UNIVERSITI PUTRA MALAYSIA

STABILITY OF SOME MODELS IN MATHEMATICAL BIOLOGY

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STABILITY OF SOME MODELS IN MATHEMATICAL BIOLOGY

By

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IN THE NAME OF ALLAH MOST
BENEFICENT MOST MERCIFUL

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Praise is for *Allah*, Lord of the Worlds. And may blessings and peace be upon the Master of the Messengers, the seal of the Prophets, the Leader of the Pure, *Muhammad* , who was sent just mercy to the Worlds, and upon his goodly Family, Pure Companions, Pious *Umma* and Pure wives, the Mothers of the Believers.

My deepest gratitude and love to my parents, my wife and all family members for every thing, specially for their prays for my success, for which the doors of heaven are always opened.

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I wish also to express my thanks to all my friends in Libya and Malaysia, I cannot specify one or some, because every one of them has special elevated place in my heart, and so that I may not forget any one, I commit them to whom never ever forget,

" That Allah may give them the best reward of what they have done, and give them more out of his grace; and Allah gives sustenance to whom he pleased without measure".

The financial support was provided by Omar Elmuktar University, El-Beida, Libya which is highly appreciated and gratefully acknowledged.

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STABILITY OF SOME MODELS IN MATHEMATICAL BIOLOGY

By

ABDALSALAM B. H. ALDIAKH

December 1998

Chairman: Associate Professor Harun Bin Budin, Ph.D.

Faculty: Science and Environmental Studies

Lately there has been an increasing awareness of the adverse side effect from the use of pesticides on the environment and on human health. As an alternative solution attention has been directed to the so-called "Biological Control" where pests are removed from the environment by the use of another living but harmless organism.

A detailed study of biological control requires a clear understanding on the types of interaction between the species involved. We have to know exactly the conditions under which the various species achieve stability and live in coexistence. It is here that mathematics can contribute in understanding and solving the problem.

A number of models for single species are presented as an introduction to the study of two species interaction. Specifically the following interactions are studied:

-Competition

-Predation

-Symbiosis.

All the above interactions are modelled based on ordinary differential equations. But such models ignore many complicating factors. The presence of delays is one such factor. In the usual models it is tacitly assumed that the coefficients of change for a given species depend only on the instantaneous conditions.

However biological processes are not temporally isolated, and the past influences the present and the future. In the real world the growth rate of a species does not respond immediately to changes in the population of interacting species, but rather will do so after a time lag. This concept should be taken into account, and this leads to the study of delay differential equations. However the mathematics required for the detailed analysis of the behaviour of such a model can be formidable, especially for biologists who share the subject. By the aid of computer and using *Mathematica software (version 3.0)*, the main properties of the solutions of many models related to the various interactions can be clarified.

Abstrak tesis yang dikemukakan kepada Senat Universiti Putra Malaysia
sebagai memenuhi keperluan untuk ijazah Master Sains.

KESTABILAN BEBERAPA MODEL DALAM BIOLOGI MATEMATIK

Oleh

ABDASALAM B. H. ALDIAKH

Disember 1998

Pengerusi: **Professor Madya Harun bin Budin, Ph.D.**

Fakulti: **Sains dan Pengajian Alam Sekitar**

Kebelakangan ini telah timbul kesedaran tentang kesan sampingan yang tidak diingini hasil dari penggunaan racun perosak kepada alam sekitar, dan seterusnya kepada kesihatan manusia. Sebagai penyelesaian alternatif kepada penggunaannya perhatian telah tertumpu kepada apa yang disebut sebagai "Kawalan Biologi" di mana sesuatu makhluk perosak dihapuskan daripada persekitaran dengan penggunaan hidupan lain yang tidak berbahaya.

Kajian mendalam tentang kawalan biologi memerlukan kefahaman yang jelas tentang bentuk interaksi di antara spesis-spesis yang terlibat dalam kawalan biologi tersebut. Kita perlu mengetahui syarat yang diperlukan bagi spesis-spesis berlainan ini mencapai kestabilan dan dapat saling hidup bersama. Di sini matematik dapat menyumbang bagi memahami dan seterusnya menyelesaikan masalah yang timbul.

Beberapa model bagi satu spesis dipersembahkan, sebagai pengenalan kepada kajian mengenai interaksi dua spesis. Secara khususnya interaksi berikut telah dikaji:

-Persaingan

-Pemangsaan

-Simbiosis.

Kesemua interaksi di atas dimodel berasaskan kepada persamaan pembezaan biasa. Tetapi model sedemikian mengabaikan banyak faktor-faktor yang merumitkan. Kehadiran lengahan adalah satu contoh. Dalam model biasa pekali ubahan secara senyapnya akan dianggap bergantung hanya pada keadaan seketika.

Namun begitu proses biologi tidaklah terpisah dari segi masa, kejadian lampau mempengaruhi apa yang berlaku sekarang dan pada masa akan datang. Dalam dunia nyata kadar pertumbuhan sesuatu spesis tidak menunjukkan respon secara serta merta kepada perubahan populasi spesis-spesis yang berinteraksi, tetapi akan menunjukkan respon selepas lengahan masa. Konsep sedemikian perlu diambil kira dan dari sini terbit kajian tentang persamaan pembezaan lengah. Walau bagaimanapun matematik yang diperlukan dalam analisis terperinci mengenai gelagat model sedemikian adalah rumit, terutama bagi ahli biologi yang berkongsi sama dalam kajian ini. Dengan bantuan komputer dan penggunaan *Perisian Mathematica (versi 3.0)*, ciri-ciri utama penyelesaian bagi banyak model yang berkaitan dengan berbagai interaksi dapat diperjelaskan.

CHAPTER I

INTRODUCTION

No creature was created without propose. But every species was created for function does it very well, irrespective of its kind or size. In spite of what many people believe about the interactions of the species, they obey in such interactions high precision mathematical laws, and in many occasions require the development of some subjects and discover new mathematical ideas, which reflects their power, by what *Allah S. W.* has granted them of properties, to coexist and keep the stability of the world far from the pollution and the other environmental problems, even without interference of the Man with his chemical pesticides which are not, usually, free from poisons. Indeed any treatment to change the system of the world, by gaining the species instincts are not created on, will not be safe results, if we do not say reverse. Any interference of the Man should be to support their good properties. This, of course, requires a very good understanding of the species interaction styles and the mathematics related to them.

"The increasing study of realistic mathematical models in ecology (basically the study of the relation between species and their environment) is a reflection of their use in helping to understand the dynamic processes involved in such areas as predator-prey and competing interactions, renewable resource management, ecological control of pests and so on"(Murray, 1989).

As a contribution to achieve such a target, this work is presented, which is divided into six chapters:

Chapter I : Introduction.

Chapter II : Literature Review.

Chapter III : Single Species Population.

Chapter IV : Two Species Interactions.

Chapter V : Some Delay Models in Population Dynamics.

Chapter VI : Summary and Conclusion

All the figures were generated by using *Mathematica* software version 2.2 and then refined by version 3.0. The related computer programmes are listed in the appendix, and the *Mathematica* recent books by Martha and Braselton, Stephen and Kevin et al. are cited in the bibliography for easy referencing .

As many authors did, there is no different sections on mathematical and biological models. That is because of their complementary nature. "Mathematical and biological models complement one another; in the absence of biological models, the mathematical treatment would tend to become more and more abstract and general, and to that extent more difficult to apply. In the absence of mathematical treatment, it would be difficult to see general relevance of particular biological models"(Smith, 1974).

Chapter III : Investigates some single species models starting with *Malthusian* model. Its features are presented. But population cannot grow exponentially for all time, as the model suggests, because the nature resources are limited. This problem is overcome by logistic model which is also discussed with some modifications in order to get more realistic model for single species population.

Chapter **IV** : Investigates the following kinds of interactions between two species :

1. Competition

"Competition is among the most important factors in population dynamics of many species. It often determines the upper limit of fluctuations of population numbers. Also it determines conditions for coexistence of ecologically similar species"(Sharov,1997).

By introducing the system

$$\frac{dx}{dt} = x(t)(b_1 - a_{11}\, x(t) - a_{12}\, y(t))$$

$$\frac{dy}{dt} = y(t)\big(b_2 - a_{22} y(t) - a_{21} x(t)\big)$$

four cases describe the different competition ways between two species are investigated to obtain the conditions under which they can coexist.

2. Predation

"Predation and parasitism are examples of antagonistic ecological interactions in which one species takes advantage of another.

The study of predation parasitism is very important for the following reasons:

1. In many species predation and parasitism are dominating among ecological processes. Dynamics of these populations cannot be predicted and understood without considering natural enemies.
2. Pest species of insects and weeds can be suppressed by introduction of natural enemies or by inundation release nature enemies (biological control).
3. Natural enemies may cause side effects in pesticide applications. The numbers of arthropod natural enemies may be reduced due to pesticide treatment which may result in increasing of pest populations." (Sharov,1996).

Lotka-Volterra model

$$x'(t) = x(t)(b_1 - a_{12}y)$$

$$y'(t) = y(t)(-b_2 + a_{21}x)$$

is investigated. According to this model in the absence of predator the prey will grow unboundedly. This is an unrealistic assumption , therefor, the logistic term should be take into account, which leads to the following model:

$$\frac{dx}{dt} = x\ (b_1 - a_{11}\,x - a_{12}\,y)$$

$$\frac{dy}{dt} = y\ (-b_2 + a_{21}\,x)$$

more modification is introduced in Leslie's model

$$x'(t) = ax - bx^2 - cxy$$

$$y'(t) = ey - fy^2 / x$$

The previous models assume that the predation is proportional with the number of predators as well as the number of prey available, that is the term $a_{12}xy$, however, the consumption rate of predator is limited because a predator spends its time on two kinds of activities :

1. Searching for prey.
2. Prey handling which includes ; chasing, killing, eating and digesting.

This means that even if prey is so abundant, that no time is needed for search, a predator still needs to spend time on prey handling. This phenomenon is called the functional response, and there is another one called numerical response, which means that predators become more abundant as prey density increases. Incorporating these concepts into Leslie's model leads to Holling-Tanner model:

$$dx/dt = rx[(1 - x/k) - wy/(D + x)]$$

$$dy/dt = sy(1 - jy/x)$$

On the other hand, this model introduces an important concept, whom was not available from the previous models, that is the concept of the limit cycle which is more suitable on describing the stable biological cycles than the Lotka-Volterra closed trajectories.

3. Mutualism or Symbiosis:

Here the interactions between two or more species are to the advantage of all, or at least of some. Three mathematical models are investigated to illustrate the three kinds of symbiosis and obtain the necessary conditions for two species to coexist.

Chapter V : Investigates the effect of time delay on the biological processes, by introducing some models for single and two species interactions. That is one of the deficiencies of single population models like logistic model is that the birth rate is considered to act instantaneously where may be a time delay to take account of the time to reach maturity, the finite gestation period and so on. On the other hand, in the two species interactions the response of one species does not happen immediately to changes in its own population or that of an interacting species. So some degree of delay should be contained in the modes to become more realistic.

Chapter **VI** : Gives a summary of the thesis steps and contributions, also suggestions for further research are discussed in this chapter.

CHAPTER II

LITERATURE REVIEW

Reproduction is the main feature of all living organisms. This is what distinguishes them from non-living things. Any model of population dynamics include reproduction.

Single Species Population

Malthusian (Exponential) Model

The population problem has been a world concern since the British economist Malthus (1798) proposed the following exponential model

$$dn/dt = rn(t)$$

which has the following solution

$$n(t) = n_0 \exp(rt).$$

Ever since many researches have carried out studies to justify this model. However, the very recent study about the model by Sharov (1997) will be chosen here.

Exponential model is associated with the name of Thomas Robert Malthus (1766-1834) who first realised that any species can potentially increase in numbers according to a geometric series.

Assumptions of Malthusian Model:

1. Continuous reproduction (e.g., no seasonally).
2. All organisms are identical (e.g., no age structure).
3. Environment is constant in place and time (e.g., resources are unlimited).

However, exponential model is robust; it gives reasonable precision even if these conditions do not met.

Applications of Malthusian Model

1. microbiology (growth of bacteria),
2. conservation biology (restoration of disturbed populations),
3. insect rearing (prediction of yield),
4. plant or insect quarantine (population growth of introduced species),
5. fishery (prediction of fish dynamics).

When the population gets extremely large, this model cannot be very accurate, since it cannot reflect the fact that individual members are now competing with each other limited living, natural resources and food available. Thus we must add a competition term to the linear differential equation.(Braun, 1993).

Logistic Model

A suitable choice of a competition term is $-an^2$, where a is a constant, since the statistical average of the number of encounters of two members per unit time is proportional to n^2, so the modified equation will be of the form

$$dn/dt = n(r - an)$$

or the equivalent form

$$dn/dt = rn(1 - n/k)$$

where $k = r/a$.

This equation is known as the logistic law of population growth. It was first introduced in [1837] by Dutch mathematical-biologist Verhulst (1804-1849).

Exponential and Logistic models help to solve different kinds of problems in ecology, here are some examples: (Sharov, 1996)

1. How long will it take for population to grow to a specific size?
2. What will be the population size after *n* years (or generations)?
3. How long the population can survive at non-favourable conditions?

Two Species Interactions

In [1925], Umberto d'Ancona, Vito Volterra's future son-in-law, a young zoologist and mathematician, pointed out that during World War I the proportion of predator fish caught in the Upper Adriatic Sea was up from before, whereas the proportion of the prey fish was down. Later on, Vito Volterra studied a model describing this phenomenon. Almost at the same time A. J. Lotka studied the same model without being aware of Volterra's work. (Yang H.,1993). So the study of predation has a long history, beginning with the work of Lotka and Volterra and continuing to be of interest today, and their model has been extensively studied by many authors, the following are some of them, Murdie, G. (1976), Stanley et al. (1974), Smith (1974), Ball (1985), Leah (1988), Murray (1989), Paul and Peter (1992) and William and Richard (1992).

Rosenzweig, in [1969] developed a graphical model allowing a general statement of conditions under which the prey null-cline should have positive slope, at low prey densities. This means that the prey curve should have a hump. And in [1971] he studied six models of predator-prey interaction, he included that all show extinction when the peak of the prey zero null-cline lies to the right of the predator zero null-cline. However Gilpin (1972) found that the populations do not go to extinction in this case, but to stable limit cycle.

Essentially all models that have been proposed for predator-prey systems are shown to posses either a stable point equilibrium or a stable limit cycle. This stable limit cycle, an explicitly non-linear feature, is commonly overlooked in conventional analysis of these models. Such a stable limit cycle provides a satisfying explanation for those animal communities in which populations are observed to oscillate in a rather reproducible periodic manner. (May, 1972).

Odum (1953) classified interaction between species in terms of the sings of the effects produced. He characterised the effect of species j upon species i as positive, neutral, or negative (that is, a_{ij} +, 0, or -) depending on whether the population of species i is increased, unaffected, or decreased by the presence of species j. Thus one can construct a table of all possible interaction types between two species (Table 1). This table shows that there are five distinguishably different categories of interactions between any given pair of species, namely

1. Commensalism (+0)
2. Amensalism (- 0)

3. Mutualism or symbiosis (++)

4. Competition (- -)

5. Predation (+ -).

Table 1: The Interaction Types between Two Species

	Effect of species j on i [i,e sign of a_{ij}]		
	+	0	-
Effect of species i on j [i.e sign of a_{ji}] +	++	+0	+-
0	0+	00	0-
-	-+	-0	--

Although the equations discussed here are extremely simple compared to the very complex relationships that exist in nature, so they cannot fully describe these relationships, it is still possible to acquire some insight into ecological principles from a study of such equations (models). When two species interact some times one of them gains and the other dies out, but some times they can coexist. This leads to study the equilibrium solution. Both the concepts of the stability and using geometric analysis are introduced in Chapter III. Chapter IV, however, is devoted to refine these ideas and extend the discussion to systems which deal with more than one species, of the form

$$\left.\begin{aligned} dx/dt &= f_1(x,y) \\ dy/dt &= f_2(x,y) \end{aligned}\right\} \qquad (2.1)$$

where f_1 , f_2 are continuous functions having partial derivatives with respect to x and y, this ensures existence of a unique solution given an initial values for x and y,