reproducing the results for low spin truncations previously obtained in [26]. As a warm-up, we derive analogous recursion relations in a toy model corresponding to an abstract bosonic 6d CFT, and match the results with the conformal block expansion of Witten diagrams in AdS$_7$, confirming the holographic arguments of [30]. We describe an algorithm to solve these recursion relations and compute anomalous dimensions in both the bosonic and supersymmetric theories to any desired twist and spin truncation. We note that this method for extracting anomalous dimensions is much more efficient than extracting them using a conformal block expansion of a known four-point function.

The anomalous dimensions are physically significant because they encode higher-derivative corrections to supergravity in AdS$_7\times$S^4. The number of derivatives in each term of the effective action can be read off from the large-twist behaviour of the corresponding anomalous dimensions. Furthermore, the coefficients of these higher-derivative terms correspond to free parameters of the solutions to the recursion relations. Therefore, these coefficients cannot be determined by our approach and we will discuss this further below.

Future directions

There are several interesting questions for future research related to the results discussed above.

- The first open question is the problem of fixing the coefficients of the higher-derivative corrections in the low-energy effective action. These coefficients correspond to the free parameters in the solutions of the recursion relations for anomalous dimensions derived in this chapter and are thus not fixed by our approach. In the flat space limit, the coefficients of the $\mathcal{R}^4$ and $D^6\mathcal{R}^4$ terms in the M-theory effective action have been deduced by uplifting string theory amplitudes (note that the $D^4\mathcal{R}^4$ term vanishes in 11 dimensions) [114, 136], but the coefficient of the $D^8\mathcal{R}^4$ term (which arises from a truncated spin-4 solution in our approach) is unknown. It would therefore be desirable to develop methods for fixing these coefficients using CFT techniques.

- A strategy for doing so was proposed in [28], and used to fix the coefficient of the $\mathcal{R}^4$ term and argue that the $D^4\mathcal{R}^4$ term vanishes. This was achieved by applying the chiral algebra conjecture in [132] to four-point correlators of the form $\langle kkkk\rangle$ with $k = 3$, where k refers to a half-BPS scalar operator in the k-index symmetric traceless representation of the $SO(5)$ R-symmetry group with scaling dimension $2k$ (note that $k = 2$ is the case considered in this chapter). It would therefore be interesting to find truncated spin solutions to the crossing equations for higher-charge correlators, compute the corresponding anomalous dimensions in their conformal block expansions, and ultimately fix the coefficients of higher-derivative terms in the M-theory effective action. Correlators of the form $\langle kkkk\rangle$ and $\langle n+k, n-k, k+2, k+2\rangle$ were computed in the supergravity approximation in [129, 142] before all supergravity correlators were obtained in [130]. Moreover new solutions to the conformal Ward identities in Mellin space have been found for $\langle kkkk\rangle$ with $k = 2, 3$ in [28], so it would be interesting to see how those methods are related to the ones developed in this chapter.

- Since the conformal blocks for higher-charge correlators appear to be much simpler in 4d [46, 49], it may be instructive to first carry out the analysis described above for 4d $\mathcal{N} = 4$ SYM (for which a chiral algebra description was also proposed in [143]), and use it to deduce terms in the effective action for IIB string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ (which was the focus of the previous chapter).

- Finally, it would be very interesting to explore the loop expansion in M-theory in $\mathrm{AdS}_7\times\mathrm{S}^4$ using conformal bootstrap techniques, following on from the recent success in $\mathrm{AdS}_5\times\mathrm{S}^5$ [11, 41, 84–88, 90–92, 144, 145]. Since the loop expansion in M-theory is an expansion in $1/c$, as is the expansion in higher-derivative corrections, the loop corrections have to be disentangled from the M-theoretic corrections. The first loop corrections to holographic correlators in the 6d $(2,0)$ theory have been obtained in [146].

Chapter 5

$AdS_2 \times S^2$ Correlators: Effective Action and 4d Conformal Symmetry

This chapter is based on [44], which at the time of submission of this thesis is in preparation for publication. We will study different aspects of superconformal correlators in one dimension which are dual to tree-level scattering in quantum gravity in $AdS_2 \times S^2$. The Kaluza-Klein spectrum of supergravity in $AdS_2 \times S^2$ is described in [147, 148]. In particular, in [147] the authors start from 11d supergravity (see subsection 4.1.1) and dimensionally reduce it to obtain 4d supergravity. After compactifying on $AdS_2 \times S^2$ one then obtains 4d $\mathcal{N} = 2$ supergravity. Finally, compactifying 4d supergravity on S^2 yields an infinite tower of KK modes in AdS_2 which form representations of the $SU(1,1|2)$ superalgebra. The holographic dual of $\mathcal{N} = 2$ supergravity in $AdS_2 \times S^2$ is expected to be a one-dimensional SCFT with superconformal group $SU(1,1|2)$. We do not specify the underlying theory of quantum gravity beyond its symmetries in this chapter, besides the M-theory origin described above supergravity in $AdS_2 \times S^2$ could also originate from superstring theory in $AdS_2 \times S^2 \times T^6$. This is e.g. discussed in [149], where the authors consider embedding the $AdS_2 \times S^2$ background into 10d type IIA or IIB superstring theory. The $AdS_2 \times S^2 \times T^6$ background can be obtained either from a quarter-supersymmetric type IIB background describing four intersecting D3-branes [150] or from a type IIA background e.g. describing a superposition of three D4-branes and one D0-brane. As mentioned in the introduction, the AdS$_2$×S^2 background is interesting to study because it is the near-horizon geometry of extremal black holes in four dimensions. Furthermore, superconformal correlators in 1d are in many ways simpler than higher-dimensional analogues, such as the ones considered in the previous chapters. Therefore, they are an excellent playing ground for the study of various aspects of holographic correlators.

Firstly, we investigate a hidden conformal symmetry which was first discovered in AdS$_5$×S^5 in [41] and later investigated in AdS$_3$×S^3 in [124, 125]. In the supergravity approximation, four-point tree-level correlators of half-BPS operators dual to scalars in AdS$_5$ were found to correspond to 10d supergravity scattering amplitudes in the flat space limit. Consequently, it is conjectured that all supergravity tree-level four-point correlators exhibit a surprising 10d conformal symmetry and that correlators of any spherical harmonics can be generated from a single $SO(10,2)$-invariant function. This conjecture is also true for free theory and predictions can be made for loop corrections, however we will leave the discussion of loop corrections in 1d to [44]. The higher-dimensional conformal symmetry arises because four-point supergravity correlators in AdS$_5$×S^5 correspond to superamplitudes in 10d flat space which have a dimensionless coupling and are thus conformally invariant. Hence, this symmetry can only arise for specific four-point functions and furthermore only on conformally flat backgrounds, AdS$_q$×S^q. In this chapter we aim to understand this hidden conformal symmetry more systematically and we will discuss it in the context of free theory, supergravity and higher-derivative corrections in 1d. The symmetry is generally broken for higher-derivative corrections and this was recently confirmed in [14], where the authors studied the double-trace spectrum of half-BPS correlators described by tree-level string theory in AdS$_5$×S^5. We will discuss this breaking of the symmetry and investigate higher-derivative corrections in AdS$_2$×S^2. We show that an infinite set of correlators for specific charges can be reconstructed from the higher-dimensional conformal symmetry, while the symmetry is broken for general KK modes. We will briefly discuss the implications for α'^3 corrections in $\mathcal{N} = 4$ SYM. Even though the higher-dimensional conformal symmetry is only a symmetry of specific four-point functions in $\mathrm{AdS}_q\times\mathrm{S}^q$, it is a remarkable symmetry which implies powerful constraints on four-point functions and nicely complements other methods such as the effective action approach we proposed in chapter 3. We will consider a 4d analogue of this effective action in this chapter.

We propose a scalar effective action in four dimensions similar to the one in $\mathrm{AdS}_5\times\mathrm{S}^5$. We do not prove the existence of the effective field theory but we deduce half-BPS correlators in the supergravity approximation as well as higher-derivative corrections from it and analyse the results. Where possible we compare the results to those from different methods, such as the 4d conformal symmetry, where we find agreement in the supergravity sector. The coefficients in the effective action encode the underlying quantum theory and they are unfixed in our considerations. To deduce all half-BPS correlators of any R-symmetry charge we again evaluate generalised Witten diagrams in AdS×S. These manifestly four-dimensional Witten diagrams contain all spherical harmonics and one can obtain all higher-derivative corrections and in the 1d case, also the supergravity correlators. Note that from the effective action approach in $\mathrm{AdS}_2\times\mathrm{S}^2$ we can deduce supergravity and all higher-derivative corrections but neither free theory nor loop corrections. This is nicely complemented by the 4d hidden conformal symmetry which describes free theory and supergravity but is generically broken by higher-derivative corrections (it also makes predictions for loop corrections which we will not discuss in this thesis).

We start by reviewing the ten-dimensional conformal symmetry of supergravity correlators in $\mathrm{AdS}_5\times\mathrm{S}^5$ before going on to consider holographic correlators in $\mathrm{AdS}_2\times\mathrm{S}^2$.

5.1 Review: A Hidden Ten-Dimensional Conformal Symmetry

In [41] a surprising ten-dimensional conformal symmetry was found for tree-level supergravity correlators in $AdS_5 \times S^5$. Of course, the correlators have four-dimensional conformal symmetry and are dual to bulk scattering in AdS_5 with an infinite tower of spherical harmonics from compactifying on the sphere. However, it is not expected that they also have a ten-dimensional conformal symmetry manifestly including the modes on the sphere. This is very powerful since all higher-charge supergravity correlators can be packaged into a single generating function which is related to the flat space superamplitude in ten dimensions.

The first hint towards a higher-dimensional conformal symmetry was observed in [144], where the authors perform a conformal block analysis of four-point correlators in the large-c expansion in $\mathcal{N} = 4$ SYM described by tree-level supergravity in $AdS_5 \times S^5$. They consider half-BPS operators which are dual to the infinite tower of spherical harmonics on the five-sphere and make use of a formula for correlators of general charge in Mellin space, conjectured in [81]. As mentioned in subsection 4.2.1, when performing a conformal block decomposition of correlators there are usually many operators in the spectrum which are described by the same quantum numbers (scaling dimension and spin), thus there is a degeneracy. To resolve this degeneracy is called to solve the mixing problem and we will describe this in subsection 5.5.1 before we discuss it in detail for supergravity and higher-derivative corrections in sections 5.6 and 5.8 respectively. It involves constructing matrices of conformal block coefficients for correlators with different R-symmetry representation and the same quantum numbers and solving an eigenvalue problem. In [144] the mixing problem for the anomalous dimensions of double-trace operators in the conformal block expansion of half-BPS correlators described by tree-level supergravity was solved and a remarkably simple structure was observed:

$$\gamma_{4d}^{\text{sugra}} = \frac{-2M_t M_{t+l+1}}{(l_{10d}+1)_6} = \frac{\delta^{(8)}}{(l_{10d}+1)_6} \;, \tag{5.1.1}$$

where

$$M_t = (t-1)(t+a)(t+a+b+1)(t+2a+b+2) \;, \tag{5.1.2}$$

where a, b describe the R-symmetry representation of the double-trace operators which is given by the Dynkin labels $[a, b, a]$. The spin l and twist t are the quantum numbers of the exchanged operators, where the twist is defined as the scaling dimension minus the spin (see the discussion around (2.1.12)). Note that the anomalous dimensions are rational numbers and their simple structure can be interpreted in terms of an effective ten-dimensional spin

$$l_{10d} = l + a + 2(i+r) - 1 - \frac{1+(-1)^{a+l}}{2} \;, \tag{5.1.3}$$

where i, r label the degenerate operators (see [14] for details on the definition of these labels). The numerator in (5.1.1) can be understood as the eigenvalue $\delta^{(8)}$ of a special eighth-order differential operator $\Delta^{(8)}$ with the conformal blocks as eigenfunctions. The significance of this differential operator which is derived from conformal Casimirs will become clear later in this subsection.

It is very surprising that the anomalous dimensions after unmixing are rational numbers. They are eigenvalues of non-trivial matrices of OPE coefficients and there is no reason to expect them to be rational and to show such a simple structure. This was the first hint that tree-level supergravity in $\mathrm{AdS}_5\times\mathrm{S}^5$ exhibits more symmetries than at first expected. In [41] the authors observed a striking similarity of (5.1.1) to coefficients of the partial wave expansion of the ten-dimensional supergravity scattering amplitude in flat space. This suggests that the structure in (5.1.1) arises from the conformal flatness of $\mathrm{AdS}_5\times\mathrm{S}^5$ and a ten-dimensional conformal symmetry of the tree-level supergravity scattering amplitude. Let us look at this heuristic argument in more detail. The ten-dimensional IIB supergravity amplitude in flat space is given by

$$A_{10} = \frac{G_N \delta^{16}(Q)}{stu} \to G_N \frac{s^3}{tu} \tag{5.1.4}$$

where we have taken the dilaton component. Note that $G_N\delta^{16}(Q)$ is dimensionless, which is the origin of the 10d conformal symmetry of this amplitude. The partial wave expansion in general dimensions is given by

$$A_d(s,\cos\theta) = \frac{1}{s^{(d-4)/2}} \sum_l (l+1)_{d-4} C_l(\cos\theta) \mathcal{A}_l^d(s) \ , \tag{5.1.5}$$

where the scattering angle $\cos\theta = 1 + \frac{2t}{s}$ and the partial waves C_l can be expressed in terms of Gegenbauer polynomials (see e.g. [151]). Expressing the 10d amplitude in terms of s and θ, we find

$$A_{10}(s,\cos\theta) = \frac{4G_N s}{\sin^2\theta} \ . \tag{5.1.6}$$

The single power of s in the numerator indicates two-derivative interactions, as expected for supergravity. Comparing this to (5.1.5) implies that the 10d partial wave coefficients are

$$\mathcal{A}_l^{10}(s) \sim 1 + \frac{R^8}{c} \frac{s^4}{(l+1)_6} \ , \tag{5.1.7}$$

where the one is put in by hand and we used the fact that the Newton constant $G_N \sim R^8/c$ in 10d, where R is the AdS radius and c is the central charge. Now compare this to the $\mathcal{N}=4$ SYM anomalous dimensions after unmixing (5.1.1), where we note that γ^{sugra} contributes at order $1/c$:

$$e^{\frac{1}{c}\gamma_{4d}^{\text{sugra}}} \sim 1 + \frac{1}{c} \frac{\delta^{(8)}}{(l_{10d}+1)_6} \ . \tag{5.1.8}$$

The eighth-order Casimir encodes s^4 in the flat space limit and we see that the Pochhammers precisely match, which justifies the definition of the effective 10d spin in (5.1.3). The anomalous dimensions after unmixing can be interpreted in terms of this higher-dimensional spin and we will discuss this in sections 5.6 and 5.8 for supergravity and higher-derivative corrections in AdS$_2$×S^2 respectively. The above similarity is suggestive of a direct relation between the 4d CFT and flat space 10d supergravity. Importantly, this higher-dimensional conformal symmetry allows for all the correlators of external operators with any R-symmetry charge to be combined into one big 10d object which acts as a generating function. We will briefly review the 10d conformal symmetry for correlators in the supergravity approximation and in free theory in $\mathcal{N} = 4$ SYM below, followed by some general considerations including higher-derivative corrections.

One goal of this chapter is to understand the hidden higher-dimensional conformal symmetries more systematically and to approach this, we study a 4d conformal symmetry of holographic correlators in $\mathrm{AdS}_2\times\mathrm{S}^2$. In this background the analogue of the dimensionless coupling $G_N\delta^{16}(Q)$, which is the origin of the 10d conformal symmetry above, is the dimensionless coupling $G_N\delta^4(Q)$ of 4d $\mathcal{N} = 2$ supergravity amplitudes in flat space. We will consider supergravity and free theory correlators and show that they exhibit a 4d conformal symmetry. Moreover, we extend our discussions to higher-derivative corrections for which the symmetry is generally broken, and discuss how some of the higher-dimensional conformal structure could still be intact even for higher-derivative corrections.

5.1.1 General Considerations and Higher-Derivative Corrections

First, consider four-point tree-level correlators in the supergravity limit which, as explained above, correspond to a flat space 10d supergravity amplitude with a dimensionless coupling $G_N\delta^{16}(Q)$ and are thus conformally invariant. It is then conjectured that all half-BPS correlators can be combined into a 10d generating function containing all KK modes. This conjecture can be extended to free theory. The higher-dimensional conformal symmetry arises when the correlator corresponds to a conformally invariant flat space scattering amplitude with dimensionless coupling. For tree-level supergravity this is the case for four-point functions in $\mathrm{AdS}_5\times\mathrm{S}^5$ which is conformally flat. If the correlator does not correspond to such an amplitude, which is the case for free theory, it can be rescaled by acting with differential operators of appropriate powers on the correlator. This can be understood from dimensional analysis.

As we have seen, for supergravity the object which exhibits 10d conformal symmetry is the correlator itself. In the case of AdS$_5$×S^5, supergravity comes with a factor of $G_N \sim l_P^8$. Since free theory is proportional to l_P^0, to construct an object of the right dimensions, one needs to act with a specific eighth-order differential operator on the correlator to get something of order l_P^8. The right differential operator to consider is $\Delta^{(8)}$ which is the same operator whose eigenvalue appears in the anomalous dimensions in (5.1.1). This shows the significance of $\Delta^{(8)}$ which is derived from quadratic Casimir operators. Acting with $\Delta^{(8)}$ on free theory correlators gives something that looks exactly like free theory of superconformal descendants (see [41] and section 5.3 below). Indeed, the correlator of descendants is the object that plays the leading role in the hidden conformal symmetry. In the case of supergravity, which is of order $1/c$, to uncover the 10d conformal symmetry one has to divide the tree-level superamplitude by the dimensionless coupling $G_N\delta^{16}(Q)$ which according to the discussion around (5.1.7) can be identified with $\Delta^{(8)}/c$. Hence, in the case of supergravity the correlator itself is the relevant object which exhibits 10d conformal symmetry while for free theory it is the correlator of descendants. Moreover, this discussion can be extended to loop corrections. These corrections contribute at higher negative orders in c and to obtain the corresponding 10d invariant object one would act with the appropriate negative powers of $\Delta^{(8)}$ on the relevant correlators.

In this chapter we study the hidden conformal symmetry in the context of one-dimensional SCFTs with hidden four-dimensional conformal symmetry. This case is much simpler than the 4d/10d case and therefore a good starting point to understand this surprising symmetry in more detail. Analogue to the above discussions for 10d, 4d $\mathcal{N}=2$ supergravity amplitudes in flat space have a dimensionless coupling $G_N\delta^4(Q)$. Thus, the corresponding 1d tree-level supergravity correlators of four chiral primaries dual to KK modes in AdS_2 are expected to exhibit a 4d conformal symmetry. We will show that this is indeed the case in subsection 5.4.2. Supergravity in $AdS_2 \times S^2$ comes with a factor of $G_N \sim l_P^2$, this is how one sees whether the object has the right dimensions to be invariant under 4d conformal symmetry or not. Since 4d $\mathcal{N} = 2$ supergravity is connected to a scaleless coupling in the flat space limit, whenever a correlator comes with $\sim l_P^2$ in 1d it is connected to a flat space amplitude with dimensionless coupling and thus has 4d conformal symmetry. If it does not, a 4d conformal object can be constructed by acting with differential operators derived from conformal Casimirs on the correlator. Therefore, free theory, which goes like l_P^0, needs to be acted on by a second-order differential operator $\Delta^{(2)}$ (analogue to $\Delta^{(8)}$ above) to get something invariant under 4d conformal symmetry. As in the 10d case above, $\Delta^{(2)}$ acting on the free theory correlator gives exactly free theory of superconformal descendants. We will show this in detail in section 5.3. This can be extended to loop corrections, but we leave this discussion to [44].

Finally, we attempt to extend these considerations to higher-derivative corrections, which were not a part of the considerations in [41]. Generally, the higher-dimensional conformal symmetry is broken for higher-derivative corrections, which can be seen specifically when unmixing anomalous dimensions in the double-trace spectrum of the conformal block expansion of half-BPS correlators corresponding to higher-derivative corrections. These anomalous dimensions after unmixing still show a simple structure and many of them are rational numbers as was the case in the supergravity approximation. However, there are anomalous dimensions which contain square roots and this indicates a breaking of the higher-dimensional conformal symmetry. These anomalous dimensions were obtained for $\mathcal{N} = 4$ SYM in [13, 14] up to four derivatives and in [16] for higher derivatives. We will study them in detail in section 5.8 for the four-derivative corrections in 1d. It is reasonable to expect that some of the hidden conformal symmetry is still intact for higher-derivative corrections since, as mentioned before, many of the anomalous dimensions are still rational numbers. Remarkably, for an infinite set of correlators with specific charges, one can indeed construct an object that does have higher-dimensional conformal symmetry. The breaking of the symmetry for generic higher-derivative corrections can be anticipated because when an interaction vertex contains derivatives and they are reduced on the sphere, this gives several terms with different numbers of derivatives in AdS which cannot all be rescaled simultaneously. This will be discussed further in subsection 5.7.3.

Let us now consider the dimensional analysis for higher-derivative corrections in AdS$_2$×S^2. Recall that higher-derivative corrections contribute to the low-energy effective action as interaction vertices with $k+2$ derivatives[1], where supergravity has two derivatives and we restrict to quartic interactions in this thesis. The vertices with ∂^{k+2} go like $G_N l_P^k$, so to get an interaction term with the right scaling, i.e. $G_N \sim l_P^2$, one has to multiply it by l_P^{-k}. To achieve this, one acts with $(\Delta^{(2)})^{-\frac{k}{2}}$ on the correlators. In practice this means that we consider objects which can be lifted to four dimensions to generate all higher-charge correlators (which turn out to be 4d conformal blocks) and then act with differential operators of positive order k on them to reconstruct the higher-charge versions of the higher-derivative corrections. We analyse this for the case of four-derivative corrections in subsection 5.7.3 where it turns out that this can be constructed for all correlators of four external operators with the charges $\{p_i\} = pp11$, where 1 is the lowest mode on the two-sphere and p corresponds to a general KK mode, and crossing versions but is broken for general correlators. Note that in AdS$_2$×S^2 the zero-derivative term describes supergravity but in AdS$_5$×S^5 the analogue term describes the first higher-derivative correction to supergravity arising from string theory, which has six more derivatives than supergravity itself.

To summarise, the higher-dimensional conformal symmetry can be realised for specific

[1]Note that we often count the number of derivatives not necessarily compared to supergravity but compared to the correlator dual to the zero-derivative bulk scalar interaction ϕ^4. In AdS$_2$×S^2 this is indeed supergravity but in higher dimensions, ϕ^4 corresponds to the term $\mathcal{R}^4$ in the low-energy effective action (2.3.18) which describes the first higher-derivative correction and has six more derivatives than supergravity. This correction is referred to as the zero-derivative correction and subleading corrections are interactions with $k-6$ derivatives acting on $\mathcal{R}^4$, where $k+2$ is the total number of derivatives in the interaction.

four-point correlators in conformally flat backgrounds. It is not just a symmetry in the supergravity case but can be understood from constructing interactions with the required scaling by acting on the correlator with appropriate differential operators. However, for higher-derivative corrections the symmetry breaks in general due to covariant derivatives acting on the interaction vertices. Nevertheless, there is still an infinite number of correlators with specific R-symmetry charges which can be constructed from the hidden conformal symmetry. Note that on the other hand, all higher-derivative corrections can be deduced from the 4d effective action, see subsection 5.7.2.

5.2 General Setup

In this chapter we study four-point correlators of half-BPS operators in a large central charge c expansion in 1d which correspond to tree-level quantum gravity scattering in the bulk. In the large-c limit, the half-BPS operators are dual to bulk scalars in AdS_2 described by an infinite tower of KK modes on the two-sphere. Furthermore, we consider the theory in the low-energy approximation, which corresponds to 4d $\mathcal{N} = 2$ supergravity, where we also consider subleading contributions in the form of higher-derivative corrections. Since we do not specify the bulk theory nor the boundary theory beyond its symmetries we formulate the low-energy approximation in terms of an expansion in a small parameter a, where $a \to 0$ corresponds to the strict low-energy limit. Considering this double-expansion, free theory correlators are proportional to $a^0 c^0$, supergravity correlators contribute at order $a^0 c^{-1}$ and the higher-derivative corrections to the supergravity effective action contribute at orders $c^{-1} a^{k-1}$, where $2k$ is the number of derivatives in the bulk scalar interaction and $k = 2, 3, \ldots$

5.2.1 Correlators in 1d and the Differential Operator $\Delta^{(2)}$

Let us start by looking at the general structure of four-point functions in 1d CFTs. The local operators are defined on a line and are invariant under the $SO(2,1)$ conformal group. This symmetry allows us to fix three points on the line, therefore there is only one free real parameter left, describing the position of the fourth point. As a consequence, the usual conformal cross-ratios u, v are not independent and there is only one cross-ratio, x. Starting from the usual conformal cross-ratios:

$$u = z\bar{z} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2} \ , \quad v = (1-z)(1-\bar{z}) = \frac{x_{23}^2 x_{14}^2}{x_{13}^2 x_{24}^2} \ , \tag{5.2.1}$$

where $x_{ij} = x_i - x_j$ and defining $z = \bar{z} = x$, the 1d cross-ratio x is

$$x = \frac{x_{12}x_{34}}{x_{13}x_{24}} \ , \tag{5.2.2}$$

and the usual u, v are given in terms of x by

$$u = x^2 \ , \quad v = (1-x)^2 \ . \tag{5.2.3}$$

Hence, 1d correlators correspond to a holomorphic limit.

To discuss the half-BPS correlators of interest, we follow the formalism of [49] with $m = n = 1$. The relevant superspace is the super Grassmannian $Gr(1|1, 2|2)$ of $(1|1)$ planes in $(2|2)$ dimensions. Coordinates on this Grassmannian can be given as

$$X_i = \begin{pmatrix} x_i & \theta_i \\ \bar{\theta}_i & y_i \end{pmatrix} \ , \tag{5.2.4}$$

where x_i is the 1d spacetime coordinate, y_i is a (complex) internal coordinate used to deal with the $SU(2)$ structure and $\theta_i, \bar{\theta}_i$ are Grassmann odd coordinates. Since we will be dealing with correlators of four operators on this space, we added a subscript $i = 1, 2, 3, 4$ to denote the particle number.

The field content of 4d $\mathcal{N} = 2$ supergravity is one graviton, six gravitinos, 15 vector and 10 (complex) hypermultiplets [147]. Each of these multiplets contains four bosonic and four fermionic degrees of freedom and compactifying on the two-sphere gives an infinite tower of KK modes for each multiplet. The 4d $\mathcal{N}=2$ hypermultiplet is the simplest to understand from a higher-dimensional perspective. This multiplet on an $\mathrm{AdS}_2\times\mathrm{S}^2$ background is AdS/CFT dual to an infinite tower of fermionic 1d half-BPS multiples [147]. These are fermionic superfields of scaling dimension Δ and $SU(2)$ representation Δ. They can be written on the above super Grassmannian and decompose into the following component fields

$$\Psi_\Delta(x_i,\theta_i,\bar\theta_i,y_i)=\psi_\Delta+\theta_i\phi_{\Delta+\frac12}+\bar\theta_i\bar\phi_{\Delta+\frac12}+\theta_i\bar\theta_i\lambda_{\Delta+1}\ ,\qquad \Delta=\tfrac12,\tfrac32,\tfrac52,\dots\ , \tag{5.2.5}$$

where the subscript denotes the dimension of the field. The field ψ_Δ has $SU(2)$ representation Δ, ϕ_Δ has $SU(2)$ representation $\Delta-1$, λ_Δ has $SU(2)$ representation $\Delta-2$. For the special multiplet $\Psi_{1/2}$ the descendant $\lambda_{3/2}$ is absent. Expanding in the y_i coordinates manifests the $SU(2)$ indices for these representations

$$\begin{aligned}
\psi_\Delta(x_i,y_i)&=\psi_{I_1\dots I_{2\Delta}}(x_i)y_i^{I_1}\dots y_i^{I_{2\Delta}}\ , &&\Delta=\tfrac12,\ \tfrac32,\ \dots\ ,\\
\phi_\Delta(x_i,y_i)&=\phi_{I_1\dots I_{2\Delta-2}}(x_i)y_i^{I_1}\dots y_i^{I_{2\Delta-2}}\ , &&\Delta=1,2,\dots\ ,\\
\lambda_\Delta(x_i,y_i)&=\lambda_{I_1\dots I_{2\Delta-4}}(x_i)y_i^{I_1}\dots y_i^{I_{2\Delta-4}}\ , &&\Delta=\left(\tfrac32\right),\tfrac52,\dots\ ,
\end{aligned} \tag{5.2.6}$$

where

$$y_i^I=(1,y_i)\ . \tag{5.2.7}$$

We are interested in the four-point functions of half-BPS superconformal primaries:

$$\langle\psi_{\Delta_1}(x_1,y_1)\psi_{\Delta_2}(x_2,y_2)\psi_{\Delta_3}(x_3,y_3)\psi_{\Delta_4}(x_4,y_4)\rangle\ . \tag{5.2.8}$$

However, it will also be useful to consider the four-point function of superdescendants

$$\langle\phi_{\Delta_1}(x_1,y_1)\phi_{\Delta_2}(x_2,y_2)\bar\phi_{\Delta_3}(x_3,y_3)\bar\phi_{\Delta_4}(x_4,y_4)\rangle\ . \tag{5.2.9}$$

Indeed it is the latter which plays the leading role in the hidden higher-dimensional conformal symmetry as described in section 5.1. Both correlators transform covariantly under the bosonic subgroup of the superconformal group $SU(1,1)\times SU(2)\subset$

$SU(1,1|2)$ which means they have the form

$$\begin{aligned}
&\langle\psi_{\Delta_1}(x_1,y_1)\psi_{\Delta_2}(x_2,y_2)\psi_{\Delta_3}(x_3,y_3)\psi_{\Delta_4}(x_4,y_4)\rangle = P_{\Delta_i} G_{\psi_{\Delta_i}}(x,y)\ ,\\
&\langle\phi_{\Delta_1}(x_1,y_1)\phi_{\Delta_2}(x_2,y_2)\bar{\phi}_{\Delta_3}(x_3,y_3)\bar{\phi}_{\Delta_4}(x_4,y_4)\rangle = \frac{P_{\Delta_i-\frac{1}{2}}}{x_{12}x_{34}y_{12}y_{34}} G_{\phi_{\Delta_i}}(x,y)\ ,
\end{aligned} \tag{5.2.10}$$

where the prefactor P_{Δ_i} is

$$\begin{aligned}
&P_{\Delta_i} = g_{12}^{\Delta_1+\Delta_2} g_{34}^{\Delta_3+\Delta_4}\left(\frac{g_{24}}{g_{14}}\right)^{\Delta_{21}}\left(\frac{g_{14}}{g_{13}}\right)^{\Delta_{43}}\ ,\\
&\text{with}\quad g_{ij} = \frac{y_{ij}}{x_{ij}}\ ,\quad y_{ij} = y_i - y_j\ ,
\end{aligned} \tag{5.2.11}$$

and $\Delta_{ij} = \Delta_i - \Delta_j$. This prefactor by itself transforms correctly as a $\langle\psi_{\Delta_1}\psi_{\Delta_2}\psi_{\Delta_3}\psi_{\Delta_4}\rangle$ correlator under the bosonic subgroup, leaving a remaining function $G_\psi(x,y)$ which is conformally invariant. Thus, it is a function of the conformal cross-ratios x, y, the spacetime and internal cross-ratios in one dimension:

$$x = \frac{x_{12}x_{34}}{x_{13}x_{24}}\ ,\qquad y = \frac{y_{12}y_{34}}{y_{13}y_{24}}\ . \tag{5.2.12}$$

The spacetime cross-ratio was discussed above, around (5.2.2), and equivalently for the internal cross-ratios, there is only a single independent one, y.

The two component correlators (5.2.10) are related to each other by superconformal symmetry. Indeed both can be obtained from the same supercorrelator $\langle\Psi_{\Delta_1}\Psi_{\Delta_2}\Psi_{\Delta_3}\Psi_{\Delta_4}\rangle$ as

$$\begin{aligned}
&\langle\psi_{\Delta_1}\psi_{\Delta_2}\psi_{\Delta_3}\psi_{\Delta_4}\rangle = \langle\Psi_{\Delta_1}\Psi_{\Delta_2}\Psi_{\Delta_3}\Psi_{\Delta_4}\rangle|_{\theta_i=\bar{\theta}_i=0}\ ,\\
&\langle\phi_{\Delta_1+\frac{1}{2}}\phi_{\Delta_2+\frac{1}{2}}\bar{\phi}_{\Delta_3+\frac{1}{2}}\bar{\phi}_{\Delta_4+\frac{1}{2}}\rangle = \partial_{\theta_1}\partial_{\theta_2}\partial_{\bar{\theta}_3}\partial_{\bar{\theta}_4}\langle\Psi_{\Delta_1}\Psi_{\Delta_2}\Psi_{\Delta_3}\Psi_{\Delta_4}\rangle|_{\theta_i=\bar{\theta}_i=0}\ .
\end{aligned} \tag{5.2.13}$$

Superconformal invariance implies that the full supercorrelator only depends non-trivially on two bosonic variables. This then implies that the above relations reduce to the following direct differential relation between the two component correlators

$$\langle\phi_{\Delta_1+\frac{1}{2}}\phi_{\Delta_2+\frac{1}{2}}\bar{\phi}_{\Delta_3+\frac{1}{2}}\bar{\phi}_{\Delta_4+\frac{1}{2}}\rangle = \mathcal{I}^{-1}\mathcal{C}_{1,2}^{SU(1,1|2)}\langle\psi_{\Delta_1}\psi_{\Delta_2}\psi_{\Delta_3}\psi_{\Delta_4}\rangle\ , \tag{5.2.14}$$

where we leave the derivation of this result to appendix H. Here $\mathcal{C}_{1,2}^{SU(1,1|2)}$ denotes the superconformal quadratic Casimir given in terms of the second-order differential operator $\Delta^{(2)}$ by

$$\begin{aligned}
\mathcal{C}_{1,2}^{SU(1,1|2)} &= P_{p_i} \times \frac{x-y}{xy}\Delta^{(2)}\frac{xy}{x-y}P_{p_i}^{-1} \ , \\
\Delta^{(2)} &= \mathcal{D}_x^{(p_{12},p_{43})} - \mathcal{D}_y^{(-p_{12},-p_{43})} \ , \\
\mathcal{D}_x^{(p_{12},p_{43})} &= x^2\partial_x(1-x)\partial_x + (p_{12}+p_{43})x^2\partial_x - p_{12}p_{43}x \ ,
\end{aligned} \tag{5.2.15}$$

where $p_{ij} = p_i - p_j$. For a derivation of the quadratic Casimir see appendix H. Furthermore, $\mathcal{I}$ is the special polynomial

$$\mathcal{I} = x_{12}x_{34}y_{13}y_{24} - y_{12}y_{34}x_{13}x_{24} \ , \tag{5.2.16}$$

which is completely antisymmetric under crossing symmetry.

The second-order differential operator $\Delta^{(2)}$ is the analogue of $\Delta^{(8)}$ playing an important role in the 10d conformal symmetry of $\mathcal{N}=4$ SYM correlators. We have discussed the significance of $\Delta^{(2)}$ in subsection 5.1.1. Recall that the object that has 4d conformal symmetry in $\mathrm{AdS}_2\times\mathrm{S}^2$ in the supergravity limit is the half-BPS correlator itself, whereas for free theory acting with $\Delta^{(2)}$ on the correlator yields the 10d conformally invariant object which is the correlator of descendant bosonic scalars ϕ. In 1d this is related to the correlator of chiral primaries ψ by acting on the latter with $\Delta^{(2)}$, see (5.2.14).

Throughout this chapter we will consider four-point functions of half-BPS operators and in the free theory case also the correlators of descendants. The half-BPS operators are fermionic primaries ψ_Δ with dimensions and $SU(2)$ charges $\Delta = \frac{1}{2}, \frac{3}{2}, \ldots$ However, it will turn out to be most useful to label the correlators in terms of the bosonic descendant operators $\phi_{\Delta+1/2}$ with dimensions $1, 2, \ldots$ Therefore, we define the half-BPS operators as

$$\mathcal{O}_p = (-1)^{\frac{p}{2}}\frac{\psi_{(2p-1)/2}}{\sqrt{2p-1}} \ , \tag{5.2.17}$$

where $p = 1, 2, \ldots$ In this convention the half-BPS operators have dimension and $SU(2)$ representation $(2p-1)/2$. Furthermore, it is useful to introduce a normalisation $(-1)^{p/2}\,(2p-1)^{-1/2}$ inspired by the higher-dimensional conformal symmetry [41]. It will become clear why this is useful in the discussions below (5.3.12) and (5.3.20). We then denote the descendants of $\mathcal{O}_p$ by $L_p, \bar{L}_p$:

$$\mathbf{O}_p(x,y) = \mathcal{O}_p(x,y) + \theta L_p(x,y) + \bar{\theta}\bar{L}_p(x,y) \ , \tag{5.2.18}$$

where L_p has dimension p and $SU(2)$ charge $p-1$.

5.2.2 Conformal Blocks

In this subsection we discuss the conformal blocks in 1d. We will consider four-point functions of half-BPS operators $\mathcal{O}_{p_i}$ (5.2.17) with $i = 1, \dots, 4$. In the OPE of two such half-BPS operators one finds also long multiplets. These can all be represented as $\mathcal{O}_{\Delta,p}$, where Δ is the dilatation weight and p is the $SU(2)$ representation of the exchanged operators.

The four-point function can be expanded in superconformal blocks $B_{\Delta,p}(x,y)$ as

$$\begin{aligned}
&\langle \mathcal{O}_{p_1}(x_1,y_1)\mathcal{O}_{p_2}(x_2,y_2)\mathcal{O}_{p_3}(x_3,y_3)\mathcal{O}_{p_4}(x_4,y_4)\rangle \\
&= \sum_{\Delta=1}^{\infty}\sum_{p=0}^{p_1+p_2-2} A^{p_1p_2p_3p_4}_{\Delta,p}\, g_{12}^{p_1+p_2-1} g_{34}^{p_3+p_4-1}\left(\frac{g_{24}}{g_{14}}\right)^{p_{21}}\left(\frac{g_{14}}{g_{13}}\right)^{p_{43}} B_{\Delta,p,p_{12},p_{34}}(x,y) \ ,
\end{aligned} \tag{5.2.19}$$

where the coefficient $A_{\Delta,p}$ is given in terms of a sum of squares of OPE coefficients as

$$A^{p_1p_2p_3p_4}_{\Delta,p} = \sum_{\mathcal{O}^{\Delta,p},\tilde{\mathcal{O}}^{\Delta,p}} C^{\mathcal{O}}_{p_1p_2} C^{\tilde{\mathcal{O}}}_{p_3p_4} C_{\mathcal{O}\tilde{\mathcal{O}}} \ . \tag{5.2.20}$$

Further

$$\tilde{g}_{ij} = \frac{y_{ij}}{|x_{ij}|} \tag{5.2.21}$$

is the two-point function of charge 1 half-BPS operators. Note that the two-point function $\tilde{g}_{ij}$ is antisymmetric under exchange i and j, as expected for fermions. It will also be convenient to define the bosonic two-point function

$$g_{ij} = \frac{y_{ij}}{x_{ij}} \ . \tag{5.2.22}$$

The superconformal blocks are derived from a general formula for superconformal blocks with symmetry group $SU(m,m|2n)$ in [49]. In our case $m = n = 1$ and the blocks for long multiplets are given by

$$\begin{aligned} B^{\text{long}}_{\Delta,p,p_{12},p_{34}}(x,y) = &- (x-y)(-x)^{\Delta}{}_2F_1(\Delta+1-p_{12},\Delta+1+p_{34};2\Delta+2;x) \\ &\times y^{-p-1}{}_2F_1(-p+p_{12},-p-p_{34};-2p;y) \ . \end{aligned} \tag{5.2.23}$$

For half-BPS multiplets $\Delta = p$ and the blocks are

$$\begin{aligned} B^{\text{half-BPS}}_{p,p_{12},p_{34}}(x,y) = \left(-\frac{x}{y}\right)^p \Big(1 + (x-y)\sum_{i=1}^{k}&[x^{-i}\,{}_2F_1(p+1-i-p_{12},p+1-i+p_{34};2p+2-2i;x)] \\ &\times y^{i-1}\,{}_2F_1(i-p+p_{12},i-p-p_{34};2i-2p;y)\Big) \ , \end{aligned} \tag{5.2.24}$$

where $k = \min(p-p_{12},p+p_{34})$ and the square bracket means we take the regular piece as $x \to 0$.

It is interesting to note that the block of an operator with dimension Δ and $SU(2)$ representation p contributes as follows to the four-point function

$$\begin{aligned} &\langle \mathcal{O}_{p_1}(x_1,y_1)\mathcal{O}_{p_2}(x_2,y_2)\mathcal{O}_{p_3}(x_3,y_3)\mathcal{O}_{p_4}(x_4,y_4)\rangle \\ &\sim \sum_{\Delta,p} A^{p_1p_2p_3p_4}_{\Delta,p}\, g_{12}^{p_1+p_2-1} g_{34}^{p_3+p_4-1} \left(\frac{g_{24}}{g_{14}}\right)^{p_{21}} \left(\frac{g_{14}}{g_{13}}\right)^{p_{43}} (-x)^{\Delta} y^{-p}\left(1+\mathcal{O}(x,y)\right) \ , \end{aligned} \tag{5.2.25}$$

where the higher orders in x,y correspond to spacetime or $SU(2)$ descendants.

5.2.3 4d Scalar Effective Action

In addition to studying the 4d hidden conformal symmetry in this chapter, we also obtain the correlators described by tree-level supergravity and higher-derivative corrections in $\text{AdS}_2\times\text{S}^2$ from a higher-dimensional effective action analogous to the 10d effective action proposed in chapter 3. There a 10d scalar effective action was introduced which generates all half-BPS four-point correlators in $\mathcal{N} = 4$ SYM described by tree-level string theory in AdS$_5$×S^5 [42].

For the present case, we consider the following four-field terms in an effective superpotential for supergravity linearised about flat 4d space, with the expansion parameter a (where $a \to 0$ corresponds to the low-energy limit):

$$V^{\rm flat}(\phi) = \tfrac{1}{4!}\Big(A\,\phi^4 + B\,a\,(\partial\phi.\partial\phi)^2 + D\,a^2\,(\partial\phi.\partial\phi)(\partial_\mu\partial_\nu\phi\partial^\mu\partial^\nu\phi) + \dots\Big)\,, \qquad (5.2.26)$$

where in this chapter we mainly focus on the first two terms. In subsection 5.4.2 we will see that in AdS$_2$×S^2 the zero-derivative term describes supergravity and thus the ϕ^4 term in the effective action is of order a^0. The fact that the supergravity approximation is included in the effective action is very powerful since we can compute all half-BPS supergravity correlators from the 4d effective action alongside all higher-derivative corrections, whereas in AdS$_5$×S^5 only higher-derivative corrections can be obtained from the effective action. We organise the expansion in a such that $2k$-derivative terms contribute at order a^{k-1} starting from $k = 2$, where $k = 0$ terms are excluded since they describe supergravity and not higher-derivative corrections.

To uplift the effective superpotential to an AdS$_2$×S^2 background the flat derivatives are replaced with covariant AdS×S derivatives, see (2.3.6). It is important to note that this uplift is not unique as we have seen in the AdS$_5$×S^5 case. There are ambiguities because the covariant derivatives no longer commute with each other and there could be contributions from interactions with a lower number of derivatives, compensated by the AdS radius R, which would vanish in the flat space limit. So to $\mathcal{O}(a^2)$ the superpotential translates to

$$\begin{aligned} V^{\rm AdS\times S}(\phi) = \tfrac{1}{4!}\Big[&A\phi^4 + a\Big(3B(\nabla\phi.\nabla\phi)^2 + 6C\nabla^2\nabla_\mu\phi\nabla^\mu\phi\phi^2\Big) \\ &+ a^2\Big(6D(\nabla\phi.\nabla\phi)(\nabla_\mu\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi) + 6E\nabla_\mu\nabla^2\nabla_\nu\phi\nabla^\mu\nabla^\nu\phi\phi^2\Big) + \dots\Big]\,. \end{aligned} \qquad (5.2.27)$$

As in the AdS$_5$×S^5 case (3.1.18), the coefficients $B, C, \dots$ themselves can have an expansion in the dimensionless parameter a/R^2 where R is the radius of AdS (or S). So whereas in flat space $2k$-derivative terms only contribute at order a^{k-1}, in AdS×S,

$2k$-derivative terms can contribute at a^{k-1} and all higher orders in principle. These expansions are

$$\begin{aligned} B(a) &= B_0 + B_1 \tfrac{a}{R^2} + \dots \\ C(a) &= C_0 + C_1 \tfrac{a}{R^2} + \dots \\ &\dots \end{aligned} \tag{5.2.28}$$

For simplicity, we will set $R = 1$ from now on throughout this chapter, but it will be understood that these higher-order terms vanish in the flat space limit. The supergravity coefficient A does not allow for a a/R^2 expansion because of superconformal symmetry (see also subsection 3.1.3).

5.3 Free Theory

In this section we will derive the 1d free theory correlators. As explained in subsection 5.1.1, the object that has 4d conformal symmetry in free theory is the correlator of descendants which is obtained by acting on the half-BPS correlator with the second-order differential operator $\Delta^{(2)}$. One can then construct a four-dimensional generating function which contains all 1d half-BPS correlators in free theory.

First, write the correlators according to (5.2.10) in terms of a prefactor and a function only depending on the conformal cross-ratios (5.2.12) as

$$\langle \mathcal{O}_{p_1}(x_1,y_1)\mathcal{O}_{p_2}(x_2,y_2)\mathcal{O}_{p_3}(x_3,y_3)\mathcal{O}_{p_4}(x_4,y_4)\rangle = P_{p_i} \times G_{p_1p_2p_3p_4}(x,y) \ , \tag{5.3.1}$$

where the prefactor is given by

$$P_{p_i} = g_{12}^{p_1+p_2-1} g_{34}^{p_3+p_4-1} \left(\frac{g_{24}}{g_{14}}\right)^{p_{21}} \left(\frac{g_{14}}{g_{13}}\right)^{p_{43}} \ . \tag{5.3.2}$$

It is useful to decompose the correlator in such a way that the solution to the superconformal Ward identities becomes straightforward. The superconformal Ward identities in 1d are $\partial_x G(x,y)|_{x=y} = 0$, i.e. simply that $G(x,x)$ is independent of x.

This has the straightforward solution

$$G_{p_i}(x,y) = k_{p_i}(x,y) + \frac{x-y}{xy} H_{p_i}(x,y) \;,$$
$$\text{where} \;\; k_{p_i}(x,y) = \kappa \left(\frac{x}{y}\right)^{p_{43}} \;, \tag{5.3.3}$$

with $H(x,x)$ finite and $k_{p_i}(x,y)$ is defined such that $\mathcal{C}_{1,2}^{SU(1,1|2)}(P_{p_i} k_{p_i}) = 0$ with a constant κ and the superconformal Casimir is given in (5.2.15).

In this decomposition the Casimir only sees the interacting piece of the correlator, the so-called reduced correlator H_{p_i}:

$$\mathcal{C}_{1,2}^{SU(1,1|2)} \langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle = P_{p_i} \frac{x-y}{xy} \Delta^{(2)} H_{p_i}(x,y) = P_{p_i} \frac{x-y}{xy} \tilde{H}_{p_i}(x,y) \;, \tag{5.3.4}$$

where we define

$$\tilde{H}_{p_i}(x,y) = \Delta^{(2)} H_{p_i}(x,y) \;. \tag{5.3.5}$$

To construct an object which has the right dimensions to transform like a correlator which has 4d conformal symmetry, we define the correlator of descendants in terms of $\tilde{H}$ like

$$\langle L_{p_1} L_{p_2} \bar{L}_{p_3} \bar{L}_{p_4} \rangle = \frac{P_{p_i}}{x_{12} x_{34} y_{12} y_{34}} \tilde{H}_{p_i}(x,y) \;, \tag{5.3.6}$$

where L_p is the descendant of $\mathcal{O}_p$ with dimension p and $SU(2)$ charge $p-1$, as described in (5.2.18). Comparing equations (5.3.4) and (5.3.6) we get

$$\langle L_{p_1} L_{p_2} \bar{L}_{p_3} \bar{L}_{p_4} \rangle = \mathcal{I}^{-1} \mathcal{C}_{1,2}^{SU(1,1|2)} \langle \mathcal{O}_{p_1} \mathcal{O}_{p_2} \mathcal{O}_{p_3} \mathcal{O}_{p_4} \rangle \;, \tag{5.3.7}$$

which agrees with the relation in (5.2.14).

5.3.1 4d Conformal Symmetry

Let us now construct the four-dimensional generating function for free theory correlators, starting with the correlators of half-BPS operators of equal charge. The free theory correlator for equal charge operators is given in terms of the fermionic two-point functions $\tilde{g}_{ij}$ (5.2.21) as

$$\begin{aligned}
&\langle \mathcal{O}_p(x_1,y_1)\mathcal{O}_p(x_2,y_2)\mathcal{O}_p(x_3,y_3)\mathcal{O}_p(x_4,y_4)\rangle|_{c^0} \\
&= \frac{1}{\mathcal{N}^{(0)}_{pppp}} \left[(\tilde{g}_{12}\tilde{g}_{34})^{2p-1} - (\tilde{g}_{13}\tilde{g}_{24})^{2p-1} + (\tilde{g}_{14}\tilde{g}_{23})^{2p-1} \right] \\
&= \frac{1}{\mathcal{N}^{(0)}_{pppp}} (\tilde{g}_{12}\tilde{g}_{34})^{2p-1} \left[1 + \left(\frac{x}{y}\right)^{2p-1} \left((-\mathrm{sgn} x) + \left(\frac{1-y}{1-x}\right)^{2p-1} \mathrm{sgn}\left[x\left(1-x\right)\right] \right) \right] \,, \quad (5.3.8)
\end{aligned}$$

where the normalisation

$$\mathcal{N}^{(0)}_{p_i} = (-1)^{\Sigma_p} \sqrt{(2p_1-1)(2p_2-1)(2p_3-1)(2p_4-1)} \quad (5.3.9)$$

comes from the normalisation of the half-BPS operators in (5.2.17) and recall that $\Sigma_p = \frac{1}{2}(p_1+p_2+p_3+p_4)$. Choosing $0 \le x \le 1$, we then find that

$$G^{(0)}_{pppp}(x,y) = \frac{1}{\mathcal{N}^{(0)}_{pppp}} \left[1 + \left(\frac{x}{y}\right)^{2p-1} \left(-1 + \left(\frac{1-y}{1-x}\right)^{2p-1} \right) \right] \,. \quad (5.3.10)$$

This decomposes into

$$k^{(0)}_{pppp}(x,y) = \frac{1}{\mathcal{N}^{(0)}_{pppp}} \,, \quad H^{(0)}_{pppp}(x,y) = \frac{1}{\mathcal{N}^{(0)}_{pppp}} \frac{x\,y}{x-y} \left(\frac{x}{y}\right)^{2p-1} \left[-1 + \left(\frac{1-y}{1-x}\right)^{2p-1} \right] \,. \quad (5.3.11)$$

The action of $\Delta^{(2)}$ then yields

$$\tilde{H}^{(0)}_{pppp} = \frac{x^{2p}}{y^{2p-2}} \left(1 + \frac{(1-y)^{2p-2}}{(1-x)^{2p}} \right) \,, \quad (5.3.12)$$

which indeed looks like the free theory of dimension p charge $p-1$ operators you would get as the descendant. Acting with $\Delta^{(2)}$ on H gives a factor of $(2p-1)^2$ which gets cancelled by the normalisation $1/\mathcal{N}^{(0)}_{pppp}$. This is how the normalisation in (5.2.17) was chosen, up to the sign which will become clear shortly.

Note that for the analysis of free theory in terms of the 4d conformal symmetry this decomposition is not necessary since acting with $\Delta^{(2)} \frac{xy}{x-y}$ on $G(x,y)$ only sees the interacting piece H anyway. Nevertheless, the discussion below can be written in terms of H in a simple way. Furthermore, the decomposition is useful as it will allow us to consider the $1/c$ expansion of H in (5.5.2) later on.

Finally, consider the correlator of descendants from which we construct the generating function with 4d conformal symmetry. From (5.3.6) and (5.3.12) we get

$$\langle L_p L_p \bar{L}_p \bar{L}_p \rangle_{c^0} = g_{14}^{2p-2} g_{23}^{2p-2} \frac{1}{x_{14}^2 x_{23}^2} + g_{13}^{2p-2} g_{24}^{2p-2} \frac{1}{x_{13}^2 x_{24}^2} \ . \tag{5.3.13}$$

Note that the correlator of bosonic descendants is given in terms of bosonic two-point functions g_{ij} (5.2.22). Take the lowest case $p = 1$ where we have

$$\langle L_1 L_1 \bar{L}_1 \bar{L}_1 \rangle_{c^0} = \frac{1}{x_{12}^2 x_{34}^2} \tilde{H}_{1111}^{(0)} = \frac{1}{x_{14}^2 x_{23}^2} + \frac{1}{x_{13}^2 x_{24}^2} \ . \tag{5.3.14}$$

To construct the generating function, now lift this to four dimensions by replacing $x_{ij}^2 \to x_{ij}^2 + y_{ij}^2 = x_{ij}^2(1 + g_{ij}^2)$ which gives

$$\langle L L \bar{L} \bar{L} \rangle_{c^0}^{4d} = \frac{1}{x_{14}^2 x_{23}^2} \frac{1}{(1+g_{14}^2)(1+g_{23}^2)} + \frac{1}{x_{13}^2 x_{24}^2} \frac{1}{(1+g_{13}^2)(1+g_{24}^2)} \ . \tag{5.3.15}$$

This is a 4d object which contains all 1d free theory correlators. To obtain the specific correlators Taylor expand in g_{ij} and take the appropriate coefficients. To check this for the case of equal charges expand out $(1+g^2)^{-1} = 1 - g^2 + g^4 - g^6 + \ldots$ and keep the two terms proportional to $g_{14}^{2p-2} g_{23}^{2p-2}$ and $g_{13}^{2p-2} g_{24}^{2p-2}$, this indeed reproduces the prediction in (5.3.13).

Next, let us investigate correlators of unequal charge and check that they can be predicted from the generating function (5.3.15), which is the main result of this section. First, consider $\{p_i\} = pqpq$, where $p > q$. These correlators will be important in later sections, together with the equal charge ones, when performing conformal block analyses of correlators in the supergravity approximation and at the order of higher-derivative corrections. The free theory correlator is given by

$$\langle \mathcal{O}_p(x_1,y_1) \mathcal{O}_q(x_2,y_2) \mathcal{O}_p(x_3,y_3) \mathcal{O}_q(x_4,y_4) \rangle_{c^0} = -\frac{1}{\mathcal{N}_{pqpq}^{(0)}} \tilde{g}_{13}^{2p-1} \tilde{g}_{24}^{2q-1} \ , \tag{5.3.16}$$

$$\text{and } G_{pqpq}^{(0)}(x,y) = -\frac{1}{\mathcal{N}_{pqpq}^{(0)}} \left(\frac{x}{y}\right)^{p+q-1} \ , \tag{5.3.17}$$

where the operators have dimensions and charges $(2p-1)/2$, $(2q-1)/2$ and we decompose $G^{(0)}_{pqpq}$ into

$$k^{(0)}_{pqpq}(x,y) = -\frac{1}{\mathcal{N}^{(0)}_{pqpq}}\left(\frac{x}{y}\right)^{q-p} \quad , \quad H^{(0)}_{pqpq}(x,y) = \frac{1}{\mathcal{N}^{(0)}_{pqpq}}\frac{x\,y}{x-y}\left(\frac{x}{y}\right)^{q-p}\left[1-\left(\frac{x}{y}\right)^{2p-1}\right] . \tag{5.3.18}$$

Acting on this with $\Delta^{(2)}$ gives

$$\tilde{H}^{(0)}_{pqpq}(x,y) = (-1)^{p+q}\,\frac{x^{p+q}}{y^{p+q-2}} \ , \tag{5.3.19}$$

which does look like free theory with dimensions p, q and charges $p-1$, $q-1$ as we would expect for descendants. Using (5.3.6) yields

$$\langle L_p L_q \bar{L}_p \bar{L}_q\rangle_{c^0} = (-1)^{p+q}\,g_{13}^{2p-2}g_{24}^{2q-2}\frac{1}{x_{13}^2 x_{24}^2} \ , \tag{5.3.20}$$

which agrees with the term proportional to $g_{13}^{2p-2}g_{24}^{2q-2}$ in the expansion of the generating function (5.3.15). Thus the choice of signs in our normalisation of the half-BPS operators (5.2.17) was inspired by the realisation of the 4d conformal symmetry in free theory, similar to the rest of the normalisation.

Consider one more correlator of mixed charges $\{p_i\} = pqqp$, where $p > q$ and the free theory correlator is given by

$$\begin{aligned}\langle \mathcal{O}_p(x_1,y_1)\mathcal{O}_q(x_2,y_2)\mathcal{O}_q(x_3,y_3)\mathcal{O}_p(x_4,y_4)\rangle_{c^0} &= \frac{1}{\mathcal{N}^{(0)}_{pqqp}}\,\tilde{g}_{14}^{2p-1}\tilde{g}_{23}^{2q-1}\\ \text{and } G^{(0)}_{pqqp}(x,y) = \frac{1}{\mathcal{N}^{(0)}_{pqqp}}\left(\frac{x}{y}\right)^{p+q-1}\left(\frac{1-y}{1-x}\right)^{2q-1} & .\end{aligned} \tag{5.3.21}$$

The operators again have dimensions and charges $(2p-1)/2$, $(2q-1)/2$ as expected. The correlator decomposes into

$$\begin{aligned}k^{(0)}_{pqqp}(x,y) &= \frac{1}{\mathcal{N}^{(0)}_{pqqp}}\left(\frac{x}{y}\right)^{p-q} \ ,\\ H^{(0)}_{pqqp}(x,y) &= \frac{1}{\mathcal{N}^{(0)}_{pqqp}}\frac{x\,y}{x-y}\left[\left(\frac{1-y}{1-x}\right)^{2q-1}\left(\frac{x}{y}\right)^{p+q-1}-\left(\frac{x}{y}\right)^{p-q}\right] .\end{aligned} \tag{5.3.22}$$

The action of $\Delta^{(2)}$ then yields

$$\tilde{H}^{(0)}_{pqqp}(x,y) = (-1)^{p+q}\,\frac{x^{p+q}}{y^{p+q-2}}\frac{(1-y)^{2q-2}}{(1-x)^{2q}} \ , \tag{5.3.23}$$

which as expected looks like free theory of descendants with dimensions p, q and charges $p-1$, $q-1$. Using (5.3.6) we finally get the correlator of descendants

$$\langle L_p L_q \bar{L}_q \bar{L}_p \rangle_{c^0} = (-1)^{p+q}\, g_{14}^{2p-2} g_{23}^{2q-2} \frac{1}{x_{14}^2 x_{23}^2} \,, \tag{5.3.24}$$

which agrees with the term proportional to $g_{14}^{2p-2} g_{23}^{2q-2}$ in the expansion of (5.3.15).

In this section we have shown that free theory correlators in 1d dual to quantum gravity in AdS$_2$×S^2 have four-dimensional conformal symmetry. The generating function can be constructed by uplifting the correlator of descendants with lowest charge to four dimensions. In the following section we will study correlators in the supergravity limit in the context of the four-dimensional conformal symmetry as well as the 4d scalar effective action described in subsection 5.2.3.

5.4 Supergravity

As explained in subsection 5.1.1, for supergravity the object which plays the leading role in the four-dimensional conformal symmetry is the half-BPS correlator itself. We will study correlators in the supergravity approximation in this section. To construct a 4d generating function of all the 1d half-BPS correlators dual to tree-level supergravity from the 4d conformal symmetry, we first need to determine the lowest-charge supergravity correlator. This will be derived in subsection 5.4.1 using only crossing symmetry and $x \to 0$ behaviour. In subsection 5.4.2 we will then discuss the 4d conformal symmetry of supergravity correlators. Furthermore, in subsection 5.4.3 we will obtain all half-BPS supergravity correlators from the 4d effective action (5.2.27) by evaluating generalised AdS$_2$×S^2 Witten diagrams analogous to chapter 3.

5.4.1 Lowest-Charge Supergravity Correlator

In this subsection we determine the supergravity correlator for $p_i = 1$ using crossing symmetry and the $x \to 0$ behaviour. The free disconnected correlator of operators with equal charges $p_i = 1$ is given in terms of $\tilde{g}_{ij}$ as

$$\langle \mathcal{O}_1(x_1,y_1)\mathcal{O}_1(x_2,y_2)\mathcal{O}_1(x_3,y_3)\mathcal{O}_1(x_4,y_4)\rangle|_{c^0} = \tilde{g}_{12}\tilde{g}_{34} - \tilde{g}_{13}\tilde{g}_{24} + \tilde{g}_{14}\tilde{g}_{23} \ . \tag{5.4.1}$$

At next order in the large-c expansion, which corresponds to supergravity, we would expect an expression of the form

$$\langle \mathcal{O}_1(x_1,y_1)\mathcal{O}_1(x_2,y_2)\mathcal{O}_1(x_3,y_3)\mathcal{O}_1(x_4,y_4)\rangle|_{1/c} = \tilde{g}_{12}\tilde{g}_{34}\frac{x}{y}(x-y)a(x) \ ,$$

however this does not turn out to be the most natural definition, which can be seen by considering the crossed version. Indeed, if we exchange positions 1 and 3, which takes $x \to 1-x$ and $y \to 1-y$, we get

$$\begin{aligned}\langle \mathcal{O}_1(x_3,y_3)\mathcal{O}_1(x_2,y_2)\mathcal{O}_1(x_1,y_1)\mathcal{O}_1(x_4,y_4)\rangle|_{1/c} &= -\tilde{g}_{32}\tilde{g}_{14}\frac{1-x}{1-y}(x-y)a(1-x) \\ &= \tilde{g}_{12}\tilde{g}_{34}\frac{x}{y}(x-y)a(1-x) \ .\end{aligned}$$

Setting this equal to minus the original expression (by Fermi statistics) then implies the crossing equation

$$a(1-x) = -a(x)\mathrm{sgn}\left[x\left(1-x\right)\right] \ . \tag{5.4.2}$$

For $0 < x < 1$, this reduces to $a(1-x) = -a(x)$. We expect the supergravity correlator should be given in terms of D-functions, or $\bar{D}$-functions to get concrete expressions in position space. Note that since there is only one conformal spacetime cross-ratio x in 1d, one has to consider holomorphic $\bar{D}$-functions, which we denote by $\bar{D}^{\mathrm{hol}}$. These functions can be reached by taking the limit $\bar{z} \to z$ of the usual $\bar{D}$-functions in terms of $z, \bar{z}$ (2.3.17) and setting $z = x$, see (5.2.3). However, the crossing condition (5.4.2) is not consistent with the behaviour of $\bar{D}^{\mathrm{hol}}$-functions under crossing.

This can be resolved by constructing the four-point function using bosonic two-point functions g_{ij} (5.2.22)[2], which yields

$$\langle \mathcal{O}_1(x_1,y_1)\mathcal{O}_1(x_2,y_2)\mathcal{O}_1(x_3,y_3)\mathcal{O}_1(x_4,y_4)\rangle|_{1/c} = g_{12}g_{34}\frac{x}{y}(x-y)a(x) \ . \tag{5.4.3}$$

In this case, from $1 \leftrightarrow 3$, we find the crossing constraint

$$a(1-x) = a(x) \ . \tag{5.4.4}$$

We shall take (5.4.3) as our ansatz. Before considering crossing versions of the four-point function, it is important to carefully consider some properties of four-point functions in one dimension. As mentioned in subsection 5.2.1, conformal symmetry allows us to fix three points on the line, i.e. we can set $x_1 = 0$, $x_3 = 1$ and $x_4 = \infty$ [152]. The position of the third point x_2 is then equal to x and ranges over all real numbers. Let us refer to the interacting part of the four-point function, $x^2 a(x)$, by $\mathcal{A}(x)$ for the purpose of this discussion. The function $\mathcal{A}(x)$ is singular for values of x where two points coincide, i.e. $x = 0, 1, \infty$. Thus, $\mathcal{A}(x)$ splits into three different functions for different regions of the real x line:

$$\mathcal{A}(x) = \begin{cases} \mathcal{A}^-(x) \text{ for } x \in (-\infty, 0) \ , \\ \mathcal{A}^0(x) \text{ for } x \in (0,1) \ , \\ \mathcal{A}^+(x) \text{ for } x \in (1,\infty) \ , \end{cases} \tag{5.4.5}$$

see also [153] for similar discussions. The functions are related to each other by the Fermi symmetry of the four-point function, where we have $\mathcal{A}^{0,\pm}(x) = x^2 a^{0,\pm}(x)$. As mentioned above, we choose the ordering $0 < x < 1$ where $\mathcal{A}^0$ is the relevant function and thus in all the above equations we replace $a(x)$ by $a^0(x)$. When considering crossing symmetry however, the other regions become relevant.

Let us now consider exchanging 2 with 3, which takes $x \to 1/x$ and $y \to 1/y$:

$$\langle \mathcal{O}_1(x_1,y_1)\mathcal{O}_1(x_3,y_3)\mathcal{O}_1(x_2,y_2)\mathcal{O}_1(x_4,y_4)\rangle|_{1/c} = g_{12}g_{34}\frac{x}{y}(x-y)\left(-\frac{1}{x^2}a^+(1/x)\right) \ . \tag{5.4.6}$$

[2]Although we are using bosonic two-point functions, the four-point function is still a valid solution to the Ward identities with the expected symmetries.

Equating this with minus the original expression (due to Fermi statistics) then implies

$$a^+\left(\frac{1}{x}\right) = x^2 a^0(x) \ . \tag{5.4.7}$$

Finally, for the third crossing condition, consider exchanging 1 with 3 in (5.4.6):

$$\langle \mathcal{O}_1(x_3,y_3)\mathcal{O}_1(x_1,y_1)\mathcal{O}_1(x_2,y_2)\mathcal{O}_1(x_4,y_4)\rangle|_{1/c} = g_{12}g_{34}\frac{x}{y}(x-y)\frac{1}{(1-x)^2}a^+\left(\frac{1}{1-x}\right) \ .$$

Equating this with the expression in (5.4.3) then gives the condition

$$a^+\left(\frac{1}{1-x}\right) = (1-x)^2 a^0(x) \ , \tag{5.4.8}$$

which follows from (5.4.4) and (5.4.7).

An additional constraint comes from the fact that the subleading part of the correlator in the large-c expansion should not encode exchange of the identity operator. To use this conditions, consider the conformal block expansion of the correlator

$$\begin{aligned}
&\langle \mathcal{O}_1(x_1,y_1)\mathcal{O}_1(x_2,y_2)\mathcal{O}_1(x_3,y_3)\mathcal{O}_1(x_4,y_4)\rangle \\
&= \sum_{\Delta=1}^{\infty} A^{1111}_{\Delta,0}\, g_{12}g_{34}\, B^{\text{long}}_{\Delta,0}(x,y) \\
&\sim \sum_{\Delta=1}^{\infty} A^{1111}_{\Delta,0}\, g_{12}g_{34}\, (-x)^{\Delta}\,(1+\mathcal{O}(x))
\end{aligned} \tag{5.4.9}$$

and compare the approximation in the last line to the ansatz (5.4.3) considering the $x \to 0$ limit. This implies that $a^0(x)$ must satisfy the additional constraint

$$a^0(x) = \mathcal{O}(1) \ . \tag{5.4.10}$$

With these constraints at our hand, assume that $a^0(x)$, $a^+(x)$ take the form

$$\begin{aligned}
a^0(x) &= p^0(x)\log x^2 + p^0(1-x)\log(1-x)^2 + r^0(x) \qquad &&\text{for } x\in(0,1) \ , \\
a^+(x) &= p^+(x)\log x^2 + p^+(1-x)\log(1-x)^2 + r^+(x) \qquad &&\text{for } x\in(1,\infty) \ ,
\end{aligned} \tag{5.4.11}$$

for rational functions p^0, p^+, r^0, r^+. Note that for $x \in (0,1)$ and $x\in(1,\infty)$ for $a^0(x)$ and $a^+(x)$ respectively the logarithms in the above ansatz will not pose a problem when considering crossing symmetry. Furthermore, this ansatz makes sense since we expect the correlator to be written in terms of $\bar{D}^{\text{hol}}$-functions, and these functions indeed only consist of $\log(x)$, $\log(1-x)$ and rational functions of x. Plugging this into the crossing equations (5.4.4) and (5.4.7) and the constraint (5.4.10), we get the following crossing equations for p^0, p^+ and r^0, r^+:

$$\begin{aligned}
&p^+(1/x) + p^+((x-1)/x) = -x^2 p^0(x) \ , \quad p^+((x-1)/x) = x^2 p^0(1-x) \ , \\
&p^+(x/(x-1)) = (1-x)^2 p^0(x) \ , \ p^+(1/(1-x)) + p^+(x/(x-1)) = -(1-x)^2 p^0(1-x) \ , \\
&r^0(1-x) = r^0(x) \ , \quad r^+(1/x) = x^2 r^0(x) \ , \quad r^+(1/(1-x)) = (1-x)^2 r^0(x) \ . \quad (5.4.12)
\end{aligned}$$

We then find the minimal solution:

$$a^0(x) = \frac{\log x^2}{1-x} + \frac{\log(1-x)^2}{x} = -\bar{D}^{\text{hol}}_{1111}(x) \quad \text{for } x \in (0,1) \ , \tag{5.4.13}$$

where $\bar{D}^{\text{hol}}_{1111}(x)$ is the holomorphic box function. Note that (5.4.12) yields that $a^0(x)$ and $a^+(x)$ have the same functional form:

$$a^+(x) = -\bar{D}^{\text{hol}}_{1111}(x) \quad \text{for } x \in (1,\infty) \ . \tag{5.4.14}$$

Using (5.3.3) we write the interacting theory at tree-level as

$$H^{\text{sugra}}_{1111} = -u\bar{D}^{\text{hol}}_{1111} \ . \tag{5.4.15}$$

We have now determined the lowest-charge supergravity correlator from which we will obtain all higher-charge correlators from the 4d conformal symmetry in the following subsection.

5.4.2 4d Conformal Symmetry

As was found in [41] for $\mathcal{N}=4$ SYM, the higher-dimensional conformal symmetry allows us to obtain the whole infinite tower of half-BPS correlators of all charges described by tree-level supergravity from the lowest correlator alone. The starting point is H^{sugra}_{1111} given in (5.4.15) which is lifted to four dimensions by simply lifting the 1d cross-ratios and holomorphic $\bar{D}$-functions to the usual cross-ratios and $\bar{D}$-

functions, using the relations (5.2.3) for the cross-ratios. To get an object that transforms as a 4d conformal correlator, divide by $x_{12}^2 x_{34}^2$ and then replace $x_{ij}^2 \to x_{ij}^2(1+g_{ij}^2)$ to get the generating function:

$$\frac{H_{1111}^{\text{sugra}}}{x_{12}^2 x_{34}^2}|_{4d} = \frac{-u\bar{D}_{1111}(u,v)}{x_{12}^2 x_{34}^2}|_{4d} \to \frac{-u_{4d}\bar{D}_{1111}(u_{4d},v_{4d})}{x_{12}^2 x_{34}^2}\frac{1}{(1+g_{12}^2)(1+g_{34}^2)} \,, \qquad (5.4.16)$$

where

$$u_{4d} = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}\frac{(1+g_{12}^2)(1+g_{34}^2)}{(1+g_{13}^2)(1+g_{24}^2)} \,, \quad v_{4d} = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}\frac{(1+g_{14}^2)(1+g_{23}^2)}{(1+g_{13}^2)(1+g_{24}^2)} \,. \qquad (5.4.17)$$

This 4d object includes all higher-charge 1d supergravity correlators. To obtain the specific correlators, expand in small g_{ij} and take the coefficients of the appropriate powers of g_{ij}^2. Before deriving a general formula describing all correlators, let us consider the explicit expansion for a few examples. The interacting piece of the correlators corresponds to the piece proportional to $\frac{x-y}{xy}$ and for the first few cases with charges $\{p_i\} = ppqq$ this looks like:

$$\begin{aligned}
\langle \mathcal{O}_1\mathcal{O}_1\mathcal{O}_2\mathcal{O}_2\rangle_{1/c}|_{\frac{x-y}{xy}} &= g_{12}g_{34}\times g_{34}^2\left(u\bar{D}_{1122}\right) \,, \\
\langle \mathcal{O}_1\mathcal{O}_1\mathcal{O}_3\mathcal{O}_3\rangle_{1/c}|_{\frac{x-y}{xy}} &= g_{12}g_{34}\times \tfrac{1}{2}g_{34}^4\left(-u\,\bar{D}_{1133}\right) \,, \\
\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\mathcal{O}_2\rangle_{1/c}|_{\frac{x-y}{xy}} &= g_{12}g_{34}\times \left(u\,g_{12}^2g_{34}^2 + g_{13}^2g_{24}^2 + v\,g_{14}^2g_{23}^2\right)\left(-u\,\bar{D}_{2222}\right) \,, \\
\langle \mathcal{O}_2\mathcal{O}_2\mathcal{O}_3\mathcal{O}_3\rangle_{1/c}|_{\frac{x-y}{xy}} &= g_{12}g_{34}\times \left(\tfrac{1}{2}u\,g_{12}^2g_{34}^4 + g_{13}^2g_{24}^2g_{34}^2 + v\,g_{14}^2g_{23}^2g_{34}^2\right)\left(u\,\bar{D}_{2233}\right) \,, \\
\langle \mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\mathcal{O}_3\rangle_{1/c}|_{\frac{x-y}{xy}} &= g_{12}g_{34}\times \Big(\tfrac{1}{4}u^2g_{12}^4g_{34}^4 + \tfrac{1}{4}g_{13}^4g_{24}^4 + \tfrac{1}{4}v^2g_{14}^4g_{23}^4 + u\,g_{12}^2g_{34}^2g_{13}^2g_{24}^2 \\
&\quad + u\,v\,g_{12}^2g_{34}^2g_{14}^2g_{23}^2 + v\,g_{13}^2g_{24}^2g_{14}^2g_{23}^2\Big)\left(-u\,\bar{D}_{3333}\right) \,. \qquad (5.4.18)
\end{aligned}$$

To get the one-dimensional correlators, go back to 1d by taking the holomorphic limit of the $\bar{D}$-functions and considering the 1d cross-ratios (5.2.12). The supergravity correlators can be conveniently written in terms of the decomposition in (5.3.3) using (5.2.22) in 1d:

$$\begin{aligned}
&H_{1122}^{\text{sugra}} = u\bar{D}_{1122}^{\text{hol}} \,, \quad H_{1133}^{\text{sugra}} = -\tfrac{1}{2}u\bar{D}_{1133}^{\text{hol}} \,, \quad H_{2222}^{\text{sugra}} = -2u^2\frac{1-y+y^2}{y^2}\bar{D}_{2222}^{\text{hol}} \,, \\
&H_{2233}^{\text{sugra}} = \tfrac{1}{2}u^2\frac{4-4y+3y^2}{y^2}\bar{D}_{2233}^{\text{hol}} \,, \quad H_{3333}^{\text{sugra}} = -\tfrac{3}{2}u^3\frac{(1-y+y^2)^2}{y^4}\bar{D}_{3333}^{\text{hol}} \,. \qquad (5.4.19)
\end{aligned}$$

Note that the supergravity correlators with general charges show a simple structure and are proportional to a single $\bar{D}$-function with the same indices: $H^{\text{sugra}}_{p_1p_2p_3p_4} \propto \bar{D}^{\text{hol}}_{p_1p_2p_3p_4}$ which will become obvious in the general considerations below.

The expansion in g^2_{ij} can be rewritten in terms of differential operators acting on D-functions. Therefore, to find a general expression for all higher-charge correlators let us formulate everything in terms of D-functions. These can be easily converted to $\bar{D}$-functions when needed to perform explicit calculations in position space using (2.3.17). Note that here we use a d-independent function defined in terms of the normalised D-function in (2.3.16) as $D'_{\{p_i\}} = (-2)^{-\Sigma_p} D_{\{p_i\}}$[3]. Again, start from the 4d uplift (5.4.16) and notice that it exactly corresponds to D'_{1111}:

$$\frac{H^{\text{sugra}}_{1111}}{x^2_{12}x^2_{34}}|_{4d} = \frac{-u\bar{D}_{1111}}{x^2_{12}x^2_{34}}|_{4d} = -D'_{1111}|_{4d} \ . \tag{5.4.20}$$

To get the coefficient of $\left(g^2_{ij}\right)^n$ in the g^2_{ij} expansion of the uplifted correlator with $(u,v) \to (u_{4d}, v_{4d})$, instead of performing the expansion we take

$$\frac{\left(x^2_{ij}\right)^n}{n!}\frac{d^n}{d\left(x^2_{ij}\right)^n} D'_{\Delta_1\Delta_2\Delta_3\Delta_4} \ , \tag{5.4.21}$$

where (see [60])

$$\frac{d}{d\,x^2_{12}} D'_{\Delta_1\Delta_2\Delta_3\Delta_4} = -D'_{\Delta_1+1\,\Delta_2+1\,\Delta_3\Delta_4} \ . \tag{5.4.22}$$

Thus the coefficient of $(g^2_{12})^n$ is

$$\frac{(x^2_{12})^n}{n!}\frac{d^n}{d\,(x^2_{12})^n} D'_{\Delta_1\Delta_2\Delta_3\Delta_4} = \frac{(-x^2_{12})^n}{n!} D'_{\Delta_1+n\,\Delta_2+n\,\Delta_3\Delta_4} \tag{5.4.23}$$

and similar for all x^2_{ij}. Therefore, a general coefficient of $\prod^4_{i<j}\left(g^2_{ij}\right)^{d_{ij}}$ is given by $\prod_{i<j}\frac{\left(-x^2_{ij}\right)^{d_{ij}}}{d_{ij}!} D'_{\{\Delta_i+\sum_{i<j} d_{ij}\}}$, where $0 \le d_{ij} = d_{ji}$, $d_{ii} = 0$.

[3]The factor $(-2)^{\Sigma_p}$ comes from the fact that in chapter 3 we usually factor out (-2) from $(-2X_i.X_j) = x^2_{ij}$ whereas here we work in terms of x^2_{ij} instead of $(X_i.X_j)$ and similar for the spherical coordinates.

Hence a general correlator with charges p_i is given as

$$\frac{H^{\text{sugra}}_{p_1p_2p_3p_4}}{x_{12}^2x_{34}^2}|_{4d} = \sum_{\{d_{ij}\}}\prod_{i<j}\left((-1)^{d_{ij}}\left(g_{ij}^2\right)^{d_{ij}}\frac{\left(x_{ij}^2\right)^{d_{ij}}}{d_{ij}!}\right)(-D'_{p_1p_2p_3p_4})$$
$$= \sum_{\{d_{ij}\}}\prod_{i<j}\left(\frac{\left(y_{ij}^2\right)^{d_{ij}}}{d_{ij}!}\right)(-1)^{\Sigma_p+1}D'_{p_1p_2p_3p_4}\ , \tag{5.4.24}$$

with

$$\sum_{i<j} d_{ij} = p_i - 1\ ,\ 0 \le d_{ij} = d_{ji}\ ,\ d_{ii} = 0\ . \tag{5.4.25}$$

We recognise the factor depending on the internal coordinates y_{ij} as the analogue of the Mellin transform on the sphere we introduced in chapter 3 as B-functions. Note that the relevant object here is $B'_{\{p_i\}} = (-2)^{\Sigma_p} B_{\{p_i\}}$, given in terms of the d-independent B-function defined in (3.1.31) (see footnote 3). The d_{ij} correspond to the sphere-analogues of Mellin variables and $y_{ij}^2 = (-2Y_i.Y_j)$. Equation (5.4.24) is a four-dimensional object which generates all 1d half-BPS correlators described by tree-level supergravity. To obtain general correlators in terms of D-functions it is useful to define the interacting piece of the correlator as

$$\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{int}} = \frac{\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle|_{\frac{x-y}{xy}}}{x_{12}x_{34}y_{12}y_{34}}\ , \tag{5.4.26}$$

where we divide by $x_{12}x_{34}y_{12}y_{34}$ which is the 1d analogue of the Intriligator polynomial in 4d defined in appendix B. Note that the interacting correlator in (5.4.26) agrees with the definition of the interacting correlator in (3.1.14). The lowest-charge interacting correlator is then simply

$$\langle\mathcal{O}_1\mathcal{O}_1\mathcal{O}_1\mathcal{O}_1\rangle_{\text{int}} = -D'_{1111} = \frac{-u\bar{D}_{1111}}{x_{12}^2x_{34}^2}\ . \tag{5.4.27}$$

Finally, a general supergravity correlator is given by:

$$\langle\mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle_{\text{int}} = (-1)^{\Sigma_p+1}D'_{p_1p_2p_3p_4}\times B'_{p_1-1\,p_2-1\,p_3-1\,p_4-1}\ . \tag{5.4.28}$$

To obtain explicit correlators in terms of conformal cross-ratios, rewrite the correlators in terms of $\bar{D}$-functions and take the holomorphic limit. In the next subsection we derive the same correlators from a 4d scalar effective action and compare the two results.

That all supergravity correlators can be obtained from the lowest-charge correlator alone is remarkable and shows that the higher-dimensional conformal symmetry is very powerful. Starting from just the box integral in the 1d case we can deduce the whole tower of spherical harmonics by acting with simple differential operators on it.

Decomposition into 4d conformal blocks

The four-dimensional conformal symmetry of the supergravity correlators suggests an expansion in 4d conformal blocks. This expansion will show that the lowest-charge supergravity correlator lifted to 4d (by using the usual $\bar{D}$-functions and cross-ratios, not the holomorphic ones) corresponds to a single 4d conformal block. This result is expected from the higher-dimensional conformal symmetry, there should be only a single block contributing for each spin l. Furthermore, 2d kinematics implies that only spin-0 blocks contribute and thus the 4d uplift of the lowest-charge supergravity correlator corresponds to a single 4d spin-0 block, which will be explained below.

The decomposition into 4d conformal blocks $G_{\Delta,l}(u,v)$ can be written as [154, 155],

$$\langle \mathcal{O}_1(x_1,y_1)\mathcal{O}_1(x_2,y_2)\mathcal{O}_1(x_3,y_3)\mathcal{O}_1(x_4,y_4)\rangle_{4d} = \frac{1}{x_{12}^2 x_{34}^2}\sum_{\Delta,l} A_{\Delta,l}\, G_{\Delta,l}(u,v)\ , \quad (5.4.29)$$

with

$$\begin{aligned} G_{\Delta,l}(z,\bar{z}) =& \frac{1}{z-\bar{z}} u^{\frac{1}{2}(\Delta-l)} \Big(\left(-\tfrac{1}{2}z\right)^l z\, {}_2F_1\left(\tfrac{1}{2}(\Delta+l), \tfrac{1}{2}(\Delta+l); \Delta+l; z\right) \\ & \times {}_2F_1\left(\tfrac{1}{2}(\Delta-l-2), \tfrac{1}{2}(\Delta-l-2); \Delta-l-2; \bar{z}\right) - (z\leftrightarrow\bar{z}) \Big)\ . \end{aligned} \quad (5.4.30)$$

First, expand the correlator at $\mathcal{O}(c^0)$, which is given by the 4d lift of $\tilde{H}_{1111}$ in (5.3.12). The leading scaling dimension is $\Delta_0 = 2n+l+2$, with the labels $n=0,1,\ldots$ and spin $l=0,2,\ldots$ The conformal block expansion at leading order is then given by

$$u + \frac{u}{v} = \sum_{n,l} A^{(0)}_{\Delta_0,l}\, G_{\Delta_0,l}(u,v)\ . \quad (5.4.31)$$

This gives the free OPE coefficients

$$A^{(0)}_{2,l} = \frac{2^{1+l}(l!)^2}{(2\,l)!} \ , \tag{5.4.32}$$

which only has $\Delta_0 = 2$ contributions. To decompose the 4d lift of the correlator at $\mathcal{O}(1/c)$ given in (5.4.15) we first expand the OPE data, as well as the decomposition in (5.4.29) to order $1/c$:

$$\Delta = 2n + l + 2 + \frac{1}{c}\gamma_{\Delta_0,l} + \dots \ , \quad A_{\Delta_0,l} = A^{(0)}_{\Delta_0,l} + \frac{1}{c}A^{(1)}_{\Delta_0,l} + \dots \ , \tag{5.4.33}$$

the block expansion up to $\mathcal{O}(1/c)$ is then

$$\begin{aligned}&\langle \mathcal{O}_1(x_1,y_1)\mathcal{O}_1(x_2,y_2)\mathcal{O}_1(x_3,y_3)\mathcal{O}_1(x_4,y_4)\rangle_{4d} = \\ &\frac{1}{x_{12}^2 x_{34}^2}\sum_{n,l}\left[A^{(1)}_{\Delta_0,l}\,G_{\Delta_0,l}(u,v) + \frac{1}{c}\,A^{(0)}_{\Delta_0,l}\,\gamma_{\Delta_0,l}\,\frac{1}{2}\frac{\partial}{\partial n}\,G_{\Delta_0,l}(u,v) + \dots\right] \ .\end{aligned} \tag{5.4.34}$$

We wish to extract the anomalous dimensions γ, and this can be done by noticing that $\partial_n G_{\Delta_0,l}$ gives an expression of the form ($\log u\, G_{\Delta_0,l} +$ non-log terms). The correlator which is given in terms of $\bar{D}$-functions also has a contribution proportional to $\log u$. Therefore, we can isolate the relevant contributions to solve for the $\gamma_{\Delta_0,l}$ by taking the pieces proportional to $\log u$ and this yields

$$-\,u\bar{D}_{1111}|_{\log u} = \sum_{\Delta_0,l} A^{(0)}_{\Delta_0,l}\gamma_{\Delta_0,l}\,G_{\Delta_0,l}(u,v) \ . \tag{5.4.35}$$

It turns out that the log-piece of the supergravity correlator indeed only has a single spin-0 block contribution with $\Delta_0 = 2$. The anomalous dimension is

$$\gamma_{2,0} = 1 \ , \tag{5.4.36}$$

where we divided by $A^{(0)}_{2,0}$. This can also be seen from 4d supergravity scattering in flat space, analogously to the discussion in section 5.1 where the supergravity anomalous dimension agrees with the conformal partial wave coefficient of the flat space scattering amplitude. In general dimensions the expectation is that the anomalous dimension is $\sim \frac{1}{(l+1)_{(d-4)}}$ which for 4d reduces to a constant agreeing with (5.4.36). We will adapt the discussion in section 5.1 to 4d in the following.

Physical interpretation

The 4d supergravity amplitude in flat space is given by

$$A_4 = G_N \delta^4(Q) \to G_N s \ , \tag{5.4.37}$$

where we have taken the scalar component. Note that $G_N \delta^4(Q)$ is dimensionless in 4d, so this amplitude has 4d conformal symmetry as expected. The factor of s indicates two-derivative interactions, as one would expect for supergravity. In contrast to the 10d amplitude (5.1.4), the 4d one has no θ dependence which implies that only $l = 0$ contributes in the partial wave expansion (5.1.5). This is expected for 2d massless scattering. Hence, there is only one partial wave coefficient:

$$\mathcal{A}_0^4(s) \sim 1 + \frac{R^2}{c} s \ , \tag{5.4.38}$$

where the Newton constant in 4d $G_N \sim R^2/c$, with the AdS radius R and the central charge c, and the one is put in by hand. This explains why supergravity correlators correspond to the 4d spin-0 block as was confirmed in (5.4.36). Based on this argument we can make a prediction for the supergravity anomalous dimensions obtained from a conformal block analysis in 1d and solving the mixing problem. Analogous to the $\mathcal{N} = 4$ SYM case, we expect the anomalous dimensions to be of the form

$$e^{\frac{1}{c}\gamma_{1d}^{\text{sugra}}} \sim 1 + \frac{1}{c}\delta^{(2)} \ , \tag{5.4.39}$$

where we note that γ^{sugra} contributes at order $1/c$ and $\delta^{(2)}$ is the eigenvalue of the second-order differential operator $\Delta^{(2)}$ acting on the 1d blocks. The eigenvalue $\delta^{(2)}$ encodes s and importantly there is no dependence on the spin l, which has to be the case because there is no spin in 1d CFTs. We will solve the mixing problem for anomalous dimensions of double-trace operators at $\mathcal{O}(1/c)$ and show that the result is consistent with this prediction in section 5.6. Before doing so, we will derive all half-BPS supergravity correlators from the 4d scalar effective action in the following subsection.

5.4.3 Effective Action

In general dimensions, $\mathrm{AdS}_q{\times}\mathrm{S}^q$, supergravity is not expected to be directly computed from the higher-dimensional scalar effective action, because it is not dual to contact interactions alone but rather is also described by exchange diagrams. But it turns out that in $\mathrm{AdS}_2{\times}\mathrm{S}^2$ supergravity is described by the ϕ^4 interaction in the 4d effective action (5.2.27). This can be seen from the discussions above, that the 4d uplift of the lowest-charge supergravity correlator corresponds to the 4d spin-0 block (or rather the $\log u$-piece of the correlator does). Hence, the conformal block expansion in 4d is truncated in spin and therefore supergravity can be described by contact diagrams alone, see also subsection 2.3.2. Thus, in $\mathrm{AdS}_2{\times}\mathrm{S}^2$ we can deduce all correlators at $\mathcal{O}(1/c)$ from the effective field theory.

All four-point half-BPS correlators described by tree-level supergravity can be directly computed from the 4d effective action (5.2.27) by evaluating zero-derivative 4d Witten diagrams, which is very similar to the zero-derivative corrections at order α'^3 described in section 3.2. To obtain correlators from the 4d effective action we evaluate $\mathrm{AdS}_2{\times}\mathrm{S}^2$ Witten diagrams, closely following the procedure explained in chapter 3. The supergravity term is the first term in the effective action (5.2.27) and is simply a ϕ^4 interaction:

$$S_{\text{sugra}} = \frac{1}{4!}\, A \times \int_{\mathrm{AdS}\times\mathrm{S}} d^2\hat{X} d^2\hat{Y} \phi(\hat{X},\hat{Y})^4 \ , \tag{5.4.40}$$

where we use embedding space formalism reviewed for AdS coordinates in subsection 2.3.1 and for spherical coordinates in 3.1.4. We obtain the corresponding CFT correlators by following the standard AdS/CFT procedure for computing correlators from AdS, but in a fully 4d covariant way, including the two-sphere manifestly. Using the generalised bulk-to-boundary propagators in (3.1.33) we obtain the $\mathrm{AdS}_2{\times}\mathrm{S}^2$ Witten diagram for this ϕ^4 contact interaction, leading to the following proposal for the supergravity correlators:

$$
\begin{aligned}
\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle^{\text{sugra}}_{\text{int}} &= \frac{1}{4!}\,A\,\frac{(\mathcal{C}_1)^4}{(-2)^4}\int_{\text{AdS}\times\text{S}}\frac{d^2\hat{X}\,d^2\hat{Y}}{(P_1+Q_1)(P_2+Q_2)(P_3+Q_3)(P_4+Q_4)}\\
&= \frac{1}{4!}\,A\,(\mathcal{C}_1)^4\times D^{\text{AdS}_2\times\text{S}^2}_{1111}\ .
\end{aligned}
\tag{5.4.41}
$$

Recall that $P_i = \hat{X}.X_i$, $Q_i = \hat{Y}.Y_i$ and the AdS×S D-functions are defined in (3.1.37). The correlator (5.4.41) contains all supergravity half-BPS correlators of all spherical harmonics and to extract any specific correlator one expands in the appropriate powers in Y_i using the Taylor expansion of the 4d bulk-to-boundary propagators

$$
(P_i+Q_i)^{-1} = \sum_{p=1}^{\infty}(-1)^{p-1}(P_i)^{-p}(Q_i)^{p-1}\ . \tag{5.4.42}
$$

The individual correlators are then given by[4]:

$$
\begin{aligned}
&\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle^{\text{sugra}}_{\text{int}}\\
&= \frac{1}{4!}\,A\,\frac{(\mathcal{C}_1)^4}{(-2)^4}\int_{\text{AdS}_2} d^2\hat{X}\prod_i\frac{1}{(P_i)^{p_i}}\times\int_{\text{S}^2}d^2\hat{Y}\prod_i(Q_i)^{p_i-1}\\
&= \frac{1}{4!}\,A\,(\mathcal{C}_1)^4\,D^{(1)}_{p_1p_2p_3p_4}(X_i)\times B^{(1)}_{p_1-1\,p_2-1\,p_3-1\,p_4-1}(Y_i)\\
&= A''\,(-1)^{\Sigma_p+1}\,D_{p_1p_2p_3p_4}(X_i)\times B_{p_1-1\,p_2-1\,p_3-1\,p_4-1}(Y_i)\\
&= A'\,(-1)^{\Sigma_p+1}\,D'_{p_1p_2p_3p_4}\times B'_{p_1-1\,p_2-1\,p_3-1\,p_4-1}\ ,
\end{aligned}
\tag{5.4.43}
$$

where going from the third to penultimate line we rewrote everything in terms of the normalised D- and B-functions defined in (2.3.16) and (3.1.31). Further, to compare to the results obtained from the 4d conformal symmetry in (5.4.28), rewrite D- and B-functions depending on $X_i.X_j, Y_i.Y_j$ as D'- and B'-functions in terms of x^2_{ij}, y^2_{ij} which have factors of $(-2)^{\pm\Sigma_p}$ between them. We absorb all numerical factors into the coefficient A' which is unfixed and we set it to one in subsequent calculations, in agreement with the choice of normalisation in the previous subsection. In this case the $\log u$ piece of the lowest-charge correlator is exactly equal to the holomorphic limit of the 4d spin-0 block. The above expression includes all tree-level supergravity half-BPS correlators of any charge and it agrees with what we found from the

[4]This is (3.1.40) with $\Delta_i = d = 1$ and with $p_i \to p_i - 1$ to account for the fact that the lowest correlator is labelled with $p_i = 1$ rather than $p_i = 0$. We do not need to worry about the minus signs in the factors $(-1)^p$ in (5.4.42) since $B_{p_1p_2p_3p_4} = 0$ if $p_1+p_2+p_3+p_4$ is odd.

4d conformal symmetry in (5.4.28), notably $\langle \mathcal{O}_{p_1}\mathcal{O}_{p_2}\mathcal{O}_{p_3}\mathcal{O}_{p_4}\rangle^{\text{sugra}}_{\text{int}} \propto D_{p_1p_2p_3p_4}$. To obtain explicit expressions in position space convert D' to $\bar{D}$-functions and take the holomorphic limit.

5.5 Conformal Block Analysis and Double-Trace Spectrum

In the previous sections we have determined all half-BPS correlators for disconnected free theory and in the supergravity limit. Next, we wish to perform a conformal block analysis of these correlators in the large-c expansion and specifically compute the anomalous dimensions of double-trace operators in the spectrum. Before doing so in section 5.6, in this section, we will analyse the double-trace spectrum and discuss the operators that can be exchanged in the conformal block expansion (5.2.19) for each pair of scaling dimensions and $SU(2)$ charges (Δ, p). We discuss the spectrum for free theory and supergravity, as well as for higher-derivative corrections.

A lot of progress has been made in the study of the conformal block analysis of four-point half-BPS correlators in the large-c expansion in the context of $\mathcal{N} = 4$ SYM [11, 14, 16, 84–91, 144, 145] and the analysis in this and following sections is inspired by these works. It is expected that short and long operators contribute to the OPE of the four-point functions of half-BPS operators $\mathcal{O}_q$ and that the only unprotected operators which contribute at leading order in $1/c$ are double-trace operators which consist of a product of derivatives of two of the operators $\mathcal{O}_q$. All other multi-trace operators are expected to be subleading in $1/c$. We discuss the double-trace operators which are present in the free theory and at leading order in $1/c$. We will find that only specific operators are present in the spectrum in the supergravity limit, whereas more operators contribute at the order of free theory and higher-derivative corrections.

Let us start by considering all long double-trace operators in the OPE of two half-

BPS operators:

$$\mathcal{O}_{q_1 q_2} = \mathcal{O}_{q_1} \partial^{\Delta+1-q_1-q_2} \mathcal{O}_{q_2} \ , \qquad q_1 \leq q_2 \ , \tag{5.5.1}$$

where the operators $\mathcal{O}_{q_1}$ and $\mathcal{O}_{q_2}$ have half-integer scaling dimensions and are labelled by integers $\Delta+1/2$, the dimensions of the descendants, in terms of q_1 and q_2 according to the conventions introduced in (5.2.17) and used in the previous sections. We denote the scaling dimensions and $SU(2)$ charges of exchanged operators $\mathcal{O}_{q_1 q_2}$ by Δ and p respectively. Note that from now on throughout the rest of this chapter, when writing Δ and p we will always refer to the dimensions and charges of the exchanged double-trace operators (not the external ones).

The interacting piece of the correlator, corresponding to H in the decomposition (5.3.3), can be written as a double expansion in terms of large central charge c and small expansion parameter a, where a describes the higher-derivative corrections:

$$H(x,y) = H^{(0)} + \frac{1}{c}\left(H^{\text{sugra}} + a\, H^{\text{4-deriv}} + \dots\right) + \mathcal{O}(c^{-2}) \ . \tag{5.5.2}$$

We wish to study the double-trace spectrum at different orders in c and a. From (5.5.1) together with considering the possible R-symmetry charges, one can predict which double-trace operators can in principle contribute to the spectrum at each weight (Δ, p). Subsequently, by performing the conformal block analysis for free theory at $\mathcal{O}(1)$, supergravity at $\mathcal{O}(1/c)$ and for four-derivative corrections at $\mathcal{O}(a/c)$, we can determine which operators are really present in the double-trace spectrum at (Δ, p), i.e. have non-zero conformal block coefficients at the order considered. It turns out that at $\mathcal{O}(1/c)$ only part of the operators contribute, while more operators contribute to free theory and to higher-derivative corrections. This will be explained by the fact that 1d correlators in the supergravity limit are truncated to 4d spin-0. Generally, there are many operators contributing at each weight (Δ, p) and to resolve this degeneracy we have to solve a so-called mixing problem. This will be studied in subsequent sections.

Before we describe the conformal block analysis in detail and solve the mixing problems for supergravity and four-derivative corrections in sections 5.6 and 5.8

respectively, we will first present the operators which are found to contribute to the double-trace spectrum. At the end of this section these will be interpreted in terms of an effective 4d spin similar to the one in 10d described in section 5.1. We notice that there are two classes of operators in the spectrum, we call them class A and class B. Both, class A and class B operators are present in the free theory, but only class A operators acquire anomalous dimensions and mix with other class A operators at order $\mathcal{O}(1/c)$, while class B operators decouple at $\mathcal{O}(1/c)$, i.e. they do not mix and do not acquire anomalous dimensions at the level of supergravity. Going to higher-derivative corrections, particularly at four-derivatives which is the case studied in this chapter, both class A and class B operators acquire anomalous dimensions and mix. Note that class A and class B operators only mix with operators of the same class and class A operators never mix with class B operators. Additionally, it is important to note that there are different operators present for the cases where $t = \Delta - p$ odd or $t = \Delta - p$ even. At the level of supergravity there are only operators present when t is odd, these are the class A operators. In free theory and at the level of four-derivative corrections, there are operators mixing and acquiring anomalous dimensions for both odd and even t and we distinguish between them by labelling them with B^{to} and B^{te} respectively.

The fact that only specific operators acquire anomalous dimensions and mix at the order of supergravity can be explained from the point of view of the 4d conformal symmetry. Correlators in the supergravity limit correspond to the 4d spin-0 block only, which only allows for class A operators. This can be seen from the 4d effective spin we will conjecture in (5.5.6). Whereas for the 4d uplift of correlators corresponding to four-derivative corrections, there are contributions from spin-0 as well as spin-2 blocks and the decomposition of free theory correlators is not truncated in spin at all, thus allowing for more operators with non-zero anomalous dimensions, which includes all class B operators. Note that this is specific to 1d, because the correlators at $\mathcal{O}(1/c)$ correspond to a ϕ^4 interaction. In the $\mathcal{N} = 4$ SYM case, supergravity correlators have no spin-truncated block decomposition and their spectrum contains all possible double-trace operators.

We label the allowed values of pairs of q_1 and q_2 for class A and class B operators with odd or even t with (q_1^A, q_2^A), $(q_1^{B^{\rm to}}, q_2^{B^{\rm to}})$ and $(q_1^{B^{\rm te}}, q_2^{B^{\rm te}})$ respectively. There are $d = d_A + d_{B^{\rm to}} + d_{B^{\rm te}}$ operators $\mathcal{O}_{q_1 q_2}$ labelled by sets of pairs (q_1, q_2), where d_A counts class A and $d_{B^{\rm to}}$, $d_{B^{\rm te}}$ class B operators with odd and even t respectively. The labels run over a set of operators $\mathcal{D}_{\Delta,p}$ and we parametrise this set by i_A, r_A, $i_{B^{\rm to}}$, $r_{B^{\rm to}}$, $i_{B^{\rm te}}$, $r_{B^{\rm te}}$ as follows:

$t = \Delta - p$ odd

For t odd we have both class A and class B operators:

$$\begin{aligned}
q_1^A &= 1 + i_A + r_A \ , & q_1^{B^{\rm to}} &= 1 + i_{B^{\rm to}} + r_{B^{\rm to}} \ ,\\
q_2^A &= 1 + i_A + p - r_A \ , & q_2^{B^{\rm to}} &= 1 + i_{B^{\rm to}} + (p-1) - r_{B^{\rm to}} \ ,\\
i_A &= 0, \dots, \frac{t-1}{2} \ , & i_{B^{\rm to}} &= 1, \dots, \frac{t-1}{2} \ ,\\
r_A &= 0, \dots, \frac{\mu_A - 1}{2} \ , & r_{B^{\rm to}} &= 0, \dots, \frac{\mu_{B^{\rm to}} - 1}{2} \ ,
\end{aligned} \tag{5.5.3}$$

where

$$t = \Delta - p \ , \qquad \mu_A = \begin{cases} p+1 & p \text{ even} \\ p & p \text{ odd} \end{cases} \ , \qquad \mu_{B^{\rm to}} = \begin{cases} p-1 & p \text{ even} \\ p & p \text{ odd} \end{cases} \ , \tag{5.5.4}$$

with $d_A = \frac{1}{4}(\mu_A + 1)(t+1)$ and $d_{B^{\rm to}} = \frac{1}{4}(\mu_{B^{\rm to}} + 1)(t-1)$. Class A operators have even numbers of derivatives while class B operators have odd numbers of derivatives at t odd.

$t = \Delta - p$ even

There are only class B operators for t even, meaning these operators contribute to the free theory and higher-derivative corrections but not to supergravity. These operators are parametrised as

$$q_1^{B^{\rm te}} = i_{B^{\rm te}} + \lfloor \tfrac{r_{B^{\rm te}}+1}{2} \rfloor \ ,$$

$$
\begin{aligned}
q_2^{B^{\text{te}}} &= i_{B^{\text{te}}} + p - \lfloor \tfrac{r_{B^{\text{te}}}}{2} \rfloor \ , \\
i_{B^{\text{te}}} &= 1, \ldots, \frac{t}{2} \ , \\
r_{B^{\text{te}}} &= 0, \ldots, p-1 \ ,
\end{aligned}
\tag{5.5.5}
$$

with $d_{B^{\text{te}}} = \frac{1}{2} t \, p$.

We illustrate the exchanged operators at weight (Δ, p) for odd and even t in figure 5.1 and 5.2 respectively before we interpret them in terms of the effective 4d spin. In figure 5.1 the exchanged operators with odd t are illustrated in terms of the pairs (q_1, q_2) with the parametrisation (5.5.3). The black nodes denote class A operators while the white nodes denote class B^{to} operators. In the supergravity approximation, operators which are connected by vertical lines have the same anomalous dimensions after unmixing which are all zero at $\mathcal{O}(1/c)$, except for a single operator with non-zero anomalous dimension which is the operator denoted by A. When including higher-derivative corrections, the degeneracy is broken and focussing on four-derivatives the operators which acquire non-zero (and non-equal) anomalous dimensions after unmixing are highlighted by a grey rectangle. Denoted by E is the one class B operator which acquires non-zero anomalous dimension at $\mathcal{O}(a/c)$.

In figure 5.2 exchanged operators with even t are illustrated in terms of pairs (q_1, q_2) parametrised as in (5.5.5). For even t, operators mix and acquire anomalous dimensions only starting from the order of higher-derivative corrections, thus they all belong to class B. The operators are parametrised following (5.5.5) and it turns out that they split into two groups, operators with even $r_{B^{\text{te}}}$ are denoted by black nodes while operators with odd $r_{B^{\text{te}}}$ are denoted by white nodes. The two types of operators do not mix. There is one operator with even and one with odd $r_{B^{\text{te}}}$ which has non-zero anomalous dimension after unmixing at $\mathcal{O}(a/c)$ and they are highlighted by a grey rectangle.

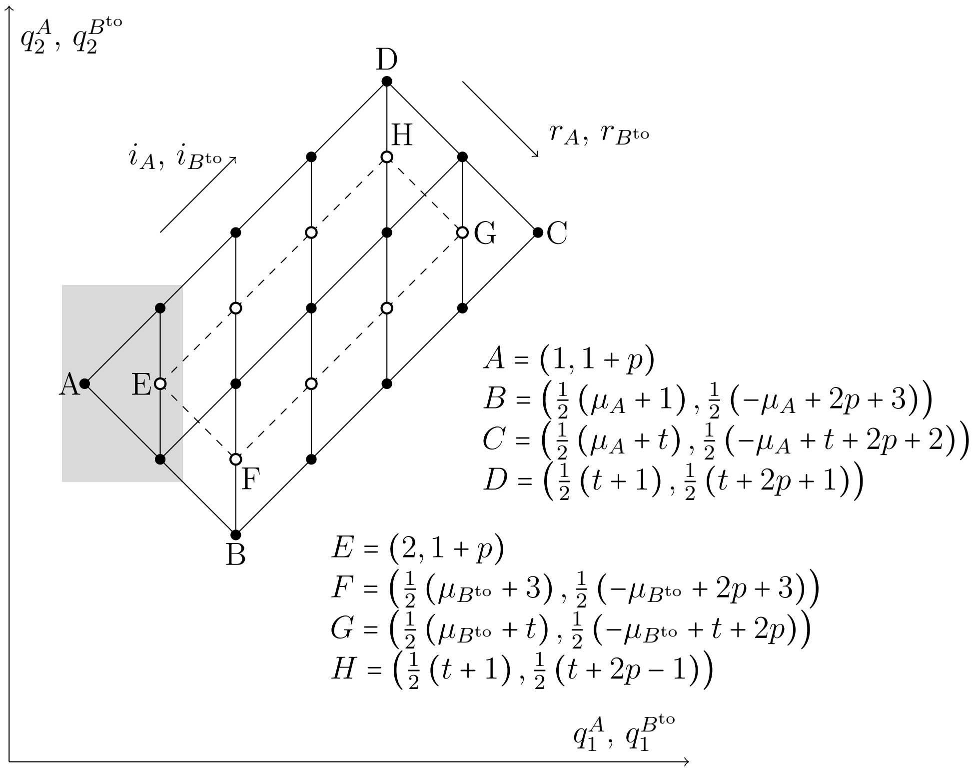


Figure 5.1: The exchanged operators $\mathcal{O}_{q_1 q_2}$ which contribute at (Δ, p) for t odd are illustrated in terms of the pairs (q_1^A, q_2^A) and $(q_1^{B^{\text{to}}}, q_2^{B^{\text{to}}})$. The operators are parametrised as described in (5.5.3). The black nodes denote operators of class A while white nodes denote operators of class B^{to}. The only operator acquiring non-zero anomalous dimension at the order of supergravity is the one denoted by A. Furthermore, the nodes in the grey rectangle correspond to operators which acquire non-zero anomalous dimensions at the order of four-derivative corrections.

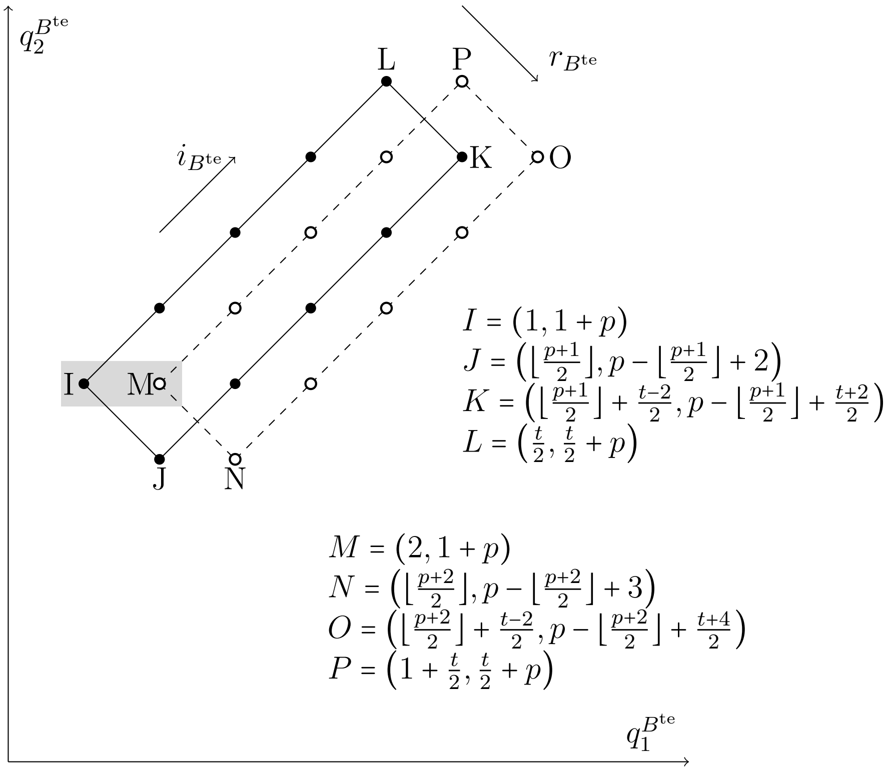


Figure 5.2: The exchanged operators $\mathcal{O}_{q_1 q_2}$ which contribute at (Δ, p) for t even are illustrated in terms of the pairs $(q_1^{B^{\rm te}}, q_2^{B^{\rm te}})$. All operators contributing at even t belong to class B. The operators are parametrised following (5.5.5) and they split into two groups which do not mix with each other. Operators with even $r_{B^{\rm te}}$ are denoted by black nodes while operators with odd $r_{B^{\rm te}}$ are denoted by white nodes. At the level of four-derivative corrections two of these operators acquire non-zero anomalous dimensions, they are highlighted by a grey rectangle.

4d effective spin

Similar to the 10d effective spin (5.1.3) which was justified from the similarity between (5.1.7) and (5.1.8), we conjecture a 4d effective spin in terms of 1d quantum numbers as follows

$$l_{4d} = 2\left(i_A + r_A + i_{B^{\mathrm{to}}} + i_{B^{\mathrm{te}}} + r_{B^{\mathrm{te}}} - \frac{1+(-1)^{r_{B^{\mathrm{te}}}+1}}{2}\right) . \tag{5.5.6}$$

This means that at each order in a, this 4d spin will predict how many and which operators have non-zero anomalous dimensions after unmixing. As we know, the conformal block expansion of the 4d lift of 1d superconformal correlators in the supergravity limit is truncated to spin zero. Therefore, the only operators with non-zero anomalous dimensions are predicted to be those whose quantum numbers give $l_{4d} = 0$ and this can only be satisfied by class A operators. Hence, for supergravity the 4d spin simplifies significantly to

$$l_{4d}^{\mathrm{sugra}} = 2\left(i_A + r_A\right) . \tag{5.5.7}$$

For higher-derivative corrections also class B operators are allowed since the dual bulk contact interactions correspond to spin-L corrections with $L \geq 2$. The four-derivative corrections whose conformal block analysis and unmixing we study in section 5.8 correspond to spin-2 corrections and thus we predict that operators whose quantum numbers satisfy $l_{4d} = 0$ and $l_{4d} = 2$ have non-zero anomalous dimensions at this order.

We will solve the mixing problem for supergravity in the next section and for four-derivative corrections in section 5.8 and we will also interpret the results in terms of l_{4d}. First, let us start by describing how to solve the mixing problem at general orders in c and a in the following subsection.

5.5.1 Solving the Mixing Problem

In [144] the authors study the double-trace spectrum of $\mathcal{N} = 4$ SYM in the supergravity limit, where the double-trace operators exhibit degeneracy, as explained in the previous subsection for 1d. To solve the mixing problem we need to consider free theory and tree-level supergravity contributions to the correlators of four half-BPS operators. In the 4d case the degeneracy can be resolved for a large family of operators, and only a small residual degeneracy is left. In 1d this residual degeneracy is not obvious since the degenerate anomalous dimensions are all zero. We will find that there is only one non-zero anomalous dimension at each weight. It was later found in [14] that the residual degeneracy in 4d is lifted when considering higher-derivative contributions to the correlators and we will come to similar conclusions for 1d in section 5.8. In this subsection we will describe how to solve the mixing problem up to $\mathcal{O}(a/c)$ which corresponds to four-derivative corrections, this discussion can be easily extended to higher orders in a to include higher-derivative corrections.

To perform the conformal block analysis we need the free theory correlator which is the leading contribution to the large-c expansion and was described in section 5.3 and the first subleading contribution in the $1/c$ expansion, which we obtained both from the 4d conformal symmetry in subsection 5.4.2 and from a 4d scalar effective action in subsection 5.4.3. Even though we have not obtained the higher-derivative correlators yet, we will assume that we have them at our disposal and describe the unmixing at $\mathcal{O}(a/c)$. We will later derive all four-derivative corrections to all half-BPS correlators described by tree-level supergravity from the 4d scalar effective action in subsection 5.7.2.

We have seen above that there can be many double-trace operators with different quantum numbers i_A, r_A, $i_{B^{\text{to}}}$, $r_{B^{\text{to}}}$, $i_{B^{\text{te}}}$, $r_{B^{\text{te}}}$ contributing at the same weight (Δ, p). To solve this mixing problem we perform the operator product expansion of $(\mathcal{O}_{p_1} \times \mathcal{O}_{p_2})$ and $(\mathcal{O}_{p_3} \times \mathcal{O}_{p_4})$, where the pairs (p_1, p_2) and (p_3, p_4) range over the same set $\mathcal{D}_{\Delta,p}$, described in (5.5.3) and (5.5.5). The conformal block expansion of the interacting piece of the correlator in terms of long blocks is

$$H(x,y) = \frac{x\,y}{x-y} \sum_{\Delta,p} A^{p_i}_{\Delta,p} B^{\text{long}}_{\Delta,p,p_{12},p_{34}}(x,y) \ , \tag{5.5.8}$$

where the superconformal blocks are given in (5.2.23). The coefficients of the decomposition are given as a sum of squares of OPE coefficients as follows

$$A^{p_i}_{\Delta,p} = \sum_{\mathcal{O}^{\Delta,p}} C_{p_1p_2\mathcal{O}}C_{p_3p_4\mathcal{O}} \ , \tag{5.5.9}$$

where the sum goes over the degenerate operators because of operator mixing. Expanding the OPE-data to order $1/c$ we get

$$\begin{aligned}\Delta_{\mathcal{O}} &= \Delta^{(0)} + \frac{1}{c}\left(\gamma^{\text{sugra}} + a\,\gamma^{\text{4-deriv}} + \dots\right) + \mathcal{O}(c^{-2}) \ ,\\ C_{pp\mathcal{O}} &= C^{(0)}_{pp\mathcal{O}} + \left(a\,C^{\text{4-deriv}}_{pp\mathcal{O}} + \dots\right) + \mathcal{O}(c^{-1}) \ ,\end{aligned} \tag{5.5.10}$$

where the anomalous dimensions γ depend on Δ, p and the degeneracy labels i and r. Plugging the expansion of the dimensions and OPE-coefficients back into (5.5.8) gives

$$\begin{aligned}H(x,y) =& \frac{x\,y}{x-y}\sum_{\Delta^{(0)},p}\Big[A^{(0)}_{\Delta^{(0)},p}B^{\text{long}}_{\Delta^{(0)},p,p_{12},p_{34}}(x,y)\\ &+\frac{1}{c}\log u\sum_{\Delta^{(0)},p}\left(M^{\text{sugra}}_{\Delta^{(0)},p} + a\,M^{\text{4-deriv}}_{\Delta^{(0)},p} + \dots\right)B^{\text{long}}_{\Delta^{(0)},p,p_{12},p_{34}}(x,y) + \dots\Big] \ ,\end{aligned} \tag{5.5.11}$$

where the dots denote terms analytic in u which do not play a role for our analysis. We define the OPE-coefficients at orders $\mathcal{O}(c^0)$, $\mathcal{O}(1/c)$ and $\mathcal{O}(a/c)$ as follows (note that from now on we denote the classical scaling dimension $\Delta^{(0)}$ of the exchanged operators by Δ for simplicity):

$$\begin{aligned}A^{(0)}_{\Delta,p} &= \sum_{\mathcal{O}^{\Delta,p}} C^{(0)}_{p_1p_2\mathcal{O}}C^{(0)}_{p_3p_4\mathcal{O}} \ , \quad M^{\text{sugra}}_{\Delta,p} = \sum_{\mathcal{O}^{\Delta,p}} \gamma^{\text{sugra}}C^{(0)}_{p_1p_2\mathcal{O}}C^{(0)}_{p_3p_4\mathcal{O}} \ ,\\ M^{\text{4-deriv}}_{\Delta,p} &= \sum_{\mathcal{O}^{\Delta,p}}\left(\gamma^{\text{4-deriv}}C^{(0)}_{p_1p_2\mathcal{O}}C^{(0)}_{p_3p_4\mathcal{O}} + \gamma^{\text{sugra}}C^{(0)}_{p_1p_2\mathcal{O}}C^{\text{4-deriv}}_{p_3p_4\mathcal{O}} + \gamma^{\text{sugra}}C^{\text{4-deriv}}_{p_1p_2\mathcal{O}}C^{(0)}_{p_3p_4\mathcal{O}}\right) \ .\end{aligned} \tag{5.5.12}$$

The left hand side of equation (5.5.11) are the explicit forms of the correlators and comparing the equation to the double-expansion in (5.5.2), it can be seen that $A^{(0)}_{\Delta,p}$ is determined from the free theory contribution, $M^{\text{sugra}}_{\Delta,p}$ is determined from the correlators in the supergravity limit and $M^{\text{4-deriv}}_{\Delta,p}$ will be determined from the four-derivative corrections to the correlators.

To decide which operators we have to consider in the conformal block analysis to solve the mixing problem, at each weight (Δ, p) let us arrange a $((d_A + d_{B^{\text{to}}}) \times (d_A + d_{B^{\text{to}}}))$ matrix of correlators running over $\mathcal{D}_{\Delta,p}$ for t odd and a $(d_{B^{\text{te}}} \times d_{B^{\text{te}}})$ matrix for t even respectively, where we use the parametrisation from earlier in this section. Note that the $((d_A + d_{B^{\text{to}}}) \times (d_A + d_{B^{\text{to}}}))$ matrix is block-diagonal, so class A and B operators can be treated independently, and similarly $(d_{B^{\text{te}}} \times d_{B^{\text{te}}})$ is block-diagonal where operators with $r_{B^{\text{te}}}$ even or odd can be treated independently. We then perform a conformal block analysis and arrange the coefficients into matrices $\hat{A}^{(0)}_{\Delta,p}$, $\hat{M}^{\text{sugra}}_{\Delta,p}$ and $\hat{M}^{\text{4-deriv}}_{\Delta,p}$ for free theory, supergravity and four-derivative corrections respectively. Comparing (5.5.11) to (5.5.2), keeping terms up to $\mathcal{O}(a/c)$ and writing (5.5.12) in matrix form then leads to the following unmixing equations:

$$
\begin{aligned}
\mathcal{O}(1): \quad & \hat{A}^{(0)}_{\Delta,p} = \mathbb{C}^{(0)} \left(\mathbb{C}^{(0)}\right)^T \ , \\
\mathcal{O}\left(1/c\right): \quad & \hat{M}^{\text{sugra}}_{\Delta,p} = \mathbb{C}^{(0)} \hat{\gamma}^{\text{sugra}} \left(\mathbb{C}^{(0)}\right)^T \ , \\
\mathcal{O}(a): \quad & 0 = \mathbb{C}^{(0)} \left(\mathbb{C}^{\text{4-deriv}}\right)^T + \mathbb{C}^{\text{4-deriv}} \left(\mathbb{C}^{(0)}\right)^T \ , \\
\mathcal{O}\left(a/c\right): \quad & \hat{M}^{\text{4-deriv}}_{\Delta,p} = \mathbb{C}^{(0)} \hat{\gamma}^{\text{4-deriv}} \left(\mathbb{C}^{(0)}\right)^T + \mathbb{C}^{(0)} \hat{\gamma}^{\text{sugra}} \left(\mathbb{C}^{\text{4-deriv}}\right)^T + \mathbb{C}^{\text{4-deriv}} \hat{\gamma}^{\text{sugra}} \left(\mathbb{C}^{(0)}\right)^T \ ,
\end{aligned}
\tag{5.5.13}
$$

where $\hat{\gamma}$ is a diagonal matrix of the anomalous dimensions. Note that $\hat{A}^{(0)}_{\Delta,p}$ is a diagonal matrix as can be easily seen from the form of the free theory correlators and $\hat{M}^{\text{sugra}}_{\Delta,p}$ and $\hat{M}^{\text{4-deriv}}_{\Delta,p}$ are symmetric matrices.

To solve the unmixing equations for the supergravity anomalous dimensions and the free three-point functions ($\mathcal{O}(1/c)$ and $\mathcal{O}(1)$) for general (Δ, p) it is useful to define the matrix $\tilde{c}$

$$
\tilde{c}\tilde{c}^T = \text{Id} \ , \quad \mathbb{C}^{(0)} = \left(\hat{A}^{(0)}\right)^{\frac{1}{2}} \cdot \tilde{c} \ . \tag{5.5.14}
$$

The unmixing equations then become:

$$
\tilde{c} \cdot \hat{\gamma}^{\text{sugra}} \cdot \tilde{c}^T = \left(\hat{A}^{(0)}\right)^{-\frac{1}{2}} \cdot \hat{M}^{\text{sugra}} \cdot \left(\hat{A}^{(0)}\right)^{-\frac{1}{2}} \ . \tag{5.5.15}
$$

The columns of $\tilde{c}$ are eigenvectors of the matrix $\left(\hat{A}^{(0)}\right)^{-\frac{1}{2}} \cdot \hat{M}^{\text{sugra}} \cdot \left(\hat{A}^{(0)}\right)^{-\frac{1}{2}}$ and the corresponding eigenvalues are the anomalous dimensions at $\mathcal{O}(1/c)$. So the mixing problem can be solved by solving the eigenvalue problem of the corresponding matrix. The anomalous dimensions after unmixing at the order of four-derivative corrections are then the $\mathcal{O}(a)$ eigenvalues of $\left(\hat{M}^{\text{sugra}} + a\hat{M}^{\text{4-deriv}}\right) \cdot \left(\hat{A}^{(0)}\right)^{-1}$. We will illustrate this with examples below.

We now know how to solve the mixing problem in the supergravity limit and for four-derivative corrections. In the following section we focus on the supergravity unmixing and we will later perform the unmixing for four-derivative corrections in section 5.8.

5.6 Unmixing Supergravity

In this section we will solve the unmixing equations for supergravity at each weight (Δ, p), where for every (Δ, p) we use (5.5.3) to determine the list of double-trace operators in the spectrum. Recall that only class A operators are in the double-trace spectrum of supergravity and thus from the 4d spin (5.5.6) we predict that there is only one non-zero anomalous dimension per weight, the one corresponding to $l_{4d} = 0$. Additionally, from the 4d conformal symmetry we predict that the value of the anomalous dimension should be the eigenvalue $\delta^{(2)}$ of the differential operator $\Delta^{(2)}$ (5.2.15) acting on the blocks. This can be seen from the arguments around (5.4.39). We determine the anomalous dimensions after unmixing for many values of (Δ, p) and finally, predict a general formula which is indeed $\delta^{(2)}$. Let us look at a few examples, starting with the sector with $SU(2)$ charge $p = 0$.

5.6.1 $p = 0$ Sector

For the $p = 0$ sector, we only need to consider correlators of the form $H_{q_1 q_1 q_2 q_2}$ and thus for free theory we only need equal charge correlators. We start with a conformal block analysis of free theory, where we also need to consider contributions from half-BPS blocks (5.2.24). Therefore, it is best to consider $G^{(0)}_{qqqq}$ (5.3.10) instead of H for free theory. $G^{(0)}_{qqqq}$ decomposes into one half-BPS block with $\Delta = p = 0$ which corresponds to the identity operator and long blocks with coefficients $A^{(0)}_{qqqq}$. Evaluating the free coefficients for many correlators and many values of Δ, p we obtain a general formula for the free coefficients $A^{(0)}_{qqqq}(\Delta, p)$ for all $\Delta,\ p,\ q$:

$$A^{(0)}_{qqqq}(\Delta,p) = \frac{(1+(-1)^{\Delta+p+1})\,\Delta!(2(p+1))!(\Delta+2q-1)!}{2\,(2\Delta)!(\Delta+p+1)p!(p+1)!(p+2q-1)!(-p+2q-2)!(1-2q)^2} \times (\Delta-p+1)_p(\Delta-2q+2)_{-p+2q-2}\ . \tag{5.6.1}$$

These coefficients are non-zero only for odd t, which is expected from the parametrisation of operators in (5.5.5) because for even t only operators $\mathcal{O}_{q_1 q_2}$ with $q_1 \neq q_2$ contribute.

For the simplest case $\Delta = 1$ there is only one exchanged operator of the form (5.5.1) contributing, $\mathcal{O}_1\mathcal{O}_1$, thus we perform the conformal block analysis of the correlator H^{sugra}_{1111}. We spell out the supergravity coefficient together with the free theory coefficient from (5.6.1):

$$A^{(0)}_{1111}(1,0) = 1\ , \quad M^{\text{sugra}}_{1111}(1,0) = \frac{4(\Delta!)^2}{(2\Delta)!}|_{\Delta=1} = 2\ , \tag{5.6.2}$$

The unmixing equations are

$$A^{(0)}_{1111}(1,0) = \left(C^{(0)}_{1,0}\right)^2\ , \quad M^{\text{sugra}}_{1111}(1,0) = \gamma^{\text{sugra}}_{1,0}\left(C^{(0)}_{1,0}\right)^2\ . \tag{5.6.3}$$

Solving these equations we get

$$\gamma^{\text{sugra}}_{1,0} = 2\ , \quad C^{(0)}_{1,0} = 1\ . \tag{5.6.4}$$

At weight $\Delta = 3$ there are two possible exchanged operators, $\mathcal{O}_2\mathcal{O}_2$ and $\mathcal{O}_1\partial^2\mathcal{O}_1$, which gives the matrices of OPE coefficients for free theory and supergravity:

$$\hat{A}^{(0)}_{3,0} = \begin{pmatrix} A^{(0)}_{1111} & 0 \\ 0 & A^{(0)}_{2222}\end{pmatrix}_{(3,0)} = \begin{pmatrix} \frac{1}{10} & 0 \\ 0 & \frac{1}{18}\end{pmatrix}\ , \quad \hat{M}^{\text{sugra}}_{3,0} = \begin{pmatrix} M^{\text{sugra}}_{1111} & M^{\text{sugra}}_{1122} \\ M^{\text{sugra}}_{1122} & M^{\text{sugra}}_{2222}\end{pmatrix}_{(3,0)} = \begin{pmatrix} \frac{1}{5} & \frac{1}{3} \\ \frac{1}{3} & \frac{5}{9}\end{pmatrix}\ , \tag{5.6.5}$$

where the new coefficients for general odd Δ are

$$M^{\text{sugra}}_{1122}(\Delta,0) = \frac{2(\Delta-1)\Delta!(\Delta+2)!}{3(\Delta+1)(2\Delta)!}\,, \quad M^{\text{sugra}}_{2222}(\Delta,0) = \frac{(\Delta-1)^2(\Delta+2)^2(\Delta!)^2}{9(2\Delta)!}\,, \tag{5.6.6}$$

the rest were already spelled out in (5.6.2). To obtain the $\mathcal{O}(1/c)$ contribution to the anomalous dimensions and the leading contributions to the three-point functions, we solve the unmixing equations in matrix form (5.5.14) and (5.5.15):

$$\hat{\gamma}^{\text{sugra}}_{3,0} = \begin{pmatrix} 12 & 0 \\ 0 & 0 \end{pmatrix}\,, \quad \mathbb{C}^{(0)}_{3,0} = \begin{pmatrix} \frac{1}{2\sqrt{15}} & \frac{1}{2\sqrt{3}} \\ \frac{\sqrt{5}}{6\sqrt{3}} & -\frac{1}{6\sqrt{3}} \end{pmatrix}\,, \tag{5.6.7}$$

where $\hat{\gamma}^{\text{sugra}}$ are the eigenvalues of the matrix $\hat{M}^{\text{sugra}} \cdot \left(\hat{A}^{(0)}\right)^{-1}$. The eigenvectors are the columns of the orthonormal matrix $\tilde{c}$ which gives the three-point functions $\mathbb{C}^{(0)} = \left(\hat{A}^{(0)}\right)^{\frac{1}{2}} \cdot \tilde{c}$, as explained above. Note that there is only one non-zero anomalous dimension at $(\Delta, p) = (3,0)$, namely $\gamma^{\text{sugra}}_{3,0} = 12$. Going to higher Δ and p this structure will continue, at each weight there is only one non-zero anomalous dimension, as expected from the higher-dimensional symmetry considerations. Let us look at a few more examples.

At weight $(5,0)$, there are three exchanged operators in the double-trace spectrum, $\mathcal{O}_3\mathcal{O}_3$, $\mathcal{O}_2\partial^2\mathcal{O}_2$ and $\mathcal{O}_1\partial^4\mathcal{O}_1$ and conformal block analysis of the appropriate correlators gives the symmetric matrices

$$\hat{A}^{(0)}_{5,0} = \begin{pmatrix} A^{(0)}_{1111} & 0 & 0 \\ & A^{(0)}_{2222} & 0 \\ & & A^{(0)}_{3333} \end{pmatrix}_{(5,0)} = \begin{pmatrix} \frac{1}{126} & 0 & 0 \\ & \frac{4}{81} & 0 \\ & & \frac{1}{75} \end{pmatrix}\,,$$

$$\hat{M}^{\text{sugra}}_{5,0} = \begin{pmatrix} M^{\text{sugra}}_{1111} & M^{\text{sugra}}_{1122} & M^{\text{sugra}}_{1133} \\ & M^{\text{sugra}}_{2222} & M^{\text{sugra}}_{2233} \\ & & M^{\text{sugra}}_{3333} \end{pmatrix}_{(5,0)} = \begin{pmatrix} \frac{1}{63} & \frac{2}{27} & \frac{1}{15} \\ & \frac{28}{81} & \frac{14}{45} \\ & & \frac{7}{25} \end{pmatrix}\,, \tag{5.6.8}$$

where the expressions for general odd Δ for the new coefficients are

$$\begin{aligned}
M^{\text{sugra}}_{1133}(\Delta,0) &= \frac{(\Delta-3)(\Delta-1)(\Delta+4)\Delta!(\Delta+2)!}{30(\Delta+1)(2\Delta)!}\,, \\
M^{\text{sugra}}_{2233}(\Delta,0) &= \frac{(\Delta-3)(\Delta-1)^2(\Delta+2)(\Delta+4)\Delta!(\Delta+2)!}{180(\Delta+1)(2\Delta)!}\,, \\
M^{\text{sugra}}_{3333}(\Delta,0) &= \frac{(\Delta-3)^2(\Delta-1)^2(\Delta+2)(\Delta+4)^2\Delta!(\Delta+2)!}{3600(\Delta+1)(2\Delta)!}\,.
\end{aligned} \tag{5.6.9}$$

Solving the unmixing equations (5.5.13) for the anomalous dimensions, we find that all are zero but one:

$$\gamma^{\text{sugra}}_{5,0} = 30\,. \tag{5.6.10}$$

Starting from 3×3 matrices as in the present case, there is a zero-degeneracy, because more than one operator have zero anomalous dimension. Due to this degeneracy the free three-point functions cannot be completely fixed but we are left with two free parameters b_1 and b_2. In particular, of the eigenvectors forming the matrix $\tilde{c}$, the two which correspond to the zero eigenvalues are not unique. Thus, the leading three-point functions is:

$$\mathbb{C}^{(0)}_{5,0} = \begin{pmatrix} \frac{1}{3\sqrt{210}} & -\frac{3\sqrt{5}\,b_1+b_2}{3\sqrt{690}\,n_{12}} & \frac{\sqrt{5}\,b_1-15\,b_2}{15\sqrt{138}\,n_{12}} \\ \frac{\sqrt{14}}{9\sqrt{15}} & \frac{\sqrt{46}\,b_2}{9\sqrt{15}\,n_{12}} & -\frac{\sqrt{46}\,b_1}{9\sqrt{15}\,n_{12}} \\ \frac{\sqrt{7}}{5\sqrt{30}} & \frac{2\sqrt{5}\,b_1-7\,b_2}{5\sqrt{690}\,n_{12}} & \frac{7\sqrt{5}\,b_1+10\,b_2}{25\sqrt{138}\,n_{12}} \end{pmatrix}, \tag{5.6.11}$$

where $n_{12}=\sqrt{b_1^2+b_2^2}$ and $b_2b_3>b_1b_4$. One of the coefficients b_1 and b_2 will be fixed when solving the mixing problem for four-derivative corrections, since the supergravity data goes into the higher-derivative corrections unmixing equations, see (5.5.13).

We solve the mixing problem for higher Δ analogously to the previous cases. Let us look at a few examples for $p=1$ before analysing the results.

5.6.2 $p = 1$ Sector

Let us start by determining the free theory coefficients. For $p > 0$ we need to consider conformal block expansions of not only G_{qqqq} but also $G_{q_1q_2q_1q_2}$ (5.3.17). These correlators decompose into one half-BPS block with $\Delta = p = q_1 + q_2 - 1$ and long blocks with coefficients $A^{(0)}_{q_1q_2q_1q_2}$. The free theory coefficients for $q_1 = q_2$ were given in (5.6.1) and the coefficients for mixed charges are:

$$
\begin{aligned}
A^{(0)}_{q_1q_2q_1q_2} = & \frac{(-1)^{q_1+q_2}\, 2^{-\left\lfloor \frac{p-|q_{12}|}{2}\right\rfloor + \frac{p+1}{2} - \frac{|q_{12}|}{2}}}{\left(\left(\sqrt{2}-1\right)(-1)^p + \left(1+\sqrt{2}\right)(-1)^{|q_{12}|}\right)(2\Delta)!(p+1)!(\Delta-p)(\Delta+p+1)} \\
& \times \frac{(2(p+1))!(\Delta-|q_{12}|)!(\Delta+|q_{12}|)!(\Delta+q_1+q_2-1)!}{(\Delta-q_1-q_2+1)!(2q_1-1)(2q_2-1)(-p+q_1+q_2-2)!(p+q_1+q_2-1)!} \\
& \times \frac{\left(\left\lfloor\frac{1}{2}(p-|q_{12}|+1)\right\rfloor + |q_{12}|\right)!\left(\left\lfloor\frac{p-|q_{12}|}{2}\right\rfloor + \left\lfloor\frac{1}{2}(p-|q_{12}|+1)\right\rfloor + |q_{12}|\right)!}{(p-|q_{12}|)!\left(\left\lfloor\frac{p-|q_{12}|}{2}\right\rfloor + |q_{12}|\right)!\left(2\left(\left\lfloor\frac{1}{2}(p-|q_{12}|+1)\right\rfloor + |q_{12}|\right)\right)!}\,, \qquad (5.6.12)
\end{aligned}
$$

where $q_{12} = q_1 - q_2$. Note that in the denominator of the first line, the eigenvalue $\delta^{(2)} = (\Delta - p)(\Delta + p + 1)$ (see (5.6.18)) appears, this was also observed in 4d free theory in [41] and suggests that $\Delta^{(2)} H^{\text{free}}$ should be a simple object as was confirmed in subsection 5.3.1.

For the simplest case $\Delta = 2$, $p = 1$ there is only one exchanged operator in the double-trace spectrum, $\mathcal{O}_1\mathcal{O}_2$, and we perform the conformal block analysis for the correlator H_{1212}. The free theory (see (5.6.12)) and supergravity coefficients are

$$
A^{(0)}_{1212}(2,1) = \frac{1}{4}\,, \qquad M^{\text{sugra}}_{1212}(2,1) = \frac{2(1+\Delta)\Delta!(1+\Delta)!}{3(2\Delta)!}\Big|_{\Delta=2} = 1 \qquad (5.6.13)
$$

and solving the unmixing equations gives

$$
\gamma^{\text{sugra}}_{2,1} = 4\,, \quad C^{(0)}_{2,1} = \frac{1}{2}\,. \qquad (5.6.14)
$$

Next, let us solve the mixing problem for $(4,1)$ where two types of operators are exchanged, $\mathcal{O}_1\partial\mathcal{O}_2$ and $\mathcal{O}_2\mathcal{O}_3$. The conformal block coefficients are:

$$\hat{A}^{(0)}_{4,1} = \begin{pmatrix} A^{(0)}_{1212} & 0 \\ 0 & A^{(0)}_{2323} \end{pmatrix}_{(4,1)} = \begin{pmatrix} \frac{5}{84} & 0 \\ 0 & \frac{1}{30} \end{pmatrix} , \; \hat{M}^{\text{sugra}}_{4,1} = \begin{pmatrix} M^{\text{sugra}}_{1212} & M^{\text{sugra}}_{1223} \\ M^{\text{sugra}}_{1223} & M^{\text{sugra}}_{2323} \end{pmatrix}_{(4,1)} = \begin{pmatrix} \frac{5}{21} & \frac{1}{3} \\ \frac{1}{3} & \frac{7}{15} \end{pmatrix} , \tag{5.6.15}$$

where the new coefficients in terms of general even Δ are given by:

$$\begin{aligned} M^{\text{sugra}}_{1223} &= \frac{(\Delta-2)(\Delta+1)(\Delta+3)\Delta!(\Delta+1)!}{15(2\Delta)!} , \\ M^{\text{sugra}}_{2323} &= \frac{(\Delta+1)\left((\Delta+3)(\Delta-2)\right)^2 \Delta!(\Delta+1)!}{150(2\Delta)!} . \end{aligned} \tag{5.6.16}$$

Solving the unmixing equations gives the anomalous dimensions and three-point functions

$$\gamma^{\text{sugra}}_{4,1} = 18 , \quad \mathbb{C}^{(0)}_{4,1} = \begin{pmatrix} \frac{\sqrt{5}}{3\sqrt{42}} & -\frac{\sqrt{5}}{6\sqrt{3}} \\ \frac{\sqrt{7}}{3\sqrt{30}} & \frac{1}{3\sqrt{15}} \end{pmatrix} . \tag{5.6.17}$$

Again there is only one non-zero anomalous dimension, $\gamma^{\text{sugra}}_{4,1} = 18$, as expected.

5.6.3 Anomalous Dimensions after Unmixing

We can solve the mixing problem for any pair (Δ, p) analogously to the above examples, where for every (Δ, p) we use (5.5.3) to determine the list of double-trace operators in the spectrum. Solving the unmixing equations for supergravity for many pairs of (Δ, p) we find that, as expected from the 4d conformal symmetry considerations in subsection 5.4.2 and the 4d effective spin conjecture (5.5.7), there is only one operator with non-zero anomalous dimension exchanged. Hence, there is no non-zero degeneracy in the supergravity limit in 1d. The value of the anomalous dimension is as expected

$$\gamma^{\text{sugra}}_{\Delta,p} = \delta^{(2)} = (\Delta-p)(\Delta+1+p) , \tag{5.6.18}$$

which is the eigenvalue of $\Delta^{(2)}$ (5.2.15) acting on the superconformal blocks:

$$\Delta^{(2)}\left(\frac{x\,y}{x-y}B^{\text{long}}_{\Delta,p,p_{12},p_{34}}\right)=\delta^{(2)}\left(\frac{x\,y}{x-y}B^{\text{long}}_{\Delta,p,p_{12},p_{34}}\right)\,. \tag{5.6.19}$$

This was predicted from the 4d conformal symmetry in (5.4.39).

We have now studied the double-trace spectrum of correlators in the supergravity limit and their anomalous dimensions in detail. We have seen that they do agree with the predictions from 4d conformal symmetry and thus that supergravity correlators in AdS$_2$×S^2 indeed have 4d conformal symmetry. In the next sections we will consider higher-derivative corrections and how the higher-dimensional conformal symmetry breaks (except for an infinite number of correlators of specific spherical harmonics). On the other hand, all higher-derivative corrections can be deduced from a higher-dimensional scalar effective field theory.

5.7 Higher-Derivative Corrections

In this section we study higher-derivative corrections described by a small-a expansion. We first derive the form of lowest-charge higher-derivative corrections, with any number of derivatives, from crossing symmetry and $x\to 0$ behaviour in subsection 5.7.1. In subsection 5.7.2 we obtain all four-derivative corrections to all half-BPS four-point correlators described by tree-level supergravity from a 4d effective action by evaluating generalised AdS$_2$×S^2 Witten diagrams with derivatives. Finally, in subsection 5.7.3 we briefly discuss how by acting with inverse higher-order differential operators on the higher-derivative corrections one can construct an object that transforms like a 4d conformal correlator in specific cases, as discussed in subsection 5.1.1. We will see that the 4d conformal symmetry is broken for general four-derivative corrections, however it is intact for an infinite tower of correlators of operators with $SU(2)$ charges $\{p_i\}=pp11$ and crossing versions.

5.7.1 Lowest-Charge Higher-Derivative Corrections

Even though we mainly focus on the four-derivative corrections in this chapter, let us also discuss the general structure of any higher-derivative corrections in 1d SCFTs. Recall from the review in subsection 2.3.2 that in [30] the authors argued that for $\mathrm{AdS}_{d>2}$ solutions to the crossing equations with a conformal block expansion truncated to spin $L = 0, 2, 4, \dots$ are dual to quartic contact interactions in the bulk. For a fixed spin L, there are $L/2+1$ independent interactions with number of derivatives running from $2L, 2L+2, ..., 3L$. In this subsection we will find that reducing these solutions to AdS_2 by taking the holomorphic limit, the $L/2+1$ solutions collapse to a single solution proportional to $\bar{D}_{jjjj}$, where $j = L/2+1$. This solution corresponds to a $2L$-derivative interaction, i.e. the interaction with the lowest number of derivatives in the spin-L tower of solutions. This is the case for the lowest-charge correlators, when going to higher charges, this degeneracy will break. In this subsection, we focus on the lowest-charge correlators but consider interactions with any number of derivatives. For later sections however, we focus on the four-derivative corrections but consider all higher-charge correlators.

Let us deduce the form of higher-derivative corrections to the $p_i = 1$ correlator. Let us start with an ansatz for a higher-derivative correction at $\mathcal{O}(a^{\#}/c)$ analogous to supergravity in (5.4.3)[5]:

$$\langle \mathcal{O}_1\mathcal{O}_1\mathcal{O}_1\mathcal{O}_1\rangle_{a^{\#}/c} = g_{12}g_{34}\frac{x}{y}(x-y)a(x) \ , \tag{5.7.1}$$

where we get the same conditions from crossing invariance and the $x \to 0$ behaviour:

$$a(1-x) = a(x) \ , \quad a\left(\frac{1}{x}\right) = x^2 a(x) \ , \quad a(x) = \mathcal{O}(1) \ . \tag{5.7.2}$$

Again we make an ansatz for $a(x)$ of the following form

$$a(x) = p(x)\log x^2 + p(1-x)\log(1-x)^2 + r(x) \ , \tag{5.7.3}$$

[5]See subsection 5.4.1 for a more careful discussion of four-point crossing in 1d which could easily be extended to higher-derivative corrections to see that $a^0(x)$ and $a^+(x)$ have the same functional form. In this subsection we will consider $a(x)$ with x in the regions $x \in (0,1)$ and $x \in (1,\infty)$.

where p, r are rational functions and $r(1-x) = r(x)$. Further we have

$$p(x) = \frac{1}{x-1}\frac{1}{x^k(1-x)^k}\sum_{i=0}^{m} b_i x^i \;, \quad r(x) = \frac{1}{x-1}\frac{1}{x^k(1-x)^k}\sum_{i=0}^{m} c_i x^i \;, \tag{5.7.4}$$

where k and m are integers. Plugging this into (5.7.2), there are only new solutions for $k = 2q$ and $m = 3k = 6q$ with an integer q. Solutions for odd k reduce to superpositions of lower-k solutions in 1d. The $q = 0$ case corresponds to supergravity. Going to higher q we find solutions that are superpositions of a new correction and lower-q solutions (supergravity and lower-derivative terms). The new corrections are of the form

$$H^q_{1111} = x^2 a(x) = -u\,(1 + u^q + v^q)\bar{D}^{\text{hol}}_{q+1\,q+1\,q+1\,q+1} + \dots \;, \tag{5.7.5}$$

where the dots denote terms with lower q. As we have seen in chapter 4 the averaged anomalous dimensions of double-trace operators in the conformal block decomposition of these correlators encode the number of derivatives of the dual quartic interactions in the bulk [30]. First, decomposing the free theory $G^{(0)}_{1111}$ (5.3.10) gives the identity operator and the coefficients of the long blocks $A^{(0)}_{1111}(\Delta, 0)$ (5.6.1):

$$A^{(0)}_{1111}(\Delta, 0) = \frac{\left(1 + (-1)^{\Delta+1}\right)(\Delta!)^2}{(2\Delta)!} \;. \tag{5.7.6}$$

Expanding the higher-derivative corrections (5.7.5) at $\mathcal{O}(1/c)$ in terms of the long blocks according to (5.5.11) we get the averaged anomalous dimensions

$$\gamma(q) = N(q) P^{(q)}(\Delta) \;, \tag{5.7.7}$$

with

$$N(q) = \frac{4^{q+1}(1+2q)!}{(2(1+2q))!} \tag{5.7.8}$$

and $P^{(q)}(\Delta)$ are polynomials of degree Δ^{4q}. For the first few cases we get

$$P^{(0)}(\Delta) = 1 \;,$$
$$P^{(1)}(\Delta) = \frac{1}{2}\left(\Delta^4 + 2\,\Delta^3 + \Delta^2 + 6\right) \;,$$

$$P^{(2)}(\Delta) = \frac{1}{8}\left(\Delta^8 + 4\,\Delta^7 + 10\,\Delta^6 + 16\,\Delta^5 + 289\,\Delta^4 + 556\,\Delta^3 + 276\,\Delta^2 + 864\right)\ \ldots \tag{5.7.9}$$

The fact that the averaged anomalous dimensions scale like Δ^{4q} in the large-Δ limit implies that the corrections (5.7.5) correspond to a $4q$-derivative interaction in the dual bulk field theory. As mentioned before, the $q = 0$ case corresponds to the supergravity correlator which is dual to a spin-0 or ϕ^4 interaction. The next case, $q = 1$, is given by $-u(1+u+v)\bar{D}^{\text{hol}}_{2222}$ plus a spin-0 term and corresponds to a spin-2 correction and is dual to a four-derivative interaction (see (5.7.17)). As predicted in the beginning of this subsection, there is only one solution per spin L in 1d. In principle there is another spin-2 solution corresponding to six derivatives and to $k = 3$, but this reduces to the four-derivative solution plus supergravity. This is only true for the lowest-charge correlators though and going to higher charges, this degeneracy will be lifted.

We have now discussed the general structure of higher-derivative corrections to the lowest-charge supergravity correlators and in the next section we will obtain all four-derivative corrections to the half-BPS correlators for all spherical harmonics.

5.7.2 Effective Action

In this section we compute all four-derivative corrections from the 4d scalar effective action (5.2.27). We follow the procedure outlined in chapter 3, in particular section 3.4 where the α'^5 corrections, which correspond to four-derivative interactions, were computed. The only independent contact terms with four derivatives one can write down are the main contribution $(\nabla\phi.\nabla\phi)^2$ and the ambiguity $\nabla^2\nabla_\mu\phi\nabla^\mu\phi\,\phi^2$. The complete effective action at this order is then

$$S_{\text{4-deriv}} = B_0 S^{\text{main}}_{\text{4-deriv}} + C_0 S^{\text{amb}}_{\text{4-deriv}}\ , \tag{5.7.10}$$

with

$$S^{\text{main}}_{\text{4-deriv}} = \frac{3}{4!}\int_{\text{AdS}\times\text{S}} d^2\hat{X} d^2\hat{Y}\,(\nabla\phi.\nabla\phi)\,(\nabla\phi.\nabla\phi)\ ,$$
$$S^{\text{amb}}_{\text{4-deriv}} = \frac{6}{4!}\int_{\text{AdS}\times\text{S}} d^2\hat{X} d^2\hat{Y}\nabla^2\nabla_\mu\phi\nabla^\mu\phi\,\phi^2\ . \tag{5.7.11}$$

Note that the ϕ^4 interaction is not considered as an ambiguity here, since it describes supergravity and not a zero-derivative correction arising from quantum gravity as in other dimensions.

The Witten diagram expression for the main correction is

$$\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle^{\text{4-deriv;main}}_{\text{int}}$$
$$= \frac{1}{4!}\frac{(\mathcal{C}_1)^4}{(-2)^4}\int_{\text{AdS}\times\text{S}} d^2\hat{X} d^2\hat{Y}\frac{N_{12}N_{34}+N_{13}N_{24}+N_{14}N_{23}}{(P_1+Q_1)^2(P_2+Q_2)^2(P_3+Q_3)^2(P_4+Q_4)^2}\ , \tag{5.7.12}$$

where

$$N_{ij} = X_i.X_j + Y_i.Y_j + P_iP_j - Q_iQ_j\ . \tag{5.7.13}$$

Furthermore, the ambiguity is

$$\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle^{\text{4-deriv;amb}}_{\text{int}}$$
$$= -\frac{1}{4!}\frac{(\mathcal{C}_1)^4}{(-2)^4}\int_{\text{AdS}\times\text{S}}\frac{d^2\hat{X} d^2\hat{Y}}{\prod_i(P_i+Q_i)}\sum_{i<j}\frac{L_{ij}}{(P_i+Q_i)(P_j+Q_j)}\ , \tag{5.7.14}$$

where

$$L_{ij} = X_i.X_j - Y_i.Y_j + P_iP_j + Q_iQ_j\ . \tag{5.7.15}$$

We get the explicit expressions in position space for the correlators at four derivatives by expanding (5.7.12) and (5.7.14) in terms of D- and B-functions, see (3.1.24) and (3.1.28). To expand a general decorated integral of the form (3.3.9) obtained from a contact interaction with any number of covariant derivatives, use (3.3.10)

$$(-2)^{2\Sigma_X+2\Sigma_Y}\sum_{p_i=0}^{\infty}\Big(\prod_{i=1}^{4}(-1)^{p_i-1}\frac{(p_i)_{\Delta_i+n_i-1}}{\Gamma(\Delta_i)}D^{(d)}_{p_i-1+\Delta_i+n_i-n_i^P}(X_i)B^{(d)}_{p_i-1+n_i^Q}(Y_i)\Big)$$
$$\times\Big(\sum_{i<j}(X_i.X_j)^{n^X_{ij}}(Y_i.Y_j)^{n^Y_{ij}}\Big)\ , \tag{5.7.16}$$

where the different labels are explained in the discussion above (3.3.10). To perform explicit calculations in 1d position space one needs to express the correlators above in terms of holomorphic $\bar{D}$-functions and we spell out a few examples here. Note that the coefficients in (5.7.10) are unfixed and we choose the overall normalisation of the $H^{\text{4-deriv}}_{p_1p_2p_3p_4}$ below such that they agree with the supergravity correlators in (5.4.28). The explicit expressions for some of the correlators are:

$$\begin{aligned}
H^{\text{4-deriv}}_{1111} &= 3u\left(\bar{D}^{\text{hol}}_{1111} - 5\,(1+u+v)\,\bar{D}^{\text{hol}}_{2222}\right)\ ,\\
H^{\text{4-deriv}}_{pp11} &= \frac{(-1)^{p+1}\,u^p}{(p-1)!}\left(f_1(p)\bar{D}^{\text{hol}}_{pp11} + f_2(p)\,u\,\bar{D}^{\text{hol}}_{p+1\,p+1\,1\,1} + f_3(p)\,(1+u+v)\,\bar{D}^{\text{hol}}_{p+1\,p+1\,2\,2}\right)\ ,\\
H^{\text{4-deriv}}_{p1p1} &= \frac{(-1)^{p+1}\,u^{\frac{p+1}{2}}}{(p-1)!\,y^{p-1}}\left(f_1(p)\bar{D}^{\text{hol}}_{p1p1} + f_2(p)\,\bar{D}^{\text{hol}}_{p+1\,1\,p+1\,1} + f_3(p)\,(1+u+v)\,\bar{D}^{\text{hol}}_{p+1\,2\,p+1\,2}\right)\ ,\\
H^{\text{4-deriv}}_{p11p} &= \frac{(-1)^{p+1}\,u^{\frac{p+1}{2}}}{(p-1)!\,y^{p-1}}\left(f_1(p)\bar{D}^{\text{hol}}_{p11p} + f_2(p)\,\bar{D}^{\text{hol}}_{p+1\,1\,1\,p+1} + f_3(p)\,(1+u+v)\,\bar{D}^{\text{hol}}_{p+1\,2\,2\,p+1}\right)\ ,\\
H^{\text{4-deriv}}_{2222} &= 2\frac{u^2}{y^2}\left(\left[-8\,C_0\,\left(1-y+y^2\right) + 2\left(41-6y+6y^2\right)\right]\bar{D}^{\text{hol}}_{2222} + 35\,u\left(y^2-1\right)\bar{D}^{\text{hol}}_{3322}\right.\\
&\quad\left. + 35\,(y-2)\,y\bar{D}^{\text{hol}}_{3223} - 63\,(1+u+v)\left(1-y+y^2\right)\bar{D}^{\text{hol}}_{3333}\right)\ , \qquad (5.7.17)
\end{aligned}$$

where

$$\begin{aligned}
&f_1(p) = 2\,C_0\,p(p-1) + p^2\,(4\,p^2 - 8\,p + 1)\ ,\ \ f_2(p) = -(1+2p)(2\,p^2 - 3\,p + 1)\ ,\\
&f_3(p) = (1+2p)(3+2p)\ . \qquad (5.7.18)
\end{aligned}$$

Note that $H^{\text{4-deriv}}_{1111}$ agrees with the prediction in the previous subsection, see (5.7.5) for $q = 1$. We investigate the correlators in terms of the 4d conformal symmetry in the next subsection before we analyse the double-trace spectrum of these correlators and solve the mixing problem in the following section.

5.7.3 Breaking of 4d Conformal Symmetry

In this subsection we investigate the higher-derivative corrections in terms of the 4d conformal symmetry. In the previous section we have seen that one can derive all four-derivative corrections to all half-BPS correlators with any charges, described by tree-level supergravity, from a four-dimensional effective field theory. Thus, higher-derivative corrections to correlators in AdS$_2$×S^2 have a four-dimensional symmetry. However, the four-dimensional *conformal* symmetry will generally be broken for higher-derivative corrections. As described in subsection 5.1.1 we will attempt to construct an object invariant under higher-dimensional conformal symmetry by acting with negative powers of differential operators related to $\Delta^{(2)}$ on the correlators. In this way, we obtain an object with the right dimensions to be invariant under 4d conformal symmetry, however, this only works for a subset of correlators with specific charge configurations, as we will explain shortly. In this subsection, we describe how this can be done for four-derivative corrections and show that there is an infinite tower of correlators of specific KK modes which can be obtained from the 4d conformal symmetry. Yet for general charges, the symmetry is broken. We will discuss the breaking of the symmetry and also briefly comment on implications for AdS$_5$×S^5.

$\langle\mathcal{O}_p\mathcal{O}_p\mathcal{O}_1\mathcal{O}_1\rangle$ and crossing versions

We start by studying correlators with external charges $\{p_i\} = pp11$ and their crossing versions. To get an object with the dimensions of a 4d conformal correlator, we have to act with negative powers of a differential operator on the correlator. By dimensional analysis, we saw in subsection 5.1.1 that for a vertex with $k+2$ derivatives (where supergravity has two derivatives) one has to act with a general $\left(\Delta^{(2)}\right)^{-k/2}$ on the corresponding correlator. Hence, for a four-derivative interaction (six derivatives in total when counting supergravity), we act with an inverse fourth-order differential operator on the correlator. In practice, we consider objects which have 4d conformal symmetry, expand them in internal coordinates to obtain all higher-charge correlators, analogous to the considerations in subsection 5.4.2, and act on them with a fourth-order differential operator to reconstruct the higher-charge higher-derivative corrections. The 4d conformal objects we have to consider are the 4d conformal blocks (5.4.30) with the appropriate twist and spin. We start from an ansatz for a general fourth-order Casimir and fix the coefficients by comparing to the explicit results obtained from the 4d effective action in (5.7.17).

By superconformal symmetry, all higher-order Casimir operators have to be of the form $\Delta^{(2)}\left(c_1\mathcal{D}_x^a + c_2\mathcal{D}_y^b\right)$ and thus a general fourth-order Casimir is given by

$$\Delta^{(4)} = c_1 + c_2\Delta^{(2)} + c_3\Delta^{(2)}\mathcal{D}_x + c_4\Delta^{(2)}\mathcal{D}_y \ , \tag{5.7.19}$$

where $\Delta^{(2)}$, $\mathcal{D}_x$, $\mathcal{D}_y$ were defined in (5.2.15) and the c_i are unfixed coefficients. As explained in subsection 5.4.2 due to the 4d conformal symmetry there is only one 4d block contributing to the 4d uplift of the 1d correlators at each spin. For supergravity it is the spin-0 block and since the four-derivative correction corresponds to a spin-2 correction, we consider the 4d spin-0 and spin-2 blocks as the objects which play the leading role in the 4d conformal symmetry. Therefore, we propose that a subset of the four-derivative corrections can be reconstructed by acting with a fourth-order Casimir of the form (5.7.19) on the holomorphic limit of the 4d spin-0 and spin-2 blocks. Note that the blocks reproduce the $\log u$ piece of the correlators rather than the full functions. The higher-charge versions of the spin-L blocks can be obtained by replacing $x_{ij}^2 \to x_{ij}^2\left(1+g_{ij}^2\right)$, expanding in g_{ij}^2 and taking the coefficients of the appropriate powers in g_{ij}^2 analogous to subsection 5.4.2. Note that higher-charge versions of the spin-0 block exactly correspond to the $\log u$ piece of the supergravity correlators. The relevant equation at different charges p_i is then

$$H^{\text{4-deriv, main}}_{p_1p_2p_3p_4}|_{\log u} = \Delta^{(4)}_{\text{spin-0}}\left(\text{4d spin-0 block}\right)^{\text{hol}}_{p_1p_2p_3p_4} + \Delta^{(4)}_{\text{spin-2}}\left(\text{4d spin-2 block}\right)^{\text{hol}}_{p_1p_2p_3p_4} \ , \tag{5.7.20}$$

where we compare our ansatz to the four-derivative corrections deduced from the effective action given in (5.7.17) at different charges. Note that we focus on the main contribution in (5.7.10) here.

Solving (5.7.20) for $H^{\text{4-deriv, main}}_{pp11}$ partially fixes the coefficients in (5.7.19) to

$$\Delta^{(4)}_{\text{spin-0}} = \frac{1}{6}\Delta^{(2)} + \frac{5}{12}\Delta^{(2)}\mathcal{D}_x + a_3\Delta^{(2)}\mathcal{D}_y \ , \quad \Delta^{(4)}_{\text{spin-2}} = -\frac{1}{45}\Delta^{(2)} + \frac{1}{90}\Delta^{(2)}\mathcal{D}_x + b_3\Delta^{(2)}\mathcal{D}_y \ , \tag{5.7.21}$$

where a_3 and b_3 remain unfixed since there is no y-dependence for $\{p_i\} = pp11$.

We can fix the remaining coefficients by considering $H^{\text{4-deriv, main}}_{p1p1}$ and $H^{\text{4-deriv, main}}_{p11p}$. We thus solve (5.7.20) by comparing to the results from (5.7.17) at the corresponding charges. This completely fixes the differential operators to

$$\Delta^{(4)}_{\text{spin-0}} = \frac{1}{12}\left(2\,\Delta^{(2)} + 5\left(\Delta^{(2)}\right)^2\right)\;,\quad \Delta^{(4)}_{\text{spin-2}} = \frac{1}{90}\left(-2\,\Delta^{(2)} + \left(\Delta^{(2)}\right)^2\right)\;. \tag{5.7.22}$$

To summarise, these Casimir operators can lift the lowest-charge four-derivative correction correctly for $\{p_i\} = pp11$ and crossing versions. So the 4d conformal symmetry is satisfied by an infinite tower of four-derivative corrections. This is highly non-trivial, since the 4d blocks themselves are not crossing symmetric and also the Casimir operators do not preserve crossing symmetry (only under exchange of $1 \leftrightarrow 2$). Hence, an operator which does not preserve crossing symmetry acts on an object which is not crossing symmetric, and one obtains a crossing symmetric correlator. However, going to different charges there are no crossing symmetric solutions to (5.7.20) and thus the symmetry is broken. We will further discuss this breaking of the hidden conformal symmetry below and comment on implications for AdS$_5$×S^5.

Breaking of the symmetry and implications for AdS$_5$×S^5

Recall that the higher-dimensional conformal symmetry arises when a correlator corresponds to a conformally invariant amplitude in flat space which is connected to a scaleless coupling. When the correlator does not correspond to such an amplitude we can rescale it by acting with differential operators of appropriate powers on it, and this works well for free theory. When the dual bulk interaction vertices contain derivatives, which is the case for general higher-derivative corrections, the higher-dimensional conformal symmetry breaks down (however, it might still be intact for an infinite tower of higher-derivative corrections with specific charges). This breaking of the symmetry can be anticipated because a reduction of derivative terms on the sphere will give a number of terms with different numbers of derivatives in AdS. Thus, we get terms which scale differently in AdS and we can therefore not rescale the corresponding correlator consistently.

We have investigated this in $AdS_2 \times S^2$ for the four-derivative corrections above. Let us now comment on the implications for the 10d conformal symmetry and its breaking in $AdS_5 \times S^5$. As discussed in subsection 5.1.1, in the supergravity approximation the correlator itself has higher-dimensional conformal symmetry whereas for free theory acting with the eighth-order differential operator $\Delta^{(8)}$ on the correlator rescales it such that it has 10d conformal symmetry. Now, consider the α'^3 corrections which correspond to a ϕ^4 interaction in the scalar effective action. We obtained all α'^3 corrections from the 10d scalar effective action in section 3.2. Based on the discussions above, since these corrections correspond to a zero-derivative interaction in the effective field theory, we expect the corresponding correlators to enjoy 10d conformal symmetry analogous to supergravity correlators. And this conjecture is indeed strongly supported by the results in [13] where the anomalous dimensions after unmixing associated with α'^3 corrections were obtained. The resulting anomalous dimensions are rational numbers which suggests that the 10d conformal symmetry is intact at this order.

To reconstruct these correlators from the 10d conformal symmetry by acting on them with inverse differential operators consider dimensional analysis. In the low-energy effective action in (2.3.18) supergravity is described by $\mathcal{R}$ which has two derivatives while the first quartic correction corresponds to $\mathcal{R}^4$ which therefore has six more derivatives than supergravity. So to obtain an object with the correct dimensions we act with an inverse sixth-order differential operator $\left(\Delta^{(6)}\right)^{-1}$ on the α'^3 corrections to the correlators. In practice, one would reconstruct the higher-charge α'^3 correlators from objects which do have 10d conformal symmetry, i.e. the 10d conformal blocks with appropriate spin and twist, and act with a differential operator $\Delta^{(6)}$ on them. Since the ϕ^4 interaction corresponds to a spin-0 correction (see subsection 2.3.2), the correct object to consider is the 10d spin-0 block uplifted to higher charges analogous to the supergravity case. We leave further discussions of this to [44]. Considering the α'^5 corrections which were obtained in section 3.4, we expect the 10d conformal symmetry to be broken. Similar to the four-derivative corrections in 1d discussed above, one might expect that by acting with a higher-order differential operator on the 10d spin-0 and spin-2 blocks uplifted to higher charges, one could reconstruct a subset of α'^5 corrections with specific charges, we do not consider this in detail here.

The breaking of the symmetry for α'^5 corrections was also observed from a different point of view in [14]. The authors found that the anomalous dimensions of operators in the OPE spectrum of α'^5 corrections contain square roots in some cases. This indicates a breaking of the 10d conformal symmetry. In more detail, consider the unmixing of the anomalous dimensions of double-trace operators in the OPE spectrum of α'^3 and α'^5 corrections. The anomalous dimensions after unmixing associated with α'^3 are rational [13], which suggests that the 10d conformal symmetry is intact, as mentioned before. An important point in these calculations is that the $\mathcal{O}(\alpha'^3 c^0)$ correction to the three-point functions is absent. In [14], the authors found that for the α'^5 corrections however, the $\mathcal{O}(\alpha'^5 c^0)$ correction to the three-point functions (analogous to the $\mathcal{O}(ac^0)$ corrections in (5.5.10)) is non-zero for some quantum numbers and this leads to square roots in some of the anomalous dimensions, which indicates breaking of the 10d conformal symmetry. We will discuss the mixing problem for four-derivative corrections in the 1d case in the following section.

5.8 Unmixing Four-Derivative Corrections

In this section, we solve the mixing problem for four-derivative corrections to half-BPS correlators in 1d SCFTs which are described by tree-level 4d $\mathcal{N} = 2$ supergravity. We have predicted all four-derivative corrections including all spherical harmonics in subsection 5.7.2. To solve the mixing problem we use the large-c and small-a expansion of the conformal block decomposition (5.5.11). We obtain a list of all operators in the double-trace spectrum at each weight (Δ, p) by using the parametrisations in (5.5.3) and (5.5.5). Further, to resolve the degeneracy of operators with different 1d quantum numbers contributing at the same weight (Δ, p), we use the unmixing equations given in (5.5.13). We perform the unmixing for many (Δ, p), predict general formulas for the anomalous dimensions and check that they agree with the prediction in terms of the 4d effective spin (5.5.6). Recall the conjecture for the 4d spin in terms of 1d quantum numbers is

$$l_{4d} = 2\left(i_A + r_A + i_{B^{\rm to}} + i_{B^{\rm te}} + r_{B^{\rm te}} - \frac{1+(-1)^{r_{B^{\rm te}}+1}}{2}\right) . \tag{5.8.1}$$

The four-derivative interaction corresponds to a spin-2 correction, thus we predict that we can have 4d spin-0 and spin-2. We therefore expect that only operators with quantum numbers satisfying $l_{4d} = 0, 2$ acquire non-zero anomalous dimensions and indeed we will find that our results agree with this prediction. We start by analysing the unmixing for $t = \Delta - p$ odd and even separately and presenting some examples. The first example where square roots appear is $(\Delta, p) = (5, 2)$ discussed around (5.8.20).

5.8.1 Unmixing for Odd t

We start by analysing the unmixing at weights (Δ, p) with $t = \Delta - p$ odd, where we have both class A and $B^{\rm to}$ operators. For each (Δ, p) use the parametrisation in (5.5.3) to list all operators in the double-trace spectrum at the given weight. Then construct $((d_A + d_{B^{\rm to}})) \times ((d_A + d_{B^{\rm to}}))$ matrices of correlators running over $\mathcal{D}_{\Delta,p}$ and solve the unmixing equations (5.5.13). Recall that class A and B operators do not mix, so they can be treated completely separately. The anomalous dimensions after unmixing are labelled as $(\gamma^A)^{\Delta,p}_{i_A,r_A}$ for class A operators and $\left(\gamma^{B^{\rm to}}\right)^{\Delta,p}_{i_{B^{\rm to}}}$ for class B. Let us discuss the same examples as for the supergravity limit in section 5.6, starting with the singlet sector.

$p = 0$ sector

For $p = 0$, there are only class A operators in the spectrum, so the operators contributing are the same as for supergravity. Starting with the simplest case $\Delta = 1$, only one exchanged operator $\mathcal{O}_1\mathcal{O}_1$ contributes, thus we perform the conformal block analysis of $H^{\text{4-deriv}}_{1111}$ and get

$$M^{\text{4-deriv}}_{1111}(1,0) = \frac{2\Delta^2(\Delta+1)\Delta!(\Delta+1)!}{(2\Delta)!}\Big|_{\Delta=1} = 4 \ , \tag{5.8.2}$$

where the free theory coefficients are given in (5.6.1) and the supergravity contributions were obtained in (5.6.2). The unmixing equations are

$$M^{\text{4-deriv}}_{1111}(1,0) = \gamma^{\text{4-deriv}}_{1,0}\left(C^{(0)}_{1,0}\right)^2 + 2\,\gamma^{\text{sugra}}_{1,0}C^{(0)}_{1,0}C^{\text{4-deriv}}_{1,0} \ , \quad 0 = C^{(0)}_{1,0}C^{\text{4-deriv}}_{1,0}. \tag{5.8.3}$$

To solve these equations, we use the supergravity results (5.6.4) which yields

$$\left(\gamma^{\text{4-deriv}}_{A}\right)^{1,0}_{0,0} = 4 \ , \quad C^{\text{4-deriv}}_{1,0} = 0 \ . \tag{5.8.4}$$

At weight $\Delta = 3$ there are two possible exchanged operators, $\mathcal{O}_2\mathcal{O}_2$ and $\mathcal{O}_1\partial^2\mathcal{O}_1$ and the matrix of OPE coefficients for the four-derivative correction is

$$\hat{M}^{\text{4-deriv}}_{3,0} = \begin{pmatrix} M^{\text{4-deriv}}_{1111} & M^{\text{4-deriv}}_{1122} \\ M^{\text{4-deriv}}_{1122} & M^{\text{4-deriv}}_{2222} \end{pmatrix}_{(3,0)} = \begin{pmatrix} \frac{72}{5} & \frac{4}{3}(15+C_0) \\ \frac{4}{3}(15+C_0) & \frac{8}{9}(38+5\,C_0) \end{pmatrix} , \tag{5.8.5}$$

where the coefficients for general Δ are

$$\begin{aligned} M^{\text{4-deriv}}_{1122}(\Delta,0) =& \frac{(\Delta+2)\,(\Delta-1)\,\Delta!(8\,C_0\,\Delta! + (\Delta-1)\Delta(\Delta+2)!)}{3(2\Delta)!} \ , \\ M^{\text{4-deriv}}_{2222}(\Delta,0) =& \frac{(\Delta+2)\,(\Delta-1)\,(\Delta!)^2}{18(2\Delta)!} \times \Big[16\,C_0\,(\Delta+2)\,(\Delta-1) \\ & + \Delta^6 + 3\Delta^5 - 3\Delta^4 - 11\Delta^3 + 26\Delta^2 + 32\Delta - 32\Big] \ , \end{aligned} \tag{5.8.6}$$

and the coefficients for free theory and supergravity were given in (5.6.5). We solve the unmixing equations in matrix form at orders $\mathcal{O}(a/c)$ and $\mathcal{O}(a)$, using the supergravity results (5.6.7):

$$\gamma^{\text{4-deriv}}_{3,0,i} = \left\{\frac{16}{3}(137+15C_0),\ \frac{64}{3}\right\} \ , \ \mathbb{C}^{\text{4-deriv}}_{3,0} = (-1+3C_0)\begin{pmatrix} \frac{-\sqrt{5}}{9\sqrt{3}} & \frac{1}{9\sqrt{3}} \\ \frac{\sqrt{5}}{27\sqrt{3}} & \frac{5}{27\sqrt{3}} \end{pmatrix} , \tag{5.8.7}$$

where $\gamma^{\text{4-deriv}}_{3,0,i}$ are the eigenvalues of the matrix $\left(\hat{M}^{\text{sugra}} + a\hat{M}^{\text{4-deriv}}\right)\cdot\left(\hat{A}^{(0)}\right)^{-1}$ at $\mathcal{O}(a)$ and i is labelling the non-zero anomalous dimensions. Recall that C_0 is the coefficient of the ambiguity in (5.7.10). While for supergravity there is only one anomalous dimension at every weight, there are two non-zero anomalous dimensions at $\mathcal{O}(a/c)$ for $(3,0)$, $(\gamma_A^{\text{4-deriv}})^{3,0}_{0,0} = \frac{16}{3}(137+15C_0)$ and $(\gamma_A^{\text{4-deriv}})^{3,0}_{1,0} = \frac{64}{3}$. We will explain the labels, interpret the result from a 4d point of view and give general formulas in subsection 5.8.3.

At weight $(5,0)$, there are three exchanged operators in the double-trace spectrum, $\mathcal{O}_3\mathcal{O}_3$, $\mathcal{O}_2\partial^2\mathcal{O}_2$ and $\mathcal{O}_1\partial^4\mathcal{O}_1$ and the conformal block analysis of the appropriate correlators gives the four-derivative contribution

$$\hat{M}^{\text{4-deriv}}_{5,0} = \begin{pmatrix} M^{\text{4-deriv}}_{1111} & M^{\text{4-deriv}}_{1122} & M^{\text{4-deriv}}_{1133} \\ & M^{\text{4-deriv}}_{2222} & M^{\text{4-deriv}}_{2233} \\ & & M^{\text{4-deriv}}_{3333} \end{pmatrix}_{(5,0)} = \begin{pmatrix} \frac{50}{7} & \frac{8}{27}(105+C_0) & \frac{4}{5}(30+C_0) \\ & \frac{32}{81}(352+7C_0) & \frac{16}{45}(321+14C_0) \\ & & \frac{24}{25}(111+7C_0) \end{pmatrix}, \tag{5.8.8}$$

where the free theory and supergravity coefficients are given in (5.6.8) and the expressions for general Δ for the new coefficients are

$$\begin{aligned} M^{\text{4-deriv}}_{1133}(\Delta,0) =& \frac{(\Delta-1)(\Delta-3)\Delta!\,(24\,C_0\,(\Delta+4)(\Delta+2)!+\Delta\,(\Delta+1)\,(\Delta-2)\,(\Delta+4)!)}{60(\Delta+1)(2\Delta)!}, \\ M^{\text{4-deriv}}_{2233}(\Delta,0) =& \frac{(\Delta+2)\,(\Delta+4)\,(\Delta-1)\,(\Delta-3)\,(\Delta!)^2}{360(2\Delta)!} \times \Big[32\,C_0\,(\Delta+2)\,(\Delta-1) \\ & + \Delta(\Delta+1)\left(\Delta(\Delta+1)\left(\Delta^2+\Delta-10\right)+88\right)-96\Big], \\ M^{\text{4-deriv}}_{3333}(\Delta,0) =& \frac{(\Delta-1)\,((\Delta+4)\,(\Delta-3))^2\,\Delta!(\Delta+2)!}{7200(\Delta+1)(2\Delta)!} \times \Big[48\,C_0\,(\Delta+2)\,(\Delta-1) \\ & + \Delta^6+3\Delta^5-11\Delta^4-27\Delta^3+226\Delta^2+240\Delta-288\Big]. \end{aligned} \tag{5.8.9}$$

We can now solve the unmixing equations (5.5.13) for the contributions to the anomalous dimensions at $\mathcal{O}(a/c)$ and the $\mathcal{O}(a)$ corrections to the three-point functions:

$$\gamma^{\text{4-deriv}}_{5,0,i} = \left\{\frac{80}{3}(424+21C_0),\ \frac{1204}{3}\right\},$$

$$\mathbb{C}^{\text{4-deriv}}_{5,0} = (-1+3C_0)\begin{pmatrix} -\frac{2\sqrt{14}}{27\sqrt{15}} & \frac{-1}{486\sqrt{42}}c & \frac{504-5c}{1458\sqrt{210}} \\ -\frac{16\sqrt{14}}{81\sqrt{15}} & \frac{-336+c}{243\sqrt{42}} & \frac{2(504-5c)\sqrt{2}}{2187\sqrt{105}} \\ \frac{4\sqrt{14}}{45\sqrt{15}} & -\frac{2016+c}{2430\sqrt{42}} & \frac{-504+5c}{1215\sqrt{210}} \end{pmatrix} , \tag{5.8.10}$$

with one unfixed parameter. Additionally, the unmixing equations at this order fix one of the coefficients b_1, b_2 in the leading three-point function (5.6.11) to

$$b_1 = -\frac{9}{4\sqrt{5}} b_2 \ . \tag{5.8.11}$$

There are two non-zero anomalous dimensions, which we will again discuss later from the point of view of the 4d conformal symmetry, they are given by $(\gamma_A^{\text{4-deriv}})^{5,0}_{0,0} = \frac{80}{3}(424+21C_0)$ and $(\gamma_A^{\text{4-deriv}})^{5,0}_{1,0} = \frac{1204}{3}$.

We solve the mixing problem for higher Δ analogously to the previous cases. Let us look at a few examples for $p = 1, 2$, where class B operators are relevant, before discussing t even.

$p = 1$ sector

Starting from $p = 1$ there are class A and class B operators exchanged, in the present case where t is odd, they are class B^{to} operators. These operators contribute from $t \geq 3$ and $p \geq 1$.

In the simplest case $\Delta = 2, p = 1$, there is only one exchanged class A operator in the double-trace spectrum, $\mathcal{O}_1\mathcal{O}_2$ and we perform the conformal block analysis at $\mathcal{O}(a/c)$ for the correlator H_{1212} and the OPE-coefficients are

$$\begin{aligned} M^{\text{4-deriv}, t\,\text{odd}}_{1212}(2,1) =& \frac{1}{3\Delta(2\Delta)!} \times \Big[\left(\Delta^5+\Delta^4-4\Delta^3-\Delta^2+7\Delta-4\right)\Delta!(\Delta+2)! \\ & + 8\,C_0\left((\Delta+1)!\right)^2\Big]_{\Delta=2} = \frac{2}{3}(11+6C_0) \ , \end{aligned} \tag{5.8.12}$$

where the free theory coefficients are given in (5.6.12) and the supergravity coefficients were obtained in (5.6.13). Solving the unmixing equations gives

$$\left(\gamma_A^{\text{4-deriv}}\right)^{2,1}_{0,0} = \frac{8}{3}(11+6C_0) \ , \quad C^{\text{4-deriv}}_{2,1} = \frac{1}{2} \ . \tag{5.8.13}$$

Next, let us solve one more mixing problem for t odd. At $(4,1)$ two class A and one class $B^{\rm to}$ operators are exchanged, $\mathcal{O}_1\partial^2\mathcal{O}_2$, $\mathcal{O}_2\mathcal{O}_3$ and $\mathcal{O}_2\partial\mathcal{O}_2$. Recall that for t odd, the class A operators have even numbers of derivatives while class B operators have odd numbers of derivatives. The conformal block analysis of the appropriate correlators gives

$$\hat{M}^{\text{4-deriv}}_{4,1} = \begin{pmatrix} M^{\text{4-deriv}}_{1212} & M^{\text{4-deriv}}_{1223} & M^{\text{4-deriv}}_{1222} \\ M^{\text{4-deriv}}_{1223} & M^{\text{4-deriv}}_{2323} & M^{\text{4-deriv}}_{2322} \\ M^{\text{4-deriv}}_{1222} & M^{\text{4-deriv}}_{2322} & M^{\text{4-deriv}}_{2222} \end{pmatrix}_{(4,1)}$$
$$= \begin{pmatrix} \frac{2}{21}(387+10C_0) & \frac{2}{15}(333+25C_0) & 0 \\ \frac{2}{15}(333+25C_0) & \frac{28}{75}(171+20C_0) & 0 \\ 0 & 0 & \frac{56}{3} \end{pmatrix} , \tag{5.8.14}$$

which is block-diagonal as expected and the new coefficients in terms of general Δ are given by:

$$\begin{aligned} M^{\text{4-deriv}, t\,\text{odd}}_{1223}(\Delta,1) =& \frac{(\Delta-2)}{30\Delta(2\Delta)!} \times \Big[20\,C_0\,\Delta(\Delta+3)((\Delta+1)!)^2 \\ &+ \left(\Delta^5+\Delta^4-7\Delta^3-\Delta^2+22\Delta-16\right)\Delta!(\Delta+3)!\Big] , \\ M^{\text{4-deriv}, t\,\text{odd}}_{2323}(\Delta,1) =& \frac{\left(\Delta^2+\Delta-6\right)^2(\Delta+1)!}{300\Delta(2\Delta)!} \times \Big[32\,C_0\,\Delta(\Delta+1)! \\ &+ \left(\Delta^6+3\Delta^5-8\Delta^4-21\Delta^3+89\Delta^2+100\Delta-128\right)\Delta!\Big] , \\ M^{\text{4-deriv}, t\,\text{odd}}_{2222}(\Delta,1) =& \frac{\Delta\left(\Delta^2+\Delta-6\right)^2\Delta!(\Delta+1)!}{3(2\Delta)!} . \end{aligned} \tag{5.8.15}$$

The free theory and supergravity coefficients are given in (5.6.15). Note that the class B operators $\mathcal{O}_2\partial\mathcal{O}_2$ have zero supergravity OPE-coefficients but their free theory coefficients are non-zero. The free theory coefficients are relevant for the unmixing at $\mathcal{O}(a/c)$, for operators $\mathcal{O}_{qq}$ they are given in (5.6.1) and the relevant coefficient at this weight is

$$A^{\text{free}}_{2222}(4,1) = \frac{2}{9} , \tag{5.8.16}$$

solving the $\mathcal{O}(1)$ unmixing equation gives the leading three-point function

$$\left(C_{4,1}^{(0)}\right)_{B^{\text{to}}} = \frac{\sqrt{2}}{3} \,. \tag{5.8.17}$$

We can now go on and solve the unmixing equations at $\mathcal{O}(a/c)$ and $\mathcal{O}(a)$, which gives the anomalous dimensions and three-point functions

$$\gamma_{4,1,i}^{\text{4-deriv}} = \left\{8(307+30C_0),\ \frac{392}{5},\ 84\right\} \ , \quad \mathbb{C}_{4,1}^{\text{4-deriv}} = \frac{1-3C_0}{27\sqrt{15}} \begin{pmatrix} 5\sqrt{14} & 10 & -\frac{\sqrt{5}}{2\sqrt{6}}\,c \\ -2\sqrt{14} & 14 & \frac{1}{\sqrt{30}}\,c \\ 0 & -c & 0 \end{pmatrix} , \tag{5.8.18}$$

where we used the supergravity results (5.6.17). Note that c is unfixed and while there is no mixing between class A and class B operators at orders $\mathcal{O}(1)$, $\mathcal{O}(1/c)$, $\mathcal{O}(a/c)$, there could potentially be mixing at the level of the $\mathcal{O}(a)$ correction to the three-point functions, depending on whether c is zero or not. This could be determined by studying higher-derivative corrections which mix with the four-derivative ones. There are three non-zero anomalous dimensions at $(4,1)$, two corresponding to class A and one to class B^{to} operators:

$$\left(\gamma_A^{\text{4-deriv}}\right)_{0,0}^{4,1} = 8(307+30C_0) \ , \quad \left(\gamma_A^{\text{4-deriv}}\right)_{1,0}^{4,1} = \frac{392}{5} \ , \quad \left(\gamma_{B^{\text{to}}}^{\text{4-deriv}}\right)_{1}^{4,1} = 84 \ , \tag{5.8.19}$$

which will be discussed in terms of the 4d spin below.

Let us look at one more example for odd t in order to understand the emergence of square roots in the anomalous dimensions, which indicates the breaking of the 4d conformal symmetry for higher-derivative corrections at higher charges. We discuss the first example where square roots appear, which is $(5,2)$. This can be seen from the parametrisation in (5.5.3) and the illustration in figure 5.1 where square roots are expected to appear once operators with $i_A = 0,\ r_A = 0$ and both $i_A = 1, r_A = 0$ and $i_A = 0, r_A = 1$ contribute. This is the case when $t \geq 3$ and $p \geq 2$. In figure 5.1 this corresponds to (q_1, q_2) where all three black nodes in the grey highlighted area are present. The grey area contains all operators which acquire non-zero anomalous dimensions at $\mathcal{O}(a/c)$ when they are present in the spectrum at the weight (Δ, p) considered. The operator at position A is the one which has non-zero anomalous dimension also in supergravity, the two black nodes in the grey area connected by a vertical line correspond to the square roots. The square roots thus lift the residual degeneracy that was there in supergravity, where the black nodes connected by a vertical line correspond to the same anomalous dimension, which in the 1d case is zero. The anomalous dimensions of class B operators, denoted by white nodes, do not acquire square roots since there is no degeneracy to lift as class B operators decouple in the supergravity limit.

We will focus on the anomalous dimensions and neglect the three-point functions for the following example since the anomalous dimensions are the relevant objects to discuss the 4d conformal symmetry. The three-point functions can be obtained analogously to the examples above and we will do so for a few more examples with even t. Note that the $\mathcal{O}(a)$ corrections to the three-point functions after unmixing are generally non-zero for higher-derivative corrections, as can be seen from the examples considered above. This agrees with the observations in [14] that for α'^5 corrections to $\mathcal{N} = 4$ SYM correlators in the supergravity approximation, the $\mathcal{O}(\alpha'^5)$ corrections to the three-point functions are non-zero for some cases which are related to square roots in the anomalous dimensions. Whereas for α'^3 it was found in [13] that the corrections to the three-point functions are absent and all anomalous dimensions are rational and thus the 10d conformal symmetry is expected to be intact. See also the discussion in subsection 5.7.3. As a consequence, we would expect that the $\mathcal{O}(a)$ corrections to the three-point functions corresponding to class B operators are absent, since there are no square roots in these cases. This is e.g. supported by (5.8.18) where the entry 2222 corresponding to the class B operator is zero and we will check this in a few more cases below.

p=2 sector

At weight $(5,2)$ there are four class A operators, $\mathcal{O}_1\partial^2\mathcal{O}_3$, $\mathcal{O}_2\mathcal{O}_4$, $\mathcal{O}_2\partial^2\mathcal{O}_2$, $\mathcal{O}_3\mathcal{O}_3$ and one class B operator, $\mathcal{O}_2\partial\mathcal{O}_3$. The class A operators will lead to square roots in the unmixed anomalous dimensions and class A and class B operators do not mix at the levels of supergravity and four-derivative correction anomalous dimensions. The conformal block analysis of the relevant correlators at $\mathcal{O}(a/c)$ then leads to the symmetric matrix:

$$\hat{M}^{\text{4-deriv}}_{5,2} = \begin{pmatrix} M^{\text{4-deriv}}_{1313} & M^{\text{4-deriv}}_{1324} & M^{\text{4-deriv}}_{1322} & M^{\text{4-deriv}}_{1333} & M^{\text{4-deriv}}_{1323} \\ & M^{\text{4-deriv}}_{2424} & M^{\text{4-deriv}}_{2422} & M^{\text{4-deriv}}_{2433} & M^{\text{4-deriv}}_{2423} \\ & & M^{\text{4-deriv}}_{2222} & M^{\text{4-deriv}}_{2233} & M^{\text{4-deriv}}_{2223} \\ & & & M^{\text{4-deriv}}_{3333} & M^{\text{4-deriv}}_{3323} \\ & & & & M^{\text{4-deriv}}_{2323} \end{pmatrix}_{(5,2)}$$
$$= \begin{pmatrix} \frac{2}{25}(774+35C_0) & \frac{6}{35}(416+35C_0) & \frac{4}{45}(942+35C_0) & \frac{36}{35}(90+7C_0) & 0 \\ & \frac{6}{245}(3910+441C_0) & \frac{36}{35}(90+7C_0) & \frac{468}{245}(62+7C_0) & 0 \\ & & \frac{8}{135}(1955+56C_0) & \frac{8}{105}(1707+112C_0) & 0 \\ & & & \frac{72}{245}(657+56C_0) & 0 \\ & & & & \frac{672}{25} \end{pmatrix}, \tag{5.8.20}$$

we spell out the coefficients for general Δ in appendix I. Furthermore, since we have not studied this example in the supergravity section 5.6, we also give the free theory and supergravity coefficients and discuss the mixing problem in the supergravity limit, which is necessary to solve the unmixing at $\mathcal{O}(a/c)$, in the appendix.

Let us solve the unmixing equations at $\mathcal{O}(a/c)$ to get the anomalous dimensions after unmixing:

$$\gamma^{\text{4-deriv}}_{5,2,i} = \left\{ \frac{32}{3}(545+51C_0),\ \frac{8}{15}(415-\sqrt{2881}),\ \frac{8}{15}(415+\sqrt{2881}),\ \frac{1008}{5} \right\}. \tag{5.8.21}$$

We then label the non-zero anomalous dimensions as follows:

$$\left(\gamma^{\text{4-deriv}}_{A}\right)^{5,2}_{0,0} = \frac{32}{3}(545+51C_0)\ , \quad \left(\gamma^{\text{4-deriv}}_{A}\right)^{5,2}_{1,0} = \frac{8}{15}(415-\sqrt{2881})\ ,$$

$$\left(\gamma_A^{\text{4-deriv}}\right)_{0,1}^{5,2} = \frac{8}{15}(415 + \sqrt{2881}) , \quad \left(\gamma_{B^{\text{to}}}^{\text{4-deriv}}\right)_{1}^{5,2} = \frac{1008}{5} . \tag{5.8.22}$$

We will give formulas for general (Δ, p) and analyse them from a 4d perspective below. Note that for the first time, square roots appear in the anomalous dimensions and this will be the case for any (Δ, p) with $p \geq 2$ and $t \geq 3$, where t is odd. These square roots resolve a residual degeneracy from supergravity, which in the 1d case is not obvious since it is a zero-degeneracy. There are no square roots for even t, which is expected because there are no supergravity contributions for even t and thus no degeneracy to resolve.

Continuing to solve the mixing problem weight by weight for many (Δ, p) for t odd, there will always be one rational anomalous dimension corresponding to the operator with $i_A = 0,\ r_A = 0$ and two square roots if both $i_A = 1,\ r_A = 0$ and $i_A = 0,\ r_A = 1$ operators are present (when both $t \geq 3$ and $p \geq 2$). This collapses to one rational anomalous dimension when only one of the two operators is present and to zero when none of the two is present. Besides class A, for class B^{to} there is one non-zero rational anomalous dimension when an operator with $i_{B^{\text{to}}} = 1$ is present (when $t \geq 3$ and $p \geq 1$). Next, let us study a few examples with t even before presenting the general formulas for all non-zero anomalous dimensions at $\mathcal{O}(a/c)$.

5.8.2 Unmixing for Even t

Let us study a few examples of unmixing for even t where only class B^{te} operators contribute to the double-trace spectrum. To get a list of exchanged operators at each weight, see (5.5.5) and figure 5.2. Class B^{to} operators only start contributing from $p = 1$, and they split into two groups, operators with even or odd $r_{B^{\text{te}}}$ respectively. These are illustrated in figure 5.2 with black and white nodes for even and odd $r_{B^{\text{te}}}$ respectively. The operators with (q_1, q_2) which are in the grey area are the ones which obtain non-zero anomalous dimensions at $\mathcal{O}(a/c)$. Let us start by looking at some examples with $p = 1$.

p=1 sector

At weight $(3,1)$ there is only one operator present, $\mathcal{O}_1\partial\mathcal{O}_2$ and performing the conformal block expansion we get the free (see (5.6.12)) and four-derivative coefficients:

$$\begin{aligned}
A^{(0)}_{1212}(3,1) &= \frac{2}{15}\ ,\\
M^{\text{4-deriv}}_{1212}(3,1) &= \frac{(\Delta^3-5\Delta+4)\,(\Delta-1)!(\Delta+2)!}{3(2\Delta)!}\Big|_{\Delta=3} = \frac{16}{9}\ ,
\end{aligned} \tag{5.8.23}$$

solving the unmixing equations (where there are no contributions from supergravity) we get the anomalous dimension

$$\left(\gamma^{\text{4-deriv}}_{B^{\text{te}}}\right)^{3,1}_{1,0} = \frac{40}{3}\ . \tag{5.8.24}$$

Next, let us obtain the three-point functions after unmixing to check whether the $\mathcal{O}(a)$ correction is zero as we would expect for class B operators. Solving the unmixing equation at $\mathcal{O}(1)$ and plugging the result into the equation at $\mathcal{O}(a)$ indeed gives:

$$C^{(0)}_{3,1} = \sqrt{\frac{2}{15}}\ ,\qquad C^{\text{4-deriv}}_{3,1} = 0\ . \tag{5.8.25}$$

At next highest weight $(5,1)$ there are two operators that contribute, $\mathcal{O}_1\partial^2\mathcal{O}_2$ and $\mathcal{O}_2\partial\mathcal{O}_3$ and performing the conformal block expansion of the corresponding correlators gives

$$\begin{aligned}
\hat{A}^{(0)}_{5,1} &= \begin{pmatrix} A^{(0)}_{1212} & 0\\ 0 & A^{(0)}_{2323}\end{pmatrix}_{(5,1)} = \begin{pmatrix} \frac{1}{42} & 0\\ 0 & \frac{9}{175}\end{pmatrix}\ ,\\
\hat{M}^{\text{4-deriv}}_{5,1} &= \begin{pmatrix} M^{\text{4-deriv}}_{1212} & M^{\text{4-deriv}}_{1223}\\ M^{\text{4-deriv}}_{1223} & M^{\text{4-deriv}}_{2323}\end{pmatrix}_{(5,1)} = \begin{pmatrix} \frac{52}{45} & \frac{104}{25}\\ \frac{104}{25} & \frac{1872}{125}\end{pmatrix}\ ,
\end{aligned} \tag{5.8.26}$$

where the new coefficients in terms of general odd Δ are

$$\begin{aligned}
M^{\text{4-deriv},\,t\,\text{even}}_{2323}(\Delta,1) &= \frac{\left(\Delta^2+\Delta-4\right)\left(\Delta^3-13\Delta+12\right)^2(\Delta-2)!(\Delta+2)!}{75(2\Delta)!}\ ,\\
M^{\text{4-deriv},\,t\,\text{even}}_{1223}(\Delta,1) &= \frac{(\Delta-3)(\Delta-1)(\Delta+4)\left(\Delta^2+\Delta-4\right)(\Delta-1)!(\Delta+2)!}{15(2\Delta)!}\ .
\end{aligned} \tag{5.8.27}$$

Solving the unmixing equations in matrix form for this case gives the anomalous dimension

$$\left(\gamma^{\text{4-deriv}}_{B^{\text{te}}}\right)^{5,1}_{1,0} = \frac{5096}{15} \ . \tag{5.8.28}$$

For this case again, as for the whole $p = 1$ sector, there is exactly one non-zero anomalous dimension, which corresponds to the operator with $i_{B^{\text{te}}} = 1,\ r_{B^{\text{te}}} = 0$. Going to $p \geq 2$, there will be two non-zero anomalous dimensions, the additional one being for the operator with $i_{B^{\text{te}}} = 1,\ r_{B^{\text{te}}} = 1$. Furthermore, solving the equations (5.5.13) at order $\mathcal{O}(1)$ and $\mathcal{O}(a/c)$ (where $\gamma^{\text{sugra}} = 0$) and plugging the result into $\mathcal{O}(a)$ we get the three-point functions:

$$\mathbb{C}^{(0)}_{5,1} = \begin{pmatrix} \frac{1}{7\sqrt{6}} & \frac{3\sqrt{6}}{35} \\ \frac{1}{7} & -\frac{3}{35} \end{pmatrix}, \quad \mathbb{C}^{\text{4-deriv}}_{5,1} = \begin{pmatrix} \frac{3}{355}\left(-97+35\sqrt{6}\right)c & c \\ -\frac{2}{355}\left(105+58\sqrt{6}\right)c & \frac{1}{426}\left(-210+97\sqrt{6}\right)c \end{pmatrix}, \tag{5.8.29}$$

where c is unfixed. However, we would expect c to be zero based on the discussions above, since for even t all operators acquire rational anomalous dimensions. This could be checked by considering higher-derivative corrections. We have now seen in a few examples that the $\mathcal{O}(a)$ corrections to the three-point functions are zero (or unfixed) for class B operators, as expected. For the following examples we will focus on the anomalous dimensions only, and the corrections to the three-point functions can be obtained analogously to the examples above. Let us now consider an example in the $p = 2$ sector where we expect two non-zero anomalous dimensions.

p=2 sector

Consider the simplest case $(4,2)$, where there are two operators in the spectrum $\mathcal{O}_1\partial\mathcal{O}_3$ and $\mathcal{O}_2\mathcal{O}_3$. Conformal block expansion of the appropriate correlators gives the coefficients:

$$\hat{A}^{(0)}_{4,2} = \begin{pmatrix} A^{(0)}_{1313} & 0 \\ 0 & A^{(0)}_{2323} \end{pmatrix}_{(4,2)} = \begin{pmatrix} \frac{3}{28} & 0 \\ 0 & \frac{2}{21} \end{pmatrix},$$

$$\hat{M}^{\text{4-deriv}}_{4,2} = \begin{pmatrix} M^{\text{4-deriv}}_{1313} & M^{\text{4-deriv}}_{1323} \\ M^{\text{4-deriv}}_{1323} & M^{\text{4-deriv}}_{2323} \end{pmatrix}_{(4,2)} = \begin{pmatrix} \frac{21}{5} & 0 \\ 0 & \frac{224}{45} \end{pmatrix} . \tag{5.8.30}$$

Note that in this case also $\hat{M}^{\text{4-deriv}}_{4,2}$ is a diagonal matrix, this is because operators with even or odd $r_{B^{\text{te}}}$ do not mix at the level of free theory and four-derivative corrections, going to higher (Δ, p) will give block-diagonal matrices where the mixing problem could be solved independently for even or odd $r_{B^{\text{te}}}$.

The $\mathcal{O}(a/c)$ conformal block coefficients for general even Δ are:

$$\begin{aligned} M^{\text{4-deriv, t even}}_{1313}(\Delta, 2) &= \frac{(\Delta-2)^3(\Delta+3)\left(\Delta^2+\Delta-8\right)(\Delta-3)!(\Delta+3)!}{20(2\Delta)!} , \\ M^{\text{4-deriv, t even}}_{2323}(\Delta, 2) &= \frac{4\left(\Delta^2+\Delta-6\right)^2\left(\Delta^2+\Delta-4\right)\Delta!(\Delta+1)!}{45\Delta(2\Delta)!} . \end{aligned} \tag{5.8.31}$$

After unmixing, there are indeed two non-zero rational anomalous dimensions as expected:

$$\gamma^{\text{4-deriv}}_{4,2,i} = \left\{ \frac{196}{5}, \frac{784}{15} \right\} , \tag{5.8.32}$$

which we label by $\left(\gamma^{\text{4-deriv}}_{B^{\text{te}}}\right)^{4,2}_{1,0} = \frac{196}{5}$ and $\left(\gamma^{\text{4-deriv}}_{B^{\text{te}}}\right)^{4,2}_{1,1} = \frac{784}{15}$ and discuss further below, see subsection 5.8.3. The two operators with non-zero anomalous dimensions correspond to the two nodes in the grey highlighted rectangle in figure 5.2. We present an additional example for even t, $(6, 2)$ where more than one operator contributes for even and odd $r_{B^{\text{te}}}$ each, in appendix I.

One can solve the mixing problem for any (Δ, p) analogous to the given examples. Solving it for many cases, we conjecture general formulas and we spell them out and discuss them from a 4d perspective in the following subsection.

5.8.3 Anomalous Dimensions after Unmixing

We present general formulas for the anomalous dimensions at $\mathcal{O}(a/c)$ for any (Δ, p) for odd and even t. From the examples discussed above, we can see that there are three different non-zero class A and one class B^{to} anomalous dimensions for odd t and two different class B^{te} ones for even t (which agrees with the predictions from the 4d spin (5.5.6)). Solving for the anomalous dimensions after unmixing for many weights up to high (Δ, p), we can conjecture general formulas. We start by presenting the results for odd t which are interesting because the breaking of the 4d conformal symmetry becomes obvious due to the appearance of square roots. Finally we also present the conjectured formulas for even t.

Anomalous dimensions for odd t

Recall that the four-derivative interaction corresponds to correlators with a conformal block expansion that is truncated to 4d spin-2. This can be understood in terms of the effective 4d spin given in (5.5.6). Firstly, analysing the 4d spin-0 sector it is easy to see that $l_{4d} = 2(i_A + r_A) = 0$ can only be satisfied by class A operators with quantum numbers $i_A = r_A = 0$, ($i_{B^{\text{to}}}$ only has values $i_{B^{\text{to}}} = 1, \ldots$ and thus does not contribute at 4d spin-0). The anomalous dimensions are

$$\left(\gamma^A\right)^{\Delta,p}_{0,0} = \delta^{(2)}\left(\frac{2}{9}\left(1 - 6\,C_0\right) + \frac{1}{18}\left(1 + 12\,C_0\right)\delta^{(2)} + \frac{5}{12}\left(\delta^{(2)}\right)^2 + \frac{1}{9}\left(1 + 12\,C_0\right)\delta^{(y)}\right)\,, \tag{5.8.33}$$

where $\delta^{(2)}$ and $\delta^{(y)}$ are the eigenvalues of the differential operators $\Delta^{(2)}$, $\mathcal{D}_y$ in (5.2.15) acting on conformal blocks

$$\delta^{(2)} = (\Delta - p)(\Delta + p + 1)\,, \quad \delta^{(y)} = p(p+1)\,. \tag{5.8.34}$$

For the 4d spin-2 sector $l_{4d} = 2\left(i_A + r_A + i_{B^{\text{to}}}\right) = 2$ there are three possible non-zero anomalous dimensions with quantum numbers $i_A = 1,\ r_A = 0$ and $i_A = 0,\ r_A = 1$ for class A and $i_A = 0,\ r_A = 0,\ i_{B^{\text{to}}} = 1$ for class B, this is the first case where class B

operators play a role. These anomalous dimensions are given by:

$$\begin{aligned}\left(\gamma^{A}\right)^{\Delta,p}_{1,0} =& \frac{1}{180}\,\delta^{(2)}\Big(4-2\,\delta^{(2)}+3\,\left(\delta^{(2)}\right)^2-4\,\delta^{(y)}\\ &-4\sqrt{8\,p^3+4\,p^4-4\,p^2\Delta(1+\Delta)-4\,p\,(1+\Delta+\Delta^2)+(-1+2\,\Delta+2\,\Delta^2)^2}\Big)\,,\end{aligned} \tag{5.8.35}$$

$$\begin{aligned}\left(\gamma^{A}\right)^{\Delta,p}_{0,1} =& \frac{1}{180}\,\delta^{(2)}\Big(4-2\,\delta^{(2)}+3\,\left(\delta^{(2)}\right)^2-4\,\delta^{(y)}\\ &+4\sqrt{8\,p^3+4\,p^4-4\,p^2\Delta(1+\Delta)-4\,p\,(1+\Delta+\Delta^2)+(-1+2\,\Delta+2\,\Delta^2)^2}\Big)\,,\end{aligned} \tag{5.8.36}$$

$$\left(\gamma^{B^{\mathrm{to}}}\right)^{\Delta,p}_{1} =\frac{1}{60}\,\delta^{(2)}\left(-2\,\delta^{(2)}+\left(\delta^{(2)}\right)^2-4\,\delta^{(y)}\right)\,. \tag{5.8.37}$$

The square roots resolve the degeneracy at supergravity, where in the 1d case several anomalous dimensions are zero (while in the 4d/10d case there is a non-zero degeneracy). The operators with the same anomalous dimensions in supergravity correspond to the black nodes (class A operators) connected by vertical lines in figure 5.1. The operators highlighted by a grey rectangle are the ones acquiring non-zero anomalous dimensions at the order of four derivatives. The two black nodes which are connected by a vertical line correspond to the operators with anomalous dimensions with square roots (5.8.35), (5.8.36) while the white node represents the operators with anomalous dimension (5.8.37).

Anomalous dimensions for even t

For even t, operators only start to contribute at 4d spin-2 and we get two different non-zero anomalous dimensions, labelled as $\left(\gamma^{B^{\mathrm{te}}}\right)^{\Delta,p}_{i_{B^{\mathrm{te}}},r_{B^{\mathrm{te}}}}$ with quantum numbers $i_{B^{\mathrm{te}}}=1,\ r_{B^{\mathrm{te}}}=0$ and $i_{B^{\mathrm{te}}}=1,\ r_{B^{\mathrm{te}}}=1$:

$$\left(\gamma^{B^{\mathrm{te}}}\right)^{\Delta,p}_{1,0}=\frac{1}{60}\left(\delta^{(2)}\right)^2\left(\delta^{(2)}-2\right)\,, \tag{5.8.38}$$

$$\left(\gamma^{B^{\mathrm{te}}}\right)^{\Delta,p}_{1,1}=\frac{1}{60}\left(\delta^{(2)}\right)^2\left(\delta^{(2)}+2\right)\,. \tag{5.8.39}$$

The operators at even t are illustrated in figure 5.2 and the two operators acquiring non-zero anomalous dimensions are highlighted by a grey rectangle, where the operator with anomalous dimension (5.8.38) is represented by the black node and the one corresponding to (5.8.39) by the white node.

Note that there are some relations between these anomalous dimensions, in particular the eigenvalues of Casimir operators in the anomalous dimensions, and the Casimir operators which act on 4d conformal objects to uplift the four-derivative corrections to higher charges in subsection 5.7.3. In particular (5.8.38) is proportional to the eigenvalue of $\Delta^{(4)}_{\text{spin-2}}$ in (5.7.22). The anomalous dimensions for odd t corresponding to 4d spin-0 and spin-2 are also partially given in terms of eigenvalues of Casimirs. However, there is no exact correspondence to eigenvalues of $\Delta^{(4)}_{\text{spin-0}}$ or $\Delta^{(4)}_{\text{spin-2}}$ due to the square roots and extra dependence on $\delta^{(y)}$. This also indicates a breaking of the 4d conformal symmetry.

We have now studied the mixing problem for four-derivative corrections in detail and have obtained general formulas for all $\mathcal{O}(a/c)$ anomalous dimensions after unmixing. These anomalous dimensions can be organised according to a 4d effective spin inspired by the 4d conformal symmetry and they indicate a breaking of the symmetry for higher-derivative corrections due to the appearance of square roots. We will conclude this chapter in the next section and discuss some interesting open questions for future research.

5.9 Conclusions and Future Directions

In this chapter we investigate holographic correlators in $\text{AdS}_2\times\text{S}^2$. We start with a review of the hidden 10d conformal symmetry discovered in [41] where it is conjectured that all half-BPS four-point correlators described by tree-level supergravity can be obtained from one single 10d conformally invariant object. This conjecture is true for the supergravity approximation and also extends to free theory, as well as loop corrections (which we do not discuss in this thesis). However, it is generally broken for higher-derivative corrections as was confirmed in [14] from a study of anomalous dimensions after unmixing. An appearance of square roots in these results indicates the breaking of the symmetry, however there are still many rational anomalous dimensions which suggests that some of the symmetry is conserved even for higher-derivative corrections. The hidden conformal symmetry arises when the corresponding flat space superamplitude is conformally invariant, hence when its coupling, i.e. $G_N\delta^{16}(Q)$ in the AdS$_5$×S^5 case, is dimensionless. Therefore, only specific four-point correlators enjoy this conformal symmetry and only on backgrounds which are conformally flat, such as AdS$_q$×S^q. We have explained that in the supergravity approximation the four-point half-BPS correlator itself enjoys the 10d conformal symmetry, while for free theory the correlator has to be rescaled by acting on it with an eighth-order differential operator derived from the superconformal Casimir. This yields the correlator of superdescendants which exhibits 10d conformal symmetry.

In addition to studying 1d correlators in the context of the 4d hidden conformal symmetry we also derive four-point half-BPS correlators from a 4d scalar effective action in AdS$_2$×S^2 similar to the one introduced in chapter 3. While in AdS$_5$×S^5 the effective action only describes higher-derivative corrections, in AdS$_2$×S^2 the effective action describes both, supergravity and higher-derivative corrections. Note that we propose the existence of this effective action and deduce the consequences without proving it. We then compare the results to those from the 4d conformal symmetry where possible. Throughout this chapter we consider both these approaches and show that they nicely complement each other.

We discuss the free disconnected correlators and show that they exhibit a 4d conformal symmetry. Recall that for free theory one has to consider the correlator of descendants which is obtained by acting with $\Delta^{(2)}$ on the correlators. This yields a generating function which contains all higher-charge free theory correlators.

Supergravity correlators in AdS$_2$×S^2 are especially interesting because they exhibit 4d conformal symmetry and can also be deduced from the proposed 4d effective action. We start by deriving the lowest-charge correlator from crossing symmetry and small x behaviour alone and then show that one can obtain all higher-charge tree-level correlators by uplifting it to 4d and expanding in the internal variables. We derive a general formula for all four-point supergravity correlators and find exact agreement when deriving the same correlators from the 4d effective action (up to an overall unfixed coefficient), which suggests that the proposed 4d effective field theory indeed generates all tree-level half-BPS four-point correlators.

The first hint towards a 10d conformal symmetry was found in [144] when solving the mixing problem for anomalous dimensions in the supergravity limit where a remarkably simple structure was uncovered. To show that a similar structure (and even simpler due to the absence of spin in 1d CFTs) is satisfied for $\mathrm{AdS}_2\times\mathrm{S}^2$ we analyse the double-trace spectrum in the conformal block decomposition of the four-point half-BPS correlators. We then explain how to solve the mixing problem for supergravity as well as higher-derivative corrections. Further, we show that the supergravity anomalous dimensions after unmixing are indeed given by a very simple formula, namely the eigenvalue $\delta^{(2)}$ of $\Delta^{(2)}$ acting on the 1d superconformal blocks. This simple structure can be explained from an effective 4d spin. Therefore, we have seen that 1d correlators in free theory and in the supergravity approximation indeed exhibit 4d conformal symmetry.

Finally, we investigate higher-derivative corrections. We start by deriving general higher-derivative corrections with lowest charge from crossing symmetry before we focus on the four-derivative corrections. We derive all four-derivative corrections to the half-BPS correlators with any charge configuration from the 4d effective field theory. Using these results, we go on to discuss the breaking of the higher-dimensional conformal symmetry. We propose that some of the correlators can be reconstructed by rescaling the higher-derivative four-point correlators by acting with an inverse fourth-order differential operator to obtain an object invariant under 4d conformal symmetry. In this way one can reproduce an infinite set of four-derivative corrections with specific charges, however the symmetry is broken for general charge configurations. This breaking can be anticipated because when dimensionally reducing interaction terms with covariant AdS×S derivatives on the sphere one gets several terms with different numbers of derivatives in AdS which then cannot all be rescaled simultaneously. This strongly suggests that the α'^3 corrections in AdS$_5$×S^5, which are described by zero-derivative interactions, have 10d conformal symmetry, which is in agreement with the rational anomalous dimensions found in [13].

Considering the four-derivative corrections we obtain from the proposed 4d effective action, we perform a conformal block analysis and solve the mixing problem for anomalous dimensions at $\mathcal{O}(a/c)$. We obtain general formulas for all anomalous dimensions and similar to [14] we find that a part of these anomalous dimensions contain square roots, whereas there are also many rational anomalous dimensions. Again we can explain the structure of these anomalous dimensions from an effective 4d spin. The fact that there are square roots in the unmixed anomalous dimensions indicates that the 4d conformal symmetry is broken for higher-derivative corrections. However, there are also many operators obtaining rational anomalous dimensions and this suggests that a part of the 4d conformal symmetry structure remains. This agrees with the findings above that a subset of correlators can still be constructed from the hidden conformal symmetry.

To summarise, both, the higher-dimensional conformal symmetry and the higher-dimensional scalar effective field theory are very powerful approaches which nicely complement each other. While the hidden conformal symmetry describes all free theory and supergravity correlators (as well as loop corrections), it is generally broken for higher-derivative corrections. On the other hand, the scalar effective action approach generates all higher-derivative corrections for any number of derivatives and any charge configuration, but does not describe free theory and loop corrections. Whereas the effective action in AdS$_5$×S^5 only describes higher-derivative corrections, the effective action in AdS$_2$×S^2 also describes supergravity.

Future directions

There are a number of interesting open questions that arise form this research.

- We have seen that the hidden conformal symmetry and the higher-dimensional effective action approach nicely complement each other and that both are very powerful in obtaining four-point half-BPS correlators at different orders in a and $1/c$. It would be very interesting to understand the breaking of the hidden conformal symmetry for higher-derivative corrections more precisely. It would be interesting to identify specific pieces of the integrands of Witten diagrams derived from the effective action that cause the breaking of the symmetry. It is not straightforward to identify these contributions from (5.7.12) and one idea might be to perform a Weyl transformation of the derivative terms in the effective action from flat space to AdS×S to see whether this could give more insight.

- Another very interesting research direction would be to study implications for 4d black holes. As mentioned before, $AdS_2 \times S^2$ is the near-horizon geometry of extremal black holes in 4d. Fixing the coefficients in the effective action would specify the underlying theory of quantum gravity and it would be interesting to see whether these coefficients could be constrained from the weak gravity conjecture [156, 157].

- It would also be interesting to consider higher-loop corrections. These can be constrained from the higher-dimensional conformal symmetry and we leave a discussion of this to [44] (also see [41]).

- Extending the conformal block analysis to higher-derivative corrections with more than four derivatives is another possible extension of our research. It would be interesting to compute the anomalous dimensions after unmixing and check that the results agree with the 4d effective spin conjecture in (5.5.6). Furthermore, one could consider constructing part of the correlators from the 4d conformal symmetry by acting with inverse differential operators on the correlators analogous to the discussion in subsection 5.7.3, where the power of the differential operators depends on the number of derivatives.

Chapter 6

Conclusion

In this thesis we have studied holographic correlators in three different examples of the AdS/CFT correspondence. In chapter 2 we started with a review of important concepts of conformal field theories, including correlators, OPE expansion and the conformal bootstrap. Furthermore, we have briefly reviewed the AdS/CFT correspondence and holographic correlators including higher-derivative corrections.

The first case we have studied was the classical canonical example of the AdS/CFT correspondence, the duality between string theory in $\mathrm{AdS}_5\times\mathrm{S}^5$ and $\mathcal{N}=4$ SYM in the boundary. The goal is to generalise the flat space Virasoro-Shapiro amplitude, describing stringy corrections to the supergravity approximation, to curved backgrounds. We have proposed a 10d scalar effective action in $\mathrm{AdS}_5\times\mathrm{S}^5$ describing α' corrections to the correlators. From there we have described a general algorithm to compute all four-point half-BPS correlators described by tree-level string theory to any order in α' by evaluating new 10d Witten diagrams which manifestly include AdS_5 and S^5 coordinates. We have justified the existence of this effective field theory by reproducing known results for α'^3 and α'^5 and have then made new predictions for α'^6 and α'^7. The coefficients of these interaction terms are fixed by comparing to the flat space VS amplitude. There are ambiguities in the AdS×S effective action which vanish in the flat space limit and can therefore not be fixed by our approach. However, most of the ambiguities at the orders considered in this thesis can be determined by comparing to results obtained from localisation.

Next, in chapter 4 we have considered holographic correlators in $AdS_7 \times S^4$ which is another canonical example of the AdS/CFT correspondence but much less understood since the bulk theory is not known beyond the supergravity approximation and the boundary theory is non-Lagrangian. Our goal was to better understand the worldvolume theory of M5-branes, the 6d $(2,0)$ theory, which is dual to M-theory in $AdS_7 \times S^4$. To study M-theory away from the supergravity approximation we consider correlators in the 6d $(2,0)$ theory away from the strict large-c limit. Since the 6d $(2,0)$ theory is non-perturbative we have approached this through conformal bootstrap methods. In particular, we have derived recursion relations for anomalous dimensions of double-trace operators in the conformal block expansion of four-point functions of primaries of the stress tensor multiplet. To do this, we have expanded the crossing equation in $1/c$ and taken the light-cone limit of the conformal cross-ratios to isolate the contributions from anomalous dimensions. Moreover, we have made use of orthonormality relations of the hypergeometric functions in the conformal blocks and have truncated the conformal block expansion to spin L. We have derived these recursion relations for a bosonic toy 6d CFT as well as for the supersymmetric 6d $(2,0)$ theory. Further, we have described an algorithm for solving the recursion relations for the anomalous dimensions at any spin-truncation and for any twist. Finally, the anomalous dimensions encode the higher-derivative corrections to the low-energy effective action of M-theory. In particular, the large-twist behaviour of the anomalous dimensions of a specific spin-L truncated solution to the crossing equation gives the number of derivatives of the corresponding interaction vertex while the coefficients of the terms in the low-energy effective action cannot be fixed from our approach.

The third case we have investigated are holographic correlators in $AdS_2 \times S^2$, where quantum gravity on this background is dual to a one-dimensional superconformal field theory with $SU(1,1|2)$ symmetry. This case is much less well understood but very interesting due to its relation to black hole physics. We have approached correlators in this example of the AdS/CFT correspondence from different points of view, the higher-dimensional conformal symmetry first observed for $\mathcal{N} = 4$ SYM and the higher-dimensional scalar effective action proposed in a previous chapter. The hidden conformal symmetry of correlators arises when the corresponding flat space superamplitude is connected to a dimensionless coupling and is thus conformally invariant. It is thus only satisfied for specific four-point correlators and moreover, requires a conformally flat background. We have seen that 1d correlators in free theory and in the supergravity approximation have 4d conformal symmetry, where one has to act with a differential operator $\Delta^{(2)}$ on the free theory correlators to obtain a 4d conformally invariant object. In these cases, all higher-charge correlators can be obtained from the lowest-charge correlator alone using the hidden conformal symmetry and we have shown that in the supergravity limit these results agree with the correlators derived from the 4d effective action by evaluating $\mathrm{AdS}_2 \times \mathrm{S}^2$ Witten diagrams. That the supergravity correlators satisfy 4d conformal symmetry was also confirmed through the analysis of anomalous dimensions of double-trace operators in the spectrum of the correlators after unmixing, which show the expected simple structure $\delta^{(2)}$.

We have then discussed higher-derivative corrections and specifically obtained all four-derivative corrections from the proposed 4d effective field theory. The higher-dimensional conformal symmetry is generally broken in this case but through rescaling the correlators by acting on them with inverse powers of differential operators we can reconstruct an infinite tower of four-derivative corrections with specific charges. We have also solved the mixing problem for anomalous dimensions of double-trace operators in the spectrum of the four-derivative corrections and have obtained general formulas. While most of the anomalous dimensions are rational, there are operators which acquire anomalous dimensions with square roots and this indicates the breaking of the 4d conformal symmetry. To conclude, the hidden conformal symmetry conjecture is valid for free theory and supergravity and is generally broken for higher-derivative corrections whereas the scalar effective field theory approach generates all higher-derivative corrections but does not describe free theory or supergravity (except in the 1d case). Hence, using both of these approaches together puts very powerful constraints on the four-point correlators of half-BPS operators in conformally flat backgrounds.

Detailed conclusions including discussions of open problems are given at the end of each chapter. There are many interesting future directions arising from the research in this thesis, such as for example proving the existence of the scalar effective action proposed in chapter 3, or deriving it directly from the CFT side of the duality. It would also be interesting to understand systematically how to fix the ambiguities and how to resum the terms in the effective action. Furthermore, it would be interesting to extend this approach to different conformally flat backgrounds, to higher-point correlators or to loop corrections. Following on from chapter 4 an interesting direction would be to fix the coefficients of the higher-derivative corrections to the M-theory low-energy effective action using the chiral algebra conjecture [132]. Related to this, it would be interesting to derive higher-charge correlators and compute the corresponding anomalous dimensions. Based on chapter 5 it would be very interesting to explore the implications for higher-derivative corrections in black hole backgrounds, as well as to extend the study to loop corrections or to interactions with more derivatives. An interesting direction to pursue, combining both chapter 3 and chapter 5, would be to study the relation between the higher-dimensional scalar effective action and the higher-dimensional conformal symmetry further and understand the breaking of the conformal symmetry for higher-derivative corrections more precisely.

Appendix A

Mellin Space

In this appendix we will review a few important concepts for the Mellin space formalism. In [57] the author proposed a representation of conformal correlators that makes their duality to scattering amplitudes more apparent, Mellin space representation. Mellin space was first brought into the context of scattering amplitudes in [158, 159], where a duality between the Mellin amplitude and flat space scattering amplitudes as a function of Mandelstam invariants was pointed out. The flat space correlator of four primary scalar operators can be written as

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\mathcal{O}_4(x_4)\rangle = \frac{\mathcal{N}}{(2\pi i)^2}\int d\delta_{ij}\prod_{i\leq j}^{n}\frac{\Gamma(\delta_{ij})}{\left(x_{ij}^2\right)^{\delta_{ij}}}M(\delta_{ij})\ , \tag{A.0.1}$$

with some normalisation $\mathcal{N}$. The integration variables satisfy the constraints

$$\sum_{i\leq j}\delta_{ij} = \Delta_i\ , \tag{A.0.2}$$

and the integrand is conformal with scaling dimension Δ_i at the point x_i. $M(\delta_{ij})$ is the Mellin amplitude and the integration contour runs parallel to the imaginary axis with $\mathrm{Re}(\delta_{ij}) > 0$.

In [57], the author then proposed that the Mellin amplitude is the scattering amplitude also in AdS spacetime (see (2.3.14)) and we will discuss this further in the following.

Contact Witten diagrams and Mellin space

In this subsection we include additional information about contact diagrams in Mellin space and their relations to the position space versions, which is useful in the context of chapter 3. Note that we do not discuss Mellin space in the context of 1d CFTs in chapter 5 because in 1d the Mandelstam variables are not independent (since $u = 0$) and thus the Mellin transforms are not unique (see also [160]).

In [57] the author computes the Mellin amplitude $M(\delta_{ij})$ for different Witten diagrams and finds that they resemble to flat space scattering amplitudes. Specifically contact Witten diagrams, which are the relevant cases for our considerations, give polynomial Mellin amplitudes which agree with flat space scattering amplitudes as functions of the Mandelstam invariants. The minimal contact diagram with no covariant derivatives always has constant Mellin amplitude[1] whereas derivative contact terms give polynomial Mellin amplitudes. More precisely a AdS contact interaction with $2k$ covariant derivatives corresponds to a Mellin amplitude which is a polynomial of degree k.

As we have seen in subsection 2.3.1 the contact Witten diagrams for interactions with no derivatives, which are described by D-functions, can be written in Mellin space as (see [57]):

$$D^{(d)}_{p_1p_2p_3p_4}(X_i) = \mathcal{N}^{\mathrm{AdS}_{d+1}}_{p_i} \times \int \frac{d\delta_{ij}}{(2\pi i)^2} \prod_{i<j} \frac{\Gamma(\delta_{ij})}{(X_i.X_j)^{\delta_{ij}}} \,, \qquad \text{with } \sum_{i<j} \delta_{ij} = p_j \,, \tag{A.0.3}$$

where the normalisation is given in (2.3.15).

The constraints $\sum_i \delta_{ij} = p_j$ can be solved most naturally by

$$\delta_{ij} = \tfrac{1}{2}(p_i + p_j - s_{ij}) \,, \tag{A.0.4}$$

where $s_{12} = s_{34} = s$, $s_{14} = s_{23} = t$, $s_{13} = s_{24} = u = p_1 + p_2 + p_3 + p_4 - s - t$ which can be interpreted as kinematic invariants of an auxiliary momentum space k_i with

[1] In any dimension d there is a contact diagram with constant Mellin transform, namely the one with $\Delta = d$ for $D_{\Delta\Delta\Delta\Delta}$. In 4d this special case is D_{4444} which corresponds to the α'^3 correction, while in 1d it is D_{1111} corresponding to supergravity.

$s_{ij} = -(k_i + k_j)^2$. We can use this to rewrite the normalised D-functions (2.3.16) in terms of $\bar{D}$-functions with a simple relation to the Mellin transform as:

$$\begin{aligned} P' \times D_{p_1p_2p_3p_4}(X_i) &= \int \frac{ds\,dt}{(2\pi i)^2} u^{\frac{s}{2}} v^{\frac{t}{2}} \prod_{i<j} \Gamma(\delta_{ij}) \\ &= u^{\frac{1}{2}(p_1+p_2)} v^{\frac{1}{2}(p_2+p_3)} \bar{D}_{p_1p_2p_3p_4}(u,v)\ , \end{aligned} \tag{A.0.5}$$

where the prefactor is

$$P' = \frac{(X_1.X_2)^{\frac{1}{2}(p_1+p_2)}(X_1.X_4)^{\frac{1}{2}(p_1+p_4)}(X_2.X_3)^{\frac{1}{2}(p_2+p_3)}(X_3.X_4)^{\frac{1}{2}(p_3+p_4)}}{(X_1.X_3)^{\frac{1}{2}(p_2+p_4)}(X_2.X_4)^{\frac{1}{2}(p_1+p_3)}} \tag{A.0.6}$$

and to go from the normalised D to $\bar{D}$ use (2.3.16) and (2.3.17), where $x_{ij}^2 = -2X_i.X_j$. Thus, one can easily convert between position space in terms of $\bar{D}$-functions and Mellin space. Note that in chapter 3 we give all correlators in Mellin space, nevertheless we can directly expand the 10d Witten diagrams in terms of D- and B-functions to get the correlators in position space (analogously for 4d Witten diagrams in chapter 5). Hence, we can get the correlators both in position space and in Mellin space directly from the effective action, yet it is interesting to understand how to convert between them.

Shifting the variables $s \to s + p_1 + p_2$, $t \to t + p_2 + p_3$ in (A.0.5) we get

$$\begin{aligned} &\bar{D}_{p_1p_2p_3p_4} = \int \frac{ds\,dt}{(2\pi i)^2} u^{\frac{s}{2}} v^{\frac{t}{2}} \\ &\times \Gamma\left(\tfrac{-s}{2}\right) \Gamma\left(\tfrac{-p_1-p_2+p_3+p_4-s}{2}\right) \Gamma\left(\tfrac{-t}{2}\right) \Gamma\left(\tfrac{p_1-p_2-p_3+p_4-t}{2}\right) \Gamma\left(\tfrac{2p_2+s+t}{2}\right) \Gamma\left(\tfrac{p_1+p_2+p_3-p_4+s+t}{2}\right)\ . \end{aligned} \tag{A.0.7}$$

Conversely the Mellin transform of any function which has the form of six Gammas in this form can be written as a $\bar{D}$-function as follows:

$$\begin{aligned} &\int \frac{ds\,dt}{(2\pi i)^2} u^{\frac{s}{2}} v^{\frac{t}{2}} \Gamma\left(a_1 - \tfrac{s}{2}\right) \Gamma\left(a_2 - \tfrac{s}{2}\right) \Gamma\left(b_1 - \tfrac{t}{2}\right) \Gamma\left(b_2 - \tfrac{t}{2}\right) \Gamma\left(c_1 + \tfrac{s+t}{2}\right) \Gamma\left(c_2 + \tfrac{s+t}{2}\right) \\ &= u^{a_1} v^{b_1} \bar{D}_{a_1+b_2+c_2,\, a_1+b_1+c_1,\, a_2+b_1+c_2,\, a_2+b_2+c_1}\ . \end{aligned} \tag{A.0.8}$$

Note that this mapping is not unique because of the symmetries $a_1 \leftrightarrow a_2$, $b_1 \leftrightarrow b_2$, $c_1 \leftrightarrow c_2$, which generate various identities amongst the $\bar{D}$-functions.

Appendix B

The Polynomial $I(X_i, Y_i)$

The polynomial $I(X_i, Y_i)$, the so-called Intriligator polynomial, which factors out of all half-BPS four-point functions (3.1.14) is:

$$\begin{aligned}
I(X_i,Y_i) &= (x-y)(\bar{x}-y)(x-\bar{y})(\bar{x}-\bar{y})(X_1.X_3)^2(X_2.X_4)^2(Y_1.Y_3)^2(Y_2.Y_4)^2 \ , \\
\text{with} \quad x\bar{x} &= \frac{X_1.X_2X_3.X_4}{X_1.X_3X_2.X_4} \ , \qquad (1-x)(1-\bar{x}) = \frac{X_1.X_4X_2.X_3}{X_1.X_3X_2.X_4} \ , \\
y\bar{y} &= \frac{Y_1.Y_2Y_3.Y_4}{Y_1.Y_3Y_2.Y_4} \ , \qquad (1-y)(1-\bar{y}) = \frac{Y_1.Y_4Y_2.Y_3}{Y_1.Y_3Y_2.Y_4} \ .
\end{aligned} \tag{B.0.1}$$

I is crossing symmetric under simultaneously permuting X_i, Y_i with X_j, Y_j . It is also a polynomial and written out fully in terms of the $SO(2,4)$ and $SO(6)$ invariants $X_i.X_j$ and $Y_i.Y_j$ is given as

$$\begin{aligned}
I(X_i,Y_i) = &(X_1.X_4)^2(X_2.X_3)^2 Y_1.Y_2Y_1.Y_3Y_2.Y_4Y_3.Y_4 \\
&+ X_1.X_2X_1.X_4X_3.X_4X_2.X_3(Y_1.Y_3)^2(Y_2.Y_4)^2 \\
&+ X_1.X_3X_1.X_4X_2.X_4X_2.X_3(Y_1.Y_2)^2(Y_3.Y_4)^2 \\
&- X_1.X_2X_1.X_4X_3.X_4X_2.X_3Y_1.Y_3Y_1.Y_4Y_2.Y_3Y_2.Y_4 \\
&- X_1.X_3X_1.X_4X_2.X_4X_2.X_3Y_1.Y_2Y_1.Y_4Y_2.Y_3Y_3.Y_4 \\
&- X_1.X_3X_1.X_4X_2.X_4X_2.X_3Y_1.Y_2Y_1.Y_3Y_2.Y_4Y_3.Y_4 \\
&- X_1.X_2X_1.X_4X_3.X_4X_2.X_3Y_1.Y_2Y_1.Y_3Y_2.Y_4Y_3.Y_4 \\
&+ X_1.X_2X_1.X_3X_2.X_4X_3.X_4(Y_1.Y_4)^2(Y_2.Y_3)^2 + (\dots)
\end{aligned}$$

$$
\begin{aligned}
&(\dots) + (X_1.X_2)^2 (X_3.X_4)^2 Y_1.Y_3 Y_1.Y_4 Y_2.Y_3 Y_2.Y_4 \\
&- X_1.X_2 X_1.X_3 X_2.X_4 X_3.X_4 Y_1.Y_3 Y_1.Y_4 Y_2.Y_3 Y_2.Y_4 \\
&+ (X_1.X_3)^2 (X_2.X_4)^2 Y_1.Y_2 Y_1.Y_4 Y_2.Y_3 Y_3.Y_4 \\
&- X_1.X_2 X_1.X_3 X_2.X_4 X_3.X_4 Y_1.Y_2 Y_1.Y_4 Y_2.Y_3 Y_3.Y_4 \ . \qquad \text{(B.0.2)}
\end{aligned}
$$

Appendix C

Contact Diagrams in AdS×S and AdS, and Supergravity

Although we will not use this fact in the rest of the thesis, it is worth pointing out an intriguing relation between the AdS×S contact diagrams and the better known standard AdS contact diagrams. This relation can be seen by comparing their respective Mellin transforms (3.1.41) and (3.1.25).

First, consider the special case $\Sigma_\Delta = d+1$. In this case the final Pochhammer in (3.1.41) is absent and the Mellin transform becomes proportional to

$$\begin{aligned}
&\sum_{p_i=0}^{\infty}(-1)^{\Sigma_p}\int\frac{d\delta_{ij}}{(2\pi i)^2}\sum_{\{d_{ij}\}}\left(\prod_{i<j}\frac{(Y_i.Y_j)^{d_{ij}}}{(X_i.X_j)^{\delta_{ij}}}\frac{\Gamma(\delta_{ij})}{\Gamma(d_{ij}+1)}\right)\\
&=\int\frac{d\boldsymbol{\delta}_{ij}}{(2\pi i)^2}\left(\prod_{i<j}\frac{\Gamma(\boldsymbol{\delta}_{ij})}{(X_i.X_j+Y_i.Y_j)^{\boldsymbol{\delta}_{ij}}}\right),\\
&\text{where}\qquad \sum_{i<j}\boldsymbol{\delta}_{ij}=\Delta_j\ ,
\end{aligned}\tag{C.0.1}$$

where the equality is obtained by performing the sums over p_i and then changing variables from $\delta_{ij}\to\boldsymbol{\delta}_{ij}=\delta_{ij}-d_{ij}$ and $\boldsymbol{\delta}_{ij}$ contains AdS and S Mellin variables.

Comparing this with the Mellin transform of the AdS contact term (3.1.25) we see that this is proportional to a D-function with $X_i.X_j\to X_i.X_j+Y_i.Y_j$. In other words it is proportional to a pure AdS contact term with embedding coordinates

$X_i^\mu = (X_i^A, Y_i^I)$, corresponding to a $(2d+2)$-dimensional bulk. More precisely we have the relation[1]

$$D^{\mathrm{AdS}_{d+1}\times \mathrm{S}^{d+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i, Y_i) = \frac{\pi^{d+1}}{(-2)^{\Sigma_\Delta}\prod_i \Gamma(\Delta_i)} \times D_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i, Y_i) \ , \quad \text{where } \Sigma_\Delta = d+1 \ . \tag{C.0.2}$$

Note that this case $\Sigma_\Delta = d+1$ corresponds precisely to the case of a dimensionless contact term in the flat space limit, $\int d^{2d+2}x\phi_{\Delta_1}\dots\phi_{\Delta_4}$. The above relation (C.0.2) is an example of the enhanced higher-dimensional conformal symmetry observed in [41]. We will look at this explicitly for $\mathcal{N}=4$ SYM in the next subsection.

Now let us modify the above discussion for the case with $\Sigma_\Delta \neq d+1$. Here the direct relation between AdS×S and AdS contact terms is spoiled by the presence of the Pochhammer at the end of (3.1.41) which depends on Σ_p that we are summing over. A simple way of reproducing this Pochhammer whilst still having a summed up formula is then to rescale all the Y variables and differentiate. Concretely, we can write

$$D^{\mathrm{AdS}_{d+1}\times \mathrm{S}^{d+1}}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i, \sqrt{r}Y_i) = \frac{2}{\Gamma(\Sigma_\Delta - d - 1)}\frac{1}{r^{d/2}}\left(\frac{d}{dr}\right)^{\Sigma_\Delta - d - 1} r^{\Sigma_\Delta - d/2 - 1} D^{(2d+2)}_{\Delta_1\Delta_2\Delta_3\Delta_4}(X_i, \sqrt{r}Y_i) \ , \tag{C.0.3}$$

where the D-function is for a $(2d+2)$-dimensional bulk.

Tree-level supergravity in $\mathcal{N}=4$ SYM

While the main focus in chapter 3 is obtaining tree-level string corrections to $\mathcal{N}=4$ SYM correlators from an effective action involving massless scalars in 10d, it is interesting to also look at the tree-level supergravity prediction following the approach described in this appendix. While we do not expect this to arise from an effective superpotential, all single-trace half-BPS correlators were shown in [41] to possess a 10d conformal structure and in particular can be obtained by expanding

[1] We here compare with the dimension-independent, normalised D-function (3.1.26), since in $2d+2$ dimensions the $D^{(d)}$-function itself diverges when $\Sigma_\Delta = d+1$ due to the Γ-function in the numerator of (2.3.15).

out D_{2422}. For D_{2422} the simple relation between AdS×S and standard AdS contact diagrams described above is valid because $\Sigma_\Delta = 5 = d+1$ and hence (C.0.2) applies. The tree-level supergravity result can be written [41]

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle_{\text{sugra}} \propto \frac{1}{(X_1.X_3+Y_1.Y_3)}\frac{1}{(X_1.X_4+Y_1.Y_4)}\frac{1}{(X_3.X_4+Y_3.Y_4)}D_{2422}(X_i,Y_i)\ . \tag{C.0.4}$$

Inserting the Mellin representation of D_{2422} (3.1.25) and changing variables $\delta_{ij} \to \delta_{ij} - 1$ for $i,j = 1,3,4$ and δ_{ij} unchanged otherwise, this can be written in the form (3.1.42) with $\Delta_i = 4$ with the Mellin amplitude

$$\mathcal{M}_{\text{sugra}} \propto \frac{1}{(\boldsymbol{\delta}_{13}-1)(\boldsymbol{\delta}_{14}-1)(\boldsymbol{\delta}_{34}-1)} = \frac{1}{(\delta_{13}-d_{13}-1)(\delta_{14}-d_{14}-1)(\delta_{34}-d_{34}-1)}\ . \tag{C.0.5}$$

The denominator in the above equations can be understood from the supergravity piece of the Virasoro-Shapiro amplitude which is $\frac{1}{STU}$, which acts like inverse derivatives on D_{2422}. Acting with STU on (C.0.4) gives back the zero-derivative contact term D_{4444} which has Mellin transform 1 and can be obtained from a 10d superpotential.

Appendix D

α'^7 Ambiguities in $\mathbf{AdS}_5\times\mathbf{S}^5$

The ambiguities at order α'^7 were introduced in (3.6.2) and we spell out their Witten diagram expressions and the corresponding Mellin amplitudes in the following.

The first ambiguity at α'^7 contributes to the effective action with

$$S^{\text{amb}_1}_{\alpha'^7} = \frac{6}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X}d^5\hat{Y}\,(\nabla^\mu\nabla^\nu\nabla_\mu\nabla^\rho\nabla^\sigma\nabla_\rho\phi)\,(\nabla_\nu\nabla_\sigma\phi)\,\phi^2\ , \tag{D.0.1}$$

it corresponds to a four-derivative interaction and its contribution to the half-BPS correlator is given as

$$\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^7;\text{amb}_1} = \frac{1}{4!}\frac{(\mathcal{C}_4)^4}{(-2)^{16}}\int_{\text{AdS}\times\text{S}}\frac{d^5\hat{X}d^5\hat{Y}}{\prod_i(P_i+Q_i)^4}\sum_{i<j}\frac{K^{\text{amb}_1}_{ij}}{(P_i+Q_i)^2(P_j+Q_j)^2}\times 4^3\times 5\ , \tag{D.0.2}$$

where

$$\begin{aligned}K^{\text{amb}_1}_{ij} =& 45\left[(X_i.X_j+P_iP_j)^2+(Y_i.Y_j+Q_iQ_j)^2\right]-9\,(P_iP_j+Q_iQ_j)^2\\ &-180\,Q_iQ_j\,Y_i.Y_j+26\,P_iP_j\,Q_iQ_j\ .\end{aligned} \tag{D.0.3}$$

We write the corresponding Mellin amplitude as

$$\mathcal{M}^{\text{amb}_1}_{\alpha'^7} = \hat{\mathcal{M}}^{\text{amb}_1}_{\alpha'^7}+204\mathcal{M}^{\text{amb}}_{\alpha'^5}+12288\mathcal{M}^{\text{main}}_{\alpha'^3}\ , \tag{D.0.4}$$

where

$$\begin{aligned}\hat{\mathcal{M}}^{\text{amb}_1}_{\alpha'^7} = 288\Big[& (\Sigma_p{-}1)_5 \left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right)\\ & + (\Sigma_p{-}1)_4\,\frac{1}{2}\left(\mathbf{s}\,c_s^2+\mathbf{t}\,c_t^2+\mathbf{u}\,c_u^2\right)\\ & - (\Sigma_p{-}1)_4\,(\Sigma_p+3)\left[2\left(\mathbf{s}\,\tilde{s}+\mathbf{t}\,\tilde{t}+\mathbf{u}\,\tilde{u}\right)+\left(\mathbf{s}\,c_s+\mathbf{t}\,c_t+\mathbf{u}\,c_u\right)\right]\\ & + (\Sigma_p{-}1)_3\left(\frac{1}{12}\left(c_s^4+c_t^4+c_u^4\right)-\Sigma_p\left(c_s^2\,\tilde{s}+c_t^2\,\tilde{t}+c_u^2\,\tilde{u}^2\right)-\frac{1}{2}\Sigma_p\left(c_s^3+c_t^3+c_u^3\right)\right)\\ & + (\Sigma_p{-}1)_3\left(+\frac{29}{36}\Sigma_p^2\left(c_s^2+c_t^2+c_u^2\right)+\frac{1}{18}\left(c_s^2\,c_t^2+c_s^2\,c_u^2+c_t^2\,c_u^2\right)-\frac{5}{6}\,c_s\,c_t\,c_u\,\Sigma_p\right)\\ & + (\Sigma_p{-}1)_3\left(2\left(\Sigma_p^2+6\right)\left[\left(\tilde{s}^2+\tilde{t}^2+\tilde{u}^2\right)+\left(\tilde{s}\,c_s+\tilde{t}\,c_t+\tilde{u}\,c_u\right)\right]-\frac{1}{6}\Sigma_p^2\left(\Sigma_p^2+72\right)\right)\Big]\,.\end{aligned} \tag{D.0.5}$$

The next three ambiguities also correspond to four-derivative terms. The contribution to the effective action from the second ambiguity is

$$S^{\text{amb}_2}_{\alpha'^7} = \frac{6}{4!}\int_{\text{AdS}\times\text{S}} d^5\hat{X}d^5\hat{Y}\left(\nabla^2\nabla^\mu\nabla^\nu\nabla^\rho\nabla_\nu\phi\right)\left(\nabla_\mu\nabla_\rho\phi\right)\phi^2\ , \tag{D.0.6}$$

which corresponds to the correlator

$$\begin{aligned}&\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^7;\text{amb}_2}\\ &= \frac{1}{4!}\frac{(\mathcal{C}_4)^4}{(-2)^{16}}\int_{\text{AdS}\times\text{S}}\frac{d^5\hat{X}d^5\hat{Y}}{\prod_i\left(P_i+Q_i\right)^4}\sum_{i<j}\frac{K^{\text{amb}_2}_{ij}}{\left(P_i+Q_i\right)^2\left(P_j+Q_j\right)^2}\times 4^3\times 5^2\times 2\ ,\end{aligned} \tag{D.0.7}$$

where

$$\begin{aligned}K^{\text{amb}_2}_{ij} = & 5\left[\left(X_i.X_j+P_iP_j\right)^2+\left(Y_i.Y_j+Q_iQ_j\right)^2\right]-\left(P_iP_j+Q_iQ_j\right)^2\\ & -20\,Q_iQ_j\,Y_i.Y_j+2\,P_iP_j\,Q_iQ_j\ .\end{aligned} \tag{D.0.8}$$

The Mellin amplitude is

$$\mathcal{M}^{\text{amb}_2}_{\alpha'^7} = \hat{\mathcal{M}}^{\text{amb}_2}_{\alpha'^7}+248\mathcal{M}^{\text{amb}}_{\alpha'^5}+14336\mathcal{M}^{\text{main}}_{\alpha'^3}\ , \tag{D.0.9}$$

where

$$\begin{aligned}\hat{\mathcal{M}}^{\text{amb}_2}_{\alpha'^7} = 320\Big[& (\Sigma_p{-}1)_5 \left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right)\\ & + (\Sigma_p{-}1)_4\,\frac{1}{2}\left(\mathbf{s}\,c_s^2+\mathbf{t}\,c_t^2+\mathbf{u}\,c_u^2\right)\\ & - (\Sigma_p{-}1)_4\,(\Sigma_p+3)\left[2\left(\mathbf{s}\,\tilde{s}+\mathbf{t}\,\tilde{t}+\mathbf{u}\,\tilde{u}\right)+\left(\mathbf{s}\,c_s+\mathbf{t}\,c_t+\mathbf{u}\,c_u\right)\right]\end{aligned}$$

$$
\begin{aligned}
&+ (\Sigma_p{-}1)_3 \left(\frac{3}{40}\left(c_s^4 + c_t^4 + c_u^4\right) - \Sigma_p \left(c_s^2\, \tilde{s} + c_t^2\, \tilde{t} + c_u^2\, \tilde{u}^2\right)\right) \\
&+ (\Sigma_p{-}1)_3 \left(\frac{1}{20}\left(c_s^2\, c_t^2 + c_s^2\, c_u^2 + c_t^2\, c_u^2\right) - \frac{1}{2}\Sigma_p \left(c_s^3 + c_t^3 + c_u^3\right) + \frac{4}{5}\Sigma_p^2 \left(c_s^2 + c_t^2 + c_u^2\right)\right) \\
&+ (\Sigma_p{-}1)_3 \left(2\left(\Sigma_p^2 + 6\right)\left[\left(\tilde{s}^2 + \tilde{t}^2 + \tilde{u}^2\right) + \left(\tilde{s}\, c_s + \tilde{t}\, c_t + \tilde{u}\, c_u\right)\right]\right) \\
&+ (\Sigma_p{-}1)_3 \left(-\frac{9}{10}\, c_s\, c_t\, c_u\, \Sigma_p - \frac{1}{40}\Sigma_p^2 \left(7\,\Sigma_p^2 + 480\right)\right)\Bigg] \,. \qquad \text{(D.0.10)}
\end{aligned}
$$

The third ambiguity contributes to the effective action with

$$
S_{\alpha'^7}^{\mathrm{amb}_3} = \int_{\mathrm{AdS}\times\mathrm{S}} d^5\hat{X} d^5\hat{Y} \left(\nabla^2\nabla^\mu\nabla^\nu\nabla^\rho\nabla_\nu\phi\right)\left(\nabla_\rho\phi\right)\left(\nabla_\mu\phi\right)\phi \;, \qquad \text{(D.0.11)}
$$

and the prediction for its contribution to the half-BPS correlator is given by

$$
\begin{aligned}
&\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^7;\mathrm{amb}_3} = \\
&\frac{1}{4!}\frac{(\mathcal{C}_4)^4}{(-2)^{16}} \int_{\mathrm{AdS}\times\mathrm{S}} \frac{d^5\hat{X} d^5\hat{Y}}{\prod_i \left(P_i + Q_i\right)^4}\left[\frac{K_{123}^{\mathrm{amb}_3}}{\left(P_1 + Q_1\right)^2\left(P_2 + Q_2\right)\left(P_3 + Q_3\right)} + \mathrm{perms}\right] \times 4^4 \times 10 \;, \qquad \text{(D.0.12)}
\end{aligned}
$$

where we sum over all permutations and

$$
\begin{aligned}
K_{ijk}^{\mathrm{amb}_3} = {}& P_i^2\left(4\, P_j P_k - X_j.X_k\right) + Q_i^2\left(4\, Q_j Q_k + Y_j.Y_k\right) + 5\, P_i\left(P_k\, X_i.X_j + P_j\, X_i.X_k\right) \\
&- 5\, Q_i\left(Q_k\, Y_i.Y_j + Q_j\, Y_i.Y_k\right) + 5\left(X_i.X_j\, X_i.X_k + Y_i.Y_j\, Y_i.Y_k\right) \;. \qquad \text{(D.0.13)}
\end{aligned}
$$

The corresponding Mellin amplitude is

$$
\mathcal{M}_{\alpha'^7}^{\mathrm{amb}_3} = \hat{\mathcal{M}}_{\alpha'^7}^{\mathrm{amb}_3} - 704\mathcal{M}_{\alpha'^5}^{\mathrm{amb}} - 32768\mathcal{M}_{\alpha'^3}^{\mathrm{main}} \;, \qquad \text{(D.0.14)}
$$

where

$$
\begin{aligned}
\hat{\mathcal{M}}_{\alpha'^7}^{\mathrm{amb}_3} = -640\Big[&(\Sigma_p{-}1)_5 \left(\mathbf{s}^2 + \mathbf{t}^2 + \mathbf{u}^2\right) \\
&+ (\Sigma_p{-}1)_4\, \frac{1}{2}\left(\mathbf{s}\, c_s^2 + \mathbf{t}\, c_t^2 + \mathbf{u}\, c_u^2\right) \\
&- (\Sigma_p{-}1)_4\, \left(\Sigma_p + 3\right)\left[2\left(\mathbf{s}\,\tilde{s} + \mathbf{t}\,\tilde{t} + \mathbf{u}\,\tilde{u}\right) + \left(\mathbf{s}\, c_s + \mathbf{t}\, c_t + \mathbf{u}\, c_u\right)\right] \\
&+ (\Sigma_p{-}1)_3 \left(\frac{1}{20}\left(c_s^4 + c_t^4 + c_u^4\right) - \Sigma_p \left(c_s^2\, \tilde{s} + c_t^2\, \tilde{t} + c_u^2\, \tilde{u}^2\right)\right) \\
&+ (\Sigma_p{-}1)_3 \left(-\frac{1}{10}\left(c_s^2\, c_t^2 + c_s^2\, c_u^2 + c_t^2\, c_u^2\right) - \frac{1}{2}\Sigma_p \left(c_s^3 + c_t^3 + c_u^3\right)\right) \\
&+ (\Sigma_p{-}1)_3 \left(\frac{13}{20}\Sigma_p^2 \left(c_s^2 + c_t^2 + c_u^2\right) - \frac{3}{10}\, c_s\, c_t\, c_u\, \Sigma_p\right)
\end{aligned}
$$

$$+\left(\Sigma_p{-}1\right)_3 \left(2\left(\Sigma_p^2+6\right)\left[\left(\tilde{s}^2+\tilde{t}^2+\tilde{u}^2\right)+\left(\tilde{s}\,c_s+\tilde{t}\,c_t+\tilde{u}\,c_u\right)\right]-\frac{1}{5}\Sigma_p^2\left(\Sigma_p^2+60\right)\right)\Big] \,. \tag{D.0.15}$$

The next ambiguity contributes to the effective action as

$$S_{\alpha'^7}^{\mathrm{amb}_4}=\int_{\mathrm{AdS}\times\mathrm{S}} d^5\hat{X}d^5\hat{Y}\left(\nabla^\mu\nabla^\nu\nabla^\rho\nabla_\nu\nabla_\rho\phi\right)\left(\nabla^\sigma\nabla_\mu\phi\right)\left(\nabla_\sigma\phi\right)\phi\;, \tag{D.0.16}$$

and the corresponding correlator is given by

$$\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^7;\mathrm{amb}_4}=$$
$$\frac{1}{4!}\frac{(\mathcal{C}_4)^4}{(-2)^{16}}\int_{\mathrm{AdS}\times\mathrm{S}}\frac{d^5\hat{X}d^5\hat{Y}}{\prod_i\left(P_i+Q_i\right)^4}\left[\frac{K_{123}^{\mathrm{amb}_4}}{\left(P_1+Q_1\right)^3\left(P_2+Q_2\right)^2\left(P_3+Q_3\right)}+\mathrm{perms}\right]\times 4^4\times 10\;, \tag{D.0.17}$$

where

$$\begin{aligned}
K_{ijk}^{\mathrm{amb}_4}=&\,P_i^2Q_i\left[5\left(5P_j+Q_j\right)\left(X_j.X_k+Y_j.Y_k\right)-4Q_jQ_k\left(6P_j+Q_j\right)+20P_j^2P_k\right]\\
&-P_iQ_i^2\left[5\left(P_j+5Q_j\right)\left(X_j.X_k+Y_j.Y_k\right)+4P_jP_k\left(P_j+6Q_j\right)-20Q_j^2Q_k\right]\\
&-P_i^2\left[Q_j\left(P_j+Q_j\right)Y_i.Y_k+5Y_i.Y_j\left(X_j.X_k+Y_j.Y_k+P_jP_k-Q_jQ_k\right)\right]\\
&+Q_i^2\left[P_j\left(P_j+Q_j\right)X_i.X_k-5X_i.X_j\left(X_j.X_k+Y_j.Y_k+P_jP_k-Q_jQ_k\right)\right]\\
&+P_iQ_i\left[25\left(P_jP_k-Q_jQ_k\right)\left(X_i.X_j+Y_i.Y_j\right)+\left(P_j+Q_j\right)\left(5Q_j\,Y_i.Y_k-5P_j\,X_i.X_k\right)\right.\\
&\qquad\left.+25\left(X_i.X_j+Y_i.Y_j\right)\left(X_j.X_k+Y_j.Y_k\right)\right]\;.
\end{aligned} \tag{D.0.18}$$

The contribution of this ambiguity to the Mellin amplitude is

$$\mathcal{M}_{\alpha'^7}^{\mathrm{amb}_4}=\hat{\mathcal{M}}_{\alpha'^7}^{\mathrm{amb}_4}-128\mathcal{M}_{\alpha'^5}^{\mathrm{main}}\;, \tag{D.0.19}$$

where

$$\begin{aligned}
\hat{\mathcal{M}}_{\alpha'^7}^{\mathrm{amb}_4}=32\Big[&\left(\Sigma_p{-}1\right)_5\,\Sigma_p^2\left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right)\\
&+\left(\Sigma_p{-}1\right)_5\left(\left(\mathbf{s}^2\,c_s^2+\mathbf{t}^2\,c_t^2+\mathbf{u}^2\,c_u^2\right)+\left[\mathbf{s}^2\left(c_t^2+c_u^2\right)+\mathbf{t}^2\left(c_s^2+c_u^2\right)+\mathbf{u}^2\left(c_s^2+c_t^2\right)\right]\right)\\
&+\left(\Sigma_p{-}1\right)_4\left(-5\left(\mathbf{s}\,c_s^3+\mathbf{t}\,c_t^3+\mathbf{u}\,c_u^3\right)-10\left(\mathbf{s}\,c_s^2\,\tilde{s}+\mathbf{t}\,c_t^2\,\tilde{t}+\mathbf{u}\,c_u^2\,\tilde{u}\right)\right)\\
&+\left(\Sigma_p{-}1\right)_4\left(-10\Sigma_p^2\left(\mathbf{s}\,\tilde{s}+\mathbf{t}\,\tilde{t}+\mathbf{u}\,\tilde{u}\right)-5\Sigma_p^2\left(\mathbf{s}\,c_s+\mathbf{t}\,c_t+\mathbf{u}\,c_u\right)\right)\\
&+\left(\Sigma_p{-}1\right)_4\left(-10\left[\mathbf{s}\,\tilde{s}\left(c_t^2+c_u^2\right)+\mathbf{t}\,\tilde{t}\left(c_s^2+c_u^2\right)+\mathbf{u}\,\tilde{u}\left(c_s^2+c_t^2\right)\right]\right)\\
&+\left(\Sigma_p{-}1\right)_4\left(-5\left[\mathbf{s}\,c_s\left(c_t^2+c_u^2\right)+\mathbf{t}\,c_t\left(c_s^2+c_u^2\right)+\mathbf{u}\,c_u\left(c_s^2+c_t^2\right)\right]\right)
\end{aligned}$$

$$+ (\Sigma_p{-}1)_3 \left(4\left(c_s^4 + c_t^4 + c_u^4\right) + 20\left(\tilde{s}^2\, c_s^2 + \tilde{t}^2\, c_t^2 + \tilde{u}^2\, c_u^2\right) + 8\left(c_s^2\, c_t^2 + c_s^2\, c_u^2 + c_t^2\, c_u^2\right)\right)$$
$$+ (\Sigma_p{-}1)_3 \left(20\left(c_s^3\, \tilde{s} + c_t^3\, \tilde{t} + c_u^3\, \tilde{u}\right) + 20\,\Sigma_p^2\left(\tilde{s}^2 + \tilde{t}^2 + \tilde{u}^2\right)\right)$$
$$+ (\Sigma_p{-}1)_3 \left(-8\,\Sigma_p^2\left(c_s^2 + c_t^2 + c_u^2\right) + 20\,\Sigma_p^2\left(\tilde{s}\, c_s + \tilde{t}\, c_t + \tilde{u}\, c_u\right)\right)$$
$$+ (\Sigma_p{-}1)_3 \left(20\left[\tilde{s}^2\left(c_t^2 + c_u^2\right) + \tilde{t}^2\left(c_s^2 + c_u^2\right) + \tilde{u}^2\left(c_s^2 + c_t^2\right)\right]\right)$$
$$+ (\Sigma_p{-}1)_3 \left(20\left[\tilde{s}\, c_s\left(c_t^2 + c_u^2\right) + \tilde{t}\, c_t\left(c_s^2 + c_u^2\right) + \tilde{u}\, c_u\left(c_s^2 + c_t^2\right)\right] - 12\,\Sigma_p^4\right)\Bigg] \,. \tag{D.0.20}$$

Finally, the fifth ambiguity contributes to the effective action with

$$S_{\alpha'^7}^{\mathrm{amb}_5} = \int_{\mathrm{AdS}\times\mathrm{S}} d^5\hat{X}\, d^5\hat{Y}\, \left(\nabla^\mu\nabla^\nu\nabla^\rho\nabla^\sigma\nabla_\rho\phi\right)\left(\nabla_\mu\nabla_\sigma\phi\right)\left(\nabla_\nu\phi\right)\phi \;, \tag{D.0.21}$$

which corresponds to a six-derivative interaction and its contribution to the half-BPS correlator is

$$\langle\mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O}\rangle|_{\alpha'^7;\mathrm{amb}_5} =$$
$$\frac{1}{4!}\frac{(\mathcal{C}_4)^4}{(-2)^{16}}\int_{\mathrm{AdS}\times\mathrm{S}} \frac{d^5\hat{X}\, d^5\hat{Y}}{\prod_i (P_i + Q_i)^4}\left[\frac{K_{123}^{\mathrm{amb}_5}}{(P_1+Q_1)^3\,(P_2+Q_2)^2\,(P_3+Q_3)} + \mathrm{perms}\right] \times \left(-4^4\right)\;, \tag{D.0.22}$$

$$\begin{aligned}
K_{ijk}^{\mathrm{amb}_5} = {}& P_i^3\left[P_j\left(4\left(Q_j + 21P_j\right)P_k - 45X_j.X_k\right)\right] + Q_i^3\left[Q_j\left(4\left(P_j + 21Q_j\right)Q_k + 45Y_j.Y_k\right)\right] \\
& + P_i^2 Q_i\left[8\left(Q_j - 4P_j\right)P_jP_k + 4\left(Q_j - 5P_j\right)\left(Q_j + 6P_j\right)Q_k - 40P_jX_j.X_k\right. \\
& \left.-5Q_jY_j.Y_k\right] + P_iQ_i^2\left[8\left(P_j - 4Q_j\right)Q_jQ_k + 4\left(P_j - 5Q_j\right)\left(P_j + 6Q_j\right)P_k\right. \\
& \left.+40Q_jY_j.Y_k + 5P_jX_j.X_k\right] + P_i^2\left[P_j\left(129P_j + 4Q_j\right)X_i.X_k + 120P_j^2Y_i.Y_k\right. \\
& +15X_i.X_j\left(17P_jP_k - 3X_j.X_k\right) + Q_j\left(-5Q_kY_i.Y_j + \left(-4P_j + Q_j\right)Y_i.Y_k\right) \\
& \left.+5Y_i.Y_jY_j.Y_k\right] + Q_i^2\left[-Q_j\left(129Q_j + 4P_j\right)Y_i.Y_k - 15Y_i.Y_j\left(17Q_jQ_k + 3Y_j.Y_k\right)\right. \\
& \left.-120Q_j^2X_i.X_k - P_j\left(-5P_kX_i.X_j + \left(-4Q_j + P_j\right)X_i.X_k\right) + 5X_i.X_jX_j.X_k\right] \\
& + P_iQ_i\left[8\left(P_j + Q_j\right)\left(P_jX_i.X_k - Q_jY_i.Y_k\right)\right. \\
& \left.+20\left(-X_i.X_j\left(P_j\left(2P_k + 15Q_k\right) + 2X_j.X_k\right) + Y_i.Y_j\left(Q_j\left(2Q_k + 15P_k\right) - 2Y_j.Y_k\right)\right)\right] \\
& + P_i\left[150P_k\left(\left(X_i.X_j\right)^2 - \left(Y_i.Y_j\right)^2\right) + 300P_jX_i.X_j\left(X_i.X_k + Y_i.Y_k\right)\right] \\
& + Q_i\left[-150Q_k\left(\left(X_i.X_j\right)^2 - \left(Y_i.Y_j\right)^2\right) + 300Q_jY_i.Y_j\left(X_i.X_k + Y_i.Y_k\right)\right] \\
& + 150\left(\left(X_i.X_j\right)^2 - \left(Y_i.Y_j\right)^2\right)\left(X_i.X_k + Y_i.Y_k\right)\;.
\end{aligned} \tag{D.0.23}$$

The corresponding Mellin amplitude is

$$\begin{aligned}\mathcal{M}^{\mathrm{amb}_5}_{\alpha'^7} =& \hat{\mathcal{M}}^{\mathrm{amb}_5}_{\alpha'^7} - \frac{11}{2}\mathcal{M}^{\mathrm{amb}_1}_{\alpha'^7} + 5\mathcal{M}^{\mathrm{amb}_2}_{\alpha'^7} + \frac{1}{8}\mathcal{M}^{\mathrm{amb}_3}_{\alpha'^7} - \mathcal{M}^{\mathrm{amb}_4}_{\alpha'^7} + 64\mathcal{M}^{\mathrm{main}}_{\alpha'^5}\\ &+ 66\mathcal{M}^{\mathrm{amb}}_{\alpha'^5} + 4096\mathcal{M}^{\mathrm{main}}_{\alpha'^3}\,, \end{aligned} \tag{D.0.24}$$

where

$$\begin{aligned}\hat{\mathcal{M}}^{\mathrm{amb}_5}_{\alpha'^7} = 128\Big[&(\Sigma_p{-}1)_6\left(\mathbf{s}^3+\mathbf{t}^3+\mathbf{u}^3\right)\\
&+(\Sigma_p{-}1)_5\left(\frac{1}{2}\left(\mathbf{s}^2\,c_s^2+\mathbf{t}^2\,c_t^2+\mathbf{u}^2\,c_u^2\right)+\frac{1}{2}\,\Sigma_p\,(\Sigma_p+8)\left(\mathbf{s}^2+\mathbf{t}^2+\mathbf{u}^2\right)\right)\\
&+(\Sigma_p{-}1)_5\left(-\left(\Sigma_p+7\right)\left[2\left(\mathbf{s}^2\,\tilde{s}+\mathbf{t}^2\,\tilde{t}+\mathbf{u}^2\,\tilde{u}\right)+\left(\mathbf{s}^2\,c_s+\mathbf{t}^2\,c_t+\mathbf{u}^2\,c_u\right)\right]\right)\\
&+(\Sigma_p{-}1)_4\left(-\frac{5}{2}\left(\mathbf{s}\,c_s^3+\mathbf{t}\,c_t^3+\mathbf{u}\,c_u^3\right)-5\left(\mathbf{s}\,c_s^2\,\tilde{s}+\mathbf{t}\,c_t^2\,\tilde{t}+\mathbf{u}\,c_u^2\,\tilde{u}\right)\right)\\
&+(\Sigma_p{-}1)_4\left(6\,(4\,\Sigma_p+7)\left[\left(\mathbf{s}\,\tilde{s}^2+\mathbf{t}\,\tilde{t}^2+\mathbf{u}\,\tilde{u}^2\right)+\left(\mathbf{s}\,\tilde{s}\,c_s+\mathbf{t}\,\tilde{t}\,c_t+\mathbf{u}\,\tilde{u}\,c_u\right)\right]\right)\\
&+(\Sigma_p{-}1)_4\left(-\Sigma_p\,(13\,\Sigma_p+24)\left[\left(\mathbf{s}\,\tilde{s}+\mathbf{t}\,\tilde{t}+\mathbf{u}\,\tilde{u}\right)+\frac{1}{2}\left(\mathbf{s}\,c_s+\mathbf{t}\,c_t+\mathbf{u}\,c_u\right)\right]\right)\\
&+(\Sigma_p{-}1)_4\left(\frac{1}{2}\,(14\,\Sigma_p+19)\left(\mathbf{s}\,c_s^2+\mathbf{t}\,c_t^2+\mathbf{u}\,c_u^2\right)\right)\\
&+(\Sigma_p{-}1)_3\left(\frac{17}{8}\left(c_s^4+c_t^4+c_u^4\right)-60\,\Sigma_p\left(\tilde{s}^3+\tilde{t}^3+\tilde{u}^3\right)-\frac{17}{2}\,\Sigma_p\left(c_s^3+c_t^3+c_u^3\right)\right)\\
&+(\Sigma_p{-}1)_3\left(10\left(c_s^3\,\tilde{s}+c_t^3\,\tilde{t}+c_u^3\,\tilde{u}\right)+10\left(\tilde{s}^2\,c_s^2+\tilde{t}^2\,c_t^2+\tilde{u}^2\,c_u^2\right)\right)\\
&+(\Sigma_p{-}1)_3\left(-90\,\Sigma_p\left(\tilde{s}^2\,c_s+\tilde{t}^2\,c_t+\tilde{u}^2\,c_u\right)-47\,\Sigma_p\left(c_s^2\,\tilde{s}+c_t^2\,\tilde{t}+c_u^2\,\tilde{u}\right)\right)\\
&+(\Sigma_p{-}1)_3\left(\frac{51}{4}\,\Sigma_p^2\left(c_s^2+c_t^2+c_u^2\right)+50\,\Sigma_p^2\left(\tilde{s}^2+\tilde{t}^2+\tilde{u}^2\right)\right)\\
&+(\Sigma_p{-}1)_3\left(50\,\Sigma_p^2\left(\tilde{s}\,c_s+\tilde{t}\,c_t+\tilde{u}\,c_u\right)-\frac{1}{8}\,\Sigma_p^2\left(81\,\Sigma_p^2+352\right)\right)\Big]\,. \end{aligned} \tag{D.0.25}$$

Appendix E

Conformal Blocks in 6d

Conformal blocks for four-point correlators of scalar operators of arbitrary scaling dimensions Δ_i, $i = 1, \ldots, 4$, in any even dimension were derived by Dolan and Osborn in [46]. In 6d, the blocks are given by

$$
\begin{aligned}
&G^{\mathrm{DO}}\left(\Delta, l, \Delta_{12}, \Delta_{34}\right) = \\
&\quad \mathcal{F}_{00} - \frac{l+3}{l+1}\mathcal{F}_{-11} \\
&\quad - \frac{\Delta-4}{\Delta-2}\,\frac{\left(\Delta+l-\Delta_{12}\right)\left(\Delta+l+\Delta_{12}\right)\left(\Delta+l+\Delta_{34}\right)\left(\Delta+l-\Delta_{34}\right)}{16\left(\Delta+l-1\right)\left(\Delta+l\right)^2\left(\Delta+l+1\right)}\mathcal{F}_{11} \\
&\quad + \frac{\left(\Delta-4\right)\left(l+3\right)}{\left(\Delta-2\right)\left(l+1\right)} \\
&\quad \times \frac{\left(\Delta-l-\Delta_{12}-4\right)\left(\Delta-l+\Delta_{12}-4\right)\left(\Delta-l+\Delta_{34}-4\right)\left(\Delta-l-\Delta_{34}-4\right)}{16\left(\Delta-l-5\right)\left(\Delta-l-4\right)^2\left(\Delta-l-3\right)}\mathcal{F}_{02} \\
&\quad + 2\left(\Delta-4\right)\left(l+3\right)\frac{\Delta_{12}\Delta_{34}}{\left(\Delta+l\right)\left(\Delta+l-2\right)\left(\Delta-l-4\right)\left(\Delta-l-6\right)}\mathcal{F}_{01}\ ,
\end{aligned}
\tag{E.0.1}
$$

where (Δ, l) are the scaling dimension and spin of a primary operator in the conformal block expansion, $\Delta_{ij} = \Delta_i - \Delta_j$, and

$$
\begin{aligned}
\mathcal{F}_{ab} = &\frac{\left(z\bar{z}\right)^{\frac{1}{2}\left(\Delta-l\right)}}{\left(z-\bar{z}\right)^3}\Big\{ z^{l+a+3}\bar{z}^b \\
&\times {}_2F_1\left(\tfrac{1}{2}(\Delta+l-\Delta_{12})+a, \tfrac{1}{2}(\Delta+l+\Delta_{34})+a; \Delta+l+2a, z\right) \\
&\times {}_2F_1\left(\tfrac{1}{2}(\Delta-l-\Delta_{12})-3+b, \tfrac{1}{2}(\Delta-l+\Delta_{34})-3+b; \Delta-l-6+2b; \bar{z}\right) \\
&- z \leftrightarrow \bar{z}\Big\}\ .
\end{aligned}
\tag{E.0.2}
$$

For the toy model analysed in section 4.2, the blocks are given by

$$G^{\mathrm{B}}_{\Delta,l}(z,\bar{z}) = (l+1)\, G^{\mathrm{DO}}(\Delta, l, 0, 0)\ , \tag{E.0.3}$$

where $\Delta = 2n + l + 2\Delta_0 + \mathcal{O}(1/c)$. Moreover, for the 6d $(2,0)$ theory analysed in section 4.3, the blocks are given by [25, 47]

$$G^{\mathrm{S}}_{\Delta,l}(z,\bar{z}) = \frac{4\,(l+1)}{(l+2)^2 - \Delta^2}\, \frac{(z-\bar{z})^3}{u^5}\, G^{\mathrm{DO}}(\Delta+4, l, 0, -2)\ , \tag{E.0.4}$$

where $\Delta = 2n + l + 8 + \mathcal{O}(1/c)$ with $n \geq 0$.

Appendix F

Orthogonality of Hypergeometrics

In this appendix we derive orthogonality relations for hypergeometric functions used in chapter 4, explicating a brief argument in [30] which then allows us to obtain a new case relevant for the supersymmetric 6d theory. Our starting point will be the differential operator[1]

$$D_z = z^2\,(1-z)\,\partial_z^2 - (a+b+1)z^2\partial_z - a\,b\,z\ . \tag{F.0.2}$$

This operator has eigenfunctions satisfying

$$D_z H_m(z) = m(m-1)H_m(z)\ , \tag{F.0.3}$$

where

$$H_m(z) = z^m{}_2F_1(m+a, m+b; 2m; z)\ . \tag{F.0.4}$$

First consider $a = b = 0$. In this case, the differential operator in (F.0.2) reduces to $D_z = z^2\partial_z\,(1-z)\partial_z$. Let us look at the object $H_m H_{1-m'}$ (we will omit the arguments (z) in the following). Using the symmetry of the differential operator D_z, after

[1]Note that this operator is closely related to the conformal Casimir. In d dimensions this is [46]

$$\begin{aligned} D_\epsilon &= z^2\,(1-z)\,\partial_z^2 + \bar{z}^2\,(1-\bar{z})\,\partial_{\bar{z}}^2 - (a+b+1)\,(z^2\partial_z + \bar{z}^2\partial_{\bar{z}}) \\ &\quad - a\,b\,(z+\bar{z}) + \epsilon\,\frac{z\,\bar{z}}{z-\bar{z}}\left((1-z)\,\partial_z - (1-\bar{z})\,\partial_{\bar{z}}\right)\ , \end{aligned} \tag{F.0.1}$$

where a, b are arbitrary constants and $\epsilon = d-2$. The non-interacting part (i.e ϵ-independent part) reduces to $D_z + D_{\bar{z}}$.

integrating by parts twice and using (F.0.3) we find that

$$
\begin{aligned}
0 =& \oint \frac{dz}{2\pi i} \frac{1}{z^2} \left[(D_z H_m) H_{1-m'} - H_m (D_z H_{1-m'}) \right] \\
=& [m(m-1) - m'(m'-1)] \oint \frac{dz}{2\pi i} \frac{1}{z^2} H_m H_{1-m'} \,,
\end{aligned} \tag{F.0.5}
$$

where the contour encircles the origin. It follows that H_m and $H_{1-m'}$ are orthogonal with respect to the inner product defined above if $m \neq m'$. Plugging in (F.0.4) and shifting (m, m') to $(m+2, m'+2)$ then implies the inner product in (4.2.16), where we fix the normalisation by noting that ${}_2F_1(\alpha, \beta, \gamma, z) = 1 + \mathcal{O}(z)$ and evaluating the residue at $z = 0$. This relation was first obtained in [30].

Next, consider $a = 0$, $b = -1$, in which case (F.0.2) reduces to $D_z = z^2 (1-z) \partial_z^2$. Following the same arguments as above we find that

$$
\begin{aligned}
0 =& \oint \frac{dz}{2\pi i} \frac{1}{z^2(1-z)} \left[(D_z H_m) H_{1-m'} - H_m (D_z H_{1-m'}) \right] \\
=& [m(m-1) - m'(m'-1)] \oint \frac{dz}{2\pi i} \frac{1}{z^2(1-z)} H_m H_{1-m'} \,,
\end{aligned} \tag{F.0.6}
$$

so H_m and $H_{1-m'}$ are orthogonal with respect to the inner product defined above if $m \neq m'$. Plugging in (F.0.4) and shifting (m, m') to $(m+2, m'+2)$ then proves the inner product in (4.3.17), where the normalisation is once again fixed by evaluating the residue at $z = 0$.

Appendix G

Tree-Level Supergravity in the 6d $(2,0)$ Theory

In this appendix we will discuss the 6d $(2,0)$ tree-level supergravity prepotential and derive the anomalous dimensions of the double-trace operators contributing to its conformal block expansion. For large c, the four-point prepotential can be derived from tree-level Witten diagrams for supergravity in $AdS_7 \times S^4$. In [139] the authors showed that the supergravity contribution consists of a contribution from free theory (including c^0 and $1/c$ terms) plus the remaining supergravity contribution, which we call dynamical contribution and which goes like $1/c$. We denote the corresponding prepotential terms by F^{free} and F^{sugra} respectively. Note that in section 4.3 we only considered the leading disconnected free contribution at c^0, now we have to consider an extra piece. The free contribution is then given by:

$$F^{\text{free}} = 1 + \frac{1}{u^2} + \frac{1}{v^2} + \frac{1}{c}\left(\frac{1}{u} + \frac{1}{v} + \frac{1}{u\,v}\right) , \tag{G.0.1}$$

which satisfies the crossing condition (4.3.3). Decomposing F^{free} according to (4.3.4) yields

$$A = 1 \ , \quad g(z) = \frac{z}{c}\left(1 + \frac{1}{1-z}\right) . \tag{G.0.2}$$

In addition to expanding G according to (4.3.5) one can expand g in terms of short conformal blocks as follows

$$g(x)=\sum_{m=0}^{\infty}B_m\,g_m(x)\ ,\quad g_m(x)=x^{m+1}{}_2F_1(m+2,m+1,2m+4,x)\ ,\qquad\text{(G.0.3)}$$

where the first contribution with $m=0$ corresponds to the conformal block of the stress tensor supermultiplet (see [26] for more details on this). However, we focus on the expansion of G in terms of long blocks which gives

$$\begin{aligned}A^{\text{free}}_{n,l}=&\frac{(l+2)(n+3)!(n+4)!(l+2n+9)(l+n+5)!(l+n+6)!}{(2n+5)!(2l+2n+9)!}\\&\times\left(\frac{1}{72}(l+2n+10)+\frac{1}{c}\frac{(-1)^n}{2(l+1)(l+3)(l+2n+8)}\right)\ ,\end{aligned}\qquad\text{(G.0.4)}$$

for l even and zero otherwise. Note that the c^0-piece was already obtained in (4.3.8).

Next, we consider the dynamical supergravity contribution at $\mathcal{O}(1/c)$. This is given by the following prepotential derived from AdS contact diagrams:

$$F^{\text{sugra}}=-\frac{1}{c}\frac{(z-\bar{z})^2}{u\,v}\bar{D}_{3337}\ .\qquad\text{(G.0.5)}$$

Decomposing this according to (4.3.4) gives $A=0$ and

$$g(z)=\frac{1}{c}\left(2\,z\,{}_2F_1(2,1,4,z)-z\left(1+\frac{1}{1-z}\right)\right)\ .\qquad\text{(G.0.6)}$$

The first term corresponds to the conformal block of the stress tensor supermultiplet, which can be seen from (G.0.3) with $m=0$. Importantly, the second term cancels with the free theory contribution (G.0.2). These terms correspond to twist-4 states (see [26] for more details). This cancellation is required because it is expected that only operators in the singlet representation with twist eight or higher develop anomalous dimensions. The twist-4 states which contribute in the free theory should be absent in the supergravity limit because from the AdS/CFT correspondence it is known that there are no supergravity states with twist four, which corresponds to the minimal twist representation.

Finally, the OPE coefficients and anomalous dimensions of the long operators in the conformal block expansion of the dynamical supergravity contribution can be obtained from an expansion of the form (4.3.10) but with $A^{\text{free}}_{n,l}$ which depends on $1/c$ instead of $A^{(0)}_{n,l}$ and without the crossing version. Let us look more closely at the contributions from the dynamical part. Since the conformal block expansion at order $1/c$ is proportional to $\log u$ (see (4.3.10)) it is useful to decompose G^{sugra} as follows:

$$G^{\text{sugra}}(z,\bar{z}) = \log u \; G_{\text{log}}(z,\bar{z}) + G_{\text{non-log}}(z,\bar{z}) \;, \tag{G.0.7}$$

where only the $\log u$ piece is important for the computation of the $\gamma_{n,l}$ and the non-log piece is analytic as $u \to 0$. Now performing the conformal block expansion of G_{log} gives the anomalous dimensions

$$\gamma^{\text{sugra}}_{n,l} = -\frac{3}{c}\left(\frac{n(n+3)}{2(l+3)(l+2n+8)}+1\right)\frac{(n+1)(n+2)(n+3)(n+4)(n+5)(n+6)}{(l+1)(l+2)(l+2n+9)(l+2(n+5))} \;, \tag{G.0.8}$$

where we divided by $A^{\text{free}}_{n,l}$. The supergravity OPE coefficients can then be computed from (4.2.9).

Note that in the large-n limit $\gamma^{\text{sugra}}_{n,l}$ goes like n^5. In subsection 4.3.2 we compare the large-twist behaviour of the anomalous dimensions from spin-L truncations to this scaling in order to deduce the additional number of derivatives the spin-L contact interactions obtain compared to supergravity.

Appendix H

Quadratic Super Casimir and Correlator of Descendants

In this appendix we will derive the quadratic super Casimir of $SU(1,1|2)$. This contains the second-order differential operator $\Delta^{(2)}$ which plays a leading role in the hidden four-dimensional conformal symmetry. Furthermore, we will sketch the calculation of the correlator of descendants introduced in subsection 5.2.1.

The superblocks (see (5.2.23) and (5.2.24)) are eigenfunctions of the quadratic super Casimir at points 1 and 2 acting on the correlator. Note that the formalism and super Casimir outlined below generalise naturally from the supergroup $SU(1,1|2)$ to any supergroup of the form $SU(m,m|2n)$. This was done in the bosonic case $SU(m,m)$ in [49].

Consider the super Grassmannian $Gr(1|1,2|2)$, the space of $(1|1)\times(2|2)$ matrices $u^{\alpha}{}_{A}$. Here the small Greek indices refer to the local isotropy group $GL(1|1)$ whilst the big Latin indices refer to the global group $GL(2|2)$. Explicitly, one can put coordinates on this Grassmannian as

$$(u_i)^{\alpha}{}_{B} = \left(\delta^{\alpha}_{\beta}, (X_i)^{\alpha}{}_{\dot\beta}\right) \ , \quad (\bar u_i)^{B}{}_{\dot\alpha} = \begin{pmatrix} -(X_i)^{\beta}{}_{\dot\alpha} \\ \delta^{\dot\beta}_{\dot\alpha} \end{pmatrix} , \tag{H.0.1}$$

where X_i is a matrix containing the spacetime and internal coordinates x, y, and

Grassmann odd variables $\theta, \bar{\theta}$:

$$(X_i)^{\alpha}{}_{\dot{\beta}} = \begin{pmatrix} x_i & \theta_i \\ \bar{\theta}_i & y_i \end{pmatrix} \quad \text{and} \quad X_{ij} = X_i - X_j \ . \tag{H.0.2}$$

We thus have $(u_i)^{\alpha}_{B}\,(\bar{u}_j)^{B}_{\dot{\alpha}} = (X_{ij})^{\alpha}_{\dot{\alpha}}$. Then the generators of the superconformal group $SU(1,1|2)$ at point i are given as

$$D^{A}_{i\,B} = (u_i)^{\alpha}_{A}\,\frac{\partial}{\partial\,(u_i)^{\alpha}_{B}} \ , \tag{H.0.3}$$

The quadratic Casimir operator acting on the four-point function at points 1 and 2 is then given as

$$\mathcal{C}^{SU(1,1|2)}_{1,2} = \frac{1}{2}(D^{A}_{1B} + D^{A}_{2B})(D^{B}_{1A} + D^{B}_{2A}) \ . \tag{H.0.4}$$

Superconformal symmetry $SU(1,1|2)$ fixes the correlator in terms of a conjugation invariant function of a cross-ratio matrix (see [49, 161]) to the form

$$\langle \Psi_{p_1}\Psi_{p_2}\Psi_{p_3}\Psi_{p_4}\rangle = P_{p_i} \times f(Z) \tag{H.0.5}$$

where the matrix of cross-ratios is given by

$$Z := X_{12}X_{24}^{-1}X_{43}X_{31}^{-1} = \begin{pmatrix} x & \xi \\ \bar{\xi} & y \end{pmatrix} \ , \tag{H.0.6}$$

with Grassmann odd variables $\xi, \bar{\xi}$ and note that x, y here are different to the cross-ratios used in the main text, see [49, 161] for more details. That $f(Z)$ is a conjugation invariant function of the cross-ratio matrix means that $f(Z) = f(G^{-1}ZG)$ where $G \in GL(2,2)$. Therefore, it can be diagonalised and the correlator can be written in terms of a function $f(\hat{x}, \hat{y})$ of the eigenvalues of Z, $\hat{x}, \hat{y}$ only. These are given by

$$\hat{x} = x + \frac{\xi\bar{\xi}}{x-y} \qquad \hat{y} = y + \frac{\xi\bar{\xi}}{x-y} \ . \tag{H.0.7}$$

That the eigenvalues of Z are given by (H.0.7) can be checked by verifying that the diagonal matrix of eigenvalues gives the same supertrace and superdeterminant as the matrix of cross-ratios Z:

$$\mathrm{str}(Z) = x - y = \hat{x} - \hat{y}\ , \qquad \mathrm{sdet}(Z) = \frac{x - \xi\bar{\xi}/y}{y} = \frac{\hat{x}}{\hat{y}}\ . \tag{H.0.8}$$

Now, acting with the Casimir on the supercorrelator in terms of a function depending on the eigenvalues of Z, $f(\hat{x},\hat{y})$ (H.0.5), and commuting through the prefactor gives

$$\begin{aligned}&\mathcal{C}^{SU(1,1|2)}_{1,2}\langle \Psi_{p_1}\Psi_{p_2}\Psi_{p_3}\Psi_{p_4}\rangle \\ &= P_{p_i} \times \left[\left((p_{12}{-}p_{34})\left(x^2\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}\right) + p_{34}p_{12}(x-y)\right) f + \frac{1}{2}\mathcal{C}^{SU(1,1|2)}_{1,2} f\right]\ ,\end{aligned} \tag{H.0.9}$$

with $p_{ij} = p_i - p_j$. To obtain this equation we used that $\mathrm{sdet}(M) = \exp(\mathrm{str}\log(M))$ to deal with differentiating the propagators $g_{ij} \coloneqq \mathrm{sdet}(X^{-1}_{ij})$ and then applied the double derivative directly, using $D_{12}{}^A{}_B\,(u_i)^\alpha_C = (u_i)^\alpha_B\,\delta^A_C$ and $D_{12}{}^B{}_A\,(\bar{u}_i)^C_{\dot\delta} = -\delta^C_A\,(\bar{u}_i)^B_{\dot\delta}$ for $i = 1$ or 2.

Next, consider the Casimir acting on the conjugation invariant function $f(Z) = f(x,y)$ of the cross-ratio matrix Z, which is a function of the eigenvalues x,y of Z only, as discussed above, where we drop the hat for simplicity. By examining the action of the Casimir on arbitrary products of traces of powers of Z, $\prod_i \mathrm{tr}(Z^i)^{a_i}$, and the corresponding expressions as polynomials of eigenvalues we find that

$$\begin{aligned}\mathcal{C}^{SU(1,1|2)}_{1,2} f(x,y) &= \left[\left(x^2\frac{\partial}{\partial x} - \frac{2xy}{x-y}\right)(1-x)\frac{\partial}{\partial x} - \left(y^2\frac{\partial}{\partial y} - \frac{2yx}{y-x}\right)(1-y)\frac{\partial}{\partial y}\right] f(x,y) \\ &= \frac{x-y}{xy}\left(x^2\partial_x(1-x)\partial_x - y^2\partial_y(1-y)\partial_y\right)\frac{xyf(x,y)}{x-y}\ .\end{aligned} \tag{H.0.10}$$

Finally, combining equations (H.0.9) and (H.0.10), we obtain the action of the quadratic super Casimir on the correlator

$$\begin{aligned}\mathcal{C}^{SU(1,1|2)}_{1,2} &= P_{p_i} \times \frac{x-y}{xy}\Delta^{(2)}\frac{xy}{x-y}P^{-1}_{p_i}\ , \\ \Delta^{(2)} &= \mathcal{D}^{(p_{12},p_{43})}_x - \mathcal{D}^{(-p_{12},-p_{43})}_y\ , \\ \mathcal{D}^{(p_{12},p_{43})}_x &= x^2\partial_x(1-x)\partial_x + (p_{12}+p_{43})x^2\partial_x - p_{12}p_{43}x\ .\end{aligned} \tag{H.0.11}$$

The second-order differential operator $\Delta^{(2)}$ is the 1d analogue of $\Delta^{(8)}$ in $\mathcal{N} = 4$ SYM and is essential for the higher-dimensional conformal symmetry.

Performing similar, simpler computations one can also obtain the Casimirs of the subgroups $SU(1|1)$ and $SU(2)$ acting on the correlator as

$$\mathcal{C}_{1,2}^{SU(1,1)} = P_{p_i} \times \mathcal{D}_x^{(p_{12},p_{43})} P_{p_i}^{-1} \,,$$
$$\mathcal{C}_{1,2}^{SU(2)} = P_{p_i} \times \mathcal{D}_y^{(-p_{12},-p_{43})} P_{p_i}^{-1} \,. \tag{H.0.12}$$

The action of the superconformal Casimir on the correlator can then be written directly in terms of those of the subgroups as:

$$\mathcal{C}_{1,2}^{SU(1,1|2)} = \frac{x-y}{xy}\left(\mathcal{C}_{1,2}^{SU(1|1)} - \mathcal{C}_{1,2}^{SU(2)}\right)\frac{xy}{x-y} \,. \tag{H.0.13}$$

Correlator of descendants

We will now compute the correlator of superconformal descendants

$$\partial_{\theta_1}\partial_{\theta_2}\partial_{\bar\theta_3}\partial_{\bar\theta_4}\langle\Psi_{p_1}\Psi_{p_2}\Psi_{p_3}\Psi_{p_4}\rangle|_{\theta_i=\bar\theta_i=0} \,, \tag{H.0.14}$$

introduced in subsection 5.2.1. As explained in (H.0.5) the supercorrelator is fixed by superconformal symmetry in terms of $f(\hat x,\hat y)$ which only depends on the eigenvalues of the cross-ratio matrix [49, 161]. To compute the derivatives acting on the correlator of the form (H.0.5) using the definitions (H.0.7) one can use Mathematica with the help of the grassmann package [162] to deal with the Grassmann odd variables. One then finds that the Grassmann odd derivatives acting on f are consistent with the following differential operator:

$$\partial_{\theta_1}\partial_{\theta_2}\partial_{\bar\theta_3}\partial_{\bar\theta_4} f(\hat x,\hat y)|_{\theta_i=\bar\theta_i=0} = \frac{1}{x_{12}x_{34}y_{12}y_{34}}\left(\mathcal{D}_{\hat x}^{(0,0)} - \mathcal{D}_{\hat y}^{(0,0)}\right)\frac{\hat x\hat y}{\hat x-\hat y} f(\hat x,\hat y) \,. \tag{H.0.15}$$

Pulling this through the prefactor then gives the action on the correlator itself:

$$\begin{aligned}
&\partial_{\theta_1}\partial_{\theta_2}\partial_{\bar\theta_3}\partial_{\bar\theta_4}\langle\Psi_{p_1}\Psi_{p_2}\Psi_{p_3}\Psi_{p_4}\rangle|_{\theta_i=\bar\theta_i=0}\\
&= \frac{P_{p_i}}{x_{12}x_{34}y_{12}y_{34}}\left(\mathcal{D}_x^{(p_{12},p_{43})} - \mathcal{D}_y^{(-p_{12},-p_{43})}\right)P_{p_i}^{-1}\frac{xy}{x-y}\langle\Psi_{p_1}\Psi_{p_2}\Psi_{p_3}\Psi_{p_4}\rangle|_{\theta_i=\bar\theta_i=0}\\
&= \mathcal{I}^{-1}\mathcal{C}_{1,2}^{SU(1,1|2)}\langle\Psi_{p_1}\Psi_{p_2}\Psi_{p_3}\Psi_{p_4}\rangle|_{\theta_i=\bar\theta_i=0} \,,
\end{aligned} \tag{H.0.16}$$

with $\mathcal{I}$ defined in (5.2.16). The final line above relates the descendant correlator to the action of the superconformal Casimir and comes directly from (5.2.15). We thus obtain

$$\begin{aligned}\langle\phi_{\Delta_1+\frac{1}{2}}\phi_{\Delta_2+\frac{1}{2}}\phi_{\Delta_3+\frac{1}{2}}\phi_{\Delta_4+\frac{1}{2}}\rangle &= \mathcal{I}^{-1}\mathcal{C}_{1,2}^{SU(1,1|2)}\langle\psi_{\Delta_1}\psi_{\Delta_2}\psi_{\Delta_3}\psi_{\Delta_4}\rangle \\ &= \frac{1}{x_{12}x_{34}y_{12}y_{34}}\left(\mathcal{C}_{1,2}^{SU(1|1)} - \mathcal{C}_{1,2}^{SU(2)}\right)\frac{xy}{x-y}\langle\psi_{\Delta_1}\psi_{\Delta_2}\psi_{\Delta_3}\psi_{\Delta_4}\rangle\ ,\end{aligned} \tag{H.0.17}$$

as written in (5.2.14).

Appendix I

Further Results for Unmixing of Four-Derivative Corrections

In this appendix we spell out some additional results relevant for section 5.8.

Unmixing at odd t for $(5,2)$

We start by considering the OPE coefficients for $(\Delta, p) = (5,2)$, where the operators in the double-trace spectrum of the OPE are $\mathcal{O}_1\partial^2\mathcal{O}_3$, $\mathcal{O}_2\mathcal{O}_4$, $\mathcal{O}_2\partial^2\mathcal{O}_2$, $\mathcal{O}_3\mathcal{O}_3$ and $\mathcal{O}_2\partial\mathcal{O}_3$. Performing a conformal block analysis for free theory and supergravity gives the coefficients:

$$\hat{A}^{(0)}_{5,2} = \begin{pmatrix} A^{(0)}_{1313} & 0 & 0 & 0 & 0 \\ & A^{(0)}_{2424} & 0 & 0 & 0 \\ & & A^{(0)}_{2222} & 0 & 0 \\ & & & A^{(0)}_{3333} & 0 \\ & & & & A^{(0)}_{2323} \end{pmatrix}_{(5,2)} = \begin{pmatrix} \frac{7}{120} & 0 & 0 & 0 & 0 \\ & \frac{5}{168} & 0 & 0 & 0 \\ & & \frac{5}{27} & 0 & 0 \\ & & & \frac{1}{7} & 0 \\ & & & & \frac{2}{15} \end{pmatrix},$$

$$\hat{M}^{\text{sugra}}_{5,2} = \begin{pmatrix} M^{\text{sugra}}_{1313} & M^{\text{sugra}}_{1324} & M^{\text{sugra}}_{1322} & M^{\text{sugra}}_{1333} & M^{\text{sugra}}_{1323} \\ & M^{\text{sugra}}_{2424} & M^{\text{sugra}}_{2422} & M^{\text{sugra}}_{2433} & M^{\text{sugra}}_{2423} \\ & & M^{\text{sugra}}_{2222} & M^{\text{sugra}}_{2233} & M^{\text{sugra}}_{2223} \\ & & & M^{\text{sugra}}_{3333} & M^{\text{sugra}}_{3323} \\ & & & & M^{\text{sugra}}_{2323} \end{pmatrix}_{(5,2)} = \begin{pmatrix} \frac{7}{30} & \frac{3}{10} & \frac{14}{45} & \frac{2}{5} & 0 \\ & \frac{27}{70} & \frac{2}{5} & \frac{18}{35} & 0 \\ & & \frac{56}{135} & \frac{8}{15} & 0 \\ & & & \frac{24}{35} & 0 \\ & & & & 0 \end{pmatrix} . \tag{I.0.1}$$

We spell out formulas for the coefficients at the order of supergravity and four-derivative corrections for general odd Δ (see (5.6.12) for general $A^{(0)}_{q_1q_2q_1q_2}$):

$$\begin{aligned}
M^{\text{sugra}}_{1313}(\Delta,2) &= \frac{(\Delta+1)(\Delta+2)\Delta!(\Delta+2)!}{30(2\Delta)!} , \\
M^{\text{sugra}}_{1324}(\Delta,2) &= \frac{(\Delta-3)(\Delta+1)(\Delta+2)(\Delta+4)\Delta!(\Delta+2)!}{420(2\Delta)!} , \\
M^{\text{sugra}}_{1322}(\Delta,2) &= \frac{(\Delta-1)(\Delta+2)\Delta!(\Delta+2)!}{15(2\Delta)!} , \\
M^{\text{sugra}}_{1333}(\Delta,2) &= \frac{(\Delta-3)(\Delta-1)(\Delta+2)(\Delta+4)\Delta!(\Delta+2)!}{210(2\Delta)!} , \\
M^{\text{sugra}}_{2424}(\Delta,2) &= \frac{(\Delta-3)^2(\Delta+1)(\Delta+2)(\Delta+4)^2\Delta!(\Delta+2)!}{5880(2\Delta)!} , \\
M^{\text{sugra}}_{2422}(\Delta,2) &= \frac{(\Delta-3)(\Delta-1)(\Delta+2)(\Delta+4)\Delta!(\Delta+2)!}{210(2\Delta)!} , \\
M^{\text{sugra}}_{2433}(\Delta,2) &= \frac{(\Delta-3)^2(\Delta-1)(\Delta+2)(\Delta+4)^2\Delta!(\Delta+2)!}{2940(2\Delta)!} , \\
M^{\text{sugra}}_{2222}(\Delta,2) &= \frac{2(\Delta-1)^2(\Delta+2)^2(\Delta!)^2}{15(2\Delta)!} , \\
M^{\text{sugra}}_{2233}(\Delta,2) &= \frac{(\Delta-3)(\Delta-1)^2(\Delta+2)(\Delta+4)\Delta!(\Delta+2)!}{105(\Delta+1)(2\Delta)!} , \\
M^{\text{sugra}}_{3333}(\Delta,2) &= \frac{(\Delta+2)\left(\Delta^3-13\Delta+12\right)^2\Delta!(\Delta+2)!}{1470(\Delta+1)(2\Delta)!} ,
\end{aligned} \tag{I.0.2}$$

$$\begin{aligned}
M^{\text{4-deriv},\,t\,\text{odd}}_{1313}(\Delta,2) &= \frac{1}{60(2\Delta)!} \times \Big[24\,C_0\,((\Delta+2)!)^2 \\
&\quad (\Delta+3)\left(\Delta(\Delta+1)\left(\Delta^2+\Delta-5\right)+24\right)(\Delta-2)^3(\Delta-3)!(\Delta+3)!\Big] , \\
M^{\text{4-deriv},\,t\,\text{odd}}_{1324}(\Delta,2) &= \frac{(\Delta-3)(\Delta+4)!}{840(2\Delta)!} \times \Big[\frac{40\,C_0\,(\Delta+2)!}{\Delta+3} \\
&\quad + (\Delta-2)\left(\Delta(\Delta+1)\left(\Delta(\Delta+1)\left(\Delta^2+\Delta-15\right)+118\right)-400\right)(\Delta-2)!\Big] ,
\end{aligned}$$

$$\begin{aligned}
M_{1322}^{\text{4-deriv},\,t\,\text{odd}}(\Delta,2) =& \frac{\Delta!(\Delta+2)!}{30(2\Delta)!} \times \Big[20\,C_0\,\left(\Delta^2+\Delta-2\right) \\
&+ (\Delta-2)(\Delta+3)\left(\Delta(\Delta+1)\left(\Delta^2+\Delta-10\right)+28\right)\Big]\,, \\
M_{1333}^{\text{4-deriv},\,t\,\text{odd}}(\Delta,2) =& \frac{(\Delta-3)\Delta!(\Delta+4)!}{620(\Delta+3)(2\Delta)!} \times \Big[36\,C_0\,\left(\Delta^2+\Delta-2\right) \\
&+ (\Delta-2)(\Delta+3)\left(\Delta(\Delta+1)\left(\Delta^2+\Delta-14\right)+60\right)\Big]\,, \\
M_{2424}^{\text{4-deriv},\,t\,\text{odd}}(\Delta,2) =& \frac{(\Delta-3)^2}{35280(2\Delta)!} \times \Big[\frac{168\,C_0\,((\Delta+4)!)^2}{(\Delta+3)^2} + (\Delta+4)^2(\Delta-2)!(\Delta+2)! \\
&\times \left(\Delta(\Delta+1)\left(\Delta(\Delta+1)\left(3\Delta(\Delta+1)\left(\Delta^2+\Delta-25\right)\right.\right.\right. \\
&\left.\left.\left.+1192\right)-8220\right)+20000\right)\Big]\,, \\
M_{2422}^{\text{4-deriv},\,t\,\text{odd}}(\Delta,2) =& \frac{(\Delta-3)\Delta!(\Delta+4)!}{420(\Delta+3)(2\Delta)!} \times \Big[36\,C_0\,\left(\Delta^2+\Delta-2\right) \\
&+ (\Delta-2)(\Delta+3)\left(\Delta(\Delta+1)\left(\Delta^2+\Delta-14\right)+60\right)\Big]\,, \\
M_{2433}^{\text{4-deriv},\,t\,\text{odd}}(\Delta,2) =& \frac{(\Delta-3)^2(\Delta+4)\Delta!(\Delta+4)!}{5880(\Delta+3)(2\Delta)!} \times \Big[52\,C_0\,\left(\Delta^2+\Delta-2\right) \\
&+ \Delta(\Delta+1)\left(\Delta(\Delta+1)\left(\Delta^2+\Delta-24\right)+280\right)-904\Big]\,, \\
M_{2222}^{\text{4-deriv},\,t\,\text{odd}}(\Delta,2) =& \frac{\left(\Delta^2+\Delta-2\right)(\Delta!)^2}{15(2\Delta)!} \times \Big[16\,C_0\,\left(\Delta^2+\Delta-2\right) \\
&+ \Delta(\Delta+1)\left(\Delta(\Delta+1)\left(\Delta^2+\Delta-15\right)+74\right)-80\Big]\,, \\
M_{2233}^{\text{4-deriv},\,t\,\text{odd}}(\Delta,2) =& \frac{(\Delta-3)(\Delta-1)(\Delta+4)\Delta!(\Delta+2)!}{210(\Delta+1)(2\Delta)!} \times \Big[32\,C_0\,\left(\Delta^2+\Delta-2\right) \\
&+ \Delta(\Delta+1)\left(\Delta(\Delta+1)\left(\Delta^2+\Delta-19\right)+130\right)-144\Big]\,, \\
M_{3333}^{\text{4-deriv},\,t\,\text{odd}}(\Delta,2) =& \frac{(\Delta-3)^2(\Delta-1)(\Delta+4)^2\Delta!(\Delta+2)!}{2940(\Delta+1)(2\Delta)!} \times \Big[48\,C_0\,\left(\Delta^2+\Delta-2\right) \\
&+ \Delta(\Delta+1)\left(\Delta(\Delta+1)\left(\Delta^2+\Delta-23\right)+330\right)-432\Big]\,, \\
M_{2323}^{\text{4-deriv},\,t\,\text{odd}}(\Delta,2) =& \frac{(\Delta-1)^3(\Delta+2)\left(\Delta^2+\Delta-12\right)^2(\Delta-2)!(\Delta+2)!}{45(2\Delta)!}\;.
\end{aligned} \tag{I.0.3}$$

Now we can solve the mixing problem in the supergravity limit. This yields the anomalous dimension

$$\gamma_{5,2}^{\text{sugra}} = 24\;, \tag{I.0.4}$$

which is $\delta_{5,2}^{(2)}$ as expected. The mixing problem at $\mathcal{O}(a/c)$ is solved in section 5.8.

Unmixing at even t for $(6,2)$

We present an additional example for even t here. At weight $(6,2)$ there are four operators in the spectrum, $\mathcal{O}_1\partial^3\mathcal{O}_3$, $\mathcal{O}_2\partial\mathcal{O}_4$, $\mathcal{O}_2\partial^2\mathcal{O}_3$ and $\mathcal{O}_3\mathcal{O}_4$, two with even and odd $r_{B^{\mathrm{te}}}$ respectively. Performing a conformal block analysis of the relevant correlators gives the following free theory and four-derivative coefficients:

$$\hat{A}^{(0)}_{6,2} = \begin{pmatrix} A^{(0)}_{1313} & 0 & 0 & 0 \\ & A^{(0)}_{2424} & 0 & 0 \\ & & A^{(0)}_{2323} & 0 \\ & & & A^{(0)}_{3434} \end{pmatrix}_{(6,2)} = \begin{pmatrix} \frac{14}{495} & 0 & 0 & 0 \\ & \frac{10}{189} & 0 & 0 \\ & & \frac{35}{297} & 0 \\ & & & \frac{5}{126} \end{pmatrix},$$
$$\hat{M}^{\text{4-deriv}}_{6,2} = \begin{pmatrix} M^{\text{4-deriv}}_{1313} & M^{\text{4-deriv}}_{1324} & M^{\text{4-deriv}}_{1323} & M^{\text{4-deriv}}_{1334} \\ & M^{\text{4-deriv}}_{2424} & M^{\text{4-deriv}}_{2423} & M^{\text{4-deriv}}_{2434} \\ & & M^{\text{4-deriv}}_{2323} & M^{\text{4-deriv}}_{2334} \\ & & & M^{\text{4-deriv}}_{3434} \end{pmatrix}_{(6,2)} = \begin{pmatrix} \frac{1224}{275} & \frac{408}{35} & 0 & 0 \\ \frac{408}{35} & \frac{1496}{49} & 0 & 0 \\ 0 & 0 & \frac{304}{55} & \frac{456}{35} \\ 0 & 0 & \frac{456}{35} & \frac{7524}{245} \end{pmatrix}. \tag{I.0.5}$$

As expected, $\hat{M}^{\text{4-deriv}}_{6,2}$ is a block-diagonal matrix, because operators with even or odd $r_{B^{\mathrm{te}}}$ do not mix. Note that each block is symmetric, as it should be from the structure of the correlators. The free coefficients for general Δ, p, q_1, q_2 are given in (5.6.12), some $\mathcal{O}(a/c)$ coefficients for general even Δ were given in (5.8.31) and the additional ones are

$$\begin{aligned}
M^{\text{4-deriv}, t\,\text{even}}_{1324}(\Delta,2) &= \frac{(\Delta-4)(\Delta-2)^3(\Delta+3)(\Delta+5)(\Delta-3)!(\Delta+3)!\,(\Delta^2+\Delta-8)}{168(2\Delta)!}\,, \\
M^{\text{4-deriv}, t\,\text{even}}_{2424}(\Delta,2) &= \frac{5(\Delta-2)^3(\Delta+3)(\Delta-3)!(\Delta+3)!\,(\Delta^2+\Delta-20)^2\,(\Delta^2+\Delta-8)}{7056(2\Delta)!}\,, \\
M^{\text{4-deriv}, t\,\text{even}}_{2334}(\Delta,2) &= \frac{(\Delta-4)(\Delta+5)\Delta!(\Delta+1)!\,(\Delta^2+\Delta-6)^2\,(\Delta^2+\Delta-4)}{105\Delta(2\Delta)!}\,, \\
M^{\text{4-deriv}, t\,\text{even}}_{3434}(\Delta,2) &= \frac{(\Delta-4)^2(\Delta-2)^2(\Delta+3)^2(\Delta+5)^2\Delta!(\Delta+1)!\,(\Delta^2+\Delta-4)}{980\Delta(2\Delta)!}\,.
\end{aligned} \tag{I.0.6}$$

Solving the unmixing equations, as expected, one finds two non-zero anomalous dimensions:

$$\gamma^{\text{4-deriv}}_{4,2,i} = \left\{\frac{3672}{5}, \frac{4104}{5}\right\} \,, \tag{I.0.7}$$

which are labelled by $\left(\gamma^{\text{4-deriv}}_{B^{\text{te}}}\right)^{6,2}_{1,0} = \frac{3672}{5}$ and $\left(\gamma^{\text{4-deriv}}_{B^{\text{te}}}\right)^{6,2}_{1,1} = \frac{4104}{5}$. Conjectured formulas for general (Δ, p) are discussed in subsection 5.8.3.

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