UNIVERSITI PUTRA MALAYSIA

THE MELLIN TRANSFORM OF GENERALIZED FUNCTIONS AND SOME APPLICATIONS

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By

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Abstract of the thesis presented to the Senate of Universiti Putra Malaysia in fulfillment of the requirements for the degree of Master of Science

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February 2002

Chairman : **Adem Kilicman, Ph.D.**

Faculty : **Faculty of Science and Environmental Studies**

From the space of testing functions $E_{p,q}$ and the semi-norms induced on this space and from the fact that the testing function

$$\phi(x) = \begin{cases} x^{s-1} & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$$

is a member of the space $E_{p,q}$ for $p < \mathrm{Re}(s) < q$, made it possible for us to choose from the space $E'_{p,q}$ generalized functions that are Mellin transformable. Extending the convolution in the Mellin sense to the generalized functions gives rise to the definition of the convolution in the Mellin sense of generalized functions in $E'_{p,q}$ by

$$\langle f(w), \langle g(u), \phi(uw) \rangle \rangle$$

Thus, this made it essential for us to discuss the characteristics to be imposed on $\phi(uw)$ so that $\Theta(w) = \langle g(u), \phi(uw) \rangle$ would also be a member of $E_{p,q}$ hence giving meaning to the definition of the convolution in the Mellin sense

$E_{p,q}$ hence giving meaning to the definition of the convolution in the Mellin sense of generalized functions in $E'_{p,q}$. Accomplishing the above gave us the result that the exchange formula also holds for the Mellin transform of the convolution in the Mellin sense for generalized functions in $E'_{p,q}$.

From the research due to Butzer and Jansche it had discussed the Mellin transform in an independent method and most importantly the Mellin translation operator $\tau_h^c f(x) = h^c f(hx)$ which is of different characteristic from the classical translation operator $T_h^c f(x) = h^c f(x+h)$, or in the particular case for $c = 0$, $T_h f(x) = f(x+h)$, it gave rise to another scope of research regarding the differential equations in the Mellin sense (i.e. differentiating functions using the Mellin translation operator) with polynomial coefficients. By considering the differential equations in the Mellin sense with polynomial coefficients, and solving it using the Mellin transform, we are able to put into order and solve various differential equations which could be shown to belong to the space X_c^k, thus generalizing the work by Sasiela.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia sebagai memenuhi keperluan untuk Ijazah Master Sains.

JELMAAN MELLIN FUNGSI TERITLAK DAN BEBERAPA APLIKASI

Oleh

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Berasaskan ruang fungsi ujian $E_{p,q}$ serta semi-norma yang ditetapkan ke atas ruang tersebut dan bersandarkan fakta bahawa fungsi ujian

$$\phi(x) = \begin{cases} x^{s-1} & ; x > 0 \\ 0 & ; x \leq 0 \end{cases}$$

adalah suatu unsur dalam ruang $E_{p,q}$ untuk $p < \mathrm{Re}(s) < q$, memungkinkan kami untuk memilih dari ruang $E'_{p,q}$ fungsi teritlak yang boleh jelma secara Mellin. Memperluaskan konvolusi secara Mellin kepada fungsi teritlak memberi takrif konvolusi secara Mellin bagi fungsi teritlak dalam $E'_{p,q}$ yang diberi oleh

$$\langle f(w), \langle g(u), \phi(uw) \rangle \rangle.$$

Oleh yang demikian, adalah perlu bagi kita untuk membincangkan ciri-ciri yang dikenakan ke atas $\phi(uw)$ supaya $\Theta(w) = \langle g(u), \phi(uw) \rangle$ juga menjadi ahli bagi $E_{p,q}$, dan justeru itu memberi makna kepada takrif konvolusi secara Mellin bagi fungsi teritlak

dalam $E'_{p,q}$. Bersandarkan di atas, ia memberi kita keputusan di mana rumus penukaran adalah benar untuk jelmaan Mellin bagi konvolusi secara Mellin untuk fungsi teritlak dalam $E'_{p,q}$.

Berasaskan kajian oleh Butzer dan Jansche yang telah membincangkan jelmaan Mellin dalam suatu kaedah bebas dan paling penting pengoperasi translasi Mellin $\tau_h^c f(x) = h^c f(hx)$ yang berbeza ciri dari pengoperasi translasi klasik $T_h^c f(x) = h^c f(x+h)$, atau untuk kes khusus $c = 0$, $T_h f(x) = f(x+h)$, ianya telah membuka satu lagi ruang kajian, mengenai persamaan pembezaan secara Mellin (i.e. membezakan fungsi menggunakan pengoperasi translasi Mellin) dengan pekali polinomial. Berasaskan persamaan pembezaan secara Mellin dengan pekali polinomial, dan menyelesaikannya menggunakan jelmaan Mellin, kami boleh menyusun serta menyelesaikan beberapa persamaan pembezaan dalam ruang X_c^k, yang akhirnya telah mengitlakkan hasil yang diperolehi oleh Sasiela.

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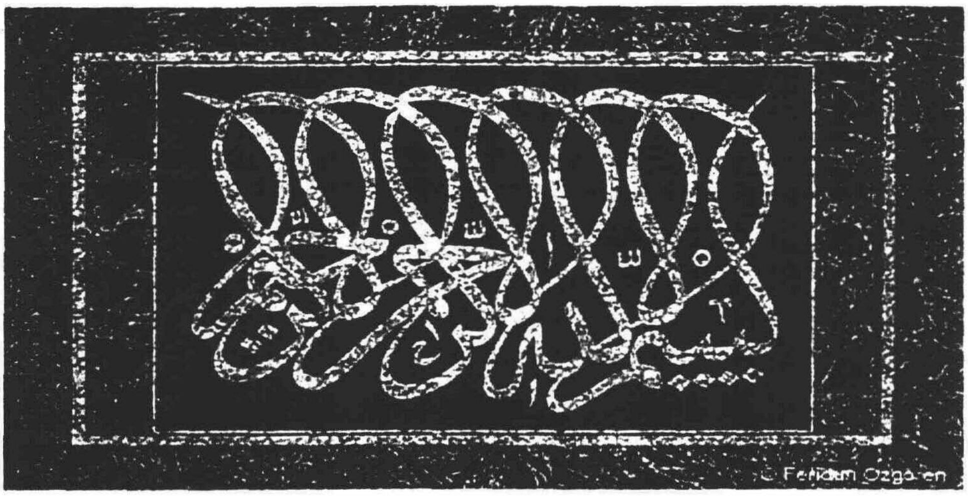

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I certify that an Examination Committee met on 6th February 2002 to conduct the final examination of Muhammad Rezal bin Kamel Ariffin on his Master of Science thesis entitled "Mellin Transform of Generalized Functions and Some Applications" in accordance Universiti Pertanian Malaysia (Higher Degree) Act 1980 and Universiti Pertanian Malaysia (Higher Degree) Regulations 1981. The committee recommends that candidate be awarded the relevant degree. Members of the Examination Committee are as follows:

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I hereby declare that the thesis is based on my original work except for quotations and citations which have been duly acknowledged. I also declare that it has not been previously or concurrently submitted for any other degree at UPM or other institutions.

MUHAMMAD REZAL B. KAMEL ARIFFIN

Date: 13 MARCH 2002

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CHAPTER 1

INTRODUCTION

1.1 Notations and Definitions

In this work we will use the usual notation, N will denote the set of natural numbers, Z the set of integers, Q the set of rational numbers, R the set of real numbers and C the set of complex numbers. The null set will be represented by $\varnothing$.

The interior point of a set E is a point $p \in E$ if there exists $\varepsilon > 0$ such that $N_\varepsilon(p) \subset E$, where $N_\varepsilon(p) = \{x \in R : |x - p| < \varepsilon\}$ and is called the epsilon-neighborhood of the point p. A point $p \in R$ is a limit point of E if every epsilon-neighborhood $N_\varepsilon(p)$ of p contains a point $q \in E$ with $q \neq p$. The subset O of R will be open if every point of O is an interior point of O. The subset M of R is closed if and only if it contains all its limit points.

Supremum of a set E denoted by sup E, defines an element which is the least upper bound of E. Infimum of a set E denoted by inf E, defines an element which is the greatest lower bound of E.

Definition 1.1.1

The space X is called a linear (vector) space if for any $x, y, z \in X$ and $\alpha, \beta \in C$ it is non-empty, closed under 'addition', closed under 'multiplication' by scalars and the following axioms hold.

i. $x + y = y + x$

ii. $x + (y + z) = (x + y) + z$

iii. $x + 0 = x$

iv. $x + (-x) = 0$

v. $\alpha(\beta x) = (\alpha\beta)x$

vi. $(\alpha + \beta)x = \alpha x + \beta x$

vii. $\alpha(x + y) = \alpha x + \alpha y$

vii. $1 \cdot x = x$

Definition 1.1.2

A linear transformation f which maps a vector space X into a vector space Y, i.e. $f : X \to Y$, is called an isomorphism if the linear transformation is one to one and onto.

Definition 1.1.3

Let X be a vector space on R. A function $\| \, \| : X \to R$ satisfying the following conditions

i. $\|x\| \geq 0$ for all $x \in X$

ii. $\|x\| = 0$ if and only if $x = 0$

iii. $\|\alpha x\| = |\alpha| \|x\|$ for all $\alpha \in C$ and $x \in X$

iv. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$

is called a norm on X, The pair $(X, \| \ \|)$ is called a normed space.

Example 1.1.1

a. For any $x \in R$, the norm induced on R is given by $\|x\| = |x|$.

b. Let $X = C[a,b]$ be the space of continuous functions on the closed interval $[a,b]$. For $f \in X$, the norm induced on this space is given by $\|f\| = \max_{x \in [a,b]} |f(x)|$

Definition 1.1.4

Let X be a vector space of R. A function $\rho : X \to R$ satisfying i., iii. and iv. in definition 1.1.2 is called a semi-norm on X. The pair (X, ρ) is called a semi-normed space.

Example 1.1.2

Let D be the space of infinitely differentiable functions (Gelfand, Shilov, 1964). The semi-norms induced on this space is given by $\gamma_k(\phi) = \sup_{x \in [a,b]} |\phi^{(k)}(x)|$ for $k = 0,1,2,\ldots$.

If $k = 0$, $\gamma_0(\phi)$ is a norm.

Definition 1.1.5

a. A sequence $\{x_n\}$ in a normed linear space $(X,\|\ \|)$ is a Cauchy sequence if for every $\varepsilon > 0$, there exists a positive integer n_0 such that $\|x_n - x_m\| < \varepsilon$ for all integers $n, m \geq n_0$.

b. A normed linear space $(X,\|\ \|)$ is complete if every Cauchy sequence in X converges in norm to an element in X.

Definition 1.1.6

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions defined on a set E. The sequence $\{f_n\}$ converges pointwise on E to f if for any $\varepsilon > 0$ and for every $x \in E$ there exists a positive integer $n_0 = n_0(x,\varepsilon)$ such that $|f(x) - f_n(x)| < \varepsilon$ for all $n \geq n_0$.

Definition 1.1.7

Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of real-valued functions defined on a set E. The sequence $\{f_n\}$ converges uniformly on E to f if for any $\varepsilon > 0$ there exists a positive integer n_0 such that $|f(x) - f_n(x)| < \varepsilon$ for all $n \geq n_0$ for every $x \in E$, where $n_0 = n_0(\varepsilon)$ (i.e. $n_0(\varepsilon)$ depends only on ε).

Example 1.1.3

a. Let $E=[0,1]$ and for each $x \in E$, $n \in N$ let $f_n(x)=x^n$. Each f_n is continuous on E. Since $f_n(1)=1$ for all n, $\lim_{n\to\infty} f_n(1)=1$. If $0 \le x < 1$ then $\lim_{n\to\infty} f_n(x)=0$. Thus we may say that the sequence of functions $\{f_n\}_{n=1}^{\infty}$ converges pointwise to the limit function $f(x)=\begin{cases}1 & ;x=1\\ 0 & ;0\le x<1\end{cases}$. That is, for any $\varepsilon>0$ there exists a positive integer n_0 depending on both ε and x such that $|f(x)-x^n|<\varepsilon$ for all $n \ge n_0(x,\varepsilon)$. If $x=1$, $|f_n(x)-f(x)|=|1-1|=0<\varepsilon$. Now let $x\in[0,1)$. In order for $\{f_n\}$ to converge to f, $|x^{n_0}|<\varepsilon$ must be true for all $x\in[0,1)$ for a positive integer n_0. Let $\varepsilon<1$ and $n\ge n_0(x,\varepsilon)=\dfrac{\log\varepsilon}{\log x}$.

Then

$$n\log x < \log\varepsilon$$

$$x^n < \varepsilon$$

b. We shall now show that the sequence in a. converges uniformly to 0 on the interval $[0,a]$ for $0<a<1$. For any $\varepsilon>0$ there exists a positive integer $n_0=\dfrac{\log\varepsilon}{\log a}$ such that for every $n\ge n_0$ we have

$$n > \frac{\log\varepsilon}{\log a}$$

Thus, $|x^n| < a^n < \varepsilon$ for every $x \in [0,a]$. This shows that the sequence of functions $f_n(x) = x^n$ converges to 0 uniformly.

Definition 1.1.8

Let X be a normed linear space. If X is a complete normed linear space then X is said to be a Banach space.

Example 1.1.4

Let us consider the space $X = C[a,b]$ of functions that are continuous on the closed interval $[a,b]$. For any $f \in X$, let the norm be defined as $\|f\| = \max_{x \in [a,b]} |f(x)|$. According to definition 1.1.5, a sequence of functions $\{f_n\}$ on X is a Cauchy sequence, if for any $\varepsilon > 0$ there exists an integer N such that for any $n,m > N$ $\max_{x \in [a,b]} |f_n(x) - f_m(x)| < \varepsilon$ for all $x \in [a,b]$.

Any Cauchy sequence in $X = C[a,b]$, with the above norm, converges uniformly to some function in $X = C[a,b]$ (Stoll, 1997). Since the convergence is uniform, then the limit function f must also be continuous. Thus, $X = C[a,b]$ is a Banach space with the above mentioned norm.

Definition 1.1.9

Let X be a Banach space, which admits scalar multiplication, vector addition and vector multiplication. If for any $x, y \in X$, $\|xy\| \leq \|x\| \cdot \|y\|$

then X is called Banach Algebra.

Let $[a,b]$, $a < b$, be a closed and bounded interval on R. By a partition P of $[a,b]$ we mean a finite set of points $P = \{x_0, x_1, \ldots, x_n\}$ such that $a = x_0 < x_1 < \ldots < x_n = b$. There is no requirement that the points x_i be equally spaced. For each $i = 1,2,\ldots,n$, set $\Delta x_i = x_i - x_{i-1}$.

Now let,

$$m_i = \inf\{f(t) : x_{i-1} \leq t \leq x_i\}$$

$$M_i = \sup\{f(t) : x_{i-1} \leq t \leq x_i\}$$

and let us define the upper sum $U(P,f)$ for the partition P and function f by $U(P,f) = \sum_{i=1}^{n} M_i \Delta x_i$ and the lower sum $L(P,f)$ by $L(P,f) = \sum_{i=1}^{n} m_i \Delta x_i$.

Definition 1.1.10

Let f be a bounded real-valued function on the closed interval $[a,b]$. The upper and lower integral of f is defined by

$$\overline{\int_a^b} f = \inf\{U(P,f)\}$$

$$\int_{\underline{a}}^{b} f = \sup\{L(P,f)\}$$

Definition 1.1.11

A function f is said to be Riemann integrable (i.e. integrable) on $[a,b]$ if the limit $\lim_{\max \Delta x \to 0} \sum_{k=1}^{n} f(x_k^*)\Delta x = \int_a^b f(x)dx$ exists, and does not depend on choice of the partitions $[x_{i-1}, x_i] \subset [a,b]$ for $i = 1,2,3,\ldots$ or on the points x_k^* in the partitions.

Theorem 1.1.1 (Stoll, 1997)

A bounded real-valued function is Riemann integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition P of $[a,b]$ such that

$$U(P,f) - L(P,f) < \varepsilon$$

Definition 1.1.12

A subset $E \subset R$ has measure zero, if for any $\varepsilon > 0$ there exists a countable collection of $\{I_n\}_{n\in A}$ of open intervals such that $E \subset \bigcup_{n\in A} I_n$ and $\sum_{n\in A} l(I_n) < \varepsilon$, where $l(I_n)$ denotes the length of the interval I_n.

Theorem 1.1.2 (Lebesgue) (Stoll, 1997)

A bounded real-valued function f on $[a,b]$ is Riemann integrable if and only if the set of discontinuities has measure zero.

Thus, when a function is said to be Riemann integrable, the integral is given by $\int_a^b f(x)dx$ which is evaluated in the usual way.

Definition 1.1.13

If J is an interval we define the measure of J, denoted by $m(J)$ to be the length of J. If J is (a,b), $[a,b)$, $(a,b]$ or $[a,b]$ then $m(J)=b-a$. If J is (a,∞), $[a,\infty)$, $(-\infty,b)$ or $(-\infty,b]$ then $m(J)=\infty$.

Definition 1.1.14

Let E be a subset of R. A collection $\{O_\alpha\}_{\alpha\in A}$ of open subsets in R is an open cover if $E\subset\bigcup_{\alpha\in A}O_\alpha$.

Definition 1.1.15

A subset K of R is compact if $\{O_\alpha\}_{\alpha\in A}$ is an open cover of K, and there exists $\alpha_1,\alpha_2,\alpha_3,\ldots,\alpha_n\in A$ such that $K\subset\bigcup_{j=1}^{n}O_{\alpha_j}$.

Definition 1.1.16

Let E be a subset of R. The Lebesgue outer measure of E is defined by $\lambda^*=\inf\{m(U):E\subset U\}$ where U is open. The Lebesgue inner measure of E is defined by $\lambda_*=\sup\{m(K):K\subset E\}$ where K is compact.

Definition 1.1.17

a. A bounded subset E of R is said to be measurable (Lebesgue measurable) if $\lambda^*(E) = \lambda_*(E)$. In this case the measure of E is defined by $\lambda(E) = \lambda^*(E) = \lambda_*(E)$.

b. An unbounded set subset E of R is said to be measurable if $E \cap [a,b]$ is measurable for every closed and bounded interval $[a,b]$. By choosing a and b sufficiently large such that $E \subset [a,b]$ we can define the measure of E as $\lambda(E) = \lim_{k \to \infty} \lambda(E \cap [-k,k])$.

The two separate definitions are required due to the existence of unbounded non measurable sets E for which $\lambda^*(E) = \lambda_*(E) = \infty$.

Example 1.1.5

a. Let $E=[a,b]$ be a bounded subset of R. Since $\lambda^*(E) = \lambda_*(E) = b - a$, then $\lambda(E) = b - a$.

b. Let $E = (c,d)$ be an unbounded subset of R. Since $\lambda^*(E) = \lim_{k \to \infty} \lambda^*(E \cap [-k,k]) = \lambda_*(E) = \lim_{k \to \infty} \lambda_*(E \cap [-k,k]) = d - c$, then $\lambda(E) = d - c$.

Definition 1.1.18

Let f be a real-valued function on $[a,b]$. The function f is said to be measurable if for every $s \in R$, the set $\{x \in [a,b] : f(x) > s\}$

generally, if E is a measurable set of R, a function $f : E \rightarrow R$ is measurable if the set $\{x \in E : f(x) > s$

Example 1.1.6

Let $f : [0,1] \rightarrow R$ be defined by $f(x) = \begin{cases} 0 & ; x \in Q \cap [0,1] \\ x & ; x \in [0,1] \setminus Q \end{cases}$ then we can

have the following set,

$$\{x \in [0,1] : f(x) > s\} = \begin{cases} [0,1] & ; s \\ Q^c \cap (s,1) & ; s \\ \varnothing & ; s \end{cases}$$

Since the sets $[0,1]$, $Q^c \cap (s$

measurable.

Definition 1.1.19

Let $\{I_n\}$ be a collection of open intervals. The family $\{I_n\}$ is pairwise disjoint if $I_n \cap I_m = \varnothing$ whenever $n \neq m$.